**A fractional approach to minimum rank and zero forcing**

by

**Kevin Francis Palmowski**

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Program of Study Committee:
Leslie Hogben, Major Professor
Fritz Keinert
Ryan Martin
Jennifer Newman
Namrata Vaswani

Iowa State University

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# DEDICATION

Hey Mom, Dad, Kimberly, Erica, Jason, and Megan,

I did it!

Thanks for all of the love and support that you've given me over the years, especially while I've been away at grad school. I've learned a lot at Iowa State, but perhaps the most important lesson of all is that our family is amazing and we're all darned lucky to have each other. No one deserves this dedication as much as you guys.

Love,

Kevin

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# ABSTRACT

This thesis applies techniques from fractional graph theory to develop fractional versions of graph parameters related to minimum rank and zero forcing. Projective rank, a graph parameter with applications to quantum information, is formally related to $r$-fold generalizations of orthogonal representations for graphs. Using similar techniques, fractional minimum positive semidefinite rank is defined via $r$-fold generalizations of faithful orthogonal representations and $r$-fold minimum positive semidefinite rank, and it is shown that the fractional minimum positive semidefinite rank of any graph equals the projective rank of the complement of the graph. An alternate characterization of $r$-fold minimum positive semidefinite rank that considers the ranks of certain Hermitian matrices is also presented. Motivated by the connections between zero forcing games and minimum rank problems, an $r$-fold analogue of the positive semidefinite zero forcing process is introduced and used to define the fractional positive semidefinite forcing number of a graph. An analysis of the $r$-fold positive semidefinite forcing game leads to a three-color forcing game that allows computation of fractional positive semidefinite forcing number without appealing to the $r$-fold game. The three-color approach is applied to the standard zero forcing game and it is shown that the skew zero forcing number of a graph is exactly the parameter obtained by applying the fractionalization technique to the standard zero forcing game. Graphs whose skew zero forcing number equals zero are characterized via the three-color approach and an algorithm.

# CHAPTER 1.   INTRODUCTION

This thesis applies techniques from fractional graph theory [4] to develop fractional versions of graph parameters related to minimum rank and zero forcing. While the overall theme of introducing and analyzing fractional versions of previously-studied graph parameters is graph theoretic in nature, linear algebra also plays a key role in the development of these new parameters. The underlying theory related to this work has connections to quantum information, control of quantum systems, and modeling the spread of disease in a network, as described in Chapters 2 and 3.

Briefly, the "fractionalization" process that we implement begins by defining an "$r$-fold" version of a graph parameter. The $r$-fold parameter should be an extension of the regular parameter to some "higher dimension;" for example, if the regular parameter is related to single objects, then the $r$-fold version may consider sets of $r$ objects. Creating a "reasonable" definition for the $r$-fold parameter is an important and challenging aspect of this process; there may be many ways to define an $r$-fold parameter, so a definition that imparts properties similar to those of the regular graph parameter is desirable. The fractional graph parameter is then defined to be the infimum (or, if appropriate, supremum) over the natural numbers of the ratio of the $r$-fold parameter to $r$. Depending on the properties of the $r$-fold parameter, there may be equivalent ways to express or define the fractional parameter.

As described in Section 1.1, defining fractional minimum positive semidefinite rank and exploring its connection with projective rank were our initial motivations. Since zero forcing processes are related to minimum rank problems, a natural application of some

of the theory developed to derive fractional minimum positive semidefinite rank was to define $r$-fold and fractional zero forcing processes.

As a final note, we emphasize that the term "fractional graph parameter" is a nod to the method with which our new parameters are developed, and not a claim that any particular parameter is rational-valued. Indeed, we will see that proving rationality of one parameter is an open problem in quantum information, and the other parameters considered turn out to be integer-valued.

## 1.1  Overview

In Chapter 2, a fractional analogue of minimum positive semidefinite rank is developed. This investigation was motivated by a desire to develop more theory related to projective rank, a parameter that was introduced in 2012 and has connections to problems arising in quantum information (see, for example, [3]). There are numerous open questions related to projective rank, most notably whether there exists a graph whose projective rank is irrational; if so, then the infamous Tsirelson's problem can be answered in the negative, thus solving an important open problem in the realm of quantum information (see Chapter 2 for more information and references).

Given a graph $G = (V, E)$ with $V(G) = \{1, 2, \ldots, n\}$, a symmetric matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is said to *fit* $G$ if $a_{ij} = 0$ if and only if $ij \notin E(G)$. The minimum rank of a symmetric matrix that fits $G$ is the *minimum rank* of the graph $G$, denoted here by $\mathrm{mr}(G)$. By further restricting to positive semidefinite matrices that fit $G$, we can similarly define the *minimum positive semidefinite rank* of $G$, denoted here by $\mathrm{mr}^+(G)$. It is also possible to define minimum rank problems using matrices over fields other than $\mathbb{C}$ (specifically, $\mathbb{R}$ is more often considered in the literature); more information on minimum rank problems and an extensive bibliography can be found in [1]. Our use of

$\mathbb{C}$ is necessitated by connections to existing parameters arising from problems based in quantum physics.

An equivalent definition of minimum positive semidefinite rank is based on faithful orthogonal representations for a graph. A *faithful orthogonal representation* for a graph $G$ is a set of vectors $\{\mathbf{x}_u\}_{u \in V(G)} \subset \mathbb{C}^d$ (for some $d$) such that $\mathbf{x}_u^* \mathbf{x}_v = 0$ if and only if $uv \notin E(G)$. It can be shown that the minimum $d$ such that $G$ has a faithful orthogonal representation in $\mathbb{C}^d$ is equal to $\mathrm{mr}^+(G)$.

Though their names are similar, faithful orthogonal representations are complementary to a different type of vector representation for a graph known as an orthogonal representation. An *orthogonal representation* for a graph $G$ is a set of vectors $\{\mathbf{x}_u\}_{u \in V(G)} \subset \mathbb{C}^d$ (for some $d$) such that if $uv \in E(G)$, then $\mathbf{x}_u^* \mathbf{x}_v = 0$. Note that, in contrast to the definition of a faithful orthogonal representation, the vectors in an orthogonal representation are orthogonal when they correspond to edges in the graph, and the orthogonality condition is not an "if and only if." The *orthogonal rank* of $G$, denoted $\xi(G)$, is the minimum $d$ such that $G$ has an orthogonal representation in $\mathbb{C}^d$. Because a faithful orthogonal representation for a graph $G$ is an orthogonal representation for the graph's complement $\overline{G}$, it is clear that $\xi(\overline{G}) \leq \mathrm{mr}^+(G)$ for any graph $G$.

The *projective rank* of a graph is a parameter of interest to those working in quantum information (see Section 2.1.3 for a mathematical definition of projective rank). Previously, it was informally conjectured and assumed to be true that projective rank can be considered through the lens of fractional graph theory as "fractional orthogonal rank." In Sections 2.2.1 and 2.2.2, we develop the necessary machinery to establish this claim (Theorem 2.2.13).

Due to the relationship between orthogonal rank and minimum positive semidefinite rank, it was also conjectured that projective rank would be related to "fractional minimum positive semidefinite rank," should definition of such a parameter be feasible. In Section 2.3, we define fractional minimum positive semidefinite rank and use this new

parameter to investigate the conjecture. A main and somewhat unexpected result in that section, Theorem 2.3.21, is that the projective rank of a graph equals the fractional minimum positive semidefinite rank of its complement. This connection, as well as the theory developed in Chapter 2, should provide an avenue through which results pertaining to minimum rank can be extended and adapted to inform new developments pertaining to projective rank.

The focus of Chapter 3 is developing fractional analogues of zero forcing parameters. The *zero forcing number (positive semidefinite zero forcing number)* of a graph $G$, denoted here by $Z(G)$ $(Z^+(G))$, can be obtained by playing a vertex coloring game on the graph. By coloring some vertices of $G$ blue and the rest white and repeatedly applying a *forcing rule* (by which white vertices can be turned blue), the player seeks to color the entire graph blue; if this is possible, the initial set of blue vertices is a *forcing set*. Zero forcing numbers are of interest because of their applications to control of quantum systems and maximum nullity problems (see Chapter 3 for more information and references).

As noted in Section 3.1.2, the material in Chapter 3 was initially motivated by analysis of matrices that $r$-fit a graph, introduced in Section 2.3.3. Note that any graph $G$ is the graph associated with a matrix that fits $G$, and zero forcing processes take place on $G$. Our $r$-fold zero forcing process considers application of zero forcing rules to the graph of a matrix that $r$-fits $G$; while there can be many such matrices, Proposition 3.1.1 allows us to focus on matrices that are associated with the (independent) $r$-blowup of $G$. With an $r$-fold parameter established, we are able to define a fractional version of the positive semidefinite zero forcing number.

An examination of the forcing game played on the graph blowup reveals that $r$-fold forcing sets with certain structure must always exist (Theorem 3.2.6). This result leads to the alternate definition of fractional positive semidefinite forcing number presented in Theorem 3.2.17. Section 3.2.3 builds on these results by introducing a new forcing game that uses three colors – dark blue, light blue, and white. It is then shown that this new

game allows direct computation of the fractional positive semidefinite forcing number of a graph without appealing to the $r$-fold game, a main result of Chapter 3.

Section 3.3 applies the three-color approach considered for fractional positive semidefinite forcing to (standard) zero forcing, providing an alternate characterization of the skew zero forcing game, a specific type of zero forcing game that was previously considered in [2]. This interpretation is used throughout the chapter to gain new insight into the skew zero forcing game and to prove new results about skew zero forcing. In Section 3.3.3, we apply the fractionalization process to the standard zero forcing game, and Theorem 3.3.18 shows that the "fractional forcing number" of a graph is actually the skew zero forcing number of the graph. The three-color approach also shows its use in Section 3.3.4, where an algorithm is used to completely characterize graphs whose skew zero forcing number equals zero (Theorem 3.3.22); this is another main result of Chapter 3.

## 1.2   Organization of the thesis

This thesis is a collection of research papers submitted to journals. Chapter 1 provides an overview of the topics discussed in the remaining chapters and serves to elucidate the common themes of this work. Note that all papers follow the mathematical convention of alphabetizing the authors' names.

Chapter 2 contains the paper "Orthogonal Representations, Projective Rank, and Fractional Minimum Positive Semidefinite Rank: Connections and New Directions," which is joint work of Kevin F. Palmowski with Leslie Hogben, David E. Roberson, and Simone Severini. Kevin Palmowski was responsible for most of the research and almost all of the writing for this paper. A version of this paper was submitted to *Electronic Journal of Linear Algebra*; the differences between the submitted version and that presented in this thesis are minor, non-mathematical editorial changes.

Chapter 3 contains the paper "Fractional Zero Forcing via Three-color Forcing Games," which is joint work of Kevin F. Palmowski with Leslie Hogben, David E. Roberson, and Michael Young. The preliminary research for this paper was conducted jointly by all authors over the course of one week during a visit of David E. Roberson to Iowa State University. Results were subsequently refined and proof details were filled in by Kevin Palmowski, who was responsible for almost all of the writing of this paper. A version of this paper was submitted to *Discrete Applied Mathematics*; the differences between the submitted version and that presented in this thesis are minor, non-mathematical editorial changes.

Concluding remarks and a discussion of future avenues for research are presented in Chapter 4.

# Bibliography

[1] S. Fallat and L. Hogben, Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs, in *Handbook of Linear Algebra*, 2nd ed., L. Hogben, ed., CRC Press, Boca Raton, FL, 2013.

[2] IMA-ISU research group on minimum rank (M. Allison, E. Bodine, L. M. DeAlba, J. Debnath, L. DeLoss, C. Garnett, J. Grout, L. Hogben, B. Im, H. Kim, R. Nair, O. Pryporova, K. Savage, B. Shader, A. Wangsness Wehe), Minimum rank of skew-symmetric matrices described by a graph, *Linear Algebra Appl.*, 432: 2457–2472, 2010.

[3] D. E. Roberson and L. Mančinska, Graph Homomorphisms for Quantum Players, to appear in *J. Combin. Theory Ser. B* (2014), arXiv:1212.1724 [quant-ph], 2012.

[4] E. Scheinerman and D. Ullman, *Fractional Graph Theory*, Dover, Mineola, NY, 2011; also available online from http://www.ams.jhu.edu/~ers/fgt/.

# CHAPTER 2. ORTHOGONAL REPRESENTATIONS, PROJECTIVE RANK, AND FRACTIONAL MINIMUM POSITIVE SEMIDEFINITE RANK: CONNECTIONS AND NEW DIRECTIONS

Modified from a paper submitted to *Electronic Journal of Linear Algebra*

Leslie Hogben[*][†], Kevin F. Palmowski[†], David E. Roberson[‡], and Simone Severini[§]

## Abstract

Fractional minimum positive semidefinite rank is defined from $r$-fold faithful orthogonal representations and it is shown that the projective rank of any graph equals the fractional minimum positive semidefinite rank of its complement. An $r$-fold version of the traditional definition of minimum positive semidefinite rank of a graph using Hermitian matrices that fit the graph is also presented. This paper also introduces $r$-fold orthogonal representations for graphs and formalizes the understanding of projective rank as fractional orthogonal rank. Connections of these concepts to quantum theory, including Tsirelson's problem, are discussed.

[*]American Institute of Mathematics, 600 E. Brokaw Rd., San Jose, CA 95112, USA (hogben@aimath.org).

[†]Department of Mathematics, Iowa State University, Ames, IA 50011, USA ({LHogben,kpalmow}@iastate.edu).

[‡]Division of Mathematical Sciences, Nanyang Technological University, SPMS-MAS-03-01, 21 Nanyang Link, Singapore 637371 (droberson@ntu.edu.sg).

[§]Department of Computer Science, University College London, Gower Street, London WC1E 6BT, United Kingdom (simoseve@gmail.com).

## 2.1 Introduction

This paper deals with fractional versions of graph parameters defined by orthogonal representations, including minimum positive semidefinite rank. In Section 2.2, we extend the existing idea of an orthogonal representation for a graph via a "higher-dimensional" construction. With this, we introduce a new parameter, $r$-fold orthogonal rank, that is to orthogonal rank as $b$-fold chromatic number is to chromatic number (see Section 2.1.2 for the definition of $b$-fold chromatic number and other terms related to fractional chromatic number). This allows us to formally characterize projective rank as "fractional orthogonal rank," a concept that was previously understood (e.g., in [14, 15]) but not rigorously presented (formal definitions of projective rank and other parameters are given in Section 2.1.3).

In Section 2.3, we apply this "fractionalization" process to the minimum positive semidefinite rank problem (viewed via faithful orthogonal representations) and develop two new graph parameters, namely, $r$-fold and fractional minimum positive semidefinite rank. We also provide an alternate definition of $r$-fold minimum positive semidefinite rank that is based on the minimum rank of a matrix that "$r$-fits" a graph, allowing us to view the "higher-dimensional" problem through either of the two viewpoints traditionally associated with the classical minimum positive semidefinite rank problem.

Our final result, found in Section 2.3.5, shows that the fractional minimum positive semidefinite rank of a graph is equal to the projective rank of the complement of the graph. This result serves to connect the two seemingly different problems; moving forward, this will allow the extensive existing literature on minimum positive semidefinite rank to be used to inform new developments in the more recently introduced area of projective rank.

In the remainder of this introduction we discuss applications of the fractional parameters discussed (Section 2.1.1), give a brief introduction to the fractional approach

to chromatic number to motivate our definitions (Section 2.1.2), and provide necessary notation and terminology (Section 2.1.3).

### 2.1.1  Applications

Linear algebraic structures and associated graph theoretic frameworks have recently become more important tools to study the fundamental differences that characterize theories of nature, like classical mechanics, quantum mechanics, and general probabilistic theories. Matrices, graphs, and their related combinatorial optimization techniques turn out to provide a surprisingly general language with which to approach questions connected with foundational ideas, such as the analysis of contextual inequalities and non-local games [2, 3], and with concrete aspects, such as quantifying various capacities of entanglement-assisted channels [6, 10], and the overhead needed to classically simulate quantum computation [9].

A point of strength of such frameworks is their ability to reformulate mathematical questions in a coarser manner that is nonetheless effective, in some cases, to single out specific facts. Tsirelson's problem [17] provides a remarkable example: deciding whether the mathematical models of non-relativistic quantum mechanics, where observers have linear operators acting on a finite dimensional tensor product space, and algebraic quantum field theory, where observers have commuting linear operators on a single (possibly infinite dimensional) space, produce the same set of correlations. We know that if Tsirelson's problem has a positive answer then the notorious Connes' Embedding conjecture [4, 11], originally concerned with an approximation property for finite von Neumann algebras, is true.

Tsirelson's problem can be seen from a combinatorial matrix point of view by working with graphs and their associated algebraic structures [12]. Roughly speaking, instead of constructing sets of correlation matrices, we can try looking for various patterns of zeroes in the sets, as in the spirit of combinatorial matrix theory. The projective rank, denoted

$\xi_f$, is a recently introduced graph parameter with the potential for settling the above discussion. Indeed, it has been shown that if there exists a graph whose projective rank is irrational, then Tsirelson's problem has a negative answer [13].

Projective representations and projective rank were originally defined in [15] as a tool for studying quantum colorings and quantum homomorphisms of graphs. Quantum colorings and the quantum chromatic number give quantitative measures of the advantage that quantum entanglement provides in performing distributed tasks and in distinguishing scenarios related to classical and quantum physics, respectively. In fact, the existence of a quantum $n$-coloring for a given graph is equivalent to the existence of a projective representation of value $n$ for the Cartesian product of the graph with a complete graph on $n$ vertices.

It was also shown in [15] that projective rank is monotone with respect to quantum homomorphisms, i.e., if there exists a quantum homomorphism from a graph $G$ to a graph $H$, then $\xi_f(G) \le \xi_f(H)$. This shows that projective rank is a lower bound for quantum chromatic number, and more generally provides a method for forbidding the existence of quantum homomorphisms. Indeed, this approach was used to determine the quantum odd girth of the Kneser graphs in [14]. Projective rank has also been studied from a purely graph theoretic point of view, and in [5] it was shown that this parameter is multiplicative with respect to the lexicographic and disjunctive graph products. Using this fact the authors were able to find a separation between quantum chromatic number and a recently defined semidefinite relaxation of this parameter, answering a question posed in [12].

This paper takes a linear algebraic approach to these questions, building connections between recent graph theoretical approaches to quantum questions and existing literature on orthogonal representations and minimum positive semidefinite rank.

### 2.1.2    A fractional approach

To demonstrate the fractional approach that we use with orthogonal representations and minimum positive semidefinite rank, consider the following derivation of the fractional chromatic number as found in [16]. The *chromatic number* $\chi(G)$ of a graph $G$ is the least number $c$ such that $G$ can be colored with $c$ colors; that is, we can assign to each vertex of $G$ one of $c$ colors in such a way that adjacent vertices receive different colors. A coloring with $c$ colors can be generalized to a *b-fold coloring with c colors*, or a *c:b*-coloring: from a palette of $c$ colors, assign $b$ colors to each vertex of $G$ such that adjacent vertices receive disjoint sets of colors. For a fixed $b$, the *b-fold chromatic number* of $G$, $\chi_b(G)$, is the smallest $c$ such that G has a *c:b*-coloring. With this, we can define the *fractional chromatic number* of $G$ as

$$\chi_f(G) = \inf_b \frac{\chi_b(G)}{b} \, .$$

While it is not obvious, it can be shown that $\chi_f(G)$ is always a rational number, as there is an alternative linear programming formulation for the parameter for which strong duality holds. For further information on fractional coloring, including a time-scheduling interpretation of the problem, see the discussions in the Preface and Chapter 3 of [16].

The process of assigning objects to the vertices of a graph, subject to certain constraints, is a key element common to the problems we examine in this work, and the procedure of generalizing from assigning one object to assigning $b$-many objects (or, in our case, $b$-dimensional or rank-$b$ objects) is an underlying theme. At each stage of the process, we are interested in graph parameters that give information about the "most efficient" set of objects we can use, with the end goal of developing fractional versions of existing parameters (in the spirit of [16]) and connecting the more recent work on projective rank with existing ideas from the realm of minimum positive semidefinite rank.

Rather than the colors used for coloring problems, the objects that we assign to the vertices of a graph are vectors and matrices, which adds a distinctly linear algebraic flavor to both the problems and the constraints: the idea of "different colors" translates to orthogonality conditions on our objects. As such, our results often see linear algebra and graph theory working hand-in-hand, with structure found in one discipline influencing results that are based in the other.

### 2.1.3   Background, definitions, and notation

The natural numbers, $\mathbb{N}$, start at 1. We use the notation $[a:b]$ to denote the set of integers $\{a, a+1, \ldots, b-1, b\}$. Throughout, $d$ and $r$ are used to represent natural numbers. Vectors are denoted by boldface font, typically $\mathbf{x}$, and matrices are capital letters, typically $A$, $B$, $P$, or $X$, depending on context. The symbol $0$ denotes either the scalar zero or a zero matrix, and an identity matrix is denoted by $I$; any of these may be subscripted to clarify their sizes. We follow the usual convention of denoting the $j^{th}$ standard basis vector in $\mathbb{C}^d$ (for some $d$) as $\mathbf{e}_j$. Rows and columns of matrices may be indexed either by natural numbers or by vertices of a graph, depending on context. The elements of a matrix $A$ are denoted $a_{ij}$; if $A$ is a block matrix, then its blocks are denoted $A_{ij}$. Graphs are usually denoted by $G$ or $H$, vertices by $u, v$ or $i, j$, and edges by $uv$ or $ij$.

