A polychromatic approach to a Turán-type problem in finite groups

by

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DEDICATION

I would like to dedicate this thesis to and thank the creators of the online news and social networking service Twitter. Even though they have nothing to do with the creation or writing of this work I don't think they're thanked or acknowledged enough.

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ABSTRACT

Let G be a finite abelian group. Given $S \subseteq G$, $a \in G$, any set of the form $a + S = \{a\} + S$ is called a *translate* of S. A coloring of the elements of G is $S-$*polychromatic* if every translate of S contains an element of each color. The largest number of colors allowing an $S-$polychromatic coloring of the translates of S is known as the *polychromatic number* of S, denoted $p_G(S)$. Determining the polychromatic number of finite abelian groups is a relatively new and unexplored method that can be used to solve the following problem: What is the maximum number of elements in a subset of G which does not contain a translate of S? This type of problem called a Turán-type problem is common in extremal graph theory, but is new to the realm of algebra. This dissertation aims to determine bounds on the desired maximum number of elements, referred to as the *Turán number*, using polychromatic colorings on the desired maximum number of elements within the context of the well known abelian group the integers modulo n, denoted $\mathbb{Z}_n$ for all $n \geq 3$. The problem is also redefined and explored within the context of nonabelian groups such as the dihedral group and the dicyclic group.

Trivial bounds are first presented on the Turán number for any group. Results on the improvement of the trivial lower bound are then presented. The results involve determining the polychromatic number for various subsets. The polychromatic number of any subset of cardinality two of any group is determined. Results related to the polychromatic number of any subset of odd prime cardinality of $\mathbb{Z}_n$ are presented. Results related to the polychromatic number of subsets of cardinality three and n of D_{2n} and subsets of cardinality three of Dic_n are also presented.

CHAPTER 1. INTRODUCTION

1.1 Review of Literature

1.1.1 Basic Graph Theory Terms

A *graph* is a pair $G = (V, E)$ of disjoint sets where the elements of E are $2-$element subsets of the set V. The elements of V are the *vertices* (or *nodes*, or *points*) of the graph G, the elements of E are its *edges* (or *lines*). Two vertices x, y (i.e. $x, y \in V$) of G are said to be *adjacent* or *neighbors* if $e = \{x, y\}$ is an edge of G (i.e. $\{x, y\} \in E$). Two edges $e \neq f$ are *adjacent* if they have a vertex in common. A vertex x is *incident* with an edge if $x \in e$. The usual way to picture a graph is by drawing a dot for each vertex and a line between two vertices if there is an edge containing said vertices. An example of such a structure is a graph called a *cycle* on n vertices, denoted C_n which is formed by connecting the n vertices in a closed chain. The figure below is the graph C_4 [8].

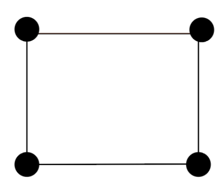

Figure 1.1 The graph of C_4

If, $V_0 \subseteq V$ and $E_0 \subseteq E$ such that $H = (V_0, E_0)$, H is a *subgraph* of a graph G, denoted $H \subseteq G$. In particular, $C_4 \subseteq Q_3$ where Q_3 is the $3-$dimensional hypercube. An *$n-$dimensional hypercube*, denoted Q_n is a graph whose vertices are $n-$tuples with entries in $\{0, 1\}$ and whose edges are pairs of $n-$tuples that differ in exactly one position. The figure below is the graph Q_3 [18].

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. G and G' are said to be *isomorphic* i.e. $G \cong G'$, if there is a bijection $\phi : V \to V'$ such that $\{xy\} \in E \iff \{\phi(x), \phi(y)\} \in E'$ for all

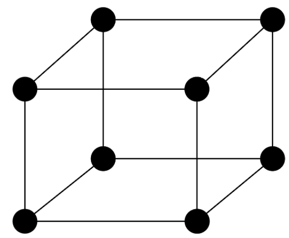

Figure 1.2 The graph of Q_3

$x, y \in V$. Such a bijection ϕ is called an *isomorphism* [8]. Note that C_4 is a subgraph of Q_3 as all of the faces of the cube shape formed by Q_3 are isomorphic to C_4. Said another way, Q_3 contains multiple "copies" of C_4.

Another notable type of graph is the *complete graph* on n vertices, denoted K_n, which is a graph such that all vertices are pairwise adjacent. The below figure is the graph K_2.

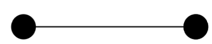

Figure 1.3 The graph of K_2

Yet another type of graph is a *hypergraph* which is a pair (V, E) of disjoint sets where the elements of E are non-empty subsets of any cardinality of V. Therefore, graphs are special types of hypergraphs where all the elements of E are subsets of V of cardinality 2 [8].

A problem common to graph theory involves assigning labels to the edges of a graph so that any adjacent edges are labeled distinctly. An *edge coloring* of a graph $G = (V, E)$ is a map $\chi : E \to S$ with $\chi(e) \neq \chi(f)$ for any adjacent edges e and f. The elements of the set S are called *colors* [8].

Another common problem that can be posed with respect to graphs is: what is the greatest number of edges that a graph on n vertices can possess without containing a copy of a certain graph H as a subgraph? A famous result of Pál Turán both answers this question when H is the complete graph on r vertices and is also viewed as the origin of a branch of mathematics known as extremal graph theory. The Turán graph, $T_{n,r}$, is formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge if and only if they belong to different subsets. The graph therefore has $n \bmod r$ subsets of size $\lceil \frac{n}{r} \rceil$ and $r - (n \bmod r)$ subsets of size $\lfloor \frac{n}{r} \rfloor$. The below figure is the graph $T_{8,3}$ [8].

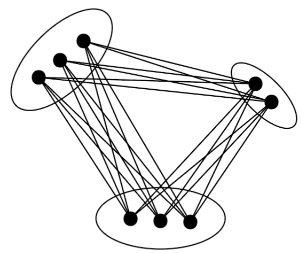

Figure 1.4 The graph of $T_{8,3}$

Turán's famous result is as follows.

THEOREM 1.1.1. *[8] For all integers r, n with $r > 1$, every n-vertex graph G such that $K_r \not\subseteq G$ and possesses the maximum number of edges is the* Turán graph $T_{n,r-1}$.

1.1.2 Basic Group Theory Terms

A *group* is an ordered pair $(G, \star)$ where G is a set and $\star$ is a function $\star : G \times G \to G$ called a *binary operation* on G satisfying the following axioms:

(i) (associativity) $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.

(ii) (existence of an identity element) There is an element $e \in G$ called an *identity* of G such that for all $a \in G$, $a \star e = e \star a = a$.

(iii) (existence of inverses) For each $a \in G$ there is an element $a^{-1} \in G$ called an *inverse* of a such that $a \star a^{-1} = a^{-1} \star a = e$. [10]

A group is called *abelian* if $a \star b = b \star a$ for all $a, b \in G$. Usually if a group is abelian the binary operation is denoted by "$+$". The integers under addition, $(\mathbb{Z}, +)$, is an example of an abelian group. However, $(\mathbb{Z}, +)$ is what is called an *infinite group* as there are infinitely many elements

in the set of integers. The groups in question that are of interest for most of this dissertation are *finite groups*, that is, groups whose underlying sets consist of only finitely many elements. An example of such a finite group which is also abelian that will be explored extensively in this dissertation is known as the *integers modulo* n denoted $\mathbb{Z}_n$ (where $n \geq 2$) under an operation known as *modular arithmetic*. Let n be a fixed positive integer. A relation can be defined on $\mathbb{Z}$ by $a \sim b$ if and only if $n|(b-a)$. This relation is an equivalence relation as $\sim$ is reflexive, symmetric, and transitive. If $a \sim b$, $a \equiv b \bmod n$ is written and read a is *congruent* to $b \bmod n$. For any $k \in \mathbb{Z}$, the equivalence class of a is denoted $\overline{a}$ and is called the *congruence class* or *residue class* of $a \bmod n$ and consists of the integers which differ from a by an integer multiple of n. That is, $\overline{a} = \{a + kn | k \in \mathbb{Z}\} = \{a, a \pm n, a \pm 2n, a \pm 3n\}$. These are the n distinct equivalence classes $\bmod n$ and thus the elements of $\mathbb{Z}_n$ are denoted $\overline{0}, \overline{1}, \ldots, \overline{n-1}$. The elements are determined by the possible remainders after division by n and these residue classes partition the integers $\mathbb{Z}$ [10].
The process of finding the equivalence class $\bmod n$ of some integer a is often referred to as *reducing* a mod n which refers to finding the smallest nonnegative integer congruent to a mod n. An addition and multiplication for the elements on $\mathbb{Z}_n$ can be defined by defining an operation called modular arithmetic. Suppose for $\overline{a}, \overline{b} \in \mathbb{Z}_n$, define their sum and product by

$$\overline{a} + \overline{b} = \overline{a+b} \text{ and } \overline{a} \cdot \overline{b} = \overline{ab}.$$

For example, consider $\mathbb{Z}_4$. The elements are $\overline{0}$, $\overline{1}$, $\overline{2}$, $\overline{3}$. Some examples which arise from modular arithmetic are:

$$\overline{2} + \overline{3} = \overline{5} \equiv \overline{1} \text{ and } \overline{2} \cdot \overline{3} = \overline{6} \equiv \overline{2}.$$

For convenience, when the integers modulo n are used, the bar notation will be dropped.

A group is called *nonabelian* if it is not true that $a \star b = b \star a$ for all $a, b \in G$ [10]. An example of such a group that will be explored extensively is the *dihedral group of order* $2n$, denoted D_{2n} with $n \geq 3$. This group is formed as follows. The elements of D_{2n} are given as

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}.$$

That is, each element can be written uniquely in terms of two generators in the form $s^k r^i$ for some $k \in \{0,1\}$ and $0 \leq i \leq n-1$. The group also requires the following relations to hold:

(i) $|r| = n$ and $|s| = 2$;

(ii) $s \neq r^i$ for any i;

(iii) $sr^i \neq sr^j$ for all $0 \leq i, j \leq n-1$ with $i \neq j$;

(iv) $sr = r^{-1}s$;

(v) $sr^i = r^{-i}s$ for all $0 \leq i \leq n$ [10].

It is also necessary to introduce topics related to substructures of groups. Let G be a group. A subset H of the group is a *subgroup* if H is nonempty, closed under the operation and inverses i.e. if $x, y \in H$, then $x^{-1} \in H$ and $x \star y \in H$. In this case, $H \leq G$ is written. Suppose $G = \mathbb{Z}_4$, consider $H = \{0, 2\}$. The inverses of 0 and 2 are themselves respectively, while $0+2 = 2 \in H$. Thus, $H \leq G$. A group is *cyclic* if it can be generated by a single element. That is, there is some element $x \in H$ so that $H = \{x^n | n \in \mathbb{Z}\}$. For example, the element 1 generates $\mathbb{Z}_n$ for all values of $n \geq 2$. Note that it is possible to form *cyclic subgroups* of any group G with any element x. That is, for any $x \in G$ where G is a group the *cyclic subgroup generated by element* x is given by $\langle x \rangle = \{x^n | n \in \mathbb{Z}\}$. For any $N \leq G$ and any $g \in G$, let $gN = \{gn | n \in N\}$ and $Ng = \{ng | n \in N\}$ called respectively a *left coset* and a *right coset* of N in G. Any element of a coset is called a *representative* for that coset. The left cosets and right cosets of $H = \{0, 2\} \leq G = \mathbb{Z}_4$ are equivalent and are $0+H = H = H+0$, $1 + H = \{1, 3\} = H + 1$, $2 + H = H = H + 2$, $3 + H = \{1, 3\} = H + 3$ [10]. Similarly, some important terms are those which show how different groups are related to each other. Let $(G, \star)$ and $(H, \diamond)$ be groups. A map $\varphi : G \to H$ such that

$$\varphi(x \star y) = \varphi(x) \diamond \varphi(y), \qquad \text{for all } x, y \in G$$

is called a *homomorphism*. The map φ is called an *isomorphism* and G and H are said to be *isomorphic*, written $G \cong H$, if

(i) φ is a homomorphism and

(ii) φ is a bijection.

For example, consider the mapping φ which sends an element in the subgroup generated by any element x in any group G to the integers modulo $|\langle x\rangle|$:

$$\varphi : \langle x\rangle \to \mathbb{Z}_{|\langle x\rangle|}; x^j \mapsto j \text{ for any nonnegative integer } j.$$

Clearly, the group is injective and surjective. It is also a homomorphism as

$$\varphi(x^j \star x^i) = \varphi(x^{j+i}) = j + i = \varphi(x^j) + \varphi(x^i).$$

1.1.3 Background of the Problem

As was mentioned in Subsection 1.1.1, Turán's theorem resolved the following problem: What is the maximum number of edges that a simple undirected graph on n vertices can contain so that the graph does not possess a complete subgraph on r vertices? This is the first instance of question called a Turán-type problem being employed. Such problems have since become commonplace in extremal graph theory as well as extremal combinatorics and generally follow the outline: maximize a specified value while averting a given illegal situation. Being as that they are such popular problems in discrete mathematics, Turán-type problems have been very well examined when it comes to graphs. However, with respect to mathematical structures such as groups, there are only a few known results. In this section a Turán-type problem in the hypercube Q_n is introduced as well a coloring technique known as polychromatic coloring and other related results and information. The coloring in question will be used extensively in this dissertation. It is important to note that polychromatic colorings can provide bounds for Turán-type problems and thus this is a motivation for studying such colorings. In addition to this, various coloring results on groups are also given in order to demonstrate versatility and usefulness of coloring problems on groups.

For graphs G and H, let $ex(G, H)$, the *Turán number*, denote the maximum number of edges in a subgraph of G which does not contain a copy of H. In [3], $ex(G, H)$ is studied when the base graph G is the $n-$dimensional hypercube Q_n. That is, the analogue of Turán's theorem in the hypercube

Q_n is the motivation. This setting was introduced by Erdős who asked how many edges can a C_4-free subgraph of the hypercube contain [11]. However, in [3] a generalization of the C_4-free subgraph problem is taken in a different direction. That is, for arbitrary d, bounds on $ex(Q_n, Q_d)$ are given.

Firstly, the trivial bounds on $ex(Q_n, Q_d)$ are explained. For convenience, the complementary problem is introduced. That is, let $f(n, d)$ denote the minimum number of edges one must delete from the $n-$cube to make it $d-$cube-free. Note that $f(n, d) = e(Q_n) - ex(Q_n, Q_d)$. A simple averaging argument yields that for any fixed d the function $\frac{f(n,d)}{e(Q_n)}$ is non-decreasing in n, so a limit c_d exists. Next, trivially, $f(d, d) = 1$, and so $c_d \geq \frac{1}{d2^{d-1}}$. Otherwise, if one deletes edges of the hypercube on every d^{th} level, one obtains a Q_d-free subgraph. This can be seen as every $d-$dimensional subcube must span $d + 1$ levels. Therefore, $c_d \leq \frac{1}{d}$. The main result in [3] presented in regards to c_d is

THEOREM 1.1.2. *[3]*

$$\Omega\left(\frac{\log d}{d2^d}\right) = c_d \leq \begin{cases} \frac{4}{(d+1)^2} & \textit{if } d \textit{ is odd} \\ \frac{4}{d(d+2)} & \textit{if } d \textit{ is even.} \end{cases}$$

All of the arguments in [3] are presented in Ramsey-type framework. A coloring of the edges of Q_n is called $d-$polychromatic if every subcube of dimension d has all the colors represented on its edges. Let $pc(n, d)$ be the largest integer p such that there exists a $d-$polychromatic coloring of the edges of Q_n in p colors. As with c_d, $pc(n, d) \leq d2^{d-1}$ and $f(n, d) \leq \frac{e(Q_n)}{pc(n,d)}$. Since $pc(n, d)$ is a non-increasing function in n, it stabilizes for large n. Let p_d be this limit, then $c_d \leq \frac{1}{p_d}$ is obtained. The main result in [3] presented in regards to p_d is

THEOREM 1.1.3. *[3]*

$$\binom{d+1}{2} \geq p_d \geq \begin{cases} \frac{(d+1)^2}{4} & \textit{if } d \textit{ is odd} \\ \frac{d(d+2)}{4} & \textit{if } d \textit{ is even.} \end{cases}$$

Note that the lower bound in Theorem 1.1.3 implies the upper bound in Theorem 1.1.2. In [16], Theorem 1.1.3 is improved upon by showing the proposed lower bound is actually the value of p_d.

THEOREM 1.1.1. *[16]*

$$p_d = \begin{cases} \frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d+2)}{4} & \text{if } d \text{ is even.} \end{cases}$$

Continuing with such work, a new asymptotic bound is given for p_d when the hypercube being examined is Q_3 in [17].

PROPOSITION 1.1.1. *[17] Let G be the graph obtained by removing two parallel edges of Q_3 that are not incident to any common edges. Then $p_d \geq 3$.*

A related problem considers coloring multiple subgraphs within a given graph G. In [5], the concept of coloring more than one subgraph of a given graph G with respect to polychromatic colorings is introduced. If G is a graph and $\mathcal{H}$ is a set of subgraphs of G, an edge-coloring of G is *$\mathcal{H}-$polychromatic* if every graph from $\mathcal{H}$ receives all colors on its edges. The *$\mathcal{H}-$polychromatic number* of G, denoted $poly_{\mathcal{H}}(G)$, is the largest number of colors allowing an $\mathcal{H}-$polychromatic coloring to exist. If such an $H-$polychromatic coloring of G uses $poly_{\mathcal{H}}(G)$ colors, this coloring is called an *optimal $H-$polychromatic coloring* of G. With these definitions, the work done in [3] yields that with $G = Q_n$, $\mathcal{H}$ is the family of subgraphs of G isomorphic to Q_d, and if d is fixed and n is large, $\lfloor \frac{(d+1)^2}{4} \rfloor \leq poly_{\mathcal{H}}(Q_n) \leq \binom{d+1}{2}$. The results of interest in [5] concern G being a complete graph and $\mathcal{H}$ being a family of spanning subgraphs. To this end, let $F_1 = F_1(n)$ be the family of all $1-$factors of K_n, $F_2 = F_2(n)$ be the family of all $2-$factors of K_n, and $HC = HC(n)$ be the family of all Hamiltonian cycles of K_n. The following results are then given.

THEOREM 1.1.4. *[5] If n is an even positive integer, then $poly_{F_1}(K_n) = \lfloor \log_2 n \rfloor$.*

THEOREM 1.1.5. *[5] There exists a constant c such that $\lfloor \log_2(n+1) \rfloor \leq poly_{F_2}(K_n) \leq poly_{HC}(K_n) \leq \lfloor \log_2 n \rfloor + c$. Moreover, $\lfloor \log_2 \frac{8(n-1)}{3} \rfloor \leq poly_{HC}(K_n)$.*

It is possible, however, to convert the question of finding the polychromatic number of a given graph to one of coloring a rectangular grid. In [12], the problem of determining the polychromatic number of a subgraph G of Q_n is transformed. The set of color classes are put into a rectangular grid with the ith row containing the color classes (a, b) with $a+b = i$ and the ith column containing classes of the form (i, j) as pictured in the following figure.

$(0,0)$						
$(0,1)$	$(1,0)$					
$(0,2)$	$(1,1)$	$(2,0)$				
$(0,3)$	$(1,2)$	$(2,1)$	$(3,0)$			
$(0,4)$	$(1,3)$	$(2,2)$	$(3,1)$	$(4,0)$		
$(0,5)$	$(1,4)$	$(2,3)$	$(3,2)$	$(4,1)$	$(5,0)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 1.5 Color classes in a rectangular grid

Firstly, special notation for hypercubes is employed. The n coordinates of a vertex of the hypercube are referred to as *bits*, and given an edge $\{x, y\}$, the unique bit where $x_i \neq y_i$ is called the *flip bit*. An edge of Q_n is represented by an $n-$bit vector with a star in the flip bit. Similarly, an embedding of Q_d in Q_n is given by an $n-$bit vector with stars in d coordinates [12].

Next, the desired grid is constructed. A *region* of the grid is all color classes contained in some consecutive rows and columns. A *shape* is a finite set of elements of the grid. Two shapes are *congruent* if one is a translation of the other. That is, if $S = \{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$ then $S' \cong S$" if and only if $S' = \{(a_1 + i, b_1 + j), (a_2 + i, b_2 + j), \ldots, (a_k + i, b_k + j)\}$ for some $i, j \in \mathbb{Z}$. A *shape list* is a finite list of shapes $S_1, \ldots, S_k$ such that if $i < j$, then S_i is not to the right of S_j. Two shape lists are *congruent* if each contains the same number of shapes, and corresponding shapes in the lists are congruent and are horizontal translations of each other. A *shape sequence* $\mathcal{S}$ is the set of all shape lists congruent to a specific list. An *instance* of a shape sequence $\mathcal{S}$ is one particular list. Define an $i \times j$ *parallelogram* to be a set of color classes of the following form: $\{(a + \alpha, b + \beta) : 0 \leq \alpha < j, 0 \leq \beta < i\}$ [12].

LEMMA 1.1.1. *[12] Consider a shape sequence $\mathcal{S}$ of shapes $S_1, \dots, S_k$, with elements in rows $i_1, \dots, i_\ell$. Let X^i_j be the number of elements in S_j in row i, and let $X^i = \max_j X^i_j$. Then*

$$p(\mathcal{S}) \leq \sum_{i=i_s}^{i_\ell} X^i.$$

Therefore, the question of determining polychromatic colorings of the hypercube so that every subgraph G contains every color is changed into a question of coloring the above grid of color classes so that every *shape sequence* corresponding to G contains every color.

FACT 1.1.1. *Let $n \geq d \geq 1$. Every shape sequence for an embedding of Q_d in Q_n consists of d shapes $S_1, \dots, S_d$ where S_i is a $(d-i+1) \times i$ parallelogram, and each shape occupies the same d rows. The color classes in S_i correspond to the edges using the ith star from the left. Conversely, every instance of such a shape sequence corresponds to some embedding of Q_d in Q_n.*

The information from [12] is of special interest as grid colorings inspired by the grid colorings of [12] are used in this dissertation to yield new results on finite groups.
Also, of interest are plane graphs and how one can apply polychromatic colorings to their faces. Let $F(G)$ denote the set of faces of G. The *size* of a face $f \in F(G)$ is the number of vertices on its boundary. For a plane graph G, which is a graph G together with an embedding of G into the plane, let $g(G)$ denote the size of the smallest face in G. For a vertex $k-$coloring, a face $f \in F(G)$ is *polychromatic* if all k colors appear on the vertices of f. A vertex $k-$coloring of G is *polychromatic* if every face of G is polychromatic. The *polychromatic number* of G, denoted $p(G)$, is the largest number of colors k such that there is a polychromatic vertex $k-$coloring of G. Define $p(g) = \min\{p(G) | G \text{ plane graph}, g(G) = g\}$. The main result of [1] bounds the minimum possible polychromatic number for plane graphs G with $g(G) = g$.

THEOREM 1.1.2. *[1] $p(1) = p(2) = 1$, $p(3) = p(4) = 2$, and for $g \geq 3$,*

$$\left\lfloor \frac{3g-5}{4} \right\rfloor \leq p(g) \leq \left\lfloor \frac{3g+1}{4} \right\rfloor.$$

Again, with the motivation for studying polychromatic numbers still being that they provide bounds for Turán-type problems, it is not just to graphs that polychromatic colorings can be applied.

In [6] polychromatic colorings are for the first time applied to abelian groups. Let G denote an arbitrary abelian group. Given $S, T \subseteq G$ and $n \in G$, define the sets $S + T = \{s + t : s \in S, t \in T\}$ and $n + S = \{n\} + S$. Any set of the form $n + S$ is called a *translate* of S. Given a subset S of G, a coloring of the elements of G is $S-$*polychromatic* if every translate of S contains an element of each color. The *polychromatic number* of S, denoted $p_G(S)$ or $p(S)$ when the choice of G is clear is defined to be the largest number of colors allowing an $S-$polychromatic coloring of the elements of G to exist. Several examples are provided.

EXAMPLE 1.1.1. *If $G = \mathbb{Z}$ and $|S| = 1$ or $|S| = 2$, then $p(S) = |S|$. If $|S| = 1$, then its easy to show that $p(S) = 1$ as simply assign every element one color. If $p(S) = 2$, $S = \{a, a + b\}$ where $a, b \in \mathbb{Z}$. Then, a translate of S is $\{0, b\}$. If b is odd, simply assign one color to all even integers and another color to all odd integers. If b is even, then any integer can be written as $mb + i$ for any $m \in \mathbb{Z}$ and $0 \leq i \leq b - 1$, so any translate can be written as $\{mb + i, (m + 1)b + i\}$ and thus the assignment is c_0 if m is even and c_1 if m is odd where c_0 and c_1 denote distinct colors.*

EXAMPLE 1.1.2. *If $G = \mathbb{Z}$ and $|S| = 3$, then $p(S) = 2$ or $p(S) = 3$. For instance, suppose $S = \{0, 1, 5\}$. Then, every translates of S consists of elements from three distinct modulus classes $\pmod 3$. Therefore, an $S-$polychromatic coloring with 3 colors can be constructed by assigning 3 distinct colors to each of the 3 congruence classes $\pmod 3$. That is, $p(\{0, 1, 5\}) = 3$.*
If $S = \{0, 1, 3\}$, however, then $p(\{0, 1, 3\}) = 2$. The proof that $p(\{0, 1, 3\}) < 3$ follows by way of contradiction. Suppose there is an $S-$polychromatic coloring of $\mathbb{Z}$ with 3 colors. Call it χ. Then, $\chi(0)$, $\chi(1)$, $\chi(3)$ are all distinct. For any $s \in \{0, 1, 3\}$ there is a translate that contains both s and 2, so $\chi(s) = \chi(2)$ and therefore any such coloring is impossible.

It is interesting to note that translates of the integers can be written as hypergraphs. In [2], a given subset S of the integers and the translates of S are thought of as a hypergraph. More explicitly, for a set of integers S, let $H = H(S)$ denote the infinite hypergraph whose set of vertices is the set of integers $\mathbb{Z}$ and whose set of edges is the set of all translates of S. That is, $V(H) = \mathbb{Z}$ and $E(H) = \{x + S : x \in \mathbb{Z}\}$. Such a hypergraph H is called a shift hypergraph of S. With these definitions, [2] explores polychromatic colorings on the integers, however, such colorings are referred to as "good" colorings. One of the main results shows that if $S \subset \mathbb{Z}$ of cardinality at least $4k^2$, then there exists an $S-$polychromatic coloring for the shift hypergraph $H(S)$. Also, for large subsets S, it can be shown that $p(S) \geq \frac{(1+O(1))|S|}{3 \ln |S|}$ [2].

The main result presented in [6] with respect to polychromatic colorings are subsets of $\mathbb{Z}$ of cardinality 4.

THEOREM 1.1.6. *[6] If $S \subseteq \mathbb{Z}$ and $|S| = 4$, then $p(S) \geq 3$.*

The proof of Theorem 1.1.6 is quite long. Of particular interest is a small result that reduces the problem of finding a polychromatic coloring of $\mathbb{Z}$ for a specified S of order 4 to finding a polychromatic coloring of $\mathbb{Z}_m = \{0, 1, \dots m-1\}$ for a particular value of m.

LEMMA 1.1.2. *[6] Let $a, b, c, k, q \in \mathbb{Z}$ with $0 < a < b < c$, $\gcd(a, b, c) = 1$, $k, q \geq 1$, and $m = c - a + b$. Let $S = \{0, ka, kb, kc\}$, $S_1 = \{0, a, b, c\}$, $S_2 = \{0, b-a, b, 2b-a\}$. Then*

(i) $p_{\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$.

(ii) $p_{\mathbb{Z}}(S_1) \geq p_{\mathbb{Z}_m}(S_1)$.

(iii) $p_{\mathbb{Z}_m}(S_1) = p_{\mathbb{Z}_m}(S_2)$.

(iv) If $\gcd(k, q) = 1$, *then* $p_{\mathbb{Z}_q}(S) = p_{\mathbb{Z}_q}(S_1)$.

The significance of Lemma 2.0.10 is not merely because of its usefulness in proving Theorem 1.1.6, but because it provides the first instance of information of polychromatic colorings on the integers modulo n - one of the large focuses of this dissertation.

In addition to the exploration of polychromatic colorings on the integers modulo n, [6] defines the relationship between the polychromatic number and a Turán-type problem in an abelian group. A subset $T \subseteq G$ is a *blocking set* for subset S if $G \setminus T$ contains no translate of S. That is, for all $n \in G$, $n + S \not\subseteq G \setminus T$. A Turán-type problem defined within the context of an abelian group G therefore asks: what is the smallest blocking set for a given set S? This is the main question that concerns this dissertation with respect to finite groups both abelian and nonabelian, however in [6] the question is relegated to the case where S is finite and $G = \mathbb{Z}$. In this case, any blocking set is countably infinite and so instead of the cardinality of a blocking set, the interest is in the density of a blocking set. To this end, for any set $T \subseteq \mathbb{Z}$, the upper density $\overline{d}(T)$ and lower density $\underline{d}(T)$ of T are defined as

$$\overline{d}(T) = \limsup_{n\to\infty} \frac{|T \cap [-n,n]|}{2n+1} \text{ and } \underline{d}(T) = \liminf_{n\to\infty} \frac{|T \cap [-n,n]|}{2n+1}.$$

If $\overline{d}(T) = \underline{d}(T)$, this common value is called the density of T and denoted $d(T)$. Next, the parameter $\alpha(S)$ measures how small the density of a blocking set for a subset S can be. That is,

$$\alpha(S) = \inf\{d(T) : T \text{ is a blocking set for } S \text{ and } d(T) \text{ exists}\}.$$

The following result gives the relationship between polychromatic colorings and blocking sets.

LEMMA 1.1.3. *[6] For any finite set* $S \subseteq \mathbb{Z}, \alpha(S) \leq \frac{1}{p(S)}$.

The proof of Lemma 1.1.3 relies on the fact that any color class in an $S-$polychromatic coloring of the integers forms a blocking set. It is important to note that [6] displays the relationship between polychromatic colorings and tiling an abelian group by translation which is relevant to original results on the integers modulo n within this dissertation. Firstly, given a set $S \subseteq G$, a subset $T \subseteq G$ is a *complement set* for S if $S + T = G$. If it is the case that this sum is unique for every group element, then S *tiles* G *by translation* i.e. if $s_1, s_2 \in S$, $t_1, t_2 \in T$, such that $s_1 + t_1 = s_2 + t_2$, then $s_1 = s_2$ and $t_1 = t_2$. If S tiles G by translation, this is denoted by $S \oplus T = G$. For example, if $S = \{0, 1, 5\}$, then S tiles $\mathbb{Z}$ with complement set $T = \{3n : n \in \mathbb{Z}\}$. However, if $S = \{0, 1, 3\}$, there is no such T. A relationship between complement sets and blocking sets can be established as well.

LEMMA 1.1.4. *[6] Let* G *be an abelian group and* $S \subseteq G$. *Then* $T \subseteq G$ *is a complement set for* S *if and only if* $-T$ *is a blocking set for* S.

It is essential to note that for any subset T of an abelian group G, let $-T$ denote the set $\{-t : t \in T\}$. A result that is particularly important on which a couple of the results presented in this dissertation rest upon is the following.

THEOREM 1.1.7. *[6] Let G be any abelian group. A finite set $S \subseteq G$ tiles G by translation if and only if $p(S) = |S|$. Moreover, if χ is an $S-$polychromatic coloring of G with $|S|$ colors and T is the set of elements of G colored by χ with any given color, then $S \oplus T = G$.*

The following corollary guarantees that given a finite set tiles an abelian group, it is not the only set that tiles the group.

Corollary 1.1.1. *[6] If a finite set S tiles an abelian group G by translation, then any polychromatic coloring of G with $|S|$ colors is also a $(-S)-$polychromatic coloring.*

Additionally, it suffices to show that a subset tiles a nontrivial subset of an abelian group in order to show it tiles the entire group.

LEMMA 1.1.5. *[6] If a set $S \subseteq G$ tiles a nontrivial subgroup H of G, then S tiles G.*

Given Theorem 1.1.7 in [6], there is clearly a connection between tiling by translation and determining the polychromatic number of a given abelian group. In [9], multiple terms and interesting results related to tiling by translation which is simply referred to as "tiling" are introduced. Suppose G is a finite abelian group. A *factorization* of G is a collection $(A_1, \ldots, A_k)$ of subsets such that every $g \in G$ can be uniquely represented as $a_1 + \ldots a_k$, where $a_i \in A_i$. A factorization is *normed* if every subset in the factorization contains 0. A *tiling* is a special case of a normed factorization in which there are only two subsets (denoted V and A). Any subset V for which there exists a subset A such that (V, A) is a tiling of G is called a *tile* of G. A group G posses the *Rédei property* if in every tiling (V, A) of G either V or A is contained in a proper subgroup of G. If G does not possess the Rédei property then there is some tiling (V, A) of G in which $\langle V \rangle = \langle A \rangle = G$, where $\langle S \rangle$ denotes the subgroup generated by S for any $S \subseteq G$ such tilings are said to be *full rank* [9].

Let $V - V$ denote the set $\{v_1 - v_2 : v_1, v_2 \in V\}$. The following results demonstrate the connection between subgroups and tiling by translation.

PROPOSITION 1.1.2. *[9] Let $V, A \subseteq G$ with $0 \in V$ and $0 \in A$. Then, (V, A) is a tiling of G if and only if $(V - V) \cap (A - A) = \{0\}$ and $|V||A| = |G|$.*

PROPOSITION 1.1.3. *[9] A subset $V \subseteq G$ is a tile of G if and only if it is a tile of $\langle V \rangle$.*

Tilings (V, A) with the property that $\langle V \rangle = G$ are called *proper tilings*.

