Extraordinary variability and sharp transitions in a maximally frustrated dynamic network

Wenjia Liu[1], B. Schmittmann[1,2] and R.K.P. Zia[1,2]

[1] *Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA*
[2] *Department of Physics and Astronomy, Iowa State University, Ames, IA 50011, USA*

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Abstract –Using Monte Carlo and analytic techniques, we study a minimal dynamic network involving two populations of nodes, characterized by different preferred degrees. Reminiscent of introverts and extroverts in a population, one set of nodes, labeled *introverts* (I), prefers fewer contacts (a lower degree) than the other, labeled *extroverts* (E). As a starting point, we consider an *extreme* case, in which an I simply cuts one of its links at random when chosen for updating, while an E adds a link to a random unconnected individual (node). The model has only two control parameters, namely, the number of nodes in each group, N_I and N_E). In the steady state, only the number of crosslinks between the two groups fluctuates, with remarkable properties: Its average (X) remains very close to 0 for all $N_I > N_E$ or near its maximum ($\mathcal{N} \equiv N_I N_E$) if $N_I < N_E$. At the transition ($N_I = N_E$), the fraction $X/\mathcal{N}$ wanders across a substantial part of $[0, 1]$, much like a pure random walk. Mapping this system to an Ising model with spin-flip dynamics and unusual long-range interactions, we note that such fluctuations are far greater than those displayed in either first or second order transitions of the latter. Thus, we refer to the case here as an 'extraordinary transition.' Thanks to the restoration of detailed balance and the existence of a 'Hamiltonian,' several qualitative aspects of these remarkable phenomena can be understood analytically.

Introduction. – Though their significance may not be understood at first glance, network structures can often be easily recognized in nature, from microscopic neurons to galactic filaments [1–5]. While these natural phenomena existed for ages and eons, more recently humans started building artificial counterparts in widely distinct arenas, e.g., in social [6, 7], infrastructural [8, 9], economic [10], and political [11] contexts. Their importance for modern societies cannot be understated. Meanwhile, quantitative efforts to characterize and model networks emerged even more recently, involving developments in many branches of science and engineering, including graph theory, statistical physics, neuroscience, computer science, etc. While these efforts led to much progress, many aspects of networks remain to be explored and/or modeled. For example, much of the literature focuses on *static* characteristics. While many situations may be well-served by a static model of networks (e.g., highways on timescales of days or months), there are many others for which a *dynamic* network description would be more appropriate. In particular, social contacts are generally in a state of flux, as new friendships or alliances are formed or existing links are severed. Our goal here is to study such evolving networks, to seek steady states (if any) and characterize their statistical properties. Are they like random Erdös-Rényi graphs [12], with Gaussian degree distributions? Are there strong or weak clustering and/or modularity characteristics? To make the network model easier to understand, we use the language of social network, so that nodes and links represent individuals and contacts (between pairs of persons), respectively. We will model the interactions between nodes *stochastically and dynamically*, i.e., through probabilistic evolution rules for adding/cutting links. In the language of graphs, the degree of each node will typically change, with a set of prescribed rates.

Inspired by the facts that an individual tends to *prefer* a certain number of contacts in social networks, and that as individuals adapt to changing circumstances, the network structure will fluctuate and drift [13], we have in mind a dynamically evolving network with preferred degree(s). Of course, the preferred number of contacts will typically differ from person to person *and* will change with time. However, our goal here is not to model any type of real social networks, but attempt to explore general statistical properties of evolving networks from a highly simplified model of social network. Therefore, let us begin with a homogeneous population of N individuals, all preferring the same degree, κ, and describe the rules for how individuals add or cut connections, in an attempt to reach their preferred degree from some other initial value. It is easy to imagine that this system will reach a steady state, in which everyone is mostly satisfied, with statistically 'normal' fluctuations around some 'happy medium.' Reminiscent of the presence of extroverts and introverts in the general population, we consider a system with two such groups (or communities), characterized by $\kappa_{1,2}$. One group of nodes prefers lower degree than the other. If $\kappa_1 \ll \kappa_2$, it is natural to adopt the psychological/sociological terminology and to refer to these two groups as introverts (I) and extroverts (E) [14,15]. Of course, the number of individuals in these groups can differ, denoted here by N_I and N_E respectively (with $N \equiv N_I + N_E$). For example, it was widely believed that $N_I/N_E \sim 1/3$ in the US population, though a more recent survey [16] suggests it is closer to 1/1. Our focus here will not be an attempt to predict complex human behavior. Instead, we are interested in general properties of stochastic, far-from-equilibrium systems which one might glean from investigating simple mathematical models of complex settings such as social interactions. A few preliminary results of this study were already reported in [15].

