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Moving Pearl Vortices in Thin-Film Superconductors

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Abstract: The magnetic field h_z of a moving Pearl vortex in a superconducting thin-film in (x, y) plane is studied with the help of the time-dependent London equation. It is found that for a vortex at the origin moving in $+x$ direction, $h_z(x, y)$ is suppressed in front of the vortex, $x > 0$, and enhanced behind ($x < 0$). The distribution asymmetry is proportional to the velocity and to the conductivity of normal quasiparticles. The vortex self-energy and the interaction of two moving vortices are evaluated.

Keywords: thin films; Pearl vortex

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1. Introduction

The time-dependent Ginzburg–Landau equations (GL) are the major tool in modeling vortex motion. Although this approach is applicable only for gapless systems near the critical temperature [1], it is gauge invariant and reproduces correctly major features of the vortex motion.

A simpler linear London approach has been employed through the years to describe static or nearly static vortex systems. The London equations express the basic Meissner effect and can be used at any temperature for problems where vortex cores are irrelevant. Moving vortices are commonly considered the same as static which are displaced as a whole.

However, recently it has been shown that this is not the case for moving vortex-like topological defects in, e.g., neutral superfluids or liquid crystals [2]. This is not so in superconductors within the time-dependent London theory (TDL) which takes into account normal currents, a necessary consequence of moving magnetic structure of a vortex [3,4]. In this paper, the magnetic field distribution of moving Pearl vortices in thin films is considered. It is shown that the self-energy of a moving vortex decreases with increasing velocity. The interaction energy of two vortices moving with the same velocity becomes anisotropic; it is enhanced when the vector $\boldsymbol{R}$ connecting vortices is parallel to the velocity $\boldsymbol{v}$ and suppressed if $\boldsymbol{R} \perp \boldsymbol{v}$. The magnetic flux carried by moving vortex is equal to flux quantum, but this flux is redistributed so that the part of it in front of the vortex is depleted, whereas the part behind it is enhanced.

In time-dependent situations, the current consists, in general, of normal and superconducting parts:

$$\boldsymbol{J} = \sigma \boldsymbol{E} - \frac{2e^2|\Psi|^2}{mc}\left(\boldsymbol{A} + \frac{\phi_0}{2\pi}\boldsymbol{\nabla}\chi\right), \tag{1}$$

where $\boldsymbol{E}$ is the electric field and Ψ is the order parameter.

The conductivity σ approaches the normal state value σ_n when the temperature T approaches T_c; in s-wave superconductors it vanishes fast with decreasing temperature along with the density of normal excitations. This is not the case for strong pair breaking when superconductivity becomes gapless, and the density of states approaches the normal state value at all temperatures. Unfortunately, there is not much experimental information about the T dependence of σ. Theoretically, this question is still debated; e.g., [5] discusses the possible enhancement of σ due to inelastic scattering.

Within the London approach $|\Psi|$ is a constant Ψ_0, and Equation (1) becomes:

$$\frac{4\pi}{c}\boldsymbol{J} = \frac{4\pi\sigma}{c}\boldsymbol{E} - \frac{1}{\lambda^2}\left(\boldsymbol{A} + \frac{\phi_0}{2\pi}\boldsymbol{\nabla}\chi\right), \tag{2}$$

where $\lambda^2 = mc^2/8\pi e^2|\Psi_0|^2$ is the London penetration depth. By acting on this via curling, one obtains:

$$-\nabla^2\boldsymbol{h} + \frac{1}{\lambda^2}\boldsymbol{h} + \frac{4\pi\sigma}{c^2}\frac{\partial\boldsymbol{h}}{\partial t} = \frac{\phi_0}{\lambda^2}\boldsymbol{z}\sum_\nu \delta(\boldsymbol{r} - \boldsymbol{r}_\nu), \tag{3}$$

where $\boldsymbol{r}_\nu(t)$ is the position of the ν-th vortex; $\boldsymbol{z}$ is the direction of vortices. Equation (3) can be considered as a general form of the time-dependent London equation.