THEOREM 1.1.8. *[9] Let V be a tile of G with $\langle V \rangle \neq G$. The $z = \frac{|G|}{|V|}$, and let $m = \frac{|G|}{|\langle V \rangle|}$. The pair (V, A) is a tiling of G if and only if A has the following form:*

1. *For $i = 0, 1, \ldots, m-1$, let $A_i \subset \langle V \rangle$ be such that (V, A_i) is a tiling of $\langle V \rangle$.*
2. *Let $c_0 = 0, c_1, \ldots, c_{m-1}$ be a set of coset representatives for $\frac{G}{\langle V \rangle}$.*

Then

$$A = A_0 \cup (c_1 + A_1) \cup \cdots \cup (c_{m-1} + A_{m-1}).$$

Theorem 1.1.8 implies that all of the tilings of G can be constructed if all of the proper tilings of the subgroups of G are known. For any $A \subseteq G$, let $A_0 = \{g \in G : g + A = A\}$. A_0 is the set of periodic points of A and is sometimes called the *kernel* of A. If $0 \in A$, then $A_0 \subseteq A$.

PROPOSITION 1.1.4. *[9] If $0 \in A$, then A_0 is a subgroup of G contained in A and A is the union of disjoint cosets of A_0.*

THEOREM 1.1.3. *[9] Let (V, A) be a tiling of G, and let A_0 be the kernel of A. Then, $(\frac{V}{A_0}, \frac{A}{A_0})$ is a tiling of $\frac{G}{A_0}$.*

There is not much already known about the polychromatic number or Turán number with respect to abelian groups and nothing in the context of nonabelian groups. However, coloring techniques have been applied to groups in the past. For example, coloring sets of natural numbers is tackled in [4] in order to resolve a conjecture given in [14]. A subset is *monochromatic* if all its elements have the same colors and is *rainbow* if all its elements have distinct colors. Assume that $\{1, \ldots, n\}$ is colored into r colors. Can an arithmetic progression of length k be found so that all its elements are colored distinct colors? Such an a colored arithmetic progression is called a *rainbow* $AP(k)$. The answer to this question is "no" in general. In [4], conditions on a coloring of $[n] = \{1, \ldots, n\}$ forcing rainbow $AP(3)$ are explored. Let $c : [n] \to \{A, B, C\}$ be a coloring of

$[n]$ in three colors. Let $M(c)$ be the cardinality of the smallest color class in c. The value $M(n)$ is defined to be the largest $M(c)$ over all colorings c of $[n]$ in three colors with no rainbow $AP(3)$.

THEOREM 1.1.4. *[4]* $M(n) \leq \frac{n+4}{6}$.

Determining how to color arithmetic progressions in other groups is taken further in [7]. Let S be a finite nonempty subset of G an abelian group. A $k-$*term arithmetic progression* ($k-$AP) in S is a set of distinct elements of the form $a, a+d, a+2d, \dots, a+(k-1)d$ where $d \geq 1$ and $k \geq 2$. An $r-$coloring of S is a function $c : S \to [r]$ where $[r] = \{1, \dots, r\}$. Such a coloring is *exact* if c is surjective. Given an $r-$coloring c of S, the i^{th} *color class* is $C_i = \{x \in S : c(x) = i\}$. An arithmetic progression is called *rainbow* if the image of the progression under the $r-$coloring is injective. Formally, given $c : S \to [r]$, a $k-$term arithmetic progression is rainbow if $\{c(a+id) : i = 0, 1, \dots, k-1\}$ has k distinct values. The *anti-van der Waerden number* $aw(S, k)$ is the smallest r such that every exact $r-$coloring of S contains a rainbow $k-$term arithmetic progression. Some results in [7] include logarithmic bounds on $aw([n], 3)$ and semi-linear bounds on $aw([n], k)$. However, of special interest is when the group G in question is the integers modulo n and $k = 3$. These results include the following:

PROPOSITION 1.1.5. *[7] Let m and s be positive integers with s odd. Then*

$$aw(\mathbb{Z}_{2^m s}, 3) \leq aw(\mathbb{Z}_s, 3) + 1.$$

PROPOSITION 1.1.6. *[7] For positive n and k, $aw(\mathbb{Z}_n, k) = n$ if and only if $k = n$.*

PROPOSITION 1.1.7. *[7] For positive $n \geq 5$, if n is prime then $aw(\mathbb{Z}_n, n-2) = n-2$; otherwise $aw(\mathbb{Z}_n, n-2) = n-1$.*

Similarly, in [19], it is shown that for a finite abelian group G, $aw(G, 3)$ is determined by the order of G and the number of groups with even order in a direct sum isomorphic to G. The *unitary anti-van der Waerden number* of a group is also defined and determined. An $r-$coloring of G is *unitary* if there is an element of G that is uniquely colored, which will be referred to as a *unitary color*. The smallest r such that every exact $r-$coloring of G that is unitary contains a rainbow $k-$term arithmetic progression is denoted by $aw_u(G, k)$.

PROPOSITION 1.1.8. *[19] For all positive integers n,*

$$aw_u(\mathbb{Z}_n, 3) = aw(\mathbb{Z}_n, 3).$$

PROPOSITION 1.1.9. *[19] For all positive integers n,*

$$aw(G, 3) + aw(\mathbb{Z}_n, 3) - 2 \leq aw(G \times \mathbb{Z}_n, 3).$$

THEOREM 1.1.5. *[19] If G is a finite abelian group and n is an odd positive integer, then*

$$aw(G \times \mathbb{Z}_n, 3) = aw(G, 3) + aw(\mathbb{Z}_n, 3) - 2.$$

THEOREM 1.1.6. *[19] For $1 \leq i \leq s$, let mi be a positive integer. Then*

$$aw(\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3) = 2s + 1.$$

Thus, studying colorings in groups in a Ramsey-type framework has become quite interesting to the realm of discrete mathematics. This is continued in [20] where Ramsey-type functions for symmetric subsets in finite abelian groups are studied. More specifically, formula are derived for computing the number of symmetric $r-$colorings of a finite group G and the number of equivalence classes of symmetric $r-$colorings of G.

Let G be a finite group. Given an element $g \in G$, the *symmetry* on G with the *center* g is the mapping $\eta_g : G \to G; x \mapsto gx^{-1}g$. A subset $S \subseteq G$ is *symmetric* if it is invariant with respect to some symmetry on G. Equivalently, S is *symmetric* if there exists an element $g \in G$ (center of symmetry) such that $gS^{-1}g = S$. Given $r \in \mathbb{N}$, an $r-$coloring of G is any mapping $\chi : G \to \{1, \ldots, r\}$. Define the number $s_r(G)$ to be the greatest number of the form $\frac{k}{|G|}$, where $k \in \mathbb{N}$ such that for every $r-$coloring of G there exists a monochrome symmetric subset of cardinality k. The number $\sigma_r(G)$ is the greatest number of the form $\frac{k}{|G|}$, where $k \in \mathbb{N}$ such that for every $r-$coloring χ of G there exists a subset $X \subseteq G$ of cardinality k and element g such that $\chi(x) = \chi(gx^{-1}g)$ for all $x \in X$. The results of [21] include determining bounds on these values and characterizing certain finite groups which have desired values for these numbers.

THEOREM 1.1.7. *[21] Let G be a finite group of odd order or any finite abelian group, let $r \in \mathbb{N}$. Then*

$$\sigma_r(G) \geq \frac{1}{r} \text{ and consequently } s_r(G) \geq \frac{1}{r^2}.$$

THEOREM 1.1.8. *[21] $\sigma_r(G) = \frac{1}{r}$ if and only if r divides $|2G|$.*

THEOREM 1.1.9. *[21] $\sigma_r(G) = 1$ if and only if one of the following cases holds:*

(1) $r = 1$;

(2) $r = 2$ and G is a cyclic group of order either 3 or 5;

(3) G is a Boolean group.

THEOREM 1.1.10. *[21] Let G be a finite Abelian group and $n \in \mathbb{N}$.*

(1) If G contains subgroup $\bigoplus_n \mathbb{Z}_4$, then $s_{2^n}(G) = \frac{1}{4^n}$;

(2) If G does not contain subgroup $\mathbb{Z}_4$, then $s_2(G) > \frac{1}{4}$.

Thus, coloring the integers modulo n is not merely a problem relegated to this dissertation, but has been explored in various other circumstances. Finally, and interestingly, in [13] rather than color group elements directly, determining how to color Cayley tables of finite groups is explored. A *latin square* of order n is an $n \times n$ array of cells containing entries from an alphabet of size n (whose elements are called symbols) in which no entry appears more than once in any row or column. Given a group $G = \{g_0, g_1, \ldots, g_{n-1}\}$, the *Cayley table* of G, denoted $L(G)$, is the $n \times n$ array in which the cell $L_{i,j}$ contains the group element $g_i g_j$. From the group axioms it follows that $L(G)$ is a latin square. If the set $T \subseteq L$ (a) intersects each row and each column of L exactly once and (b) contains exactly one occurrence of each symbol in L, T is a *transversal* of L. A *partial transversal* is a collection of cells in a Latin square that intersects each row, column, and symbol class at most once. Define a $k-$*coloring* of a latin square L as a partition of its cells into k partial transversals, and the *chromatic number* of L, denoted $\chi(L)$, as the minimum k for which L is $k-$colorable. Of a group G, where p is prime and $|G| = p^k m$ with $\gcd(p^k, m) = 1$, a subgroup of order p^k. The famous Sylow Theorems state that, for any p dividing $|G|$, there exists a Sylow $p-$subgroup, and furthermore any two Sylow $p-$subgroups of G are isomorphic. A *complete mapping* of a group G is a bijection $\theta : G \to G$ such that the derived mapping $\eta : G \to G$ defined by $\eta(g) = g \cdot \theta(g)$ is also a bijection.

THEOREM 1.1.11. *[13] Let G be a group of order n. Then the following are equivalent:*

1. *$\chi(G) = n$.*
2. *$\chi(G) \leq n + 1$.*
3. *$L(G)$ has a transversal.*
4. *G has a complete mapping.*
5. *There is an ordering of the element of G, say $g_1, g_2, \ldots, g_n$, such that $g_1 g_2 \cdots g_n = e$ where e is the identity of G.*
6. *$Syl_2(G)$ (Given a group G, the isomorphism class of its Sylow $2-$subgroups) is either trivial or non-cyclic.*

THEOREM 1.1.12. *[13] Let G be an abelian group of order n. Then*

$$\chi(G) \begin{cases} n & \textit{if } Syl_2(G) \textit{ is either trivial or non-cyclic,} \\ n+2 & \textit{otherwise.} \end{cases}$$

There are many ways to color groups in order to solve a plethora of problems. Now, attention should be drawn to the task at hand which will encompass the remainder of this dissertation. This task is using a coloring technique on finite groups in order to improve on the bounds of a Turán-type problem adjusted to be set in group theoretic terms.

CHAPTER 2. RESULTS ON FINITE ABELIAN GROUPS

In 1941, Hungarian mathematician Pál Turán proved what is now known as Turán's theorem which resolved the following problem: What is the maximum number of edges that a simple undirected graph on n vertices can contain so that the graph does not possess a complete subgraph on k vertices? This is the first instance of question called a Turán-type problem being employed. Such problems have since become commonplace in the branches of mathematics known as extremal graph theory and extremal combinatorics and generally follow the outline: maximize a specified value while averting a given illegal situation. As mentioned previously, results in [3] seek to determine bounds on this maximum value when the base graph is the hypercube Q_n and as such a coloring called polychromatic coloring was introduced to determine said bounds in a Ramsey-type framework. This coloring technique was then applied to the integers, an infinite abelian group in [6]. The integers can be used to construct a family of finite groups known as the integers modulo n as described in Chapter 1 by partitioning the infinite abelian group into finitely many equivalence classes which consist of infinitely many integers. This finite abelian group is a principal focal point of this chapter.

What does this Turán-type problem within the context of finite abelian groups look like? The answer can be posed in the form of the following question.

QUESTION 2.0.1. *For a given finite abelian group G, and an arbitrary subset $S \subset G$, what is the maximum number of elements in a subset of G which does not contain a translate of S?*

Firstly, by the word "translate" what is meant is given $S \subseteq G$ where G is an arbitrary finite abelian group, for any $a \in G$, any set of the form $a + S = \{a\} + S$ is called a *translate* of S. The maximum number of elements in a subset of G which does not contain a translate of that subset whether G is abelian or not will hereafter be referred to as the *Turán number*, denoted $ex(G, S)$. It should also be noted that the question can be reformed into a Ramsey-type question:

QUESTION 2.0.2. *For a given finite abelian group G, and an arbitrary subset $S \subset G$, what is the minimum number of elements one must delete from G so that it does not contain a translate of S?*

This desired complementary value will hereafter be denoted by $f(G, S)$ and note that $f(G, S) = |G| - ex(G, S)$. Answering Question 2.0.1 requires improving on a trivial lower bound and trivial upper bound on $ex(G, S)$ using $f(G, S)$. The trivial lower bound in question is given by $|S| - 1$. This is due to the fact that any collection of $|S| - 1$ elements of the finite group G does not contain a translate of S. The trivial upper bound is given by $|G| - 1$. This is because the collection of all elements in the finite group will contain all translates of S unless, the collection is made smaller by at least one. That is, for any finite group G, the following Remark can be made.

REMARK 2.0.1. *For any $S \subset G$, $|S| - 1 \leq ex(G, S) \leq |G| - 1$ and so $1 \leq f(n, d) \leq n + 1 - |S|$.*

It's also worth noting that if $S = G$ for any finite group, $ex(G, S) = |G| - 1$ due to Remark 2.0.1. Thus, all subsets considered in the work that follows are proper.

This chapter concerns itself with the lower bound which is done by improving the upper bound on $f(G, S)$. The method employed to augment the lower bound comes in the form of polychromatic colorings which of course were originally applied to a type of graph known as the $n-$dimensional hypercube in [3]. With respect to finite abelian groups the following are definitions that will be used in great detail from this point forward.

Given a subset S of a finite abelian group G, a coloring of the elements of G is $S-$*polychromatic* if every translate of S contains an element of each color. The *polychromatic number* given a finite abelian group G and a subset of G called S, denoted $p_G(S)$, is the largest number of colors allowing an $S-$polychromatic coloring of the translates of S. The notation $p(S)$ is used when the choice of G is clear from context. These definitions lay the groundwork for partitioning the elements of the finite group so that a construction of a set which contains the minimum number of elements that need to be deleted so that no translate of the given subset can be formed. This is explained in more detail with the following figure and reasoning.

If it is supposed there is a polychromatic coloring of G with respect to subset S and all of the

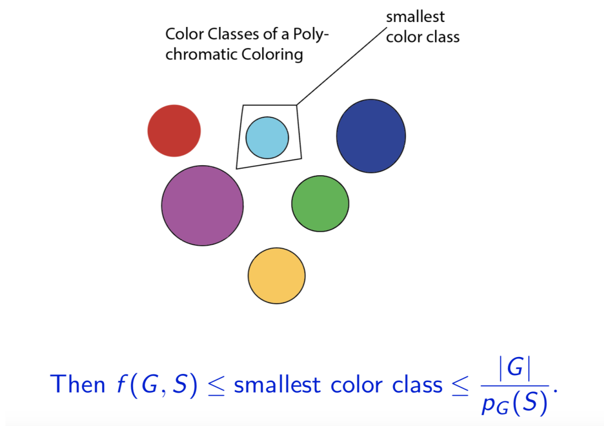

Figure 2.1 The relationship between the polychromatic number and the complementary value

elements belonging to the smallest color class are removed, then G does not possess a translate of S, so $f(G,S)$ is less than or equal to the cardinality of the smallest color class. Given a simple averaging argument, that is distributing the elements of the finite group among the color classes, the smallest color class is less than or equal to $\frac{|G|}{p_G(S)}$. Hence, $f(G,S) \leq \frac{|G|}{p_G(S)}$ and so $n - \frac{|G|}{p_G(S)} \leq ex(G,S)$. Figure 2 pictorially demonstrates how exactly the polychromatic number yields an upperbound on $f(G,S)$. From the trivial bounds on the Turán number, it is known that $1 \leq f(G,S) \leq |G|-|S|+1$. Therefore, finding the sets S so that $p_G(S)$ as large as possible i.e. $p_G(S) = |S|$ will yield a tighter upper bound on $f(G,S)$ than $|G|-|S|+1$. Thus, these are the subsets that the following work will largely concentrate on. Similarly, there are many finite abelian groups to consider, however, firstly the focus will be on the well-known integers modulo n, denoted $\mathbb{Z}_n$. Therefore, the discussion in this chapter will begin with the integers modulo n, and the size of S will be restricted to less than or equal to 3 and then larger sizes of S will be considered.

2.0.1 The Integers Modulo n, $\mathbb{Z}_n$ for $n \geq 3$

There are a few small groups whose Turán number is known.

EXAMPLE 2.0.1. *Let $G = \mathbb{Z}_3$. The elements of this group are $0, 1, 2$. So, the only subsets of interest are two element subsets. There are three of them: $\{0,1\}$, $\{1,2\}$, and $\{0,2\}$. These sets are* *also translates of each other. For example, let* $S = \{0, 1\}$*, then* $1 + S = \{1, 2\}$ *and* $2 + S = \{2, 0\}$*. The trivial bounds reveal* $1 \leq ex(G, S) \leq 2$*. However* $ex(G, S)$ *must be 1 because any two element subset of* $\mathbb{Z}_3$ *obviously contains one of the translates.*

EXAMPLE 2.0.2. *Let* $G = \mathbb{Z}_4$*. The elements of this group are* $0, 1, 2, 3$*. The subsets of interest are two and three element subsets. The possible two element subsets are* $\{0, 1\}$*,* $\{0, 2\}$*,* $\{0, 3\}$*,* $\{1, 2\}$*,* $\{1, 3\}$*,* $\{2, 3\}$*. Note that with* $S = \{0, 1\}$*,* $1+S = \{1, 2\}$*,* $2+S = \{2, 3\}$*,* $3+S = \{3, 0\}$*. In this case, the trivial bounds yield* $1 \leq ex(G, S) \leq 3$*. However,* $p_G(S) = 2$ *under the coloring* $\chi(0) = c_0 = \chi(2)$ *and* $\chi(1) = c_1 = \chi(3)$ *and so* $4 - \frac{4}{2} = 4 - 2 = 2 \leq ex(G, S)$*. Still,* $ex(G, S) = 2$ *as any three element subset of* $\mathbb{Z}_4$ *contains a translate. If* $S = \{0, 2\}$*, then* $1 + S = \{1, 3\}$*,* $2 + S = \{2, 0\}$*,* $3 + S = \{3, 1\}$ *and* $p_G(S) = 2$ *under the coloring* $\chi(0) = c_0 = \chi(1)$ *and* $\chi(2) = c_1 = \chi(3)$*. Nevertheless,* $ex(G, S)$ *as any three element subset of* $\mathbb{Z}_4$ *contains a translate. Next, the possible three element subsets are* $\{0, 1, 2\}$*,* $\{1, 2, 3\}$*,* $\{2, 3, 0\}$*,* $\{3, 0, 1\}$ *and these sets are also translates of each other. For example, let* $S = \{0, 1, 2\}$*, then* $1 + S = \{1, 2, 3\}$*,* $2 + S = \{2, 3, 0\}$*,* $3 + S = \{3, 0, 1\}$*. The trivial bounds give that* $2 \leq ex(G, S) \leq 3$*. However,* $ex(G, S) = 2$ *as any set of three elements is exactly one of the translates.*

Of course, $\mathbb{Z}_n$ for larger values of n are the finite groups of interest. So, with this in mind first what will be examined is improving the lower bound on the Turán number of this family of finite abelian groups with any subset of size two or three. In the following results $\langle a \rangle$ denotes the subgroup generated by element a belonging to the specified group. Note that the result which immediately follows is one that can be applied to subsets of $\mathbb{Z}_n$ of any size.

THEOREM 2.0.1. *Let* $t \geq 2$*,* $m_1, m_2, \ldots, m_{t-1} \in \mathbb{Z}$ *are distinct, and* $a \in \mathbb{Z}_n \setminus \{0\}$*. Then* $p_{\mathbb{Z}_n}(\{0, m_1 a, m_2 a, \ldots, m_{t-1} a\}) = p_{\mathbb{Z}_{|\langle a \rangle|}}(\{0, m_1, m_2, \ldots, m_{t-1}\})$*.*

Proof. For any $x \in \mathbb{Z}_n$, $x + \{0, m_1 a, m_2 a, \ldots, m_{t-1} a\} \subseteq x + \langle a \rangle$. Therefore, partition the translates of $\{0, m_1 a, m_2 a, \ldots, m_{t-1} a\}$ among the cosets of $\langle a \rangle$. If $r = [\mathbb{Z}_n : \langle a \rangle]$, then the r collections of translates are colored identically due to the following isomorphism between elements in the translates: $\phi : y + \langle a \rangle \to z + \langle a \rangle; y + ia \mapsto z + ia$ for any nonnegative integer i. In particular, color the translates whose elements belong to $\langle a \rangle$. These translates are of the form $\{ja, (m_1 + j)a, (m_2+j)a, \ldots, (m_{t-1}+j)a\}$. Coloring these translates in $\mathbb{Z}_n$ is identical to coloring the translates $\{j, m_1 + j, m_2 + j, \ldots, m_{t-1} + j\}$ in $\mathbb{Z}_{|\langle a \rangle|}$ via the following isomorphism between elements in the translates: $\langle a \rangle \to \mathbb{Z}_{|\langle a \rangle|}; ja \mapsto j$ for any nonnegative integer j. □

Simply put, Theorem 2.0.1 says that if all of the elements belonging to S also belong to the subgroup generated by an element, the problem of determining the polychromatic number can be reduced. Next, a complete characterization of $p_{\mathbb{Z}_n}(S)$ when $|S| = 2$ is given.

LEMMA 2.0.1.

$$p_{\mathbb{Z}_n}(\{0,1\}) = \begin{cases} 1 & \textit{if } n \textit{ is odd} \\ 2 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Since $S = \{0, 1\}$ and all of its translates consist of two elements each, it is only feasible to either assign one or two colors to S and its translates. Clearly, it is easy to create an $S-$polychromatic coloring consisting of only one color simply by assigning the same color to every element of $\mathbb{Z}_n$. Next, assume n is even. Note that all of the translates of S are of the form $\{i, i+1\}$ for all $i \in \mathbb{Z}_n$. That is, consecutive elements appear together in the translates of S. Let $\chi : \mathbb{Z}_n \to \{c_0, c_1\}$ denote a coloring defined as follows:

$$\chi(i) = c_{i \bmod 2} \text{ for all } i \in \mathbb{Z}_n$$

Since n is even, every translate consists of two consecutive and therefore distinctly colored elements. Now, assume n is odd and assume by way of contradiction that χ' is an $S-$polychromatic coloring consisting of two colors. Then, any such coloring of the translates of S in G ensures without loss of generality $\chi'(0) = c_0 \neq c_1 = \chi'(1) = \chi'(n-1)$. Note that $n-1$ is even and that $\chi'(n-1) = \chi'(n-1-2j)$ for $0 \leq j \leq \frac{n-1}{2} - 1$. However, then $\chi'(1) = c_1 = \chi'(2)$ but $\{1, 2\}$ is a translate. □

THEOREM 2.0.2. *For all $a \neq b \in \mathbb{Z}_n$ and $n \geq 3$ where $a' \in \mathbb{Z}_n$ such that $\{0, a'\}$ is a translate of $\{a, b\}$,*

$$p_{\mathbb{Z}_n}(\{a,b\}) = \begin{cases} 1 & \textit{if } |\langle a' \rangle| \textit{ is odd} \\ 2 & \textit{if } |\langle a' \rangle| \textit{ is even.} \end{cases}$$

Proof. Let $S = \{a, b\}$ for any $a \neq b \in \mathbb{Z}_n$ and $n \geq 3$. Then, consider either $n-a+S$ or $n-b+S$ and so let a' be either $n-a+b$ or $n-b+a$. By Theorem 2.0.1, $p_{\mathbb{Z}_n}(S) = p_{\mathbb{Z}_n}(\{0, a'\}) = p_{\mathbb{Z}_{|a'|}}(\{0, 1\})$. By Lemma 2.0.1,

$$p_{\mathbb{Z}_n}(S) = \begin{cases} 1 & \text{if } |\langle a' \rangle| \text{ is odd} \\ 2 & \text{if } |\langle a' \rangle| \text{ is even.} \end{cases}$$

□

Thus, the entire proof of Theorem 2.0.2 involves coloring a simple set in $\mathbb{Z}_n$ and then extending this coloring using Theorem 2.0.1. The following result gives the corresponding lower bound on the Turán number.

Corollary 2.0.1. *For all $a \neq b \in \mathbb{Z}_n$ and $n \geq 3$ where $a' \in \mathbb{Z}_n$ such that $\{0, a'\}$ is a translate of $\{a, b\}$,*

$$ex(\mathbb{Z}_n, \{a, b\}) \geq \begin{cases} 0 & \textit{if } |\langle a' \rangle| \textit{ is odd} \\ n - \frac{n}{2} & \textit{if } |\langle a' \rangle| \textit{ is even.} \end{cases}$$

The proof of Theorem 2.0.3 is long, and is thus divided among the following several results.

THEOREM 2.0.3. *For all $a \neq b \neq c, d, m_a, m_b \in \mathbb{Z}_n$, $n \geq 3$, and $j \geq 0$ where $a', b', d \in \mathbb{Z}_n$ such that $\{0, a', b'\}$ is a translate of $\{a, b, c\}$,*

$$p_{\mathbb{Z}_n}(\{a, b, c\}) = \begin{cases} 1 & \textit{if } n \equiv 0 \bmod 7 \textit{ and } a' = d, b' = 3d \textit{ or } a' = d, b' = 5d \textit{ and } |\langle d \rangle| = 7 \\ 2 & \textit{otherwise} \\ 3 & \textit{if } n \equiv 0 \bmod 3^{j+1}, a' = 3^j m_a, b' = 3^j m_b, m_a \equiv 1 \bmod 3, \textit{ and } m_b \equiv 2 \bmod 3. \end{cases}$$

Corollary 2.0.2. *For all $a \neq b \neq c, d, m_a, m_b \in \mathbb{Z}_n$, $n \geq 3$, and $j \geq 0$ where $a', b', d \in \mathbb{Z}_n$ such that $\{0, a', b'\}$ is a translate of $\{a, b, c\}$,*

$$ex(\mathbb{Z}_n, \{a, b, c\}) \geq \begin{cases} 0 & \textit{if } n \equiv 0 \bmod 7 \textit{ and } a' = d, b' = 3d \textit{ or } a' = d, b' = 5d \textit{ and } |\langle d \rangle| = 7 \\ n - \frac{n}{2} & \textit{otherwise} \\ n - \frac{n}{3} & \textit{if } n \equiv 0 \bmod 3^{j+1}, a' = 3^j m_a, b' = 3^j m_b, m_a \equiv 1 \bmod 3, \textit{ and } m_b \equiv 2 \bmod 3. \end{cases}$$

Recall the following definition from [6]. Given a set $S \subset G$, a set $T \subset G$ is a *complement set* for S if $S+T=G$. If S has a complement set T such that if $s_1, s_2 \in S$, $t_1, t_2 \in T$, then $s_1+t_1 = s_2+t_2$ implies $s_1 = s_2$ and $t_1 = t_2$, then S *tiles* G *by translation.* The notation $S \oplus T$ is written when S tiles G by translation. Without loss of generality, $0 \in S, T$ for all of the following arguments. These definitions along with the following result from [6] are indispensable in proving Theorem 2.0.3.
In [6] the following result was proven which yields a divisibility requirement on subsets of $\mathbb{Z}_n$ so that $p(S) = |S|$.

THEOREM 2.0.4. *[6] Let G be any finite abelian group. A finite set $S \subset G$ tiles G by translation if and only if $p(S) = |S|$. Moreover, if χ is an $S-$polychromatic coloring of G with $|S|$ colors and T is the set of elements of G colored by χ with any given color, then $S \oplus T = G$.*

REMARK 2.0.2. *Therefore, $p(S) = |S|$ is equivalent to $G = S \oplus T$ for some subset $T \subset G$. That is, $|G| = |S| \cdot |T|$ and hence $|S|||G|$.*

The aim of the following few results is to equivalently characterize the subsets of $\mathbb{Z}_n$ of order three with polychromatic number also equal to three using Theorem 2.0.4, Remark 2.0.2, and the previous definitions.

LEMMA 2.0.2. *Suppose $S = \{0, a, b\}$, $i \in S$, $S \oplus T = G$. If $x \in i+T$, then $x+a+b \in i+T$.*

Proof. Note that becuase $S \oplus T = G$, every element of G belongs to exactly one of the sets T, $a+T$, $b+T$.

Case 1: Suppose $x \in T$. If $x+a+b \in b+T$, then $x+a \in T$. However, $x+a \in a+T$. If $x+a+b \in a+T$, then $x+b \in T$. However, $x+b \in b+T$. So, $x+a+b \in T$.

Note that If $x \in T$, then $x+m(a+b) \in T$ for any nonnegative integer m.

Base Case: If $m = 0$ the statement is clearly true, if $m = 1$ this follows by the argument given in Case 1.

Induction Step: Assume $x \in T \implies x+m(a+b) \in T$. Notice $x+(m+1)(a+b) = x+m(a+b)+a+b$. By the induction hypothesis, $x+m(a+b) \in T$. Let $x_1 = x+m(a+b)$. Then, $x_1+a+b \in T$.

Case 2: Suppose $x \in a+T$. If $x+a+b \in b+T$, then $x+a \in T$. Notice $x+a-(a+b) = x-b$

is either in $a+T$, $b+T$, or T. However, if $x-b \in a+T$, then $x \in a+b+T$, but as was assumed $x \in a+T \implies x = a+t_1$ for some $t_1 \in T$. Therefore, $x \in a+b+T \implies t_1 \in b+T$ which is not possible. If $x-b \in b+T$, then $x-2b \in T$ and by case 1 and because $0 \in T$, $x-2b+a+b \in T \implies x+a-b \in T \implies x+a \in b+T$, but it was assumed $x+a \in T$. So, $x-b \in T \implies x \in b+T$, a contradiction.

If $x+a+b \in T$, let $k = |\langle a+b \rangle|$. By the above induction argument, $x+a+b+(k-1)(a+b) = x \in T$, however this is a contradiction. So, $x+a+b \in a+T$.

Case 3: Suppose $x \in b+T$. If $x+a+b \in a+T$, then $x+b \in T$. Notice $x+b-(a+b) = x-a$ is either in $b+T$, $a+T$, or T. However, if $x-a \in b+T$, then $x \in a+b+T$, but as was assumed $x \in b+T \implies x = b+t_1$ for some $t_1 \in T$. Therefore, $x \in a+b+T \implies t_1 \in a+T$ which is not possible. If $x-a \in a+T$, then $x-2a \in T$ and by case 1 and because $0 \in T$, $x-2a+a+b \in T \implies x-a+b \in T \implies x+b \in a+T$, but it was assumed $x+b \in T$. So, $x-a \in T \implies x \in a+T$, a contradiction.

If $x+a+b \in T$, let $k = |\langle a+b \rangle|$. By the above induction argument, $x+a+b+(k-1)(a+b) = x \in T$, however this is a contradiction. So, $x+a+b \in b+T$.

$\square$

REMARK 2.0.3. *By case 1 of the proof of Lemma 2.0.2, since* $0 \in T$, $\langle a+b \rangle \subseteq T$.

LEMMA 2.0.3. *If* $k \in \mathbb{Z}_n$ *and* $gcd(k,n) = 1$ *such that* $a_1, \ldots, a_{\ell-1}, ka_1, \ldots, ka_{\ell-1} \neq 0$ *and* $ka_i \neq ka_j$ *for all* $i \neq j$, *then* $p(\{0, a_1, a_2, \ldots, a_{\ell-1}\}) = p(\{0, ka_1, ka_2, \ldots, ka_{\ell-1}\})$.

Proof. Let χ_k be a $r-$polychromatic coloring of any translate of $\{0, ka_1, ka_2, \ldots, ka_{\ell-1}\}$ in $\mathbb{Z}_n$. Set $\chi(x) = \chi_k(kx)$. Then, $\chi(x) = \chi_k(kx)$, $\chi(x+a_1) = \chi_k(kx+ka_1)$,..., $\chi(x+a_{\ell-1}) = \chi_k(kx+ka_{\ell-1})$. Note that r of these colors are distinct as $\{kx, kx+ka_1, \ldots, kx+ka_{\ell-1}\}$ is a translate of $\{0, ka_1, \ldots, ka_{\ell-1}\}$ and χ_k is an $r-$polychromatic coloring for $\{0, ka_1, ka_2, \ldots, ka_{\ell-1}\}$. Thus, χ is an $r-$polychromatic coloring of any translate of $\{0, a_1, a_2, \ldots, a_{\ell-1}\}$ in $\mathbb{Z}_n$.