In the next section, we present the specifications of our model and surprising results from Monte Carlo simulations. In particular, we discover an extraordinarily sharp transition as the ratio N_I/N_E crosses unity, as well as $O\left(N^2\right)$ fluctuations and slow dynamics when $N_I = N_E$. Section 3 is devoted to theoretical approaches which offer some insight into these remarkable properties. Setting up the associated master equation, we find that the rates obey detailed balance and provide the exact stationary distribution. As will be shown, this system can be regarded as an equilibrium Ising model in two dimensions with (highly unusual) long-range interactions, evolving with Glauber spin-flip dynamics. Since an exact solution of such a model is beyond reach, we introduce several approximate approaches, with which much of the system's behavior can be reasonably well understood. The main challenge is the $N_I = N_E$ system, which displays many characteristics of critical phenomena. We end with a summary and outlook for future research.

Model specifications and simulation results. – Though our primary interest here will be a system with two different types of individuals, let us first consider a homogeneous population, which facilitates the description of the rules of evolution for a network with a *preferred degree*: κ [17]. While the number of nodes is fixed, the links do change: In a discrete time step (an attempt), a random node is chosen and its degree, k, is noted. For simplicity, we restrict ourselves to undirected links, i.e., identical in/out degrees. If $k > \kappa$, the node cuts a randomly chosen existing link. If $k < \kappa$, it adds a link to a random node to which it is not already connected. To avoid possible ambiguity, we can let κ be a half integer. With these simple rules, it is clear how κ plays the roles of a preferred degree. Obviously, this dynamics is not realistic; chosen for simplicity, it can be generalized to account for a variety of human preferences through, e.g., suitably chosen rate functions $w_{\pm}(k|\kappa)$ for adding/cutting links [17].

The standard description of a network is to specify the (symmetric) adjacency matrix $\mathbb{A}$, with elements $A_{ij} = 0/1$ representing the absence/presence of the link between nodes i and j. Since there are $\mathcal{L} \equiv N(N-1)/2$ links, the configuration space spans the $2^{\mathcal{L}}$ vertices of a unit cube in $\mathcal{L}$ dimensions, while adding/cutting a link corresponds to traversing along an edge of this $\mathcal{L}$-cube. Since the action on the links is random, this dynamics is ergodic. Thus, a finite system will settle into a unique stationary state. However, despite the simplicity, the transition rates do not satisfy detailed balance. Though straightforward, the proof is tedious and will be provided elsewhere [18]. As a result, the stationary state will be a non-equilibrium steady state (NESS), in the sense that persistent probability currents will prevail [19]. Though the network in the NESS resembles a random one [12], the (average) degree distribution is not a Gaussian, but instead, a Laplacian: $\propto e^{-\mu|k-\kappa|}$ [17]. This behavior can be understood from a simple argument [18], but let us turn to the more interesting problem of inhomogeneous populations.