The time-dependent version of London equation (3) is valid only outside vortex cores, similarly to the static London approach. As such, it may give useful results for materials with large GL parameter κ values in fields away from the upper critical field H_{c2}. On the other hand, Equation (3) is a useful, albeit approximate tool for low temperatures where GL theory does not work and the microscopic theory is forbiddingly complex.

2. Thin Films

Let the film of thickness d be in the xy plane. Integration of Equation (3) over the film thickness gives, for the z component of the field, a Pearl vortex moving with velocity $\boldsymbol{v}$:

$$\frac{2\pi\Lambda}{c}\text{curl}_z\boldsymbol{g} + h_z + \tau\frac{\partial h_z}{\partial t} = \phi_0\delta(\boldsymbol{r} - \boldsymbol{v}t). \tag{4}$$

Here, ϕ_0 is the flux quantum; $\boldsymbol{g}$ is the sheet current density related to the tangential field components at the upper film face by $2\pi\boldsymbol{g}/c = \hat{\boldsymbol{z}} \times \boldsymbol{h}$; $\Lambda = 2\lambda^2/d$ is the Pearl length; and $\tau = 4\pi\sigma\lambda^2/c^2$. With the help of $\text{div}\boldsymbol{h} = 0$, this equation is transformed to:

$$h_z - \Lambda\frac{\partial h_z}{\partial z} + \tau\frac{\partial h_z}{\partial t} = \phi_0\delta(\boldsymbol{r} - \boldsymbol{v}t). \tag{5}$$

As was shown by Pearl [6], a large contribution to the energy of a vortex in a thin film comes from stray fields. In fact, the problem of a vortex in a thin film is reduced to that of the field distribution in free space subject to the boundary condition supplied by solutions of Equation (4) at the film's surface. Outside the film $\text{curl}\boldsymbol{h} = \text{div}\boldsymbol{h} = 0$, one can introduce a scalar potential for the *outside* field:

$$\boldsymbol{h} = \boldsymbol{\nabla}\varphi, \qquad \nabla^2\varphi = 0\,. \tag{6}$$

The general form of the potential satisfying Laplace equation that vanishes at $z \to \infty$ of the empty upper half-space is

$$\varphi(\boldsymbol{r}, z) = \int \frac{d^2\boldsymbol{k}}{4\pi^2}\varphi(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r} - kz}\,. \tag{7}$$

Here, $\boldsymbol{k} = (k_x, k_y)$, $\boldsymbol{r} = (x, y)$, and $\varphi(\boldsymbol{k})$ is the two-dimensional Fourier transform of $\varphi(\boldsymbol{r}, z = 0)$. In the lower half-space, one has to replace $z \to -z$ in Equation (7).

As is done in [3], one applies the 2D Fourier transform to Equation (5) to obtain a linear differential equation for $h_{z\boldsymbol{k}}(t)$. Since $h_{z\boldsymbol{k}} = -k\varphi_{\boldsymbol{k}}$, we obtain:

$$\varphi_{\boldsymbol{k}} = -\frac{\phi_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}{k(1 + \Lambda k - i\boldsymbol{k}\cdot\boldsymbol{v}\tau)}\,. \tag{8}$$

In fact, this gives distributions for all field components outside the film, its surface included. In particular, h_z at $z = +0$ (the upper film face) is given by

$$h_{z\boldsymbol{k}} = -k\varphi_{\boldsymbol{k}} = \frac{\phi_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}{1+\Lambda k - i\boldsymbol{k}\cdot\boldsymbol{v}\tau}\,. \tag{9}$$

We are interested in the vortex's motion with constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$, so that we can evaluate this field in real space for the vortex at the origin at $t = 0$:

$$h_z(\boldsymbol{r}) = \frac{\phi_0}{4\pi^2}\int \frac{d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{1+\Lambda k - ik_x v\tau}\,. \tag{10}$$

It is convenient in the following to use Pearl Λ as the unit length and measure the field in units $\phi_0/4\pi^2\Lambda^2$:

$$h_z(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{1+k - ik_x s}\,, \quad s = \frac{v\tau}{\Lambda}. \tag{11}$$

(we left the same notations for $\boldsymbol{h}_z$ and $\boldsymbol{k}$ in new units; when needed, we indicate formulas written in common units).

