Parameters Related to Tree-Width, Zero Forcing, and Maximum Nullity of a Graph

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Abstract: Tree-width, and variants that restrict the allowable tree decompositions, play an important role in the study of graph algorithms and have application to computer science. The zero forcing number is used to study the maximum nullity/minimum rank of the family of symmetric matrices described by a graph. We establish relationships between these parameters, including several Colin de Verdière type parameters, and introduce numerous variations, including the minor monotone floors and ceilings of some of these parameters. This leads to new graph parameters and to new characterizations of existing graph parameters. In particular, tree-width, largeur d'arborescence, path-width, and proper path-width are each characterized in terms of a minor monotone floor of a certain zero forcing parameter defined by a color change rule. J. Graph Theory XX: 1–32, 2012

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1. INTRODUCTION

This paper introduces and studies several new graph parameters that are motivated by the maximum nullity/minimum rank of the family of symmetric matrices described by a graph. These new parameters are related to known parameters, including tree-width, zero forcing number, and Colin de Verdière type parameters. We also obtain new characterizations of existing parameters such as tree-width, largeur d'arborescence, path-width, and proper path-width.

A *graph* $G = (V_G, E_G)$ means a simple undirected graph (no loops, no multiple edges) with a finite nonempty set of vertices V_G and edge set E_G (an edge is a two-element subset of vertices). All matrices discussed are real and symmetric; the set of $n \times n$ real symmetric matrices is denoted by $S_n(\mathbb{R})$. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph* of A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The *set of symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. The *maximum nullity of* G is $\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}$, and the *minimum rank* of G is $\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}$. Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices of G. See Fallat and Hogben [17] for a survey of results and discussion of the motivation for the minimum rank/maximum nullity problem.

The *maximum positive semidefinite nullity of* G is $\mathrm{M}_+(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G), A$ is positive semidefinite$\}$, and the *minimum positive semidefinite rank* of G is $\mathrm{mr}_+(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G), A$ is positive semidefinite$\}$. Clearly $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|$. See, for example, Booth et al. [10] and Hackney et al. [19], and references therein, for more information on the minimum positive semidefinite rank problem.

Observation 1.1. *For every graph* G, $\mathrm{M}_+(G) \leq \mathrm{M}(G)$ *and* $\mathrm{mr}(G) \leq \mathrm{mr}_+(G)$.

The zero forcing number is a useful tool for determining the minimum rank of structured families of graphs and small graphs, and is motivated by simple observations about null vectors of matrices (see [2] where this parameter was introduced). Zero forcing is the same as graph infection used by physicists to study control of quantum systems [12, 30]. Let $G = (V_G, E_G)$ be a graph. A subset $Z \subseteq V_G$ defines an initial set of black vertices (and all the vertices not in Z white). The *color change rule* is to change the color of a white vertex w to black if w is the unique white neighbor of a black vertex u; in this case we say *u forces w*. A *zero forcing set* for G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, applying the color change rule until no more changes are possible turns all the vertices black. The *zero forcing number*, $\mathrm{Z}(G)$, is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V_G$. For any graph G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$ [2, Proposition 2.4].

A *tree decomposition* of a graph $G = (V_G, E_G)$ is a pair $(T, \mathcal{W})$, where T is a tree and $\mathcal{W} = \{W_t : t \in V_T\}$ is a collection of subsets of V_G with the following properties:

(1) $\cup\{W_t : t \in V_T\} = V_G$.
(2) Every edge of G has both ends in some W_t.
(3) If $t_1, t_2, t_3 \in V_T$ and t_2 lies on a path from t_1 to t_3, then $W_{t_1} \cap W_{t_3} \subseteq W_{t_2}$.

The *bags* of the tree decomposition are the subsets W_t. The *width* of a tree decomposition is $\max\{|W_t| - 1 : t \in V_t\}$, and the *tree-width* $\mathrm{tw}(G)$ of G is the minimum width of any tree decomposition of G.

A *k-tree* is constructed inductively by starting with a complete graph on $k + 1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices. Every clique in a k-tree is part of a maximal clique of order $k + 1$, and the k-clique subgraphs of a maximal clique are called its *facets*. We say that two maximal cliques are *adjacent* if they share a facet. A k-tree is a k-connected chordal graph with maximum clique size $k + 1$. It is known (see, e.g., [8]) that for a graph G, $\mathrm{tw}(G)$ is the minimum k such that G is a subgraph of a k-tree.

Section 2 defines additional existing parameters related to maximum nullity, zero forcing number, and tree-width, introduces related new parameters and new characterizations of existing parameters, and establishes some of their properties. In particular, tree-width, largeur d'arborescence, path-width, and proper path-width are each characterized in terms of a minor monotone floor of a certain zero forcing number. The new characterizations may assist in the computation of these parameters. Open questions are presented in Section 3. Appendix A contains many additional examples, including examples to show that all but possibly one of the new inequalities established in Section 2 are strict, thereby establishing that certain parameters are distinct, and examples showing noncomparability of parameters.

Because there are many parameters and the relationships are quite complicated, Table I summarizes the notation for the parameters discussed, and Fig. 1 describes the relationships between these parameters, for graphs that have at least one edge. For each parameter, the third column of Table I gives the location in this paper of the definition; when a parameter was discussed in prior works, a reference in which it was defined is also listed. A parameter that is not equal to another parameter in the table and for which no external reference is given is new in this paper. Where parameters are described as equal in the table or figure, this is for graphs that contain at least one edge.

When a parameter is listed with two names in Fig. 1, e.g., $\mathrm{la}(G) = \lfloor \mathrm{Z}_+ \rfloor(G)$, the theorem that justifies the equality is listed in Table I in the entry for the second named parameter, e.g., $\lfloor \mathrm{Z}_+ \rfloor(G)$. In Fig. 1, a line between two parameters q, p means that for all graphs G, $q(G) \le p(G)$, where q is below p in the diagram. Each line has two numbers adjacent to it. The upper number is the result that justifies the line, i.e., the parameter bound, except in the case of the dashed line of small triangles, which is Conjecture 2.13. In all but one case, the lower number references an example from Section A of the Appendix showing that the inequality can be strict. In that one case, the notation ?3.5 references Question 3.5, since the strictness of that inequality is an open question. When there is not a monotone sequence of lines between q and p in Fig. 1, then in most cases these parameters are not comparable. These noncomparability results are established in Section B of the Appendix.

Parameters that involve matrices, such as the maximum nullity $\mathrm{M}(G)$, require taking a maximum or minimum over an infinite family of matrices, and are thus usually more difficult to compute than purely combinatorial parameters, such as the zero forcing number $\mathrm{Z}(G)$. Software implementing known bounds on minimum rank is available [13], but may not produce the exact value unless the upper and lower bounds coincide. This software includes functions to compute $\mathrm{Z}(G)$ and $\mathrm{Z}_+(G)$.

We need the following additional graph terminology. We denote the complete graph on n vertices by K_n and the cycle on n vertices by C_n. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two-element sets from V that are not in E. The *union* of $G_i = (V_i, E_i)$ is $\cup_{i=1}^h G_i = (\cup_{i=1}^h V_i, \cup_{i=1}^h E_i)$; a disjoint union is denoted $\dot\cup_{i=1}^h G_i$. The *intersection* of $G_i = (V_i, E_i)$ is $\cap_{i=1}^h G_i = (\cap_{i=1}^h V_i, \cap_{i=1}^h E_i)$ (provided the intersection of the vertex sets is nonempty). If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are disjoint graphs, the *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph having vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E$, where E consists of all the edges $\{u, v\}$ with $u \in V(G_1)$, $v \in V(G_2)$. A subgraph $H = (V_H, E_H)$ of a graph $G = (V_G, E_G)$ is a *spanning subgraph* of G if $V_H = V_G$. For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ is usually denoted by $G - W$, or in the case W is a singleton $\{v\}$, by $G - v$. Vertex v is a *neighbor*

TABLE I. Summary of graph parameter definitions.

Symbol	Name	Definition #, Section §#, or external reference [#]
$\mathrm{Z}(G)$	Zero forcing number	[2], §1
$\widehat{\mathrm{Z}}(G)$	Enhanced zero forcing number	2.23
$\mathrm{Z}_\ell(G)$	Loop zero forcing number	2.28
$\mathrm{Z}_+(G)$	Positive semidefinite zero forcing number	[3], §2.E
$\mathrm{ppw}(G)$	Proper path-width	[31], §D
$\lfloor \mathrm{Z} \rfloor(G)$	Minor monotone floor of zero forcing number $=$ proper path-width $\mathrm{ppw}(G)$, Theorem 2.39	§1, 2.6
CCR-$\lfloor \mathrm{Z} \rfloor(G)$	Defined by zero forcing rule for minor monotone floor of zero forcing number $=$ proper path-width $\mathrm{ppw}(G)$, Theorem 2.38	2.23
$\mathrm{lc}(G)$	Largeur de chemin $=$ proper path-width $\mathrm{ppw}(G)$, Theorem 2.18	2.15
$\lfloor \widehat{\mathrm{Z}} \rfloor(G)$	Minor monotone floor of enhanced zero forcing number	F and 2.23
$\mathrm{pw}(G)$	Path-width	[16], §2.D
$\lfloor \mathrm{Z}_\ell \rfloor(G)$	Minor monotone floor of loop zero forcing number $=$ path-width $\mathrm{pw}(G)$, Theorem 2.45	2.6 and 2.28
CCR-$\lfloor \mathrm{Z}_\ell \rfloor(G)$	Defined by zero forcing rule for minor monotone floor of loop zero forcing number $=$ path-width $\mathrm{pw}(G)$, Theorem 2.44	2.41
$\mathrm{la}(G)$	Largeur d'arborescence	[15], §2.D
$\mathrm{tstw}(G)$	Two-sided (straight) tree-width $=$ largeur d'arborescence $\mathrm{la}(G)$, [33]	[33], §2.D
$\lfloor \mathrm{Z}_+ \rfloor(G)$	Minor monotone floor of positive semidefinite zero forcing number $=$ largeur d'arborescence $\mathrm{la}(G)$, Theorem 2.51	2.6 and §2.E
CCR-$\lfloor \mathrm{Z}_+ \rfloor(G)$	Defined by zero forcing rule for minor monotone floor of positive semidefinite zero forcing number $=$ largeur d'arborescence $\mathrm{la}(G)$, Theorem 2.50	2.48
$\mathrm{tw}(G)$	Tree-width	[16], §1, §2.D
CCR-$\mathrm{tw}(G)$	Defined by zero forcing rule for tree-width $=$ tree-width $\mathrm{tw}(G)$, Corollary 2.57	2.55
$\mathrm{M}(G)$	Maximum nullity	[28], §1
$\mathrm{M}_+(G)$	Positive semidefinite maximum nullity	[34], §1
$\lfloor \mathrm{M} \rfloor(G)$	Minor monotone floor of maximum nullity	§2.C
$\lfloor \mathrm{M}_+ \rfloor(G)$	Minor monotone floor of positive semidefinite maximum nullity	§2.C
$\xi(G)$	Colin de Verdière type analog of $\mathrm{M}(G)$	[6], §2.B
$\mu(G)$	Colin de Verdière number	[14], §2.B
$\nu(G)$	Colin de Verdière type analog of $\mathrm{M}_+(G)$	[15], §2.B
$\lceil \delta \rceil(G)$	Minor monotone ceiling of minimum degree	[18], §2.C
$\lceil \kappa \rceil(G)$	Minor monotone ceiling of vertex connectivity	[18], §2.C
$\delta(G)$	Minimum degree	[16], §1
$\kappa(G)$	Vertex connectivity	[16], §1
$\mathrm{P}(G)$	Path cover number	[21], §2.A
$\mathrm{h}(G) - 1$	Hadwiger number minus one	[16], §2.B