The above picture becomes considerably more complex if we allow some diversity, e.g., a distribution of κ's. Indeed, even for a system with just two groups (i.e., I's and E's, with $\kappa_I < \kappa_E$), several new and puzzling features emerge [18]. Needless to say, an introvert/extrovert will typically find himself (herself) with more/less contacts than the preferred degree and so, tends to cut/add links. Such activities may be characterized quantitatively and regarded as a form of 'frustration,' a concept we will not discuss here. A systematic study requires scanning much of the κ_I-κ_E plane, a serious task which we initiate here by considering the *minimal* case. This serves as a baseline study, with an unexpected bonus of providing some exact analytic results. In this spirit, we consider a population with *extreme* preferences: $\kappa_I = 0$ and $\kappa_E = \infty$. In other words, if possible, the I's/E's always cut/add links. We coin this 'maximally frustrated' case the XIE (eXtreme Introverts/Extroverts) model and show that it provides some insight into the puzzles discovered in the general case of moderate I's and E's.

One significant simplification associated with this model is immediately apparent: Since the I's/E's always cut/add links, our system quickly evolves into a state where all I-

I/E-E links are absent/present. Only the crosslinks (I-E) are dynamic, changing according to which one of its associated nodes is chosen for update. Thus, instead of having to account for $\mathcal{L}$ links, we need to consider only $\mathcal{N} \equiv N_I N_E$ crosslinks and focus on $\mathbb{N}$, the appropriate rectangular sector of $\mathbb{A}$. Let us denote the elements of this $N_I \times N_E$ matrix by n_{ij}. Note that the first/second index is associated with an introvert/extrovert, so that $i \in [1, N_I]$, $j \in [1, N_E]$.

To re-emphasize, there are no degrees of freedom associated with our nodes. Each keeps its preassigned κ for the run, and only its links are cut or created. Note also that there are no spatial structures or built-in correlations – other aspects of realistic social networks which should be incorporated in future studies. Given the fixed preferences and rules of evolution, there are only two control parameters, (N_I, N_E), in our model. Furthermore, we can regard it as an Ising model on a square lattice of size $N_I \times N_E$ with 'spin' $\sigma_{ij} = \pm 1$. Clearly, the correspondence is $n_{ij} = (1 + \sigma_{ij})/2$. Of course, the dynamics of XIE is *not* governed by the Hamiltonian of the standard Ising model, $-J\Sigma\sigma_{ij}\sigma_{i'j'}$. Indeed, there is no *a priori* reason to believe that the stationary state of the XIE model can be characterized as an equilibrium system with a Boltzmann distribution. However, detailed balance is restored in this limit, a property we will exploit in our analysis.

For simulations, we start our runs mostly with an empty network. After about N Monte Carlo steps (MCS, with 1 MCS defined as N update attempts), we find that the E-E links are mostly filled. Thus we believe it is adequate to discard just the first 10^4 MCS before taking data. Our runs are mostly 10^6 - 10^7 MCS long, as we measure quantities of interest every 100 MCS. At first sight, this model appears to be entirely trivial, with random connections and no spatial structure. For example, we may expect more or fewer crosslinks, distributed homogeneously among the nodes, as the difference $H \equiv N_E - N_I$ is varied with fixed N. While some of these expectations are indeed confirmed qualitatively, many quantitative aspects are quite surprising. Here, we report findings from simulating a system with $N = 200$ and various H. We will comment briefly on preliminary results for other N's in the last section.

Despite the minimal structure of the XIE model, interesting quantities can be measured, e.g., degree distributions, correlations, and time dependence. The most immediate and natural quantity to consider is just X, the total number number of crosslinks. Since $X = \Sigma_{i,j} n_{ij}$, it plays precisely the same role as the total magnetization, $M = \Sigma_{i,j}\sigma_{ij}$, in the Ising model. As X is a time-dependent, stochastic variable, it corresponds to M in the kinetic Ising model [20]. Thus, questions about how M behaves as a function of the control parameters (temperature, magnetic field, system size, ...) can be immediately translated into ones for X. While we do not have a temperature, it is clear that our H plays a role similar to the magnetic field in the Ising model. Certainly, the average $\langle X \rangle$ in a *stationary state* should be a monotonically increasing function of H. Pursuing this analogy, let us define a magnetization-like quantity, $m \equiv 2\langle X \rangle / \mathcal{N} - 1 \in [-1, 1]$. Similarly, we define $h \equiv H/N = (N_E - N_I)/(N_E + N_I) \in [-1, 1]$, which plays the role of (say, the hyperbolic tangent of) the magnetic field. It is now natural to ask: How does $m(h)$ compare to the Ising equation of state?