If $A \in \mathbb{C}^{p \times p}$ and $B \in \mathbb{C}^{q \times q}$, then the *direct sum* of $A$ and $B$, denoted $A \oplus B$, is the block diagonal matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{C}^{(p+q) \times (p+q)}.$$

We denote the conjugate transpose of $A$ by $A^*$. A *Hermitian* matrix satisfies $A = A^*$. A Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is *positive semidefinite*, denoted $A \succeq 0$, if $\mathbf{x}^* A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{C}^n$, or equivalently, if all of its eigenvalues are nonnegative.

Typically, $G = (V, E)$ will denote a simple undirected graph on $n$ vertices, where $V = V(G)$ is the set of vertices of $G$ and $E = E(G)$ is the set of edges of $G$. An *isolated vertex* is a vertex that is not adjacent to any other vertex of $G$. A *subgraph* of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* of a graph $G$, denoted $G[W]$ for some set $W \subseteq V(G)$, is a subgraph with vertex set $W$ such that if $u, v \in W$ and $uv \in E(G)$, then $uv \in E(G[W])$. The *union* of graphs $G$ and $H$, denoted $G \cup H$, is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then this union is *disjoint* and denoted $G \dot\cup H$. The *complement* of $G$, denoted $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u \neq v, uv \notin E(G)\}$. An *independent set* in $G$ is a set $W \subseteq V(G)$ such that if $u, v \in W$, then $uv \notin E(G)$. The *independence number* of $G$, denoted $\alpha(G)$, is the largest possible cardinality of an independent set in $G$. A *clique* in $G$ is an induced subgraph $H$ that is a complete graph, i.e., $uv \in E(H)$ for every $u, v \in V(H)$. The *clique number* of $G$, denoted $\omega(G)$, is the largest possible order of a clique in $G$. A *clique-sum* of graphs $G$ and $H$ on $K_t$, i.e., the graph $G \cup H$ where $G \cap H = K_t$, is denoted by $G \langle K_t \rangle H$; this is also called a $t$-clique-sum of $G$ and $H$. A *chordal graph* is a graph that does not have any induced cycles of length greater than 3; any chordal graph can be constructed as clique-sum(s) of complete graphs. A *perfect graph* is a graph $G$ for which every induced subgraph $H$ of $G$ satisfies $\omega(H) = \chi(H)$. A *cut-vertex* of a connected graph $G$ is a vertex whose deletion disconnects $G$. A graph with a cut-vertex can be viewed as a 1-clique-sum.

We work in the vector space $\mathbb{C}^d$ for some $d \in \mathbb{N}$. We use $S$ to denote a subspace of a vector space. A *basis matrix* for an $r$-dimensional subspace $S$ of $\mathbb{C}^d$ is a matrix $X \in \mathbb{C}^{d \times r}$ that has orthonormal columns and satisfies $S = \text{range}(X)$. We say that two subspaces $S_1$ and $S_2$ of $\mathbb{C}^d$ are *orthogonal*, denoted $S_1 \perp S_2$, if $\mathbf{u}_1^* \mathbf{u}_2 = 0$ for all $\mathbf{u}_1 \in S_1$ and all $\mathbf{u}_2 \in S_2$; an equivalent condition is that $X_1^* X_2 = 0$, where $X_1$ and $X_2$ are basis matrices for $S_1$ and $S_2$, respectively.

Given some graph $G$ and $d \in \mathbb{N}$, an *orthogonal representation* in $\mathbb{C}^d$ for $G$ is a set of unit vectors $\{\mathbf{x}_u\}_{u \in V(G)} \subset \mathbb{C}^d$ such that $\mathbf{x}_u^* \mathbf{x}_v = 0$ if $uv \in E(G)$. It is clear that such a representation always exists for $d = |V(G)|$. Provided that $G$ has at least one edge, it is clear that such a representation cannot be made for $d = 1$. We define the *orthogonal rank* of $G$ to be

$$\xi(G) = \min \left\{ d : G \text{ has an orthogonal representation in } \mathbb{C}^d \right\}.$$

Let $d, r \in \mathbb{N}$ with $r \leq d$. A *$d/r$-projective representation*, or *$d/r$-representation*, is an assignment of matrices $\{P_u\}_{u \in V(G)}$ to the vertices of $G$ such that

- for each $u \in V(G)$, $P_u \in \mathbb{C}^{d \times d}$, $\operatorname{rank} P_u = r$, $P_u^* = P_u$, and $P_u^2 = P_u$; and

- if $uv \in E(G)$, then $P_u P_v = 0$.

In words, a $d/r$-representation is an assignment of rank-$r$ ($d \times d$) orthogonal projection matrices (projectors) to the vertices of $G$ such that adjacent vertices receive projectors that are orthogonal. The *projective rank* of $G$ is defined as

$$\xi_f(G) = \inf_{d,r} \left\{ \frac{d}{r} : G \text{ has a } d/r\text{-representation} \right\}.$$

Projective rank was first introduced in 2012 by Roberson and Mančinska, where it is noted that $\xi_f(G) \leq \xi(G)$; see [14] and [15] for additional information, properties, and applications.

Complementary to the idea of an orthogonal representation is that of a *faithful* orthogonal representation (here we follow the complementary usage in the minimum rank literature). In order for the definitions given next to coincide with those in the minimum rank literature, we must assume that the graph $G$ has no isolated vertices. A *faithful orthogonal representation* in $\mathbb{C}^d$ for a graph $G$ is a set of unit vectors $\{\mathbf{x}_u\}_{u \in V(G)} \subset \mathbb{C}^d$ such that $\mathbf{x}_u^* \mathbf{x}_v = 0$ if and only if $uv \notin E(G)$. We define the *minimum positive semidefinite rank* of $G$ as

$$\operatorname{mr}^+(G) = \min \left\{ d : G \text{ has a faithful orthogonal representation in } \mathbb{C}^d \right\}. \tag{2.1}$$

We say that a matrix $A \in \mathbb{C}^{n \times n}$ *fits* the order-$n$ graph $G$ if $a_{ii} = 1$ for all $i \in [1:n]$, and for all $i \neq j$, we have $a_{ij} = 0$ if and only if $ij \notin E(G)$. Let $\mathcal{H}^+(G) = \{A \in \mathbb{C}^{n \times n} : A \succeq 0 \text{ and } A \text{ fits } G\}$. A faithful orthogonal representation in $\mathbb{C}^d$ for $G$ corresponds to a matrix $A \in \mathcal{H}^+(G)$ with rank $A \leq d$, and a matrix $A \in \mathcal{H}^+(G)$ with rank $d$ can be factored as $A = B^*B$ for some $B \in \mathbb{C}^{d \times n}$. Thus an alternate characterization (see, e.g., [7]) of $\mathrm{mr}^+(G)$ is

$$\mathrm{mr}^+(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{H}^+(G)\},$$

(and in fact, this is the customary definition of this parameter).

The definitions and explanation given here coincide with those in the literature provided that the graph $G$ has no isolated vertices. The most common definition of $\mathcal{H}^+(G)$ in the literature does not contain the assumption that $a_{ii} = 1$. If vertex $i$ is adjacent to at least one other vertex, then properties of positive semidefinite matrices require $a_{ii} > 0$, and so $A$ can be scaled by a positive diagonal congruence to a matrix of the same rank and nonzero pattern that has all diagonal entries equal to one. However, consider the case where $G$ consists of $n$ isolated vertices (no edges): then as defined in [1, 7], etc., $\mathrm{mr}^+(G) = 0$, whereas with our definition $\mathrm{mr}^+(G) = n$. The two definitions of minimum positive semidefinite rank coincide precisely when $G$ has no isolated vertices. Our definition facilitates connections to the use of orthogonal rank in the study of quantum issues, and the assumption of no isolated vertices is needed only when connecting to the minimum rank literature, so we omit it except when discussing connections to such work (where we state either this assumption or one that implies it, such as the graph being connected and of order at least two). We also note that for any graph the values of the parameters studied can be computed from their values on the connected components of the graph (see Section 2.3), which facilitates handling cases with isolated vertices separately.

## 2.2 Orthogonal subspace representations and projective rank

In this section, we introduce and discuss $(d; r)$ orthogonal subspace representations for a graph $G$, which are extensions of orthogonal representations in the spirit of fractional graph theory [16]. The $r$-fold orthogonal rank of a graph, $\xi_{[r]}(G)$, is defined and some properties of this quantity are examined. We then relate these representations to $d/r$-projective representations and tie projective rank into the new theory, formalizing the existing understanding that projective rank and "fractional orthogonal rank" are one and the same.

Unless otherwise specified, all matrices and vectors in this section are assumed to be complex-valued.

### 2.2.1 Orthogonal subspace representations and $r$-fold orthogonal rank

Let $G$ be a graph and let $d, r \in \mathbb{N}$ with $d \geq r$. A $(d; r)$ *orthogonal subspace representation*, or $(d; r)$-OSR, for $G$ is a set of subspaces $\{S_u\}_{u \in V(G)}$ such that

- for each $u \in V(G)$, $S_u$ is an $r$-dimensional subspace of $\mathbb{C}^d$; and

- if $uv \in E(G)$, then $S_u \perp S_v$.

The *$r$-fold orthogonal rank* of a graph $G$ is defined by

$$\xi_{[r]}(G) = \min \{d : G \text{ has a } (d; r) \text{ orthogonal subspace representation}\}.$$

An orthogonal representation in $\mathbb{C}^d$ naturally generates a $(d; 1)$ orthogonal subspace representation, and vice versa, so $\xi(G) = \xi_{[1]}(G)$.

We now explore some properties of $\xi_{[r]}(G)$.

**Lemma 2.2.1.** $\xi_{[r]}$ *is a subadditive function of* $r$, *i.e., for every graph* $G$ *and all* $r, s \in \mathbb{N}$,

$$\xi_{[r+s]}(G) \leq \xi_{[r]}(G) + \xi_{[s]}(G).$$

*Proof.* Let $d_r = \xi_{[r]}(G)$ and $d_s = \xi_{[s]}(G)$. Then $G$ has a $(d_r; r)$ orthogonal subspace representation containing $r$-dimensional subspaces of $\mathbb{C}^{d_r}$, say $\{S_u^r\}_{u \in V(G)}$, and a $(d_s; s)$ orthogonal subspace representation containing $s$-dimensional subspaces of $\mathbb{C}^{d_s}$, say $\{S_u^s\}_{u \in V(G)}$. We show by construction that there exists an orthogonal subspace representation for $G$ containing $(r + s)$-dimensional subspaces of $\mathbb{C}^{d_r + d_s}$.

For each $u \in V(G)$, let $X_u^r \in \mathbb{C}^{d_r \times r}$ and $X_u^s \in \mathbb{C}^{d_s \times s}$ be basis matrices for $S_u^r$ and $S_u^s$, respectively. Define

$$X_u = \begin{bmatrix} X_u^r & 0_{d_r \times s} \\ 0_{d_s \times r} & X_u^s \end{bmatrix} \in \mathbb{C}^{(d_r + d_s) \times (r+s)}$$

and let $S_u = \text{range}(X_u)$. We immediately see that $S_u$ is a subspace of $\mathbb{C}^{d_r + d_s}$, $X_u$ is a basis matrix for $S_u$, and $\dim(S_u) = \text{rank } X_u = \text{rank } X_u^r + \text{rank } X_u^s = r + s$.

Suppose $u, v \in V(G)$ and let $X_u^r, X_v^r, X_u^s, X_v^s, X_u$, and $X_v$ be as above; then

$$X_u^* X_v = \begin{bmatrix} (X_u^r)^*(X_v^r) & 0 \\ 0 & (X_u^s)^*(X_v^s) \end{bmatrix}.$$

Suppose $uv \in E(G)$. Since $\{S_u^r\}$ is an orthogonal subspace representation, we have $(X_u^r)^*(X_v^r) = 0$; similarly, $(X_u^s)^*(X_v^s) = 0$, so $X_u^* X_v = 0$. Since $X_u$ and $X_v$ are basis matrices for $S_u$ and $S_v$, respectively, we conclude that if $uv \in E(G)$, then $S_u \perp S_v$.

Thus $\{S_u\}_{u \in V(G)}$ is a $(d_r + d_s; r + s)$ orthogonal subspace representation for $G$, so $\xi_{[r+s]}(G) \leq d_r + d_s = \xi_{[r]}(G) + \xi_{[s]}(G)$. $\qquad \square$

**Corollary 2.2.2.** *For every graph* $G$ *and all* $r \in \mathbb{N}$, $\frac{\xi_{[r]}(G)}{r} \leq \xi(G)$.

*Proof.* Since $\xi_{[1]}(G) = \xi(G)$, we have

$$\xi_{[r]}(G) \leq \xi_{[r-1]}(G) + \xi(G) \leq \ldots \leq r \cdot \xi(G). \qquad \square$$

**Observation 2.2.3.** *For every graph $G$ and all $r \in \mathbb{N}$, $\xi_{[r]}(G) \geq r \cdot \omega(G)$.*

**Proposition 2.2.4.** *Let $r \in \mathbb{N}$ and let $H$ be a subgraph of $G$. Then $\xi_{[r]}(H) \leq \xi_{[r]}(G)$.*

*Proof.* Since every edge of $H$ is an edge of $G$, any $(d; r)$ orthogonal subspace representation for $G$ provides a $(d; r)$ orthogonal subspace representation for $H$, and the result is immediate. $\square$

**Proposition 2.2.5.** *Suppose $r \in \mathbb{N}$ and $G = \dot{\bigcup}_{i=1}^{t} G_i$ for some graphs $\{G_i\}_{i=1}^{t}$. Then $\xi_{[r]}(G) = \max_i \{\xi_{[r]}(G_i)\}$.*

*Proof.* Since each $G_i$ is an induced subgraph of $G$, we have $\xi_{[r]}(G_i) \leq \xi_{[r]}(G)$ for each $i$, so $\max_i \{\xi_{[r]}(G_i)\} \leq \xi_{[r]}(G)$.

For each $i \in [1 : t]$, let $d_i = \xi_{[r]}(G_i)$ and let $d = \max_i\{d_i\}$. Let $\{S_u^i\}_{u \in V(G_i)}$ be a $(d_i; r)$ orthogonal subspace representation for $G_i$ and for each vertex $u \in V(G_i)$ let $X_u^i \in \mathbb{C}^{d_i \times r}$ be a basis matrix for $S_u^i$. For each $u \in V(G)$, we have $u \in V(G_i)$ for some $i$; define

$$S_u = \text{range} \begin{bmatrix} X_u^i \\ 0_{(d-d_i) \times r} \end{bmatrix}.$$

Each $S_u$ is an $r$-dimensional subspace of $\mathbb{C}^d$, and if $uv \in E(G)$, then $uv \in E(G_k)$ for some $k$, so $S_u^k \perp S_v^k$, which implies that $S_u \perp S_v$ (by construction). Therefore, $\{S_u\}_{u \in V(G)}$ is a $(d; r)$-OSR for $G$, so $\xi_{[r]}(G) \leq d = \max_i\{\xi_{[r]}(G_i)\}$ and equality follows. $\square$

This result does not hold for arbitrary graph unions, as the following example for the $r = 1$ case shows.

**Example 2.2.6.** Let $G = C_5$ with $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{12, 23, 34, 45, 51\}$. Define $G_1 = P_4$ with $V(G_1) = \{1, 2, 3, 4\}$ and $E(G_1) = \{12, 23, 34\}$ and define $G_2 = P_3$ with $V(G_2) = \{4, 5, 1\}$ and $E(G_2) = \{45, 51\}$. We see that $G = G_1 \cup G_2$, but since $\xi(P_3) = \xi(P_4) = 2$ and $\xi(C_5) = 3$, it is not true that $\xi(G) = \max\{\xi(G_1), \xi(G_2)\}$.

While the maximum property observed in Proposition 2.2.5 may not carry over to the case when $G$ is a nondisjoint union of graphs, we are still able to obtain a weaker result, which follows.

**Proposition 2.2.7.** *Suppose* $r \in \mathbb{N}$ *and* $G = \bigcup_{i=1}^{t} G_i$, *where* $G_i$ *is an induced subgraph of* $G$ *for each* $i$. *Then* $\xi_{[r]}(G) \leq \sum_{i=1}^{t} \xi_{[r]}(G_i)$.

*Proof.* We prove the result for the case where $t = 2$ and note that recursive application of this case will prove the more general one.

For each $i \in \{1, 2\}$, let $d_i = \xi_{[r]}(G_i)$ and $\{S_u^i\}_{u \in V(G_i)}$ be a $(d_i; r)$-OSR for $G_i$, and for each $u \in V(G_i)$, let $X_u^i \in \mathbb{C}^{d_i \times r}$ be a basis matrix for $S_u^i$.

We partition $V(G) = V(G_1) \cup V(G_2)$ into three disjoint sets and consider vertices in each set. If $u \in V(G_1) \setminus V(G_2)$, let

$$X_u = \begin{bmatrix} X_u^1 \\ 0_{d_2 \times r} \end{bmatrix};$$

if $u \in V(G_2) \setminus V(G_1)$, let

$$X_u = \begin{bmatrix} 0_{d_1 \times r} \\ X_u^2 \end{bmatrix};$$

and if $u \in V(G_1) \cap V(G_2)$, let

$$X_u = \begin{bmatrix} X_u^1 \\ X_u^2 \end{bmatrix}.$$

For each $u \in V(G)$, let $S_u = \mathrm{range}(X_u)$. Each $S_u$ is an $r$-dimensional subspace of $\mathbb{C}^{d_1 + d_2}$.

We consider multiple cases to show that if $uv \in E(G)$, then $X_u^* X_v = 0$, so $S_u \perp S_v$. Throughout, we assume that $uv \in E(G)$.

First, suppose that $u \in V(G_1) \setminus V(G_2)$; then either $v \in V(G_1) \setminus V(G_2)$ or $v \in V(G_1) \cap V(G_2)$. In either case, $uv \in E(G_1)$ (since $G_1$ is an induced subgraph), and block multiplication yields $X_u^* X_v = (X_u^1)^* X_v^1$. Since $S_u^1 \perp S_v^1$, this quantity equals the zero matrix, so $S_u \perp S_v$. The case where $u \in V(G_2) \setminus V(G_1)$ is similar.

If $u, v \in V(G_1) \cap V(G_2)$, then $uv \in E(G_1) \cap E(G_2)$ since $G_1$ and $G_2$ are induced subgraphs. Then $X_u^* X_v = (X_u^1)^* X_v^1 + (X_u^2)^* X_v^2$. Since $S_u^1 \perp S_v^1$ and $S_u^2 \perp S_v^2$, this quantity is again the zero matrix, so $S_u \perp S_v$.

Therefore, $\{S_u\}_{u \in V(G)}$ is a $(d_1 + d_2; r)$-OSR for $G$, so $\xi_{[r]}(G) \leq d_1 + d_2 = \xi_{[r]}(G_1) + \xi_{[r]}(G_2)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

**Lemma 2.2.8.** *Suppose that the complete graph $K_t$ is a subgraph of $G$ with $V(K_t) = [1:t]$ and $G$ has a $(d; r)$ orthogonal subspace representation. Then $d \geq rt$ and $G$ has a $(d; r)$ orthogonal subspace representation in which the vertex $i \in V(K_t)$ is represented by*

$$\text{span}\left\{\mathbf{e}_{(i-1)r+1}, \ldots, \mathbf{e}_{(i-1)r+r-1}, \mathbf{e}_{ir}\right\}.$$

*Proof.* By Observation 2.2.3, $d \geq r \cdot \omega(G) \geq rt$.

If $M \in \mathbb{C}^{d \times \ell}$ for some $\ell \leq d$ and the columns of $M$ are orthonormal, then by a change of orthonormal basis there exists a unitary matrix $U \in \mathbb{C}^{d \times d}$ such that $UM = [\mathbf{e}_1, \ldots, \mathbf{e}_\ell]$.

Let $\{S_u\}_{u \in V(G)}$ be a $(d; r)$ orthogonal subspace representation for $G$ and for each $u \in V(G)$ let $X_u$ be a basis matrix for $S_u$. Define $M = [X_1, \ldots, X_t]$ and choose $U$ so that $UM = [\mathbf{e}_1, \ldots, \mathbf{e}_{tr}]$. Define $S_u' = \text{range}(UX_u)$. Then $\{S_u'\}_{u \in V(G)}$ is a $(d; r)$ orthogonal subspace representation for $G$ with the desired property. $\qquad\qquad\qquad\qquad\quad \square$

**Theorem 2.2.9.** *If $G = G_1 \langle K_t \rangle G_2$ and $r \in \mathbb{N}$, then $\xi_{[r]}(G) = \max\left\{\xi_{[r]}(G_1), \xi_{[r]}(G_2)\right\}$.*

*Proof.* Without loss of generality, let $d_1 = \xi_{[r]}(G_1) \geq d_2 = \xi_{[r]}(G_2)$ and $V(K_t) = [1:t]$. Then by Lemma 2.2.8, for $i = 1, 2$, each $G_i$ has a $(d_1; r)$ orthogonal subspace represen-tation, $\{S_u^i\}_{u \in V(G)}$, in which vertex $v \leq t$ is represented by $S_v^i = \text{span}\left\{\mathbf{e}_{(v-1)r+1}, \ldots, \mathbf{e}_{(v-1)r+r-1}, \mathbf{e}_{vr}\right\}$. Thus for $v \in [1:t]$, $S_v^1 = S_v^2$; denote this common subspace by $S_v$.

For vertices $u \in V(G_i) \setminus [1:t]$, define $S_u = S_u^i$ (observe that $u > t$ is in only one of $V(G_1)$ or $V(G_2)$). Then $\{S_u\}_{u \in V(G)}$ is a $(d_1; r)$ orthogonal subspace representation for $G$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

**Proposition 2.2.10.** *If $G$ is a graph with $\omega(G) = \chi(G)$, then $\xi_{[r]}(G) = r \cdot \omega(G)$ for every $r \in \mathbb{N}$.*

*Proof.* It is well-known that $\xi(G) \leq \chi(G)$ (see, e.g., [14]). Therefore,

$$r \cdot \omega(G) \leq \xi_{[r]}(G) \leq r \cdot \xi(G) \leq r \cdot \chi(G) = r \cdot \omega(G)$$

and thus equality holds throughout. $\square$

We note that perfect graphs and chordal graphs are among those that satisfy $\omega(G) = \chi(G)$, and so Proposition 2.2.10 applies to these classes.