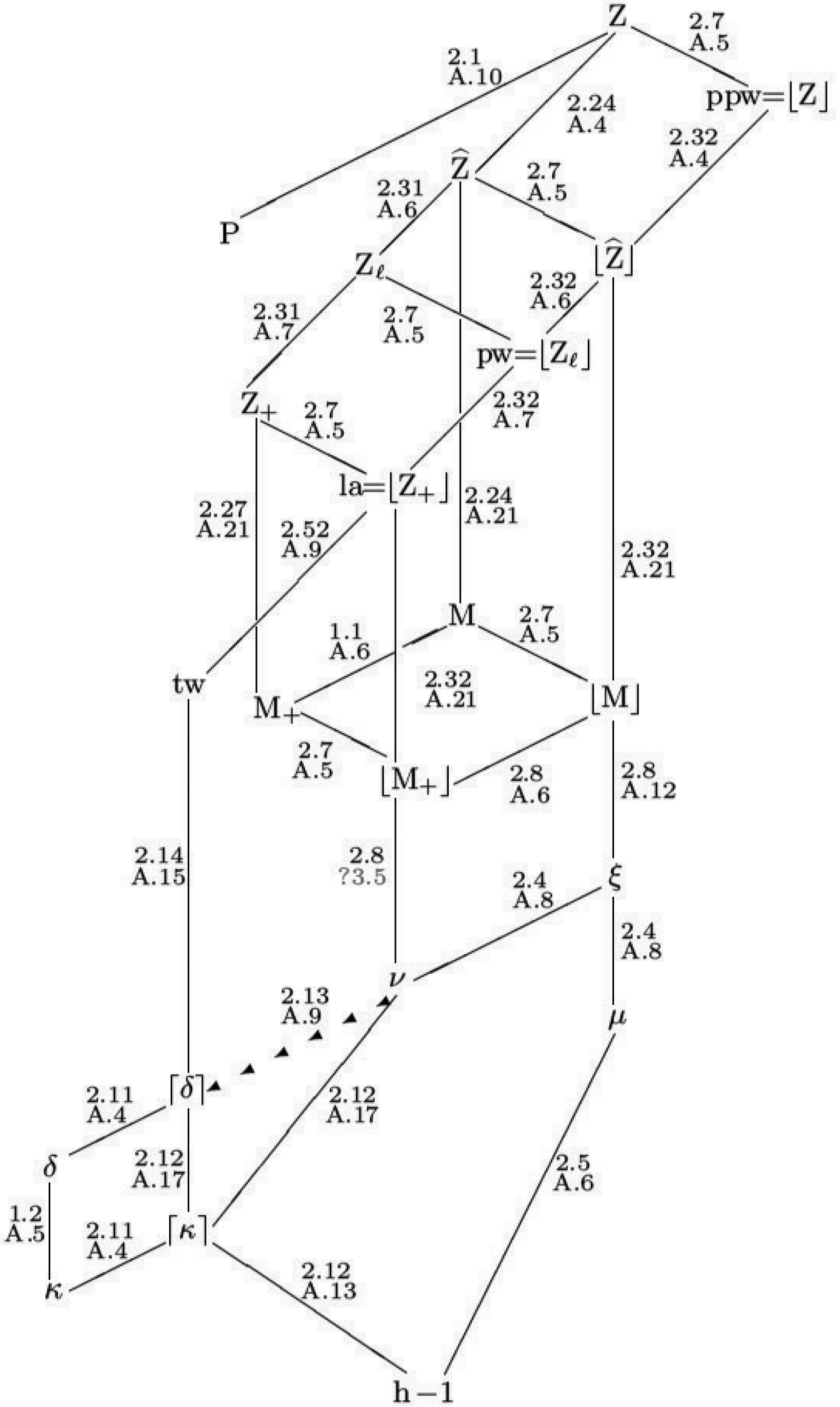

FIGURE 1. Relationships between parameters related to maximum nullity.

of u (and vice versa) if $\{u, v\}$ is an edge of G; this is denoted by $v \sim u$. The *degree* of vertex v in graph G, $\deg_G v$, is the number of neighbors of v; a *leaf* is a vertex of degree 1. Let $\delta(G)$ denote the minimum degree of the vertices of G. Let $\kappa(G)$ denote the *vertex connectivity of* G, i.e., if G is not complete, the smallest number k such that there is a set of vertices S, with $|S| = k$, for which $G - S$ is disconnected. By convention, $\kappa(K_r) = r - 1$. If $\kappa(G) \geq k$, then G is *k-connected.*

Observation 1.2. *It is well-known that for every graph G, $\kappa(G) \leq \delta(G)$ (see, e.g., [16, p. 20]). Results of Lovász et al. [25] and [26] imply that $\kappa(G) \leq \mathrm{M}_+(G)$.*

A graph is *planar* if it can be drawn in the plane without crossing edges. Given two graphs G and H, the *Cartesian product* of G and H, denoted by $G \,\square\, H$, is the graph whose vertex set is the Cartesian product of V_G and V_H, with an edge between two vertices exactly when they are identical in one coordinate and adjacent in the other.

2. PARAMETERS

A. Path Cover Number

The *path cover number* $\mathrm{P}(G)$ of G is the smallest positive integer m such that there are m vertex-disjoint induced paths $P_1, \ldots, P_m$ in G that cover all the vertices of G (i.e., $V_G = \dot\bigcup_{i=1}^{m} V_{P_i}$). Path cover number was first used in the study of minimum rank and maximum multiplicity of an eigenvalue in Johnson and Leal Duarte [21] (since the matrices in $\mathcal{S}(G)$ are symmetric, algebraic, and geometric multiplicities of eigenvalues are the same, and since the diagonal is free, maximum multiplicity is the same as maximum nullity).

In Johnson and Leal Duarte [21] it was shown that for a tree T, $\mathrm{P}(T) = \mathrm{M}(T)$. In Barioli et al. [5] it was shown that for graphs in general, $\mathrm{P}(G)$ and $\mathrm{M}(G)$ are not comparable (see Examples A.2 and A.3).

Proposition 2.1. *[3, Proposition 2.10] For any graph G,* $\mathrm{P}(G) \leq \mathrm{Z}(G)$.

B. Colin De Verdière Type Parameters

In 1990 Colin de Verdière ([14], in English) introduced the graph parameter μ that is equal to the maximum nullity among all matrices satisfying several conditions including the Strong Arnold Hypothesis (defined below). The parameter μ, which is used to characterize planarity, is the first of several parameters (called *Colin de Verdière type parameters*) that require the Strong Arnold Hypothesis and that bound the maximum nullity from below.

The *contraction* of an edge $e = \{u, v\}$ of G, denoted by G/e, is obtained by identifying the vertices u and v, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A *minor* of G arises by performing a sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges. The notation $H \preceq G$ means that H is a minor of G, and $H \prec G$ means that H is a *proper minor* of G, i.e., $H \preceq G$ and $H \neq G$. A graph parameter β is *minor monotone* if for any minor H of G, $\beta(H) \leq \beta(G)$. All the Colin de Verdière type parameters we discuss have been shown to be minor monotone.

A real symmetric matrix A satisfies the *Strong Arnold Hypothesis* provided there does not exist a nonzero real symmetric matrix X satisfying $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where $\circ$ denotes the Hadamard (entrywise) product and I is the identity matrix. The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally (see [36]). The Colin de Verdière number $\mu(G)$ is defined to be the maximum nullity among symmetric matrices $A = [a_{ij}]$ such that: $A \in \mathcal{S}(G)$; A satisfies the Strong Arnold Hypothesis; for all $i \neq j$, $a_{ij} \leq 0$; and A has exactly one negative eigenvalue (counting multiplicity). Another minor monotone parameter, introduced by Colin de Verdière in [15], is denoted by $\nu(G)$ and defined to be the maximum nullity among matrices A such that: $A \in \mathcal{S}(G)$; A satisfies the Strong Arnold Hypothesis; and A is positive semidefinite. It is evident that for any graph G, $\nu(G) \leq \mathrm{M}_+(G)$. As noted in van der Holst [35], the following result can be derived from Lovász et al. [25].

Theorem 2.2. *[25, 35] For every graph G, $\kappa(G) \leq \nu(G)$.*

It is easy to see that the zero forcing number is an upper bound for the minimum degree, i.e., $\delta(G) \leq \mathrm{Z}(G)$ [7, Proposition 4.1]. It was conjectured [1] that for any graph G, $\delta(G) \leq M(G)$, or equivalently $\mathrm{mr}(G) \leq |G| - \delta(G)$. This conjecture, often referred to as the *delta conjecture*, was proved for bipartite graphs in Berman et al. [7] but remains open in general. In Lovász et al. [25] it is reported that in 1987, Maehara made a stronger conjecture equivalent to $\delta(G) \leq \mathrm{M}_+(G)$. Here we conjecture an even stronger claim.

Conjecture 2.3. *For every graph G, $\delta(G) \leq \nu(G)$.*

The parameter $\xi(G)$ was introduced in Barioli et al. [6] as a Colin de Verdière type parameter intended for use in computing maximum nullity and minimum rank, by removing any unnecessary restrictions while preserving minor monotonicity. Define $\xi(G)$ to be the maximum nullity among real symmetric matrices such that $A \in \mathcal{S}(G)$ and A satisfies the Strong Arnold Hypothesis. In Barioli et al. [6] it is shown that the parameter $\xi(G)$ is minor monotone.

Observation 2.4. *For every graph G, $\nu(G) \leq \xi(G)$, $\mu(G) \leq \xi(G)$, and $\xi(G) \leq \mathrm{M}(G)$.*

The *Hadwiger number* of a graph G is the largest k for which K_k is a minor of G, and is denoted by $\mathrm{h}(G)$ (see [16]).

Observation 2.5. *For every graph G,* $\mathrm{h}(G) - 1 \leq \mu(G)$ *and* $\mathrm{h}(G) - 1 \leq \nu(G)$. *Consequently,* $\mathrm{h}(G) - 1 \leq \mathrm{M}_+(G) \leq \mathrm{M}(G)$.

C. Minor Monotone Floor and Ceiling

As seen in Section B, Colin de Verdière's work led to the study of numerous parameters involving the Strong Arnold Hypothesis, since a variety of matrix-based graph parameters acquire the very nice property of being minor monotone if one restricts consideration to matrices that satisfy the Strong Arnold Hypothesis.

For a minor monotone graph parameter β and nonnegative integer k, define $\mathrm{Forb}_k(\beta)$ to be the set of all graphs G such that $\beta(G) > k$, but every proper minor H of G has $\beta(H) \leq k$. Then $\beta(G) \leq k$ if and only if there is no H in $\mathrm{Forb}_k(\beta)$ such that $H \preceq G$. Robertson and Seymour's famous result tells us that for any minor monotone graph parameter β and any k, $\mathrm{Forb}_k(\beta)$ is a finite set (see, for example, [16, Corollary 12.5.3]). For every minor monotone parameter β that we discuss, it will be seen that $\mathrm{Forb}_k(\beta)$

contains K_{k+2}. Since $\mathrm{Forb}_k(\mathrm{h}-1) = \{K_{k+2}\}$, all the minor monotone parameters that we discuss can be viewed as generalizations of $\mathrm{h}(G) - 1$.

For any graph parameter taking values in the natural numbers (or in fact any well-ordered set), a direct conversion to a minor monotone parameter can be made in two possible ways, by taking a minimum over all graphs H that contain G as a minor, called the minor monotone floor (this relates naturally to the Colin de Verdière type parameters), or by taking a maximum over all graphs H that are minors of G, called the minor monotone ceiling. Although the terms "minor monotone floor" and "minor monotone ceiling" are not used, the definitions given below of these terms appear in Fijavž and Wood [18], p. 80. The minor monotone floor of the crossing number, called the minor crossing number, was introduced and studied in Bokal et al. [9].

Definition 2.6. Let p be a graph parameter whose range is well-ordered. The *minor monotone floor* of p is $\lfloor p \rfloor(G) = \min\{p(H) : G \preceq H\}$.

Observation 2.7. *For any graph parameter p whose range is well-ordered, the minor monotone floor $\lfloor p \rfloor$ is characterized by the following three properties:*

(1) If $G \preceq H$, then $\lfloor p \rfloor(G) \le \lfloor p \rfloor(H)$.
(2) For any graph G, $\lfloor p \rfloor(G) \le p(G)$.
(3) For any graph G, there exists a graph H such that $G \preceq H$ and $\lfloor p \rfloor(G) = p(H)$.

We consider the minor monotone floors $\lfloor \mathrm{M} \rfloor(G)$ and $\lfloor \mathrm{M}_+ \rfloor(G)$ (and in Section F also the minor monotone floors of several zero forcing parameters).

Observation 2.8. *If p and q are graph parameters such that for all graphs G, $q(G) \le p(G)$, then $\lfloor q \rfloor(G) \le \lfloor p \rfloor(G)$. If q is a minor monotone graph parameter, then $q(G) = \lfloor q \rfloor(G)$. In particular, for all graphs G,*

(1) $\lfloor \mathrm{M}_+ \rfloor(G) \le \lfloor \mathrm{M} \rfloor(G)$, since $\mathrm{M}_+(G) \le \mathrm{M}(G)$;
(2) $\xi(G) \le \lfloor \mathrm{M} \rfloor(G)$, since $\xi(G) \le \mathrm{M}(G)$;
(3) $\nu(G) \le \lfloor \mathrm{M}_+ \rfloor(G)$, since $\nu(G) \le \mathrm{M}_+(G)$.

Example 2.9. Let G_6 and G_6' be the graphs shown in Fig. 2.

It is easy to see that $\mathrm{M}_+(G_6) = \mathrm{M}(G_6) = 3$, and $\mathrm{M}(G_6') = 2$ because G_6' is a 2-connected partial linear 2-tree [20] (linear k-trees are defined in Section D below). Since G_6 is a minor of G_6', $\lfloor \mathrm{M} \rfloor(G_6) \le \mathrm{M}(G_6') \le 2$. Since G_6 has a K_3 minor, $2 = \nu(K_3) \le \lfloor \mathrm{M}_+ \rfloor(K_3) \le \lfloor \mathrm{M}_+ \rfloor(G_6) \le \lfloor \mathrm{M} \rfloor(G_6)$. Thus, $\lfloor \mathrm{M}_+ \rfloor(G_6) = \lfloor \mathrm{M} \rfloor(G_6) = 2$. For use in Section B of the Appendix, we also note that $\mathrm{P}(G_6) = 3$.

Note that it is not always useful to consider the minor monotone floor for a graph parameter. For example, $\lfloor \delta \rfloor(G) = 0$ because $H = G \dot\cup K_1$ has G as a minor and $\delta(H) = 0$.