Before presenting our findings, let us discuss what might be expected. Starting with a population with no links, the extroverts quickly fill all E-E links, while also making E-I contacts. Of course, when chosen to update, an introvert will cut one of its (necessarily I-E) links. Since every node is equally likely to be chosen, we may naïvely expect the ratio of link creations to deletions to be just N_E/N_I, leading to $N_E/N_I = \langle X \rangle / (\mathcal{N} - \langle X \rangle)$, i.e., $m = h$. This expectation is shown as the dashed line in Fig. 1.

Our simulation data (red diamonds in Fig. 1) paint an entirely different picture: For XIE, $m(h)$ is reminiscent of the equation of state in a ferromagnet at temperatures far below criticality. In particular, when just two introverts 'change sides' (from 101/200 to 99/200), the fraction $\langle X \rangle / \mathcal{N}$ jumps from $\sim 15\%$ to $\sim 85\%$. A jump of 70% (i.e., m jumping from -0.7 to $+0.7$) is extraordinary indeed! What causes such a sharp transition? and how is $\langle X \rangle = \mathcal{N}/2$ realized in the $N_E = N_I$ case? To answer these questions, we probe beyond simple averages, by studying the time traces, $X(t)$. In Fig. 2, the data for $N_I - N_E = \pm 2$ (green and blue lines) show that X hovers around $\langle X \rangle$ with fluctuations of $O(100)$ (i.e., $O\left(\sqrt{\mathcal{N}}\right)$ here), as one would expect for a 'non-critical' system. By contrast, in the $N_I = N_E$ case (red line), X wanders over a major portion of the allowed range, $[0, \mathcal{N}]$, evolving exceedingly slowly. Large fluctuations and slow dynamics are typical of ordinary equilibrium systems at a second-order phase transition. For example, in a standard Ising model at criticality, we have $\Delta M^2 \propto \mathcal{N}\chi \sim O\left(\mathcal{N}^{1+\gamma/d\nu}\right)$ [21]. Here, the fluctuations appear to be even larger: $\Delta X^2 \sim O\left(\mathcal{N}^2\right)$. Of course, a firm conclusion can only be drawn after a detailed finite-size scaling analysis is carried out.

From the time traces in the stationary state, we compile histograms for the full distribution of X: $P(X)$. Shown in Fig. 3, these reveal the expected (Gaussian-like for $H \neq 0$) and the unexpected – an *essentially flat* distribution over most of the full range (for $H = 0$). We should remind the reader that the standard Ising model does not exhibit such 'extreme' variability in $P(M)$. In particular, if we study very long runs of a finite system with $T \ll T_{Onsager}$ and zero magnetic field, then we expect $M(t)$ to hover around one of two values, $\pm m_{sp}\mathcal{N}$ (m_{sp} being the spontaneous magnetization), for extremely long periods, with infrequent yet very rapid transits from one to the other. Consequently, $P(M)$ will display two sharp (Gaussian, width $O\left(\mathcal{N}^{1/2}\right)$) peaks, with a deep valley in between. Returning to the XIE model, we recall that a flat stationary distribution is related to an unbiased sim-

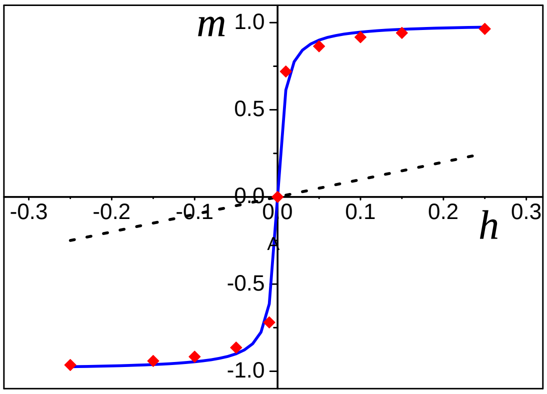

Fig. 1: The behavior of the average number of crosslinks for various N_I and N_E, displayed in terms of $m(h)$. Data points (red diamonds) are associated with $(N_I, N_E) = (125, 75), (110, 90), (101, 99), (100, 100)$, etc. The dashed line is the prediction from an 'intuitively reasonable' argument. A mean field approach leads to the solid (blue) line.