Suppose χ is an $r-$polychromatic coloring of any translate of $\{0, a_1, a_2, \ldots, a_{\ell-1}\}$ in $\mathbb{Z}_n$. Since $gcd(k,n) = 1$, k is an invertible element of $\mathbb{Z}_n$. So, set $\chi_k(x) = \chi(k^{-1}x)$. Then, $\chi_k(x) = \chi(k^{-1}x)$,

$\chi_k(x+ka_1) = \chi(k^{-1}x+k^{-1}ka_1) = \chi(k^{-1}x+a_1)$, ..., $\chi_k(x+ka_{\ell-1}) = \chi(k^{-1}x+k^{-1}ka_{\ell-1}) = \chi(k^{-1}x+a_{\ell-1})$. Note that r of these colors are distinct as $\{k^{-1}x, k^{-1}x+a_1, \ldots, k^{-1}x+a_{\ell-1}\}$ is a translate of $\{0, a_1, a_2, \ldots, a_{\ell-1}\}$ and χ is an $r-$polychromatic coloring for $\{0, a_1, a_2, \ldots, a_{\ell-1}\}$. Thus, χ_k is an $r-$polychromatic coloring of any translate of $\{0, ka_1, ka_2, \ldots, ka_{\ell-1}\}$ in $\mathbb{Z}_n$. □

REMARK 2.0.4. *Note that the first direction of Lemma 2.0.3 holds for all $k \in \mathbb{Z}_n$ such that $a_1, \ldots, a_{\ell-1}, ka_1, ka_2, \ldots, ka_{\ell-1} \neq 0$.*

REMARK 2.0.5. *If $k = n-1$, then k is relatively prime to n for all $n \geq 3$ and therefore is an invertible element in $\mathbb{Z}_n$ and so Lemma 2.0.3 yields $p(S) = p(-S)$.*

Lemma 2.0.3 states that if the polychromatic number of a subset can be determined, then this also gives the polychromatic number of certain multiples of this subset as well. This result is actually quite powerful because of its ability to be applied to larger subsets of $\mathbb{Z}_n$ and because it will be used in the proof of Lemma 2.0.4.

LEMMA 2.0.4. *There is a $\{0, a, b\}-$polychromatic coloring of $\mathbb{Z}_n$ with three colors if and only if $n \equiv 0 \bmod 3^{j+1}$, $a = 3^j m_a$, $b = 3^j m_b$, $m_a \equiv 1 \bmod 3$, $m_b \equiv 2 \bmod 3$, and $j \geq 0$.*

Proof. Suppose $n \equiv 0 \bmod 3^{j+1}$, $a = 3^j m_a$, $b = 3^j m_b$, $m_a \equiv 1 \bmod 3$, $m_b \equiv 2 \bmod 3$, and $j \geq 0$. Every element of $\mathbb{Z}_n$ can be written in the form $t \cdot 3^j + r$ where t is some positive integer such that $0 \leq t \leq |\langle 3^j \rangle| - 1$ and $0 \leq r \leq 3^j - 1$. Let χ be the coloring given by $\chi(t \cdot 3^j + r) = c_{t \bmod 3}$. Under this coloring every translate of $\{0, 3^j m_a, 3^j m_b\}$ contains three distinct colors as every translate is of the form $\{t \cdot 3^j + r, (t+m_a) \cdot 3^j + r, (t+m_b) \cdot 3^j + r\}$ and $\chi(t \cdot 3^j + r) = c_{t \bmod 3}$, $\chi((t+m_a) \cdot 3^j + r) = c_{t+1 \bmod 3}$, and $\chi((t+m_b) \cdot 3^j + r) = c_{t+2 \bmod 3}$. If instead $m_a \equiv 2 \bmod 3$, and $m_b \equiv 1 \bmod 3$, the result is proven analogously.

Next, assume the $p(\{0, a, b\}) = 3$. The condition $p(\{0, a, b\}) = 3$ is equivalent to S tiling G, by Theorem 2.0.4. Suppose $T \subset \mathbb{Z}_n$ so that $\mathbb{Z}_n = \{0, a, b\} \oplus T$. Then $n = 3|T|$ and so $n \equiv 0 \bmod 3$. Let $x \in T$. By Lemma 2.0.2, $x + a + b \in T$ for any $x \in T$. If $x \in T$, then the coset $x + \langle a+b \rangle$ is a subset of T. Therefore, there is some integer k such that $k|\langle a+b \rangle| = |T| = \frac{n}{3}$. Also, $|\langle a+b \rangle| = \frac{n}{gcd(a+b,n)}$. Thus, $k\frac{n}{gcd(a+b,n)} = \frac{n}{3}$ implies $3k = gcd(a+b, n)$. Thus, $3|(a+b)$. Therefore, $a \equiv 1 \bmod 3$ and

$b \equiv 2 \bmod 3$, $a \equiv 2 \bmod 3$ and $b \equiv 1 \bmod 3$, or $a \equiv 0 \bmod 3$ and $b \equiv 0 \bmod 3$.

If it is the case that $a \equiv 1 \bmod 3$ and $b \equiv 2 \bmod 3$ or $a \equiv 2 \bmod 3$ and $b \equiv 1 \bmod 3$, then the proof is complete for $j = 0$. If $a \equiv 0 \bmod 3$ and $b \equiv 0 \bmod 3$, then $a = 3^j m_a$ and $b = 3^i m_b$ for integers j, i, m_a, m_b where $1 \leq j, i$ and $m_a, m_b \not\equiv 0 \bmod 3$. Without loss of generality let $j \leq i$. Note that since $p(\{0, 3^j m_a, 3^i m_b\}) = 3$, $p(\{0, m_a, 3^{i-j} m_b\}) = 3$ by Remark 2.0.4 with a multiple of 3^j. Therefore, $m_a + 3^{i-j} m_b \equiv 0 \bmod 3$. Since $m_a \not\equiv 0 \bmod 3$, $i - j = 0$ and $m_b \equiv -m_a \bmod 3$. So, $m_a \equiv 1 \bmod 3$ and $m_b \equiv 2 \bmod 3$ or vice versa.

The translates of $\{0, 3^j m_a, 3^j m_b\}$ consisting entirely of elements in the subgroup generated by 3^j are of the form $\{3^j y, 3^j (y + m_a), 3^j (y + m_b)\}$ where $y \in \mathbb{Z}_n$. Since $p(\{0, 3^j m_a, 3^j m_b\}) = 3$ each of these translates consists of three distinct colors. Coloring these translates is isomorphic to coloring the translates of $\{0, m_a, m_b\}$ in $\mathbb{Z}_{|\langle 3^j \rangle|}$ by Theorem 2.0.1. Since there is a $\{0, m_a, m_b\}-$polychromatic coloring consisting of three colors, $3 || \langle 3^j \rangle|$. Since $3 || \langle 3^j \rangle|$ and $|\langle 3^j \rangle| = \frac{n}{\gcd(3^j, n)}$, this implies $\gcd(3^j, n) = 3^j$. Therefore, $n = 3^j |\langle 3^j \rangle| \equiv 0 \bmod 3^{j+1}$. Hence, the proof is complete for $j \geq 1$. $\square$

The proof of Lemma 2.0.4 quite aptly demonstrates the usefulness and power of Theorem 2.0.4. The argument given here has been used to extend to subsets of larger size.

Next, in the proof of Theorem, it will be shown that in some cases the polychromatic number is only one.

LEMMA 2.0.5. *If $n \equiv 0 \bmod 7$, $S = \{0, c, 3c\}$ or $S = \{0, c, 5c\}$ and $|\langle c \rangle| = 7$ where $c \in \mathbb{Z}_n$, then $p(S) = 1$.*

Proof. Without loss of generality, let $S = \{0, 1, 3\}$. First, it can be shown that $\mathbb{Z}_7$ cannot be $S-$polychromatically colored with two colors through an exhaustive search. Now, let $S = \{0, c, 3c\}$. Assume there is a $S-$polychromatic coloring with two colors, χ, of $\mathbb{Z}_n$ where $n \equiv 0 \bmod 7$ and $|\langle c \rangle| = 7$. Consider the translates of S which consist only of the elements belonging to the subgroup generated by c. These translates are of the form $\{jc, (j+1)c, (j+3)c\}$. Coloring these translates in $\mathbb{Z}_n$ is isomorphic to coloring $\{j, j+1, j+3\}$ in $\mathbb{Z}_7$ by Theorem 2.0.1; however, this is not possible. $\square$

Much like the proof of Theorem 2.0.2, something simple is colored or rather in this case, is shown can not be colored and then the coloring or lack thereof is extended using Theorem 2.0.1. A large part of the proof of Theorem 2.0.3 is showing that the polychromatic number of subsets of the form $\{0, 1, \ell\}$ is at least two. In some cases, explicit colorings are constructed as in Lemma 2.0.7 and Theorem 2.0.7 and in other cases, such colorings are just proven to exist as in Theorems 2.0.5 and 2.0.6. First, however, a lemma notes that in any subset of the form $\{0, 1, \ell\}$, it is only necessary to consider certain elements for the choice of ℓ.

LEMMA 2.0.6. *For all $\ell \in \mathbb{Z}_n$ with $n \geq 3$ there exists an $\ell' \in \mathbb{Z}_n$ so that $\ell' \leq \frac{n}{2}$ and $p(\{0, 1, \ell\}) = p(\{0, 1, \ell'\})$.*

Proof. As noted in Remark 2.0.5, $p(S) = p(-S)$. If $\ell > \frac{n}{2}$, then $\ell' = n - \ell + 1 \leq \frac{n}{2}$. Therefore, $p(\{0, 1, \ell\}) = p(\{0, -1, -\ell\}) = p(\{0, n-1, n-\ell\}) = p(\{0, 1, n-\ell+1\})$. □

Lemma 2.0.6 simply states that if for some of the subsets of the form $\{0, 1, \ell\}$ an $\{0, 1, \ell\}-$polychromatic coloring consisting of two colors can be found, then for all such subsets such a coloring can be found. An *interval* of $\mathbb{Z}_n$, denoted $[a, b]$, consists of the collection of consecutive elements of $\mathbb{Z}_n$ from a to b i.e. $a, a+1, a+2, \ldots, b-1, b \mod n$. An *$r-$polychromatic precoloring* of $\mathbb{Z}_n$ with respect to S is a coloring of a proper subset of $\mathbb{Z}_n$ such that for every translate of S it is the case that either all elements are distinctly colored, two elements are distinctly colored and the third is left uncolored, two elements are the same color and the third is another color, or one element is colored one color and the other two elements are left uncolored. The number of elements not colored must also sum to at least r.

Let χ' be an $r-$polychromatic precoloring of $\mathbb{Z}_n$ and let j be an element of $\mathbb{Z}_n$ that is not colored by the precoloring. The element j has an *option* if there exists at least two colors that can be assigned to $\chi'(j)$ and χ' is still an $r-$polychromatic precoloring. An *$r-$polychromatic interval precoloring* is an $r-$polychromatic precoloring such that there exist elements j and k such that $[j, k]$ are left uncolored. These definitions will be used in the proofs of Theorem 2.0.5 and Theorem 2.0.6.

THEOREM 2.0.5. *Let $S = \{0, 1, \ell\}$ in $\mathbb{Z}_n$ such that $n \equiv r \bmod \ell - 2$ where $r = 0, 1, 2,$ or 3 and ℓ is odd. In each case, there is an $S-$polychromatic coloring of $\mathbb{Z}_n$ with two colors.*

Proof. Let $n = m(\ell - 2) + r$. By Lemma 2.0.6, $\ell \leq \frac{n}{2}$. So, $2 \leq m$.

In each of the following cases, the n elements of $\mathbb{Z}_n$ are broken up the into m consecutive collections of $\ell - 2$ elements and a final collection of r elements.

Firstly, let $r = 0$. Each of the collections of $\ell - 2$ elements is colored by an $S-$polychromatic coloring, χ_0, which consists of two colors and is defined as follows

$$\chi_0(x) = \begin{cases} c_0 & \text{if } x = i(\ell - 2) \\ c_0 & \text{if } x = i(\ell - 2) + 2k + 1 \text{ where } 0 \leq k \leq \frac{\ell - 5}{2} \\ c_1 & \text{if } x = i(\ell - 2) + 2k \text{ where } 1 \leq k \leq \frac{\ell - 3}{2} \end{cases}$$

where $0 \leq i \leq m - 1$. Note that any translate $\{x, x + 1, x + \ell\}$ contains two distinct colors if $x = i(\ell - 2) + 2k + 1$ where $\ell - 3 \geq k \geq 0$ or $x = i(\ell - 2) + 2k$ where $\ell - 3 \geq k \geq 1$. If $x = i(\ell - 2)$, it is colored c_0, and so $x + 1$ is also colored c_0. The third element in the translate, $x + \ell$, is $(i + 1)(\ell - 2) + 2$ which is colored c_1.

If $r = 1$, the n elements of $\mathbb{Z}_n$ are colored by a $2-$polychromatic precoloring given by χ_0 when $0 \leq i \leq m - 2$ as in the previous case except for the last collection of $\ell - 2$ elements and the remaining one element. These elements are colored by a $2-$polychromatic interval precoloring, χ_1, as follows

$$\chi_1(x) = \begin{cases} c_0 & \text{if } x = n - (2k + 1) \text{ where } 0 \leq k \leq \frac{\ell - 1}{2} \\ c_1 & \text{if } x = n - (2k) \text{ where } 2 \leq k \leq \frac{\ell - 1}{2} \\ c_1 & \text{if } x = n - 2 \end{cases}$$

The two translates to check are $\{n - 2, n - 1, \ell - 2\}$ and $\{n - \ell, n - \ell + 1, 0\}$. Still, $\chi_1(n - 2) = c_1$, $\chi_1(n - 1) = c_1$, $\chi_1(\ell - 2) = c_0$, $\chi_1(n - \ell) = c_1$, $\chi_1(n - \ell + 1) = c_1$, and $\chi_1(0) = c_0$.

If $r = 2$, the n elements of $\mathbb{Z}_n$ are colored by a $2-$polychromatic precoloring given by χ_0 when $0 \leq i \leq m - 2$ as in the previous cases except for the last collection of $\ell - 2$ elements and the remaining two elements. These elements are colored by a $2-$polychromatic interval precoloring,

χ_2, as follows

$$\chi_2(x) = \begin{cases} c_0 & \text{if } x = n-\ell \\ c_1 & \text{if } x = n-\ell+1 \\ c_0 & \text{if } x = n-\ell+2k+1 \text{ where } 1 \leq k \leq \frac{\ell-3}{2} \\ c_1 & \text{if } x = n-\ell+2k \text{ where } 1 \leq k \leq \frac{\ell-1}{2} \end{cases}$$

There is only one translate to check: $\{n-\ell+1, n-\ell+2, 1\}$. Still, $\chi_2(n-\ell+1) = c_1$, $\chi_2(n-\ell+2) = c_1$, and $\chi_2(1) = c_0$.

If $r = 3$, the n elements of $\mathbb{Z}_n$ are colored by a 2−polychromatic precoloring given by χ_0 when $0 \leq i \leq m-2$ as in the previous cases except the last collection of $\ell - 2$ elements and the remaining three elements. These elements are colored by a 2−polychromatic interval precoloring, χ_3, as follows

$$\chi_3(x) = \begin{cases} c_0 & \text{if } x = n-\ell-1 \\ c_1 & \text{if } x = n-1 \\ c_1 & \text{if } x = n-\ell+2k+1 \text{ where } 0 \leq k \leq \frac{\ell-3}{2} \\ c_0 & \text{if } x = n-\ell+2k \text{ where } 0 \leq k \leq \frac{\ell-3}{2} \end{cases}$$

The two translates to check are $\{n-\ell-1, n-\ell, n-1\}$ and $\{n-2, n-1, \ell-2\}$ which contain two distinctly colored elements as $\chi_3(n-\ell-1) = c_0$, $\chi_3(n-\ell) = c_0$, $\chi_3(n-1) = c_1$, $\chi_3(n-2) = c_1$, and $\chi_3(\ell-2) = c_0$. □

THEOREM 2.0.6. *Let ℓ be odd. Let $S = \{0, 1, \ell\}$ in $\mathbb{Z}_n$ such that $n \equiv r \bmod \ell - 2$ and $r \geq 4$. There is a 2−polychromatic interval precoloring and at least two options which extend to an S−polychromatic coloring of $\mathbb{Z}_n$ with two colors.*

Proof. First, by Lemma 2.0.6, assume $\ell \leq \frac{n}{2}$. Therefore, $\ell - 2 \leq \frac{n}{2}$. Consider the following 2−polychromatic interval precoloring where the n elements of $\mathbb{Z}_n$ are broken up into m consecutive collections of $\ell - 2$ elements and a final collection of r elements:

$$\chi(x) = \begin{cases} c_0 & \text{if } x = i(\ell-2) \\ c_0 & \text{if } x = i(\ell-2)+2k+1 \text{ where } 0 \leq k \leq \frac{\ell-5}{2} \\ c_1 & \text{if } x = i(\ell-2)+2k \text{ where } 1 \leq k \leq \frac{\ell-3}{2} \end{cases}$$

where $0 \leq i \leq m-2$. The final collection of $\ell - 2$ elements and the remaining r elements are left uncolored save for the following assignments $\chi(n-r-(\ell-2)) = c_0$, $\chi(n-r-(\ell-3)) = c_0$, $\chi(n-r-(\ell-4)) = c_1$, and $\chi(n-r+2) = c_1$. The aim is to show there is an option between $n-r-(\ell-4)$ and $n-r+2$ and an option between $n-r+2$ and 0 and use these options so that χ can be extended to a $S-$polychromatic coloring with two colors.

First, note that if $\{x, x+1, x+\ell\}$ is a translate with $x = i(\ell-2)+j$ where $0 \leq i \leq m-2$ and $0 \leq j \leq \ell-3$, then this translate consists of two distinctly colored elements. Next, note that $n-1$ must be colored c_1 as $\{n-1, 0, \ell-1\}$ is a translate of S with $\chi(0) = c_0$ and $\chi(\ell-1) = c_0$. The element $n-2$ has an option as the only translate containing $n-2$ which also contains two colored elements is $\{n-2, n-1, \ell-2\}$ with $\chi(n-1) = c_1$ and $\chi(\ell-2) = c_0$. Now, assigning a color to $n-2$ depends on the parity of the number of elements between $n-r+2$ and $n-2$. If the number of elements is even, $\chi(n-2) = c_0$ and $n-j$ such that $3 \leq j \leq r-3$ are assigned colors c_1 if j is odd and c_0 if j is even. If the number of elements is odd, $\chi(n-2) = c_1$ and $n-j$ such that $3 \leq j \leq r-3$ are assigned c_0 if j is odd and c_1 if j is even. In either case, the translates $\{n-r+2, n-r+3, \ell-r+2\}$ and $\{n-2, n-1, \ell-2\}$ always contain two distinct colors. This also holds true for the first two elements of the translates $\{n-j, n-j+1, n-j+\ell\}$ such that $3 \leq j \leq r-3$.

Next, consider the translates $\{n-\ell, n-\ell+1, 0\}$ and $\{n-\ell+1, n-\ell+2, 1\}$. It is the case that one of $n-\ell$ or $n-\ell+1$ will always have an option. Since $r \geq 4$, $n-(\ell-2)-r \leq n-\ell-2 \leq n-\ell$. Suppose $\chi(n-\ell+2) = c_1$. Then $\chi(1) = c_0$, so $n-\ell+1$ has an option. On the other hand, if the element $n-\ell+2$ is assigned color c_0, the element 1 is assigned color c_0, so $n-\ell+1$ must be assigned color c_1. Next, using these latter assignments the translate $\{n-\ell, n-\ell+1, 0\}$ must yield that there is an option at $n-\ell$.

If $r=4$, then $n-(r-2)=n-2$ and $n-1$ must be assigned color c_1 as noted before. Next, $n-r-(\ell-4)=n-4-(\ell-4)=n-\ell$ which is the third element of the last collection of $\ell-2$ elements and is therefore colored c_1. Since either $n-\ell$ or $n-\ell+1$ has the option, $n-\ell+1$ must have the option. Then the translate $\{n-\ell, n-\ell+1, 0\}$ contains two distinct colors regardless of the color that is assigned to $n-\ell+1$. Suppose $n-\ell+1$ is colored some fixed color. If $n-\ell+1$ is colored c_0, then the translate $\{n-\ell+1, n-\ell+2, 1\}$ consists of two distinct colors as $n-\ell+2$ must be colored c_1. If $n-\ell+1$ is colored c_1, then the translate $\{n-\ell, n-\ell+1, 0\}$ consists of two distinct colors.

If $r\geq 5$, and there is an odd number of elements between either $n-\ell$ or $n-\ell+1$, color the option c_1 and color the elements of the form $n-\ell+i+k$ with c_0 if k is odd with $i\in\{0,1\}$ (depending on which element has the option) and c_1 if k is even. If there is an even number of elements between either $n-\ell$ or $n-\ell+1$, color the option c_0 and color the elements of the form $n-\ell+i+k$ with c_0 if k is even with $i\in\{0,1\}$ and c_1 if k is odd.

Next, the remaining elements between $n-\ell$ or $n-\ell+1$ and $n-r+2$ must be colored. Consider $n-r+1$ and the translate $\{n-r+1, n-r+2, n-r+\ell+1\}$. Either $n-r+1$ is an option or not. If it is not, the other two elements in this translate are colored the same color, so $n-r+1$ must be a different color. Next, the same reasoning is applied to $n-r$. This process continues in this way until the elements $n-\ell+1$ and $n-\ell$ are reached corresponding to translates $\{n-\ell+1, n-\ell+1, 1\}$ and $\{n-\ell, n-\ell+1, 0\}$. Hence, χ can be extended to an $S-$polychromatic coloring with two colors. □

The following results, Lemma 2.0.7 and Theorem 2.0.7 conclude the argument for finding the polychromatic number of subsets of the form $\{0, 1, \ell\}$.

LEMMA 2.0.7. *For all $n\geq 9$ or $n=5$ with n odd, $p_{\mathbb{Z}_n}(\{0,1,3\})=2$.*

Proof. Note that Theorem 2.0.6 does not apply to this case as $\ell - 2 = 1$. For the lower bound, if $n \equiv 1 \bmod 4$ consider the following coloring

$$\chi(i) = \begin{cases} c_0 & \text{if } i = 0,3,4,5 \text{ or if } i \equiv 0,1 \bmod 4 \text{ and } i \neq 0,1,2,3,4,5 \\ c_1 & \text{if } i = 1,2 \text{ or if } i \equiv 2,3 \bmod 4 \text{ and } i \neq 0,1,2,3,4,5. \end{cases}$$

First, if i is any of the elements $0,1,2,3,4,$ or 5, the translate $\{i, i+1, i+3\}$ consists of two distinct colors. If the translate in question consists of consecutive elements and $i \geq 6$ then $i \equiv j \bmod 4$, $i+1 \equiv j+1 \bmod 4$, $i+3 \equiv j+3 \bmod 4$, and so two of these elements will be differently colored. There are three translates to check: $\{n-3, n-2, 0\}, \{n-2, n-1, 1\}, \{n-1, 0, 2\}$. Then, $n-3 \equiv 2 \bmod 4$, $n-2 \equiv 3 \bmod 4$, and $n-1 \bmod 0 \bmod 4$. Therefore, $\chi(n-3) = c_1$, $\chi(n-2) = c_1$, $\chi(n-1) = c_0$. So, all translates contain two distinct colors.

If $n \equiv 3 \bmod 4$ consider the following coloring

$$\chi(i) = \begin{cases} c_0 & \text{if } i = 0,3,4,5 \text{ or if } i \equiv 1,2 \bmod 4 \text{ and } i \neq 0,1,2,3,4,5,6 \\ c_1 & \text{if } i = 1,2,6 \text{ or if } i \equiv 0,3 \bmod 4 \text{ and } i \neq 0,1,2,3,4,5,6. \end{cases}$$

First, if i is any of the elements $0,1,2,3,4,5,$ or 6, the translate $\{i, i+1, i+3\}$ consists of two distinct colors. If the translate in question consists of consecutive elements and $i \geq 7$ then $i \equiv j \bmod 4$, $i+1 \equiv j+1 \bmod 4$, $i+3 \equiv j+3 \bmod 4$, so two of these elements will be differently colored. There are three translates to check: $\{n-3, n-2, 0\}, \{n-2, n-1, 1\}, \{n-1, 0, 2\}$. Then, $n-3 \equiv 0 \bmod 4$, $n-2 \equiv 1 \bmod 4$, $n-1 \bmod 2 \bmod 4$. So, $\chi(n-3) = c_1$, $\chi(n-2) = c_0$, $\chi(n-1) = c_0$ and all translates contain two distinct colors.

If $n = 5$, consider the coloring $\chi(0) = c_0, \chi(1) = c_1, \chi(2) = c_1, \chi(3) = c_0, \chi(4) = c_0$. Then, the translates $\{0,1,3\}, \{1,2,4\}, \{2,3,0\}, \{3,4,1\}, \{4,0,2\}$ all contain two distinct colors. □

THEOREM 2.0.7. *There is a $\{0,1,\ell\}$-polychromatic coloring of $\mathbb{Z}_n$ with two colors for $\ell \neq 0,1 \in \mathbb{Z}_n$ if $\ell \neq 3,5$ and $n \neq 7$.*

Proof. First, $\ell \leq \frac{n}{2}$ by Lemma 2.0.6. Note that if n is even, there is a $\{0,1,\ell\}-$polychromatic coloring of $\mathbb{Z}_n$ for any $\ell \in \mathbb{Z}_n$ by assigining $\chi(i) = c_{i \bmod 2}$. This can always be guaranteed as the first two elements of each translate are differently colored.

If n is odd and ℓ is even, there is a $\{0, 1, \ell\}-$polychromatic coloring of $\mathbb{Z}_n$ by the following coloring:

$$\chi(i) = \begin{cases} c_{i \bmod 2} & \text{if } i \leq 2 \\ c_0 & \text{if } i \geq 3 \text{ and } i \text{ is odd} \\ c_1 & \text{if } i \geq 4 \text{ and } i \text{ is even.} \end{cases}$$

Suppose $\ell = 2m$ for some positive integer m. Then, all translates are of the form $\{i, i+1, i+2m\}$. First, assume $i = 0$, then $i+1 = 1$ and so this translate consists of two distinctly colored elements. If $i = 1$, then $i+1 = 2$ and so this translate consists of two distinctly colored elements. If $i = 2$, then $i + 2m$ is an even number and so is colored with c_1. If $\ell = n-1$, then $i + \ell = 1$ which is also colored with c_1. If $i = 3$, then $i+1 = 4$ and this translate consists of two distinctly colored elements. If $i \geq 4$, then either i or $i+1$ will be assigned c_0 and the other will be assigned c_1.
If n is odd and ℓ is odd the result follows from Theorems 2.0.5 and 2.0.6 and Lemma 2.0.7. $\square$

The following results wrap up the argument for subsets of size three with polychromatic number equal to two. The methods used in in the following results are a little different to those used previously. First, the notion of a specialized coloring on a grid is introduced with results on said coloring being presented. Then, this grid coloring is utilized to color the remaining subsets of the integers modulo n that are not captured by the previous results.

Consider the k by k' grid below

(1,1)	(1,2)	$\cdots$	(1,k')
(2,1)	(2,2)	$\cdots$	(2,k')
$\vdots$	$\vdots$	$\ddots$	$\vdots$
(k,1)	(k,2)	$\cdots$	(k,k')

Figure 2.2 k by k' grid

An *ell - tile* is a subset of a k by k' grid consisting of a 2 by 2 array of entries with the lower right entry removed:

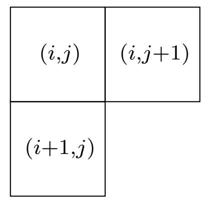

Figure 2.3 ell - tile

Note that when either $i = k$ or $j = k'$ respectively $i + 1 \equiv 1$ or $j + 1 \equiv 1$.
An *ell - tile 2-coloring* of a grid is a mapping of the entries in the collection of all possible ell - tiles in a grid to a set of two colors such that the three entries in each ell - tile consist of two colors.

LEMMA 2.0.8. *If $k, k' \geq 2$, then every $k \times k'$ grid has an ell - tile 2-coloring.*

Proof. The entries of a grid (i, j) are labeled such that i denotes the row position and j denotes the column position of the entry.
Case 1: If k and k' are even, then the entries are colored as follows:

$$\chi((i, j)) = c_{i+j \bmod 2} \text{ for any } (i, j).$$

First, consider an ell - tile of this grid consisting of $(i, j), (i, j + 1), (i + 1, j)$. If $i + j$ is even, then either both i and j are even or both are odd. If both are even, then $i + (j + 1)$ and $(i + 1) + j$ are odd. So, the ell - tile contains two distinct colors. If both are odd, $i + (j + 1)$ and $(i + 1) + j$ are even. So, the ell - tile contains two distinct colors. If $i + j$ is odd, then i and j are of opposite parity. So, $i + (j + 1)$ is even and $(i + 1) + j$ is even. Therefore, the ell - tile contains two distinct colors.
Case 2: If k and k' are odd, then the entries are colored as follows

$$\chi((i, j)) = \begin{cases} c_{i+j \bmod 2} & \text{if } (i, j) \neq (1, k'), (k, 1) \\ c_1 & \text{if } (i, j) = (1, k'), (k, 1). \end{cases}$$

First, consider an ell - tile of this grid so that none of $(i, j), (i, j + 1), (i + 1, j)$ are $(1, k')$ or $(k, 1)$. Then, the argument is the same as in case 1.

Now, assume one of $(i,j),(i,j+1),(i+1,j)$ is $(1,k')$ or $(k,1)$.

If $i=1, j=k'$, then (i,j) is colored c_1 and $(i,j+1)$ is colored c_0.

If $i=k, j=1$, then (i,j) is colored c_1 and $(i+1,j)$ is colored c_0.

If $i=1, j+1=k'$, then $(i,j+1)$ is colored c_1 and $(i+1,j)$ is colored c_0.

If $i=k, j+1=1$, then $(i,j+1)$ is colored c_1 and (i,j) is colored c_0.

If $i+1=1, j=k'$, then $(i+1,j)$ is colored c_1 and (i,j) is colored c_0.

If $i+1=k, j=1$, then $(i+1,j)$ is colored c_1 and $(i,j+1)$ is colored c_0.

Case 3: If exactly one of k and k' are odd, then without loss of generality, let k' be odd. The entries are colored as follows:

$$\chi((i,j)) = \begin{cases} c_{i+j \bmod 2} & \text{if } (i,j) \neq (\ell,k') \text{ for any } \ell \\ c_0 & \text{if } (i,j) = (\ell,k') \text{ for odd values of } \ell \\ c_1 & \text{if } (i,j) = (\ell,k') \text{ for even values of } \ell. \end{cases}$$

First, consider an ell - tile of this grid so that none of $(i,j),(i,j+1),(i+1,j)$ are (ℓ,k') for any value of ℓ. Then, the argument is the same as in case 1.

Now, assume one of $(i,j),(i,j+1),(i+1,j)$ is (i,k') for some i.

If i is even and $j=k'$, then (i,j) is colored c_1 and $(i+1,j)$ is colored c_0.

If i is odd and $j=k'$, then (i,j) is colored c_0 and $(i+1,j)$ is colored c_1.

If i is even and $j+1=k'$, then $(i,j+1)$ is colored c_1 and (i,j) is colored c_0.

If i is odd and $j+1=k'$, then $(i,j+1)$ is colored c_0 and (i,j) is colored c_1.

If $i+1$ is even and $j=k'$, then $(i+1,j)$ is colored c_1 and (i,j) and $(i,j+1)$ are both colored c_0.

If $i+1$ is odd and $j=k'$, then $(i+1,j)$ is colored c_0 and (i,j) and $(i,j+1)$ are both colored c_1.

Hence, every ell - tile contains two distinct colors.

$\square$

LEMMA 2.0.9. *$p_{\mathbb{Z}_n}(\{0,a,b\})=2$ if and only if it is not the case that $n \equiv 0 \bmod 7$ and without loss of generality $b=3a$ or $5a$ and $|\langle a\rangle|=7$.*

Proof. **Case 1:** If $b \in \langle a \rangle$,

then $b = ma$ for some positive integer m and let $k = |\langle a \rangle|$. So, $\{0, a, b\} = \{0, a, ma\}$. Thus, $p_{\mathbb{Z}_n}(\{0, a, b\}) = p_{\mathbb{Z}_n}(\{0, a, ma\}) = p_k(\{0, 1, m\}) = 2$ by Theorem 2.0.7.

Case 2: If $b \notin \langle a \rangle$, then consider the following cases.

If $|\langle a \rangle||\langle b \rangle| < n$, then there is some positive integer m so that $|\langle a \rangle||\langle b \rangle|m = n$. Consider the grid whose dimensions are given by $|\langle a \rangle| \times |\langle b \rangle|$ and whose entries of the first row are the elements of $\langle b \rangle$ arranged in the order $0, b, 2b, \ldots$ and whose entries of the first column are the elements $0, a, 2a, \ldots$ arranged in order. The remaining entries in the grid are given by $(i + 1, j + 1) = ja + ib$ where $0 \leq i \leq k' - 1$ and $0 \leq j \leq k - 1$ and $k = |\langle a \rangle|$ and $k' = |\langle b \rangle|$. Note that any translate of S is of the form $\{i, i + a, i + b\}$ and every ell - tile in the grid is of the form

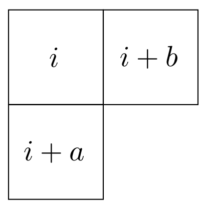

Figure 2.4 ell - tile containing i, $i + a$, $i + b$

By Lemma 2.0.8, there is a coloring for this grid. To color all of the elements in the group, $m - 1$ more grids are formed of the same dimension. The entries of these grids are obtained from the entries of the grid described above by adding an element x to each of the above entries $(i + 1, j + 1) = ja + ib$ where $0 \leq i \leq k' - 1$ and $0 \leq j \leq k - 1$ such that $1 \leq x \leq m - 1$. These grids are then colored identically.

If $|\langle a \rangle||\langle b \rangle| = n$, then the grid whose dimensions are given by $|\langle a \rangle| \times |\langle b \rangle|$ contains all group elements and is formed and colored as above.