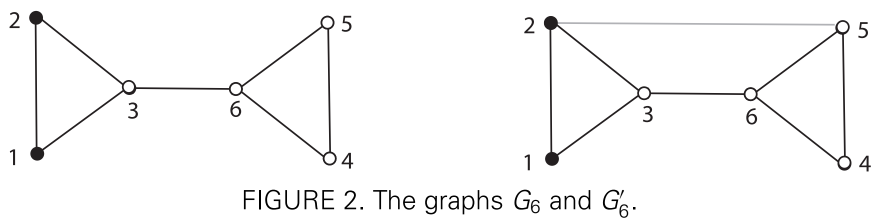

FIGURE 2. The graphs G_6 and G_6'.

Less trivially, for every graph G, $\lfloor \mathrm{P} \rfloor(G) \le 2$. Consider the join of two disjoint copies of a path on n vertices, denoted by $P_n \vee P_n$. Clearly $\mathrm{P}(P_n \vee P_n) \le 2$. Denote the vertices of the two paths by $U = \{u_1, \ldots, u_n\}$ and $W = \{w_1, \ldots, w_n\}$, where each path has vertices in numerical order. Contracting the edges $\{u_i, w_i\}$, $i = 1, \ldots, n$ produces the complete graph K_n. Any graph G of order n is a minor of K_n, so $\lfloor \mathrm{P} \rfloor(G) \le 2$. Thus, $\lfloor \mathrm{P} \rfloor(G)$ is uninteresting and will not be discussed further.

Definition 2.10. Let p be a graph parameter with a totally ordered range. The *minor monotone ceiling* of p is $\lceil p \rceil(G) = \max\{p(H) : H \preceq G\}$.

Observation 2.11. *For any graph parameter p, the minor monotone ceiling $\lceil p \rceil$ is characterized by three properties:*

(1) If $H \preceq G$, then $\lceil p \rceil(H) \le \lceil p \rceil(G)$.
(2) For any graph G, $p(G) \le \lceil p \rceil(G)$.
(3) For any graph G, there exists $H \preceq G$ such that $\lceil p \rceil(G) = p(H)$.

We consider the minor monotone ceilings $\lceil \delta \rceil(G)$ and $\lceil \kappa \rceil(G)$. Note that it is not useful to consider the minor monotone ceiling for maximum nullity, since $\lceil \mathrm{M} \rceil(G) = |G|$ because the discrete graph $\overline{K_{|G|}}$ is a minor of G.

Observation 2.12. *If p and q are graph parameters such that for all graphs G, $q(G) \le p(G)$, then $\lceil q \rceil(G) \le \lceil p \rceil(G)$. If q is a minor monotone graph parameter, then $q(G) = \lceil q \rceil(G)$. In particular:*

(1) $\lceil \kappa \rceil(G) \le \lceil \delta \rceil(G)$, since $\kappa(G) \le \delta(G)$;
(2) $\mathrm{h}(G) - 1 \le \lceil \kappa \rceil(G)$, since the $K_{\mathrm{h}(G)}$ minor of G implies $\kappa(K_{\mathrm{h}(G)}) \le \lceil \kappa \rceil(G)$;
(3) $\lceil \kappa \rceil(G) \le \nu(G)$, since $\kappa(G) \le \nu(G)$.

Since ν is minor monotone, Observation 2.12 implies that Conjecture 2.3 is equivalent to the following:

Conjecture 2.13. *For every graph G, $\lceil \delta \rceil(G) \le \nu(G)$.*

Since $\kappa(G) \le \nu(G)$, and ν is minor monotone, any counterexample to Conjecture 2.13 would necessarily have $\lceil \delta \rceil(G) > \lceil \kappa \rceil(G)$, which is unusual in small graphs. The Mader graph M_{12} in Example A.16 has $\lceil \delta \rceil(M_{12}) = 5 > 4 = \lceil \kappa \rceil(M_{12})$, and we show $\nu(M_{12}) = 5$.

D. Tree-width

Tree-width is one of the most widely studied minor monotone graph parameters and also plays an important role in the Graph Minor Theorem. It is clear from the constructive definition of a k-tree that if G is a subgraph of a k-tree, then $\delta(G) \le k$. Thus, for every graph G, $\delta(G) \le \mathrm{tw}(G)$. The next result now follows from Observation 2.12 and the fact that tree-width is minor monotone [16, Proposition 12.3.6].

Corollary 2.14. *For every graph G, $\lceil \delta \rceil(G) \le \mathrm{tw}(G)$.*

Tree-width can be viewed as a zero forcing parameter, as defined in Section E below, and several zero forcing parameters can be characterized as the minimum k for which the graph is a subgraph of a certain type of k-trees. Here we define the types of k-trees needed for these characterizations.

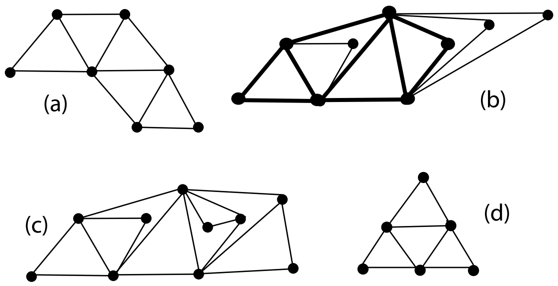

FIGURE 3. (a) a linear 2-tree, (b) a 2-caterpillar that is not a linear 2-tree, (c) a two-sided 2-tree that is not a 2-caterpillar, and (d) a 2-tree (the supertriangle T_3) that is not a two-sided 2-tree.

A *linear k-tree* is constructed inductively by starting with K_{k+1} and connecting each new vertex to the vertices of an existing K_k that includes a vertex of degree k (it can be assumed that the new vertex is adjacent to the vertex just added); an example with $k = 2$ is shown in Fig. 3(a). Equivalently, a linear k-tree is either K_{k+1} or a k-tree in which exactly two vertices have degree k. Another equivalent definition of a linear k-tree is a k-tree in which no more than two maximal cliques are adjacent along any given facet and every maximal clique is adjacent to at most two other maximal cliques. A *partial linear k-tree* is a subgraph of a linear k-tree. Partial linear k-trees have been used to characterize forbidden minors for certain values of $\xi(G)$ [20,22,35]. The *proper path-width* of a graph G, denoted by $\mathrm{ppw}(G)$, is the minimum k for which G is a partial linear k-tree. Proper path-width was introduced in Takahashi et al. [31], where acyclic forbidden minors for proper path-width k were characterized.

A *k-caterpillar* is constructed inductively by starting with K_{k+1} and at each stage adding a new maximal clique by adjoining a new vertex to the k vertices of some facet of the maximal clique that was added at the previous stage; an example with $k = 2$ is shown in Fig. 3(b). A k-caterpillar can also be constructed by first constructing a linear k-tree L, and then possibly adding extra vertices, with each new vertex joined to a facet that is shared by two maximal cliques in L; L is called an *underlying* linear k-tree of the k-caterpillar. In Fig. 3(b), the thicker lines and larger dots are used for the edges and vertices of an underlying linear k-tree. A k-tree is a k-caterpillar if and only if (1) each maximal clique is adjacent to other maximal cliques along at most two facets; and (2) all but at most two of those adjacent maximal cliques have a vertex of degree k. The *path-width* of a graph G, denoted by $\mathrm{pw}(G)$, is the minimum width of a tree decomposition of G for which the tree is a path. Equivalently, the path-width of a graph G is the minimum k for which G is a subgraph of a k-caterpillar.

A *two-sided k-tree* is constructed inductively by starting with K_{k+1} and connecting each new vertex to the vertices of an existing K_k that either includes a vertex of degree k or is the same as the K_k to which some previous vertex was connected; an example with $k = 2$ is shown in Fig. 3(c). A two-sided k-tree is also called a *straight k-tree* (there is a distinction between these two terms for graphs with multiple edges that is irrelevant here). A k-tree is a two-sided k-tree if and only if each maximal clique has a maximum of two facets along which it is adjacent to any other maximal clique. Since K_2 has only two K_1-facets, every tree is a two-sided 1-tree. The *two-sided tree-width* (also called the *straight tree-width*) of a graph G is the minimum k for which G is a subgraph of a two-sided k-tree, and is denoted by $\mathrm{tstw}(G)$. Equivalently, the two-sided tree-width of a graph G is the minimum width of a two-sided tree decomposition of G (see [33] for the definition of two-sided tree decomposition).

Clearly any linear k-tree is a k-caterpillar, any k-caterpillar is a two-sided k-tree, and any two-sided k-tree is a k-tree, but not vice versa, as illustrated in Fig. 3.

The *largeur d'arborescence* of G, $\mathrm{la}(G)$, was defined by Colin de Verdière in [15] as the minimum k for which G is a minor of the Cartesian product $K_k \square T$ of a complete graph on k vertices with a tree. The largeur d'arborescence is equal to two-sided (straight) tree-width, i.e., for any graph G that has at least one edge, $\mathrm{tstw}(G) = \mathrm{la}(G)$ [33]. We now introduce the largeur de chemin as an obvious analogy to largeur d'arborescence for path-width.

Definition 2.15. The *largeur de chemin* of G, denoted by $\mathrm{lc}(G)$, is the minimum k for which G is a minor of the Cartesian product $K_k \square P$ of a complete graph on k vertices with a path.

We will see that largeur de chemin is identical to proper path-width (Theorem 2.18).

Lemma 2.16. *If G is a partial linear k-tree (respectively, partial k-caterpillar) of order at least $k + 1$, then G is a spanning subgraph of a linear k-tree (respectively, k-caterpillar).*

Proof. Since G is a partial linear k-tree (respectively, partial k-caterpillar), G is a subgraph of a linear k-tree (k-caterpillar) L'. If L' contains one or more vertices not in G, we show how to produce a linear k-tree (k-caterpillar) L'' still containing G as a subgraph and having one fewer vertex. Repeated application of this process yields a linear k-tree (k-caterpillar) L such that $V_L = V_G$.

We first consider a linear k-tree, and number the vertices $1, \ldots, k+1, k+2, \ldots, t$ as follows: vertex $k + 1 + i$ is the vertex added at the ith stage in the construction of L (from the definition of linear k-tree). The vertices $1, \ldots, k + 1$ of the base maximal clique are numbered such that they "drop off" the current K_{k+1} in order as we move through the stages (it need not be the case that a vertex from the base maximal clique drops off every time we move to the next stage). That is, if $1 \leq r < q \leq k + 1$ and $v_q \not\sim v_\ell$, then $v_r \not\sim v_\ell$. If $k + 1 < p < t$, denote by $m(p)$ the unique vertex such that $m(p) < p$, $p \sim m(p)$ and $p + 1 \not\sim m(p)$; for $1 < p \leq k + 1$, let $m(p) = 1$.

Suppose $p \in V_{L'}$ and $p \notin V_G$. If $p = 1$ or $p = t$, then just delete p; the result is the linear k-tree L''. For $1 < p < t$, let L'' be the linear k-tree obtained from L' by contracting $\{m(p), p\}$. Then L'' is a linear k-tree that contains G as a subgraph and has one fewer vertex not in G.

For the case of a k-caterpillar L', number all the vertices of an underlying linear k-tree $1, \ldots, s$ as above, and the remaining vertices $s + 1, \ldots, t$. If $\deg_{L'} p = k$, simply delete p; otherwise, follow the procedure for a linear k-tree to select the edge to contract. ■

Observation 2.17. *Any minor of a partial linear k-tree is a partial linear k-tree, because proper path-width is minor monotone [31]. Any minor of a partial k-caterpillar is a partial k-caterpillar, because path-width is minor monotone [8]. Any minor of a partial two-sided k-tree is a partial two-sided k-tree, because two-sided tree-width is minor monotone [33].*

Theorem 2.18. *For any graph G that has at least one edge,* $\mathrm{ppw}(G) = \mathrm{lc}(G)$.

Proof. It is sufficient to show that for any path P, $K_k \square P$ is a minor of a linear k-tree, and that every linear k-tree is a minor of $K_k \square P$ for some path P. For the special case where P is a path on two vertices, call $K_k \square P$ a *k-prism*, with the two copies of K_k called the *ends* of the k-prism. For longer paths P, $K_k \square P$ is obtained by "gluing" multiple k-prisms along their ends. To each K_{k+1} in a linear k-tree we also assign two ends, a pair of facets that include any K_k along which it is adjacent to another maximal clique. One direction of inclusion of minors comes from the fact that each k-prism can be triangulated to (and is thus a minor of) a linear k-tree having the same ends as the original k-prism—the triangulation consists of k maximal cliques glued along $k-1$ facets. Conversely, a maximal clique with two facets specified as ends is a minor of a k-prism with the same ends, and hence a linear k-tree with n maximal cliques is a minor of $K_k \square P$ with n k-prisms. ■

A *smooth* tree decomposition is one in which all bags contain the same number of vertices. If a two-sided tree decomposition of width k exists, then a smooth two-sided tree decomposition of width k exists [33].

Proposition 2.19. *If G is a partial two-sided k-tree of order at least $k+1$, then G is a spanning subgraph of a two-sided k-tree.*

Proof. Choose a smooth two-sided tree decomposition for G, and insert any additional edges needed so that the vertices in each bag become a clique. ■

E. Zero Forcing Parameters

In this section, we discuss several graph parameters that can be viewed as generalizations of the zero forcing number. In each case, starting with a set of vertices colored black (the remainder starting out white) there are certain moves (governed by a color change rule) that allow white vertices to be changed to black. The value of the parameter is the minimum size of a black set that eventually allows all vertices to become black. Since we wish to generalize the zero forcing number, we restate the definition of the zero forcing number $\mathrm{Z}(G)$ in parts, with the color change rule, which varies with the parameter, separated from other parts of the definition.