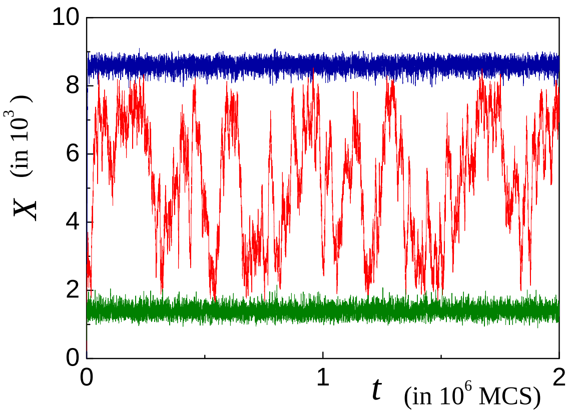

Fig. 2: Time traces of X for three cases: $N_I = 101$ (green, bottom), 100 (red, middle), and 99 (blue, top).

ple random walk. To confirm this expectation, we constructed the power spectrum from $X(t)$ and found that it is entirely consistent with $1/f^2$ (for $1/f \lesssim 10^6$ MCS, the time for X to traverse the observed range). This behavior can be roughly understood as X increasing or decreasing by unity (with equal probability, due to $N_I = N_E$) at each update attempt. Since X wanders over $O(\mathcal{N})$, the time scale associated with a traverse is $O(\mathcal{N}^2)$ attempts, i.e., $O(\mathcal{N}^2/N) = O(N^3)$ MCS, consistent with the $\sim 10^6$ MCS observed. While all of these arguments need to be tightened quantitatively, they do paint a plausible picture, namely, that X performs an unbiased random walk in case of a 50-50 split between introverts and extroverts.

Although a sizable jump in $\langle X \rangle$ suggests a first order phase transition, the standard associated characteristics are *absent* here. For example, in an Ising system at $T \ll T_c$, these include a well-separated bimodal $P(M)$, limited fluctuations, phase coexistence, hysteresis, metastability, etc. In addition to observing the flat $P(X)$ and $O(\mathcal{N})$ fluctuations, we tested for hysteresis and metastability. Starting a system in steady state with $N_E/N_I = 99/101$, we suddenly set $N_E/N_I = 101/99$. Preliminary data for $X(t)$ show that it promptly embarks on a *biased* random walk, with average velocity 2/MCS, until it reaches the appropriate $\langle X \rangle$. We are not aware of any other system, except for the Poland-Sheraga model [22], displaying such extraordinary behavior at a 'first order transition.' In the next section, we will present some analytic approaches for understanding these phenomena.

Analytic approaches. – A complete analytical description of the discrete-time stochastic XIE model is given by $\mathcal{P}(\mathbb{N}, t\,|\mathbb{N}_0, 0)$, the probability of finding configuration $\mathbb{N}$, t steps after some initial configuration $\mathbb{N}_0$. Writing down the rates which dictate the evolution for $\mathcal{P}$ is an easy first step. Solving it, even for the stationary $\mathcal{P}^*(\mathbb{N})$, is not so facile. In this section, we will present the master equation and the rates, as well as an explicit $\mathcal{P}^*$. From here, finding averages such as $\langle X \rangle$ is challenging enough, let alone computing full distributions such as $P(X)$. To make some progress, we exploit a mean field approach and gain some insights into the phenomena described above.