**Remark 2.2.11.** Since $\xi_{[1]}(G) = \xi(G)$ for every graph $G$, the previous properties of $r$-fold orthogonal rank also apply to orthogonal rank, where appropriate.

### 2.2.2 Projective rank as fractional orthogonal rank

It is easy to see that $(d; r)$ orthogonal subspace representations are closely related to $d/r$-representations; in fact, they are in one-to-one correspondence.

**Proposition 2.2.12.** *A graph $G$ has a $(d; r)$ orthogonal subspace representation if and only if $G$ has a $d/r$-representation.*

*Proof.* Suppose that $G$ has a $(d; r)$ orthogonal subspace representation $\{S_u\}_{u \in V(G)}$, so each $S_u$ is an $r$-dimensional subspace of $\mathbb{C}^d$. For each $u \in V(G)$, define $P_u = X_u X_u^*$, where $X_u \in \mathbb{C}^{d \times r}$ is a basis matrix for $S_u$. It is then easy to verify that $P_u \in \mathbb{C}^{d \times d}$, $\operatorname{rank} P_u = \operatorname{rank} X_u = r$, $P_u^* = P_u$, and $P_u^2 = P_u$.

Let $uv \in E(G)$, so $S_u \perp S_v$. We see that

$$S_u \perp S_v \iff X_u^* X_v = 0 \iff X_u X_u^* X_v X_v^* = 0 \iff P_u P_v = 0.$$

Thus if $uv \in E(G)$, then $P_u P_v = 0$. We conclude that $\{P_u\}_{u \in V(G)}$ is a $d/r$-representation for $G$.

For the converse, suppose that $\{P_u\}_{u \in V(G)}$ is a $d/r$-representation for $G$. For each $u \in V(G)$, let $P_u = X_u I_r X_u^*$ be a reduced singular value decomposition of the projector $P_u$ (where $X_u \in \mathbb{C}^{d \times r}$) and define $S_u = \text{range}(P_u) = \text{range}(X_u)$. Clearly $S_u$ is an $r$-dimensional subspace of $\mathbb{C}^d$. If $uv \in E(G)$, then $P_u P_v = 0$, so by the above chain of equivalences $S_u \perp S_v$. Therefore, $\{S_u\}_{u \in V(G)}$ is a $(d; r)$ orthogonal subspace representation for $G$. $\qquad\square$

With this in mind, we obtain the following "fractional" definition of projective rank.

**Theorem 2.2.13.** *For every graph $G$,*

$$\xi_f(G) = \inf_r \left\{ \frac{\xi_{[r]}(G)}{r} \right\}.$$

*Proof.*

$$\inf_r \left\{ \frac{\xi_{[r]}(G)}{r} \right\} = \inf_r \left\{ \frac{\min\{d : G \text{ has a } (d; r)\text{-OSR}\}}{r} \right\}$$
$$= \inf_r \left\{ \min_d \left\{ \frac{d}{r} : G \text{ has a } (d; r)\text{-OSR} \right\} \right\}$$
$$= \inf_{d,r} \left\{ \frac{d}{r} : G \text{ has a } (d; r)\text{-OSR} \right\}$$
$$= \inf_{d,r} \left\{ \frac{d}{r} : G \text{ has a } d/r\text{-representation} \right\}$$
$$= \xi_f(G). \qquad\square$$

Given that this expression of $\xi_f(G)$ is similar to that of $\chi_f(G)$ given in [16], it is not unreasonable to hope that this could shed some light on the question of the rationality of $\xi_f(G)$ for all graphs.[1] Unfortunately, finding a $b$-fold coloring with $c$ colors for $G$ is ultimately a far different problem from finding a $(d; r)$ orthogonal subspace representation for $G$. In the $b$-fold coloring problem, we have a restriction on the number of available colors, which adds a certain finiteness to the problem: each vertex is assigned a subset of the available $c < \infty$ colors. In contrast, restricting the subspaces to lie in $\mathbb{C}^d$ in the

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[1]Recall that $\chi_f(G)$ is rational for any graph $G$.

orthogonal subspace representation problem does not impose this same type of finiteness: each vertex is assigned a finite dimensional subspace of $\mathbb{C}^d$, and $d < \infty$, but there are infinitely many subspaces that can be assigned to each vertex.

We provide one additional equivalent definition of projective rank, for which we need the following utility result from [16], also commonly known as Fekete's Lemma.

**Lemma 2.2.14** ([16], Lemma A.4.1). *Suppose $g : \mathbb{N} \to \mathbb{R}$ is subadditive and $g(n) \geq 0$ for all $n$. Then the limit*

$$\lim_{n \to \infty} \frac{g(n)}{n}$$

*exists and is equal to the infimum of $g(n)/n$ $(n \in \mathbb{N})$.*

Since $\xi_{[r]}$ is subadditive, this yields the following corollary to the previous theorem.

**Corollary 2.2.15.** *For every graph $G$,*

$$\xi_f(G) = \inf_r \left\{ \frac{\xi_{[r]}(G)}{r} \right\} = \lim_{r \to \infty} \frac{\xi_{[r]}(G)}{r} ,$$

*and this limit exists.*

With this result, we see that many of the properties of $\xi_{[r]}(G)$ also apply to $\xi_f(G)$.

**Theorem 2.2.16.** *For every graph $G$:*

*i)* [14, 15] $\xi_f(G) \geq \omega(G)$.

*ii) If $H$ is a subgraph of $G$, then $\xi_f(H) \leq \xi_f(G)$.*

*iii) If $G = \dot{\bigcup}_{i=1}^t G_i$ for some graphs $\{G_i\}_{i=1}^t$, then $\xi_f(G) = \max_i \{\xi_f(G_i)\}$.*

*iv) If $G = \bigcup_{i=1}^t G_i$ for some induced subgraphs $\{G_i\}_{i=1}^t$, then $\xi_f(G) \leq \sum_{i=1}^t \xi_f(G_i)$.*

*v) If $G = G_1 \langle K_t \rangle G_2$, then $\xi_f(G) = \max \{\xi_f(G_1), \xi_f(G_2)\}$.*

*vi) If $G$ satisfies $\omega(G) = \xi(G)$, then $\xi_f(G) = \omega(G)$.*

*Proof.* Consider the second claim. By Proposition 2.2.4, for any $r \in \mathbb{N}$, $\xi_{[r]}(H) \leq \xi_{[r]}(G)$, so $\frac{\xi_{[r]}(H)}{r} \leq \frac{\xi_{[r]}(G)}{r}$. Taking the limit as $r$ approaches $\infty$ and applying Corollary 2.2.15, we have $\xi_f(H) \leq \xi_f(G)$.

The remaining claims follow by applying similar arguments to the corresponding $r$-fold results. $\qquad\square$

## 2.3 Fractional minimum positive semidefinite rank

In this section, we introduce $(d; r)$ faithful orthogonal subspace representations, $r$-fold minimum positive semidefinite rank, and fractional minimum positive semidefinite rank, extending the definitions of faithful orthogonal representations and minimum positive semidefinite rank. We then introduce faithful $d/r$-projective representations and connect everything to projective rank. A connection to positive semidefinite matrices is explored, and properties of our new quantities are proven.

Unless otherwise specified, all matrices and vectors in this section are assumed to be complex-valued (the literature on minimum positive semidefinite rank is mixed, with both real and complex cases studied).

### 2.3.1 Faithful orthogonal subspace representations and fractional minimum positive semidefinite rank

Given a graph $G$ and $d, r \in \mathbb{N}$ with $r \leq d$, a $(d; r)$ *faithful orthogonal subspace representation*, or $(d; r)$-FOSR, for $G$ is a set of subspaces $\{S_u\}_{u \in V(G)}$ where

- for each $u \in V(G)$, $S_u$ is an $r$-dimensional subspace of $\mathbb{C}^d$; and

- $S_u \perp S_v$ if and only if $uv \notin E(G)$.

A faithful orthogonal representation (as defined in Section 2.1.3) generates a $(d; 1)$ faithful orthogonal subspace representation, and vice versa. Further, a $(d; r)$-FOSR for a graph $G$ is a $(d; r)$-OSR for its complement $\overline{G}$, but the reverse statement is not true in general.

Now that we have defined an $r$-fold analogue of a faithful orthogonal representation, it is natural to consider a corresponding version of $\mathrm{mr}^+(G)$. The *$r$-fold minimum positive semidefinite rank* of $G$ is

$$\mathrm{mr}^+_{[r]}(G) = \min\{d : G \text{ has a } (d; r) \text{ faithful orthogonal subspace representation}\}.$$

In particular, we have $\mathrm{mr}^+_{[1]}(G) = \mathrm{mr}^+(G)$, using definition (2.1) of $\mathrm{mr}^+$; we caution the reader that this coincides with the definitions of faithful orthogonal representation and minimum positive semidefinite rank in the literature (e.g. [1, 7]) if and only if $G$ has no isolated vertices.

We note that $\mathrm{mr}^+_{[r]}(G)$ is subadditive. The proof is analogous to the proof of Lemma 2.2.1 and is omitted, as are the proofs for other results in this section that parallel those for the non-faithful case (i.e., the $\xi$-family of parameters).

**Lemma 2.3.1.** $\mathrm{mr}^+_{[r]}$ *is a subadditive function of $r$, i.e., for every graph $G$ and all $r, s \in \mathbb{N}$,*

$$\mathrm{mr}^+_{[r+s]}(G) \leq \mathrm{mr}^+_{[r]}(G) + \mathrm{mr}^+_{[s]}(G).$$

As in the non-faithful case, an immediate corollary relates $\mathrm{mr}^+_{[r]}$ to $\mathrm{mr}^+$.

**Corollary 2.3.2.** *For every graph $G$ and all $r \in \mathbb{N}$,*

$$\frac{\mathrm{mr}^+_{[r]}(G)}{r} \leq \mathrm{mr}^+(G).$$

For any graph $G$, we define the *fractional minimum positive semidefinite rank* of $G$ as

$$\mathrm{mr}^+_f(G) = \inf_r \left\{ \frac{\mathrm{mr}^+_{[r]}(G)}{r} \right\}.$$

Notice that if $G$ has a $(d; r)$ faithful orthogonal subspace representation, then $\mathrm{mr}^+_{[r]}(G) \leq d$, so $\mathrm{mr}^+_f(G) \leq \frac{d}{r}$.

We can upper bound fractional minimum positive semidefinite rank by the non-fractional version by using Corollary 2.3.2. Again, recall that this coincides with the literature if and only if the graph $G$ has no isolated vertices.

**Corollary 2.3.3.** *For every graph $G$,*

$$\mathrm{mr}_f^+(G) \leq \mathrm{mr}^+(G).$$

Since $\mathrm{mr}_{[r]}^+(G)$ is subadditive, we have the following corollary, which follows from Lemma 2.2.14 ([16], Lemma A.4.1).

**Corollary 2.3.4.** *For every graph $G$,*

$$\mathrm{mr}_f^+(G) = \lim_{r \to \infty} \frac{\mathrm{mr}_{[r]}^+(G)}{r},$$

*and this limit exists.*

We conclude this section with an example that gives further insight into these new parameters.

**Example 2.3.5.** Let $r \in \mathbb{N}$ and consider the graph $G = P_4$ with $V(P_4) = \{1, 2, 3, 4\}$ and $E(P_4) = \{12, 23, 34\}$. With $\mathbf{e}_i$ as the $i^{th}$ standard basis vector in $\mathbb{C}^{2r+1}$, we can verify that the following is a valid $(2r + 1; r)$-FOSR for $P_4$: $S_1 = \mathrm{range}([\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r])$, $S_2 = \mathrm{range}([\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_{r+1}])$, $S_3 = \mathrm{range}([\mathbf{e}_{r+1}, \mathbf{e}_{r+2}, \ldots, \mathbf{e}_{2r}])$, $S_4 = \mathrm{range}([\mathbf{e}_{r+2}, \mathbf{e}_{r+3}, \ldots, \mathbf{e}_{2r+1}])$. Therefore, $\mathrm{mr}_{[r]}^+(P_4) \leq 2r + 1$. Suppose that $\{Q_u\}_{u \in V(P_4)}$ is a $(2r; r)$-FOSR for $P_4$; we show that such a representation cannot exist. Since $13, 14 \notin E(P_4)$, $Q_1 \perp Q_3$ and $Q_1 \perp Q_4$. The underlying space is $\mathbb{C}^{2r}$ and each subspace $Q_i$ is $r$-dimensional, so we must therefore have $Q_3 = Q_4 = Q_1^\perp$. Now, $23 \in E(P_4)$, so $Q_2 \not\perp Q_3$, but $24 \notin E(P_4)$, so it also follows that $Q_2 \perp Q_4$. Since $Q_3 = Q_4$, this is a contradiction; thus there is no $(2r; r)$-FOSR for $P_4$, and so $\mathrm{mr}_{[r]}^+(P_4) = 2r + 1$. Using the limit characterization of $\mathrm{mr}_f^+$, it follows that $\mathrm{mr}_f^+(P_4) = \lim_{r \to \infty} \frac{2r+1}{r} = 2$.

This example demonstrates that the infimum in the definition of the fractional minimum positive semidefinite rank cannot be replaced with a minimum, even when $\mathrm{mr}_f^+$ is a rational number. Additionally, since $\mathrm{mr}^+(P_4) = 3$, the graph $G = P_4$ satisfies $\mathrm{mr}_f^+(G) < \mathrm{mr}^+(G)$.

### 2.3.2 Faithful $d/r$-projective representations

Let $G$ be a graph and $d, r \in \mathbb{N}$ with $r \leq d$. A *faithful $d/r$-projective representation*, or faithful $d/r$-representation for short, is an assignment of matrices $\{P_u\}_{u \in V(G)}$ to the vertices of $G$ such that

- for each $u \in V(G)$, $P_u \in \mathbb{C}^{d \times d}$, rank $P_u = r$, $P_u^* = P_u$, and $P_u^2 = P_u$; and

- $P_u P_v = 0$ if and only if $uv \notin E(G)$.

A faithful $d/r$-representation for $G$ is a $d/r$-representation for $\overline{G}$, but the reverse is not necessarily true.

It is convenient to note that a $(d; r)$ faithful orthogonal subspace representation for $G$ is equivalent to a faithful $d/r$-representation. The proof is analogous to that of Proposition 2.2.12; as before, we will omit such parallel proofs.

**Proposition 2.3.6.** *A graph $G$ has a $(d; r)$ faithful orthogonal subspace representation if and only if $G$ has a faithful $d/r$-representation.*

An immediate corollary gives an alternate definition for $\mathrm{mr}_f^+(G)$.

**Corollary 2.3.7.** *For every graph $G$,*

$$\mathrm{mr}_f^+(G) = \inf_{d,r} \left\{ \frac{d}{r} : G \text{ has a faithful } d/r\text{-representation} \right\}.$$

**Corollary 2.3.8.** *For any graph $G$ with complement $\overline{G}$,*

$$\xi_f(\overline{G}) \leq \mathrm{mr}_f^+(G) \leq \mathrm{mr}^+(G).$$

*Proof.* This follows from the fact that any faithful $d/r$-representation for $G$ is also a $d/r$-representation for $\overline{G}$, as well as from Corollary 2.3.3. $\square$

### 2.3.3 Relation to positive semidefinite matrices

In this section, we connect $(d; r)$ faithful orthogonal subspace representations to positive semidefinite matrices, thus generalizing the known results for the $r = 1$ case (when the graph in question has no isolated vertices) and connecting $\mathrm{mr}^+_{[r]}(G)$ to the rank of a positive semidefinite matrix.

We begin with some definitions. Let $G$ be a graph on $n$ vertices and suppose that $V(G) = [1 : n]$. For some $r \in \mathbb{N}$, let $A \in \mathbb{C}^{nr \times nr}$ be partitioned into an $n \times n$ block matrix $[A_{ij}]$, where $A_{ij}$ is the $r \times r$ submatrix in (block) row $i$ and (block) column $j$ of $A$. We say that the matrix $A$ $r$-*fits* $G$ if $A_{ii} = I_r$ for each $i \in V(G)$ and $A_{ij} = 0$ if and only if $ij \notin E(G)$, and define the set

$$\mathcal{H}^+_{[r]}(G) = \left\{ A \in \mathbb{C}^{nr \times nr} : A \succeq 0 \text{ and } A \ r\text{-fits } G \right\}.$$

**Example 2.3.9.** We provide a simple example for the $r = 2$ case. Let $G = P_3$, the path on 3 vertices, with $V(G) = \{1, 2, 3\}$ and $E(G) = \{12, 23\}$. Choosing $X = [\mathbf{e}_1 \, \mathbf{e}_2 \,|\, \mathbf{e}_1 \, \mathbf{e}_4 \,|\, \mathbf{e}_3 \, \mathbf{e}_4]$, where $\mathbf{e}_j$ is the $j^{th}$ standard basis vector in $\mathbb{C}^4$, we can verify that

$$A = X^*X = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \in \mathcal{H}^+_{[2]}(P_3).$$

This constructive example gives an intuitive feel for one direction of the proof of the main result of this section.

**Theorem 2.3.10.** *For every graph $G$ on $n$ vertices and any $r \in \mathbb{N}$,*

$$\mathrm{mr}^+_{[r]}(G) = \min\left\{\mathrm{rank}\, A : A \in \mathcal{H}^+_{[r]}(G)\right\}.$$

*Proof.* Let $d = \mathrm{mr}^+_{[r]}(G)$ and let $\ell = \min\left\{\mathrm{rank}\, A : A \in \mathcal{H}^+_{[r]}(G)\right\}$.

First, assume that $\{S_i\}$ is a $(d; r)$ faithful orthogonal subspace representation for $G$ and for each $i \in V(G)$ let $X_i \in \mathbb{C}^{d \times r}$ be a basis matrix for $S_i$. Define $X = [X_1 \,|\, X_2 \,|\, \cdots \,|\, X_n] \in \mathbb{C}^{d \times nr}$ and let $B = X^*X \in \mathbb{C}^{nr \times nr}$. We see immediately that $B \succeq 0$ and $\mathrm{rank}\, B = \mathrm{rank}\, X \leq d$. Partitioning $B$ into an $n \times n$ block matrix with blocks $[B_{ij}]$ of size $r \times r$, we have $B_{ij} = X_i^* X_j$. Since $S_i \perp S_j$ if and only if $X_i^* X_j = 0$, we have $B_{ij} = 0$ if and only if $S_i \perp S_j$, which occurs if and only if $ij \notin E(G)$. Additionally, since $X_i$ has orthonormal columns, we have $B_{ii} = I_r$ for each $i$. Therefore, $B \in \mathcal{H}^+_{[r]}(G)$, so $\min\left\{\mathrm{rank}\, A : A \in \mathcal{H}^+_{[r]}(G)\right\} \leq \mathrm{rank}\, B \leq d = \mathrm{mr}^+_{[r]}(G)$.

For the reverse inequality, suppose that $B \in \mathcal{H}^+_{[r]}(G)$ and $\mathrm{rank}\, B = \ell$. Then there exists a matrix $X \in \mathbb{C}^{\ell \times nr}$ such that $B = X^*X$. Partition $B$ into $r \times r$ blocks $[B_{ij}]$ and partition $X$ into $\ell \times r$ blocks as $X = [X_1 \,|\, X_2 \,|\, \cdots \,|\, X_n]$. For each vertex $i \in V(G)$, let $S_i = \mathrm{range}(X_i) \subseteq \mathbb{C}^\ell$. Since $X_i^* X_i = I_r$, we have $\mathrm{rank}\, X_i = r$, so each $S_i$ is an $r$-dimensional subspace of $\mathbb{C}^\ell$. Additionally, $X_i^* X_j = B_{ij} = 0$ if and only if $ij \notin E(G)$, so $S_i \perp S_j$ if and only if $ij \notin E(G)$. Therefore, $\{S_i\}$ is an $(\ell; r)$ faithful orthogonal subspace representation for $G$, so $\mathrm{mr}^+_{[r]}(G) \leq \ell = \min\left\{\mathrm{rank}\, A : A \in \mathcal{H}^+_{[r]}(G)\right\}$ and thus equality holds. $\square$

This matrix-based representation is a powerful theoretical tool that allows us to simplify the proofs of some properties of $r$-fold minimum positive semidefinite rank, as well as to more clearly draw parallels to the existing and well-established $r = 1$ case (although again, the connection to the literature requires that the graph in question has no isolated vertices).

The condition that $A_{ii} = I_r$ if $A$ $r$-fits a graph $G$ is a strong one, so we conclude this section with a weaker condition that will be used to further simplify proofs without

sacrificing utility. We say that $A$ *weakly $r$-fits $G$* if $A_{ii}$ is a diagonal matrix with strictly positive diagonal entries for each $i \in V(G)$ and $A_{ij} = 0$ if and only if $ij \notin E(G)$. Clearly, any matrix that $r$-fits $G$ also weakly $r$-fits $G$.

**Remark 2.3.11.** Suppose that $A$ weakly $r$-fits a graph $G$ and let $D = D_1 \oplus \cdots \oplus D_n$, where each $D_i$ is the inverse of the positive square root of $A_{ii}$, i.e., $D_i = A_{ii}^{-\frac{1}{2}}$. Then the matrix $B = DAD$ $r$-fits $G$, since $D$ is a diagonal matrix with strictly positive diagonal entries, so multiplication by $D$ does not change the zero pattern of $A$. Further, rank $B$ = rank $A$, since $D$ has full rank.