Definition 2.20. Let G be a graph (or loop graph, see below).

(1) A subset $Z \subseteq V_G$ defines a coloring by coloring all vertices in Z black and all the vertices not in Z white.
(2) A color change rule is a rule describing conditions on a vertex u and its neighbors under which u can cause the color of a white vertex w to change to black. In this case we say u forces w and write $u \rightarrow w$.
(3) Given a coloring of G and a color change rule CCR-p, a CCR-p *derived set* is a set of black vertices obtained by applying CCR-p until no more changes are possible.
(4) A CCR-p *zero forcing set* for G is a subset Z of vertices such that if initially the vertices in Z are colored black and the remaining vertices are colored white, V_G is a CCR-p derived set.
(5) The CCR-p *zero forcing parameter* is the minimum of $|Z|$ over all CCR-p zero forcing sets $Z \subseteq V_G$.

We can recast the definition given in Section 1 of the zero forcing number $\mathrm{Z}(G)$ in this generalized form. The color change rule CCR-Z is as follows:

CCR-Z If u is a black vertex, and exactly one neighbor w of u is white, then change the color of w to black.

In order to obtain an improved bound on M, we consider graphs that allow loops. A *loop graph* is a graph that allows loops, i.e., $\widehat{G} = (V_{\widehat{G}}, E_{\widehat{G}})$ where $V_{\widehat{G}}$ is the set of vertices of $\widehat{G}$ and the set of edges $E_{\widehat{G}}$ is a set of two-element multisets. A vertex u is a *neighbor* of vertex v in $\widehat{G}$ if $\{u, v\} \in E_{\widehat{G}}$; note that u is a neighbor of itself if and only if the loop $\{u, u\}$ is an edge. The *underlying simple graph* of a loop graph $\widehat{G}$ is the graph G obtained from $\widehat{G}$ by deleting all loops. The *set of symmetric matrices described by a loop graph* $\widehat{G}$ is $\mathcal{S}(\widehat{G}) = \{A = [a_{ij}] \in S_n(\mathbb{R}) : a_{ij} \neq 0 \Leftrightarrow \{i, j\} \in E_{\widehat{G}}\}$, and the *maximum nullity of* G is $\mathrm{M}(\widehat{G}) = \max\{\operatorname{null} A : A \in \mathcal{S}(\widehat{G})\}$. Note that a loop graph $\widehat{G}$ constrains the zero–nonzero pattern of the main diagonal entries of matrices described by $\widehat{G}$. There is a distinction between a graph, i.e., a simple graph, and a loop graph that has no loops—the latter forces the matrices to have zero diagonal, whereas the former does not.

Definition 2.21. The CCR-Z($\widehat{G}$) is:

CCR-Z($\widehat{G}$) If exactly one neighbor w of u is white, then change the color of w to black.

The *zero forcing number of a loop graph* $\widehat{G}$, denoted by $\mathrm{Z}(\widehat{G})$, is the zero forcing parameter for CCR-Z($\widehat{G}$).

In a simple graph, a vertex must be black to force another vertex because it is unknown whether the diagonal entry is zero or nonzero, whereas in a loop graph it is known whether the diagonal entry is zero or nonzero, and this information is used in the color change rule. The proof of the following theorem is similar to that of Barioli et al. [2, Proposition 2.4].

Theorem 2.22. *For any loop graph* $\widehat{G}$, $\mathrm{M}(\widehat{G}) \leq \mathrm{Z}(\widehat{G})$.

The zero forcing number for digraphs (which allow loops) was introduced in Barioli et al. [4]. The zero forcing number of a loop graph is the same as the zero forcing number of the associated doubly directed digraph. The *nonzero pattern* of a digraph (or loop graph) with vertex set $\{1, \ldots, n\}$ is the $n \times n$ matrix having (u, v)-entry equal to $*$ if uv is an arc (edge) of the digraph and zero otherwise (* indicates a nonzero entry in the associated family of matrices). As shown in Barioli et al. [4], $|\widehat{G}| - \mathrm{Z}(\widehat{G})$ is equal to the triangle number (size of the largest triangle) of the pattern of $\widehat{G}$ for a loop graph $\widehat{G}$.

Definition 2.23. The *enhanced zero forcing number* of a graph G, denoted by $\widehat{\mathrm{Z}}(G)$, is the maximum of $\mathrm{Z}(\widehat{G})$ over all loop graphs $\widehat{G}$ such that the underlying simple graph of $\widehat{G}$ is G.

Corollary 2.24. *For any graph* G, $\mathrm{M}(G) \leq \widehat{\mathrm{Z}}(G) \leq \mathrm{Z}(G)$.

Proof. Let G be a graph. If $\widehat{G}$ is a loop graph having G as its underlying simple graph, then any CCR-Z zero forcing set Z for G is a CCR-Z($\widehat{G}$) zero forcing set for $\widehat{G}$, so $\mathrm{Z}(\widehat{G}) \leq \mathrm{Z}(G)$. Thus, $\widehat{\mathrm{Z}}(G) \leq \mathrm{Z}(G)$. Let $A \in \mathcal{S}(G)$ be such that $\operatorname{null} A = \mathrm{M}(G)$. Let $\widehat{G}$ be the loop graph of A; the underlying simple graph of $\widehat{G}$ is G. Clearly $\operatorname{null} A \leq \mathrm{M}(\widehat{G}) \leq \mathrm{M}(G)$, so $\mathrm{M}(G) = \mathrm{M}(\widehat{G})$. By Theorem 2.22, $\mathrm{M}(\widehat{G}) \leq \mathrm{Z}(\widehat{G}) \leq \widehat{\mathrm{Z}}(G)$. ■

The next example illustrates the computation of $\widehat{\mathrm{Z}}(G)$.

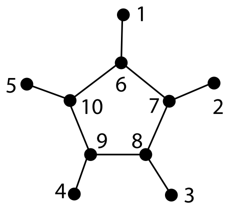

FIGURE 4. The pentasun H_5.

Example 2.25. For the pentasun H_5 depicted in Fig. 4, it is shown in Barioli et al. [5] that $\mathrm{mr}(H_5) = 8$ and thus $\mathrm{M}(H_5) = 2$. There are 2^{10} possible loop graphs having underlying simple graph H_5, but we can reduce this number by symmetry and by grouping graphs by the loop configuration on the leaves (degree 1 vertices).

(1) Let $\widehat{G}_0$ be a loop graph that has no loops on leaves. $\mathrm{Z}(\widehat{G}_0) = 0$ since each leaf can force its adjacent cycle vertex ($i \to i+5, i = 1, \ldots, 5$) and then each cycle vertex can force its adjacent leaf ($i \to i-5, i = 6, \ldots, 10$).
(2) Let $\widehat{G}_1$ be a loop graph that has at least one loop on a leaf, say, on vertex 1. $\mathrm{Z}(\widehat{G}_1) \leq 2$ since $\{2, 3\}$ is a zero forcing set with the following forces $2 \to 7,\ 3 \to 8,\ 7 \to 6,\ 8 \to 9,\ 1 \to 1,\ 6 \to 10,\ 9 \to 4,\ 10 \to 5$.

Thus, $\widehat{\mathrm{Z}}(H_5) \leq 2$. Since $\widehat{\mathrm{Z}}(H_5) \geq \mathrm{M}(H_5) = 2$, $\widehat{\mathrm{Z}}(H_5) = 2$.

Zero forcing for positive semidefinite matrices was defined in Barioli et al. [3], using the following color change rule.

CCR-Z_+ Let B be the set consisting of all the black vertices. Let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$ (note that it is possible that $k = 1$). Let $w \in W_i$. If $u \in B$ and w is the only white neighbor of u in $G[W_i \cup B]$, then change the color of w to black.

The *positive semidefinite zero forcing number of a graph G*, denoted by $\mathrm{Z}_+(G)$, is the CCR-Z_+ zero forcing parameter.

Forcing using color change rule CCR-Z_+ can be thought of as decomposing the graph into a union of certain induced subgraphs and using CCR-Z on each of these induced subgraphs. The application of CCR-Z_+ to a specific graph is illustrated in the next example.

Example 2.26. The tree Y_2 shown in Fig. 5 has $\mathrm{Z}_+(Y_2) = 1$, because any one vertex is a zero forcing set for CCR-Z_+.

Theorem 2.27. *[3, Theorem 3.5] For any graph G,* $\mathrm{M}_+(G) \leq \mathrm{Z}_+(G)$.

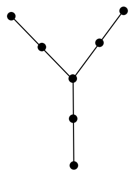

FIGURE 5. Y_2, the second long-Y.

Note that $M_+(G)$ is denoted in Barioli et al. [3] by $M_+^{\mathbb{R}}(G)$. In order to see in a natural manner that $\widehat{Z}$ bounds Z_+ from above, we introduce another parameter.

Definition 2.28. The *loop zero forcing number* of a graph G, denoted by $Z_\ell(G)$, is $Z(\widehat{G})$ where $\widehat{G}$ is the loop graph whose underlying simple graph is G, and such that $\widehat{G}$ has a loop at $v \in V_G$ if and only if $\deg_G v \geq 1$.

Although Z_ℓ is already defined as Z evaluated on a specific loop graph, we can see that Z_ℓ is a zero forcing parameter, which aids in computing the value of this parameter.

CCR-Z_ℓ If u is black and exactly one neighbor w of u is white, then change the color of w to black. If w is white, w has a neighbor, and every neighbor of w is black, then change the color of w to black.

Each of the two parts of CCR-Z_ℓ is merely the simple graph interpretation of CCR-$Z(\widehat{G})$ where $\widehat{G}$ is the loop graph constructed from G in Definition 2.28.

Example 2.29. Any one vertex of the complete bipartite graph $K_{1,3}$ is a CCR-Z_ℓ zero forcing set and the empty set is not a CCR-Z_ℓ zero forcing set, so $Z_\ell(K_{1,3}) = 1$. It is well-known (and obvious) that $Z(K_{1,3}) = M(K_{1,3}) = 2$, and $\widehat{Z}(K_{1,3}) = 2$. For use in Section B of the Appendix, we also note that $\mu(K_{1,3}) = 2$ since $K_{1,3}$ is a tree that is not a path [36].

Example 2.30. The tree Y_2 shown in Fig. 5 with $Z_+(Y_2) = 1$ has $Z_\ell(Y_2) = 2$, since any two leaves are a CCR-Z_ℓ zero forcing set, and no one vertex can force all the others using CCR-Z_ℓ.

For any graph G that is the disjoint union of connected components G_i, $i = 1, \ldots, k$, $\widehat{Z}(G) = \sum_{i=1}^k \widehat{Z}(G_i)$, $Z_\ell(G) = \sum_{i=1}^k Z_\ell(G_i)$, and $Z_+(G) = \sum_{i=1}^k Z_+(G_i)$ (the analogous results for M, M_+ and Z are well-known).

Theorem 2.31. *For any graph G, $Z_+(G) \leq Z_\ell(G) \leq \widehat{Z}(G)$.*

Proof. It is immediate from the definition that $Z_\ell(G) \leq \widehat{Z}(G)$.

To show that $Z_+(G) \leq Z_\ell(G)$, assume first that G is connected and $|G| \geq 2$. Let $\widehat{G}$ be the graph having a loop at every vertex whose underlying simple graph is G, and let Z be a CCR-$Z(\widehat{G})$ zero forcing set for $\widehat{G}$ such that $|Z| = Z(\widehat{G})$. We show Z is a CCR-Z_+ zero forcing set for G. Suppose u forces w using CCR-$Z(\widehat{G})$, so w is the unique white neighbor of u. Either u is black and $w \neq u$, or $w = u$. If u is black then u forces w using CCR-Z_+. If $w = u$, then let B be the set consisting of all the black vertices (at the stage at which w forces itself). The component containing w in $G - B$ is $\{w\}$. Since G is connected and $|G| \geq 2$, w has a (necessarily black) neighbor v in $G[B \cup \{w\}]$. So v forces w using CCR-Z_+. Thus, Z is a zero forcing set for G using CCR-Z_+, and so $Z_+(G) \leq |Z| = Z(\widehat{G})$.

The result for arbitrary G is obtained by summing over connected components, noting that $Z_+(K_1) = 1 = Z_\ell(K_1)$. ■

F. Tree-width and Minor Monotone Floors of Zero Forcing Parameters

We now turn our attention to the minor monotone floors of zero forcing parameters. Although $\lfloor Z \rfloor$, $\lfloor Z_\ell \rfloor$ and $\lfloor Z_+ \rfloor$ are already defined as minor monotone floors, we show that they are also zero forcing parameters by exhibiting appropriate color change rules, and show that they are equal, respectively, to proper path-width, path-width, and largeur d'arborescence. We also show that tree-width itself is a zero forcing parameter.

The next observation follows from Observation 2.8 together with Corollary 2.24 and Theorems 2.27 and 2.31.