Master equation and stationary $\mathcal{P}^*$. Suppressing the reference to $\mathbb{N}_0$, the master equation provides the change over one time step (one attempt), $\mathcal{P}(\mathbb{N}, t+1) - \mathcal{P}(\mathbb{N}, t)$, as $\sum_{\{\mathbb{N}'\}} [W(\mathbb{N}, \mathbb{N}')\mathcal{P}(\mathbb{N}', t) - W(\mathbb{N}', \mathbb{N})\mathcal{P}(\mathbb{N}, t)]$. where $W(\mathbb{N}', \mathbb{N})$ specifies the rate for $\mathbb{N}$ to change to $\mathbb{N}'$. As noted above, a link being added/cut corresponds to a simple spin flip in Ising language. However, the rates for XIE are more involved, especially since we have no *a priori* knowledge of either a Hamiltonian or detailed balance. Here, letting $\bar{n} \equiv 1 - n$, we define

$$k_i \equiv \Sigma_j n_{ij}; \quad \bar{p}_j \equiv \Sigma_i \bar{n}_{ij} \tag{1}$$

which are, respectively, the degree of node i and the *complement* of the degree of node j. Note that $k_i \in [0, N_E]$ and $\bar{p}_j \in [0, N_I]$. In the lattice gas language of the Ising model, k_i and $\bar{p}_j$ are the number of particles in row i and holes in column j, respectively. Similar to the Ising case, a key symmetry here is $n_{ij} \Leftrightarrow \bar{n}_{ji} \oplus N_I \Leftrightarrow N_E$, which we will refer to as 'particle-hole symmetry.' Using (1), $W(\mathbb{N}', \mathbb{N})$ can be easily written:

$$\sum_{i,j} \frac{\Delta}{N} \left[\frac{\Theta(k_i)}{k_i} \bar{n}'_{ij} n_{ij} + \frac{\Theta(\bar{p}_j)}{\bar{p}_j} n'_{ij} \bar{n}_{ij} \right] \tag{2}$$

where $\Theta(x)$ is the Heavyside function (i.e., 1 if $x > 0$ and 0 if $x \leq 0$) and $\Delta \equiv \Pi_{k\ell \neq ij} \delta(n'_{k\ell}, n_{k\ell})$ insures that *only* n_{ij} may change.

Besides a reduction of the configuration space (to the vertices of an $\mathcal{N}$-cube), the XIE model enjoys another major simplification: the restoration of detailed balance. Our proof invokes the Kolmogorov criterion [23]: Detailed

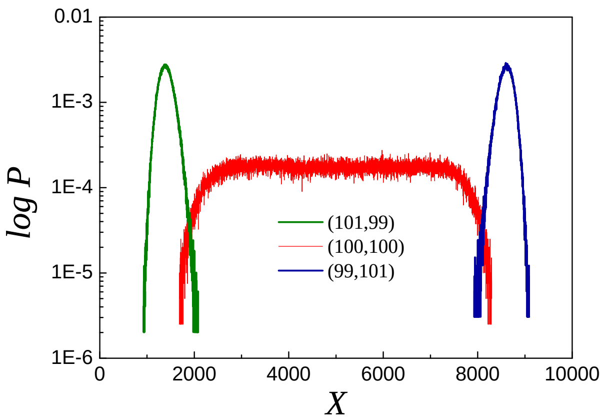

Fig. 3: Histograms of X for three cases: $N_I = 101$ (green, left), 100 (red, middle), and 99 (blue, right).

balance holds if the product of transition rates around every closed loop in configuration space is independent of the direction in which the loop is traversed. It is straightforward to show this for every face of our $\mathcal{N}$-cube, and so around all closed loops. As a consequence, in the $t \to \infty$ limit, $\mathcal{P}$ approaches the stationary distribution $\mathcal{P}^* (\mathbb{N})$ with no probability currents, just like a system in thermal equilibrium. Further, $\mathcal{P}^*$ can be obtained from repeated use of the detailed balance condition $\mathcal{P}^* (\mathbb{N}) = \mathcal{P}^* (\mathbb{N}') W (\mathbb{N}, \mathbb{N}') / \ W (\mathbb{N}', \mathbb{N})$. Imposing normalization, we find explicitly

$$\mathcal{P}^* (\mathbb{N}) = \frac{1}{\Omega} \prod_{i=1}^{N_I} (k_i!) \prod_{j=1}^{N_E} (\bar{p}_j!) \tag{3}$$

where $\Omega = \Sigma_{\{\mathbb{N}\}} \Pi (k_i!) \Pi (\bar{p}_j!)$ is a 'partition function.' Note that the particle-hole symmetry is manifest here.