This remark yields an immediate corollary to the previous theorem.

**Corollary 2.3.12.** *For every graph $G$ on $n$ vertices and any $r \in \mathbb{N}$,*

$$\mathrm{mr}_{[r]}^+(G) = \min \left\{ \mathrm{rank}\, A : A \in \mathbb{C}^{nr \times nr},\ A \succeq 0 \ \text{and } A \text{ weakly } r\text{-fits } G \right\}.$$

## 2.3.4 Properties of $\mathrm{mr}_{[r]}^+(G)$ and $\mathrm{mr}_f^+(G)$

In this section, we prove numerous results regarding properties of $r$-fold and fractional minimum positive semidefinite rank, many of which extend known properties of $\mathrm{mr}^+$ to the new parameters.

**Observation 2.3.13.** *For every graph $G$ and all $r \in \mathbb{N}$, $\mathrm{mr}_{[r]}^+(G) \geq r \cdot \alpha(G)$.*

**Proposition 2.3.14.** *Let $r \in \mathbb{N}$ and let $H$ be an induced subgraph of $G$. Then $\mathrm{mr}_{[r]}^+(H) \leq \mathrm{mr}_{[r]}^+(G)$.*

*Proof.* For any $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$, since $H$ is induced. Therefore any $(d; r)$ faithful orthogonal subspace representation for $G$ provides a $(d; r)$ faithful orthogonal subspace representation for $H$, and the result follows immediately. $\square$

**Proposition 2.3.15.** *If* $G = \dot{\bigcup}_{i=1}^{t} G_i$ *for some graphs* $\{G_i\}_{i=1}^{t}$, *then* $\mathrm{mr}_{[r]}^{+}(G) = \sum_{i=1}^{t} \mathrm{mr}_{[r]}^{+}(G_i)$ *for each* $r \in \mathbb{N}$.

*Proof.* Suppose that $V(G) = [1:n]$ and that $|V(G_i)| = n_i$ for $i = 1, 2, \ldots, t$. Further assume that $V(G_i) = \left[ 1 + \sum_{j=1}^{i-1} n_j : \sum_{j=1}^{i} n_j \right]$, so that if $A \in \mathcal{H}_{[r]}^{+}(G)$, then $A = A_1 \oplus A_2 \oplus \cdots \oplus A_t$, where $A_i \in \mathcal{H}_{[r]}^{+}(G_i)$ for each $i$. Note that $\mathrm{rank}\, A = \sum_{i=1}^{t} \mathrm{rank}\, A_i$. We therefore have

$$
\begin{aligned}
\mathrm{mr}_{[r]}^{+}(G) &= \min \left\{ \mathrm{rank}\, A : A \in \mathcal{H}_{[r]}^{+}(G) \right\} \\
&= \min \left\{ \sum_{i=1}^{t} \mathrm{rank}\, A_i : A_i \in \mathcal{H}_{[r]}^{+}(G_i) \text{ for each } i \right\} \\
&= \sum_{i=1}^{t} \min \left\{ \mathrm{rank}\, A_i : A_i \in \mathcal{H}_{[r]}^{+}(G_i) \right\} \\
&= \sum_{i=1}^{t} \mathrm{mr}_{[r]}^{+}(G_i). \qquad \square
\end{aligned}
$$

**Theorem 2.3.16.** *If* $G = \bigcup_{i=1}^{t} G_i$ *for some graphs* $\{G_i\}_{i=1}^{t}$, *then* $\mathrm{mr}_{[r]}^{+}(G) \leq \sum_{i=1}^{t} \mathrm{mr}_{[r]}^{+}(G_i)$ *for each* $r \in \mathbb{N}$.

*Proof.* We prove the result for the case where $t = 2$ and note that recursive application of this case will prove the more general one.

Let $V(G) = [1:n]$ where $n > 0$ and assume that $V(G_1) \setminus V(G_2) = [1:n_1]$, $V(G_1) \cap V(G_2) = [n_1 + 1 : n_1 + c]$, and $V(G_2) \setminus V(G_1) = [n_1 + c + 1 : n_1 + c + n_2]$, where $n_1, n_2,$ $c \geq 0$ (it is not assumed that each of these is strictly nonzero). Note that $n = n_1 + c + n_2$, and this ordering asserts that the first $n_1$ vertices (enumerating in the natural order) lie exclusively in $G_1$, the next $c$ are common to both graphs, and the last $n_2$ lie exclusively in $G_2$.

For $i = 1, 2$, let $\mathrm{mr}_{[r]}^{+}(G_i) = d_i$ and let $A_i \in \mathcal{H}_{[r]}^{+}(G_i)$ be chosen so that $\mathrm{rank}\, A_i = d_i$. Notice that $A_1 \in \mathbb{C}^{(n_1+c)r \times (n_1+c)r}$ has its rows and columns indexed by $V(G_1) = [1 : n_1 + c]$ and $A_2 \in \mathbb{C}^{(n_2+c)r \times (n_2+c)r}$ has its rows and columns indexed by $V(G_2) = [n_1 + 1 : n]$.

Let

$$\hat{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{nr \times nr}, \qquad \hat{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{C}^{nr \times nr}$$

and define $A = \hat{A}_1 + \beta \hat{A}_2 \in \mathbb{C}^{nr \times nr}$, where $\beta > 0$ is chosen so that if $A$, $\hat{A}_1$, and $\hat{A}_2$ are partitioned into $n \times n$ block matrices with block size $r \times r$, then $A_{ij} = 0$ if and only if $(\hat{A}_1)_{ij} = 0$ and $(\hat{A}_2)_{ij} = 0$ (i.e., no cancellation of an entire block occurs).

Since $A$ is a positive linear combination of positive semidefinite matrices, $A \succeq 0$, and by our choice of $\beta$ we see that $A$ weakly $r$-fits $G$. Therefore,

$$\mathrm{mr}^+_{[r]}(G) \le \mathrm{rank}\, A \le \mathrm{rank}\, \hat{A}_1 + \mathrm{rank}\, \hat{A}_2 = d_1 + d_2 = \mathrm{mr}^+_{[r]}(G_1) + \mathrm{mr}^+_{[r]}(G_2). \qquad \square$$

All of the results we have proven for $r$-fold minimum positive semidefinite rank can be extended to results for fractional minimum positive semidefinite rank. The proof is analogous to that of Theorem 2.2.16 and is omitted.

**Theorem 2.3.17.** *For every graph $G$:*

*i)* $\mathrm{mr}^+_f(G) \ge \alpha(G)$.

*ii) If $H$ is an induced subgraph of $G$, then $\mathrm{mr}^+_f(H) \le \mathrm{mr}^+_f(G)$.*

*iii) If $G = \dot{\bigcup}_{i=1}^{t} G_i$ for some graphs $\{G_i\}_{i=1}^{t}$, then $\mathrm{mr}^+_f(G) = \sum_{i=1}^{t} \mathrm{mr}^+_f(G_i)$.*

*iv) If $G = \bigcup_{i=1}^{t} G_i$ for some graphs $\{G_i\}_{i=1}^{t}$, then $\mathrm{mr}^+_f(G) \le \sum_{i=1}^{t} \mathrm{mr}^+_f(G_i)$.*

Let $G$ be a connected graph of order at least two. A standard technique for computing the minimum positive semidefinite rank of $G$ is cut-vertex reduction [1, 7, 18]: Suppose that $v \in V(G)$ is a cut-vertex and $(G - v)$ has connected components $\{H_i\}_{i=1}^{t}$. For each $i$, let $G_i$ be the subgraph of $G$ induced by the union of the vertices of $H_i$ with $v$, that is, $G_i = G[V(H_i) \cup \{v\}]$. Then $\mathrm{mr}^+(G) = \sum_{i=1}^{t} \mathrm{mr}^+(G_i)$. Unfortunately, this technique does not carry over to the $r$-fold case when $r > 1$, as the following example shows.

**Example 2.3.18.** Consider the graph $G = P_4$, the path on 4 vertices, with $V(G) = \{x, y, v, z\}$ in path order; recall from Example 2.3.5 that $\text{mr}^+_{[r]}(G) = 2r + 1$ for any $r \in \mathbb{N}$. Taking $v$ as a cut-vertex, we have $G_1 = P_3$ with $V(G_1) = \{x, y, v\}$ and $G_2 = P_2$ with $V(G_2) = \{v, z\}$. Fix $r > 1$. Since $\alpha(G_1) = 2$, any valid $(d; r)$-FOSR for $G_1$ must have $d \geq 2r$. Further, it is easy to see that $\text{mr}^+(G_1) = 2$, so $4 \leq \text{mr}^+_{[r]}(G_1) \leq 2 \cdot \text{mr}^+(G_1) = 2r$. Hence equality holds and $\text{mr}^+_{[r]}(G_1) = 2r$. Next, since $\text{mr}^+(G_2) = 1$ and $d \geq r$ for any valid $(d; r)$-FOSR, we have $r \leq \text{mr}^+_{[r]}(G_2) \leq r \cdot \text{mr}^+(G_2) = r$, so $\text{mr}^+_{[r]}(G_2) = r$. Hence if $r > 1$, then $\text{mr}^+_{[r]}(G) = 2r + 1 < 2r + r = \text{mr}^+_{[r]}(G_1) + \text{mr}^+_{[r]}(G_2)$, so cut-vertex reduction does not apply.

### 2.3.5 Fractional minimum positive semidefinite rank and projective rank

Recall that any $(d; r)$-FOSR for $G$ is a $(d; r)$-OSR for $\overline{G}$, but the reverse statement does not apply in general. It thus follows that $\xi_{[r]}(\overline{G}) \leq \text{mr}^+_{[r]}(G)$ for any graph $G$ and $r \in \mathbb{N}$, and the next example demonstrates that this inequality can be strict.

**Example 2.3.19.** Consider the graph $G = P_4$ with $V(P_4) = \{1, 2, 3, 4\}$ and $E(P_4) = \{12, 23, 34\}$ and fix $r \in \mathbb{N}$. Since $\omega(P_4) = 2$, we have $\xi_{[r]}(P_4) \geq 2r$. With $\mathbf{e}_i$ as the $i^{th}$ standard basis vector for $\mathbb{C}^{2r}$, it is easy to verify that the following is a $(2r; r)$-OSR for $P_4$: $S_1 = S_3 = \text{range}([\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r])$, $S_2 = S_4 = \text{range}([\mathbf{e}_{r+1}, \mathbf{e}_{r+2}, \ldots, \mathbf{e}_{2r}])$. Therefore, $\xi_{[r]}(P_4) = 2r$. Since $\overline{P_4} = P_4$ and $\text{mr}^+_{[r]}(P_4) = 2r + 1$ (Example 2.3.5), we have $2r = \xi_{[r]}(\overline{P_4}) < \text{mr}^+_{[r]}(P_4) = 2r + 1$.

Recall from Corollary 2.3.8 that $\xi_f(\overline{G}) \leq \text{mr}^+_f(G)$ for any graph $G$. While strict inequality may hold in the $r$-fold case for an arbitrary graph $G$, we now demonstrate that equality always holds in the "fractional case" for any graph $G$. For this result, we require the following lemma.

**Lemma 2.3.20.** *Let $G$ be a graph with complement $\overline{G}$. Let $\{P_u\}_{u \in V(G)}$ be a $d/r$-representation for $\overline{G}$ and let $\{R_u\}_{u \in V(G)}$ be a faithful $b/1$-representation for $G$. Then for any $k \in \mathbb{N}$, $G$ has a faithful $(kd+b)/(kr+1)$-representation $\{Q_u\}_{u \in V(G)}$. Further, given any $\varepsilon > 0$, $k$ can be chosen such that $\left| \frac{d}{r} - \frac{kd+b}{kr+1} \right| < \varepsilon$, i.e., the value of the faithful representation $\{Q_u\}$ for $G$ is within $\varepsilon$ of the value of the (non-faithful) representation $\{P_u\}$ for $\overline{G}$.*

*Proof.* Since $\{P_u\}$ is a $d/r$-representation for $\overline{G}$, we have $P_u \in \mathbb{C}^{d \times d}$ with rank $P_u = r$ for each $u \in V(\overline{G}) = V(G)$, and $P_u P_v = 0$ if $uv \in E(\overline{G})$, so $P_u P_v = 0$ if $uv \notin E(G)$.

Let $\varepsilon > 0$ be arbitrary and choose $k > \left( \frac{|d - rb|}{r^2 \varepsilon} - \frac{1}{r} \right)$.

For each vertex $u \in V(G)$, let $Q_u \in \mathbb{C}^{(kd+b) \times (kd+b)}$ be the block diagonal matrix constructed from $k$ copies of $P_u$ and one copy of $R_u$, i.e.,

$$Q_u = \left( \bigoplus_{i=1}^{k} P_u \right) \oplus R_u.$$

We see immediately that rank $Q_u = kr + 1$, and since $P_u$ and $R_u$ are projectors, so is $Q_u$. Since $P_u P_v = 0$ if $uv \notin E(G)$ and $R_u R_v = 0$ if *and only if* $uv \notin E(G)$, we conclude that $Q_u Q_v = 0$ if and only if $uv \notin E(G)$. Therefore, $\{Q_u\}_{u \in V(G)}$ is a faithful $(kd+b)/(kr+1)$-representation for $G$, which verifies the first claim.

By choice of $k$, we have $kr + 1 > \frac{|d-rb|}{r\varepsilon}$. Consider

$$\left| \frac{d}{r} - \frac{kd+b}{kr+1} \right| = \left| \frac{d(kr+1) - r(kd+b)}{r(kr+1)} \right|$$
$$= \frac{|d-rb|}{r} \cdot \frac{1}{kr+1}$$
$$< \frac{|d-rb|}{r} \cdot \frac{r\varepsilon}{|d-rb|}$$
$$= \varepsilon,$$

which verifies the second claim. $\qquad\square$

It was previously noted that any faithful $d/r$-representation for $G$ is also $d/r$-representation for $\overline{G}$. Lemma 2.3.20 is a partial converse in the sense that, given any $d/r$-

representation for $\overline{G}$, we can construct a faithful $d_1/r_1$-representation for $G$ such that the two representations have essentially the same value. This yields the next result.

**Theorem 2.3.21.** *For every graph $G$ with complement $\overline{G}$,*

$$\xi_f(\overline{G}) = \mathrm{mr}_f^+(G).$$

*Proof.* Let

$$R = \left\{ \frac{d}{r} : \overline{G} \text{ has a } d/r\text{-representation} \right\},$$

$$F = \left\{ \frac{d}{r} : G \text{ has a faithful } d/r\text{-representation} \right\}.$$

For any $\frac{d}{r} \in R$ and $\varepsilon > 0$, Lemma 2.3.20 asserts that there exists some $\frac{d_1}{r_1} \in F$ such that $\left| \frac{d}{r} - \frac{d_1}{r_1} \right| < \varepsilon$. It follows that $\inf R = \inf F$, i.e., $\xi_f(\overline{G}) = \mathrm{mr}_f^+(G)$. $\qquad\square$

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# Bibliography

[1] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, T. D. Lenker, L. H. Mitchell, S. K. Narayan, A. Pascoe, and B. D. Sutton, On the minimum semidefinite rank of a simple graph, *Linear Multilinear Algebra*, 59: 483–506, 2011.

[2] A. Cabello, S. Severini, and A. Winter, Graph-Theoretic Approach to Quantum Correlations, *Phys. Rev. Lett.*, 112: 040401, 2014.

[3] P. J. Cameron, A. Montanaro, M. W. Newman, S. Severini, and A. Winter, On the quantum chromatic number of a graph, *Electron. J. Combin.*, 14: Research Paper #R81, 2007.

[4] V. Capraro and M. Lupini, *Introduction to Sofic and Hyperlinear Groups and Connes' Embedding Conjecture*, Lecture Notes in Mathematics 2136, 2015.

[5] T. Cubitt, L. Mančinska, D. E. Roberson, S. Severini, D. Stahlke, and A. Winter, Bounds on entanglement-assisted source-channel coding via the Lovász theta number and its variants, *IEEE Trans. Inform. Theory*, 60: 7330–7344, 2014.

[6] R. Duan, S. Severini, and A. Winter, Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovász function, *IEEE Trans. Inform. Theory*, 59: 1164–1174, 2013.

[7] S. Fallat and L. Hogben, Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs, in *Handbook of Linear Algebra*, 2nd ed., L. Hogben, ed., CRC Press, Boca Raton, FL, 2013.

[8] S. Fallat and L. Hogben, The Minimum Rank of Symmetric Matrices Described by a Graph: A Survey, *Linear Algebra Appl.*, 426: 558–582, 2007.

[9] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the 'magic' for quantum computation, *Nature*, 510: 351–355, 2014.

[10] D. Leung, L. Mančinska, W. Matthews, M. Ozols, and A. Roy, Entanglement can Increase Asymptotic Rates of Zero-Error Classical Communication over Classical Channels, *Comm. Math. Phys.*, 311: 97–111, 2012.

[11] N. Ozawa, About the Connes embedding conjecture: Algebraic approaches, *Jpn. J. Math.*, 8: 147–183, 2013.

[12] V. I. Paulsen and I. G. Todorov, Quantum chromatic numbers via operator systems, arXiv:1311.6850 [math.OA], 2013.

[13] V. I. Paulsen, S. Severini, D. Stahlke, I. G. Todorov, and A. Winter, Estimating quantum chromatic numbers, arXiv:1407.6918 [math.OA], 2014.

[14] D. E. Roberson, Variations on a Theme: Graph Homomorphisms, Ph.D. thesis, University of Waterloo, 2013.

[15] D. E. Roberson and L. Mančinska, Graph Homomorphisms for Quantum Players, to appear in *J. Combin. Theory Ser. B* (2014), arXiv:1212.1724 [quant-ph], 2012.

[16] E. Scheinerman and D. Ullman, *Fractional Graph Theory*, Dover, Mineola, NY, 2011; also available online from `http://www.ams.jhu.edu/~ers/fgt/`.

[17] B. S. Tsirelson, Some results and problems on quantum Bell-type inequalities, *Hadronic J. Suppl.*, 8: 329–345, 1993.

[18] H. van der Holst, On the maximum positive semi-definite nullity and the cycle matroid of graphs, *Electron. J. Linear Algebra*, 18: 192–201, 2009.

# CHAPTER 3.   FRACTIONAL ZERO FORCING VIA THREE-COLOR FORCING GAMES

Leslie Hogben[*][†], Kevin F. Palmowski[†], David E. Roberson[‡], and Michael Young[†]

## Abstract

An $r$-fold analogue of the positive semidefinite zero forcing process that is carried out on the $r$-blowup of a graph is introduced and used to define the fractional positive semidefinite forcing number. Properties of the graph blowup when colored with a fractional positive semidefinite forcing set are examined and used to define a three-color forcing game that directly computes the fractional positive semidefinite forcing number of a graph. We develop a fractional parameter based on the standard zero forcing process and it is shown that this parameter is exactly the skew zero forcing number with a three-color approach. This approach and an algorithm are used to characterize graphs whose skew zero forcing number equals zero.

[*]American Institute of Mathematics, 600 E. Brokaw Rd., San Jose, CA 95112, USA. `hogben@aimath.org`.

[†]Department of Mathematics, Iowa State University, Ames, IA 50011, USA. {`LHogben, kpalmow, myoung`}`@iastate.edu`.

[‡]Division of Mathematical Sciences, Nanyang Technological University, SPMS-MAS-03-01, 21 Nanyang Link, Singapore 637371. `droberson@ntu.edu.sg`.

## 3.1 Introduction

This paper studies fractional versions (in the spirit of [12]) of the standard and positive semidefinite zero forcing numbers and introduces three-color forcing games to compute these parameters. The three-color approach allows simpler proofs of some results and yields new results about existing parameters (see, e.g., Section 3.3.4).

The zero forcing process was introduced independently in [1] as a method of forcing zeros in a null vector of a symmetric matrix described by a graph, which yields an upper bound to the nullity of the matrix, and in [6] for control of quantum systems. There are potential applications to the spread of rumors or diseases (see, e.g., [5]); one of the original names of zero forcing was "graph infection." Despite the fact that when studied as a graph parameter there are no zeros involved, the name "zero forcing number" has become the standard term in the literature. The original zero forcing number has since spawned numerous variants (see, e.g., [3, 4, 11]). The speed with which the zero forcing process colors all vertices has also been studied (see, e.g., [9, 14]).

### 3.1.1 Zero forcing games

In this section, we introduce zero forcing, which can be described as a coloring game [4], and the terminology used. Abstractly, a *forcing game* is a type of coloring game that is played on a simple graph $G$. First, a "target color," typically blue or dark blue, is designated. Each vertex of the graph is then colored the target color, white, or possibly some other color (in prior work, only white and the target color have been used). A *forcing rule* is chosen: this is a rule that describes the conditions under which some vertex can cause another vertex to change to the target color. If vertex $u$ causes a neighboring vertex $w$ to change color, we say that $u$ *forces* $w$ and write $u \to w$. The forcing rule is repeatedly applied until no more forces can be performed, at which point the game ends; the coloring at the end is called the *final coloring*. An ordered list of the

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forces performed is referred to as a *chronological list of forces*. Note that there is usually some choice as to which forces are performed, as well as the order in which these forces occur. As such, a single forcing set may generate many different chronological lists of forces; however, the final coloring is unique for all of the games discussed herein. If the graph is totally colored with the target color at the end of the game, then we say that $G$ has been *forced*. The goal of the game is to force $G$. If this is possible, then the initial set of non-white vertices is called a *forcing set*.

The *(standard) zero forcing game* uses only the colors blue (the target color) and white. The *(standard) zero forcing rule* is as follows:

If $w$ is the only white neighbor of a blue vertex $u$, then $u$ can force $w$.