If $|\langle a \rangle||\langle b \rangle| > n$, then the grid whose dimensions are given by $|\langle a \rangle| \times |\langle b \rangle|$ is formed as above. Next, this grid is broken up into $\gcd(|\langle a \rangle|, |\langle b \rangle|) \times \gcd(|\langle a \rangle|, |\langle b \rangle|)$ blocks of dimensions $\frac{|\langle a \rangle|}{\gcd(|\langle a \rangle|, |\langle b \rangle|)} \times \frac{|\langle b \rangle|}{\gcd(|\langle a \rangle|, |\langle b \rangle|)}$. By Lemma 2.0.8, there exists a coloring for each of these blocks. The only translates that must be checked are translates that consist of one of i, $i + a$, or $i + b$ in one block and the other elements in other blocks and translates such that either two of i, $i + a$, or $i + b$ are colored the same color in one block and the third element appears in another block. Let $r' = \frac{|\langle b\rangle|}{\gcd(|\langle a\rangle|,|\langle b\rangle|)}$ and $r = \frac{|\langle a\rangle|}{\gcd(|\langle a\rangle|,|\langle b\rangle|)}$.

If r' is odd and r is even, the two translates that must be checked are when i corresponds to entry $(r, r'-1)$ or (r, r'). If i corresponds to entry $(r, r'-1)$, its color is c_0 and $i+a$ corresponds to entry $(1, r'-1)$ which is colored c_1. If i corresponds to entry (r, r'), its colored c_0 and $i+b$ corresponds to entry $(r, 1)$ which is colored c_1. This is approached similarly if k' is even and k is odd.

If r and r' are both odd, the three translates to check are when i corresponds to entry $(1, r')$ or $(r, 1)$, or (r, r'). If i corresponds to entry $(1, r')$, its color is c_1 and $i+b$ corresponds to entry $(1, 1)$ which is colored c_0. If i corresponds to entry $(r, 1)$, its color is c_1 and $i+a$ corresponds to entry $(1, 1)$ which is colored c_0. If i corresponds to (r, r'), its color is c_0 and $i+b$ corresponds to entry $(r, 1)$ which is colored c_1.

If r and r' are both even, the only translate to check is when i corresponds to entry (r, r') which is colored c_0 and in which case $i+b$ corresponds to entry $(r, 1)$ and is colored c_1. □

Now, that the examination of the polychromatic number of three elements subsets is complete, what can be said of four element subsets and subsets of larger size as well? As mentioned in Chapter 1, there are some partial results on such subsets, however as the above argument for sets of cardinality three suggests, the problem becomes much more difficult as the size of the set increases. So, to ease into the journey of examining larger subsets subsets consisting of arithmetic progressions will first be explored.

First, consider Proposition 2.0.1 which explores the polychromatic number of a subset of size four with the progression a, $a+1$, $a+2$, $a+3$.

PROPOSITION 2.0.1. *Suppose $a \in \mathbb{Z}_n$ such that $n \geq 5$. Then,*

$$p_{\mathbb{Z}_n}(\{a, a+1, a+2, a+3\}) = \begin{cases} 2 & \text{if } n = 5 \\ 3 & \text{if } n \not\equiv 0 \bmod 4 \text{ and } n \neq 5 \\ 4 & \text{if } n \equiv 0 \bmod 4 \end{cases}$$

Proof. Note that a translate of $\{a, a+1, a+2, a+3\}$ is $\{0, 1, 2, 3\}$. So, all translates of $\{a, a+1, a+2, a+3\}$ are the translates of $\{0, 1, 2, 3\}$ and therefore, without loss of generality the following argument will be given with respect to $S = \{0, 1, 2, 3\}$.

If $n = 5$, note that the translates of S can be colored based on parity i.e. $\chi(i) = c_{i \bmod 2}$ for any $i \in \mathbb{Z}_5$. Each translate will consist of two distinct colors as the first two elements always have opposite parity since they are consecutive with the exception of $4 + S$, however this translate contains 1 and so contains two distinct colors. Next, by way of contradiction suppose there is an $S-$polychromatic coloring χ consisting of three colors. The translates of S are $S = \{0, 1, 2, 3\}$, $1+S = \{1, 2, 3, 4\}$, $2+S = \{2, 3, 4, 0\}$, $3+S = \{3, 4, 0, 1\}$, $4+S = \{4, 0, 1, 2\}$. Since there are three colors assigned to S, it is the case that either either $0, 1, 2$; $0, 1, 3$; $0, 2, 3$; or $1, 2, 3$ are distinctly colored. These cases are considered as follows.

Case 1: $\chi(0) = c_0$,$\chi(1) = c_1$,$\chi(2) = c_2$.

Then, $1 + S$ contains colors c_1 and c_2 and uncolored elements $3, 4$; $2 + S$ contains colors c_0 and c_2 and uncolored elements $3, 4$; and $3 + S$ contains colors c_0 and c_1 and uncolored elements $3, 4$. So, there is no way colors are assigned to 3 and 4 so that each translate consists of three distinct colors.

Case 2: $\chi(0) = c_0$,$\chi(1) = c_1$,$\chi(3) = c_2$.

Then, $1 + S$ contains colors c_1 and c_2 and uncolored elements $2, 4$; $2 + S$ contains colors c_0 and c_2 and uncolored elements $2, 4$; and $4 + S$ contains colors c_0 and c_1 and uncolored elements $2, 4$. So, there is no way colors are assigned to 2 and 4 so that each translate consists of three distinct colors.

Case 3: $\chi(0) = c_0$,$\chi(2) = c_1$,$\chi(3) = c_2$.

Then, $1 + S$ contains colors c_1 and c_2 and uncolored elements $1, 4$; $3 + S$ contains colors c_0 and c_2 and uncolored elements $1, 4$; and $4 + S$ contains colors c_0 and c_1 and uncolored elements $1, 4$. So, there is no way colors are assigned to 1 and 4 so that each translate consists of three distinct colors.

Case 4: $\chi(1) = c_0$,$\chi(2) = c_1$,$\chi(3) = c_2$.

Then, $2 + S$ contains colors c_1 and c_2 and uncolored elements $0, 4$; $3 + S$ contains colors c_0 and c_2 and uncolored elements $0, 4$; and $4 + S$ contains colors c_0 and c_1 and uncolored elements $0, 4$. So, there is no way colors are assigned to 0 and 4 so that each translate consists of three distinct colors.

Hence, $p_G(S) = 2$.

Next, by Proposition 2.0.2 with $n \not\equiv 0 \bmod 4$ and $n \neq 5$, the Frobenius number of $A_0(4)+A_1(3) = n$ is 5, and so every such n can be written in the form $n = A_0(4) + A_1(3)$ and thus $p_{\mathbb{Z}_n}(S) \geq 3$. The upper bound is also 3 by the fact that $4 \nmid n$ as n is not a multiple of 4 and this is a requirement by Remark 2.0.2.

Finally, in the case that $n \equiv 0 \bmod 4$ and $n \neq 5$, $p(S) = 4$ follows by Proposition 2.0.3. $\square$

This result rests upon the following two results. Proposition 2.0.2 gives a lower bound for the polychromatic number of any subset of any size with arithmetic progression $a, a+1, a+2, \ldots a+m-1$ as its elements and Proposition 2.0.3 yields precisely which such subsets have $p(S) = |S|$. Before the proof of Proposition 2.0.2 can be presented, there are a few definitions needed for the argument that will be introduced.

Let $S = \{0, 1, 2, \ldots, m-1\}$ where $S \subset \mathbb{Z}_n$. Let $T_m(0) = S$ and for any $j \in \mathbb{Z}_n$, $T_m(j) = \{j, j+1, j+2, \ldots, j+(m-1)\}$ represents any translate of S. Consider the following coloring of the elements of $\mathbb{Z}_n$ which will be referred to as a *block coloring of* $\mathbb{Z}_n$. First, break the n elements of $\mathbb{Z}_n$ up into a collection of disjoint sets consisting each of a nonnegative number of elements. These sets must be broken up in such a way that the elements of $\mathbb{Z}_n$ appear in consecutive order $0, 1, 2, 3, \ldots, n-1$. Also, in these sets the position of each element is unique. Finally each set consists of either $m-x, m-(x-1), m-(x-2), \ldots, m-1, m$ elements of $\mathbb{Z}_n$. These sets will hereafter be referred to as *blocks of length* i where $m-x \leq i \leq m$.

Next, the elements of these blocks of length i are colored. Starting with the blocks of length $m-x$, assign colors $c_0, c_1, c_2, \ldots, c_{m-(x+1)}$ in order respectively to the $m-x$ elements of all such blocks. Then, in the blocks of length $m-(x-1)$, choose one of the elements to remain uncolored (this element will be colored later). After this element is chosen, assign colors $c_0, c_1, c_2, \ldots, c_{m-(x+1)}$ in order respectively to the remaining elements of all such blocks. Next, in the blocks of length $m-(x-2)$, choose two elements to remain uncolored (these elements will be colored later) and one of these uncolored elements must be in the same position as the uncolored element in blocks of length $m-(x-1)$. After these two elements have been specified, assign colors $c_0, c_1, c_2, \ldots, c_{m-(x+1)}$

in order respectively to the $m-x$ remaining elements of all such blocks. Continue in this way coloring blocks of length $i = m-(x-3), \ldots, m-1, m$ so that in each block, the position of any uncolored element is also the position of one of the x uncolored elements in blocks of length m. That is, the collection of positions of uncolored elements in blocks with lengths smaller than m are proper subsets of the collection of positions of uncolored elements in blocks with length m. The remaining $m-x$ elements are always assigned colors $c_0, c_1, c_2, \ldots, c_{m-(x+1)}$ in order respectively. Finally, assign the color c_0 to all the uncolored elements. When a block and a translate $T_m(j)$ share elements in common, the block *covers* some of the elements of $T_m(j)$.

PROPOSITION 2.0.2. *If there exist* $0 \leq A_0, A_1, \ldots, A_x \in \mathbb{Z}$ *such that* $n = A_0m + A_1(m-1) + A_2(m-2) + \ldots + A_x(m-x)$ *where* $m-1 \geq x \geq 0$ *is the least such number so that* n *can be decomposed into such a linear combination, then the block coloring of* $\mathbb{Z}_n$ *is yields* $p_{\mathbb{Z}_n}(\{a, a+1, a+2, \ldots, a+m-1\}) \geq m-x$.

Proof. First, note that a translate of $\{a, a+1, a+2, \ldots, a+m-1\}$ is $\{0, 1, 2, \ldots, m-1\}$. Also, $m-1 \leq n$ otherwise there are not enough elements to put in S.

So, without loss of generality the following argument will be given so that $S = \{0, 1, 2, \ldots m-1\}$. Suppose $n = A_0m + A_1(m-1) + A_2(m-2) + \ldots + A_x(m-x)$ where $m-1 \geq x \geq 0$ is the least such number so that n can be decomposed into such a linear combination. Then, it is possible to break the elements of $\mathbb{Z}_n$ up into blocks of length i for $m-x \leq i \leq m$. For all $j \in \mathbb{Z}_n$, $T_m(j)$ is the translate $\{j, j+1, j+2, j+3, j+4, \ldots, j+i-1, j+i, j+i+1, \ldots, j+m-x, j+m-(x-1), j+m-(x-2), \ldots, j+m-2, j+m-1\}$. So, what must be shown is that the elements of each $T_m(j)$ can be colored with $m-x$ distinct colors. Consider the colors $c_0, c_1, c_2, c_3, \ldots, c_{j-1}, c_j, c_{j+1}, \ldots, c_{m-(x+1)}$ and blocks of lengths $m, m-1, m-2, \ldots m-x$. Align the blocks by listing the zeroth position to be the left endpoint, the next position over being the first position, and so on until the right endpoint is the $m-(x+1)$th, $m-(x+2)$th, $\ldots$, or $m-1$ th position.

Now, color the elements as per the definition of block coloring leaving the first uncolored element in the blocks of length $m-(x-1)$ to be the right endpoint position, the uncolored elements of blocks of length $m-(x-2)$ to be in the right endpoint position and one over to the left from the right endpoint, and so on until the x uncolored elements of blocks of length m are the x elements in from the right endpoint. These elements are resultantly colored c_0. The block coloring is as pictured below.

element:	d	$d+1$	$d+2$		$d+m-(x+1)$	$d+m-(x+2)$		$d+m-1$
color:	c_0	c_1	c_2	$\cdots$	$c_{m-(x+1)}$	c_0	$\cdots$	c_0
position:	0th	1st	2nd		$m-(x+1)$th	$m-(x+2)$th		$m-1$th

Figure 2.5 Block Coloring of $\mathbb{Z}_n$

If all of the elements of $T_m(j)$ appear in one entire block together, then $T_m(j)$ most assuredly contains $m-x$ distinctly colored elements. If all of the elements of $T_m(j)$ appear in a block except for one entry, then this missing entry because of the alignment must be the right or left endpoint. If it is the right endpoint then all $m-x$ colors appear. If it is the left endpoint, all colors but c_0 appear. However, the next block over that includes elements of $T_m(j)$ which starts covering as soon as the first block ends must have the left endpoint in place which is colored with c_0 and so color c_0 appears and thus all $m-x$ colors appear. Similarly, without loss of generality, say a block covering $T_m(j)$ misses its first i slots. Then all the colors $c_0, c_1, \ldots, c_{m-i-1}$ are guaranteed to appear in the remaining $m-i$ elements of $T_m(j)$ assuming that $x \geq i$ as in this case $m-x \leq m-i$. If $x < i$ and consequently $m-x > m-i$, there are $i-x$ colors missing : $c_{m-i}, \ldots, c_{m-(x+1)}$. However, a block must cover the first i slots and in the worst case scenario this block has length m. In this case, the positions in the block that are covering the elements in the translate are the $m-1-i$th, $m-i$th, $m-i+1$th, …, $m-(x+1)$th, …, $m-3$th, $m-2$th, and $m-1$th positions. These elements are colored with $c_{m-(i+1)}$, c_{m-i}, …, $c_{m-(x+1)}$, c_0, …, c_0 respectively and so all $m-x$ colors must appear in this translate. If $T_m(j)$ is covered by three blocks or more, then this means at least one entire block corresponds with the entries in $T_m(j)$ and so $m-x$ distinct colors must appear. □

PROPOSITION 2.0.3. *For any $a \in \mathbb{Z}_n$, $p_{\mathbb{Z}_n}(\{a, a+1, a+2, \ldots, a+m-1\}) = m$ if and only if $n \equiv 0 \bmod m$.*

Proof. First, note that $\{0, 1, 2, \ldots, m-1\}$ is a translate of $\{a, a+1, a+2, \ldots, a+m-1\}$, so without loss of generality let $S = \{0, 1, 2, \ldots, m-1\}$. Also, $m - 1 \leq n$ otherwise there are not enough elements to put in S.

Let χ be a an $S-$polychromatic coloring of $\mathbb{Z}_n$. Next, what will be show is that $p_G(S) = m \implies \chi(i) = \chi(i+m)$. So, suppose $p_G(S) = m$, then every translate of S which is of the form $i + S = \{i, i+1, i+2, \ldots, i+(m-1)\}$ consists of m distinct colors under χ. Let $c_0, c_1, \ldots, c_{m-1}$ denote these m distinct colors. Without loss of generality, assign $\chi(i+j) = c_j$ for all $0 \leq j \leq m-1$. Note that $\{i+1, i+2, i+3, \ldots, i+(m-1), i+m\}$ is a translate. Then, the elements $i+1, i+2, i+3, \ldots, i+(m-1)$ are colored with the colors c_1,c_2, c_3, $\ldots$, c_{m-1} respectively. This forces $i+m$ to be colored with color c_0. Therefore, the n elements of G are being broken up into m color classes with the same number of elements in each class. That is $m|n$. This also follows by Remark 2.0.2. Next, assume $n \equiv 0 \bmod m$. Define

$$\chi(i) = c_{i \bmod m}$$

to be a coloring which assigns the colors $c_0, \ldots, c_{m-1}$ to the elements of G. The translates of S are of the form $i+S = \{i, i+1, i+2, \ldots, i+(m-1)\}$ for all $i \in G$ and obviously every element belongs to a distinct congruence class mod$m$. So, under χ with all translates consisting of consecutive elements mod$m$, each translate will consist of m distinct colors. Hence, χ is an $S-$polychromatic coloring and the result follows. □

Corollary 2.0.3. *For any $a \in \mathbb{Z}_n \setminus \{0\}$, $b \in \mathbb{Z}_n$, $p_{\mathbb{Z}_n}(\{b, b+a, b+2a, \ldots, b+(m-1)a\}) = m$ if and only if $|\langle a \rangle| \equiv 0 \bmod m$.*

Proof. First, note that $\{0, a, 2a, \ldots, (m-1)a\}$ is a translate of {b, b+a,b+2a, ..., b+(m-1)a}, so without loss of generality, let $S = \{0, a, 2a, \ldots, (m-1)a\}$. Also, $m-1 \leq n$ otherwise there are not enough elements to put in S.

By Theorem 2.0.1, $p_{\mathbb{Z}_n}(S) = p_{|\langle a \rangle|}(\{0, 1, 2, \ldots, (m-1)\})$. The result then follows by Proposition 2.0.3. □

On a final note for subsets consisting of arithmetic progressions in $\mathbb{Z}_n$, these subsets can also be formed in the integers. Recall the following result from [6].

LEMMA 2.0.10. *[6] Let* $a, b, c, k, q \in \mathbb{Z}$ *with* $0 < a < b < c$, $\gcd(a, b, c) = 1$, $k, q \geq 1$, *and* $m = c - a + b$. *Let* $S = \{0, ka, kb, kc\}$, $S_1 = \{0, a, b, c\}$, $S_2 = \{0, b - a, b, 2b - a\}$. *Then*

(i) $p_{\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$.

(ii) $p_{\mathbb{Z}}(S_1) \geq p_{\mathbb{Z}_m}(S_1)$.

(iii) $p_{\mathbb{Z}_m}(S_1) = p_{\mathbb{Z}_m}(S_2)$.

(iv) If $\gcd(k, q) = 1$, *then* $p_{\mathbb{Z}_q}(S) = p_{\mathbb{Z}_q}(S_1)$.

Part (ii) of this theorem can be generalized to arithmetic progressions of any size in $\mathbb{Z}_n$ and this is what is accomplished in Theorem 2.0.8.

THEOREM 2.0.8. $p_{\mathbb{Z}_n}(S) \leq p_{\mathbb{Z}}(S)$ *for any* $S = \{0, 1, 2, \ldots, m - 1\}$ *with* $m \geq 2$.

Proof. Note that an $S-$polychromatic coloring of $\mathbb{Z}_n$ involves coloring elements with more restrictions than coloring the translates of S with respect to $\mathbb{Z}$. That is, an $S-$polychromatic coloring of $\mathbb{Z}_n$ involves coloring n translates in which every element appears at total of m times while an $S-$polychromatic coloring of $\mathbb{Z}$ involves coloring a countably infinite number of translates where each element appears in m translates total and each translate is consecutive. Now, consider the following coloring with colors $c_0, c_1, c_2, \ldots, c_{m-1}$:

$$\chi(i) = c_j \text{ for all } j \in \mathbb{Z} \text{ if and only if } j \equiv i \bmod m.$$

Since all translates of $\mathbb{Z}$ are consecutive and consist of m elements each translates will always have m distinct values each of a different residue modulus m and so each translate will contain m distinct colors. Also, it is not possible to color the translates with more than m colors as each translate only contains m elements. Thus, $m \leq p_{\mathbb{Z}}(S) \leq m \implies p_{\mathbb{Z}}(S) = m$.

Finally, it can be seen easily that $p_{\mathbb{Z}_n}(S) \leq m$ as each of the translates of S only contains m elements. Therefore, $p_{\mathbb{Z}_n}(S) \leq p_{\mathbb{Z}}(S) = m$. □

Another subset of interest in $\mathbb{Z}_n$ are subsets with $p(S) = |S|$. Recall that Theorem 2.0.4 gives a divisibility requirement. For subsets of odd prime size it is possible to classify these subsets completely, for subsets of composite size the problem is much more complicated and remains open. The proof of the structure of subsets of odd prime size with $p(S) = |S|$ follows and requires the following notation and Lemma 2.0.11.

DEFINITION 2.0.1. *Let $S = \{0, a_1, \ldots, a_{\ell-1}\} \subset \mathbb{Z}_n$. Let e_i denote the equivalence classes* $\bmod\, m$ *for all $0 \leq i \leq m-1$ for some integer m. Let $|S|_{e_i}$ denote the number of elements in S that belong to class e_i.*

LEMMA 2.0.11. *If $p(S) = |S|$ for $S = \{0, a_1, \ldots, a_{\ell-1}\}$, then $|S|_{e_0} = |S|_{e_i}$ for all $0 \leq i \leq |S|-1$*

Proof. By way of contradiction, there is some m so that $|S|_{e_0} \neq |S|_{e_m}$. First, let $|S|_{e_0} < |S|_{e_m}$. Let $|S|_{e_0} = k$ and $|S|_{e_m} = k + r$ for some $r \geq 1$. Because $p(S) = |S|$, $S \oplus T = \mathbb{Z}_n$ for some $T \subset \mathbb{Z}_n$. So, $n = |S|j$ for some integer j where $|T| = j$. If any element in S_{e_i} is added to any element t_z in T for all $0 \leq z \leq j-1$, k elements of the form $t_z + i$ result for all $0 \leq i \leq |S|-1$ which does not include $i = m$. That is, $(|S|-1)(k)(j)$ elements result. If any element in S_{e_m} is added to any element t_z in T for all $0 \leq z \leq j-1$, $k+r$ elements of the form $t_z + m$ result. That is, $(k+r)(j)$ elements result. Again because $S \oplus T = \mathbb{Z}_n$, the sum $(|S|-1)(k)(j) + (k+r)(j)$ should be n exactly, however, the following is obtained:

$$\begin{aligned}(|S|-1)(k)(j) + (k+r)(j) &= |S|kj - kj + kj + rj \\ &= |S|kj + rj \\ &= (|S|k + r)j.\end{aligned}$$

So, $n = (|S|k + r)j$ implies $|S| = |S|k + r$ which in turn implies $k = 1$ and $r = 0$ which is a contradiction.

Similarly, let $|S|_{e_0} > |S|_{e_m}$. Let $|S|_{e_0} = k$ and $|S|_{e_m} = k - r$ for some $r \geq 1$. Because $p(S) = |S|$, $S \oplus T = \mathbb{Z}_n$ for some $T \subset \mathbb{Z}_n$. So, $n = |S|j$ for some integer j where $|T| = j$. If any element in S_{e_i} is added to any element t_z in T for all $0 \leq z \leq j-1$, k elements of the form $t_z + i$ result for all $0 \le i \le |S|-1$ which does not include $i = m$. That is, $(|S|-1)(k)(j)$ elements result. If any element in S_{e_m} is added to any element t_z in T for all $0 \le z \le j-1$, $k-r$ elements of the form $t_z + m$ result. That is, $(k-r)(j)$ elements result. Again because $S \oplus T = \mathbb{Z}_n$, the sum $(|S|-1)(k)(j) + (k-r)(j)$ should be n exactly, however, the following is obtained:

$$\begin{aligned}(|S|-1)(k)(j) + (k-r)(j) &= |S|kj - kj + kj - rj \\ &= |S|kj - rj \\ &= (|S|k - r)j.\end{aligned}$$

So, $n = (|S|k-r)j$ implies $|S| = |S|k - r$ which in turn implies $k = 1$ and $r = 0$ which is a contradiction. □

One can make the observation that if the order of set S is an odd prime and $p(S) = |S|$, then $|S|$ divides the sum of the elements in S.

OBSERVATION 2.0.1. *If $S = \{0, a_1, \ldots, a_{\ell-1}\} \subset \mathbb{Z}_n$ such that ℓ is an odd prime and $p(S) = \ell$, then $\ell | a_1 + \ldots + a_{\ell-1}$.*

Proof. By Lemma 2.0.11, it is either the case that $|S|_{e_0} = 1$ or $|S|_{e_0} = \ell$. If the latter of these options is true, then a factor of ℓ can be pulled out of every element in S and so clearly the desired result hold.

If the former of the above options is true, without loss of generality let $a_i \equiv e_i \bmod \ell$. Then, $e_1 + \ldots + e_{\ell-1} = \sum_{i=1}^{\ell-1} i = \frac{(\ell-1)(\ell)}{2}$. Since ℓ is odd, $\ell - 1$ must be even and so ℓ must divide this sum. □

THEOREM 2.0.9. *Let $S \subset \mathbb{Z}_n$ and $|S| = \ell$ be an odd prime so that $p(S) = \ell$ if and only if for any $a \in \mathbb{Z}_n$, $S = \{a, a + \ell^j m_1, a + \ell^j m_2, \ldots, a + \ell^j m_{\ell-1}\}$ where $\ell^{j+1} | n$, $j \ge 0$, and $0 \not\equiv m_1 \not\equiv m_2 \not\equiv \ldots \not\equiv m_{\ell-1} \bmod \ell$.*

Proof. First, note that $\{0, \ell^j m_1, \ell^j m_2, \ldots, \ell^j m_{\ell-1}\}$ is a translate of S. So, without loss of generality, the following argument will be given as if $S = \{0, \ell^j m_1, \ell^j m_2, \ldots, \ell^j m_{\ell-1}\}$.

Every element of $\mathbb{Z}_n$ can be written in the form $t \cdot \ell^j + r$ where t is some positive integer such that $0 \le t \le |\langle \ell^j \rangle| - 1$ and $0 \le r \le \ell^j - 1$. Let χ be the coloring function given by $\chi(t \cdot \ell^j + r) = c_{t \bmod \ell}$. Under this coloring every translate of S contains ℓ distinct colors as every translate is of the form

$$\{t \cdot \ell^j + r, (t + m_1) \cdot \ell^j + r, (t + m_2) \cdot \ell^j + r, \ldots, (t + m_{\ell-1}) \cdot \ell^j + r\}$$

and $\chi((t + m_i) \cdot \ell^j + r) = c_{t+m_i \bmod \ell}$ for $1 \le i \le \ell - 1$.

Next, assume $p(S) = \ell$ and $S = \{0, a_1, \ldots, a_{\ell-1}\}$. Then, Lemma 2.0.11, $|S|_{e_0}||S|$ i.e. $\ell = |S|_{e_0} k$ for some integer k. Since ℓ is an odd prime either $|S|_{e_0} = 1$ or $|S|_{e_0} = \ell$.

If $|S|_{e_0} = 1$, then each element in S belongs to a distinct class mod$|S|$, without loss of generality, set $a_1 = \ell^0 m_1, \ldots a_{\ell-1} = \ell^0 m_{\ell-1}$ with $\ell^{0+1}|n$.

If $|S|_{e_0} = \ell$, then every element in S belongs to e_0 and the highest power of ℓ is pulled from each element to obtain the following:

$$0 < a_1 = \ell^{b_1} c_1 < a_2 = \ell^{b_2} c_2 < \ldots < a_{\ell-1} = \ell^{b_{\ell-1}} c_{\ell-1} \le n - 1.$$

Let $b_q = \min\{b_1, \ldots, b_{\ell-1}\}$. By Remark 2.0.4 with a multiple of ℓ^{b_q},

$$\ell = p(\{0, \ell^{b_1} c_1, \ell^{b_2} c_2, \ldots, \ell^{b_{\ell-1}} c_{\ell-1}\}) \le p(\{0, \ell^{b_1 - b_q} c_1, \ell^{b_2 - b_q} c_2, \ldots, \ell^{b_{\ell-1} - b_q} c_{\ell-1}\}) = \ell.$$

By Lemma 2.0.11, either $|S|_{e_0} = 1$ or $|S|_{e_0} = \ell$, however the latter option is impossible because at least one element is not divisible by ℓ. This means $b_q = b_1 = \ldots = b_{\ell-1}$.

Note that the translates of $\{0, \ell^{b_q} c_1, \ldots, \ell^{b_q} c_{\ell-1}\}$ that belong to the subgroup generated by ℓ^{b_q} are of the form:

$$\{\ell^{b_q} y, \ell^{b_q}(y + c_1), \ell^{b_q}(y + c_2), \ldots, \ell^{b_q}(y + c_{\ell-1})\}$$

for $0 \le y \le |\langle \ell^{b_q} \rangle|$ and of course $p(\{\ell^{b_q} y, \ell^{b_q}(y + c_1), \ell^{b_q}(y + c_2), \ldots, \ell^{b_q}(y + c_{\ell-1})\}) = \ell$. By Theorem 2.0.1, coloring these translates in $\mathbb{Z}_n$ is equivalent to coloring $\{0, c_1, \ldots, c_{\ell-1}\}$ in $\mathbb{Z}_{|\langle \ell^{b_q} \rangle|}$. Since there is a $\{0, c_1, \ldots, c_{\ell-1}\}-$polychromatic coloring consisting of ℓ colors, it must be that $\ell||\langle \ell^{b_q} \rangle|$. Also, with $\ell||\langle \ell^{b_q} \rangle|$ and $|\langle \ell^{b_q} \rangle| = \frac{n}{\gcd(\ell^{b_q}, n)}$, this implies $\gcd(\ell^{b_q}, n) = \ell^{b_q}$ as the highest power of ℓ that can be factored from n must be larger than b_q as ℓ divides n, ℓ^{b_q}, and $|\langle \ell^{b_q} \rangle|$. So, $n = \ell^{b_q} |\langle \ell^{b_q} \rangle| \equiv 0 \bmod \ell^{b_q+1}$. Hence, the proof is complete. □

REMARK 2.0.6. *The backward direction is repeatable if ℓ is a composite number, however the forward direction is not necessarily repeatable.*

Corollary 2.0.4. *Let $S \subset \mathbb{Z}_n$ such that $n \geq 3$, and $|S| = \ell$ is an odd prime. Then, $n - r \leq ex(\mathbb{Z}_n, S)$ if and only if*

$S = \{a, a + \ell^j m_1, a + \ell^j m_2, \ldots, a + \ell^j m_{\ell-1}\}$ where $0 \not\equiv m_1 \not\equiv m_2 \not\equiv \ldots \not\equiv m_{\ell-1} \bmod \ell$ where $n = \ell r$.

An interesting problem to consider in the future is what subsets of $\mathbb{Z}_n$ have $p(S) = |S|$ where $|S|$ is a composite number. Lemma 2.0.11 yields that such a subset must contains an equal number of elements per each equivalence class mod$|S|$. So, one case is if $|S|_{e_i} = 1$ for all $0 \leq i \leq |S| - 1$. This case is explored in the following result.

PROPOSITION 2.0.4. *Suppose $S = \{b, b + a_1, b + a_2, \ldots, b + a_{\ell-1}\} \subset \mathbb{Z}_n$ where $b, a_1 \neq a_2 \neq \ldots \neq a_{\ell-1} \in \mathbb{Z}_n$, $|S|_{e_i} = 1$ for all $0 \leq i \leq |S| - 1$, and $|S|$ is a composite number. Then, $p(S) = |S|$.*

Proof. Since $|S|_{e_i} = 1$ for all $0 \leq i \leq |S| - 1$, $b \not\equiv b + a_1 \not\equiv b + a_2 \not\equiv \ldots \not\equiv b + a_{\ell-1} \bmod |S|$. Consider the assignment

$$\chi(j) = c_{j \bmod |S|}.$$

Since every translate is of the form $d + S = \{d + b, d + b + a_1, d + b + a_2, \ldots, d + b + a_{\ell-1}\}$, every translate contains $|S|$ distinct colors as the elements all belong to distinct congruence classes mod$|S|$. □

If $|S|$ is composite, $|S|_{e_0} > 1$, and $p(S) = |S|$, then finding the form of such subsets is still an open problem.

It is also worth noting the polychromatic number when S is a subgroup of $\mathbb{Z}_n$.

PROPOSITION 2.0.5. *If $S \leq \mathbb{Z}_n$, then $p_{\mathbb{Z}_n} = |\langle a \rangle|$ where $S = \langle a \rangle$.*

Proof. Since $\mathbb{Z}_n$ is a cyclic group, so are all of its subgroups. That is, there is some $a \in \mathbb{Z}_n$ so that $S = \langle a \rangle$. So, $S = \{0, a, 2a, \ldots, (\langle a \rangle - 1)a\}$. Then, by Theorem 2.0.1 $p_{\mathbb{Z}_n}(S) = p_{\mathbb{Z}_{|\langle a \rangle|}}(\{0, 1, 2, \ldots, |\langle a \rangle| - 1\})$. Finally, by Proposition 2.0.4, $p_{\mathbb{Z}_{|\langle a \rangle|}}(\{0, 1, 2, \ldots, |\langle a \rangle| - 1\}) = |\langle a \rangle|$.

Note that this also follows from Lemma 1.1.5. □

2.0.2 Results on General Finite Abelian Groups

First, note that if there is an $S-$polychromatic coloring of any finite abelian group consisting of any value of colors between 1 and $p_G(S)$.

OBSERVATION 2.0.2. *If $S \subset G$ and $p_G(S) = m$ then there exists an $S-$polychromatic coloring with $m - j$ colors for all $0 \leq j \leq m - 1$.*

Proof. Every translate can be divided up so that at least m elements belong to m distinct color classes. Remove j distinct color classes by taking all the elements in these color classes and putting them in one of the remaining $m - j$ color classes. □

Also, if the polychromatic number of a subset of G is known, then an upper bound on the polychromatic number for any subset of this original subset is obtained.

THEOREM 2.0.10. *Let G be a finite abelian group. Let $S \subset G$. Then, $p(S') \leq p(S)$ for any $S' \subseteq S$.*

Proof. If $S = S'$, the statement is obviously true. Without loss of generality, suppose the identity 1 belongs to S and S', then $S' = \{1, a_{j_1}, \ldots, a_{j_{\ell-1}}\} \subset S = \{1, a_1, a_2, \ldots, a_{k-1}\}$ and let χ' be an $S-$polychromatic coloring of S' with ℓ colors, then

$$S = \{\underbrace{\underbrace{1}_{c_0}, \underbrace{a_{j_1}}_{c_1}, \ldots, \underbrace{a_{j_{\ell-1}}}_{c_{\ell-1}}}_{S'}, \underbrace{\_, \_, \_, \ldots, \_}_{\text{remaining elements}}\}.$$

With this partitioning, χ' extends to an $S-$polychromatic coloring of S and its translates by simply assigning all remaining elements to at least one color class that has already been used. □

The following results generalize Theorem 2.0.1, Theorem 2.0.2, and Corollary 2.0.1 to any finite abelian group.