Observation 2.32. For all graphs G,

(1) $\lfloor \mathrm{M} \rfloor(G) \leq \lfloor \widehat{\mathrm{Z}} \rfloor(G) \leq \lfloor \mathrm{Z} \rfloor(G)$.
(2) $\lfloor \mathrm{M}_+ \rfloor(G) \leq \lfloor \mathrm{Z}_+ \rfloor(G) \leq \lfloor \mathrm{Z}_\ell \rfloor(G) \leq \lfloor \widehat{\mathrm{Z}} \rfloor(G)$.

Definition 2.33.

CCR-$\lfloor \mathrm{Z} \rfloor$ If u is black and w is the only white neighbor of u, then change the color of w to black. If u is black, all neighbors of u are black, and u has not yet performed a force, then change the color of any white vertex w to black; in this case we say that *u hops* to w or *u forces w by a hop*. For u to hop, it is not required that u have any neighbors.

We will denote the value on G of the zero forcing parameter associated with CCR-$\lfloor \mathrm{Z} \rfloor$ by CCR-$\lfloor \mathrm{Z} \rfloor(G)$ until we have proved that CCR-$\lfloor \mathrm{Z} \rfloor(G) = \lfloor \mathrm{Z} \rfloor(G)$ (see Theorem 2.39 below).

Observation 2.34. *Since a vertex v must be black and have all but at most one neighbor black in order for v to perform a* CCR-$\lfloor \mathrm{Z} \rfloor$ *color change,* $\delta(G) \leq$ CCR-$\lfloor \mathrm{Z} \rfloor(G)$ *for any graph G.*

It is convenient to define an *active* vertex to be a black vertex that has not performed a force; an *inactive* vertex is a black vertex that has performed a force. Thus, the subset $Z \subseteq V_G$ that defines a coloring is active initially. Only an active vertex can force another vertex using CCR-$\lfloor \mathrm{Z} \rfloor$ (or CCR-$\lfloor \mathrm{Z}_\ell \rfloor$ defined below), because a vertex can force at most one other vertex. When vertex u forces vertex w, u becomes inactive and w becomes active. Thus, the number of active vertices remains constant. The application of CCR-$\lfloor \mathrm{Z} \rfloor$ to a specific graph is illustrated in the next example.

Example 2.35. Recall that G_6 and G_6' are the graphs shown in Fig. 2. The set $\{1, 2\}$ is a CCR$-\lfloor \mathrm{Z} \rfloor$ zero forcing set for G_6, because:

- 1 forces 3; 2 and 3 are active;
- 2 forces 5 by a hop; 3 and 5 are active;
- 3 forces 6; 5 and 6 are active;
- 5 forces 4.

Thus, CCR-$\lfloor \mathrm{Z} \rfloor(G_6) \leq 2$. Also $2 = \delta(G_6) \leq$ CCR-$\lfloor \mathrm{Z} \rfloor(G_6)$, so CCR-$\lfloor \mathrm{Z} \rfloor(G_6) = 2$. Performing the hop from 2 to 5 is like adding edge $\{2, 5\}$ to obtain the graph G_6' shown on the right in Fig. 2. Clearly $G_6 \prec G_6'$ and $\{1, 2\}$ is a CCR-Z zero forcing set for G_6', so $\lfloor \mathrm{Z} \rfloor(G_6) \leq 2$ (as a consequence of Theorem 2.39 below, $\lfloor \mathrm{Z} \rfloor(G_6) =$ CCR-$\lfloor \mathrm{Z} \rfloor(G_6) = 2$).

As noted in Barioli et al. [2], the CCR-Z derived set of a given set of black vertices is unique. However, a derived set need not be unique when hopping is involved, as the next example shows.

Example 2.36. The graph G in Fig. 6 has more than one CCR-$\lfloor \mathrm{Z} \rfloor$ derived set for coloring $\{1, 2, 3\}$: the set $\{1, 2, 3\}$ is a CCR-$\lfloor \mathrm{Z} \rfloor$ zero forcing set for G, with the forces:

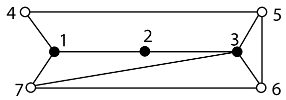

FIGURE 6. The graph G for Example 2.36.

- 2 forces 4 by a hop; 1, 3 and 4 are active;
- 4 forces 5; 1, 3 and 5 are active;
- 5 forces 6; 1, 3 and 6 are active;
- 6 forces 7.

But $\{1, 2, 3, 6\}$ is also a CCR-$\lfloor Z \rfloor$ derived set for the same coloring, with the force

- 2 forces 6 by a hop; 1, 3 and 6 are active.

At this point no further forces are possible.

Lemma 2.37. *Let H be a linear k-tree with $k \geq 1$. Then $\mathrm{Z}(H) = k$ and a set consisting of a vertex of degree k and all but one of its neighbors is a* CCR-Z *zero forcing set.*

Proof. Since H is a k-tree, $k = \delta(H) \leq \mathrm{Z}(H)$. We show by induction on $|H|$ that a set consisting of a vertex of degree k and all but one of its neighbors is a zero forcing set. The result is clear if H is K_{k+1}. Assume Lemma 2.37 to be true for all linear k-trees of order less than $|H|$. Let Z be a vertex of degree k and all but one vertex to which it is adjacent. The degree k vertex will then force its remaining neighbor. Remove the vertex that just performed the force to obtain a smaller linear k-tree and set of black vertices consisting of a set containing a vertex of degree k and all but one of its neighbors. ■

Note that T_3 shown in Fig. 3(d) is a 2-tree that is not linear, and $\mathrm{Z}(T_3) = 3$.

Theorem 2.38. *For any graph G having at least one edge,* CCR-$\lfloor \mathrm{Z} \rfloor(G)$ *is the minimum k for which G is a partial linear k-tree, i.e.,* CCR-$\lfloor \mathrm{Z} \rfloor(G)$ *is equal to the proper path-width of G.*

Proof. Let $\ell = \mathrm{ppw}(G)$, and let H be a linear ℓ-tree containing G as a subgraph. By Lemma 2.16, we may assume $V_G = V_H$. By Lemma 2.37 there is a zero forcing set Z of order ℓ for H. Then Z is a CCR-$\lfloor \mathrm{Z} \rfloor$ zero forcing set for G (a hop can be used in G whenever v forces u in H and $\{v, w\} \notin E_G$). Thus, CCR-$\lfloor \mathrm{Z} \rfloor(G) \leq \mathrm{Z}(H) \leq \ell = \mathrm{ppw}(G)$.

Now let $k =$ CCR-$\lfloor \mathrm{Z} \rfloor(G)$, and let Z be a CCR-$\lfloor \mathrm{Z} \rfloor$ zero forcing set of G with $|Z| = k$. Let $V_G \setminus Z = \{u_1, \ldots, u_{n-k}\}$ where u_i is the ith vertex to turn black (under one particular forcing order that turns all the vertices black). Construct an increasing sequence of linear k-trees H_i on subsets of the vertices of G as follows. Let H_1 be the complete graph on $Z \cup \{u_1\}$. For each $i = 2, \ldots, n-k$, construct H_i from H_{i-1} by adding u_i to the vertex set and adding all the edges between u_i and the k vertices that were active when u_i was forced. It is clear that H_1 is a linear k-tree, and the active vertices after H_1 is constructed include a vertex of degree k. If H_{i-1} is a linear k-tree and the active vertices after H_{i-1} is constructed include a vertex of degree k, then the same is true of H_i. Thus, each H_i is a linear k-tree. Since H_{n-k} contains G as a subgraph, $\mathrm{ppw}(G) \leq k =$ CCR-$\lfloor \mathrm{Z} \rfloor(G)$. ■

Theorem 2.39. $\lfloor \mathrm{Z} \rfloor$ *is the* CCR-$\lfloor \mathrm{Z} \rfloor$ *zero forcing parameter. In other words, for every graph G having at least one edge,* $\mathrm{ppw}(G) = \lfloor \mathrm{Z} \rfloor(G) =$ CCR-$\lfloor \mathrm{Z} \rfloor(G)$.

Proof. We show that for every graph G, $\lfloor Z \rfloor(G) \leq$ CCR-$\lfloor Z \rfloor(G)$ and CCR-$\lfloor Z \rfloor(G) \leq \lfloor Z \rfloor(G)$.

Let G be a graph and let Z be a CCR-$\lfloor Z \rfloor$ zero forcing set such that $|Z| =$ CCR-$\lfloor Z \rfloor(G)$. Construct a graph G' by adding the edge $\{u, w\}$ for each hop from u to w. Then Z is a CCR-Z zero forcing set for G', so $Z(G') \leq |Z| =$ CCR-$\lfloor Z \rfloor(G)$. Since $G \preceq G'$, $\lfloor Z \rfloor(G) \leq$ CCR-$\lfloor Z \rfloor(G)$.

Let $k = \lfloor Z \rfloor(G)$. There exists G' such that $G \preceq G'$ and $Z(G') = k$. Since CCR-$\lfloor Z \rfloor(G') \leq Z(G')$, G' is a partial linear k-tree by Theorem 2.38. Since $G \preceq G'$, G is a partial linear k-tree by Observation 2.17. Then by Theorem 2.38, CCR-$\lfloor Z \rfloor(G) \leq k = \lfloor Z \rfloor(G)$. ■

Whereas M, M_+, Z, $\widehat{Z}$, Z_ℓ, and Z_+ take the sum over connected components, many minor monotone parameters take the maximum. For example, if G is the disjoint union of connected components G_i, $i = 1, \ldots, k$, then $\mu(G) = \max_{i=1}^{k} \mu(G_i)$ [36], $\nu(G) = \max_{i=1}^{k} \nu(G_i)$ [15], $\xi(G) = \max_{i=1}^{k} \xi(G_i)$ [6], and $\text{tw}(G) = \max_{i=1}^{k} \text{tw}(G_i)$ (clear from the definition). Theorem 2.39 can be used to establish a similar result for $\lfloor Z \rfloor$.

Corollary 2.40. *If G_i, $i = 1, \ldots, k$ are the connected components of a graph G, then* $\lfloor Z \rfloor(G) = \max_{i=1}^{k} \lfloor Z \rfloor(G_i)$.

Proof. Order the components so that $\lfloor Z \rfloor(G_i) \geq \lfloor Z \rfloor(G_{i+1})$, $i = 1, \ldots, k-1$. Let Z_i be a CCR-$\lfloor Z \rfloor$ zero forcing set of minimum size for G_i. Let $Z = Z_1$. Then Z can force all the vertices in G_1. There are still $|Z|$ active vertices, which can force the vertices in Z_2 by hopping. Continuing in this manner, all vertices in G can be forced. Thus, $\lfloor Z \rfloor(G) \leq \lfloor Z \rfloor(G_1)$. Since G_1 is a minor of G, $\lfloor Z \rfloor(G_1) \leq \lfloor Z \rfloor(G)$. ■

The color change rule for $\lfloor Z_\ell \rfloor$ can be thought of as CCR-Z_ℓ with hopping.

Definition 2.41.

CCR-$\lfloor Z_\ell \rfloor$ If u is black and exactly one neighbor w of u is white, then change the color of w to black. If w is white, w has a neighbor, and every neighbor of w is black, then change the color of w to black. If u is black, all neighbors of u are black, and u has not yet performed a force, then change the color of any white vertex w to black (this does not require that u have any neighbors).

The color change rule CCR-$\lfloor Z_\ell \rfloor$ gives an associated zero forcing parameter whose value on a graph G is denoted by CCR-$\lfloor Z_\ell \rfloor(G)$, although by Theorem 2.45 below this may be shortened to $\lfloor Z_\ell \rfloor(G)$. As with CCR-$\lfloor Z \rfloor$, an active vertex is a black vertex that has not performed a force and the number of active vertices remains constant.

Observation 2.42. *For any graph G, $\delta(G) \leq$ CCR-$\lfloor Z_\ell \rfloor(G)$.*

The proofs of Lemma 2.43 and Theorems 2.44 and 2.45 are very similar to the proofs of Lemma 2.37 and Theorems 2.38 and 2.39, and are omitted.

Lemma 2.43. *Let H be a k-caterpillar with $k \geq 1$ constructed inductively by starting with K_{k+1} and at each stage adding a new maximal clique by adjoining a new vertex to the k vertices of some facet of the maximal clique added at the previous stage. Then $Z_\ell(H) = k$ and a set consisting of all but one of the vertices in the last maximal clique added is a CCR-Z_ℓ zero forcing set.*

Theorem 2.44. *For any graph G having at least one edge,* CCR-$\lfloor Z_\ell \rfloor(G)$ *is the minimum k for which G is a subgraph of a k-caterpillar, i.e.,* CCR-$\lfloor Z_\ell \rfloor(G)$ *is equal to the path-width of G.*

Theorem 2.45. $\lfloor Z_\ell \rfloor$ *is the CCR-*$\lfloor Z_\ell \rfloor$ *zero forcing parameter. Thus, for every graph having at least one edge,* $\text{pw}(G) = \text{CCR-}\lfloor Z_\ell \rfloor(G) = \lfloor Z_\ell \rfloor(G)$.

Corollary 2.46. *If* $G_i, i = 1, \ldots, k$ *are the connected components of a graph G, then* $\lfloor Z_\ell \rfloor(G) = \max_{i=1}^{k} \lfloor Z_\ell \rfloor(G_i)$.