A *(standard) zero forcing set* is an initial set of blue vertices that can force $G$ using this rule. The *(standard) zero forcing number* of $G$, denoted $Z(G)$, is the minimum cardinality of a zero forcing set for $G$. We present an illustrative example in Figure 3.1.



(a) Graph $G$     (b) Initial forcing set     (c) First three forces     (d) Final forces
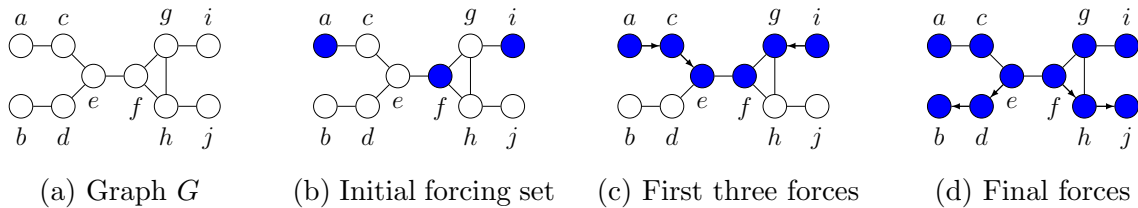
Figure 3.1: Standard zero forcing game example

From this point forward, we will omit the word "standard" when referring to the standard zero forcing game, its forcing rule, or zero forcing sets whenever there is no risk of ambiguity.

The *positive semidefinite zero forcing game* is a modification of the zero forcing game used to force zeros in a null vector of a positive semidefinite matrix described by a graph [3]. Like the zero forcing game, positive semidefinite zero forcing uses only the colors

blue (target) and white. The *positive semidefinite zero forcing rule* is the same as the standard zero forcing rule, except that this rule also features a *disconnect rule*:

> Remove all blue vertices from the graph, leaving a set of connected components. To each connected component (of white vertices) in turn, add the blue vertices, the edges among the blue vertices, and any edges between the blue vertices and that component, and perform forces via the standard rule: If $w$ is the only white neighbor of a blue vertex $u$ in this induced subgraph, then $u$ can force $w$.

It is not assumed that disconnection occurs; if there is only one component, then we simply force via the standard forcing rule. If disconnection does occur, then after the force the graph is "reassembled" prior to applying the rule again. As one would expect, a *positive semidefinite zero forcing set* is an initial set of blue vertices that can force $G$ using this rule, and the *positive semidefinite zero forcing number* of $G$, denoted $Z^+(G)$, is the minimum cardinality of a positive semidefinite zero forcing set for $G$. In Figure 3.2 we illustrate the positive semidefinite zero forcing process on the graph from Figure 3.1a.



(a) Connected components     (b) Forcing in each component     (c) Reassembled graph
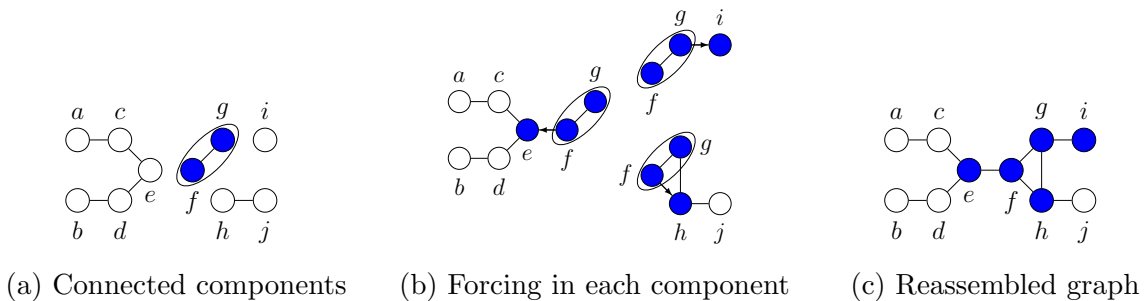
Figure 3.2: Positive semidefinite zero forcing game example (first steps)

The *skew zero forcing game*, another variant on zero forcing that uses the colors white and blue (target), was first considered in [11] to force zeros in a null vector of a skew symmetric matrix described by a graph. The *skew zero forcing rule* is as follows:

If $w$ is the only white neighbor of any vertex $u$, then $u$ can force $w$.

Skew zero forcing removes the standard requirement that the forcing vertex $u$ be blue; as a result, skew zero forcing allows *white vertex forcing*, i.e., a white vertex is allowed to force its only white neighbor. A *skew zero forcing set* is an initial set of blue vertices that can force $G$ using this rule, and the *skew zero forcing number* of $G$, denoted $Z^-(G)$, is the minimum cardinality of a skew zero forcing set for $G$. Figure 3.3 demonstrates skew zero forcing; notice that the initial forcing set contains no blue vertices.



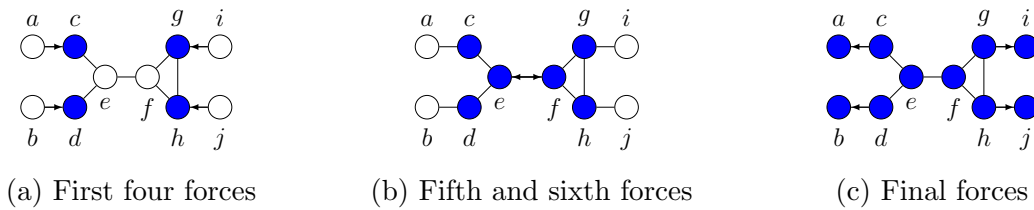(a) First four forces      (b) Fifth and sixth forces      (c) Final forces

Figure 3.3: Skew zero forcing game example

### 3.1.2    Motivation and method

This paper focuses on fractional versions of the standard and positive semidefinite zero forcing numbers. We first present the construction of fractional chromatic number found in [12] as an example of the method used to define a fractional graph parameter. A *proper coloring* of a graph $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices receive different colors. The *chromatic number* of $G$, denoted $\chi(G)$, is the least number of colors required to properly color $G$. We can generalize a proper coloring of $G$ using $c$ colors to a *proper $r$-fold coloring with $c$ colors*, or a *$c$:$r$-coloring*: from a total of $c$ colors, we assign $r$ colors to each vertex of $G$ such that adjacent vertices receive disjoint sets of colors. The *$r$-fold chromatic number* of $G$, denoted $\chi_r(G)$, is the smallest value of $c$ such that G has a $c$:$r$-coloring; we emphasize that to compute $\chi_r(G)$ we fix $r$ and minimize the value of $c$. The *fractional chromatic number* of $G$ is then

defined as

$$\chi_f(G) = \inf_{r \in \mathbb{N}} \left\{ \frac{\chi_r(G)}{r} \right\}.$$

The interested reader is referred to [12] for an in-depth treatment of fractional chromatic number, as well as other fractional graph parameters. For this paper, defining an $r$-fold version of a graph parameter and then defining the fractional parameter as the infimum of the ratios of the $r$-fold parameter to $r$ are key ideas.

Suppose that $G$ is a simple graph on $n$ vertices with $V(G) = [n]$. We say that a symmetric matrix $A \in \mathbb{C}^{nr \times nr}$ $r$-*fits* $G$ if, after partitioning $A$ as a block $n \times n$ matrix, block $A_{ii} = I_r$ for each $i$ and for all $i, j$ with $i \neq j$, block $A_{ij} = 0_{r \times r}$ if and only if $ij \notin E(G)$ [10]. While there may be many such matrices for a given graph, the following result shows that certain structure can be chosen.

**Proposition 3.1.1.** *Suppose that $A \in \mathbb{C}^{nr \times nr}$ $r$-fits a graph $G$ on $n$ vertices. We can construct a unitary matrix $U$ such that $U^*AU$ $r$-fits $G$ and if $ij \in E(G)$, then every entry of block $(U^*AU)_{ij}$ is nonzero.*

*Proof.* Assume that $V(G) = [n]$ and partition $A = [A_{ij}]$ as an $n \times n$ block matrix with $A_{ij} \in \mathbb{C}^{r \times r}$. By definition, we have $A_{ii} = I_r$ for each $i \in [n]$, and for $i, j \in [n]$ with $i \neq j$ we have $A_{ij} = 0_{r \times r}$ if and only if $ij \notin E(G)$.

For each $i \in [n]$, let $U_i \in \mathbb{C}^{r \times r}$ be a random unitary matrix with $U_i$ and $U_j$ chosen independently if $i \neq j$. Define $U = \text{blockdiag}(U_1, \ldots, U_n)$ and let $C = U^*AU$. Partitioning $C$ conformally with $A$, we have $C_{ij} = U_i^* A_{ij} U_j$. Notice that $C_{ii} = U_i^* I_r U_i = I_r$ and if $ij \notin E(G)$ (for $i \neq j$), then $C_{ij} = U_i^* 0_{r \times r} U_j = 0_{r \times r}$.

Suppose $ij \in E(G)$ and consider the product $A_{ij} U_j$. Since $U_j$ is random, with high probability no column of $U_j$ lies in $\ker A_{ij}$, so no column of $A_{ij} U_j$ is a zero vector. Let $\mathbf{z}$ be any column of $A_{ij} U_j$ (so with high probability $\mathbf{z} \neq \mathbf{0}$) and consider $(U_i^* \mathbf{z})_k$. If $(U_i^* \mathbf{z})_k = 0$, then $\mathbf{z}$ is orthogonal to the $k^{th}$ column of $U_i$. Since $U_i$ is a random unitary matrix, with high probability this does not happen. We conclude that if $ij \in E(G)$, then

with high probability no entry of $C_{ij}$ is zero. Thus there exists a matrix that $r$-fits $G$ and has the desired structure. □

Let $G$ be a graph and choose $r \in \mathbb{N}$. The $r$-*blowup* of $G$ is the graph $G^{(r)}$ constructed by replacing each vertex of $u \in V(G)$ with an independent set of $r$ vertices, denoted $R_u$, and replacing each edge $uw \in E(G)$ by the edges of a complete bipartite graph on partite sets $R_u$ and $R_w$.[1] We call the set $R_u$ a *cluster*. Note that $V(G^{(r)}) = \bigcup_{u \in V(G)} R_u$ and if $uw \in E(G)$ then every vertex of $R_u$ is adjacent to every vertex of $R_w$ in $G^{(r)}$.

Suppose that $A \in \mathbb{C}^{nr \times nr}$ is positive semidefinite and $r$-fits a graph $G$ on $n$ vertices with $V(G) = [n]$. As a result of Proposition 3.1.1 (by replacing $A$ with $U^*AU$), we can assume that if $ij \in E(G)$, then block $A_{ij}$ has no zero entries. Consider the graph of such a matrix $A$, namely, the simple graph with vertex set $[nr]$ and with an edge between vertices $k$ and $\ell$ if $k \neq \ell$ and the entry in row $k$ and column $\ell$ of $A$ is nonzero. Since $A_{ii} = I_r$, the vertices of $G$ will map to independent sets (clusters) of size $r$; let $R_i$ denote the cluster associated with vertex $i \in V(G)$. Since each entry of $A_{ij}$ is nonzero, every vertex in $R_i$ will be adjacent to every vertex in $R_j$, and vice versa. Hence the graph of $A$ is exactly $G^{(r)}$, the $r$-blowup of $G$.

The positive semidefinite zero forcing number of a graph is an upper bound on the maximum positive semidefinite nullity of the graph, which equals the order of the graph minus its minimum positive semidefinite rank [3, 7]. The authors of [10] define an $r$-fold analogue of minimum positive semidefinite rank and use this new parameter to define fractional minimum positive semidefinite rank. A key element of this treatment is that the $r$-fold minimum positive semidefinite rank of a graph can be expressed as the rank of a positive semidefinite matrix that $r$-fits the graph [10, Theorem 3.10]. Our previous discussion allows us to assume that the graph of such a matrix is $G^{(r)}$.

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[1]Given graphs $G$ and $H$, the *lexicographic product* of $G$ with $H$, denoted $G \times_L H$, is the graph with $V(G \times_L H) = V(G) \times V(H)$ and $(g,h)(i,j) \in E(G \times_L H)$ if $gi \in E(G)$ or if $g = i$ and $hj \in E(H)$. We can also define the $r$-blowup of $G$ as $G^{(r)} = G \times_L \overline{K_r}$, where $\overline{K_r}$ denotes the empty graph on $r$ vertices.

As mentioned in Section 3.1.1, playing the positive semidefinite zero forcing game can be interpreted as forcing zeros in a null vector of a positive semidefinite matrix whose graph is $G$, hence the connection to maximum positive semidefinite nullity and minimum positive semidefinite rank. Since the $r$-fold minimum positive semidefinite rank is defined in terms of matrices that $r$-fit the original graph, an $r$-fold analogue of positive semidefinite zero forcing number would naturally be associated with a game played on the graph of a positive semidefinite matrix that $r$-fits $G$. To this end, our $r$-fold forcing parameters will be defined in terms of forcing games played on $G^{(r)}$.

### 3.1.3   Definitions and notation

Throughout this paper, all graphs are simple. We use $|G|$ to denote the order of a graph $G$, i.e., $|G| = |V(G)|$. If $G$ is a graph and $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of $G$ *induced* by $S$, namely, the graph with $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. We use $G - S$ as shorthand for the induced subgraph $G[V(G) \setminus S]$. The *neighborhood* of a vertex $u \in V(G)$, denoted $N(u)$, is the set of vertices adjacent to $u$. The *degree* of a vertex $u$, $\deg(u)$, is the number of neighbors of $u$, i.e., $|N(u)|$. A *leaf* is a vertex of degree one. We use $\delta(G)$ to denote the minimum of the degrees of the vertices of $G$. We write $S \dot\cup T$ to denote the union of disjoint sets $S$ and $T$.

Throughout, $B$ will be used to denote a set of blue vertices associated with a two-color forcing game. We emphasize that in a two-color forcing game the target color is blue. For three-color forcing games, we use two non-white colors: dark blue, which is our target color, and light blue. A set of colored vertices associated with a three-color forcing game with be denoted by $\mathcal{B}$. Given such a set $\mathcal{B}$, we let $\mathcal{D}$ be the set of dark blue vertices and $\mathcal{L}$ be the set of light blue vertices. Since $\mathcal{D} \cap \mathcal{L} = \emptyset$, we have $\mathcal{B} = \mathcal{D} \dot\cup \mathcal{L}$. While $\mathcal{B}$ is a set, we will abuse notation and write $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ to emphasize the decomposition of $\mathcal{B}$ into its component sets.

### 3.1.4 Contribution and organization of the paper

In Section 3.2 we introduce and examine the fractional positive semidefinite forcing number of a graph. An $r$-fold extension of the positive semidefinite zero forcing number, based on graph blowups, is introduced and used to define the fractional positive semidefinite forcing number of a graph $G$, denoted $Z_f^+(G)$. We also introduce a three-color forcing game played on $G$ called the fractional positive semidefinite forcing game and prove a main result of that section:

**Theorem** (Theorem 3.2.19). *For any graph $G$, $Z_f^+(G)$ is the minimum number of dark blue vertices in a (three-color) fractional positive semidefinite forcing set for $G$.*

This result allows us to determine the fractional positive semidefinite forcing number of a graph by playing the fractional positive semidefinite forcing game, as opposed to computation via the $r$-fold approach. We prove numerous results pertaining to fractional positive semidefinite forcing number and the structure of optimal fractional positive semidefinite forcing sets and apply these results to compute the fractional positive semidefinite forcing number for some common graph families. We also prove that any graph has an ordinary (two-color) minimum positive semidefinite zero forcing set such that the first force in the forcing process can be done without using the disconnect rule.

In Section 3.3 we introduce a three-color forcing game that is equivalent to the skew zero forcing game. The three-color approach is used to prove numerous results pertaining to skew zero forcing. We define an $r$-fold analogue of the (standard) zero forcing game and using this to define the fractional forcing number of a graph, denoted $Z_f(G)$. A main result of that section shows that skew zero forcing number and fractional zero forcing number of a graph are the same:

**Theorem** (Theorem 3.3.18). *For any graph $G$, $Z_f(G) = Z^-(G)$.*

We conclude the section by introducing an algorithm that is used to characterize graphs that satisfy $Z^-(G) = 0$.

## 3.2   Fractional positive semidefinite forcing

In this section, we introduce the $r$-fold and fractional positive semidefinite forcing numbers of a graph, as well as a three-color forcing game that relates to the fractional parameter.

### 3.2.1   The $r$-fold positive semidefinite forcing game and fractional positive semidefinite forcing number

Let $G$ be a graph and for $r \in \mathbb{N}$ consider the following $r$-*fold positive semidefinite forcing game*, which is a two-color forcing game played on $G^{(r)}$. As in any forcing game, we initially color some set $B \subseteq V(G^{(r)})$ blue and then try to force $G^{(r)}$ through repeated application of the following $r$-fold positive semidefinite forcing rule:

**Definition 3.2.1** ($r$-fold positive semidefinite forcing rule)**.** Let $B_t$ denote the set of blue vertices of $G^{(r)}$ at some step $t$ of the $r$-fold positive semidefinite forcing process[2] and let $W_1, \ldots, W_h$ denote the sets of vertices of the connected components of $G^{(r)} - B_t$. If $u \in B_t$ and $|N(u) \cap W_i| \leq r$, then $u$ can force $N(u) \cap W_i$, i.e., all white neighbors of $u$ in $G^{(r)}[B_t \cup W_i]$ can be simultaneously colored blue.

The $r$-fold positive semidefinite forcing game can be thought of as a generalization of the positive semidefinite zero forcing game: instead of forcing one white neighbor in a component after applying the disconnect rule, a vertex forces up to $r$ white neighbors in a component. This is a positive semidefinite analogue of the $r$-forcing process described in [2], but we apply this process only to the blowup of the graph.

If $G^{(r)}$ can be forced, then the initial set of blue vertices is called an $r$-*fold positive semidefinite (PSD) forcing set* for $G$. An $r$-fold PSD forcing set $B$ is *minimum* if there is no $r$-fold PSD forcing set of smaller cardinality than $B$. The cardinality of a minimum

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[2]We caution the reader that a chronological list of forces is not a propagating process and $B_t$ here has different meaning than that used in the study of propagation.

$r$-fold PSD forcing set is called the *$r$-fold positive semidefinite forcing number* of $G$ and is denoted $Z_{[r]}^+(G)$. We define the *fractional positive semidefinite forcing number* of a graph $G$ as

$$Z_f^+(G) = \inf_{r \in \mathbb{N}} \left\{ \frac{Z_{[r]}^+(G)}{r} \right\}.$$

Note that $G^{(1)} = G$ and a 1-fold PSD forcing set is exactly a positive semidefinite zero forcing set, so $Z_{[1]}^+(G) = Z^+(G)$. Any positive semidefinite zero forcing set $B$ can be converted into an $r$-fold PSD forcing set (for $r \geq 2$) by the following rule: If $u \in B$, then color every vertex in $R_u \in V(G^{(r)})$ blue. This creates an $r$-fold PSD forcing set that contains $r \cdot Z^+(G)$ blue vertices, so $Z_{[r]}^+(G) \leq r \cdot Z^+(G) = r \cdot Z_{[1]}^+(G)$. We conclude that $Z_f^+(G) \leq Z^+(G)$ and that

$$Z_f^+(G) = \inf_{r \geq 2} \left\{ \frac{Z_{[r]}^+(G)}{r} \right\}.$$

### 3.2.2   Global interpretation of $r$-fold positive semidefinite forcing

In this section, we assume that $r \geq 2$ and utilize the global structure of a graph $r$-blowup, namely, clusters joined by edges. Three specific types of cluster are of particular interest. An *All cluster* is a cluster in which all vertices are colored blue. A *One cluster* is a cluster in which exactly one vertex is colored blue and the rest are colored white. A *None cluster* is a cluster in which all vertices are colored white. We define a *All-One-None (minimum) $r$-fold positive semidefinite forcing set $B$* for a graph $G$ to be a (minimum) $r$-fold PSD forcing set in which each cluster of $G^{(r)}$ is either an All, One, or None cluster when $G^{(r)}$ is colored with $B$. For the sake of brevity, we will hereafter shorten All-One-None to AON.

We say that a cluster $R_u$ is *forced into* when any vertex in $R_u$ is forced. Once a cluster changes from a non-All to an All cluster, we say that the cluster has been *forced*. Any cluster that is forced into becomes an All cluster after the forcing operation, so forcing into a cluster and forcing the cluster are equivalent.

**Remark 3.2.2.** At some stage of the $r$-fold positive semidefinite forcing process using a particular chronological list of forces, let $B_t$ denote the set of blue vertices in $G^{(r)}$. Assume that $R_u \not\subseteq B_t$ for some $u \in V(G)$. Suppose that the next force in the process is done by $x \in R_u$, so $x$ has at most $r$ white neighbors. Since $R_u \not\subseteq B_t$, there exists at least one white vertex $w \in R_u$. Because $x$ and $w$ have the same neighbors and $w$ is white, all white neighbors of $x$ are connected through $w$ and lie in the same connected component. Hence, after $x$ forces, all neighbors of every vertex in $R_u$ must be blue, so without loss of generality $R_u$ can be forced in the next step of the forcing process.

**Definition 3.2.3.** If at any stage of the $r$-fold positive semidefinite forcing process a vertex in any partially-filled cluster performs a force, then that cluster can itself be forced at the next forcing step. We refer to this process as *backforcing*.

Remark 3.2.2 asserts that requiring backforcing does not affect whether or not a set is an $r$-fold PSD forcing set, so we will always assume that backforcing is used when performing the $r$-fold positive semidefinite forcing process.

**Definition 3.2.4.** Let $R_{u_1}, R_{u_2}, \ldots, R_{u_m}$ be "partially-filled" clusters (i.e., no cluster is an All or a None) in $G^{(r)}$ that together contain $pr + q$ blue vertices for some $0 \le p < m$ and $0 \le q < r$. We define the process of *consolidation* as follows: Use $pr$ of the blue vertices to convert $R_{u_1}, \ldots, R_{u_p}$ into All clusters and move the remaining $q$ blue vertices into $R_{u_{p+1}}$.