THEOREM 2.0.11. *The Reduction Theorem for Finite Abelian Groups. Let $t \geq 2$, $m_1, m_2, \ldots, m_{t-1} \in \mathbb{Z}$ are distinct, and $a \in G \setminus \{0\}$ where G is any finite abelian group.*
Then $p_G(\{0, m_1a, m_2a, \ldots, m_{t-1}a\}) = p_{\mathbb{Z}_{|\langle a \rangle|}}(\{0, m_1, m_2, \ldots, m_{t-1}\})$.

Proof. For any $x \in G$, $x + \{0, m_1a, m_2a, \ldots, m_{t-1}a\} \subseteq x + \langle a \rangle$. Therefore, partition the translates of $\{0, m_1a, m_2a, \ldots, m_{t-1}a\}$ among the cosets of $\langle a \rangle$. If $r = [G : \langle a \rangle]$, then the r collections of translates are colored identically due to the following isomorphism between elements in the translates: $\phi : y + \langle a \rangle \to z + \langle a \rangle; y + ia \mapsto z + ia$ for any nonnegative integer i. In particular, color the translates whose elements belong to $\langle a \rangle$. These translates are of the form $\{ja, (m_1 + j)a, (m_2+j)a, \ldots, (m_{t-1}+j)a\}$. Coloring these translates in G is identical to coloring the translates $\{j, m_1 + j, m_2 + j, \ldots, m_{t-1} + j\}$ in $\mathbb{Z}_{|\langle a \rangle|}$ via the following isomorphism between elements in the translates: $\langle a \rangle \to \mathbb{Z}_{|\langle a \rangle|}; ja \mapsto j$ for any nonnegative integer j. □

THEOREM 2.0.12. *Let G be any finite abelian group. For all $a \neq b \in G$ where $a' \in G$ such that $\{0, a'\}$ is a translate of $\{a, b\}$,*

$$p_G(\{a,b\}) = \begin{cases} 1 & \text{if } |\langle a' \rangle| \text{ is odd} \\ 2 & \text{if } |\langle a' \rangle| \text{ is even.} \end{cases}$$

Proof. Let $S = \{a, b\}$ for any $a \neq b \in G$. Then, consider either $-a + S = S - a$ or $-b + S = S - b$ and so let a' be either $b - a$ or $a - b$. By Theorem 2.0.11, $p_G(S) = p_G(\{0, a'\}) = p_{\mathbb{Z}_{|\langle a' \rangle|}}(\{0, 1\})$. By Lemma 2.0.1,

$$p_{\mathbb{Z}_n}(S) = \begin{cases} 1 & \text{if } |\langle a' \rangle| \text{ is odd} \\ 2 & \text{if } |\langle a' \rangle| \text{ is even.} \end{cases}$$

□

Corollary 2.0.5. *Let G be any finite abelian group. For all $a \neq b \in G$ where $a' \in G$ such that $\{0, a'\}$ is a translate of $\{a, b\}$,*

$$ex(G, \{a,b\}) \geq \begin{cases} 0 & \text{if } |\langle a' \rangle| \text{ is odd} \\ |G| - \frac{|G|}{2} & \text{if } |\langle a' \rangle| \text{ is even.} \end{cases}$$

A group G is *finitely generated* if there is a finite subset A of G such that $G = \langle A \rangle$. Note that any finite abelian group is finitely generated as one can simply take all group elements as the set of generators. Consider the result from abstract algebra known as the fundamental Theorem of Finitely Generated Abelian Groups.

THEOREM 2.0.13. *[10] Let G be a finitely generated abelian group. Then*

(i)

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s},$$

for some integers $r, n_1, n_2, \ldots, n_s$ satisfying the following conditions:

(i) $r \geq 0$ *and* $n_j \geq 2$ *for all* j, *and*

(ii) $n_{i+1} | n_i$ *for* $1 \leq i \leq s-1$

(ii) the expression in (i) is unique: if $G \cong \mathbb{Z}^r \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s}$, *where* t *and* $m_1, m_2, \ldots, m_u$ *satisfy* $t \geq 0$, $m_j \geq 2$ *for all* j *and* $m_{i+1} | m_i$ *for* $1 \leq i \leq u-1$, *then* $t = r$, $u = s$, *and* $m_i = n_i$ *for all* i.

Essentially, what this result states is that any finite abelian group is a product of copies of the integers and various sizes of the integers modulo n. Next, it should be noted that any results on the integers modulo n can be extended to direct products of the integers modulo n and specific subsets.

Corollary 2.0.6. *Suppose* $G \cong \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_j} \times \cdots \times \mathbb{Z}_{q_r}$, *and one of the following is true*

(i) Each q_j *is distinct and*

$$S = \{\underbrace{(0, 0, \ldots, \underbrace{0}_{jth\ component}, \ldots, 0)}_{r\ components}, \underbrace{(0, 0, \ldots, \underbrace{a_1}_{jth\ component}, \ldots, 0)}_{r\ components}, \ldots, \underbrace{(0, 0, \ldots, \underbrace{a_{t-1}}_{jth\ component} \ldots, 0)}_{r\ components}\},$$

then $p_G(S) = p_{\mathbb{Z}_{q_j}}(\{0, a_1, \ldots, a_{t-1}\})$

(ii) $q_1 = \ldots = q_j = \ldots = q_r$ *and*

$$S = \{ \underbrace{(0, 0, \ldots, \underbrace{0}_{jth\ component}, \ldots, 0)}_{r\ components}, \underbrace{(a_1, a_1, \ldots, \underbrace{a_1}_{jth\ component} \ldots, a_1)}_{r\ components},$$
$$\ldots, \underbrace{(a_{t-1}, a_{t-1}, \ldots, \underbrace{a_{t-1}}_{jth\ component} \ldots, a_{t-1})}_{r\ components} \},$$

then $p_G(S) = p_{\mathbb{Z}_{q_j}}(\{0, a_1, \ldots, a_{t-1}\})$

(iii) There is some subcollection of positive integers $i_1, \dots, i_n \in \{q_1, q_2, \dots, q_j, \dots, q_r\}$ *so that* $i_1 = \dots = i_n$ *and*

$$S = \{\underbrace{(0, 0, \dots, 0, \dots, 0)}_{r\,components},$$

$$\underbrace{(0, \dots, 0, \underbrace{a_1}_{i_1 th\ component}, 0, \dots, 0, \underbrace{a_1}_{i_2 th\ component}, 0, \dots, 0, \underbrace{a_1}_{i_n th\ component}, 0, \dots, 0)}_{r\,components}, \dots,$$

$$\underbrace{(0, \dots, 0, \underbrace{a_{t-1}}_{i_1 th\ component}, 0, \dots, 0, \underbrace{a_{t-1}}_{i_2 th\ component}, 0, \dots, 0, \underbrace{a_{t-1}}_{i_n th\ component}, 0, \dots, 0)\}}_{r\ components}$$

then $p_G(S) = p_{\mathbb{Z}_{q_j}}(\{0, a_1, \dots, a_{t-1}\})$

Proof. For (i), the group isomorphism $\varphi : G \to \mathbb{Z}_{q_j}; \underbrace{(0, 0, \dots, \underbrace{a_k}_{j\text{th component}}, \dots, 0)}_{r\text{components}} \to a_k$ yields the result. For (ii), the group isomorphism $\varphi : G \to \mathbb{Z}_{q_j}; \underbrace{(a_k, a_k, \dots, \underbrace{a_k}_{j\text{th component}} \dots, a_k)}_{r\text{components}} \to a_k$ yields the result. For (iii), the group isomorphism

$$\varphi : G \to \mathbb{Z}_{q_j}; \underbrace{(0, \dots, 0, \underbrace{a_k}_{i_1\text{th component}}, 0, \dots, 0, \underbrace{a_k}_{i_2\text{th component}}, 0, \dots, 0, \underbrace{a_k}_{i_n\text{th component}}, 0, \dots, 0)}_{r\text{components}} \to a_k$$ yields the result. □

Next, the last result in this chapter begins the exploration of finding the polychromatic number of subsets taken from a direct product of copies of the integers modulo n. If the direct product in question includes the integers modulo 2 and the integers modulo n for any even n, the following result is obtained.

PROPOSITION 2.0.6. *If* $S = \{(x, a), (y, b), (z, c)\} \subset \mathbb{Z}_2 \times \mathbb{Z}_{2m}$ *for any* $m \geq 1$ *and any* $x, y \in \mathbb{Z}_2$

$$p(S) = \begin{cases} p_{2m}(\{0, a', b'\}) & \text{if } a' \not\equiv b' \bmod 2m, \text{ and } a' \not\equiv 0 \bmod 2m, \text{ and } b' \not\equiv 0 \bmod 2m \\ 2 & \text{otherwise} \end{cases}$$

where $S' = \{(0, 0), (x', a'), (y', b')\}$ *is a translate of* S *with* $x', y' \in \mathbb{Z}_2$ *and* $a', b' \in \mathbb{Z}_{2m}$.

Proof. Suppose $S = \{(x,a),(y,b),(z,c)\}$, Then, any of the translates $(2-x, 2m-a)+S$, $(2-y, 2m-b)+S$, or $(2-z, 2m-c)+S$, yield a translate of the form $S' = \{(0,0),(x',a'),(y',b')\}$. Next, it is either the case that S' is of the form $\{(0,0),(0,a'),(0,b')\}$, $\{(0,0),(0,a'),(1,b')\}$, $\{(0,0),(1,a'),(0,b')\}$, $\{(0,0),(1,a'),(1,b')\}$, however in any case the set of translates can be arranged into two distinct collections as follows.

Case 1: $S' = \{(0,0),(0,a'),(0,b')\}$. In which case the two collections of translates are disjoint: $(0,d)+S' = \{(0,d),(0,a'+d),(0,b'+d)\}$ and $(1,d)+S' = \{(1,d),(1,a'+d),(1,b'+d)\}$. Clearly, $(0,d)$ and $(1,d)$ never appear in the same translate.

Case 2: $S' = \{(0,0),(0,a'),(1,b')\}$. In which case one of the two collects contains two elements with at 0 in the first component and the other contains two elements with a 1 in the second component: $(0,d)+S' = \{(0,d),(0,a'+d),(1,b'+d)\}$ and $(1,d)+S' = \{(1,d),(1,a'+d),(0,b'+d)\}$. Note also that $(0,d)$ and $(1,d)$ never appear in the same translate as $(1,d)+S = \{(1,d),(1,d+a'),(0,d+b')\}$, $(0,d)+S = \{(0,d),(0,d+a'),(1,d+b')\}$, $(0,d-a')+S = \{(0,d-a'),(0,d),(1,d-a'+b')\}$, $(0,d-b')+S = \{(0,d-b'),(0,d-b'+a'),(1,d)\}$, $(1,d-a')+S = \{(1,d-a'),(1,d),(0,d-a'+b')\}$, $(1,d-b')+S = \{(1,d-b'),(1,d-b'+a'),(0,d)\}$.

Case 3: $S' = \{(0,0),(1,a'),(0,b')\}$. In which case one of the two collects contains two elements with at 0 in the first component and the other contains two elements with a 1 in the second component: $(0,d)+S' = \{(0,d),(1,a'+d),(0,b'+d)\}$ and $(1,d)+S' = \{(1,d),(0,a'+d),(1,b'+d)\}$. Note also that $(0,d)$ and $(1,d)$ never appear in the same translate as $(1,d)+S = \{(1,d),(0,d+a'),(1,d+b')\}$, $(0,d)+S = \{(0,d),(1,d+a'),(0,d+b')\}$, $(0,d-a')+S = \{(0,d-a'),(1,d),(0,d-a'+b)\}$, $(0,d-b')+S = \{(0,d-b'),(1,d-b'+a'),(0,d)\}$, $(1,d-a')+S = \{(1,d-a'),(0,d),(1,d-a'+b')\}$, $(1,d-b')+S = \{(1,d-b'),(0,d-b'+a'),(1,d)\}$.

Case 4: $S' = \{(0,0),(1,a'),(1,b')\}$. In which case one of the two collects contains two elements with at 1 in the first component and the other contains two elements with a 0 in the second component: $(0,d)+S' = \{(0,d),(1,a'+d),(1,b'+d)\}$ and $(1,d)+S' = \{(1,d),(0,a'+d),(0,b'+d)\}$. Note also that $(0,d)$ and $(1,d)$ never appear in the same translate as $(1,d)+S = \{(1,d),(0,d+a'),(0,d+b')\}$, $(0,d)+S = \{(0,d),(1,d+a'),(1,d+b')\}$, $(0,d-a')+S = \{(0,d-a'),(1,d),(1,d-a'+$

$b')\}$, $(0, d-b')+S = \{(0, d-b'), (1, d-b'+a'), (1, d)\}$, $(1, d-a')+S = \{(1, d-a'), (0, d), (0, d-a'+b')\}$, $(1, d-b') + S = \{(1, d-b'), (0, d-b'+a'), (0, d)\}$.

In any case, if χ is a color assignment to the elements of $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$, then making the distinction $\chi((0, i)) = \chi((1, i))$ will ensure that only one collection of translates in each above case must be colored and the other collection will be colored simultaneously. Thus, the isomorphism $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_{2m} \to \mathbb{Z}_{2m} : (w, i) \mapsto i$ yields that coloring the translates of S' in $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$ is isomorphic to coloring the translates of $\{0, a', b'\}$ in $\mathbb{Z}_{2m}$.

Next, consider the cases where either $a' \equiv b' \bmod 2m$, or $a' \equiv 0 \bmod 2m$, or $b' \equiv 0 \bmod 2m$.

Case 1: $a' \equiv b' \bmod 2m$. Then $S' = \{(0, 0), (0, b'), (1, b')\}$. However, consider the translates $(1, 0) + S = \{(1, 0), (1, b'), (0, b')\}$ and $(0, 2m - b') + S = \{(0, 2m - b'), (0, 0), (1, 0)\}$. Now, suppose χ is an $S'-$polychromatic coloring with three colors, such that without loss of generality $\chi((0, 0)) = c_0$, $\chi((0, b')) = c_1$, $\chi((1, b')) = c_2$. Then, $\chi((1, 0)) = c_0$, however $(0, 2m - b') + S$ can not be colored with three colors. So, $p_{\mathbb{Z}_n}(S') < 3$.

Case 2: Without loss of generality $a' \equiv 0 \bmod 2m$. Then $S' = \{(0, 0), (1, 0), (1, b)\}$ or $S' = \{(0, 0), (1, 0), (0, b)\}$. If it is the former case, then translates $(1, 0) + S = \{(1, 0), (0, 0), (0, b)\}$ and $(0, b) + S = \{(0, b), (1, b), (1, 2b)\}$ ensure any assignment of three colors to S' will force such assignment of three colors to fail. If it is the latter case, then the translates $(1, 0)+S = \{(1, 0), (0, 0), (1, b)\}$ and $(0, b) + S = \{(0, b), (1, b), (0, 2b)\}$ ensure any assignment of three colors to S' will force such assignment of three colors to fail. Thus, $p_{\mathbb{Z}_n}(S') < 3$.

Finally, notice that the translates of $S' = \{(0, 0), (0, b'), (1, b')\}$ are of the form $(0, d) + S' = \{(0, d), (0, d + b'), (1, d + b')\}$ and $(1, d) + S' = \{(1, d), (1, d + b'), (0, d + b')\}$ while the translates of $S' = \{(0, 0), (1, 0), (1, b)\}$ are $(0, d) + S' = \{(0, d), (1, d), (1, d + b')\}$ and $(1, d) + S' = \{(1, d), (0, d), (0, d+b')\}$ and those of $S' = \{(0, 0), (1, 0), (0, b)\}$ are $(0, d)+S' = \{(0, d), (1, d), (0, d+b')\}$ and $(1, d) + S' = \{(1, d), (0, d), (1, d + b')\}$. So, the assignment of two colors to the elements of $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$ via $\chi((0, i)) = c_0$ and $\chi((1, i)) = c_1$ ensures an $S'-$polychromatic coloring with two colors in any case. □

REMARK 2.0.7. *If* $\mathbb{Z}_2 \times \mathbb{Z}_{2m+1}$ *where* $m \geq 1$, *then by the Chinese remainder theorem,* $\mathbb{Z}_2 \times \mathbb{Z}_{2m+1} \cong \mathbb{Z}_{4m+2}$. *Determining the polychromatic number and the Turán number of this direct product is therefore isomorphic to one copy of the integers modulo* n.

An exploration of more generalized finite abelian groups is a topic that is still open.

CHAPTER 3. RESULTS ON FINITE NONABELIAN GROUPS

3.1 Introduction

It is also worth noting that not all finite groups can be classified as abelian. For all groups which are finite and nonabelian, an analogous question can be introduced:

QUESTION 3.1.1. *For a given finite nonabelian group G, and an arbitrary subset $S \subset G$, what is the maximum number of elements in a subset of G which does not contain a translate of S?*

The problem appears to be the same, however, due to the fact that the group operation is not necessarily commutative, it is possible to create almost twice as many translates as before ($2|G|-1$ translates to be exact). Suppose G is a finite nonabelian group and $S \subset G$. Then, the *left-translate* of S by any group element a of G is $a * S$. The *right-translate* of S by any group element a of G is $S*a$. Therefore "translate" in the context of this question means any left- or right-translate. As was the case with the finite abelian group work, the Turán number, $ex(G, S)$, and the complementary value, $f(G, S)$, are defined as previously. Similarly, the idea behind tackling the question is to divide the work into improving on a trivial lower bound and trivial upper bound which are again given by $|S| - 1$ and $|G| - 1$ with the same reasoning as previously given. This chapter concerns itself with working on the lower bound by examining the complementary problem which comes in the form of a Ramsey-type question: For a given finite nonabelian group G, and an arbitrary subset $S \subset G$, what is the minimum number of elements one must delete from G so that it does not contain a translate of S? Of course, because the problem is a little different, the method used to solve it must change as well. Therefore, the type of coloring that I have employed to augment the lower bound with respect to finite abelian groups, must be redefined. Given a subset S of a finite nonabelian group G, a coloring of the elements of G is S^L*−polychromatic* if every left-translate of S contains an element of each color. The *left-polychromatic number* given a finite nonabelian group G and a subset of G called S, denoted $p^L_G(S)$ or $p^L(S)$ when the choice of G is clear, is the largest number of colors allowing an S^L-polychromatic coloring of the left-translates of S. A coloring of the elements of G is S^R-*polychromatic* if every right-translate of S contains an element of each color. The *right-polychromatic number* given a finite nonabelian group G and a subset of G called S, denoted $p_G^R(S)$ or $p^R(S)$ when the choice of G is clear, is the largest number of colors allowing an S^R-polychromatic coloring of the left-translates of S. When referring to all translates left and right, the following definitions will be used. A coloring of the elements of G is $S-$*polychromatic* if every left- and right-translate of S contains an element of each color. The *polychromatic number* given a finite nonabelian group G and a subset of G called S, denoted $p_G(S)$ or $p(S)$ when the choice of G is clear, is the largest number of colors allowing an $S-$polychromatic coloring of the translates of S.

When the subset S of a finite nonabelian group G is a normal subgroup, it is worth noting that left- and right-translates are equal.

OBSERVATION 3.1.1. *Suppose $S \subset G$ where G is a finite nonabelian group is a normal subgroup, then for any $a \in G$, $a * S = S * a$.*

However, a big question that is only partially answered in this chapter is whether in general there is a bijection between the left- and right-translates of any given subset S of any finite nonabelian group G and more importantly whether it is the case that $p^L(G) = p^R(G)$.

3.2 Results on General Finite Nonabelian Groups

First and foremost, there are results that apply to any finite nonabelian group in general.

OBSERVATION 3.2.1. *If $S \subset G$ and $p^L(S) = m$ (or $p^R(S) = m$), then there exists an S^L-polychromatic coloring of S and its left-translates (S^R-polychromatic coloring of S and its right-translates) with $m - j$ colors for all $0 \leq j \leq m-1$.*

Proof. Every left-translate can be divided up so that at least m elements belong to m distinct color classes. Remove j distinct color classes by taking all the elements in these color classes and putting them in one of the remaining $m - j$ color classes.

Similarly, every right-translate can be divided up so that at least m elements belong to m distinct color classes. Remove j distinct color classes by taking all the elements in these color classes and putting them in one of the remaining $m - j$ color classes.

□

THEOREM 3.2.1. *Let G be a finite nonabelian group. Let $S \subset G$. Then, $p(S) \leq \min\{p^L(S), p^R(S)\}$.*

Proof. Suppose χ is an $S-$polychromatic coloring of G, then, χ is also an S^L-polychromatic and an S^R-polychromatic coloring. So, $p(S) \leq p_G^L(S)$ and $p(S) \leq p_G^R(S)$. Since $p(S)$ is less than or equal to both $p_G^L(S)$ and $p_G^R(S)$, $p(S) \leq \min\{p_G^L(S), p_G^R(S)\}$. □

It is worth noting that the inequality in Theorem 3.2.1 can be strict.

PROPOSITION 3.2.1. *It is not necessarily the case that $p_G^L(S) = p_G(S) = p_G^R(S)$.*

Proof. Suppose $G = S_3$, the symmetric group of degree 3. The group consists of elements $\{(1), (123), (132), (12), (13), (23)\}$. Suppose further that $S = \{(1), (12), (123)\}$. It will be shown that it is impossible to color the elements of S_3 so that an $S-$polychromatic coloring consisting of three colors exists but that S^L- and S^R-polychromatic colorings consisting of three colors do exist. Note that S is both a left- and right-translate. Suppose that χ is an $S-$polychromatic coloring consisting of three colors, then the three elements of S must be distinctly colored. Without loss of generality assign $\chi((1)) = c_0$, $\chi((12)) = c_1$, and $\chi((123)) = c_2$. Since a left-translate is $\{(12), (1), (23)\}$, it must be the case that $\chi((23)) = c_2$, however $\{(23), (123), (12)\}$ is a right translate which means $\chi((23)) = c_0$. This a contradiction.

An $S-$polychromatic coloring in two colors can be constructed by the assignment $\chi((1)) = c_0$, $\chi((12)) = c_1$, $\chi((123)) = c_1$, $\chi((13)) = c_0$, $\chi((132)) = c_1$, $\chi((23)) = c_0$.

An S^L-polychromatic coloring in three colors is the assignment $\chi((1)) = c_0$, $\chi((12)) = c_1$, $\chi((123)) = c_2$, $\chi((23)) = c_2$, $\chi((132)) = c_1$, $\chi((13)) = c_0$.

An S^R-polychromatic coloring in three colors is the assignment $\chi((1)) = c_0$, $\chi((12)) = c_1$, $\chi((123)) = c_2$, $\chi((23)) = c_0$, $\chi((132)) = c_1$, $\chi((13)) = c_2$.

Therefore, $p_G(S) = 2 < p_G^L(S) = p_G^R(S) = 3$. □

Before launching into the following examination of the problem in specific finite nonabelian groups, the examination of the problem in the integers modulo n can actually provide general results for all finite nonablelian groups.

THEOREM 3.2.2. *The Reduction Theorem for Finite Nonabelian Groups. Let G be a finite nonabelian group. If $S = \{1, a^{j_1}, a^{j_2}, \ldots, a^{j_{t-1}}\} \subset G$ such that $j_1 < j_2 < \ldots < j_{t-1}$ are distinct integers and $1 \neq a \in G$, then $p_G(S) = p_G^L(S) = p_G^R(S) = p_{\mathbb{Z}_{|\langle a \rangle|}}(\{0, j_1, j_2, \ldots, j_{t-1}\})$.*

Proof. The argument that was applied to Theorem 2.0.1 and Theorem 2.0.11 is essentially repeated. This is due to the fact that the left and right translates, xS and Sx, of S are equal when $x \in \langle a \rangle$. The coloring assignment $\chi(a^{j_\ell}) = \chi(xa^{j_\ell}) = \chi(a^{j_\ell}x)$ for $x \notin \langle a \rangle$ is then made. □

THEOREM 3.2.3. *Let G be a finite group. Let $S = \{a, b\} \subset G$ such that $a \neq b \in G$. Then,*

$$p(S) = \begin{cases} 1 & \text{if } |a^{-1}b| = |ba^{-1}| \text{ is odd} \\ 2 & \text{if } |a^{-1}b| = |ba^{-1}| \text{ is even.} \end{cases}$$

Proof. Note that $|a^{-1}b| = |ba^{-1}|$. Therefore, coloring, without loss of generality, the left-translates of S is equivalent to coloring the left-translates of $\{1, a^{-1}b\}$. By Theorem 3.2.2, $p^L(\{1, a^{-1}b\}) = P^R(\{1, a^{-1}b\}) = p_{\mathbb{Z}_{|\langle a^{-1}b \rangle|}}(\{0, 1\})$. □

LEMMA 3.2.1. *If $p_G^L(S) = |S| = k$ or if $p_G^R(S) = |S| = k$, then $k||G|$.*

Proof. There are $|G|$ translates with k positions in each translate. Each position ranges through all elements of G. Let $|c_i| = r$ denote the number of elements in the same color class colored c_i where i is any value $0 \leq i \leq k - 1$. Since each element appears in positions 1 through k in some translate and each translate contains one single element of each color, there are r translates where the color c_i is used to color the element in the first position, r translates where the color c_i is used to color the element in the second position and so on up to and including the kth position. So, every color class must contain the same number of elements and $|G| = kr$ as desired. □

REMARK 3.2.1. *If G is a finite abelian group, then by Theorem 2.0.4 in [6], $p(S) = |S|$ is equivalent to S tiling G by translation and so $G = S \oplus T$ for some subset $T \subset G$. Therefore, $|G| = |S| \cdot |T|$. Hence $|S| \,|\, |G|$.*

THEOREM 3.2.4. *Let G be a finite nonabelian group. Let $S \subset G$. Then, $p^L(S') \leq p^L(S)$ or $p^R(S') \leq p^R(S)$ for any $S' \subseteq S$.*

Proof. If $S = S'$, the statement is obviously true. Without loss of generality, suppose the identity 1 belongs to S and S', then $S' = \{1, a_{j_1}, \ldots, a_{j_{\ell-1}}\} \subset S = \{1, a_1, a_2, \ldots, a_{k-1}\}$ and let χ' be an S^L-polychromatic coloring of S' with ℓ colors, then

$$S = \{\underbrace{\underbrace{1}_{c_0}, \underbrace{a_{j_1}}_{c_1}, \ldots, \underbrace{a_{j_{\ell-1}}}_{c_{\ell-1}}}_{S'}, \underbrace{\_, \_, \_, \ldots, \_}_{\text{remaining elements}}\}.$$

With this partitioning, χ' extends to a S^L-polychromatic coloring of S and its left-translates by simply assigning all remaining elements to at least one color class that has already been used.
Similarly, let χ'' be a S^R-polychromatic coloring of S' with ℓ colors, then

$$S = \{\underbrace{\underbrace{1}_{c_0}, \underbrace{a_{j_1}}_{c_1}, \ldots, \underbrace{a_{j_{\ell-1}}}_{c_{\ell-1}}}_{S'}, \underbrace{\_, \_, \_, \ldots, \_}_{\text{remaining elements}}\}.$$

With this partitioning, χ'' extends to a S^R-polychromatic coloring of S and its right-translates by simply assigning all remaining elements to at least one color class that has already been used.

$\square$

3.2.1 The Dihedral Group, D_{2n} with $n \geq 2$

There are many finite nonabelian groups that one could examine, however, it is not always the case that one can represent elements of such groups easily. This is not the case with the dihedral group, denoted D_{2n} for all $n \geq 2$ [10]. The dihedral group is the symmetry group of an $n-$sided regular polygon, however this family of finite groups can be given by the following presentation

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}.$$

That is, each element can be written uniquely in the form $s^k r^i$ for some $k \in \{0,1\}$ and $0 \leq i \leq n-1$. Superscripts of r are understood to be taken modulo n. In addition to the generators r and s, the group is generated by the following relations.

(i) The generators of the group have orders: $|r| = n$ and $|s| = 2$;

(ii) $s \neq r^i$ for any i;

(iii) $sr^i \neq sr^j$ for all $0 \leq i, j \leq n-1$ with $i \neq j$;

(iv) $sr = r^{-1}s$;

(v) $sr^i = r^{-i}s$ for all $0 \leq i \leq n$.

It should be noted that in the following examination $\circ$ is used to denote the group operation on D_{2n}. Also, Elements of the form sr^i are called *reflections* and elements of the form r^i are called *rotations* for all $0 \leq i \leq n-1$.

EXAMPLE 3.2.1. *Note that for any two element subset* S *of* D_4, $ex(D_4, S) = 2$. *This is because there are four elements of this group* $1, r, s, sr$ *and all possible two element subsets are* $\{1, r\}$, $\{s, sr\}$, $\{1, s\}$, $\{r, sr\}$, $\{1, sr\}$, *and* $\{r, s\}$. *The two element subsets* $\{1, r\}$ *and* $\{s, sr\}$ *are the left- and right-translates of each other; the two element subsets* $\{1, s\}$ *and* $\{r, sr\}$ *are the left- and right-translates of each other; and the two element subsets* $\{1, sr\}$ *and* $\{r, s\}$ *are the left- and right-translates of each other. The trivial bounds on the Turán number are* $1 \leq ex(D_4, S) \leq 3$. *An* $S-$*polychromatic coloring with two colors can be constructed in each case* $\chi_0(1) = c_0 = \chi_0(s)$ *and* $\chi_0(r) = c_1 = \chi_0(sr)$; $\chi_1(1) = c_0 = \chi_1(r)$ *and* $\chi_1(s) = c_1 = \chi_(sr)$; *and* $\chi_2(1) = c_0 = \chi_2(s)$ *and* $\chi_2(r) = c_1 = \chi_2(sr)$. *So, the bounds can be improved to* $2 \leq ex(D_4, S) \leq 3$. *The three element subsets are* $\{1, r, s\}$, $\{1, r, sr\}$, $\{1, s, sr\}$, $\{r, s, sr\}$. *Any of these three element subsets contains one of the translates in any of the aforementioned cases and so it must be that* $ex(D_4, S) = 2$.

Note that all three element subsets are translates of each other. Let $S = \{1, r, s\}$, *then* $r \circ S = \{r, 1, sr\}$, $s \circ S = \{s, sr, 1\}$, $sr \circ S = \{sr, s, r\}$, $\{r, 1, sr\} = S \circ r$, $\{s, sr, 1\} = S \circ s$, *and* $\{sr, s, r\} = S \circ sr$. *So,* $ex(D_4, S) = 2$ *if* S *contains three elements as any three element subset of this group* *most assuredly contains one of these three element subsets and so the trivial bound is the value of the Turán number.*

If all of the elements in S are rotations or reflections, then following lemma establishes the equivalence between this situation and one in the integers modulo n.

LEMMA 3.2.2. *If $S \subset D_{2n}$ such that $S = \{r^{i_1}, r^{i_2}, \ldots, r^{i_m}\}$ or $S = \{sr^{i_1}, sr^{i_2}, \ldots, sr^{i_m}\}$ with $i_m \leq n-1$, then $p_{D_{2n}}(S) = p_{\mathbb{Z}_n}(\{i_1, i_2, \ldots, i_m\}) = p^L_{D_n}(S) = p^R_{D_n}(S)$.*

Proof. If $S = \{r^{i_1}, r^{i_2}, \ldots, r^{i_m}\}$, then result follows from Theorem 3.2.2 as all elements belong to the subgroup generated by element r. If $S = \{sr^{i_1}, sr^{i_2}, \ldots, sr^{i_m}\}$, then some left-translates of S are of the form: $sr^i \circ S = \{r^{i_1-i}, r^{i_2-i}, \ldots, r^{i_j-i}, \ldots, r^{i_m-i}\}$. Coloring $sr^i \circ S$ is isomorphic to coloring $\{i_1 - i, i_2 - i, \ldots, i_j - i, \ldots, i_m - i\}$ in $\mathbb{Z}_n$ since the superscripts of r are taken modulo n. Some right-translates of S are of the form: $S \circ sr^i = \{r^{i-i_1}, r^{i-i_2}, \ldots, r^{i-i_j}, \ldots, r^{i-i_m}\}$. Coloring $S \circ sr^i$ is isomorphic to coloring $\{i-i_1, i-i_2, \ldots, i-i_j, \ldots, i-i_m\}$ in $\mathbb{Z}_n$ since the superscripts of r are taken modulo n. Recall by Remark 2.0.5 $p_{\mathbb{Z}_n}(S) = p_{\mathbb{Z}_n}(-S)$. So, $p^L_{D_{2n}}(S) = p^L_{D_{2n}}(sr^i \circ S) = p_{\mathbb{Z}_n}(\{i_1 - i, i_2 - i, \ldots, i_j - i, \ldots, i_m - i\}) = p_{\mathbb{Z}_n}(\{i - i_1, i - i_2, \ldots, i - i_j, \ldots, i - i_m\}) = p^R_{D_{2n}}(S \circ sr^i) = p^R_{D_{2n}}(S)$. Also, $p_{\mathbb{Z}_n}(\{i_1 - i, i_2 - i, \ldots, i_j - i, \ldots, i_m - i\}) = p_{\mathbb{Z}_n}(\{i_1, i_2, \ldots, i_m\})$. Therefore, coloring left- and right-translates simultaneously is possible. □

So, if $S \subset D_{2n}$ such that all of the elements in S are rotations or reflections, the polychromatic number will be the size of the set S if and only if the superscripts of r tile $\mathbb{Z}_n$. If $p(S) = |S|$, by Lemma 3.2.1, then $|S|||G| = 2n$. So, either $|S||2$ or $|S||n$. If this is the first case, then without loss of generality assume the identity 1 is in the the two element subset and so $p^L_{D_{2n}}(\{1, r^i\}) = 2$ if and only if $|r^i|$ is even and $p^L_{D_{2n}}(\{1, sr^i\}) = 2$ always because $|sr^i| = 2$ for any i by Theorem 3.2.3. So, the interesting and definitely nontrivial case is if $|S||n$ which is the focus of the below work. However, it should be noted that most of the results on this group hinge upon the ability to construct isomorphisms between the left- and right-translates which do not necessarily indicate anything about the value of the polychromatic number. They only yield that the left- and right-polychromatic numbers are equal and so this is most of the work that follows. In the results that follow the identity 1 is without loss of generality always assumed to be in subset S.