2.1. Evaluation of $\boldsymbol{h_z(r)}$

With the help of identity

$$(1+k-ik_x s)^{-1} = \int_0^\infty e^{-u(1+k-ik_x s)}du\,, \tag{12}$$

one rewrites the field as

$$h_z(\boldsymbol{r}) = \int_0^\infty du\, e^{-u}\int d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}-uk},$$
$$\boldsymbol{\rho} = (x+us, y). \tag{13}$$

To evaluate the last integral over $\boldsymbol{k}$, we note that the three-dimensional (3D) Coulomb Green's function is

$$\frac{1}{4\pi R} = \frac{1}{(2\pi)^3}\int \frac{d^3\boldsymbol{q}}{q^2}\, e^{i\boldsymbol{q}\cdot\boldsymbol{R}} = \frac{1}{8\pi^2}\int \frac{d^2\boldsymbol{k}}{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}-kz}. \tag{14}$$

To do here the last step, we used $\boldsymbol{R} = (\boldsymbol{r}, z)$, $\boldsymbol{q} = (\boldsymbol{k}, q_z)$ and

$$\int_{-\infty}^{\infty} dq_z \frac{e^{iq_z z}}{k^2+q_z^2} = \frac{\pi\, e^{-k|z|}}{k}\,. \tag{15}$$

It follows from Equation (14)

$$\int d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}-kz} = -2\pi\frac{\partial}{\partial z}\frac{1}{\sqrt{r^2+z^2}} = \frac{2\pi z}{(r^2+z^2)^{3/2}}. \tag{16}$$

Replace now $\boldsymbol{r} \to \boldsymbol{\rho}$, $z \to u$, $R \to \sqrt{\rho^2+u^2}$ to obtain instead of Equation (13):

$$h_z(\boldsymbol{r}) = 2\pi\int_0^\infty du \frac{u\, e^{-u}}{(\rho^2+u^2)^{3/2}}\,. \tag{17}$$

After integrating by parts, one obtains:

$$h_z = 2\pi\left[\frac{1}{r} - \int_0^\infty \frac{du\, e^{-u}}{\sqrt{\rho^2+u^2}}\left(1+\frac{s(x+su)}{\rho^2+u^2}\right)\right]. \tag{18}$$

For the Pearl vortex at rest $s = 0$, $\rho = r$, and the known result follows; see, e.g., [7]:

$$h_z(\boldsymbol{r}) = 2\pi\left\{\frac{1}{r} + \frac{\pi}{2}[Y_0(r) - \boldsymbol{H}_0(r)]\right\}, \tag{19}$$

Y_0 and $\boldsymbol{H}_0$ are second-kind Bessel and Struve functions.

Hence, we succeeded in reducing the double integral (11) to a single integral over u. Besides, the singularity at $\boldsymbol{r} = 0$ is now explicitly represented by $1/r$, whereas the integral over u is convergent and can be evaluated numerically.

The results are shown in Figure 1.