Our goal for the remainder of this section is to use these tools and definitions to develop an equivalent characterization of the $r$-fold positive semidefinite forcing game that relies only upon a particular type of AON $r$-fold PSD forcing set.

**Remark 3.2.5.** Suppose that $r \ge 3$. If $B$ is an AON $r$-fold PSD forcing set, then from a global perspective exactly one cluster in $G^{(r)}$ is forced at each step of the forcing process. This is because the vertex that performs the force can only force into One or None

clusters, and if this vertex were adjacent to more than one of these (in any combination), then it would have more than $r$ white neighbors and could not actually perform a force.

The case when $r = 2$ is slightly different. In this case, it is possible for a vertex to force two One clusters at the same forcing step (see Example 3.2.10 below). Every 2-fold PSD forcing set is automatically an AON set, so we cannot claim that if $G^{(r)}$ has a global AON structure, then exactly one cluster will be forced at each forcing step. However, Theorem 3.2.6 uses consolidation to show that even though every AON PSD forcing set need not have this property, there always exist an AON minimum PSD forcing set and forcing process that do.

**Theorem 3.2.6.** *Let $G$ be a graph and suppose $r \geq 2$. Then there exists an AON minimum $r$-fold PSD forcing set for $G$. For all $r \geq 3$, exactly one cluster of $G^{(r)}$ will be forced at each step of any forcing process that begins with any such set. For $r = 2$, there exists a forcing process for the set constructed such that exactly one cluster of $G^{(r)}$ is forced at each step.*

*Proof.* We first consider the case where $r \geq 3$. Let $B$ be a minimum $r$-fold PSD forcing set for $G$ and assume that $B$ is not AON. Write a chronological list of the forces performed using the forcing set $B$, assuming the use of backforcing, and let $B_t$, $t \geq 0$, denote the set of blue vertices after step $t$ of this forcing process, where $B_0 = B$.

Suppose that a vertex $x \in R_u$ performs a force at step $\ell \geq 1$ of the forcing process and $R_u \not\subseteq B_{\ell-1}$, implying that $R_u$ was not forced into at any step prior to step $\ell$. Since we assume backforcing and $R_u$ contains at least one white vertex, $R_u$ was not used to force any other cluster prior to step $\ell$, and $R_u$ will be forced in step $\ell + 1$. Thus if $R_u$ is not a One cluster, we can uncolor every blue vertex in $R_u$ except for $x$ without changing the ability of $x$ to force or the ability of $R_u$ to be backforced at step $\ell + 1$; since $R_u$ is not involved in any forces prior to step $\ell$, we can make this change in the original set $B$ and obtain a forcing set with fewer blue vertices, contradicting the assumption that $B$

was a minimum forcing set. We conclude that every cluster in a minimum $r$-fold PSD forcing set that is not an All cluster and contains a vertex that performs a force must be a One cluster.

Now, suppose that at step $\ell \geq 1$ we have $x \to W \subseteq (R_{u_1} \cup R_{u_2} \cup \cdots \cup R_{u_m})$ for some $m \geq 2$, where each $R_{u_j}$ contains at least one white vertex. Since $x$ is performing a force, it has at most $r$ white neighbors in the component containing $\bigcup_{j=1}^{m} R_{u_j}$, so there are at least $r(m-1)$ blue vertices in $\bigcup_{j=1}^{m} R_{u_j}$. Each cluster $R_{u_j}$ is an All cluster after step $\ell$, and no $R_{u_j}$ was forced into prior to step $\ell$. Since we assume backforcing and each of the $R_{u_j}$ clusters contains at least one white vertex, none of the $R_{u_j}$ clusters contains a vertex that was used to force at a step prior to step $\ell$. Analogous to Remark 3.2.2, removing blue vertices from any of the $R_{u_j}$ will not affect the application of the disconnect property, as each $R_{u_j}$ contains at least one white vertex. Similarly, adding blue vertices to convert an $R_{u_j}$ into an All cluster may make available additional disconnects (which we do not use), but these would not affect any previous forces. Therefore, we can consolidate the (at least $r(m-1)$) blue vertices in $\bigcup_{j=1}^{m} R_{u_j}$ without affecting the ability to perform any previous force.

Without loss of generality, suppose that $R_{u_1}, \ldots, R_{u_{m-1}}$ become All clusters after the consolidation and any remaining blue vertices are left in $R_{u_m}$. After consolidation, the new force at step $\ell$ will be $x \to R_{u_m}$; after this point, the state of the system is the same as it would have been had we not consolidated (i.e., every $R_{u_j}$ is an All cluster), so future forces are unaffected by consolidation. Furthermore, after consolidation, exactly one cluster ($R_{u_m}$) is forced at step $\ell$. Since the consolidation process does not affect any of the forces before or after the force at step $\ell$, we are free to perform the consolidation on the original set $B$ to obtain a new minimum $r$-fold PSD forcing set $\widetilde{B}$ and the sequence of vertices that perform forces remains unchanged. Note that since $\widetilde{B}$ is minimum, $R_{u_m}$ must necessarily be a None cluster.

By repeated application of the consolidation process, we are able to convert every non-One cluster into an All cluster or a None cluster. By Remark 3.2.5, any AON forcing process for $r \geq 3$ must necessarily consist of forcing only one cluster at each step, which proves the claim for $r \geq 3$.

Now, suppose that $r = 2$. Every minimum 2-fold PSD forcing set for $G$ is automatically an AON set. Suppose that, at step $\ell \geq 1$ of the forcing process, more than one cluster must be forced. Since any vertex can force at most 2 of its neighbors, it must be the case that two One clusters are forced at this step. For the reasons described in the $r \geq 3$ case, we can consolidate these two One clusters into one All cluster and one None cluster without affecting any previous or future forces; after this consolidation, only one cluster is forced at step $\ell$. As before, we can modify our original minimum forcing set and the result follows for the $r = 2$ case (using the forcing process to which consolidation was applied). $\qquad\square$

We call the type of AON minimum $r$-fold PSD forcing set guaranteed to exist by Theorem 3.2.6 an *optimal AON $r$-fold PSD forcing set*. We emphasize that an optimal AON $r$-fold PSD forcing set is minimum by definition, and given an optimal AON $r$-fold PSD forcing set there is a corresponding forcing process in which exactly one cluster is forced at each step. Further, the set of blue vertices at each step of the forcing process associated with an optimal AON $r$-fold PSD forcing set will always create a global AON structure in $G^{(r)}$.

Suppose that $B$ is an AON $r$-fold PSD forcing set for a graph $G$ and color $G^{(r)}$ with $B$. We use $a(B)$ to denote the number of All clusters in $G^{(r)}$ and $\ell(B)$ to denote the number of One clusters in $G^{(r)}$, so $|B| = r \cdot a(B) + \ell(B)$. This new terminology yields a corollary to Theorem 3.2.6.

**Corollary 3.2.7.** *For every graph $G$ and $r \geq 2$, there exists an optimal AON $r$-fold PSD forcing set for $G$, and for any such set $B$, we have $Z_{[r]}^+(G) = |B| = r \cdot a(B) + \ell(B)$.*

**Definition 3.2.8.** Let $r, s \geq 2$ with $s \neq r$ and suppose that $B$ is an AON $r$-fold PSD forcing set for $G$. Copy the AON structure of $G^{(r)}$ when colored with $B$ onto $G^{(s)}$ to create a new AON set of blue vertices of cardinality $s \cdot a(B) + \ell(B)$. This process is called *replication*.

**Remark 3.2.9.** Let $B$ be a 2-fold PSD forcing set for $G$ and suppose that two One clusters are forced simultaneously at some step of the forcing process on $G^{(2)}$. In this case, replicating $B$ onto $G^{(s)}$ for $s > 2$ will not yield a valid forcing set (see Example 3.2.10, next). However, if $B$ is an optimal AON 2-fold PSD forcing set, then Theorem 3.2.6 guarantees that there is a forcing process in which exactly one force occurs at each step, so replication will yield a valid forcing set. As we see in Example 3.2.11, however, the replicated set may not be minimum and hence not optimal.



(a) (Minimum) AON 2-fold PSD forcing set      (b) Optimal AON 2-fold PSD forcing set
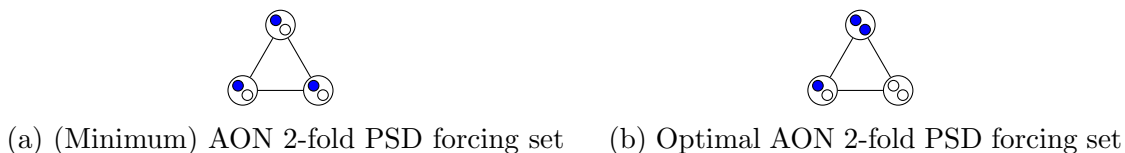
Figure 3.4: AON 2-fold PSD forcing sets for $K_3$

**Example 3.2.10.** Consider the (minimum) 2-fold PSD forcing sets for $K_3$ shown in Figure 3.4. For simplicity, the edges in the figure represent the complete bipartite graphs between the clusters at their endpoints. The first forcing step in Figure 3.4a would consist of forcing two of the One clusters simultaneously. This set is no longer a forcing set when replicated onto $K_3^{(s)}$ for $s \geq 3$, as each of the blue vertices will have too many white neighbors to perform a force. The optimal AON PSD forcing set shown in Figure 3.4b, however, can be replicated successfully, as only one cluster must be forced at any step of the forcing process.

(a) Graph $G$       (b) $r = 2$       (c) $r = 3$
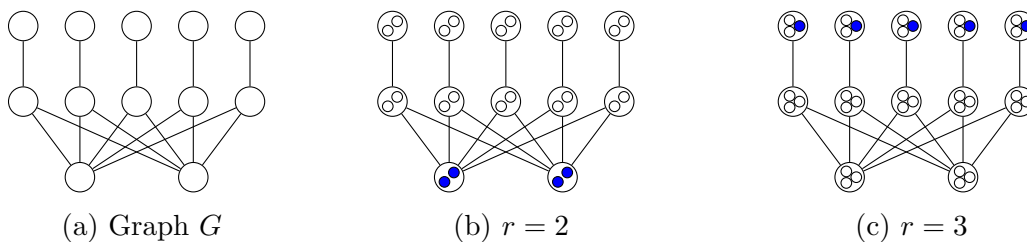
Figure 3.5: Optimal AON $r$-fold PSD forcing sets

**Example 3.2.11.** Suppose that we have the complete bipartite graph $K_{5,2}$ and let $G$ be the graph formed by attaching one leaf to each of the vertices in the partite set containing five vertices (Figure 3.5a). Consider the (unique) optimal AON $r$-fold PSD forcing sets for $G$ shown in Figures 3.5b and 3.5c. When $r = 2$, the forcing set has two All clusters, so $Z_{[2]}^+(G) = 4$. When $r = 3$, the forcing set has five One clusters, so $Z_{[3]}^+(G) = 5$. Replicating either optimal forcing set onto the other blowup will generate a forcing set that is not minimum, hence not optimal.

We now prove further properties of AON $r$-fold PSD forcing sets and use these results to provide an alternate definition of the fractional PSD forcing number.

**Lemma 3.2.12.** *Let $G$ be a graph on $n$ vertices and fix $r \geq n$. Let $B$ be an optimal AON $r$-fold PSD forcing set for $G$ and let $B'$ be an AON $r$-fold PSD forcing set for $G$. Then $a(B) \leq a(B')$.*

*Proof.* Assume first that $\ell(B') < n$. Since $B$ is optimal, it is minimum, so $r \cdot a(B) + \ell(B) = |B| \leq |B'| = r \cdot a(B') + \ell(B')$. Dividing through by $r$ and manipulating this inequality yields

$$a(B) - a(B') \leq \frac{\ell(B') - \ell(B)}{r} < \frac{n}{r} \leq 1.$$

Since $a(B) - a(B')$ is an integer, we must have $a(B) - a(B') \leq 0$, which proves the claim when $\ell(B') < n$. Now suppose that $\ell(B') = n$, so $a(B') = 0$. Since $r \geq n$, at most one force happens at each step, so we can replace the first cluster forced with a None

cluster to obtain a new AON $r$-fold PSD forcing set $B''$ with $a(B'') = a(B') = 0$ and $\ell(B'') = n - 1 < n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

**Corollary 3.2.13.** *Let $G$ be a graph on $n$ vertices and fix $r \geq n$. If $B$ and $B'$ are optimal AON $r$-fold PSD forcing sets for $G$, then $a(B) = a(B')$.*

Thus for a fixed "large enough" $r$, every optimal AON $r$-fold PSD forcing set for $G$ must contain the same number of All clusters (and, consequently, One clusters). Of particular interest is the case $r = n = |G|$. We define $a_\star^+(G)$ to be the unique number of All clusters created in $G^{(n)}$ by any optimal AON $n$-fold PSD forcing set for $G$, and define $\ell_\star^+(G)$ to be the (unique) number of One clusters created in this manner.

**Proposition 3.2.14.** *Let $G$ be a graph on $n$ vertices. For all $r \geq n$, if $B$ is an optimal AON $r$-fold PSD forcing set for $G$, then $a(B) = a_\star^+(G)$.*

*Proof.* Let $\widetilde{B}$ be the AON $n$-fold PSD forcing set formed by replicating $B$ onto $G^{(n)}$. By Lemma 3.2.12, $a_\star^+(G) \leq a(\widetilde{B}) = a(B)$. Similarly, let $B'$ be the AON $r$-fold PSD forcing set formed by replicating any optimal AON $n$-fold PSD forcing set onto $G^{(r)}$. By Lemma 3.2.12, $a(B) \leq a(B') = a_\star^+(G)$, and thus equality holds. $\qquad\qquad\square$

**Corollary 3.2.15.** *Let $G$ be a graph on $n$ vertices. For all $r \geq n$, $Z_{[r]}^+(G) = r \cdot a_\star^+(G) + \ell_\star^+(G)$. Additionally,*

$$\lim_{r \to \infty} \frac{Z_{[r]}^+(G)}{r} = a_\star^+(G).$$

Before we can prove the final result of this section, which ties the fractional positive semidefinite forcing number into the machinery just developed, we require one final utility result.

**Lemma 3.2.16.** *Let $G$ be a graph on $n$ vertices and choose $r \geq 2$. Then for any optimal AON $r$-fold PSD forcing set $B$, $\frac{|B|}{r} \geq a_\star^+(G)$.*

*Proof.* First, suppose that $2 \leq r < n$. Let $\widetilde{B}$ be the AON $n$-fold PSD forcing set obtained by replicating $B$ onto $G^{(n)}$. Then $a(B) = a(\widetilde{B})$ and $\ell(B) = \ell(\widetilde{B})$, so

$$\frac{|B|}{r} = a(B) + \frac{\ell(B)}{r} = a(\widetilde{B}) + \frac{\ell(\widetilde{B})}{r} \geq a(\widetilde{B}) + \frac{\ell(\widetilde{B})}{n} = \frac{|\widetilde{B}|}{n}.$$

Let $B'$ be any optimal AON $n$-fold PSD forcing set for $G$. Since $B'$ is optimal, it is minimum, hence $|\widetilde{B}| \geq |B'|$. Therefore,

$$\frac{|B|}{r} \geq \frac{|\widetilde{B}|}{n} \geq \frac{|B'|}{n} = a_\star^+(G) + \frac{\ell_\star^+(G)}{n} \geq a_\star^+(G),$$

which proves the claim when $r < n$.

If $r \geq n$, then Proposition 3.2.14 shows that $|B| = r \cdot a_\star^+(G) + \ell_\star^+(G)$ and the conclusion follows. $\square$

We conclude this section with an alternate characterization of fractional positive semidefinite forcing number.

**Theorem 3.2.17.** *For every graph $G$,*

$$Z_f^+(G) = a_\star^+(G).$$

*Proof.* Recall that $Z_f^+ = \inf_{r \geq 2} \left\{ \frac{Z_{[r]}^+(G)}{r} \right\}$. By Corollary 3.2.15, $Z_f^+(G) \leq a_\star^+(G)$. Let $B$ be an optimal AON $r$-fold PSD forcing set for $G$ for some $r \geq 2$. Then by Corollary 3.2.7 and Lemma 3.2.16, $\frac{Z_{[r]}^+(G)}{r} = \frac{|B|}{r} \geq a_\star^+(G)$, and thus equality holds. $\square$

This shows that the fractional positive semidefinite forcing number of a graph is always a nonnegative integer, an interesting result in light of its fractional construction.

### 3.2.3 Three-color interpretation of fractional positive semidefinite forcing

Motivated by the AON interpretation of the $r$-fold positive semidefinite forcing game, we consider a three-color forcing game that allows us to compute the fractional positive semidefinite forcing number for any graph without playing the $r$-fold game.

Let $G$ be a graph and consider the following *fractional positive semidefinite forcing game*, which is a three-color forcing game that uses the colors dark blue (target), light blue, and white. Assign to each vertex of $G$ one of these colors and let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$, where $\mathcal{D}$ denotes the set of dark blue vertices and $\mathcal{L}$ denotes the set of light blue vertices.[3] We repeatedly apply the following *fractional positive semidefinite forcing rule*:

**Definition 3.2.18** (fractional positive semidefinite forcing rule)**.** Let $\mathcal{B}_t = (\mathcal{D}_t, \mathcal{L}_t)$ denote the set of colored vertices of a graph $G$ at some step of the fractional positive semidefinite forcing process and let $W_1, \ldots, W_h$ denote the sets of vertices of the connected components of $G - \mathcal{D}_t$. If $u \in \left( \mathcal{D}_t \,\dot{\cup}\, (\mathcal{L}_t \cap W_i) \right)$ and $w \in W_i$ is the only light blue or white neighbor of $u$ in $G[\mathcal{D}_t \cup W_i]$, then $u$ can force $w$, i.e., $w$ can be colored dark blue.

Loosely speaking, we apply the disconnect rule from positive semidefinite zero forcing using the dark blue vertices of $G$, and then in each augmented component any dark or light blue vertex can force its only light blue or white neighbor. As usual, the goal of this forcing game is to choose the initial set $\mathcal{B}$ in such a way that by repeated application of this rule the entire graph can be forced (i.e., turned dark blue). If $G$ can be forced, then we say that the initial set $\mathcal{B}$ is a *fractional positive semidefinite (PSD) forcing set* for $G$. The *(three-color) fractional positive semidefinite forcing number* of $G$, denoted $\hat{Z}_f^+(G)$, is then defined as

$$\hat{Z}_f^+(G) = \min\left\{ |\mathcal{D}| : (\mathcal{D}, \mathcal{L}) \text{ is a fractional PSD forcing set for } G, \text{ for some } \mathcal{L} \right\}.$$

---

[3]Recall from Section 3.1.3 that this is equivalent to writing $\mathcal{B} = \mathcal{D} \,\dot{\cup}\, \mathcal{L}$.

We say that a fractional PSD forcing set $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ for $G$ is *optimal* if $|\mathcal{D}| = \hat{Z}_f^+(G)$ and no fractional PSD forcing set for $G$ with $|\mathcal{D}| = \hat{Z}_f^+(G)$ has fewer than $|\mathcal{L}|$ light blue vertices. We use $\hat{\ell}_\star^+(G)$ to denote the number of light blue vertices in any optimal fractional PSD forcing set for $G$, i.e., $\hat{\ell}_\star^+(G) = |\mathcal{L}|$.

The process of backforcing described for the $r$-fold positive semidefinite forcing game applies to the fractional positive semidefinite forcing game, albeit with a three-color modification. After a light blue vertex $u$ performs a force, all of its neighbors must necessarily be dark blue, and so we can backforce $u$ at the next forcing step.

The observant reader will notice that we have defined "fractional positive semidefinite forcing number" twice: here, and in Section 3.2.1. The final result of this section shows that this is not an error: the parameter $Z_f^+$, defined via an $r$-fold two-color game, is equal to the parameter $\hat{Z}_f^+$, defined via a three-color game.

**Theorem 3.2.19.** *For any graph $G$, $Z_f^+(G) = \hat{Z}_f^+(G)$.*

*Proof.* Let $|G| = n$ and let $B$ be an optimal AON $n$-fold PSD forcing set for $G$. By Theorem 3.2.17, we have $a(B) = a_\star^+(G) = Z_f^+(G)$. Color $G^{(n)}$ with $B$ and color $G$ with $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{L}})$, defined as follows: Let $\widetilde{\mathcal{D}} = \{u : R_u \text{ is an All cluster in } G^{(n)}\}$ and let $\widetilde{\mathcal{L}} = \{u : R_u \text{ is a One cluster in } G^{(n)}\}$. Since $B$ is an optimal AON $n$-fold PSD forcing set, exactly one cluster is forced at each step of the forcing process using $B$, and $G^{(n)}$ can be forced. Further, backforcing is applied to One clusters in $G^{(n)}$, and One clusters correspond to light blue vertices, to which backforcing can also be applied. Therefore, the forcing process used on $G^{(n)}$ can be used to force $G$, so $\widetilde{\mathcal{B}}$ is a fractional PSD forcing set for $G$ and $\hat{Z}_f^+(G) \leq |\widetilde{\mathcal{D}}| = a(B) = Z_f^+(G)$.

Let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be an optimal fractional PSD forcing set for $G$. The reverse inequality easily follows by associating elements of $\mathcal{D}$ with All clusters in $G^{(n)}$ and elements of $\mathcal{L}$ with One clusters and applying Lemma 3.2.12 and arguments similar to those above. $\square$

**Corollary 3.2.20.** *For any graph $G$, $\ell_\star^+(G) = \hat{\ell}_\star^+(G)$.*

As a consequence of these results, the $\hat{Z}_f^+$ and $\hat{\ell}_\star^+$ notations will be suppressed in favor of the simpler $Z_f^+$ and $\ell_\star^+$.