A more general $S \subset D_{2n}$ has the form $S = \{1, s^{k_1}r^{i_1}, s^{k_2}r^{i_2}, \ldots, s^{k_m}r^{i_m}\}$. If $S \subset D_{2n}$ contains both rotations and reflections, its form is given as $S = \{1, r^{i_1}, r^{i_2}, \ldots, r^{i_k}, sr^{j_1}, \ldots, sr^{j_t}\}$.

The following result ensures that every subsest of D_{2n} of size at least two containing at least one rotation and one reflection has a polychromatic number of at least two.

PROPOSITION 3.2.2. *If $S = \{1, r^{i_1}, r^{i_2}, \ldots, r^{i_k}, sr^{j_1}, \ldots, sr^{j_t}\}$ such that $k, t \geq 1$, $p(S) \geq 2$.*

Proof. Note that the left-translates of S are $r^i \circ S = \{r^i, r^{i+i_1}, r^{i+i_2}, \ldots, r^{i+i_k}, sr^{j_1-i}, \ldots, sr^{j_t-i}\}$ and $sr^i \circ S = \{sr^i, sr^{i+i_1}, sr^{i+i_2}, \ldots, sr^{i+i_k}, r^{j_1-i}, \ldots, r^{j_t-i}\}$ for all $0 \leq i \leq n-1$. The right-translates of S are $S \circ r^i = \{r^i, r^{i_1+i}, r^{i_2+i}, \ldots, r^{i_k+i}, sr^{j_1+i}, \ldots, sr^{j_t+i}\}$ and
$S \circ sr^i = \{sr^i, sr^{i-i_1}, sr^{i-i_2}, \ldots, sr^{i-i_k}, r^{i-j_1}, \ldots, r^{i-j_t}\}$ for all $0 \leq i \leq n-1$. Then, each translate contains at least one rotation and one reflection. Therefore, assigning one color to all rotations and another color to all reflections ensures that every translate contains two colors. □

Corollary 3.2.1. *If $S = \{1, r^{i_1}, r^{i_2}, \ldots, r^{i_k}, sr^{j_1}, \ldots, sr^{j_t}\}$ such that $k, t \geq 1$, $ex(D_{2n}, S) \geq 2n - \frac{2n}{2} = n$.*

3.2.1.1 Subsets S with $p^L(S) = p^R(S)$

Excluding the situation of Lemma 3.2.2, it is not the case that the left- and right-translates of any subset S are the same.

OBSERVATION 3.2.2. *For any $S \subset D_{2n}$, it is not necessarily the case that any right- and left-translate of S by the same element are equal.*

Proof. The general subset S has the form $S = \{1, s^{k_1}r^{i_1}, s^{k_2}r^{i_2}, \ldots, s^{k_j}r^{i_j}, s^{k_m}r^{i_m}\}$. So, an element in a left-translate of S with application of element $s^k r^i$ is of the form $s^k r^i \circ s^{k_j} r^{i_j}$ and in a right-translate of S with application of element $s^k r^i$ is of the form $s^{k_j} r^{i_j} \circ s^k r^i$. Now, consider the following cases:

Case 1: $k = 0$, $k_j = 0$. Then, $s^{k_j} r^{i_j} \circ s^k r^i = r^{i_j+i}$ and $s^k r^i \circ s^{k_j} r^{i_j} = r^{i+i_j}$. These elements are equal.

Case 2: $k = 0$, $k_j = 1$. Then, $s^{k_j}r^{i_j} \circ s^k r^i = sr^{i_j+i}$ and $s^k r^i \circ s^{k_j}r^{i_j} = r^i sr^{i_j} = sr^{i_j-i}$. These elements are equal if and only if $i \equiv -i \bmod n$.

Case 3: $k = 1$, $k_j = 0$. Then, $s^{k_j}r^{i_j} \circ s^k r^i = r^{i_j}sr^i = sr^{i-i_j}$ and $s^k r^i \circ s^{k_j}r^{i_j} = sr^{i+i_j}$. These elements are equal if and only if $i_j \equiv -i_j \bmod n$.

Case 4: $k = 1$, $k_j = 1$. Then, $s^{k_j}r^{i_j} \circ s^k r^i = sr^{i_j}sr^i = r^{i-i_j}$ and $s^k r^i \circ s^{k_j}r^{i_j} = sr^i sr^{i_j} = r^{i_j-i}$. These elements are equal if and only if $i - i_j \equiv i_j - i \bmod n \iff 2i \equiv 2i_j \bmod n$.

Therefore, left- and right-translates differ in their elements unless special conditions hold. □

The rest of the results in this section give the structure of some subsets of D_{2n} that have $p^L_{D_{2n}}(S) = p^R_{D_{2n}}(S)$. The results on the dihedral group which actually determine these values heavily depend on the results of this section.

LEMMA 3.2.3. *If there is only one rotation in S i.e. $S = \{1, sr^{j_1}, \ldots, sr^{j_t}\}$, then $p^L_{D2n}(S) = p^R_{D2n}(S)$.*

Proof. All left-translates of S are of the form either $r^i \circ S = \{r^i, sr^{j_1-i}, sr^{j_2-i}, \ldots, sr^{j_t-i}\}$ or $sr^i \circ S = \{sr^i, r^{j_1-i}, r^{j_2-i}, \ldots, r^{j_t-i}\}$ for all $0 \le i \le n-1$. All right-translates of S are of the form either $S \circ r^{n-i} = \{r^{n-i}, sr^{j_1-i}, sr^{j_2-i}, \ldots, sr^{j_t-i}\}$ or $S \circ sr^i = \{sr^i, r^{i-j_1}, r^{i-j_2}, \ldots, r^{i-j_t}\}$ for all $0 \le i \le n-1$. Next suppose χ^L is an S^L-polychromatic coloring of the left-translates. Consider the mapping $\phi : D_{2n} \to D_{2n}$ such that $\phi(r^j) = r^{n-j}$ and $\phi(sr^j) = sr^j$ for all $0 \le j \le n-1$. This mapping ensures all translates contain the desired amount of colors as

$$\phi(r^i \circ S) = \{\phi(r^i), \phi(sr^{j_1-i}), \phi(sr^{j_2-i}), \ldots, \phi(sr^{j_t-i})\} \mapsto$$

$$S \circ r^{n-i} = \{r^{n-i}, sr^{j_1-i}, sr^{j_2-i}, \ldots, sr^{j_t-i}\}$$

and

$$\phi(sr^i \circ S) = \{\phi(sr^i), \phi(r^{j_1-i}), \phi(r^{j_2-i}), \ldots, \phi(r^{j_t-i})\} \mapsto$$

$$S \circ sr^i = \{sr^i, r^{i-j_1}, r^{i-j_2}, \ldots, r^{i-j_t}\}.$$

Therefore, the left- and right-polychromatic numbers are equal. □

LEMMA 3.2.4. *If* $S = \{1, r^{i_1}, sr^{i_2}, \ldots, sr^{i_{m-1}}\} \subset D_{2n}$*, then* $p^L(S) = p^R(S)$*.*

Proof. Consider the mapping $\phi : D_{2n} \to D_{2n}$ such that $\phi(r^i) = r^{n-i}$ and $\phi(sr^i) = sr^{i+n-i_1}$ for all $0 \le i \le n-1$. All left - translates are of the form

$$r^k \circ S = \{r^k, r^{k+i_1}, sr^{i_2-k}, \ldots, sr^{i_{m-1}-k}\}$$

and

$$sr^k \circ S = \{sr^k, sr^{k+i_1}, r^{i_2-k}, \ldots, r^{i_{m-1}-k}\}$$

for all $0 \le k \le n-1$. Next, consider the right - translates

$$\{r^{n-k-i_1}, r^{n-k}, sr^{n-k+i_2-i_1}, \ldots, sr^{n-k+i_{m-1}-i_1}\} = S \circ r^{n-k-i_1}$$

and

$$\{sr^{n+k}, sr^{n+k-i_1}, r^{n-(i_2-k)}, \ldots, r^{n-(i_{m-1}-k)}\} = S \circ sr^{n+k}$$

for all $0 \le k \le n-1$. Also,

$$\phi(r^k \circ S) = \{\phi(r^{k+i_1}), \phi(r^k), \phi(sr^{i_2-k}), \ldots, \phi(sr^{i_{m-1}-k})\} = S \circ r^{n-k-i_1}$$

and

$$\phi(sr^k \circ S) = \{\phi(sr^{k+i_1}), \phi(sr^k), \phi(r^{i_2-k}), \ldots, \phi(r^{i_{m-1}-k})\} = S \circ sr^{n+k}$$

for all $0 \le k \le n-1$. □

LEMMA 3.2.5. *If* $2 \le |S| \le 5$ *and* $S \subset D_{2n}$*, then* $p^L(S) = p^R(S)$*.*

Proof. If $n = 2$. This follows from Theorem 3.2.3.

If $n = 3$, then S is of the form $S_1 = \{1, r^{i_1}, sr^{j_1}\}$ or $S_2 = \{1, r^{i_1}, r^{i_2}\}$ or $S_3 = \{1, sr^{j_1}, sr^{j_2}\}$. Note that by Lemma 3.2.2, $p^L_{D_{2n}}(S_2) = p_{\mathbb{Z}_n}(\{0, i_1, i_2\}) = p^R_{D_{2n}}(S_2)$. If S is of the form S_1, then $sr^{j_1} \circ S = \{1, r^{i_1}, sr^{j_1}\} = \{sr^{j_1}, sr^{j_1+i_1}, 1\}$ is a left-translate. So, determining the left-polychromatic number of S_1 and its translates is equivalent to determining the left-polychromatic number of S_3. By Lemma 3.2.3, it follows that $p^L_{D2n}(S_1) = p^R_{D2n}(S_1)$ and $p^L_{D2n}(S_3) = p^R_{D2n}(S_3)$.

If $n = 4$, then S is of the form $S_1 = \{1, sr^{j_1}, sr^{j_2}, sr^{j_3}\}$, $S_2 = \{1, r^{i_1}, sr^{j_2}, sr^{j_3}\}$, $S_3 = \{1, r^{i_1}, r^{i_2}, sr^{j_1}\}$,

$S_4 = \{1, r^{i_1}, r^{i_2}, r^{i_3}\}$. First, by Lemma 3.2.2, $p^L_{D2n}(S_4) = p^R_{D2n}(S_4) = p_{\mathbb{Z}_n}(\{0, i_1, i_2, i_3\})$. Also, $sr^{j_1} \circ S_3 = \{sr^{j_1}, sr^{j_1+i_1}, sr^{j_1+i_2}, 1\}$ is a left-translate. So, determining the left-polychromatic number of S_3 and its translates is equivalent to determining the left-polychromatic number of S_1. By Lemma 3.2.3, it follows that $p^L_{D2n}(S_1) = p^R_{D2n}(S_1)$ and $p^L_{D2n}(S_3) = p^R_{D2n}(S_3)$. By Lemma 3.2.4, $p^L_{D2n}(S_2) = p^R_{D2n}(S_2)$.

If $n = 5$, then S is of the form $S_1 = \{1, sr^{j_1}, sr^{j_2}, sr^{j_3}, sr^{j_4}\}$, $S_2 = \{1, r^{i_1}, sr^{j_1}, sr^{j_2}, sr^{j_3}\}$, $S_3 = \{1, r^{i_1}, r^{i_2}, sr^{j_1}, sr^{j_2}\}$, $S_4 = \{1, r^{i_1}, r^{i_2}, r^{i_3}, sr^{j_1}\}$, $S_5 = \{1, r^{i_1}, r^{i_2}, r^{i_3}, r^{i_4}\}$. First by Lemma 3.2.2, $p^L_{D2n}(S_5) = p^R_{D2n}(S_5)$. Also, $sr^{j_1} \circ S_4 = \{sr^{j_1}, sr^{j_1+i_1}, sr^{j_1+i_2}, sr^{j_1+i_3}, 1\}$ is a left-translate of S_4 so by Lemma 3.2.3, it follows that $p^L_{D2n}(S_1) = p^R_{D2n}(S_1)$ and $p^L_{D2n}(S_4) = p^R_{D2n}(S_4)$. Also, $sr^{j_1} \circ S_3 = \{sr^{j_1}, sr^{j_1+i_1}, sr^{j_1+i_2}, 1, r^{j_2-j_1}\}$ is a left-translate of S_3 so by Lemma 3.2.4, $p^L_{D2n}(S_2) = p^R_{D2n}(S_2)$ and $p^L_{D2n}(S_3) = p^R_{D2n}(S_3)$. □

LEMMA 3.2.6. *If* $S = \{1, r^{i_1}, r^{2i_1}, \ldots, r^{(m-1)i_1}, sr^{j_1}, \ldots, sr^{j_t}\} \subset D_{2n}$*, then* $p^L(S) = p^R(S)$ *for any* $0 \le i_1, j_1, \ldots, j_t \le n-1$.

Proof. Consider the mapping $\phi : D_{2n} \to D_{2n}$ such that $\phi(r^i) = r^{-i+(m-1)i_1}$ and $\phi(sr^i) = sr^i$ for all $0 \le i \le n-1$. The translates map thusly:

$$\phi(r^k \circ S) = \{\phi(r^k), \phi(r^{k+i_1}), \ldots, \phi(r^{k+(m-1)i_1}), \phi(sr^{j_1-k}), \ldots, \phi(sr^{j_t-k})\} \mapsto$$

$$S \circ r^{-k} = \{r^{-k}, r^{-k+i_1}, \ldots, r^{-k+(m-1)i_1}, sr^{j_1-k}, \ldots, sr^{j_t-k}\}$$

and

$$\phi(sr^k \circ S) = \{\phi(sr^k), \phi(sr^{k+i_1}), \ldots, \phi(sr^{k+(m-1)i_1}), \phi(r^{j_1-k}), \ldots, \phi(r^{j_t-k})\} \mapsto$$

$$S \circ sr^{k+(m-1)i_1} = \{sr^{k+(m-1)i_1}, sr^{k+(m-2)i_1}, \ldots, sr^{k+i_1}, sr^k, r^{k+(m-1)i_1-j_1}, \ldots, r^{k+(m-1)i_1-j_t}\}$$

for all $0 \le k \le n-1$. □

3.2.1.2 The Polychromatic Number of Subsets of Size 3 or 4

Results on small subsets of the dihedral group are obtained by reimagining the problem in a geometric setting. That is, drawing from [12] where the problem of finding the polychromatic number of a given graph was reinterpreted to one of coloring a rectangular grid and shapes within that grid, a similar rearranging is done for $S \subset D_{2n}$ such that $|S| = 3$ or 4 and its translates.

DEFINITION 3.2.1. *A $2 \times M$ grid will hereafter be depicted as*

$(2,0)$	$\cdots$	$(2,i)$	$(2,i+1)$	$\cdots$	$(2,M-1)$
$(1,0)$	$\cdots$	$(1,i)$	$(1,i+1)$	$\cdots$	$(1,M-1)$

Figure 3.1 $2 \times M$ *grid*

where $0 \leq i \leq M-1$ correspond to the columns in the grid. Note that when $i = M-1$, $i+1 = 0$.

DEFINITION 3.2.2. *A 2×2 picture frame is a subset of a $2 \times M$ grid such that 2×2 box with the lower right corner removed (called a lower box) whose top two row entries always appear in the first row of the $2 \times M$ grid and whose bottom entry always belongs to the second row of the grid overlaying a 2×2 box with the upper left corner removed (called an upper box) whose top entry always appears in the first row of the grid and whose bottom two entries always belong to the second row of the grid as pictured.*

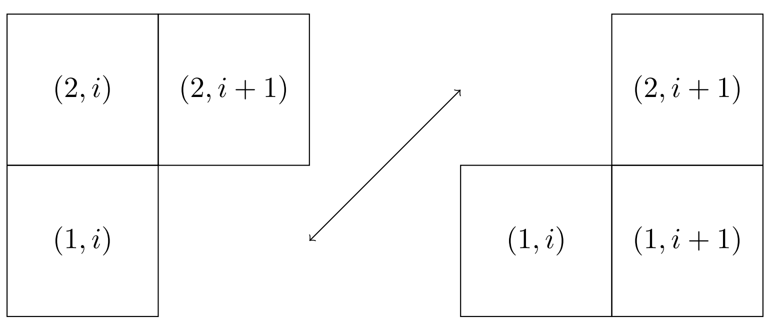

Figure 3.2 *A* 2×2 *picture frame*

DEFINITION 3.2.3. *A $k - 2 \times 2$ frame coloring is an assignment of the entries in a $2 \times M$ grid so that the entries in every lower box contain k colors and the entries in every upper box contain k colors.*

OBSERVATION 3.2.3. *By the above definition, the upper right entry in the lower box and the upper right in the upper box are the same entry and thus colored the same color. Similarly, the lower left entry in the lower box and the lower left in the upper box are the same entry and thus colored the same color. So, if there is a $3 - 2 \times 2$ frame coloring, the upper left in the lower box and the lower right in the upper box which are different entries must be colored the same color. That is, indexing the $2 \times M$ grid, $\chi((2, i)) = \chi((1, i + 1))$.*

PROPOSITION 3.2.3. *There is a $3 - 2 \times 2$ frame coloring of a $2 \times M$ grid if and only if $3|M$*

Proof. If $3|M$, make the following assignment

$$\chi((1, i)) = \begin{cases} c_0 & \text{if } i \equiv 1 \bmod 3 \\ c_1 & \text{if } i \equiv 0 \bmod 3 \\ c_2 & \text{if } i \equiv 2 \bmod 3 \end{cases}$$

$$\chi((2, i)) = \begin{cases} c_0 & \text{if } i \equiv 2 \bmod 3 \\ c_1 & \text{if } i \equiv 1 \bmod 3 \\ c_2 & \text{if } i \equiv 0 \bmod 3. \end{cases}$$

Then every lower and upper box contains three colors. Next, assume $3 \nmid M$ and there is a $3 - 2 \times 2$ frame coloring χ'. Without loss of generality make the assignment $\chi'((1, 0)) = c_0$, $\chi'((2, 0)) = c_1$, and $\chi'((2, 1)) = c_2$ since these entries constitute an lower box. Because every upper and lower box in the grid must contain three distinct colors, entries must be colored as follows

$$\chi'((1, i)) = \begin{cases} c_0 & \text{if } 1 + i \equiv 1 \bmod 3 \\ c_1 & \text{if } 1 + i \equiv 2 \bmod 3 \\ c_2 & \text{if } 1 + i \equiv 0 \bmod 3 \end{cases}$$

$$\chi'((2,i)) = \begin{cases} c_0 & \text{if } 2+i \equiv 1 \bmod 3 \\ c_1 & \text{if } 2+i \equiv 2 \bmod 3 \\ c_2 & \text{if } 2+i \equiv 0 \bmod 3. \end{cases}$$

So, if $3 \nmid M$, then either $M \equiv 1 \bmod 3$ or $M \equiv 2 \bmod 3$. If $M \equiv 1 \bmod 3$, $M-1 \equiv 0 \bmod 3$ and so $\chi'((2, M-1)) = c_1$, $\chi'((2,0)) = c_1$, and $\chi'((1, M-1)) = c_0$ which means this lower box does not contain three distinct colors and a contradiction is reached. If $M \equiv 2 \bmod 3$, $M-1 \equiv 1 \bmod 3$ and so $\chi'((2, M-1)) = c_2$, $\chi'((2,0)) = c_1$, and $\chi'((1, M-1)) = c_1$ which means this lower box does not contain three distinct colors and a contradiction is reached once again. □

LEMMA 3.2.7. *Any subset $S = \{1, sr^j, sr^m\} \subset D_{2n}$ and its left-translates can be written in* $\gcd(n, m-j)$ *many disjoint* $2 \times \frac{n}{\gcd(n,m-j)}$ *grids.*

Proof. Note that left-translates $sr^j \circ S = \{sr^j, 1, r^{m-j}\}$ are of the form consisting of two rotations and one reflection. That is, the problem is isomorphic to putting a set of this form $\{1, r^k, sr^\ell\}$ and all of its left-translates in a grid. Let $g = \gcd(n,k)$ for simplicity. The left-translates of S can be represented as 2×2 picture frames in the following $2 \times \frac{n}{g}$ grids

$sr^{\ell-(n-(k+i))}$	$sr^{\ell-i}$	$sr^{\ell-k-i}$	$sr^{\ell-2k-i}$	$\cdots$	$sr^{\ell-(m-1)k-i}$	$sr^{\ell-mk-i}$	$\cdots$	$sr^{\ell-\left(\frac{n}{g}-2\right)k-i}$
r^i	r^{k+i}	r^{2k+i}	r^{3k+i}	$\cdots$	r^{mk+i}	$r^{(m+1)k+i}$	$\cdots$	$r^{\left(\frac{n}{g}-1\right)k+i}$

Figure 3.3 Left-translates arranged into grid entries

for $0 \le i \le g-1$ and $0 \le m \le \frac{n}{g} - 1$. Note that $sr^{\ell-(n-(k+i))} = sr^{k+\ell-i}$ and $sr^{\ell-\left(\frac{n}{g}-2\right)k-i} = sr^{\ell+2k-i}$. □

LEMMA 3.2.8. *If $S = \{1, sr^j, sr^m\} \subset D_{2n}$, then $p^L(S) = 3 = p^R(S)$ if and only if $3 | \frac{n}{\gcd(n,m-j)}$.*

Proof. First, note that by Lemma 3.2.3, the left-translates and the right-translates can be simultaneously colored. Next, by Proposition 3.2.3 and Lemma 3.2.7 the result follows. □

THEOREM 3.2.5. *If $S \subset D_{2n}$ so that $|S| = 3$, then $p^L(S) = 3 = p^R(S)$ if and only if one of the following is true:*

(1) $S = \{r^j, r^k, r^\ell\}$ *and* $3^{m+1}|n$, $k' = 3^m m_k$, $\ell' = 3^m m_\ell$, *and without loss of generality* $m_k \equiv 1 \bmod 3$, $m_\ell \equiv 2 \bmod 3$*;*

(2) $S = \{sr^j, sr^k, sr^\ell\}$ *and* $3^{m+1}|n$, $k' = 3^m m_k$, $\ell' = 3^m m_\ell$, *and without loss of generality* $m_k \equiv 1 \bmod 3$, $m_\ell \equiv 2 \bmod 3$*;*

(3) $S = \{r^j, r^k, sr^\ell\}$ *and* $3|\frac{n}{\gcd(n,n-k+j)}$*;*

(4) $S = \{r^j, sr^k, sr^\ell\}$ *and* $3|\frac{n}{\gcd(n,n-k+\ell)}$.

Proof. If (1) or (2), then these cases are isomorphic to coloring $\{j, k, \ell\}$ in $\mathbb{Z}_n$ by Lemma 3.2.2 and the desired result follows. If (3) or (4) the result follows from Lemma 3.2.8 as there is always a translate of the form $\{1, sr^a, sr^b\}$ where $a, b \in \mathbb{Z}_n$.

The other way,

Case 1: S only contains rotations or only reflections. Then, this case is again isomorphic to coloring $\{j, k, \ell\}$ in $\mathbb{Z}_n$ by Lemma 3.2.2 and the result follows.

Case 2: S contains rotations and reflections. Since $p^L(S) = 3 = p^R(S)$, $3|\frac{n}{\gcd(n,n-k+j)}$ by Lemma 3.2.8. □

THEOREM 3.2.1. *If* $|S| \subset D_{2n}$ *such that* $|S| = 3$*, then*

$$p^R_{D_{2n}}(S) = p^L_{D_{2n}}(S) = \begin{cases} 3 & \textit{if } S = \{r^j, r^k, sr^\ell\} \textit{ and } 3|\frac{n}{\gcd(n,n-k+j)} \\ & \textit{or } S = \{r^j, sr^k, sr^\ell\} \textit{ and } 3|\frac{n}{\gcd(n,n-k+\ell)}; \\ p_{\mathbb{Z}_n}(S) & \textit{if } S = \{r^j, r^k, r^\ell\} \textit{ or } S = \{sr^j, sr^k, sr^\ell\}; \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. The result follows by Theorem 3.2.5 and Lemma 3.2.2. Also, since there is always a translate of $S = \{r^j, r^k, sr^\ell\}$ or $S = \{r^j, sr^k, sr^\ell\}$ of the form $\{1, sr^a, sr^b\}$ where $a, b \in \mathbb{Z}_n$, by Lemma 3.2.7, by assigning colors to the entries of the grid as

$$\chi((i,j)) = \begin{cases} c_0 & \text{if } i + j \equiv 0 \bmod 2 \\ c_1 & \text{if } i + j \equiv 1 \bmod 2 \end{cases}$$

every translate of S consists of 2 colors. The $S-$polychromatic coloring with two colors also follows by Proposition 3.2.2. $\square$

THEOREM 3.2.2. *If $|S| \subset D_{2n}$ such that $|S| = 3$, then*

$$p_{D_{2n}}(S) = \begin{cases} p_{\mathbb{Z}_n}(S) & \text{if } S = \{r^j, r^k, r^\ell\} \text{ or } S = \{sr^j, sr^k, sr^\ell\}; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Firstly, in the case that S contains solely rotations or solely reflections, the argument follows directly from the fact that these cases are isomorphic to coloring $\{j, k, \ell\}$ in $\mathbb{Z}_n$ by Lemma 3.2.2. If $\{a, b, c\}$ contains both rotations and reflections, by Proposition 3.2.2, $p_{D_{2n}}(\{a, b, c\}) \geq 2$. Note that any subset of order three which contains both rotations and reflections has a translate of the form $\{1, r^k, sr^\ell\}$. Next, note that a couple left-translates are $sr^\ell \circ \{1, r^k, sr^\ell\} = \{sr^\ell, sr^{\ell+k}, 1\}$ and $sr^{\ell-k} \circ \{1, r^k, sr^\ell\} = \{sr^{\ell-k}, sr^\ell, r^k\}$ and a couple right-translates are $\{1, r^k, sr^\ell\} \circ sr^\ell = \{sr^\ell, sr^{\ell-k}, 1\}$ and $\{1, r^k, sr^\ell\} \circ sr^{\ell+k} = \{sr^{\ell+k}, sr^\ell, r^k\}$. If without loss of generality, 1, r^k and sr^ℓ are assigned three distinct colors as follows: $\chi(1) = c_0$, $\chi(r^k) = c_1$ and $\chi(sr^\ell) = c_2$, then $\chi(sr^{\ell+k}) = c_1$ and $\chi(sr^{\ell-k}) = c_0$ for the constraints from the left-translates to be satisfied, however for the constraints from the right-translates it must be the case that $\chi(sr^{\ell+k}) = c_0$ and $\chi(sr^{\ell-k}) = c_1$. This is a contradiction. So, if $\{a, b, c\}$ contains both rotations and reflections, $p_{D_{2n}}(\{a, b, c\}) < 3$. $\square$

Corollary 3.2.2. *If $|S| \subset D_{2n}$ such that $|S| = 3$, then*

$$ex(D_{2n}, S) \geq \begin{cases} 2n - \frac{2n}{p_{\mathbb{Z}_n}(S)} & \text{if } S = \{r^j, r^k, r^\ell\} \text{ or } S = \{sr^j, sr^k, sr^\ell\}; \\ n & \text{otherwise.} \end{cases}$$

DEFINITION 3.2.4. *A 2×2 box is a sub-grid of a $2 \times M$ grid as pictured*

DEFINITION 3.2.5. *A $k - 2 \times 2$ box coloring of a $2 \times M$ grid is an assignment of the entries of a $2 \times M$ grid so that every entry in any 2×2 box in the grid contains k distinct colors.*

LEMMA 3.2.9. *If M is even, there exists a $4 - 2 \times 2$ box coloring of a $2 \times M$ grid.*

$(2,i)$	$(2,i+1)$
$(1,i)$	$(1,i+1)$

Figure 3.4 A 2×2 box

Proof. Index the entries of the first row of the $2 \times M$ grid as $(1, i)$ and the entries of the second row as $(2, i)$ for $1 \leq i \leq M$. Make the following assignments

$$\chi((1,i)) = \begin{cases} c_0 \text{ if } i \equiv 1 \bmod 2 \\ c_2 \text{ if } i \equiv 0 \bmod 2 \end{cases}$$

$$\chi((2,i)) = \begin{cases} c_1 \text{ if } i \equiv 1 \bmod 2 \\ c_3 \text{ if } i \equiv 0 \bmod 2 \end{cases}$$

Any 2×2 box contains the entries $(1, i)$, $(1, i + 1)$, $(2, i)$, $(2, i + 1)$ and by the above coloring assignment all four entries are distinctly colored. □

LEMMA 3.2.10. *There exists a $4 - 2 \times 2$ box coloring of a $2 \times M$ grid if and only if $2|M$.*

Proof. Suppose there is a $4 - 2 \times 2$ box coloring of a $2 \times M$ grid. The following entries in a grid always form a 2×2 box: $(1, i)$, $(1, i + 1)$, $(2, i)$, and $(2, i + 1)$. Also, so do the entries $(1, i + 1)$, $(1, i+2)$, $(2, i+1)$, and $(2, i+2)$. Therefore, each of these collections of four entries must be colored distinctly and since there are only four colors being used the colors assigned to the entries $(1, i)$ and $(2, i)$ and $(1, i + 2)$ and $(2, i + 2)$ must be the same. So, the columns of the $2 \times M$ grid break up evenly with half being colored two colors and the other half being colored the other two. That is, $2|M$. The other direction comes from the above Lemma 3.2.9. □

PROPOSITION 3.2.4. *If $S = \{1, r^k, sr^j, sr^\ell\} \subset D_{2n}$ where $0 \leq j < \ell \leq n - 1$, $\ell = j + k$, and $2|\frac{n}{\gcd(n,k)}$, then $p^L(S) = 4 = p^R(S)$.*

Proof. By Lemma 3.2.5, $p^L(S) = p^R(S)$. Without loss of generality, the following argument will be accomplished via left-translates. Note that left-translates are of the form $r^i \circ S = \{r^i, r^{k+i}, sr^{j-i}, sr^{\ell-i}\}$ while $sr^{j-i} \circ S = \{sr^{j-i}, sr^{j-i+k}, r^{j-(j-i)}, r^{\ell-(j-i)}\} = \{sr^{j-i}, sr^{\ell-i}, r^i, r^{k+i}\}$ for all $0 \leq i \leq n-1$. Therefore, the only left-translates that must be considered are of the form $r^i \circ S$ for all $0 \leq i \leq n-1$. Let $g = \gcd(n,k)$ for simplicity. Observe that all left-translates of S are the 2×2 boxes in the following disjoint grids.