Remark 2.47. The value a minor monotone floor takes on a single graph is a minimum over an unbounded collection of graphs, which naively requires an infinite calculation, whereas a parameter defined by a color change rule has a finite-time algorithm. Of the four zero forcing parameters Z, $\widehat{Z}$, Z_ℓ, and Z_+, we define a color change rule for the minor monotone floors of all but $\widehat{Z}$. However, since $\lfloor Z \rfloor(G) = \text{ppw}(G)$, $\lfloor Z_\ell \rfloor(G) = \text{pw}(G)$, $\lfloor Z_\ell \rfloor(G) \le \lfloor \widehat{Z} \rfloor(G) \le \lfloor Z \rfloor(G)$, and $\text{pw}(G) \le \text{ppw}(G) \le \text{pw}(G) + 1$ [32], $\lfloor \widehat{Z} \rfloor(G)$ is always equal to at least one of $\lfloor Z_\ell \rfloor(G)$ or $\lfloor Z \rfloor(G)$.

The following color change rule for $\lfloor Z_+ \rfloor$ can be thought of as forcing with decomposition and hopping.

Definition 2.48.

CCR-$\lfloor Z_+ \rfloor$ Let B be the set consisting of all the black vertices. Let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$ (note that it is possible that $k = 1$). For each component $1 \le i \le k$, let $C_i \subseteq B$ be the subset of black vertices that are considered to be "active" with regard to that component. (Initially, each $C_i = B = Z$.) If $u \in C_i$, $w \in W_i$, and u has no white neighbors in $G[W_i \cup B] - w$, then change the color of w to black. (This allows for either a normal forcing move, or a hop from u to w.) To each connected component of $G[W_i] - w$, associate a new active set equal to $(C_i \setminus \{u\}) \cup \{w\}$.

The color change rule CCR-$\lfloor Z_+ \rfloor$ gives an associated zero forcing parameter whose value on a graph G is denoted by CCR-$\lfloor Z_+ \rfloor(G)$, although by Theorem 2.51 below this may be shortened to $\lfloor Z_+ \rfloor(G)$. Note that for every component the number of active vertices in B is the same (and equal to the number of vertices in the zero forcing set).

Lemma 2.49. *Let H be a two-sided k-tree. If K is a k-clique such that K contains a vertex of degree k or* $H - V_K$ *contains more than one component, then* V_K *is a* CCR-Z_+ *zero forcing set. Thus,* $Z_+(H) = k$.

Proof. Since H is a k-tree, K_{k+1} is a subgraph of H, so $k = \nu(K_{k+1}) \le \nu(H) \le M_+(H) \le Z_+(H)$. We show by induction on $|H|$ that the set of vertices V_K of any k-clique K satisfying the hypothesis is a CCR-Z_+ zero forcing set. The result is clear if H is K_{k+1}. Assume true for all two-sided k-trees of order less than $|H|$. Let K be a k-clique such that (1) K contains a vertex of degree k or (2) $H - V_K$ contains more than one component. Case 1: If V_K contains a vertex v of degree k, then v can force its one neighbor w that is not in V_K. By the construction of a two-sided k-tree, $H - v$ is a two-sided k-tree in which $S = (V_K \setminus \{v\}) \cup \{w\}$ induces a k-clique that contains a vertex of degree k or $(H - v) - S$ contains more than one component. Case 2: If $H - V_K$ has more than one component, let $W_1, \ldots, W_r$ be the vertices of the components of $H - V_K$. Then for each $i = 1, \ldots, r$,

K has a vertex of degree k in $H[K \cup W_i]$. In either case the problem is reduced and the induction hypothesis completes the proof. ■

The proofs of Theorems 2.50 and 2.51 are very similar to the proofs of Theorems 2.38 and 2.39, and are omitted.

Theorem 2.50. *For any graph G,* $\text{CCR-}\lfloor Z_+ \rfloor(G)$ *is the minimum k for which G is a subgraph of a two-sided k-tree, i.e.,* $\text{CCR-}\lfloor Z_+ \rfloor(G) = \text{tstw}(G)$.

Theorem 2.51. $\lfloor Z_+ \rfloor$ *is the* $\text{CCR-}\lfloor Z_+ \rfloor$ *zero forcing parameter. Thus,* $\text{la}(G) = \text{tstw}(G) = \text{CCR-}\lfloor Z_+ \rfloor(G) = \lfloor Z_+ \rfloor(G)$.

Corollary 2.52. *For every graph G,* $\text{tw}(G) \le \lfloor Z_+ \rfloor(G)$.

Corollary 2.53. *If* $G_i, i = 1, \dots, k$*, are the connected components of a graph G, then* $\lfloor Z_+ \rfloor(G) = \max_{i=1}^{k} \lfloor Z_+ \rfloor(G_i)$.

We will now define the most complicated zero forcing parameter with the broadest allowable moves, thus giving the lowest numerical value. We will prove that this parameter is equivalent to tree-width. The proof is based on the cops-and-robber game [8].

In the usual description of the game, there is a single robber who moves with unlimited speed from vertex to vertex of a graph G, only along edges, while the cops are slower but move by way of helicopter (that is, they are not constrained to travel along edges). If the robber and a cop ever occupy the same vertex simultaneously, the robber is caught; in particular, at any given time the robber is confined to one component of the graph obtained by deleting the vertices occupied by cops. All parties know the positions of all others: the robber is visible to the cops, and can see in turn where a cop traveling by helicopter is going to land. The cops win the game by restricting the freedom of the robber until there is nowhere left to run. It is not hard to see that the cops gain no advantage by ever having more than one helicopter in the air at a time, and it can also be shown that without loss of generality a winning strategy for the cops monotonically restricts the freedom of the robber, or in other words never allows the robber to revisit a vertex that has once been occupied by a cop [8].

Formally, the game can be played in discrete time by two players (C and R) as follows: First, Player C places all but one of the cops on a subset Z of the vertices of G, and Player R then selects one component of $G - Z$ on which the robber is understood to be traveling freely. From then on, each complete turn proceeds as follows: Player C announces the vertex w on which the airborne cop is about to land; then Player R, whose robber is currently confined to a subgraph W of G, selects a component X of $W - w$ in which to remain once the cop has landed; then Player C chooses a vertex u from which a cop will take to the air. (Choosing the vertex u potentially increases the size of the robber's component, but only, without loss of generality, if the cops have no winning strategy [8]).

Theorem 2.54. *[8,29] The tree-width of a graph G is one less than the minimum number of cops for which Player C has a winning strategy on G.*

A winning strategy for the cops-and-robber game is equivalent to obtaining V_G as a derived set for the color change rule below. The set B of black vertices corresponds to all vertices that have ever been occupied by a cop, with the current position of the cops (other than one in the air) given by a set C_i that depends on which component W_i of $G - B$ the robber is in.

Definition 2.55.

CCR- tw Let B be the set consisting of all the black vertices. Let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$ (note that it is possible that $k = 1$). For each component $1 \leq i \leq k$, let $C_i \subseteq B$ be the subset of black vertices that are considered to be *active* with regard to that component. (Initially, each $C_i = B = Z$.) Suppose that w is a vertex in W_i such that for each component X of $G[W_i] - w$, there is a vertex $u_X \in C_i$ with no white neighbors in $G[V_X \cup B]$. Then change the color of w to black and, to each component X, associate a new active set equal to $(C_i \setminus \{u_X\}) \cup \{w\}$.

This provides for one complete turn of a winning strategy for the cops: Player C announces that the cop in the air will land at vertex w, knowing that for any component X of $G[W_i] - w$ the robber then chooses there is a vertex $u_X \in C_i$ that the robber cannot reach from X (because u_X has no white neighbors in the graph $G[V_X \cap B]$). This will allow the cop at u_X to leave and become the cop in the air.

After w is added to B, the vertex sets of the components of $G - B$ will be the same except for a partition of $W_i \setminus \{w\}$. The only difference between CCR-$\lfloor Z_+ \rfloor$ and CCR- tw is that in CCR-$\lfloor Z_+ \rfloor$ u is not allowed to depend on X, whereas in CCR- tw the forcing vertex u_X depends on what the components of $G - B$ will be at the end rather than the beginning of the step. The difference of one time step can be compensated for by allowing one more vertex in the initially black set Z; it is a consequence of Corollary 2.57 (see below) and Theorems 2.51 and Colin de Verdière's result [15] that for any graph G, $\text{tw}(G) \leq \text{la}(G) \leq \text{tw}(G) + 1$, and that CCR- tw$(G)$ is less than CCR-$\lfloor Z_+ \rfloor(G)$ by at most 1.

Lemma 2.56. *The minimum number of cops required to catch the robber in a game of cops and robber on G is equal to* CCR- tw$(G) + 1$.

Proof. We prove an inequality in each direction. Let Z be a set of k vertices of G that is a CCR- tw zero forcing set. Player C starts with k cops on the vertices of Z and one in the air, and starts at the beginning of the complete list of zero forcing moves. At the beginning of each complete turn, Player C looks through the list of zero forcing moves to the step that involves the set of vertices W_i in which the robber is currently trapped, and announces that the cop in the air will land at w. Player R must then choose which component X the robber will flee to, and Player C then removes the cop from the corresponding vertex u_X. Since V_G is a derived set of black vertices starting from Z, eventually the list W_i is empty, implying that the robber has been caught.

Suppose, on the other hand, that a winning strategy for Player C using $k + 1$ cops is known, which without loss of generality monotonically reduces the freedom of the robber. Let Z consist of the k starting positions (not including the helicopter) of the cops in this winning strategy, and color these vertices black. At any stage of the zero forcing, let B be the set of black vertices. Unless $B = V_G$, let W_i be some component of $G \setminus B$, and assume by way of induction that all vertices in the set C_i considered active for W_i are occupied by cops at a stage of the game when the robber is trapped in W_i. Let w be the next winning move (choice of vertex) that Player C should then announce. For any choice X that Player R can make of a component of $G[W_i] - w$, the strategy guarantees a vertex u_X in C_i holding a cop who will be able, once w is occupied, to take off in a helicopter without increasing the freedom of the robber. This implies that for each X, the vertex u_X has no white neighbors in $G[B \cup V_X]$, and thus it is a legal CCR-tw move to color w black and associate with each X the new active set $(C_i \setminus \{u_X\}) \cup \{w\}$. Since a cop lands at w and another cop takes off from u_X, these choices preserve the induction hypothesis, and so we may continue to color vertices black until $B = V_G$. ■

Theorem 2.54 and Lemma 2.56 together justify the name CCR- tw.

Corollary 2.57. *For any graph G,* $\text{CCR-tw}(G) = \text{tw}(G)$.

Remark 2.58. The cops-and-robber game is changed considerably if the location of the robber is not known to the cops, so that instead of adapting their strategy based on a known component W_i, they must systematically search the entire graph. Instead of requiring one more cop than the tree-width of G, it is known that for the invisible robber game the number of cops required is exactly one more than the path-width of G, and that a winning strategy is without loss of generality monotone [24]. Indeed, Theorem 2.44 can be proved either by following the proof of Theorem 2.38 or by showing, in the same way as Lemma 2.56, that having V_G as a derived set under CCR-$\lfloor Z_\ell \rfloor$ starting with k vertices is equivalent to a monotonic winning strategy for $k + 1$ cops to catch an invisible robber.

3. OPEN PROBLEMS

A major long-standing open question involves the relationship of minimum degree to maximum nullity, in various forms.

Question 3.1.

- *Is Conjecture 2.13, $\lceil\delta\rceil(G) \leq \nu(G)$, true?*
- *Is Maehara's conjecture, $\delta(G) \leq \mathrm{M}_+(G)$, true?*
- *Is the delta conjecture, $\delta(G) \leq \mathrm{M}(G)$, true?*

As noted earlier, M, Z, and many other parameters sum their values over components. However, the Colin de Verdière parameters and minor monotone floors of zero forcing parameters take the maximum over components. This leads to the related questions for minor monotone floors of maximum nullity parameters.

Question 3.2. *Suppose $G_i, i = 1, \ldots, k$ are the connected components of a graph G.*

- *Does $\lfloor \mathrm{M} \rfloor(G) = \max_{i=1,\ldots,k} \lfloor \mathrm{M} \rfloor(G_i)$?*
- *Does $\lfloor \mathrm{M}_+ \rfloor(G) = \max_{i=1,\ldots,k} \lfloor \mathrm{M}_+ \rfloor(G_i)$?*

Recall that given a graph G, we can realize $\lfloor \mathrm{Z} \rfloor(G)$ as $\mathrm{Z}(G')$ where G' is obtained from G by adding certain edges, and similarly for $\lfloor \mathrm{Z}_\ell \rfloor$ and $\lfloor \mathrm{Z}_+ \rfloor$. Whenever this is true, the minor monotone floor can be algorithmically computed.