In contrast to the process of computing the values of fractional versions of general graph parameters, computing the fractional positive semidefinite forcing number of a graph does not require any explicit knowledge of the $r$-fold analogue. If knowledge of $Z_f^+$ is all that is of interest, one can bypass the $r$-fold game and opt to play the fractional positive semidefinite forcing game instead. The benefit of taking a three-color approach is also demonstrated in Section 3.3, where a three-color interpretation is used to obtain new results pertaining to skew zero forcing.

### 3.2.4 Results for fractional positive semidefinite forcing number

The fractional positive semidefinite forcing game allows us to easily prove many interesting properties of the fractional positive semidefinite forcing number.

**Remark 3.2.21.** Any isolated vertex in a graph $G$ must be colored dark blue. Thus if $\delta(G) = 0$, then $Z_f^+(G) \geq |\{u \in V(G) : \deg(u) = 0\}| \geq 1$.

**Observation 3.2.22.** *If a graph $G$ has connected components $\{G_i\}_{i=1}^m$, then $Z_f^+(G) = \sum_{i=1}^m Z_f^+(G_i)$ and $\ell_\star^+(G) = \sum_{i=1}^m \ell_\star^+(G_i)$.*

Thus we are able to focus on connected graphs (as is customary for zero forcing).

**Remark 3.2.23.** Let $G$ be a graph and let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be a fractional PSD forcing set for $G$. The set $B = \mathcal{D} \dot\cup \mathcal{L}$ is a positive semidefinite zero forcing set for $G$, so $Z^+(G) \leq |B| = |\mathcal{D}| + |\mathcal{L}|$. If $\mathcal{B}$ is optimal, then this shows that $Z^+(G) \leq Z_f^+(G) + \ell_\star^+(G)$.

A natural question in light of this remark is whether $Z^+(G) = Z_f^+(G) + \ell_\star^+(G)$. By taking a minimum positive semidefinite zero forcing set for $G$ and changing some vertices to light blue, it may be possible to obtain an optimal fractional PSD forcing set for $G$. Even though this works for some graphs, the next example provides a graph for which this technique fails.

**Example 3.2.24.** Let $G$ be the graph shown in Example 3.2.11, a $K_{5,2}$ with one leaf appended to each vertex in the partite set on 5 vertices. By coloring each of the leaves light blue, we can force each of their neighbors, and using the disconnect rule we can subsequently backforce the leaves and force all of $G$. Thus $Z_f^+(G) = 0$ and $\ell_\star^+(G) = 5$, but it is known that $Z^+(G) = 2 < 0 + 5 = Z_f^+(G) + \ell_\star^+(G)$. The key to this example is that the set $B = \mathcal{L}$ is a minimal positive semidefinite zero forcing set for $G$, but it is not a minimum positive semidefinite zero forcing set.

**Remark 3.2.25.** If $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ is an optimal fractional PSD forcing set for a connected graph $G$, then any vertex that is colored light blue must perform a force before it is itself forced; if not, then that vertex can be colored white to obtain a fractional PSD forcing set with the same number of dark blue vertices and fewer light blue vertices, contradicting the optimality of $\mathcal{B}$. Additionally, no two light blue vertices in an optimal fractional PSD forcing set can be adjacent, as one would have to force the other before the other has performed a force. Therefore, $\mathcal{L}$ is an independent set in $G$, so $\ell_\star^+(G) \leq \alpha(G)$.

The following result pertains to the (two-color) positive semidefinite zero forcing game.

**Lemma 3.2.26** ([13], Lemma 2.1.1)**.** *Let $G$ be a graph and let $B$ be a positive semidefinite zero forcing set of $G$. If $v \in B$ is the vertex that performs the first force, $v \to w$, where $w$ is a white neighbor of $v$, then $(B \setminus \{v\}) \cup \{w\}$ is a positive semidefinite zero forcing set of $G$.*

We now present a three-color version of Lemma 3.2.26. The proof is similar to the proof of the two-color version found in [13] and is omitted.

**Lemma 3.2.27.** *Let $G$ be a graph and let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be a fractional PSD forcing set for $G$. Suppose that the first force, $v \to w$, is performed by some $v \in \mathcal{D}$ on some $w \notin \mathcal{L}$. Let $\widetilde{\mathcal{D}} = (\mathcal{D} \setminus \{v\}) \cup \{w\}$. Then $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{D}}, \mathcal{L})$ is also a fractional PSD forcing set for $G$.*

**Theorem 3.2.28.** *If $G$ is a graph with at least one edge, then $G$ has an optimal fractional PSD forcing set with which the first force can be performed by a light blue vertex.*

*Proof.* Suppose for the sake of contradiction that $G$ does not have an optimal fractional PSD forcing set with which the first force can be performed by a light blue vertex. Note that if the first force with an optimal set can be done without using the disconnect rule, then this force must be done by a light blue vertex (else the set is not optimal), so our assumption implies that the disconnect rule must be applied to perform the first force with any optimal fractional PSD forcing set. Let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be an optimal fractional PSD forcing set such that $|W_1|$ is minimum, where $W_1, W_2, \ldots, W_h$ are the sets of vertices of the connected components of $G - \mathcal{D}$ and $|W_1| \leq |W_2| \leq \cdots \leq |W_h|$. As noted in Remark 1.14 of [14], we can assume that the first vertex forced lies in $W_1$. Let $v \to w$ denote the first force, where $v \in \mathcal{D}$ and $w \in W_1$.

By Lemma 3.2.27, the set $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{D}}, \mathcal{L})$ with $\widetilde{\mathcal{D}} = (\mathcal{D} \setminus \{v\}) \cup \{w\}$ is also an optimal fractional PSD forcing set for $G$. Since $w$ must be the only non-dark-blue neighbor of $v$ in $W_1$, it must be the case that $v$ joins a component other than $W_1$ in $G - \widetilde{\mathcal{D}}$; further, in $G - \widetilde{\mathcal{D}}$, the component $W_1$ will not contain the vertex $w$, and may split into multiple smaller components. If $W_1 \neq \{w\}$, then this argument shows that there must be a component with fewer than $|W_1|$ vertices in $G - \widetilde{\mathcal{D}}$, which contradicts the choice of $\mathcal{B}$; thus we must have $W_1 = \{w\}$. However, the first force in $G$ using $\widetilde{\mathcal{B}}$ can therefore be chosen as $w \to v$, which can be done without applying the disconnect rule; by the comments above, $w$ can thus be light blue, contradicting optimality of $\mathcal{B}$. Therefore, $G$ must have an optimal fractional PSD forcing set with which the first force can be performed by a light blue vertex. $\square$

Theorem 3.2.28 yields a lower bound on $Z_f^+(G)$ as a corollary.

**Corollary 3.2.29.** *For any graph $G$, $\delta(G) - 1 \leq Z_f^+(G)$.*

*Proof.* The result is trivial for $\delta(G) \leq 1$. If $\delta(G) \geq 2$, then $G$ has an edge, so by Theorem 3.2.28 there exists some optimal fractional PSD forcing set $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ such that the first force in $G$ can be done by some $u \in \mathcal{L}$. Remark 3.2.25 asserts that $u$ has no light blue neighbors, and all white neighbors of $u$ must be in the same component of $G - \mathcal{D}$. Since $u$ can force, all but one of its neighbors must be dark blue. Thus $|\mathcal{D}| \geq |N(u)| - 1 \geq \delta(G) - 1$. □

An additional corollary to Theorem 3.2.28 gives a lower bound on $\ell_\star^+(G)$ in the case where $G$ has at least one edge.

**Corollary 3.2.30.** *If $G$ is a graph with at least one edge, then $\ell_\star^+(G) \geq 1$.*

The following result is a two-color analogue of Theorem 3.2.28 that applies to the positive semidefinite zero forcing game. The proof is similar to that of Theorem 3.2.28 and is omitted.

**Theorem 3.2.31.** *If $G$ is a graph with at least one edge, then there exists a minimum positive semidefinite zero forcing set for $G$ such the first force can be done without using the disconnect rule.*

With Theorem 3.2.31, we can obtain an improved upper bound on $Z_f^+(G)$.

**Corollary 3.2.32.** *For any graph $G$ with at least one edge, $Z_f^+(G) \leq Z^+(G) - 1$.*

*Proof.* Theorem 3.2.31 ensures that there is some minimum positive semidefinite zero forcing set $B$ such that the first force using $B$ can be done without using the disconnect rule. If $\mathcal{B}$ is obtained by coloring the vertex that performs this first force light blue and all of the other vertices in $B$ dark blue, then $\mathcal{B}$ is a fractional PSD forcing set with $Z^+(G) - 1$ dark blue vertices. □

### 3.2.5 Fractional positive semidefinite forcing numbers for graph families

In this section, we determine the fractional PSD forcing numbers for certain graph families, illustrating the utility of some of the results in Section 3.2.4.

**Example 3.2.33.** Let $n \geq 2$ and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Note that $Z^+(K_n) = n-1$ [7, Example 46.4.2]. Applying Corollaries 3.2.29 and 3.2.32, $n - 2 = \delta(K_n) - 1 \leq Z_f^+(K_n) \leq Z^+(K_n) - 1 = n - 2$ and thus equality holds. By Corollary 3.2.30, $\ell_\star^+(K_n) \geq 1$. The set $\mathcal{B} = (\{v_1, v_2, \ldots, v_{n-2}\}, \{v_{n-1}\})$ is an optimal fractional PSD forcing set for $K_n$, so $Z_f^+(K_n) = n - 2$ and $\ell_\star^+(K_n) = 1$.

In each of the next four examples, optimality of the exhibited fractional PSD forcing sets is obtained by application of Corollaries 3.2.29 and 3.2.30.

**Example 3.2.34.** For any $n \geq 2$, the set $\mathcal{B} = (\emptyset, \{v_1\})$ is an optimal fractional PSD forcing set for $P_n$, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ in path order, so $Z_f^+(P_n) = 0$ and $\ell_\star^+(P_n) = 1$.

**Example 3.2.35.** For any $n \geq 3$, the set $\mathcal{B} = (\{v_1\}, \{v_2\})$ is an optimal fractional PSD forcing set for $C_n$, where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ in cycle order, so $Z_f^+(C_n) = 1$ and $\ell_\star^+(C_n) = 1$.

**Example 3.2.36.** Let $n \geq 4$ and consider the wheel on $n$ vertices, $W_n$, which is obtained by adding a vertex $w$ adjacent to every vertex of $C_{n-1}$. If $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ is any optimal fractional PSD forcing set for $C_{n-1}$, then $\widetilde{\mathcal{B}} = (\mathcal{D} \cup \{w\}, \mathcal{L})$ is an optimal fractional PSD forcing set for $W_n$, so $Z_f^+(W_n) = 2$ and $\ell_\star^+(W_n) = 1$.

**Example 3.2.37.** Let $p \geq q \geq 1$ and consider $K_{p,q}$, the complete bipartite graph on partite sets $P$ and $Q$ with $|P| = p$ and $|Q| = q$. Let $\mathcal{D}$ be a set containing any $(q - 1)$ elements of $Q$ and let $\mathcal{L}$ be a set containing any one element of $P$; then $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ is an optimal fractional PSD forcing set for $K_{p,q}$, so $Z_f^+(K_{p,q}) = q - 1$ and $\ell_\star^+(K_{p,q}) = 1$.

As a final example, we consider the fractional PSD forcing number of a tree.

**Example 3.2.38.** Suppose that $T$ is a tree of order at least 2. We have $Z^+(T) = 1$ [7, Example 46.4.3], so Corollary 3.2.32 implies that $0 \leq Z_f^+(T) \leq Z^+(T) - 1 = 0$ and hence equality holds. If we let $\mathcal{L}$ be any leaf of $T$, then $\mathcal{B} = (\emptyset, \mathcal{L})$ is an optimal fractional PSD forcing set, so $Z_f^+(T) = 0$ and $\ell_\star^+(T) = 1$.

## 3.3   Three-color interpretation of skew zero forcing

In this section, we introduce a three-color interpretation of the skew zero forcing game and use this to show that the skew zero forcing number and "fractional (zero) forcing number" of a graph are equal. Using the three-color interpretation, we derive new results pertaining to skew zero forcing number and the associated coloring process.

### 3.3.1   The three-color skew zero forcing game

Consider the following three-color forcing game played on a graph $G$. Choose an initial set of dark blue vertices, $\mathcal{D}$, and a set of light blue vertices, $\mathcal{L}$, and let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$; color all other vertices of $G$ white. The forcing rule is as follows:

**Definition 3.3.1** (three-color skew zero forcing rule). If $w$ is the only non-dark-blue neighbor of a dark blue or light blue vertex $u$, then $u$ can force $w$.

The set $\mathcal{B}$ is a *three-color skew zero forcing set* if $G$ can be forced after repeated application of the three-color skew zero forcing rule. We define

$$\hat{Z}^-(G) = \min\left\{|\mathcal{D}| : (\mathcal{D}, \mathcal{L}) \text{ is a three-color skew zero forcing set for } G, \text{ for some } \mathcal{L}\right\}.$$

A three-color skew zero forcing set $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ is *optimal* if $|\mathcal{D}| = \hat{Z}^-(G)$ and no such forcing set for $G$ has fewer light blue vertices than $\mathcal{B}$. Let $\ell_\star^-(G)$ denote the number of light blue vertices in any optimal three-color skew zero forcing set for $G$, i.e., $\ell_\star^-(G) = |\mathcal{L}|$.

The inclusion of the word "skew" in the development of $\hat{Z}^-(G)$ is not an accident. It is easy to see that the three-color skew zero forcing game is equivalent to the (two-color) skew zero forcing game described in Section 3.1.1: dark blue vertices correspond to (regular) blue vertices in two-color skew zero forcing, light blue vertices correspond to white vertices that perform white vertex forcing, and white vertices that do not perform a white vertex force are the same in both cases. Therefore, $\hat{Z}^-(G) = Z^-(G)$, and we are free to use the more familiar notation $Z^-(G)$ when discussing the three-color game.

**Remark 3.3.2.** Notice that any three-color skew zero forcing set for a graph $G$ is also a fractional PSD forcing set for $G$: playing the three-color skew zero forcing game is equivalent to playing the fractional PSD zero forcing game without using the disconnect rule. Therefore, $Z_f^+(G) \leq Z^-(G)$.

From this point forward, since they give more information than their two-color counterparts, we will focus on three-color skew zero forcing sets, and usually omit the "three-color" descriptor for the sake of brevity.

### 3.3.2  General results for skew zero forcing

The three-color interpretation easily lends itself to making observations about skew zero forcing number of a graph. The next two results are well-known for $Z^-(G)$ using the two-color approach, where we interpret $\ell_\star^-(G)$ as the number of vertices that perform white vertex forces in that case.

**Remark 3.3.3.** Any isolated vertex in a graph $G$ must be colored dark blue, so if $\delta(G) = 0$, then $Z^-(G) \geq |\{u \in V(G) : \deg(u) = 0\}| \geq 1$.

**Observation 3.3.4.** *If a graph $G$ has connected components $\{G_i\}_{i=1}^m$, then $Z^-(G) = \sum_{i=1}^m Z^-(G_i)$ and $\ell_\star^-(G) = \sum_{i=1}^m \ell_\star^-(G_i)$.*

As is customary, we are able to focus our attention on connected graphs.

**Remark 3.3.5.** For every connected graph $G$, $\delta(G) - 1 \leq Z^-(G)$. This is because if a candidate skew zero forcing set does not contain at least $\delta(G) - 1$ dark blue vertices, then every dark blue or light blue vertex has at least two white or light blue neighbors, so the forcing process cannot start.

**Remark 3.3.6.** Suppose that $G$ is a connected graph on 2 or more vertices and color each of its vertices dark blue. Any one adjacent pair can then be re-colored white and light blue (in either order), so $Z^-(G) \leq |G| - 2$.

**Remark 3.3.7.** For every connected graph $G$, we have $Z^-(G) \leq Z(G) \leq Z^-(G) + \ell_\star^-(G)$. The first inequality follows because every zero forcing set for a graph $G$ is also a skew zero forcing set for $G$. For the second, note that if $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ is an optimal skew zero forcing set, then $B = \mathcal{D} \dot\cup \mathcal{L}$ is a (standard) zero forcing set.

The justification for the next observation is the same as that given in Remark 3.2.25.

**Observation 3.3.8.** *If $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ is an optimal skew zero forcing set for a connected graph $G$, then any vertex that is colored light blue must perform a force before it is itself forced. No two light blue vertices in an optimal skew zero forcing set can be adjacent. The set $\mathcal{L}$ is an independent set in $G$, and $\ell_\star^-(G) \leq \alpha(G)$.*

**Remark 3.3.9.** For a graph $G$, the quantity $|G| - Z^-(G)$ is the number of non-dark-blue vertices in an optimal skew zero forcing set. In the worst case, half of these vertices would need to be colored light blue to force their white neighbors, so $0 \leq \ell_\star^-(G) \leq \left\lfloor \frac{|G| - Z^-(G)}{2} \right\rfloor \leq \left\lfloor \frac{|G|}{2} \right\rfloor < |G|$.

### 3.3.3 Skew zero forcing as fractional zero forcing

In this section, we develop an $r$-fold version of the standard zero forcing game and use it to prove that the "fractional (zero) forcing number" of a graph is equal to the skew zero forcing number of the graph. This treatment is similar to the positive semidefinite case discussed in Sections 3.2.1 and 3.2.2.

Let $G$ be a graph and for some $r \in \mathbb{N}$ consider the following $r$-*fold forcing game*, which is a two-color forcing game played on $G^{(r)}$, the $r$-blowup of $G$. As in any zero forcing game, we initially color some set $B \subseteq V(G^{(r)})$ blue and then try to force $G^{(r)}$ through repeated application of the following $r$-*fold forcing rule*:

**Definition 3.3.10** ($r$-fold forcing rule)**.** At some step $t$ of the forcing process, let $B_t$ denote the set of blue vertices in $G^{(r)}$. If $u \in B_t$ and $|N(u) \setminus B_t| \leq r$, then $u$ can force $N(u) \setminus B_t$, i.e., all white neighbors of $u$ can be colored blue simultaneously.

The $r$-fold forcing rule is exactly the $r$-forcing rule found in [2], although applied to $G^{(r)}$ instead of $G$. The $r$-fold forcing game was developed in the spirit of fractional graph theory [12], while the $r$-forcing process described in [2] is more general. We have chosen to use different terminology with our treatment to emphasize this key difference.

If $G^{(r)}$ can be forced, then the initial set of blue vertices is called an $r$-*fold forcing set* for $G$. A *minimum $r$-fold forcing set* is an $r$-fold forcing set of minimum cardinality. The $r$-*fold forcing number* of $G$, $Z_{[r]}(G)$, is the cardinality of a minimum $r$-fold forcing set.[4] We define the *fractional forcing number* of $G$ as

$$Z_f(G) = \inf_{r \in \mathbb{N}} \left\{ \frac{Z_{[r]}(G)}{r} \right\}.$$

Clearly, $Z_{[1]}(G) = Z(G)$. By an argument similar to that used in Section 3.2.1, it is easy to see that $Z_{[r]}(G) \leq r \cdot Z(G)$ for $r \geq 2$, so we can equivalently define fractional forcing number as

$$Z_f(G) = \inf_{r \geq 2} \left\{ \frac{Z_{[r]}(G)}{r} \right\}.$$

Our goal in this section is to prove that $Z_f(G) = Z^-(G)$ for any graph $G$. In order to do this, we will follow an approach similar to that used in Section 3.2.2, with the noted difference that we have a three-color interpretation of skew zero forcing that can be used to simplify some of our arguments.

---

[4]Note that $Z_{[r]}(G) = F_r(G^{(r)})$, where $F_k(H)$ is the $k$-forcing number of a graph $H$; see [2].

The global view of the $r$-fold forcing game, analogous to that of the $r$-fold positive semidefinite forcing game, will also be considered. Since backforcing does not apply to this game, in addition to All, One, and None clusters in $G^{(r)}$, we consider one other type of cluster: a *Most cluster* is a cluster in which all but one vertex is colored blue. We consider Most clusters only for $r \geq 3$, as when $r = 2$ a Most cluster is equivalent to a One cluster. An *All-Most-One-None (AMON) $r$-fold forcing set* is an $r$-fold forcing set for $G$ that creates All, Most, One, and None clusters in $G^{(r)}$. As before, we let $a(B)$ denote the number of All clusters and $\ell(B)$ denote the number of One clusters created in $G^{(r)}$ by an AMON $r$-fold forcing set $B$; we introduce $m(B)$ to denote the number of Most clusters created by $B$. If $B$ is an AMON $r$-fold forcing set, then $|B| = r \cdot a(B) + (r-1) \cdot m(B) + \ell(B) = r\,(a(B) + m(B)) + \ell(B) - m(B)$.

Many of the remarks and observations from Section 3.2.2 apply to the global interpretation of the $r$-fold forcing game and we omit or reduce their proofs. As before, we note that forcing into a cluster $R_u$ is equivalent to forcing $R_u$ and a cluster that is forced becomes an All cluster. Each cluster performs at most one force.

**Theorem 3.3.11.** *For any graph $G$ and any $r \geq 2$, an AMON minimum $r$-fold forcing set for $G$ exists, as does a forcing process in which at each step either exactly one cluster is forced or a One cluster and a Most cluster (or, when $r = 2$, two One clusters) are forced simultaneously.*

*Proof.* The result is trivially true for $r = 2$, so assume that $r \geq 3$. Let $B$ be a minimum $r$-fold forcing set for $G$ and suppose that $B$ is not AMON. Create a chronological list of forces in $G^{(r)}$ and suppose that at step $\ell \geq 1$ we have $x \to R_u$ for some $u$, and $R_u$ is the only cluster forced at this step. If $R_u$ is not a One or a None cluster, then consider the set $B'$ obtained by replacing $R_u$ with a One cluster. Since $B'$ is a forcing set with fewer blue vertices than $B$, this contradicts that $B$ is minimum. Thus if a single cluster is forced at some step of the forcing process, then it is either a One or a None cluster.