Mean-field approximation. Although having an explicit $\mathcal{P}^*$ is a major step in understanding a stochastic process, we are still quite far from being able to compute macroscopic quantities of interest analytically. In analogy, the explicit $\mathcal{P}^*$ for the ordinary Ising model was proposed by Lenz in the 1920's, but nontrivial analytic results first appeared in 1944. Indeed, few such results exist for the 3D case, while even in 2D, the exact equation of state, $m(h, T)$, remains unknown. Our system poses much more serious challenges: Defining a 'Hamiltonian' by $\mathcal{H} \equiv -\ln \mathcal{P}$, we see that, being $-\sum_{i=1}^{N_I} \ln (\Sigma_j n_{ij})! - \sum_{j=1}^{N_E} \ln (\Sigma_i \bar{n}_{ij})!$ (apart from a constant), it contains peculiarly *anisotropic, long-ranged, multispin* interactions, in which a 'spin' is coupled to all others in its row and column.

To make progress, we invoke a mean-field approach and find an approximate expression for

$$P(X) \equiv \sum_{\{\mathbb{N}\}} \delta (X, \Sigma_{ij} n_{ij}) \mathcal{P}^* (\mathbb{N}) \quad . \tag{4}$$

In other words, we replace n_{ij} by $X/\mathcal{N}$ and attempt to perform the sums. Thus, $\Sigma_j n_{ij} \to N_E (X/\mathcal{N})$ and $\Sigma_i \bar{n}_{ij} \to N_I (1 - X/\mathcal{N})$, while

$$\sum_{\{\mathbb{N}\}} \delta (X, \Sigma_{ij} n_{ij}) = \binom{\mathcal{N}}{X} . \tag{5}$$

Exploiting Stirling's formula, we can compute the mean-field approximate $P_{MF}(X)$. In this spirit, it is natural to label

$$F(\rho) \equiv (-\ln P_{MF}(X)) / \mathcal{N} \tag{6}$$

as a 'Landau free energy density' for $\rho \equiv X/\mathcal{N}$. Remarkably, contributions from $\mathcal{H}$ cancel most of the entropic terms from (5), so that, to leading order in the thermodynamic limit ($X, \mathcal{N} \to \infty$ at fixed ρ), F is *linear* in ρ with slope $\ln (N_I/N_E)$. In other words, there is nothing to stabilize ρ to non-trivial values. Its behavior follows an 'all or nothing' maxim: As long as $N_I \neq N_E$, ρ can assume only the boundary values: 0 or 1. For $N_I = N_E$, F is *flat* over the entire interval. This picture fits the simulation data *qualitatively* and provides insight into the extraordinary transition observed. To avoid the extremes, we find nonlinear contributions at the next order: $-\left(\frac{\ln \rho}{N_E} + \frac{\ln[1-\rho]}{N_I}\right) \Big/ 2$. Thus, ρ settles within $O(1/N \ln [N_I/N_E])$ of the boundaries for generic (N_I, N_E). In the language of magnetism, the m-dependent part of $F(m; h)$ reads

$$\frac{m}{2} \ln \frac{1+h}{1-h} - \frac{1}{N} \left[\frac{\ln (1+m)}{1+h} + \frac{\ln (1-m)}{1-h} \right] . \tag{7}$$

From here, we can find the minimum of F and plot it as $m(h)$ for the specific case of $N = 200$. The resultant (solid blue curve in Fig. 1) is remarkably respectable. We should caution the reader, however, that such good agreement does not extend to the entire distribution, i.e., $P_{MF}(X)$ deviates considerably from the histograms of X. If the $N \to \infty$ limit is taken first, this approach will provide us with a highly singular $m(h) = sign(h)$. Such an 'extreme' equation of state occurs only at $T = 0$ in the Ising model! Of course, such predictions should be tested against simulations with other N's, along with finite-size scaling plots.