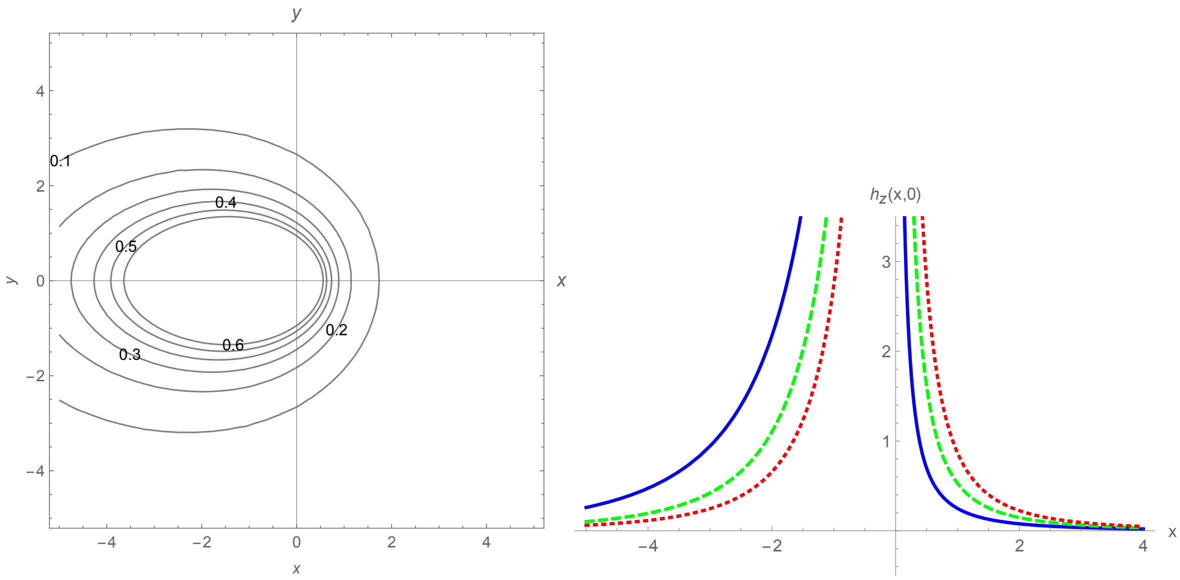

Figure 1. The left panel: contours of $h_z(x,y) = \text{const}$ (h_z is in units $\phi_0/4\pi^2\Lambda^2$ and x, y in units of Λ) for $s = 2$. **The right one**: $h_z(x,0)$ for $s = 0.5$ (dotted red), $s = 1$ (dashed green), and $s = 2$ (solid blue).

The field distribution is not symmetrical relative to the singularity position: the field in front of the moving vortex is suppressed relative to the symmetric distribution of the vortex at rest, whereas behind the vortex it is enhanced. This is an interesting consequence of our calculations: the magnetic flux of the moving vortex is redistributed so that it is depleted in front of the vortex and enhanced behind it.

We can characterize this redistribution by calculating the magnetic flux Φ_+ in front of the vortex:

$$\begin{aligned}\frac{\Phi_+}{\phi_0} &= \int_0^\infty dx \int_{-\infty}^\infty dy\, h_z(x,y) \\ &= \int_0^\infty dx \int_{-\infty}^\infty dy \int \frac{d^2\boldsymbol{k}}{4\pi^2} \frac{\phi_0 e^{i\boldsymbol{kr}}}{1 + k - ik_x s}.\end{aligned} \tag{20}$$

The integral over y gives $2\pi\delta(k_y)$, whereas when integrating over k_x we use

$$\int_0^\infty dx\, e^{ik_x x} = i\left(\frac{\mathcal{P}}{k_x} - i\pi\delta(k_x)\right), \tag{21}$$

where $\mathcal{P}$ indicates that the integral over k_x in Equation (20) should be understood as the principal value. Hence, we have

$$\frac{\Phi_+}{\phi_0} = \frac{i}{2\pi}\left(\mathcal{P}\int_{-\infty}^\infty \frac{dk_x}{k_x(1 + |k_x| - ik_x s)} - i\pi\right). \tag{22}$$

The integration now is straightforward and we obtain

$$\Phi_+ = \frac{\phi_0}{2} - \frac{\phi_0}{\pi} \arctan s. \tag{23}$$

Note that the total flux carried by vortex is given by Fourier component $h_z(\boldsymbol{k}=0) = \phi_0$; see Equation (9). I.e., $\phi_0/2$ is the flux through the half-plane $x > 0$ of the vortex at rest. The flux behind the moving vortex is therefore

$$\Phi_- = \frac{\phi_0}{2} + \frac{\phi_0}{\pi} \arctan s. \tag{24}$$

2.2. Potential and London Energy of the Moving Vortex

The potential φ introduced above is useful not only as an intermediate step in the evaluation of a magnetic field; it is directly related to the London energy (the sum of the magnetic energy outside the film and the kinetic energy of the currents inside) [8].

The potential

$$\varphi(\boldsymbol{r}) = -\frac{\phi_0}{4\pi^2\Lambda} \int \frac{d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k(1+k-ik_x s)}. \tag{25}$$

Employing again the identity (12), we have

$$\begin{aligned} \frac{4\pi^2\Lambda}{\phi_0}\varphi(\boldsymbol{r}) &= \int_0^\infty du\, e^{-u} \int \frac{d^2\boldsymbol{k}}{k} e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}-uk} \\ &= 2\pi \int_0^\infty \frac{du\, e^{-u}}{\sqrt{\rho^2+u^2}} \end{aligned} \tag{26}$$

with $\rho^2 = (x+us)^2 + y^2$.