Question 3.3. *Let G be a graph.*

- *Is G a subgraph of a graph G' with $V_G = V_{G'}$ and $\lfloor \widehat{\mathrm{Z}} \rfloor(G) = \widehat{\mathrm{Z}}(G')$?*
- *Is G a subgraph of a graph G' with $V_G = V_{G'}$ and $\lfloor \mathrm{M} \rfloor(G) = \mathrm{M}(G')$?*
- *Is G a subgraph of a graph G' with $V_G = V_{G'}$ and $\lfloor \mathrm{M}_+ \rfloor(G) = \mathrm{M}_+(G')$?*

For some parameter pairs, it is known that the discrepancy $p(G) - q(G)$ is at most 1; this is true, for example, of $\text{ppw}(G) - \text{pw}(G)$. For discrepancies between M and other parameters such as Z, it is often easy to see that if there is a graph where the discrepancy is nonzero, then the discrepancy cannot be bounded (for example, by taking disjoint unions). However, many of the minor monotone parameters take the maximum over components rather than the sum. Thus, it is not obvious that the discrepancy is unbounded, and there are several such lines in the diagram. The tree G_{13} in Example A.11 has $\lfloor \mathrm{M} \rfloor(G_{13}) - \xi(G_{13}) = 1$, and we do not know an example with a higher discrepancy.

Question 3.4. *Is* $\lfloor \mathrm{M} \rfloor(G) - \xi(G)$ *bounded?*

Our last question relates to the lower label ?3.5 on the line between $\lfloor \mathrm{M}_+ \rfloor$ and $\nu(G)$ in Fig. 1.

Question 3.5. *Does there exist a graph G for which* $\lfloor \mathrm{M}_+ \rfloor(G) > \nu(G)$*?*

It is usually difficult to compute $\lfloor \mathrm{M} \rfloor(G)$ or $\lfloor \mathrm{M}_+ \rfloor(G)$, because computing a minor monotone floor without a zero forcing rule is challenging. In Example A.11, the difference between $\lfloor \mathrm{M} \rfloor(G)$ and $\xi(G)$ occurs for very low values of these parameters, where we were able to use the characterization in Johnson et al. [22] of all graphs having $\mathrm{M}(G) \leq 2$. Since the forbidden minors for $\lfloor \mathrm{Z}_+ \rfloor(G) = \mathrm{la}(G) \leq 2$, namely K_4 and T_3, are the same as the forbidden minors for $\nu(G) \leq 2$ [23], it follows that if $\lfloor \mathrm{M}_+ \rfloor(G) > \nu(G)$, then $\nu(G) \geq 3$, and quite possibly much larger. Notice that Example A.20, which shows a difference between $\lfloor \mathrm{Z}_+ \rfloor$ and $\lfloor \mathrm{M}_+ \rfloor$, has $\lfloor \mathrm{M}_+ \rfloor(\overline{H}) = 10$, whereas the pentasun has $\lfloor \mathrm{Z} \rfloor(H_5) > \lfloor \mathrm{M} \rfloor(H_5) = 2$.

APPENDIX: EXAMPLES

Before we discuss non-equality and non-comparability of parameters, we exhibit a family of graphs for which all the parameters discussed are equal.

Example A.1. Let L be a linear 2-tree. Then $\mathrm{Z}(L) = 2$, $\kappa(L) = 2$, $\mathrm{P}(L) = 2$, and $\mathrm{h}(L) = 3$ so $\mathrm{h}(L) - 1 = 2$. Thus, all the parameters in Fig. 1 are equal.

A. Examples for Non-equality between Parameters

We now give additional examples to show that all but possibly one of the inequalities displayed in Fig. 1 are strict.

Example A.2. In K_{2s} the vertices can be covered by s disjoint edges and no three vertices can be on the same induced path, so for $s \geq 2$, $\mathrm{P}(K_{2s}) = s < 2s - 1 = \mathrm{h}(K_{2s}) - 1 = \mathrm{M}(K_{2s})$.

Example A.3. It was shown in Example 2.25 that $\widehat{\mathrm{Z}}(H_5) = 2$, where H_5 is the graph in Fig. 4. $\lfloor \mathrm{Z} \rfloor(H_5) = 3 = \mathrm{Z}(H_5)$ because any three leaves are a zero forcing set, and H_5 is not a subgraph of a linear 2-tree [22] so it has proper path-width of at least 3. Since for any graph G that is not a path, $\mathrm{M}(G) \geq 2$ and $\widehat{\mathrm{Z}}(G) \geq \mathrm{M}(G)$, $\widehat{\mathrm{Z}}(G) = 1$ if and only if G is a path. Thus, $\lfloor \widehat{\mathrm{Z}} \rfloor(H_5) = 2$, since H_5 is not a minor of a path and $\lfloor \widehat{\mathrm{Z}} \rfloor(H_5) \leq \widehat{\mathrm{Z}}(H_5)$. Since C_5 is a subgraph of H_5, $\lceil \delta \rceil(H_5) = 2 > 1 = \delta(H_5)$ and $\lceil \kappa \rceil(H_5) = 2 > 1 = \kappa(H_5)$. Since a path can contain at most two leaves, $\mathrm{P}(H_5) = 3 > \mathrm{M}(H_5) = 2$.

Observation A.4. *The graph* H_5 *satisfies* $\widehat{\mathrm{Z}}(H_5) < \mathrm{Z}(H_5)$, $\lfloor \widehat{\mathrm{Z}} \rfloor(H_5) < \lfloor \mathrm{Z} \rfloor(H_5)$, $\delta(H_5) < \lceil \delta \rceil(H_5)$, *and* $\kappa(H_5) < \lceil \kappa \rceil(H_5)$.

For the graph G_6 on the left in Fig. 2, it was shown in Example 2.9 that $\mathrm{M}_+(G_6) = 3$ and $\lfloor \mathrm{M}_+ \rfloor(G_6) = 2$, and in Example 2.35 that $\lfloor \mathrm{Z} \rfloor(G_6) \le 2$. Thus, $\lfloor \mathrm{M}_+ \rfloor(G_6) = \lfloor \mathrm{M} \rfloor(G_6) = \lfloor \mathrm{Z}_+ \rfloor(G_6) = \lfloor \mathrm{Z}_\ell \rfloor(G_6) = \lfloor \widehat{\mathrm{Z}} \rfloor(G_6) = \lfloor \mathrm{Z} \rfloor(G_6) = 2$. Furthermore, $\mathrm{Z}(G_6) \le 3$, since $\{1, 2, 4\}$ is a CCR-Z zero forcing set. Thus, $\mathrm{M}_+(G_6) = \mathrm{M}(G_6) = \mathrm{Z}_+(G_6) = \mathrm{Z}_\ell(G_6) = \widehat{\mathrm{Z}}(G_6) = \mathrm{Z}(G_6) = 3$. Observe that $\delta(G_6) = 2 > 1 = \kappa(G_6)$.

Observation A.5. *The graph G_6 satisfies $\lfloor \mathrm{Z}_+ \rfloor(G_6) < \mathrm{Z}_+(G_6)$, $\lfloor \mathrm{Z}_\ell \rfloor(G_6) < \mathrm{Z}_\ell(G_6)$, $\lfloor \widehat{\mathrm{Z}} \rfloor(G_6) < \widehat{\mathrm{Z}}(G_6)$, $\lfloor \mathrm{Z} \rfloor(G_6) < \mathrm{Z}(G_6)$, $\lfloor \mathrm{M}_+ \rfloor(G_6) < \mathrm{M}_+(G_6)$, $\lfloor \mathrm{M} \rfloor(G_6) < \mathrm{M}(G_6)$, and $\kappa(G_6) < \delta(G_6)$.*

As shown in Example 2.29, $\mathrm{M}(K_{1,3}) = \widehat{\mathrm{Z}}(K_{1,3}) = \mathrm{Z}(K_{1,3}) = 2$ and $\mathrm{Z}_\ell(K_{1,3}) = 1$. Since $1 \le \mathrm{M}_+(G) \le \mathrm{Z}_+(G) \le \mathrm{Z}_\ell(G)$ for every G, $\mathrm{M}_+(K_{1,3}) = \lfloor \mathrm{M}_+ \rfloor(K_{1,3}) = \lfloor \mathrm{Z}_\ell \rfloor(K_{1,3}) = 1$. $\lfloor \mathrm{M} \rfloor(K_{1,3}) = 2$ because $K_{1,3}$ is not a minor of a path, so $\lfloor \widehat{\mathrm{Z}} \rfloor(K_{1,3}) = 2$ also. In addition, $\mathrm{h}(K_{1,3}) - 1 = 1 < 2 = \mu(K_{1,3})$.

Observation A.6. *The star $K_{1,3}$ satisfies $\mathrm{M}_+(K_{1,3}) < \mathrm{M}(K_{1,3})$, $\mathrm{Z}_\ell(K_{1,3}) < \widehat{\mathrm{Z}}(K_{1,3})$, $\lfloor \mathrm{M}_+ \rfloor(K_{1,3}) < \lfloor \mathrm{M} \rfloor(K_{1,3})$, $\lfloor \mathrm{Z}_\ell \rfloor(K_{1,3}) < \lfloor \widehat{\mathrm{Z}} \rfloor(K_{1,3})$, and $\mathrm{h}(K_{1,3}) - 1 < \mu(K_{1,3})$.*

For the tree Y_2 in Fig. 5, $\mathrm{Z}_+(Y_2) = 1 < 2 = \mathrm{Z}_\ell(Y_2)$ was established in Examples 2.26 and 2.30. We also have $\lfloor \mathrm{Z}_+ \rfloor(Y_2) = 1$ because $1 \le \lfloor \mathrm{Z}_+ \rfloor(Y_2) \le \mathrm{Z}_+(Y_2)$, and $\lfloor \mathrm{Z}_\ell \rfloor(Y_2) = 2$ because $\lfloor \mathrm{Z}_\ell \rfloor(Y_2) \le \mathrm{Z}_\ell(Y_2) = 2$ and no one vertex can force all the others using CCR-$\lfloor \mathrm{Z}_\ell \rfloor$.

Observation A.7. *The tree Y_2 satisfies $\mathrm{Z}_+(Y_2) < \mathrm{Z}_\ell(Y_2)$ and $\lfloor \mathrm{Z}_+ \rfloor(Y_2) < \lfloor \mathrm{Z}_\ell \rfloor(Y_2)$.*

Observation A.8. *The graph G shown in Fig. A.1 establishes that the inequalities $\mu(G) \le \xi(G)$ and $\nu(G) \le \xi(G)$ can be strict, since $\xi(G) = 3 > 2 = \mu(G) = \nu(G)$ (see [6]).*

For the supertriangle T_3 in Fig. 3(d), $\lceil \delta \rceil(T_3) \le \mathrm{tw}(T_3) = 2$ since T_3 is a 2-tree, but $\lfloor \mathrm{Z}_+ \rfloor(T_3) \ge \nu(T_3) = 3$ [15].

Observation A.9. *The supertriangle T_3 satisfies $\mathrm{tw}(T_3) < \lfloor \mathrm{Z}_+ \rfloor(T_3)$ and $\lceil \delta \rceil(T_3) < \nu(T_3)$ (the latter shows that the conjectured inequality $\lceil \delta \rceil(G) \le \nu(G)$ can be strict).*

Observation A.10. *The graph K_4 satisfies $\mathrm{P}(K_4) = 2 < 3 = \mathrm{Z}(K_4)$.*

Example A.11. We show that the tree G_{13} on 13 vertices shown on the left in Fig. A.2 has $\lfloor \mathrm{M} \rfloor(G_{13}) = 3$. In Johnson et al. [22, Theorem 5.1, Lemma 3.4] it is shown that if $M(G) \le 2$, then G is a partial linear 2-tree or one of the exceptional graphs listed in Appendix B of Johnson et al. [22]. In the terminology of Takahashi et al. [31], G_{13} is the star-composition of three copies of $K_{1,3}$, and it is shown there that G_{13} is a forbidden minor

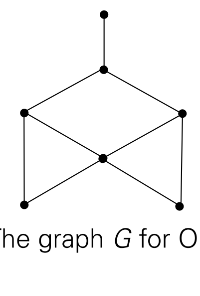

FIGURE A.1. The graph G for Observation A.8.

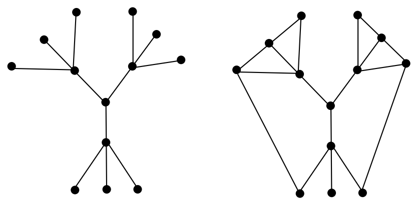

FIGURE A.2. The tree G_{13} and the graph G' for Example A.11.

for partial linear 2-trees (this also follows from the software-verifiable fact that no two vertices of G_{13} are a CCR-$\lfloor Z \rfloor$ zero forcing set). G_{13} is not a minor of any of the exceptional graphs because G_{13} has nine leaves, and the maximum number of leaves in a connected minor of any of the exceptional graphs is seven. Thus, $\lfloor M \rfloor(G_{13}) \geq 3$. Since G_{13} is a minor of the graph G' shown on the right in Fig. A.2, $\lfloor M \rfloor(G_{13}) \leq M(G') \leq Z(G') \leq 3$ (a top vertex with one of its neighbors, plus the pendant vertex, is a zero-forcing set). Thus, $\lfloor M \rfloor(G_{13}) = M(G') = 3$.

In Barioli et al. [6, Theorem 3.7] it is shown that any tree T that is not a path has $\xi(T) = 2$, so $\xi(G_{13}) = 2$.