Now, suppose that at step $\ell \geq 1$ we have $x \to W \subseteq (R_{u_1} \cup R_{u_2} \cup \cdots \cup R_{u_m})$ for some $m \geq 2$, where each $R_{u_j}$ contains at least one white vertex. Since $x$ is performing a force, it has at most $r$ white neighbors. Thus we can perform a partial consolidation on the blue vertices spread among the $R_{u_j}$ as follows: Convert $R_{u_1}, R_{u_2}, \ldots, R_{u_{m-2}}$ into All clusters, convert $R_{u_{m-1}}$ into a Most cluster, and leave the remaining blue vertices in $R_{u_m}$. If we let $\widetilde{B}$ be the set obtained by performing this particular partial consolidation on $B$, then $\widetilde{B}$ is also a minimum $r$-fold forcing set for $G$. Notice that after partial consolidation, minimality of $B$ implies that $R_{u_m}$ must be a One cluster. Therefore, after partial consolidation, $x$ will force exactly two clusters, simultaneously – a Most cluster and a One cluster.

By performing partial consolidation, each cluster will become an All, Most, One, or None cluster, and a forcing process exists with which at each step either a single One or None cluster will be forced, or a Most and a One cluster will be forced simultaneously. $\square$

The type of AMON minimum $r$-fold forcing set guaranteed by Theorem 3.3.11 is called an *optimal AMON $r$-fold forcing set* for $G$. We emphasize that optimal AMON forcing sets are minimum, so $Z_{[r]}(G)$ is the size of such a set, and there is a corresponding forcing process in which at most two clusters are forced simultaneously. Using such a set and the associated forcing process, $G^{(r)}$ will have a global AMON structure at each forcing step.

**Corollary 3.3.12.** *For every graph $G$ and $r \geq 2$, there exists an optimal AMON $r$-fold forcing set for $G$. If $B$ is any such set, then $\ell(B) \geq m(B)$.*

*Proof.* For each Most cluster in an optimal AMON $r$-fold forcing set there exists a corresponding One cluster that is forced simultaneously using the forcing process guaranteed by Theorem 3.3.11, so the number of Most clusters cannot exceed the number of One clusters. $\square$

To obtain our main results of this section, we require a way to convert an AMON $r$-fold forcing set for $G$ into a (three-color) skew zero forcing set for $G$, and vice-versa.

**Remark 3.3.13.** For $r \geq 2$, let $B$ be an optimal AMON $r$-fold forcing set for a graph $G$. Color $G^{(r)}$ with $B$ and let $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{L}})$, where $\widetilde{\mathcal{D}} = \{u : R_u$ is an All or Most cluster$\}$ and $\widetilde{\mathcal{L}} = \{u : R_u$ is a One cluster$\}$. It is easy to see that $\widetilde{\mathcal{B}}$ is a skew zero forcing set for $G$. Similarly, let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be a skew zero forcing set for $G$. Color $G^{(r)}$ according to the following rule: If $u \in \mathcal{D}$, then make $R_u$ an All cluster, and if $u \in \mathcal{L}$, then make $R_u$ a One cluster. The set $\widetilde{B}$ of blue vertices is an AMON $r$-fold forcing set for $G$ (with $m(\widetilde{B}) = 0$).

**Definition 3.3.14.** Regardless of whether we transform an $r$-fold forcing set into a three-color skew zero forcing set or a three-color skew zero forcing set into an $r$-fold forcing set, we call the process described in Remark 3.3.13 *conversion*.

When performing conversion, we will always specify which type of set is being converted.

**Proposition 3.3.15.** *Let $G$ be a graph on $n$ vertices. If $r \geq n$ and $B$ is an optimal AMON $r$-fold forcing set, then $a(B) + m(B) = Z^-(G)$.*

*Proof.* Assume the hypotheses. Converting $B$ into a skew zero forcing set $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{L}})$ yields $Z^-(G) \leq |\widetilde{\mathcal{D}}| = a(B) + m(B)$.

Now, let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be an optimal skew zero forcing set for $G$ and convert $\mathcal{B}$ into an AMON $r$-fold forcing set $\widetilde{B}$. Since $B$ is optimal, it is minimum, so $|B| \leq |\widetilde{B}|$. Thus

$$a(B) + m(B) + \frac{\ell(B) - m(B)}{r} = \frac{|B|}{r} \leq \frac{|\widetilde{B}|}{r} = |\mathcal{D}| + \frac{|\mathcal{L}|}{r} = Z^-(G) + \frac{\ell_\star^-(G)}{r} \, .$$

Since $0 \leq \ell(B) - m(B) \leq \ell(B)$ by Corollary 3.3.12, $\ell_\star^-(G) < n$ by Remark 3.3.9, and $n \leq r$ by assumption, applying the floor function through the above inequality yields $a(B) + m(B) \leq Z^-(G)$, provided that $\ell(B) < n$. This must be the case, because if

$\ell(B) = n$ then $B$ cannot be minimum ($r \geq n$ implies that the first cluster forced could be a None). $\qquad\square$

**Corollary 3.3.16.** *If $G$ is a graph on $n$ vertices, then*

$$\lim_{r \to \infty} \frac{Z_{[r]}(G)}{r} = Z^-(G).$$

**Proposition 3.3.17.** *For any $r \geq 2$ and any graph $G$, $\frac{Z_{[r]}(G)}{r} \geq Z^-(G)$.*

*Proof.* Let $B$ be an optimal AMON $r$-fold forcing set for $G$ and let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be obtained by converting $B$ into a skew zero forcing set. By Corollary 3.3.12, $\ell(B) \geq m(B)$, so

$$\frac{Z_{[r]}(G)}{r} = \frac{|B|}{r} = a(B) + m(B) + \frac{\ell(B) - m(B)}{r} \geq a(B) + m(B) = |\mathcal{D}| \geq Z^-(G). \quad \square$$

**Theorem 3.3.18.** *For any graph $G$,*

$$Z_f(G) = Z^-(G).$$

### 3.3.4 Leaf-stripping and skew zero forcing number

In this section, we prove results about graphs with leaves and show that skew zero forcing number is unchanged by removing leaves and their neighbors. A leaf-stripping algorithm is presented and used to characterize graphs $G$ that have $Z^-(G) = 0$. For convenience, we define $Z^-(\emptyset) = 0$.

**Lemma 3.3.19.** *Let $G$ be a graph with leaf $u \in V(G)$ and let $v \in V(G)$ be the neighbor of $u$. Let $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ be an optimal skew zero forcing set for $G$. i) If $u$ is either light or dark blue, then $v$ is white. ii) If $u$ is white, then $v$ is not dark blue.*

*Proof.* For the first claim, since $u \in \mathcal{B}$ and $v$ is the only neighbor of $u$, we can choose $u \to v$ as the first step in the forcing process. In this case $v$ must be white because $\mathcal{B}$ is optimal. For the second claim, if $v \in \mathcal{D}$, then the set $\widetilde{\mathcal{B}} = (\mathcal{D} \setminus \{v\}, \mathcal{L} \cup \{u\})$ has fewer dark blue vertices than $\mathcal{B}$ but is a skew zero forcing set for $G$, contradicting the optimality of $\mathcal{B}$. $\qquad\square$

**Theorem 3.3.20.** *If $G$ is a graph with leaf $u \in V(G)$ and $v \in V(G)$ is the neighbor of $u$, then $Z^-(G - \{u, v\}) = Z^-(G)$.*

*Proof.* Suppose that $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{L}})$ is an optimal skew zero forcing set for $\widetilde{G} = G - \{u, v\}$ and let $\mathcal{D} = \widetilde{\mathcal{D}}$, $\mathcal{L} = \widetilde{\mathcal{L}} \cup \{u\}$, and $\mathcal{B} = (\mathcal{D}, \mathcal{L})$. Carry out the forcing process on $G$ using $\mathcal{B}$ for the initial coloring, starting with $u \to v$. Since $v$ is then dark blue, it does not affect the ability of its neighbors to force. Thus the forcing process on $G$ can be continued until $\widetilde{G}$ is forced, since $\widetilde{\mathcal{B}} = \mathcal{B} \setminus \{u\}$ is a skew zero forcing set for $\widetilde{G}$. The final force can then be $v \to u$, which forces $G$, so $\mathcal{B}$ is a skew zero forcing set for $G$ with $Z^-(\widetilde{G})$ dark blue vertices. Thus $Z^-(G) \leq Z^-(\widetilde{G})$.

Now suppose that $\mathcal{B} = (\mathcal{D}, \mathcal{L})$ is an optimal skew zero forcing set for $G$; we consider three cases. As before, $\widetilde{G}$ will denote $G - \{u, v\}$.

First, if $u \in \mathcal{L}$, then $v$ is white by Lemma 3.3.19 and $u \to v$ can be taken as the first step of the forcing process. Without loss of generality, we can assume that $v \to u$ is the last step of the forcing process. By continuing the forcing process, we will color $\widetilde{G}$ completely dark blue, since $\mathcal{B}$ is a skew zero forcing set for $G$ and $v$ cannot force any vertex in $\widetilde{G}$; thus $\mathcal{B} \setminus \{u\}$ is a skew zero forcing set for $\widetilde{G}$ with $Z^-(G)$ dark blue vertices, so $Z^-(\widetilde{G}) \leq Z^-(G)$.

Next, suppose that $u \in \mathcal{D}$; again, by Lemma 3.3.19, $v$ is white and $u \to v$ can be chosen as the first step of the forcing process. If $v$ never subsequently forces any of its other neighbors, then $\mathcal{B}$ is not optimal, since $u$ could have been chosen as a light blue vertex instead of a dark blue vertex (and then $v \to u$ could be the final step in the new forcing process). Thus $v$ must eventually force one of its neighbors, say $w$. It must be the case that at that stage all neighbors of $v$ (except $w$) are colored dark blue, and since $v$ is itself dark blue it did not affect any of the forces that led to this state. Therefore, if we let $\widetilde{\mathcal{D}} = (\mathcal{D} \setminus \{u\}) \cup \{w\}$ and $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{D}}, \mathcal{L})$, we will have a set containing $Z^-(G)$ dark blue vertices that can color all of $\widetilde{G}$ dark blue. We see that $Z^-(\widetilde{G}) \leq Z^-(G)$.

Lastly, suppose that $u$ is white, so $v$ is not dark blue by Lemma 3.3.19. There is a point in time after which $v$ will be dark blue; all forces prior to this time (except possibly $v \to u$ in the case where $v$ is light blue) do not involve $v$ in any way, and all forces after this time (except possibly $v \to u$) can be performed regardless of the presence of $v$, as it is dark blue. Let $\widetilde{\mathcal{B}} = (\mathcal{D}, \widetilde{\mathcal{L}})$, where $\widetilde{\mathcal{L}} = \mathcal{L} \setminus \{v\}$ if $v \in \mathcal{L}$ and $\widetilde{\mathcal{L}} = \mathcal{L}$ otherwise. Then $\widetilde{\mathcal{B}}$ can completely force $\widetilde{G}$, so $Z^{-}(\widetilde{G}) \leq Z^{-}(G)$. $\qquad\square$

Motivated by this result, we present a *leaf-stripping algorithm* that can be used to reduce a graph $G$ to a smaller graph with the same skew zero forcing number. This algorithm is a modification of Algorithm 3.16 in [8].

---

**Algorithm 3.3.20:** Leaf-stripping algorithm

**Input**: Graph $G$

**Output**: Graph $\hat{G}$ with $\delta(\hat{G}) \neq 1$, or $\hat{G} = \emptyset$

$\hat{G} := G$

**while** $\hat{G}$ has a leaf $u$ with neighbor $v$ **do**

$\quad \mid \quad \hat{G} := \hat{G} - \{u, v\}$

**end**

**return** $\hat{G}$

---

**Theorem 3.3.22.** *Let $G$ be a graph and let $\hat{G}$ be the graph returned by Algorithm 3.3.20. Then*

   *i.* $Z^{-}(G) = Z^{-}(\hat{G})$*; and*

  *ii.* $Z^{-}(G) = 0$ *if and only if* $\hat{G} = \emptyset$.

*Proof.* The first claim follows by repeated application of Theorem 3.3.20. As a result, if $\hat{G} = \emptyset$, then $Z^{-}(G) = 0$, which proves one direction of the second claim. For the other direction, suppose that Algorithm 3.3.20 does not return the empty set. If $\delta(\hat{G}) = 0$,

then $1 \leq Z^-(\hat{G})$. If $\delta(\hat{G}) \geq 2$, then $1 \leq \delta(\hat{G}) - 1 \leq Z^-(\hat{G})$. In either case, $1 \leq Z^-(\hat{G}) = Z^-(G)$, which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We immediately see that if $G$ is a graph on an odd number of vertices, then $Z^-(G) > 0$. Additionally, if $G$ is a graph with $Z^-(G) = 0$, then $G$ has a unique perfect matching; if the leaf-stripping algorithm is applied to $G$, then each removed leaf and its neighbor contribute an edge to this perfect matching. Note that having a unique perfect matching is not sufficient to guarantee that $Z^-(G) = 0$, as the next example shows.

**Example 3.3.23.** Consider the graph $G$ shown in Figure 3.6. The thick edges in the figure show the unique perfect matching for $G$, but since $\delta(G) = 2$, we have $Z^-(G) \geq 2 - 1 = 1$. In fact, $Z^-(G) = 1$, and the forcing set $\mathcal{B}$ shown in Figure 3.6 is optimal.
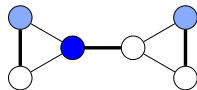


Figure 3.6: Graph $G$ with unique perfect matching and $Z^-(G) > 0$

**Remark 3.3.24.** If $G$ is a graph on $n$ vertices, then Algorithm 3.3.20 returns the graph $\hat{G}$ in at most $\lfloor \frac{n}{2} \rfloor$ leaf-stripping steps. Theorem 3.3.22 asserts that if $Z^-(\hat{G})$ is known, then the algorithm has computed $Z^-(G) = Z^-(\hat{G})$. In particular, if $G = T$ is a tree, then necessarily $\hat{G} = pK_1$ for some $p \geq 0$ and $Z^-(T) = p$.

A natural question is whether we can prove a version of Theorem 3.3.22 that applies to the fractional positive semidefinite forcing game. If Algorithm 3.3.20 returns the empty set when applied to a graph $G$, then by Remark 3.3.2 and Theorem 3.3.22 we have $0 \leq Z_f^+(G) \leq Z^-(G) = 0$, and so equality holds for one direction. The converse may fail, however: the graph $G$ in Examples 3.2.11 and 3.2.24 satisfies $Z_f^+(G) = 0$, but applying the algorithm to $G$ would return the nonempty partite set on 2 vertices. While we cannot generate a positive semidefinite analogue of Theorem 3.3.22, the result can still be a useful tool when Algorithm 3.3.20 returns the empty set.

# Acknowledgements

# Bibliography

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioaba, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness), Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.*, 428: 1628–1648, 2008.

[2] D. Amos, Y. Caro, R. Davila, and R. Pepper, Upper bounds on the $k$-forcing number of a graph, *Discrete Appl. Math.*, 181: 1–10, 2015.

[3] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, Zero forcing parameters and minimum rank problems, *Linear Algebra Appl.*, 433: 401–411, 2010.

[4] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, Parameters related to tree-width, zero forcing, and maximum nullity of a graph, *J. Graph Theory*, 72: 146–177, 2013.

[5] D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini, and M. Young, Zero forcing, linear and quantum controllability for systems evolving on networks, *IEEE Trans. Auto. Control*, 58: 2349–2354, 2013.

[6] D. Burgarth and V. Giovannetti, Full control by locally induced relaxation, *Phys. Rev. Lett.*, 99: 100501, 2007.

[7] S. Fallat and L. Hogben, Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs, in *Handbook of Linear Algebra*, 2nd ed., L. Hogben, ed., CRC Press, Boca Raton, FL, 2013.

[8] C. Grood, J. Harmse, L. Hogben, T. J. Hunter, B. Jacob, A. Klimas, and S. McCathern, Minimum rank with zero diagonal, *Electron. J. Linear Algebra*, 27: 458–477, 2014.

[9] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, S. Walker, M. Young, Propagation time for zero forcing on a graph, *Discrete Appl. Math.*, 160: 1994–2005, 2012.

[10] L. Hogben, K. Palmowski, D. Roberson, and S. Severini, Orthogonal Representations, Projective Rank, and Fractional Minimum Positive Semidefinite Rank: Connections and New Directions, arXiv:1502.00016 [math.CO], 2015.

[11] IMA-ISU research group on minimum rank (M. Allison, E. Bodine, L. M. DeAlba, J. Debnath, L. DeLoss, C. Garnett, J. Grout, L. Hogben, B. Im, H. Kim, R. Nair, O. Pryporova, K. Savage, B. Shader, A. Wangsness Wehe), Minimum rank of skew-symmetric matrices described by a graph, *Linear Algebra Appl.*, 432: 2457–2472, 2010.

[12] E. Scheinerman and D. Ullman, *Fractional Graph Theory*, Dover, Mineola, NY, 2011; also available online at http://www.ams.jhu.edu/~ers/fgt/.

[13] T. A. Peters, Positive semidefinite maximum nullity and zero forcing number, Ph.D. thesis, Iowa State University, 2012.

[14] N. Warnberg, Positive semidefinite propagation time, *Discrete Appl. Math.*, in press, doi:10.1016/j.dam.2015.04.008.

# CHAPTER 4.   CONCLUSIONS

## 4.1   General conclusions

In Chapter 2, $r$-fold orthogonal representations and $r$-fold orthogonal rank were introduced and used to formally define projective rank as "fractional orthogonal rank." The techniques used were applied to faithful orthogonal representations to generalize minimum positive semidefinite rank to an $r$-fold version and develop fractional minimum positive semidefinite rank. A main result of that chapter showed that the fractional minimum positive semidefinite rank of any graph equals the projective rank of the complement of the graph.

In Chapter 3, a fractionalization process was applied to the positive semidefinite zero forcing game and the resulting forcing game was analyzed. A three-color fractional positive semidefinite forcing game was introduced and used to prove various results about the fractional positive semidefinite forcing number. The three-color approach was applied to the existing zero forcing game and gave new insight to skew zero forcing number. In particular, graphs whose skew zero forcing number equals zero were completely characterized.

## 4.2   Recommendations for future research

There are numerous open questions related to projective rank and, by association, fractional minimum positive semidefinite rank. It was shown in Example 2.3.5 that fractional minimum positive semidefinite rank is truly an infimum rather than a min-

imum, i.e., there exist graphs $G$ for which there is no $r$-fold representation such that $\mathrm{mr}_f^+(G) = \frac{\mathrm{mr}_{[r]}^+(G)}{r}$. A natural question is whether this is also true for projective rank. Related to this is the question of rationality of projective rank: is there a graph whose projective rank is irrational? Note that if the infimum in the definition of projective rank is actually minimum, then the answer to this question is obviously "no."

The disjunctive product of two graphs $G$ and $H$, denoted here by $G * H$, is a graph with $V(G * H) = V(G) \times V(H)$ and $(g_1, h_1)(g_2, h_2) \in E(G * H)$ if and only if $g_1 g_2 \in E(G)$ or $h_1 h_2 \in E(H)$. Let $G^{*r}$ denote $r$-many copies of a graph $G$ joined by disjunctive products and consider the *asymptotic orthogonal rank* of $G$, defined by

$$\xi_\infty(G) = \inf_{r \in \mathbb{N}} \sqrt[r]{\xi(G^{*r})}.$$

An open question is whether $\xi_f(G) = \xi_\infty(G)$ for every graph $G$; an analogous equality holds, for example, for fractional chromatic number [2]. It is shown in [1] that $\xi_f(G * H) = \xi_f(G)\xi_f(H)$ for any graphs $G$ and $H$, so it is easy to verify that $\xi_f(G) \le \xi_\infty(G)$ for any graph $G$, but it is unknown whether the reverse inequality also holds. The definitions of projective rank in terms of $r$-fold orthogonal rank seen in Chapter 2 may be useful for approaching this problem; alternately, an approach using fractional minimum positive semidefinite rank may be possible.

The characterization of $r$-fold minimum positive semidefinite rank in terms of positive semidefinite matrices that $r$-fit a graph was motivated by connections to $(d; r)$ faithful orthogonal subspace representations. The standard minimum rank problem is not connected to faithful orthogonal representations, but it is described via a matrix rank minimization. Is there a way to meaningfully define $r$-fold (standard) minimum rank via a matrix characterization similar to that used for $r$-fold minimum positive semidefinite rank?

Regarding the results in Chapter 3, an open question is whether every graph has a minimum $r$-fold (standard) forcing set that induces a global AON structure (i.e., whether the Most clusters used to prove the results of Section 3.3.3 are required beyond their

theoretical use in that section). Finding applications of fractional positive semidefinite forcing number and the associated forcing game, as well as further applications of skew zero forcing – perhaps motivated by the new three-color interpretation – is also of interest.

# Bibliography

[1] T. Cubitt, L. Mančinska, D. E. Roberson, S. Severini, D. Stahlke, and A. Winter, Bounds on entanglement-assisted source-channel coding via the Lovász theta number and its variants, *IEEE Trans. Inform. Theory*, 60: 7330–7344, 2014.

[2] E. Scheinerman and D. Ullman, *Fractional Graph Theory*, Dover, Mineola, NY, 2011; also available online from http://www.ams.jhu.edu/~ers/fgt/.