2.2.1. Self-Energy of a Moving Vortex

This energy is given by

$$\epsilon_0 = -\frac{\phi_0}{4\pi}\varphi(\boldsymbol{r})|_{r\to 0} \tag{27}$$

whereas the integral (26) in this limit is logarithmically divergent. As is commonly done, we can approach the singularity at $\boldsymbol{r}=0$ from any side—e.g., setting $x=0$ and $y=\xi$, the core size is:

$$\begin{aligned} \epsilon_0 &= \frac{\phi_0^2}{8\pi^2\Lambda} \int_0^\infty \frac{du\, e^{-u}}{\sqrt{u^2s^2+\xi_c^2+u^2}} \\ &\approx \frac{\phi_0^2}{8\pi^2\Lambda\sqrt{1+s^2}} \ln \frac{\Lambda\sqrt{1+s^2}}{\xi^2} \end{aligned} \tag{28}$$

for the small dimensionless $\xi_c = \xi/\Lambda$. Compare this with the energy of a vortex at rest; see, e.g., [8]:

$$\epsilon_0 \approx \frac{\phi_0^2}{8\pi^2\Lambda} \ln \frac{\Lambda}{\xi}, \tag{29}$$

Hence, the vortex self-energy decreases with increasing velocity, a result qualitatively similar to that of moving vortices in the bulk [4].

2.2.2. Interaction of Moving Vortices

It has been shown in [8] that in infinite films the interaction is given by $\epsilon_{int} = (\phi_0/8\pi)[\varphi_1(2) + \varphi_2(1)]$; $\varphi_1(2)$ is the potential of the vortex at the origin at the position $\boldsymbol{r}$ of the second. Using Equation (26) we obtain

$$\frac{8\pi^2\Lambda}{\phi_0^2}\epsilon_{int} = \int_0^\infty du\, e^{-u}\Big(\frac{1}{\sqrt{(x+us)^2+y^2+u^2}} + \frac{1}{\sqrt{(-x+us)^2+y^2+u^2}}\Big). \quad (30)$$

Clearly, $\epsilon_{int}(x,y) = \epsilon_{int}(-x,y)$. This energy can be evaluated numerically and the result is shown in Figure 2 for $s = 2$.

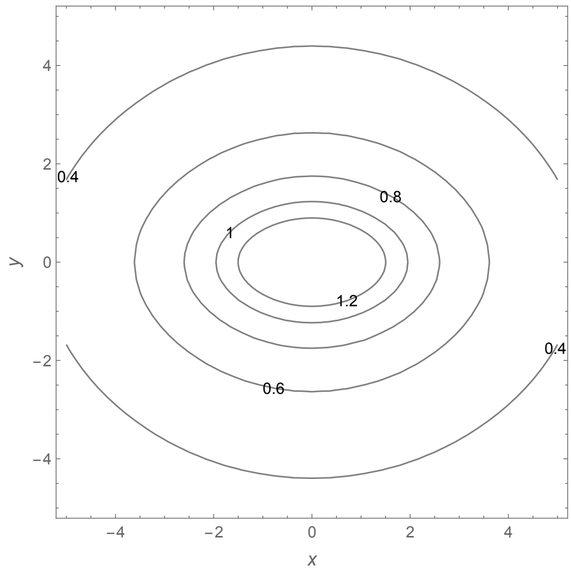

Figure 2. Contours of constant interaction energy $\epsilon_{int}(x,y)$ for $s = 2$.

It is worth noting that in thin films the interaction is not proportional to the field of one vortex at the position of the second. In our case the field of one vortex (see Figure 1) is not symmetric relative to $x \to -x$, whereas the interaction energy is.