$r^{(\frac{n}{g}-1)k+i}$	$\cdots$	r^{mk+i}	$r^{(m-1)k+i}$	$\cdots$	r^{k+i}	r^i
$sr^{j-(\frac{n}{g}-2)k-i}$	$\cdots$	$sr^{j-(m-1)k-i}$	$sr^{j-(m-2)k-i}$	$\cdots$	sr^{j-i}	$sr^{\ell-i}$

Figure 3.5 Left-translates arranged into a grid

where $0 \leq i \leq g-1$ and $0 \leq m \leq \frac{n}{g}-1$. By Lemma 3.2.10 since $2|\frac{n}{g}$, $p^L(S) = 4 = p^R(S)$. $\square$

OBSERVATION 3.2.4. *Any subset of the form $S = \{1, r^k, sr^j, sr^\ell\} \subset D_{2n}$ and its left-translates can be represented as 2×2 boxes in a $2 \times 2n$ grid.*

Proof. By Lemma 3.2.5, $p^L(S) = p^R(S)$. Without loss of generality, the following argument will be accomplished via left-translates. If n is even consider the following $2 \times 2n$ grid

$r^{-\frac{n}{2}j-(\frac{n}{2}-1)k+\frac{n}{2}\ell}$	$\cdots$	$r^{-(m+1)j-mk+(m+1)\ell}$	$r^{-mj-mk+m\ell}$	$r^{-mj-(m-1)k+m\ell}$	$\cdots$	$r^{-j-k+\ell}$	$r^{-j+\ell}$	1
$sr^{\frac{n}{2}j+\frac{n}{2}k-(\frac{n}{2}-1)\ell}$	$\cdots$	$sr^{(m+1)j+(m+1)k-m\ell}$	$sr^{(m+1)j+mk-m\ell}$	$sr^{mj+mk-(m-1)\ell}$	$\cdots$	$sr^{2j+k-\ell}$	sr^{j+k}	sr^j

r^k	$r^{j+k-\ell}$	$r^{j+2k-\ell}$	$\cdots$	$r^{(d-1)j+dk-(d-1)\ell}$	$r^{dj+dk-d\ell}$	$r^{dj+(d+1)k-d\ell}$	$\cdots$	$r^{\frac{n}{2}j+\frac{n}{2}k-\frac{n}{2}\ell}$
sr^ℓ	$sr^{-k+\ell}$	$sr^{-j-k+2\ell}$	$\cdots$	$sr^{-(d-1)j-(d-1)k+d\ell}$	$sr^{-(d-1)j-dk+d\ell}$	$sr^{-dj-dk+(d+1)\ell}$	$\cdots$	$sr^{-(\frac{n}{2}-1)j-\frac{n}{2}k+\frac{n}{2}\ell}$

Figure 3.6 The left-translates of any 4 element subset of D_{2n} arranged in a grid when n is even

where the entry containing 1 is the entry to the left of the entry containing r^k and the entry containing sr^j is the entry to the left of the entry containing sr^ℓ. Notice that in the first half of the grid, the 2×2 box containing $r^{-mj-mk+m\ell}$, $r^{-mj-(m-1)k+m\ell}$, $sr^{(m+1)j+mk-m\ell}$, $sr^{mj+mk-(m-1)\ell}$ is a left-translate of S of the form $r^i \circ S$ with $i = -mj - mk + m\ell$ and the 2×2 box containing $sr^{(m+1)j+(m+1)k-m\ell}$, $sr^{(m+1)j+mk-m\ell}$, $r^{-mj-mk+m\ell}$, $r^{-(m+1)j-mk+(m+1)\ell}$ is a left-translate of S of the form $sr^i \circ S$ with $i = (m+1)j + (m+1)k - m\ell$ for each value of $1 \le m \le \frac{n}{2} - 1$ as well as $m = 0, \frac{n}{2}$. Also, there are two 2×2 boxes for each value of $1 \le m \le \frac{n}{2} - 1$ and one for $m = 0, \frac{n}{2}$ which yields n translates. Notice that in the second half of the grid, the 2×2 box containing $r^{dj+dk-d\ell}$, $r^{dj+(d+1)k-d\ell}$, $sr^{-(d-1)j-dk+d\ell}$, $sr^{-dj-dk+(d+1)\ell}$ is a left-translate of S of the form $r^i \circ S$ with $i = dj + dk - d\ell$ and the 2×2 box containing $sr^{-(d-1)j-dk+d\ell}$, $sr^{-(d-1)j-(d-1)k+d\ell}$, $r^{dj+dk-d\ell}$, $r^{(d-1)j+dk-(d-1)\ell}$ is a left-translate of S of the form $sr^i \circ S$ with $i = -(d-1)j - dk + d\ell$ for each value of $1 \le d \le \frac{n}{2} - 1$ as well as $d = 0, \frac{n}{2}$. Also, there are two 2×2 boxes for each value of $1 \le d \le \frac{n}{2} - 1$ and one for $d = 0, \frac{n}{2}$ which yields n translates. Finally, note that the entries containing $r^{\frac{n}{2}+\frac{n}{2}k-\frac{n}{2}\ell}$, $sr^{-(\frac{n}{2}-1)j-\frac{n}{2}k+\frac{n}{2}\ell}$, $r^{-\frac{n}{2}j-(\frac{n}{2}-1)k+\frac{n}{2}\ell}$, and $sr^{\frac{n}{2}j+\frac{n}{2}-(\frac{n}{2}-1)\ell}$ form a 2×2 box with $d = \frac{n}{2}$. Also, $r^{-\frac{n}{2}j-(\frac{n}{2}-1)k+\frac{n}{2}\ell} = r^{\frac{n}{2}j+(\frac{n}{2}+1)k-\frac{n}{2}\ell}$ and $sr^{\frac{n}{2}j+\frac{n}{2}k-(\frac{n}{2}-1)\ell} = sr^{-\frac{n}{2}j-\frac{n}{2}k+(\frac{n}{2}+1)\ell}$ as $\frac{n}{2} \equiv -\frac{n}{2} \bmod n$. Similarly and clearly, with $d = 0$, the entries containing 1, sr^j, r^k, sr^ℓ form a 2×2 box.

If n is odd consider the following $2 \times 2n$ grid.

$r^{-(\frac{n-1}{2})j-(\frac{n-1}{2})k+\frac{n-1}{2}\ell}$	$\cdots$	$r^{-(m+1)j-mk+(m+1)\ell}$	$r^{-mj-mk+m\ell}$	$r^{-mj-(m-1)k+m\ell}$	$\cdots$	$r^{-j-k+\ell}$	$r^{-j+\ell}$	1
$sr^{\frac{n+1}{2}j+\frac{n-1}{2}k-\left(\frac{n-1}{2}\right)\ell}$	$\cdots$	$sr^{(m+1)j+(m+1)k-m\ell}$	$sr^{(m+1)j+mk-m\ell}$	$sr^{mj+mk-(m-1)\ell}$	$\cdots$	$sr^{2j+k-\ell}$	sr^{j+k}	sr^j

r^k	$r^{j+k-\ell}$	$r^{j+2k-\ell}$	$\cdots$	$r^{(d-1)j+dk-(d-1)\ell}$	$r^{dj+dk-d\ell}$	$r^{dj+(d+1)k-d\ell}$	$\cdots$	$r^{\frac{n-1}{2}j+\frac{n+1}{2}k-(\frac{n-1}{2})\ell}$
sr^ℓ	$sr^{-k+\ell}$	$sr^{-j-k+2\ell}$	$\cdots$	$sr^{-(d-1)j-(d-1)k+d\ell}$	$sr^{-(d-1)j-dk+d\ell}$	$sr^{-dj-dk+(d+1)\ell}$	$\cdots$	$sr^{-(\frac{n-1}{2})j-(\frac{n-1}{2})k+\frac{n+1}{2}\ell}$

Figure 3.7 The left-translates of any 4 element subset of D_{2n} arranged in a grid when n is odd

The break-down of this grid is the same as the even case with a couple exceptions. In the first half of the grid, $0 \le m \le \frac{n-3}{2}$ with two left-translates for each value of m and one left-translate corresponding to $m = \frac{n-1}{2}$ which yields n left-translates. In the second half of the grid, $1 \le d \le \frac{n-1}{2}$ with two left-translates for each value of d and one left-translate corresponding to $d = \frac{n+1}{2}$ which yields n left-translates. Notice that with $d = \frac{n+1}{2}$ in a 2×2 box with entries $r^{(d-1)j+dk-(d-1)\ell}$, $sr^{-(d-1)j-(d-1)k+d\ell}$, $r^{dj+dk-d\ell}$, $sr^{-(d-1)j-dk+d\ell}$, then these entries become the elements $r^{\frac{n-1}{2}j+\frac{n+1}{2}k-(\frac{n-1}{2})\ell}$, $sr^{-(\frac{n-1}{2})j-(\frac{n-1}{2})k+\frac{n+1}{2}\ell}$, $r^{-(\frac{n-1}{2})j-(\frac{n-1}{2})k+\frac{n-1}{2}\ell}$, and $sr^{\frac{n+1}{2}j+\frac{n-1}{2}k-(\frac{n-1}{2})\ell}$ with $\frac{-n-1}{2} \equiv \frac{n-1}{2} \bmod n$ and $\frac{-n+1}{2} \equiv \frac{n+1}{2} \bmod n$.

□

REMARK 3.2.2. *Note that rotations can be repeated in entries in the first row while reflections can also be repeated in entries in the second row.*

REMARK 3.2.3. *By Proposition 3.2.2, the polychromatic number of any subset of the form $S = \{1, r^k, sr^j, sr^\ell\} \subset D_{2n}$ is at least 2, so the question becomes how to determine all subsets of D_{2n} whose translates have an S-polychromatic coloring with three or four colors. The problem of coloring the above grids with three or four colors is more challenging because elements can be repeated in the entries.*

DEFINITION 3.2.6. *A $m-n$-tooth-comb is a collection of entries in a $k \times k'$ grid where $k \ge n, k' \ge m$ such that m entries make up the stem of the comb and belong to a single column while n entries make the teeth of the comb and belong to the next column over. Consider the following figure.*

where $1 \le d \le \lfloor \frac{m}{2} \rfloor$.

LEMMA 3.2.11. *For any two positive integers j and k there is a positive integer d so that $dj \equiv dk \bmod n$ for any $n \ge 2$.*

Proof. Let $d = n$. Then, $d(j-k) \equiv 0 \bmod n \implies dj - dk \equiv 0 \bmod n$. □

OBSERVATION 3.2.5. *Any subset of the form $S = \{1, r^k, r^j, sr^\ell\} \subset D_{2n}$ and its left-translates can be represented as $3-1$-tooth-combs in $2(j-k)$ $3 \times m+1$ grids where m is the least positive integer such that $(m+1)j \equiv (m+1)k \bmod n$.*

(i,j)	
$\vdots$	$\vdots$
$(i+m-2d,j)$	$(i+m-2d,j+1)$
$\vdots$	$\vdots$
$(i+m-5,j)$	
$(i+m-4,j)$	$(i+m-4,j+1)$
$(i+m-3,j)$	
$(i+m-2,j)$	$(i+m-2,j+1)$
$(i+m-1,j)$	

Figure 3.8 A $m-n$-tooth-comb

Proof. By Lemma 3.2.5, $p^L(S) = p^R(S)$. Without loss of generality, the following argument will be accomplished via left-translates. The left-translates of S are of the form $r^t \circ S = \{r^t, r^{t+k}, r^{t+j}, sr^{\ell-t}\}$ for all $0 \le t \le n-1$. These left-translates are in the below $j-k$ $3 \times m+1$ grids as $3-1$-tooth-combs

$sr^{\ell-i}$	$sr^{\ell-(k+i-j)}$	$sr^{\ell-(2k+i-2j)}$	$\cdots$	$sr^{\ell-(dk+i-dj)}$	$sr^{\ell-((d+1)k+i-(d+1)j)}$	$\cdots$	$sr^{\ell-(mk+i-mj)}$
r^{j+i}	r^{k+i}	r^{2k+i-j}	$\cdots$	$r^{dk+i-(d-1)j}$	$r^{(d+1)k+i-dj}$	$\cdots$	$r^{mk+i-(m-1)j}$
r^i	r^{k+i-j}	$r^{2k+i-2j}$	$\cdots$	$r^{dk+i-dj}$	$r^{(d+1)k+i-(d+1)j}$	$\cdots$	$r^{mk+i-mj}$

Figure 3.9 The left-translates of a four element subset of D_{2n} arranged in a grid

for all $0 \le i \le j-k-1$ and $0 \le d \le m$. While $sr^t \circ S = \{sr^t, sr^{t+k}, sr^{t+j}, r^{\ell-t}\}$ for all $0 \le t \le n-1$. These left-translates are in the below $j-k$ $3 \times m+1$ grids as $3-1$-tooth-combs

$r^{\ell-i}$	$r^{\ell-(k+i-j)}$	$r^{\ell-(2k+i-2j)}$	$\cdots$	$r^{\ell-(dk+i-dj)}$	$r^{\ell-((d+1)k+i-(d+1)j)}$	$\cdots$	$r^{\ell-(mk+i-mj)}$
sr^{j+i}	sr^{k+i}	sr^{2k+i-j}	$\cdots$	$sr^{dk+i-(d-1)j}$	$sr^{(d+1)k+i-dj}$	$\cdots$	$sr^{mk+i-(m-1)j}$
sr^{i}	sr^{k+i-j}	$sr^{2k+i-2j}$	$\cdots$	$sr^{dk+i-dj}$	$sr^{(d+1)k+i-(d+1)j}$	$\cdots$	$sr^{mk+i-mj}$

for all $0 \leq i \leq j-k-1$ and $0 \leq d \leq m$.

□

REMARK 3.2.4. *Note that rotations can be repeated in entries in the second two rows of the first collection of grids while reflections can also be repeated in entries in the second two rows of the second collection of grids.*

REMARK 3.2.5. *By Proposition 3.2.2, the polychromatic number of any subset of the form $S = \{1, r^k, r^j, sr^\ell\} \subset D_{2n}$ is at least two, so the question becomes determine all subsets of D_{2n} whose translates have an S-polychromatic coloring with three or four colors. The problem of coloring the above grids with three or four colors is more challenging because elements can be repeated in the entries.*

3.2.1.3 The Polychromatic Number of Subsets of Size Larger than 4

The largest that a subset of D_{2n} can possibly be made without including all of the group elements is $2n-1$. The following result gives the left- and right-polychromatic number as well as the polychromatic number for such a subset.

LEMMA 3.2.12. *If $|S| = 2n-1$ and $S \subset D_{2n}$, $p^L(S) = p^R(S) = p(S) = n$.*

Proof. First, consider the left-translates of S. There are always $\binom{2n-1}{2n-2} = 2n-1$ such subsets S to check. This can be seen as any $S \subset D_{2n}$ is a left-translate of or is in the form $\{1, r^{i_1}, \ldots, r^{i_{n-1}}, sr^{j_1}, \ldots, sr^{j_{n-1}}\}$ of which there are n or $\{1, r^{i_1}, \ldots, r^{i_{n-2}}, sr^{j_1}, \ldots, sr^{j_n}\}$ of which there are n and where $0 \leq i_k, j_\ell \leq n-1$ for $1 \leq k \leq n-1$ and $1 \leq \ell \leq n$. In fact, all sets of this form are left-translates of the set $S = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-2}\}$ as the following are the left-translates of S: $r^i \circ S = \{r^i, r^{i+1}, \ldots, r^{i+n-1}, sr^{-i}, sr^{1-i}, \ldots, sr^{n-2-i}\}$ and $sr^i \circ S = \{sr^i, sr^{i+1}, \ldots, sr^{i+n-1}, r^{-i}, r^{1-i}, \ldots, r^{n-2-i}\}$ for all $0 \leq i \leq n-1$. For the lower bound, consider the S^L-polychromatic coloring given by

$$\chi(s^k r^i) = c_i$$

for $0 \leq i \leq n-1$ and for any $k \in \{0, 1\}$. The first n elements in $r^i \circ S$ and $sr^i \circ S$ will be colored with these n distinct colors under this S^L-polychromatic coloring. For the upper bound, suppose by way of contradiction that there does exist an S^L-polychromatic coloring, χ', that assigns $n+1$ colors to the elements of D_{2n}. For every element in D_{2n}, there is one left-translate which does not contain that element as there are $2n-1$ elements in the left-translates and therefore only $2n-1$ opportunities for a single element to appear while there are $2n$ translates. So, whatever color a single element is assigned must appear in that left-translate that does not contain it. Therefore, there are two elements that are colored c_i for all $0 \leq i \leq n$, however, this means there are $2n+2$ elements which is not possible.

Next, consider the right-translates of S. There are also $2n-1$ such subsets as any $S \subset D_{2n}$ is a right-translate of or is in the form $\{1, r^{i_1}, \ldots, r^{i_{n-1}}, sr^{j_1}, \ldots, sr^{j_{n-1}}\}$ of which there are n or $\{1, r^{i_1}, \ldots, r^{i_{n-2}}, sr^{j_1}, \ldots, sr^{j_n}\}$ of which there are n and where $0 \leq i_k, j_\ell \leq n-1$ for $1 \leq k \leq n-1$ and $1 \leq \ell \leq n$. In fact, all sets of this form are right-translates of the set $S = \{1, r, r^2, \ldots, r^{n-2}, s, sr, sr^2, \ldots, sr^{n-1}\}$ as the following are the right-translates of S: $S \circ r^i = \{r^i, r^{1+i}, r^{2+i}, \ldots, r^{n-2+i}, sr^i, sr^{1+i}, sr^{2+i}, \ldots, sr^{n-1+i}\}$ and
$S \circ sr^i = \{sr^i, sr^{i-1}, sr^{i-2}, \ldots, sr^{i-n+2}, r^i, r^{i-1}, r^{i-2}, \ldots, r^{i-n+1}\}$ for all $0 \leq i \leq n-1$. For the lower bound, consider the S^R-polychromatic coloring given by

$$\chi(s^k r^i) = c_i$$

for $0 \leq i \leq n-1$ and for any $k \in \{0, 1\}$. The last n elements in $S \circ r^i$ and $S \circ sr^i$ will be colored with these n distinct colors under this S^R-polychromatic coloring. However, as has been shown this coloring is also an S^L-polychromatic coloring for any such subset S, so $p^L_{D_{2n}}(S) = p^R_{D_{2n}}(S) = p_{D_{2n}}(S) = n$. □

This raises the question as to what the left- and right-polychromatic number is of a subset that contains all rotations and only some of the reflections. From Theorems 3.2.2 and 3.2.4, the left- and right-polychromatic numbers are at least n. The value is exactly n and this is also true of the polychromatic number.

LEMMA 3.2.13. *If $S = \langle r\rangle \cup sr^i \subset D_{2n}$ for some $0 \le i \le n-1$, then $p^L(S) = n = p^R(S)$.*

Proof. First consider the left-translates of S. For the lower bound consider the assignment

$$\chi(s^k r^j) = c_j$$

for all $0 \le j \le n-1$ and any $k \in \{0, 1\}$. Since each translate contains either n rotations or n reflections, the assignment is an S^L-polychromatic coloring. For the upper bound, suppose that there does exist an S^L-polychromatic coloring with $n+1$ colors. Then, in left-translates of the form $r^k \circ S = \{r^k, r^{k+1}, \ldots, r^{k+n-1}, sr^{i-k}\}$ for all $0 \le k \le n-1$, all rotations are distinct colors and so all reflections are the same color. However, left-translates of the form $sr^k \circ S = \{sr^k, sr^{k+1}, \ldots, sr^{k+n-1}, r^{i-k}\}$ yield that such an S^L-polychromatic coloring of the left-translates with $n+1$ colors is impossible.

Now, consider the right-translates of S. For the lower bound consider the assignment

$$\chi(s^k r^j) = c_j$$

for all $0 \le j \le n-1$. Since each translate contains either n rotations or n reflections, the assignment is an S^R-polychromatic coloring. For the upper bound, suppose that there does exist an S^L-polychromatic coloring with $n+1$ colors. Then, in left-translates of the form $S \circ r^k = \{r^k, r^{k+1}, \ldots, r^{k+n-1}, sr^{i+k}\}$ for all $0 \le k \le n-1$, all rotations are distinct colors and so all reflections are the same color. However, right-translates of the form $S \circ sr^k = \{sr^k, sr^{k-1}, \ldots, sr^{k-n+1}, r^{k-i}\}$ yield that such an S^R-polychromatic coloring of the left-translates with $n+1$ colors is impossible.
Note that by the above arguments, the assignment

$$\chi(s^k r^j) = c_j$$

for all $0 \leq j \leq n-1$ colors left- and right- translates simultaneously and so an $S-$polychromatic coloring is obtained. Hence, $p^L(S) = p^R(S) = p(S) = n$. □

By the previous two results no matter how many reflections are put into a subset with the subgroup generated by r, the polychromatic is at least n. The above results also bring up a good point and that is that there is always a lower bound for the polychromatic number of a subset $S \subset D_{2n}$ given by the rotations in the subset.

REMARK 3.2.6. *Let $R = \langle r^i \rangle \subset D_{2n}$ be the largest subgroup of $\langle r \rangle$ such that $R \subset S \subset D_{2n}$, then $\frac{n}{\gcd(n,i)} \leq p(S)$.*

Proof. This follows from Theorem 3.2.2 and Theorem 3.2.4 and noting that $|\langle r^i \rangle| = \frac{n}{\gcd(n,i)}$. □

If instead of containing all of the rotations in a subset of size n of D_{2n}, one considers such a subset with one of the rotations replaced by a reflection, the left- and right-polychromatic numbers will still be n.

LEMMA 3.2.14. *If $S = (\langle r \rangle \setminus \{r^j\}) \cup \{sr^k\} \subset D_{2n}$ for some $1 \leq j \leq n-1$ and any $0 \leq k \leq n-1$, then $p^L(S) = p^R(S) = n$.*

Proof. First, note that a left-translate of S is $sr^k \circ S = \{sr^k, sr^{k+1}, \ldots, sr^{k+j-1}, sr^{k+j+1}, \ldots, sr^{k+n-1}, 1\}$. By Lemma 3.2.3, $p^L(S) = p^R(S)$, so this argument will be given with respect to left-translates. Create the following assignment of the rotations except for r^j: $\chi(r^i) = c_i$ for all $0 \leq i < j$ and $\chi(r^{j+m}) = c_{j+m-1}$ for $1 \leq m \leq n-j-1$. Also, set $\chi(sr^k) = c_{n-1} = \chi(r^j)$. Consider the left-translates of the form $r^\ell \circ S = \{r^\ell, r^{\ell+1}, r^{\ell+2}, \ldots, r^{\ell+j-1}, r^{\ell+j+1}, \ldots, r^{\ell+n-1}, sr^{k-\ell}\}$ for all $0 \leq \ell \leq n-1$. Since the element $r^{\ell+j}$ does not belong to $r^\ell \circ S$, set $\chi(sr^{k-\ell}) = \chi(r^{\ell+j})$ for all $0 \leq \ell \leq n-1$. By the above assignment, $r^\ell \circ S$ consists of n distinctly colored elements. Next, consider left-translates of the form $sr^\ell \circ S = \{sr^\ell, sr^{\ell+1}, sr^{\ell+2}, \ldots, sr^{\ell+j-1}, sr^{\ell+j+1}, \ldots, sr^{\ell+n-1}, r^{k-\ell}\}$ for all $0 \leq \ell \leq n-1$. Since all rotations have distinct colors, all reflections have distinct colors. So, all that must ensured is that $\chi(r^{k-\ell})$ is not the same color as the color of any of the elements $sr^\ell, sr^{\ell+1}, \ldots, sr^{\ell+n-1}$. By the above assignment the only element to be colored the same color as

$r^{k-\ell}$ is $sr^{\ell+j}$ as $\chi(sr^{\ell+j}) = \chi(sr^{k-(k-\ell-j)}) = \chi(r^{k-\ell-j+j}) = \chi(r^{k-\ell})$. The element $sr^{\ell+j}$ does not appear in $sr^{\ell} \circ S$, so the left-translate consists of n distinctly colored elements. $\square$

The following two results give the form of some subsets of the dihedral group whose left- and right-polychromatic numbers are the cardinality of the subset and include all of the rotations in the group except for two which are replaced by two reflections. The proofs involve supplying an S^L-polychromatic coloring and an exhaustive case analysis to show the coloring in question ensures every element in each translate is colored distinctly.

PROPOSITION 3.2.5. *If n is even and $S = \{1, r, r^2, \ldots, r^{\frac{n}{2}-2}, r^{\frac{n}{2}}, r^{\frac{n}{2}+1}, r^{\frac{n}{2}+2}, \ldots, r^{n-2}, sr^{i-\frac{n}{2}}, sr^{i}\}$ such that $\frac{n}{2} \leq i \leq n-2$, then $p^L(S) = p^R(S) = |S| = n$.*

Proof. First, note that a left-translate of S is $sr^i \circ S = \{sr^i, sr^{i+1}, sr^{i+2}, \ldots, sr^{i+\frac{n}{2}-2}, sr^{i+\frac{n}{2}}, sr^{i+\frac{n}{2}+1}, sr^{i+\frac{n}{2}+2}, \ldots, sr^{i+n-2}, r^{\frac{n}{2}}, 1\}$. By Lemma 3.2.4, $p^L(S) = p^R(S)$. Without loss of generality, the following argument will be given with respect to left-translates. Consider the coloring χ given by $\chi(r^j) = c_j$ for all $0 \leq j \leq \frac{n}{2} - 2$, $\chi(r^j) = c_{j-1}$ for all $\frac{n}{2} \leq j \leq n-2$, $\chi(r^{\frac{n}{2}-1}) = c_{n-2}$, $\chi(r^{n-1}) = c_{n-1}$, $\chi(sr^{i-\frac{n}{2}-k}) = c_{k-1}$ for all $1 \leq k \leq \frac{n}{2} - 1$, $\chi(sr^{i-k}) = c_{\frac{n}{2}+k-2}$ for all $1 \leq k \leq \frac{n}{2} - 1$, and $\chi(sr^{i-\frac{n}{2}}) = c_{n-2}$, $\chi(sr^i) = c_{n-1}$. Any left-translate is of the form

$$r^{\ell} \circ S = \{r^{\ell}, r^{\ell+1}, r^{\ell+2}, \ldots, r^{\ell+\frac{n}{2}-2}, r^{\ell+\frac{n}{2}}, r^{\ell+\frac{n}{2}+1}, r^{\ell+\frac{n}{2}+2}, \ldots, r^{\ell+n-2}, sr^{i-\frac{n}{2}-\ell}, sr^{i-\ell}\}$$

$$sr^{\ell} \circ S = \{sr^{\ell}, sr^{\ell+1}, sr^{\ell+2}, \ldots, sr^{\ell+\frac{n}{2}-2}, sr^{\ell+\frac{n}{2}}, sr^{\ell+\frac{n}{2}+1}, sr^{\ell+\frac{n}{2}+2}, \ldots, sr^{\ell+n-2}, r^{i-\frac{n}{2}-\ell}, r^{i-\ell}\}$$

where $0 \leq \ell \leq n-1$.

For left-translates of the form $r^{\ell} \circ S$, it is clear from the supplied coloring that all rotations are distinctly colored, so all that must be shown is that the two reflections are colored with two distinct colors which do not appear in the left-translate already. Note that the only rotations that do not appear in $r^{\ell} \circ S$ are $r^{\ell+\frac{n}{2}-1}$ and $r^{\ell+n-1}$. So, by definition of the coloring, it suffices to show that the colors assigned to $r^{\ell+\frac{n}{2}-1}$ and $r^{\ell+n-1}$ are the same colors assigned to $sr^{i-\frac{n}{2}-\ell}, sr^{i-\ell}$.

Case 1: $\ell+n-1 \equiv n-1 \bmod n$. Then, $\ell \equiv 0 \bmod n$. So, $\ell+\frac{n}{2}-1 \equiv \frac{n}{2}-1$. Thus, $\chi(r^{\ell+\frac{n}{2}-1}) = c_{n-2}$

and $\chi(r^{\ell+n-1}) = c_{n-1}$. Next, $sr^{i-\frac{n}{2}-\ell} = sr^{i-\frac{n}{2}}$ and $sr^{i-\ell} = sr^i$. So, $\chi(sr^{i-\ell}) = \chi(sr^i) = c_{n-1}$ and $\chi(sr^{i-\frac{n}{2}-\ell}) = \chi(sr^{i-\frac{n}{2}}) = c_{n-2}$.

Case 2: $\ell+n-1 \equiv \frac{n}{2}-1$. Then, $\ell \equiv \frac{n}{2} \bmod n$. So, $\ell+\frac{n}{2}-1 \equiv n-1 \bmod n$. Thus, $\chi(r^{\ell+\frac{n}{2}-1}) = c_{n-1}$ and $\chi(r^{\ell+n-1}) = \chi(r^{\frac{n}{2}-1}) = c_{n-2}$. Next, $sr^{i-\frac{n}{2}-\ell} = sr^{i-\frac{n}{2}+\frac{n}{2}} = sr^i$ and $sr^{i-\ell} = sr^{i-\frac{n}{2}}$. So, $\chi(sr^{i-\ell}) = c_{n-2}$ and $\chi(sr^{i-\frac{n}{2}-\ell}) = c_{n-1}$.

Case 3: $0 \bmod n \leq \ell + n - 1 \bmod n \leq \frac{n}{2} - 2 \bmod n$. Then, $\chi(r^{\ell+n-1}) = c_{\ell+n-1 \bmod n} = c_{\ell-1}$ and $-\frac{n}{2} \bmod n \leq \ell + \frac{n}{2} - 1 \bmod n \leq n - 2 \bmod n$ and so $\chi(r^{\ell+\frac{n}{2}-1}) = c_{\ell+\frac{n}{2}-2}$. Also, $1 \bmod n \leq \ell + n \bmod n \leq \frac{n}{2} - 1 \bmod n \implies 1 \bmod n \leq \ell \bmod n \leq \frac{n}{2} - 1 \bmod n$ and so $\chi(sr^{i-\frac{n}{2}-\ell}) = c_{\ell-1}$ and $\chi(sr^{i-\ell}) = c_{\frac{n}{2}+\ell-2}$.

Case 4: $\frac{n}{2} \bmod n \leq \ell + n - 1 \bmod n \leq n - 2 \bmod n$. Then, $\chi(r^{\ell+n-1}) = c_{\ell+n-1-1 \bmod n} = c_{\ell-2 \bmod n}$. Note that $0 \bmod n \leq \ell + \frac{n}{2} - 1 \bmod n \leq \frac{n}{2} - 2 \bmod n$ and $\frac{n}{2} + 1 \bmod n \leq \ell \bmod n \leq n - 1 \bmod n$ so $\ell = \frac{n}{2} + m$ for some positive integer m and so $\chi(r^{\ell+\frac{n}{2}-1}) = c_{\ell+\frac{n}{2}-1 \bmod n} = c_{\frac{n}{2}+m+\frac{n}{2}-1 \bmod n} = c_{m-1 \bmod n}$. Also, $\chi(sr^{i-\frac{n}{2}-\ell}) = c_{\frac{n}{2}+\frac{n}{2}+\ell-2 \bmod n} = c_{\ell-2 \bmod n}$ and $\chi(sr^{i-\ell}) = \chi(sr^{i-\frac{n}{2}-m}) = c_{m-1 \bmod n}$.

Similarly, for left-translates of the form $sr^\ell \circ S$, it is clear from the supplied coloring that all reflections are distinctly colored, so all that must be shown is that the two rotations are colored with two distinct colors which do not appear in the left-translate already. Note that the only reflections that do not appear in $sr^\ell \circ S$ are $sr^{\ell+\frac{n}{2}-1}$ and $sr^{\ell+n-1}$. So, by definition of the coloring, it suffices to show that the colors assigned to $sr^{\ell+\frac{n}{2}-1}$ and $sr^{\ell+n-1}$ are the same colors assigned to $r^{i-\frac{n}{2}-\ell}, r^{i-\ell}$.

Case 1: $\ell + n - 1 \equiv i \bmod n$. Then, $\chi(sr^{\ell+n-1}) = \chi(sr^i) = c_{n-1}$. Also, $\ell + n - 1 - \frac{n}{2} \bmod n \equiv i - \frac{n}{2} \bmod n$ and so $\chi(sr^{\ell+\frac{n}{2}-1}) = \chi(sr^{i-\frac{n}{2}}) = c_{n-2}$. So, $\ell + n - 1 - \ell \bmod n \equiv i - \ell \bmod n$ and thus $\chi(r^{i-\ell}) = \chi(r^{n-1}) = c_{n-1}$. Similarly, $n - 1 - \frac{n}{2} \bmod n \equiv i - \ell - \frac{n}{2} \bmod n$ and so $\chi(r^{i-\ell-\frac{n}{2}}) = \chi(r^{\frac{n}{2}-1}) = c_{n-2}$.

Case 2: $\ell + n - 1 \equiv i - \frac{n}{2} \bmod n$. Then, $\chi(sr^{\ell+n-1}) = \chi(sr^{i-\frac{n}{2}}) = c_{n-2}$. Also, $\ell + n - 1 + \frac{n}{2} \bmod n \equiv i \bmod n$ and so $\chi(sr^{\ell+\frac{n}{2}-1}) = \chi(sr^i) = c_{n-1}$. So, $\ell + \frac{n}{2} - 1 - \ell \bmod n \equiv i - \ell \bmod n$ and thus $\chi(r^{i-\ell}) = \chi(r^{\frac{n}{2}-1}) = c_{n-2}$. Similarly, $\frac{n}{2} - 1 - \frac{n}{2} \bmod n \equiv i - \ell - \frac{n}{2} \bmod n$ and so

$\chi(r^{i-\ell-\frac{n}{2}}) = \chi(r^{n-1}) = c_{n-1}$.

Case 3: $\ell + n - 1 \equiv i - \frac{n}{2} - k \bmod n$ with $1 \leq k \leq \frac{n}{2} - 1$. Then, $\chi(sr^{\ell+n-1}) = \chi(sr^{i-\frac{n}{2}-k}) = c_{k-1}$. Also, $\ell + n - 1 + \frac{n}{2} \bmod n \equiv i - \frac{n}{2} - k + \frac{n}{2} \bmod n$ and so $\chi(sr^{\ell+\frac{n}{2}-1}) = \chi(sr^{i-k}) = c_{\frac{n}{2}+k-2}$. So, $\ell + \frac{n}{2} - 1 + k - \ell \bmod n \equiv i - \ell \bmod n$ and because $1 \bmod n \leq k \bmod n \leq \frac{n}{2} - 1 \bmod n \implies \frac{n}{2} \bmod n \leq k + \frac{n}{2} - 1 \bmod n \leq n - 2 \bmod n$ and so $\chi(r^{i-\ell}) = \chi(r^{\frac{n}{2}+k-1}) = c_{\frac{n}{2}+k-2}$. Similarly, $\frac{n}{2} - 1 + k - \frac{n}{2} \bmod n \equiv i - \ell - \frac{n}{2} \bmod n$ with $1 - 1 \bmod n \leq k - 1 \bmod n \leq \frac{n}{2} - 1 - 1 \bmod n$ and so $\chi(r^{i-\ell-\frac{n}{2}}) = \chi(r^{k-1}) = c_{k-1}$.