Before ending, let us call attention to another salient feature of this F. For $h \ll 1$, it is simply $-mh - \ln (1 - m^2) / N$. Unlike the standard Landau form, $-mh - \tau m^2 + g m^4$, our F has just *one* minimum, consistent with the absence of metastability. Instead of displaying hysteresis, our system will immediately begin, when started in equilibrium state with $h < 0$ and suddenly set with $h > 0$, a constant (for $N \to \infty$) average velocity journey to the stable state. Clearly, this mean-field approach captures some key features of the XIE model.

Summary and outlook. – In this article, we considered a simple model of two groups ('extreme' introverts/extroverts) interacting via a dynamic network of links. Using Monte Carlo simulations, we discovered an extraordinarily sharp transition, as the mix of the two groups crosses 50-50. The nature of this transition is puzzling. While the large jump in $\langle X \rangle$ suggests a first-order transition, the system displays none of the standard characteristics: metastability, hysteresis, phase co-existence, etc. Instead the extensive fluctuations and slow dynamics at 50-50 remind us of those in a system at a second-order transition (but with $\delta = \infty$ to 'fit' $m = sign(h)$). Although similar features are displayed elsewhere [22], the XIE model is unique, as the transition is driven by a change in the system *geometry* (in the language of a 2D Ising model), rather than tuning the temperature or magnetic field. Starting with a master equation for this stochastic process, we find an explicit expression for the stationary distribution $\mathcal{P}^*$, which plays the role of the Boltzmann factor for systems in thermal equilibrium. A mean-field approach provides some insight into much of the surprising behavior through a Landau-like $F(m;h)$. The long-ranged, multispin interactions in our 'Hamiltonian' are reminiscent of the Kac potential [25] (for which the mean-field treatment is exact). However, there are clearly substantial differences, which are beyond the scope of this letter and will be discussed elsewhere.

These initial findings serve as an excellent starting point for pursuing many other interesting issues associated with this model. Most immediate is a finite-size scaling analysis of the critical region of the 'extraordinary' transition. Fluctuations of X , which can be related to various $\langle nn' \rangle$ correlations, as well as details of the power spectrum (associated with $X(t)$) should be explored. Some preliminary data for $Q(k)$, the degree distributions of both I's and E's, hint at a rich variety of behavior and promise a cornucopia of novel phenomena. Apart from these standard concepts from statistical mechanics, we may venture to quantify the notion of 'frustration' – ϕ. The simplest possibility is to start with Q_i, the degree distribution of an individual i, and define $\phi_i \equiv [\Sigma_{k>\kappa} - \Sigma_{k<\kappa}] Q_i(k)$. Thus, ϕ vanishes if the node has as many links above its preferred κ as below. In XIE, ϕ assumes the extremal values ± 1 for every nodes and so, the system deserves the term 'maximally frustrated.'

Beyond this simple model, there are many avenues to explore other dynamic networks with preferred degrees, such as having generic $\kappa_{I,E}$'s or a realistic distribution of κ's. 'Frustration' should be less extreme, as we expect $|\phi_i| < 1$ in general. To assess how 'frustrated' the *entire* population is, we may consider, say, $\Phi \equiv \Sigma_i \phi_i^2 / N$. Perhaps such a mathematical concept will be useful for sociologists and psychologists. There are also multiple ways to model interactions between the various groups. Looking further, we should overlay node variables on our dynamic network, e.g., the opinion, the wealth, the state of health, etc., associated with each individual. Of course, it would be interesting to see whether the features discovered in simple models are robust and appear in more realistic models of social networks. While considerations of models like ours are unlikely to predict the swing of public opinion, the widening gap between rich and poor, or the spread of epidemics in our society, they may provide some insight into certain universal aspects of collective, stochastic behavior.

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