2.3. Electric Field and Dissipation

Having the magnetic field (9) of a moving vortex, one gets for two vortices, one at the origin and the second at $\boldsymbol{R}$:

$$h_{z\boldsymbol{k}} = \frac{\phi_0(1+e^{-i\boldsymbol{kR}})e^{-ik_xvt}}{1+k\Lambda - ik_x\Lambda s} \quad (31)$$

(in common units). The moving, nonuniform vortex's magnetic field causes an electric field $\boldsymbol{E}$ out of the vortex core, which in turn causes the normal currents $\sigma\boldsymbol{E}$ and the dissipation $\sigma\boldsymbol{E}^2$. Usually this dissipation is small relative to Bardeen–Stephen core dissipation [9], but for fast vortex motion and high conductivity of normal excitations [5] it can become substantial [3].

The field $\boldsymbol{E}$ caused by known $\boldsymbol{h}(t)$ is given by Maxwell equations $i(\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k}})_z = -\partial_t h_{z\boldsymbol{k}}/c$ and $\boldsymbol{k}\cdot\boldsymbol{E}_{\boldsymbol{k}} = 0$:

$$E_{x\boldsymbol{k}} = -\frac{\phi_0 v}{c}\frac{k_xk_y(1+e^{-i\boldsymbol{kR}})}{k^2(1+k\Lambda - ik_x\Lambda s)}, \quad (32)$$

$$E_{y\boldsymbol{k}} = \frac{\phi_0 v}{c}\frac{k_x^2(1+e^{-i\boldsymbol{kR}})}{k^2(1+k\Lambda - ik_x\Lambda s)}. \quad (33)$$

For a constant velocity, one can consider the dissipation at $t = 0$. The dissipation power is:

$$W = \sigma d \int d\boldsymbol{r} E^2 = \sigma d \int \frac{d^2\boldsymbol{k}}{4\pi^2}\left(|E_{x\boldsymbol{k}}|^2 + |E_{y\boldsymbol{k}}|^2\right)$$
$$= \frac{\phi_0^2 \sigma d v^2}{\pi^2 c^2} \int d^2\boldsymbol{k} \frac{k_x^2 \cos^2(\boldsymbol{kR}/2)}{k^2[(1+k\Lambda)^2 + k_x^2\Lambda^2 s^2]}\,. \quad (34)$$

The integral here is divergent at large k, but the London theory breaks down in the vortex core of a size ξ, so one can introduce a factor $e^{-k^2\xi^2}$ to truncate this divergence. We then calculate the reduced quantity $w(x,y) = W(\pi c^2\Lambda^2/\phi_0^2\sigma d v^2)$ shown in Figure 3.

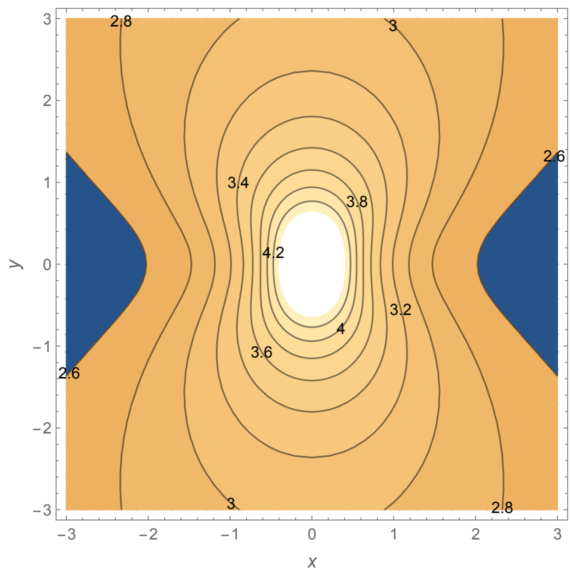

Figure 3. Contours of constant dissipation $w(x,y)$ in the system of two moving vortices for $s = 2$. At $t = 0$ one is situated at the origin and the second at (x,y).

We note that the dissipation $w(x,y)$ develops a shallow ditch along the x axis. Hence, for a fixed separation of vortices in the pair, the dissipation is minimal if they are aligned along the velocity.