Observation A.12. *The tree* G_{13} *satisfies* $\xi(G_{13}) < \lfloor M \rfloor(G_{13})$. *This also shows that the Strong Arnold Hypothesis imposes additional restrictions beyond minor monotonicity.*

For the 4-antiprism G_8 in Fig. A.3, $\lceil \kappa \rceil(G_8) \geq \kappa(G_8) = 4$ but $h(G_8) - 1 \leq 3$ because G is planar and thus $3 \geq \mu(G_8) \geq h(G_8) - 1$.

Observation A.13. *The graph* G_8 *satisfies* $h(G_8) - 1 < \lceil \kappa \rceil(G_8)$.

Example A.14. We show that the graph V_8 in Fig. A.4 (also known as Möbius ladder of order 8) has $\lceil \delta \rceil(V_8) = 3$. If $\delta(G') \geq 4$, then $|E_{G'}| \geq 2|G'|$, and this is impossible to achieve in a minor of V_8, since if $\delta(G') \geq 4$, then $|G'| \geq 5$, $|E_{V_8}| = 12$, $|V_8| = 8$, and a contraction reduces the number of edges by at least 1 and the number of vertices by 1, an edge deletion reduces the number of edges by 1 and the number of vertices stays the same, and in order to delete an isolated vertex, the edges incident with the vertex would first have to be deleted.

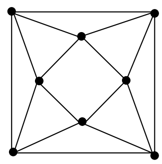

FIGURE A.3. The 4-antiprism G_8.

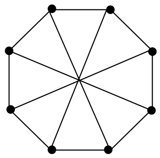

FIGURE A.4. The graph V_8 for Example A.14.

It is well-known that V_8 is a forbidden minor for $\mathrm{tw}(G) \leq 3$ (see, for example, [11, F33 p. 112]), so $\mathrm{tw}(V_8) \geq 4$.

Observation A.15. *The graph V_8 satisfies $\lceil\delta\rceil(V_8) < \mathrm{tw}(V_8)$.*

For a graph G, an *orthogonal representation of G of dimension d* is a set of vectors in $\mathbb{R}^d$, one corresponding to each vertex, with the property that if two vertices are nonadjacent, then their corresponding vectors are orthogonal. A *faithful orthogonal representation of G of dimension d* is an orthogonal representation such that if two vertices are adjacent, then their corresponding vectors are not orthogonal. In the minimum rank literature, the term "orthogonal representation" is often used for what is here called a faithful orthogonal representation, following the notation of Lovász et al. [25].

Example A.16. Mader [27, Fig. 7] exhibits the order 12 graph M_{12} shown in Fig. A.5 having $\lceil\delta\rceil(M_{12}) = 5$ and $\lceil\kappa\rceil(M_{12}) = 4$. This graph is constructed from two copies of $C_4 \vee K_2$ (octahedron-plus-axis) by attaching one vertex of each C_4 to the three remaining vertices of the opposite C_4 (in Fig. A.5, the 4-cycles are (1,2,3,4) and (7,8,9,10)).

It is not difficult to verify that the claimed values of $\lceil\delta\rceil$ and $\lceil\kappa\rceil$ hold for Mader's graph, and we also show $\lfloor Z\rfloor(M_{12}) = \nu(M_{12}) = 5$, thereby also determining the values of all the parameters in between. Every vertex has degree at least 5, so $\lceil\delta\rceil(M_{12}) \geq 5$. Since $Z = \{1, 2, 5, 6, 7\}$ is a CCR-$\lfloor Z\rfloor$ zero forcing set for M_{12}, $\lfloor Z\rfloor(M_{12}) \leq 5$. Thus, $\lceil\delta\rceil(M_{12}) = \lfloor Z\rfloor(M_{12}) = 5$. The two degree-7 vertices separate M_{12}, so any minor G' of M_{12} with $\kappa(G') > 2$ must lose all vertices to one side of the separation. Thus, G' must be a minor of one copy of $C_4 \vee K_2$ plus a single vertex attached to all of C_4. Any further contraction to remove the degree-4 vertex leaves at most six vertices, but does not leave K_6. The minor G' consisting of one copy of $C_4 \vee K_2$ plus a single vertex attached to all of C_4 has $\kappa(G') = 4$, thus $\lceil\kappa\rceil(M_{12}) = 4$.

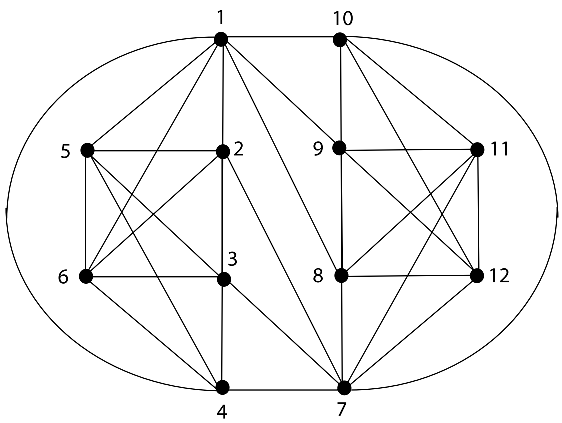

FIGURE A.5. The graph M_{12} for Example A.16.

The matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

gives a faithful orthogonal representation of M_{12}. Note that $B^T B \in \mathcal{S}(M_{12})$ and $\operatorname{null} B^T B = 5$. It is straightforward (e.g., using a computer mathematics system) to show that $B^T B$ satisfies the Strong Arnold Hypothesis. Thus, $\nu(M_{12}) \geq 5$ and since $\nu(M_{12}) \leq \lfloor \mathrm{Z} \rfloor(M_{12}) = 5$, $\nu(M_{12}) = 5$.

Observation A.17. *The graph M_{12} satisfies $\lceil \kappa \rceil(M_{12}) < \lceil \delta \rceil(M_{12})$ and $\lceil \kappa \rceil(M_{12}) < \nu(M_{12})$.*

In order to provide an example of a graph having tree-width greater than maximum nullity (Example A.20 below), we need some additional terminology and results. Let $G = (V_G, E_G)$ be a graph and let $U, W \subseteq V_G$. We say U is connected if the subgraph induced by U is connected, and U and W *touch* if $U \cap W \neq \emptyset$ or there exist $u \in U$, $w \in W$ such that $\{u, w\} \in E_G$. A *bramble* is a set of mutually touching connected subsets of vertices. A subset S is a *cover* of a bramble $\mathcal{B}$ if for all $U \in \mathcal{B}$, $S \cap U \neq \emptyset$. The *order* of a bramble $\mathcal{B}$ is the minimum number of vertices in a cover of $\mathcal{B}$. For a positive integer k, a graph G has $\operatorname{tw}(G) \geq k$ if and only if G has a bramble of order greater than k [16, Theorem 12.3.9].

Proposition A.18. *If C_4 is not induced in $\overline{G}$, then E_G is a bramble of G.*

Proof. Let $\{u, v\}, \{x, y\} \in E_G$. If $\{u, v\}$ and $\{x, y\}$ do not touch, then $G[\{u, v, x, y\}] = 2K_2$ and $\overline{G}[\{u, v, x, y\}] = C_4$, a contradiction. ■

Theorem A.19. *If* $\operatorname{girth}(\overline{G}) \geq 5$*, then* $\operatorname{tw}(G) \geq |G| - 3$.

Proof. Neither K_3 nor C_4 is induced in $\overline{G}$. By Proposition A.18, E_G is a bramble of G. Let $S \subseteq V_G$ with $|S| \leq |G| - 3$. Let $x, y, z \in V_G \backslash S$. Since K_3 is not induced in $\overline{G}$, $G[\{x, y, z\}]$ contains an edge of G. Therefore S does not cover E_G. So the order of the bramble E_G is greater than $|G| - 3$. Thus, by the bramble characterization of tree-width, $\operatorname{tw}(G) \geq |G| - 3$. ■

Example A.20. The Heawood graph H is shown in Fig. A.6; we consider its complement $\overline{H}$. It is well-known that H is the incidence graph of the Fano projective plane (the numbering in Fig. A.6 follows Barioli et al. [4, Fig. 4.1] with vertices 1–7 interpreted as the lines and vertices 8–14 interpreted as points). Any matrix in $\mathcal{S}(\overline{H})$ has the form $\left[\begin{smallmatrix} * & X_F \\ X_F^T & * \end{smallmatrix}\right]$ where * indicates every entry is nonzero (except possibly on the diagonal) and X_F is the zero–nonzero pattern of the complement of the incidence pattern in the Fano projective plane. So by Barioli et al. [4, Theorem 3.1], $\operatorname{mr}_+(\overline{H}) = \operatorname{mr}(\overline{H})$ is equal to the minimum of the ranks of the (not necessarily symmetric) matrices described by X_F, i.e., $\operatorname{mr}_+(\overline{H}) = \operatorname{mr}(\overline{H}) = 4$, so $\mathrm{M}_+(\overline{H}) = \mathrm{M}(\overline{H}) = 10$. This further implies that $\lfloor \mathrm{M} \rfloor(\overline{H}) \leq 10$ and $\lfloor \mathrm{M}_+ \rfloor(\overline{H}) \leq 10$.

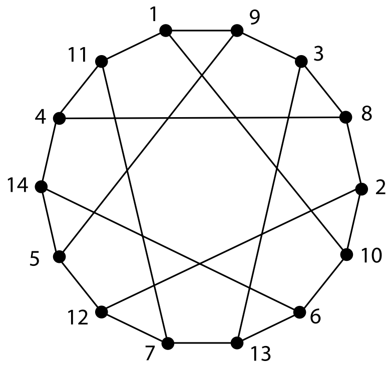

FIGURE A.6. The Heawood graph H.

Since $\mathrm{girth}(H) = 6$, $\mathrm{tw}(\overline{H}) \geq 11$ by Theorem A.19. But $\overline{H}$ is a subgraph of the 11-tree obtained from K_{14} by deleting the three edges $\{1, 9\}$, $\{1, 10\}$, and $\{2, 10\}$, so $\mathrm{tw}(\overline{H}) \leq 11$. Thus, $\mathrm{tw}(\overline{H}) = 11$. This further implies that $\lfloor \mathrm{Z}_+ \rfloor(\overline{H}) \geq 11$, $\lfloor \widehat{\mathrm{Z}} \rfloor(\overline{H}) \geq 11$, $\mathrm{Z}_+(\overline{H}) \geq 11$, and $\widehat{\mathrm{Z}}(\overline{H}) \geq 11$.

Observation A.21. *The graph $\overline{H}$ satisfies $\lfloor \mathrm{M}_+ \rfloor(\overline{H}) < \lfloor \mathrm{Z}_+ \rfloor(\overline{H})$, $\mathrm{M}_+(\overline{H}) < \mathrm{Z}_+(\overline{H})$, $\lfloor \mathrm{M} \rfloor(\overline{H}) < \lfloor \widehat{\mathrm{Z}} \rfloor(\overline{H})$, and $\mathrm{M}(\overline{H}) < \widehat{\mathrm{Z}}(\overline{H})$.*

B. Noncomparability of Parameters

In this section we show that with the exception listed in Remark A.23, if the parameters p and q are not joined by a monotone sequence of lines in Fig. 1, then they are noncomparable. Call (p, q) a *minimal pair* if

TABLE A.1. Minimal pairs.

	p	q	Graph	Example #
	P	h − 1	K_{2s}, $s \geq 2$	A.2
	P	κ	K_{2s}, $s \geq 2$	A.2
	$\widehat{\mathrm{Z}}$	P	H_5	A.3, 2.25
	$\lfloor \mathrm{Z} \rfloor$	P	G_6	2.9, 2.35
	$\widehat{\mathrm{Z}}$	$\lfloor \mathrm{Z} \rfloor$	H_5	2.25, A.4
	$\lfloor \mathrm{Z} \rfloor$	M_+	G_6	2.9, 2.35
	Z_+	$\lfloor \mathrm{Z}_\ell \rfloor$	Y_2	2.26, A.7
*	M	tw	$\overline{H}$	A.20
	tw	ν	T_3	A.9
	Z_ℓ	μ	$K_{1,3}$	2.29
	μ	κ	G_8	A.13
*	$\lceil \kappa \rceil$	δ	M_{12}	A.16
	δ	h − 1	paw	A.22

*(M,tw) and $(\lceil \kappa \rceil, \delta)$ are minimal pairs if Conjecture 2.13 is true.

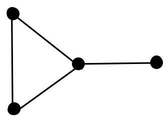

FIGURE A.7. The paw.

- p is not above q;
- for every parameter p' above p, p' is above q;
- for every parameter q' below q, q' is below p.

It is sufficient to provide an example of a graph $p(G) < q(G)$ for each minimal pair (p, q), since for any pair (a, b) with a not above b, there is a minimal pair (p, q) such that $a(G) \leq p(G)$ and $q(G) \leq b(G)$. Table A.1 below identifies examples for a complete list of minimal pairs, modulo the exceptions listed in Remark A.23. One final example is needed.

Example A.22. The paw, shown in Fig. A.7, has minimum degree equal to 1 and Hadwiger number equal to 3.

Remark A.23. Conjecture 2.13 states that for all G, $\lceil\delta\rceil(G) \leq \nu(G)$, so we do not have an example of a graph G such that $\nu(G) < \lceil\delta\rceil(G)$. Since we do not have an example of a graph G such that $\nu(G) < \lfloor \mathrm{M}_+ \rfloor(G)$, we do not have an example of a graph G such that $\xi(G) < \lfloor \mathrm{M}_+ \rfloor(G)$.

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