3. Discussion

We have shown that in thin films the magnetic structure of the moving Pearl vortex is distorted relative to the vortex at rest. A similar formal procedure can be employed for moving Abrikosov vortices in the bulk, see Appendix A. The flux quantum of a moving vortex is redistributed and the back side of the flux is enhanced, whereas the front side is depleted. Physically, the distortion is caused by normal currents arising due to changing in time magnetic field at each point in space; the electric field is induced and causes normal currents. Naturally, it leads to the suppression of the flux where it is increasing (in front of the moving vortex) and to enhancement where it is decreasing (behind the vortex). We characterize this asymmetry by the difference of fluxes behind ($x < 0$) and in front ($x > 0$) of the moving vortex $\Delta\Phi = \Phi_- - \Phi_+ = (2\phi_0/\pi)\arctan s$. For a realistic situation, $s = v\tau/\Lambda \ll 1$, although the relaxation time $\tau \propto \sigma\lambda^2$ where σ is the poorly-known conductivity of above-the-gap normal excitations. Measuring $\Delta\Phi$ one can extract σ, an important physical characteristic of superconductors. There is an experimental technique which, in principle, could probe the field distribution in moving vortices [10]. This is highly sensitive SQUID-on-tip with the loop small on the scale of possible Pearl lengths.

Recent experiments have traced vortices moving in thin superconducting films with velocities well exceeding the speed of sound [10,11]. Vortices crossing thin-film bridges being pushed by transport currents have a tendency to form chains directed along the velocity. The spacing of vortices in a chain is usually exceeded by much of the core size, so the commonly accepted reason for the chain formation, namely, the depletion of the order parameter behind moving vortices, is questionable. However, at distances $r \gg \xi$ the time-dependent London theory is applicable. Another promising technique for studying moving vortices is Tonomura's Lorentz microscopy [12].

In this paper, we consider only properties of a single vortex and of interaction between two vortices moving with the same velocity, It would be interesting to consider how these results change if the quantization of the transverse electron motion is taken into account [13]. The problem of interaction in systems of many vortices is still to be considered.

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Appendix A. Abrikosov Vortex Moving in the Bulk

The field distribution of this case has been evaluated numerically in [3]. Here, we provide this distribution in closed analytic form.

The magnetic field $\boldsymbol{h}$ has one component h_z, so we can omit the subscript z. By choosing λ as a unit length and measuring the field in units of $\phi_0/4\pi\lambda^2$, we have:

$$h(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\boldsymbol{r}}}{1+k^2-ik_xs}, \qquad s = \frac{v\tau}{\lambda}. \tag{A1}$$

First, we use the identity

$$(1+k^2-ik_xs)^{-1} = \int_0^\infty e^{-u(1+k^2-ik_xs)}du\,, \tag{A2}$$

so that

$$\begin{aligned} h(\boldsymbol{r}) &= \int_0^\infty du\, e^{-u} \int d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}-u(k^2-ik_xs)} \\ &= \int_0^\infty du\, e^{-u} \int d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}-uk^2}, \;\; \boldsymbol{\rho} = (x+us, y). \end{aligned} \tag{A3}$$

Now, integrals over k_x, k_y are doable:

$$\int_{-\infty}^{\infty} dk_x\, e^{ik_x\rho_x-uk_x^2} \int_{-\infty}^{\infty} dk_y\, e^{ik_yy-uk_y^2} = \frac{\pi}{u}e^{-\rho^2/4u} \tag{A4}$$

where $\rho^2 = (x+us)^2+y^2$. Hence, we have

$$\begin{aligned} h(\boldsymbol{r}) &= \pi \int_0^\infty \frac{du}{u}\, e^{-u-\rho^2/4u} \\ &= \pi \int_0^\infty \frac{du\, e^{-u}}{u}\, \exp\left[-\frac{(x+us)^2+y^2}{4u}\right] \\ &= \frac{\phi_0}{2\pi\lambda^2} e^{-sx/2\lambda} K_0\left(\frac{r}{2\lambda}\sqrt{4+s^2}\right). \end{aligned} \tag{A5}$$

The last line is written in common units. Note that for the vortex at rest, $s = 0$, and we get the standard result $h = (\phi_0/2\pi\lambda^2)K_0(r/\lambda)$ [14].

This section is not mandatory, but may be added if there are patents resulting from the work reported in this manuscript.

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