Case 4: $\ell + n - 1 \equiv i - k \bmod n$ with $1 \leq k \leq \frac{n}{2} - 1$. Then, $\chi(sr^{\ell+n-1}) = \chi(sr^{i-k}) = c_{\frac{n}{2}+k-2}$. Also, $\ell + n - 1 - \frac{n}{2} \bmod n \equiv i - k - \frac{n}{2} \bmod n$ and so $\chi(sr^{\ell+\frac{n}{2}-1}) = \chi(sr^{i-k-\frac{n}{2}}) = c_{k-1}$. So, $\ell + n - 1 + k - \ell \bmod n \equiv i - \ell \bmod n$ and because $1 \bmod n \leq k \bmod n \leq \frac{n}{2} - 1 \bmod n \implies 0 \bmod n \leq k - 1 \bmod n \leq \frac{n}{2} - 2 \bmod n$ and so $\chi(r^{i-\ell}) = \chi(r^{k-1}) = c_{k-1}$. Similarly, $-1 + k - \frac{n}{2} \bmod n \equiv i - \ell - \frac{n}{2} \bmod n \implies k + \frac{n}{2} - 1 \bmod n \equiv i - \ell - \frac{n}{2} \bmod n$ with $1 - 1 - \frac{n}{2} \bmod n \leq k - 1 - \frac{n}{2} \bmod n \leq \frac{n}{2} - 1 - 1 - \frac{n}{2} \bmod n \implies \frac{n}{2} \bmod n \leq k - 1 + \frac{n}{2} \bmod n \leq n - 2 \bmod n$ and so $\chi(r^{i-\ell-\frac{n}{2}}) = \chi(r^{k-1-\frac{n}{2}}) = c_{k+\frac{n}{2}-1}$.

□

PROPOSITION 3.2.6. *$n = p^L_{D_{2n}}(\{1, r, r^2, r^3, \ldots, r^{n-j-2}, r^{n-j}, \ldots, r^{n-4}, r^{n-3}, r^{n-2}, sr^{i-j}, sr^i\}) = p^R_{D_{2n}}(\{1, r, r^2, r^3, \ldots, r^{n-j-2}, r^{n-j}, \ldots, r^{n-4}, r^{n-3}, r^{n-2}, sr^{i-j}, sr^i\})$ for all $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$ and $0 \leq i \leq n-1$.*

Proof. First, note that if n is even, then n is at most $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} - \frac{1}{2} \rfloor = \frac{n}{2} - 1 = \frac{n-2}{2}$. Next, note that a left-translate of S is

$sr^i \circ S = \{sr^i, sr^{i+1}, sr^{i+2}, \ldots, sr^{i+n-j-2}, sr^{i+n-j}, \ldots, sr^{i+n-4}, sr^{i+n-3}, sr^{i+n-2}, r^{n-j}, 1\}$. By Lemma 3.2.4, $p^L_{D_{2n}}(S) = p^R_{D_{2n}}(S)$. Without loss of generality, the following argument will be given with respect to left-translates. Consider the coloring χ given by $\chi(r^k) = c_k$ for all $0 \leq k \leq n-j-2$, $\chi(r^k) = c_{k-1}$ for all $n-j \leq k \leq n-2$, $\chi(r^{n-1}) = c_{n-2}$, $\chi(r^{n-j-1}) = c_{n-1}$, $\chi(sr^{i-j-k}) = c_{k-1}$ for all $1 \leq k \leq n-1-j$, $\chi(sr^{i-j}) = c_{n-2}$, $\chi(sr^i) = c_{n-1}$, $\chi(sr^{i-k}) = c_{n+k-j-2} = c_{k-j-2}$ for all $1 \leq k \leq j-1$ and subscripts of the colorings are taken modulo n. Any left-translate is of the form

$$r^\ell \circ S = \{r^\ell, r^{\ell+1}, r^{\ell+2}, \ldots, r^{\ell+n-j-2}, r^{\ell+n-j}, \ldots, r^{\ell+n-4}, r^{\ell+n-3}, r^{\ell+n-2}, sr^{i-j-\ell}, sr^{i-\ell}\}$$

$$sr^\ell \circ S = \{sr^\ell, sr^{\ell+1}, sr^{\ell+2}, \ldots, sr^{\ell+n-j-2}, sr^{\ell+n-j}, \ldots, sr^{\ell+n-4}, sr^{\ell+n-3}, sr^{\ell+n-2}, r^{i-j-\ell}, r^{i-\ell}\}$$

where $0 \le \ell \le n-1$.

For left-translates of the form $r^\ell \circ S$, it is clear from the supplied coloring that all rotations are distinctly colored, so all that must be shown is that the two reflections are colored with two distinct colors which do not appear in the left-translate already. Note that the only rotations that do not appear in $r^\ell \circ S$ are $r^{\ell+n-j-1}$ and $r^{\ell+n-1}$. So, by definition of the coloring, it suffices to show that the colors assigned to $r^{\ell+n-j-1}$ and $r^{\ell+n-1}$ are the same colors assigned to $sr^{i-j-\ell}, sr^{i-\ell}$.

Case 1: $\ell+n-1 \equiv n-1 \bmod n$. Then, $\chi(r^{\ell+n-1}) = \chi(r^{n-1}) = c_{n-2}$. Also, $\ell+n-1-j \equiv n-1-j \bmod n$ and so $\chi(r^{\ell+n-1-j}) = \chi(r^{n-1-j}) = c_{n-1}$. So, $\ell+n-1 \equiv n-1 \bmod n \implies \ell+i \equiv i \bmod n \implies \ell+i-\ell \bmod n \equiv i-\ell \bmod n$ and so $\chi(sr^{i-\ell}) = \chi(sr^i) = c_{n-1}$. Similarly, $i-j \bmod n \equiv i-\ell-j \bmod n$ and so $\chi(sr^{i-\ell-j}) = \chi(sr^{i-j}) = c_{n-2}$.

Case 2: $\ell+n-1 \equiv n-j-1 \bmod n$. Then, $\chi(r^{\ell+n-1}) = \chi(r^{n-j-1}) = c_{n-1}$. Also $\ell+n-1-j \equiv n-1-j \bmod n \implies \ell-1 \equiv -1-j \bmod n \implies \ell \equiv -j \bmod n \implies 0 \equiv -\ell-j \bmod n \implies i \equiv i-\ell-j \bmod n$ and so $\chi(sr^{i-j-\ell}) = \chi(sr^i) = c_{n-1}$. For $r^{\ell+n-j-1}$, it is not directly clear what the element should be colored, so consider the following cases:

Subcase 1: $\ell-n-j-1 \equiv n-1 \bmod n$. Then, $\chi(r^{\ell+n-j-1}) = \chi(r^{n-1}) = c_{n-2}$. Also, $\ell-n-j-1 \equiv n-1 \bmod n \implies \ell-j \equiv 0 \bmod n \implies \ell \equiv j \bmod n$. However it has also been noted that $\ell \equiv -j \bmod n$ or equivalently $-\ell \equiv j \bmod n$. So, $-j \equiv j \bmod n$. Thus, $i-\ell \equiv i+j \bmod n \equiv i-j \bmod n$ and so $\chi(sr^{i-\ell}) = \chi(sr^{i-j}) = c_{n-2}$.

Subcase 2: $\ell+n-j-1 \equiv k \bmod n$ so that $0 \bmod n \le k \bmod n \le n-j-2 \bmod n$ and so $\chi(r^{\ell+n-j-1}) = c_k$. Next, $\ell+n-j-1 \equiv k \bmod n \implies \ell-j-1 \equiv k \bmod n \implies -j-1 \bmod -\ell+k \bmod n \implies -j-1-k \equiv -\ell \bmod n \implies i-j-1-k \equiv i-\ell \bmod n$. Also, $0 \bmod n \le k \bmod n \le n-j-2 \bmod n \implies 1 \bmod n \le k+1 \bmod n \le n-j-1 \bmod n$. Therefore, $\chi(sr^{i-\ell}) = \chi(sr^{i-j-(k+1)}) = c_k$.

Subcase 3: $\ell+n-j-1 \equiv k \bmod n$ so that $n-j \bmod n \le k \bmod n \le n-2 \bmod n$ and so $\chi(r^{\ell+n-j-1}) = c_{k-1}$. Next, $\ell+n-j-1 \equiv k \bmod n \implies -j-1 \equiv k-\ell \bmod n \implies i-j-1-k \equiv i-\ell \bmod n$. Also, $n-j \bmod n \le k \bmod n \le n-2 \bmod n \implies 1 \bmod n \le k+j+1 \bmod n \le$

$j-1 \bmod n$ and so $\chi(sr^{i-\ell}) = \chi(sr^{i-(k+j+1)}) = c_{k-1}$.

Case 3: $\ell+n-1 \equiv k$ so that $0 \bmod n \leq k \bmod n \leq n-j-2 \bmod n$. Then, $\chi(r^{\ell+n-1}) = c_k$. Also, $\ell+n-1 \equiv k \implies \ell-1 \equiv k \bmod n \implies -k-1 \equiv -\ell \bmod n \implies i-j-k-1 \equiv i-j-\ell \bmod n$ as well as $0 \bmod n \leq k \bmod n \leq n-j-2 \bmod n \implies 1 \bmod n \leq k+1 \bmod n \leq n-j-1 \bmod n$. Therefore, $\chi(sr^{i-j-\ell}) = \chi(sr^{i-j-(k+1)}) = c_k$. For $r^{\ell+n-j-1}$, it is not directly clear what the element should be colored, so consider the following cases:

Subcase 1: $\ell+n-j-1 \equiv n-1 \bmod n$ and so $\chi(r^{\ell+n-j-1}) = \chi(r^{n-1}) = c_{n-2}$. Also, $\ell+n-j-1 \equiv n-1 \bmod n \implies \ell-j \equiv 0 \bmod n \implies -j \equiv -\ell \bmod n \implies i-j \equiv i-\ell \bmod n$ and so $\chi(sr^{i-\ell}) = \chi(sr^{i-j}) = c_{n-2}$.

Subcase 2: $\ell+n-j-1 \equiv n-j-1 \bmod n$ and so $\chi(r^{\ell+n-j-1}) = \chi(r^{n-j-1}) = c_{n-1}$. Also, $\ell+n-j-1 \equiv n-j-1 \bmod n \implies \ell \equiv 0 \bmod n \implies 0 \bmod -\ell \bmod n \implies i \equiv i-\ell \bmod n$ and so $\chi(sr^{i-\ell}) = \chi(sr^{i}) = c_{n-1}$.

Subcase 3: $\ell+n-j-1 \equiv k-j \bmod n$ so that $n-j \bmod n \leq k-j \bmod n \leq n-2 \bmod n$ and so $\chi(r^{\ell+n-j-1}) = c_{k-j-1}$. Next, $\ell+n-j-1 \equiv k-j \bmod n \implies i-k-1 \equiv i-\ell \bmod n$ and with $0 \bmod n \leq k \bmod n \leq n-j-2 \bmod n$, then $1 \bmod n \leq k+1 \bmod n \leq n-j-1 \bmod n$. If $1 \bmod n \leq k+1 \bmod n \leq j-1 \bmod n$, then $\chi(sr^{i-\ell}) = \chi(sr^{i-k-1}) = c_{n-j+k+1-2} = c_{k-j-1}$. If $j \bmod n \leq k+1 \bmod n \leq n-j-1 \bmod n$, then $j-1 \bmod n \leq k \bmod n \leq n-j-2 \bmod n$ and consequently $0 \bmod n \leq k-j \bmod n \leq n-2j-2 \bmod n$ since if $k-j \equiv n-1$ this is subcase 1. However, it was assumed $n-j \bmod n \leq k-j \bmod n \leq n-2 \bmod n$ and $n-j \bmod n > n-2j-2 \bmod n$ with $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$ unless n is odd and $j = \frac{n-1}{2}$. In which case $n-j \equiv \frac{n+1}{2}$ and $n-2j-2 \equiv n-1$ and so $j \bmod k+1 \bmod n-j-1 \implies \frac{n-1}{2} \bmod n \leq k+1 \bmod n \leq \frac{n-1}{2} \implies i-(k+1) \equiv i-\frac{n-1}{2} \equiv i-j$ and so $\chi(sr^{i-\ell}) = \chi(sr^{i-k-1}) = \chi(sr^{i-j}) = c_{n-2}$. Also, $k \equiv \frac{n-3}{2}$ and so $k-j-1 \equiv n-2$ and so $\chi(r^{\ell+n-j-1}) = c_{k-j-1} = c_{n-2}$.

Subcase 4: $\ell+n-j-1 \equiv k-j \bmod n$ so that $0 \bmod n \leq k-j \bmod n \leq n-j-2 \bmod n$ and so $\chi(r^{\ell+n-j-1}) = c_{\ell-j-1}$. Note that $0 \bmod n \leq \ell+n-j-1 \bmod n \leq n-j-2 \bmod n \implies 1 \leq \ell-j \leq n-j-1$. Also, $sr^{i-\ell} = sr^{i-j+j-\ell} = sr^{i-j-(\ell-j)}$ and so $\chi(sr^{i-\ell}) = c_{\ell-j-1}$.

Case 4: $\ell+n-1 \equiv k$ so that $n-j \bmod n \leq k \bmod n \leq n-2 \bmod n$. Then, $\chi(r^{\ell+n-1}) = c_{k-1}$.

It is not immediately obvious what color should be assigned to $r^{\ell+n-j-1}$, so the following cases are considered:

Subcase 1: $\chi(r^{\ell+n-j-1}) = c_{n-1}$. Note that $\ell+n-j-1 \bmod n \equiv n-j-1 \bmod n \implies \ell \bmod n \equiv 0$ together with $\ell+n-1 \equiv k \implies k \equiv n-1$. Therefore, $\chi(r^{\ell+n-1}) = \chi(r^k) = \chi(r^{n-1}) = c_{n-2}$, $\chi(sr^{i-\ell}) = \chi(sr^i) = c_{n-1}$, and $\chi(sr^{i-j-\ell}) = \chi(sr^{i-j}) = c_{n-2}$.

Subcase 2: $\chi(r^{\ell+n-j-1}) = c_{n-2}$. Note that $\ell+n-j-1 \equiv n-1 \implies \ell-j \equiv 0 \implies \ell \equiv j$ together with $\ell+n-1 \equiv k \implies j-1 \equiv k \implies j \equiv k+1$. So, $\chi(sr^{i-\ell}) = \chi(sr^{i-j}) = c_{n-2}$ and $\chi(r^{\ell+n-1}) = c_{k-1} = c_{j-2}$. Then, it is either the case that $\chi(sr^{i-j-\ell})$ is $c_{\ell-1}$ with $1 \leq \ell \leq n-j-1$ in this case however it was assumed $n-j+1 \leq \ell = k+1 \leq n-1$ and $n-j-1 < n-j+1$ with $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$ unless $j=1$ and so $\ell = k+1 = 1$ and $k=0$ thus $\chi(sr^{i-j-\ell}) = \chi(sr^{i-1-1}) = c_0$, $\chi(sr^{i-\ell}) = \chi(sr^{i-1}) = c_{n-2}$, $\chi(r^{\ell+n-1}) = \chi(r^{1+n-1}) = \chi(1) = c_0$, and $\chi(r^{\ell+n-j-1}) = \chi(r^{1+n-1-1}) = \chi(r^{n-1}) = c_{n-2}$; or $\chi(sr^{i-j-\ell})$ is c_{n-1} which yields $-j-\ell \equiv 0 \implies j \equiv n-j$ and so $\ell+n-1 \equiv j+n-1 \equiv j-1 \equiv n-j-1$ thus $\chi(r^{\ell+n-1}) = \chi(r^{n-j-1}) = c_{n-1}$; and so $\chi(sr^{i-j-\ell}) = c_{j+\ell+n-j-2} = c_{\ell-2} = c_{j-2}$ and $1 \leq j+\ell \leq j-1$.

Subcase 3: $\chi(r^{\ell+n-j-1}) = c_{k-j-1}$ where $n-j \leq k-j \leq n-2$. However, it was assumed that $n-j \leq k \leq n-2$ and this implies $n-2j \leq k-j \leq n-j-2$. Since $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$, there is no value of j which satisfies both of these inequalities.

Subcase 4: $\chi(r^{\ell+n-j-1}) = c_{k-j}$ where $0 \leq k-j \leq n-j-2$ with $k \equiv \ell+n-1$. Note that $0 \leq k-j \leq n-j-2 \implies j+1 \leq k+1 \leq n-1 \implies 1 \leq \ell-j \leq n-j-1$ and $i-\ell \equiv i-j+j-\ell \equiv i-j-(\ell-j)$, so $\chi(sr^{i-\ell}) = c_{\ell-j-1} = c_{k-j}$ as $k \equiv \ell+n-1 \implies k-j \equiv \ell-j-1$. Also, as it was assumed $n-j \leq k \leq n-2$, then $n-j+1 \leq k+1 \leq n-1 \implies n-2j+1 \leq \ell-j \leq n-j-1 \implies 1 \leq \ell-j \leq j-1$, so $\chi(sr^{i-j-\ell}) = \chi(sr^{i-(j+\ell)}) = c_{n+j+\ell-j-2} = c_{\ell-2} = c_{k+1-2} = c_{k-1}$.

Similarly, for left-translates of the form $sr^\ell \circ S$, it is clear from the supplied coloring that all reflections are distinctly colored, so all that must be shown is that the two rotations are colored with two distinct colors which do not appear in the left-translate already. Note that the only reflections that do not appear in $sr^\ell \circ S$ are $sr^{\ell+n-j-1}$ and $sr^{\ell+n-1}$. So, by definition of the coloring, it suffices to show that the colors assigned to $sr^{\ell+n-j-1}$ and $sr^{\ell+n-1}$ are the same colors assigned to $r^{i-j-\ell}, r^{i-\ell}$.

Case 1: $i-\ell \equiv n-j-1$ and so $\chi(r^{i-\ell}) = \chi(r^{n-j-1}) = c_{n-1}$. Also, $i-j-\ell \equiv n-2j-1$. It is not readily obvious which colors to assign to $r^{i-j-\ell}$, $sr^{\ell+n-j-1}$, or $sr^{\ell+n-1}$ so the following cases are considered:

Subcase 1: $n-2j-1 \equiv n-1 \bmod n$ and so $\chi(r^{i-j-\ell}) = \chi(r^{n-2j-1}) = c_{n-2}$. So, $n-2j-1 \equiv n-1 \bmod n \implies -2j \equiv 0 \bmod n \implies 0 \equiv 2j \bmod n \implies j \equiv \frac{n}{2}$, but $j \leq \lfloor \frac{n-1}{2} \rfloor < \frac{n}{2}$.

Subcase 2: $n-2j-1 \equiv k \bmod n$ with $n-j \leq k \leq n-2$. However, since $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$, then $n-1 \geq n-j \geq n - \lfloor \frac{n-1}{2} \rfloor \implies n-2 \geq n-j-1 \geq n - \lfloor \frac{n-1}{2} \rfloor - 1 \implies n-j-2 \geq n-2j-1 \geq n - \lfloor \frac{n-1}{2} \rfloor - j - 1$. Note that $n-j > n-j-2$. So this subcase is not possible.

Subcase 3: $n-2j-1 \equiv k \bmod n$ with $0 \leq k \leq n-j-2$. So, $\chi(r^{i-j-\ell}) = \chi(r^{n-2j-1}) = c_k = c_{n-2j-1}$. Note that $\ell - n - j - 1 \equiv n-2j-1+j+\ell \equiv n-2j-1+j+i+j+1 \equiv i$ and so $\chi(sr^{\ell+n-j-1}) = \chi(sr^i) = c_{n-1}$. Note that $\ell+n-1 \equiv n-2j-1+2j+\ell \equiv n-2j-1+2j+i+j+1 \equiv i+j$. Next it is not readily obvious which colors to assign to $sr^{\ell+n-1}$ so the following cases are considered:

Subsubcase 1: $\ell+n-1 \equiv i-j$. Then, $i-j \equiv i+j \implies 2j \equiv 0 \implies j \equiv \frac{n}{2}$ which is not possible as $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$.

Subsubcase 2: $\ell+n-1 \equiv i+j \equiv i+j+n-2j-1+2j+1 \equiv k+i+j+2j+1 \equiv i-j-(n-k-4j-1)$ with $1 \leq n-k-4j-1 \leq n-j-1$. Note that $n-k-4j-1 \equiv n-(n-2j-1)-4j-1 \equiv 2j+1-4j-1 \equiv n-2j \bmod n$. So, $\chi(sr^{\ell+n-1}) = \chi(sr^{i+j}) = c_{n-2j-1}$. Also, note that since $0 \leq n-2j-1 \leq n-j-2 \implies 1 \leq n-2j \leq n-j-1$.

Subsubcase 3: $\ell+n-1 \equiv i+j \equiv i+j+n-2j-1+2j+1 \equiv i-(-k-3j-1)$ with $1 \leq n-k-3j-1 \leq j-1$. Note that $n-k-3j-1 \equiv n-(n-2j-1)-3j-1 \equiv n-j$, so $1 \leq n-j \leq j-1$. However, $0 \leq n-2j-1 \leq n-j-2 \implies j+1 \leq n-j \leq n-1$ which is a contradiction.

Case 2: $i-\ell \equiv n-1 \bmod n$. Then, $\chi(r^{i-\ell}) = \chi(r^{n-1}) = c_{n-2}$ and $i+1 \equiv \ell \bmod n$. So, $i-j-\ell \equiv n-j-1$ and thus $\chi(r^{i-j-\ell}) = \chi(r^{n-j-1}) = c_{n-1}$. Also, $\ell+n-1 \equiv i+1+n-1 \equiv i \bmod n$ and so $\chi(sr^{\ell+n-1}) = \chi(sr^i) = c_{n-1}$. Finally, $\ell+n-1-j \equiv i-j \bmod n$, so $\chi(sr^{\ell+n-j+1}) = \chi(sr^{i-j}) = c_{n-2}$.

Case 3: $i-\ell \equiv k \bmod n$ with $0 \leq k \leq n-j-2$, so $\chi(r^{i-\ell}) = \chi(r^k) = c_k$. Note that $i-\ell \equiv k \implies$

$\ell \equiv i-k \bmod n$ and so $\ell+n-j-1 \equiv i-j-(k+1)$ and $0 \leq k \leq n-j-2 \implies 1 \leq k+1 \leq n-j-1$. Thus, $\chi(sr^{\ell+n-j-1}) = \chi(sr^{i-j-(k+1)}) = c_{k+1-1} = c_k$. It is not readily obvious which colors to assign to $r^{i-j-\ell}$ or $sr^{\ell+n-1}$ so the following cases are considered:

Subcase 1: $\chi(r^{i-j-\ell}) = \chi(r^{k-j}) = c_{n-2}$ and so $k-j \equiv n-1 \implies k+1 \equiv j$. Thus, $\ell+n-1 \equiv \ell+k-j \equiv i-k+k-j \equiv i-j$ and so $\chi(sr^{\ell+n-1}) = \chi(sr^{i-j}) = c_{n-2}$.

Subcase 2: $\chi(r^{i-j-\ell}) = \chi(r^{k-j}) = c_{n-1}$ and so $k-j \equiv n-j-1 \implies k \equiv n-1$. Thus, $\ell+n-1 \equiv i-k+k \equiv i$ and so $\chi(sr^{\ell+n-1}) = \chi(sr^{i}) = c_{n-1}$.

Subcase 3: $\chi(r^{i-j-\ell}) = c_{i-j-\ell-1} = c_{i-j-(i-k)-1} = c_{k-j-1}$ with $n-j \leq i-j-\ell \leq n-2 \implies 1 \leq k \leq j-1$. Since $\ell+n-1 \equiv i-k-1$, $\chi(sr^{\ell+n-1}) = \chi(sr^{i-(k+1)}) = c_{k+1-j-2} = c_{k-j-1}$.

Subcase 4: $\chi(r^{i-j-\ell}) = c_{i-j-\ell} = c_{i-j-(i-k)} = c_{k-j}$ with $0 \leq i-j-\ell \leq n-j-2 \implies 1 \leq k-j+1 \leq n-j-1$. Since $\ell+n-1 \equiv i-k-1 \equiv i-j+j-k-1 \equiv i-j-(k-j+1)$, $\chi(sr^{\ell+n-1}) = \chi(sr^{i-j-(k-j+1)}) = c_{k-j+1-1} = c_{k-j}$.

Case 4: $i-\ell \equiv k \bmod n$ with $n-j \leq k \leq n-2$, so $\chi(r^{i-\ell}) = \chi(r^{k}) = c_{k-1}$ and $i-\ell \equiv k \bmod n \implies i-k \equiv \ell \bmod n$ and $i-j-\ell \equiv k-j \bmod n$. It is not readily obvious which colors to assign to $r^{i-j-\ell}$, $sr^{\ell+n-j-1}$, or $sr^{\ell+n-1}$ so the following cases are considered:

Subcase 1: $\chi(r^{i-j-\ell}) = \chi(r^{k-j}) = c_{n-2}$. So, $k-j \equiv n-1$. Thus, $\ell+n-1 \equiv i-k+k-j \equiv i-j \bmod n$ and so $\chi(sr^{\ell+n-1}) = \chi(sr^{i-j}) = c_{n-2}$. Also, $\ell+n-j-1 \equiv i-j-k-1 \equiv i-(k+j+1)$ and $n-j \leq k \leq n-2 \implies 1 \leq k+j+1 \leq j-1$, so $\chi(sr^{\ell+n-j-1}) = \chi(sr^{i-(k+j+1)}) = c_{k+j+1-j-2} = c_{k-1}$.

Subcase 2: $\chi(r^{i-j-\ell}) = \chi(r^{i-j-\ell}) = \chi(r^{k-j}) = c_{n-1}$. So, $k-j \equiv n-j-1 \implies k \equiv n-1$. Thus, $\ell+n-1 \equiv \ell+k \equiv i-k+k \equiv i$ and so $\chi(sr^{\ell+n-1}) = \chi(sr^{i}) = c_{n-1}$. Also, $\ell+n-j-1 \equiv i-j \bmod n$. Thus, $\chi(sr^{\ell+n-j-1}) = \chi(sr^{i-j}) = c_{n-2}$. Finally, $\chi(r^{k}) = \chi(r^{n-1}) = c_{n-2}$.

Subcase 3: $\chi(r^{i-j-\ell}) = \chi(r^{k-j}) = c_{k-j}$ with $0 \leq k-j \leq n-j-2 \implies 1k-j+1 \leq n-j-1$. Note that $n-j \leq k \leq n-2 \implies 1 \leq k+j+1 \leq j-1$ and $\ell+n-j-1 \equiv i-j-k-1 \equiv i-(k+j+1)$, so $\chi(sr^{\ell+n-j-1}) = \chi(sr^{i-(k+j+1)}) = c_{k+j+1-j-2} = c_{k-1}$. Also, $\ell+n-1 \equiv \ell-1 \equiv i-k-1 \equiv i-j+j-k-1 \equiv i-j-(k-j+1)$ and $\chi(sr^{\ell+n-1}) = \chi(sr^{i-j-(k-j+1)}) = c_{k-j+1-1} = c_{k-j}$.

Subcase 4: $\chi(r^{i-j-\ell}) = \chi(r^{k-j}) = c_{k-j-1}$ with $n-j \leq k-j \leq n-2 \implies 1 \leq k+1 \leq j-1$. Also, $n-j \leq k \leq n-2 \implies 1 \leq k+j+1 \leq j-1$. So, $\ell+n-1 \equiv \ell-1 \equiv i-k-1 \equiv i-(k+1)$ and thus

$\chi(sr^{\ell+n-1}) = \chi(sr^{i-(k+1)}) = c_{k+1-j-2} = c_{k-j-1}$. Note that $\ell+n-j-1 \equiv i-j-(k+1) \equiv i-(k+j+1)$ and so $\chi(sr^{\ell+n-j-1}) = \chi(sr^{i-(k+j+1)}) = c_{k+j+1-j-2} = c_{k-1}$. □

If instead a subset of D_{2n} contains all of the rotations except for r^{n-3}, r^{n-2}, and r^{n-1}, then $p^L(S) = n = p^R(S)$ if and only if the remaining three elements in the subset are rotations which must satisfy certain conditions.

PROPOSITION 3.2.7. *Let $S = \{1, r, r^2, \ldots, r^{n-4}, sr^i, sr^j, sr^k\} \subset D_{2n}$ with $n \geq 7$ and $p^L(S) = n = p^R(S)$ if and only if without loss of generality $j = i+1$ and $k = i+2$*

Proof. By Lemma 3.2.6, with $i_1 = 1$ and $m = n-3$, $p^L(S) = p^R(S)$. So, without loss of generality, the following argument will be given with respect to left-translates. The left-translates of S are of the form $r^\ell \circ S = \{r^\ell, r^{\ell+1}, r^{\ell+2}, \ldots, r^{\ell+n-4}, sr^{i-\ell}, sr^{j-\ell}, sr^{k-\ell}\}$ and
$sr^\ell \circ S = \{sr^\ell, sr^{\ell+1}, sr^{\ell+2}, \ldots, sr^{\ell+n-4}, r^{i-\ell}, r^{j-\ell}, r^{k-\ell}\}$ for all $0 \leq \ell \leq n-1$. Consider the following assignment $\chi(r^\ell) = c_\ell$ for $0 \leq \ell \leq n-4$ with $\chi(r^{n-3}) = c_{n-1}$, $\chi(r^{n-2}) = c_{n-2}$, $\chi(r^{n-1}) = c_{n-3}$, and $\chi(sr^{i-\ell}) = c_{\ell-1}$ for all $1 \leq \ell \leq n-3$ and/or $\chi(sr^{i+\ell}) = c_{n-\ell-1}$ for all $3 \leq \ell \leq n-1$ with $\chi(sr^i) = c_{n-3}$, $\chi(sr^j) = \chi(sr^{i+1}) = c_{n-2}$, $\chi(sr^k) = \chi(sr^{i+2}) = c_{n-1}$. The following left-translates are colored with n distinct colors:

$\{1, r, r^2, \ldots, r^{n-4}, sr^i, sr^{i+1}, sr^{i+2}\}$ colored $c_0, c_1, c_2, \ldots, c_{n-4}, c_{n-3}, c_{n-2}, c_{n-1}$ respectively

$\{r, r^2, r^3, \ldots, r^{n-4}, r^{n-3}, sr^{i-1}, sr^i, sr^{i+1}\}$ colored $c_1, c_2, c_3, \ldots, c_{n-4}, c_{n-1}, c_0, c_{n-3}, c_{n-2}$ respectively

$\{r^2, r^3, r^4, \ldots, r^{n-4}, r^{n-3}, r^{n-2}, sr^{i-2}, sr^{i-1}, sr^i\}$ colored $c_2, c_3, c_4, \ldots, c_{n-4}, c_{n-1}, c_{n-2}, c_1, c_0, c_{n-3}$ respectively

$\{r^{n-3}, r^{n-2}, r^{n-1}, 1, \ldots, r^{n-7}, sr^{i+3}, sr^{i+4}, sr^{i+5}\}$ colored $c_{n-1}, c_{n-2}, c_{n-3}, c_0, \ldots, c_{n-7}, c_{n-4}, c_{n-5}, c_{n-6}$ respectively

$\{r^{n-2}, r^{n-1}, 1, r, \ldots, r^{n-6}, sr^{i+2}, sr^{i+3}, sr^{i+4}\}$ colored $c_{n-2}, c_{n-3}, c_0, c_1, \ldots, c_{n-6}, c_{n-1}, c_{n-4}, c_{n-5}$ respectively

$\{r^{n-1}, 1, r, r^2, \ldots, r^{n-5}, sr^{i+1}, sr^{i+2}, sr^{i+3}\}$ colored $c_{n-3}, c_0, c_1, c_2, \ldots, c_{n-5}, c_{n-2}, c_{n-1}, c_{n-4}$ respectively

If $3 \leq \ell \leq n-4$, then $sr^{i-\ell}$, $sr^{i+1-\ell}$, and $sr^{i+2-\ell}$ are colored $c_{\ell-1}$, $c_{\ell-2}$, $c_{\ell-3}$ respectively. Also, there is some positive integer d so that $\ell + d = n-4$, so the rotations in $r^\ell \circ S$ are r^ℓ, $r^{\ell+1}$,

$\ldots$, $r^{\ell+d} = r^{n-4}$ which are colored c_ℓ, $c_{\ell+1}$, $\ldots$, c_{n-4} respectively, $r^{\ell+d+1} = r^{n-3}$, $r^{\ell+d+2} = r^{n-2}$, $r^{\ell+d+3} = r^{n-1}$ which are colored c_{n-1}, c_{n-2}, c_{n-3} respectively, and $r^{\ell+d+4} = 1$, $r^{\ell+d+5} = r$, $\ldots$, $r^{\ell+n-4} = r^{\ell-4}$ which are colored c_0, c_1, $\ldots$, $c_{\ell-4}$. The colors c_0, $\ldots$, $c_{\ell-4}$, $c_{\ell-3}$, $c_{\ell-2}$, $c_{\ell-1}$, c_ℓ, $c_{\ell+1}$, $\ldots$, c_{n-4}, c_{n-3}, c_{n-2}, c_{n-1} are distinct and so all left-translates of the form $r^\ell \circ S$ contains n distinct colors. Next, The following left-translates are colored with n distinct colors:

$\{sr^{i+1}, sr^{i+2}, sr^{i+3}, \ldots, sr^{i-3}, r^{n-1}, 1, r\}$ colored $c_{n-2}, c_{n-1}, c_{n-4}, \ldots, c_2, c_{n-3}, c_0, c_1$ respectively

$\{sr^{i+2}, sr^{i+3}, sr^{i+4}, \ldots, sr^{i-2}, r^{n-2}, r^{n-1}, 1\}$ colored $c_{n-1}, c_{n-4}, c_{n-5}, \ldots, c_1, c_{n-2}, c_{n-3}, c_0$ respectively

$\{sr^{i+3}, sr^{i+4}, sr^{i+5}, \ldots, sr^{i-1}, r^{n-3}, r^{n-2}, r^{n-1}\}$ colored $c_{n-4}, c_{n-5}, c_{n-6}, \ldots, c_0, c_{n-1}, c_{n-2}, c_{n-3}$ respectively

$\{sr^{i+4}, sr^{i+5}, sr^{i+6}, \ldots, sr^{i-1}, sr^i, r^{n-4}, r^{n-3}, r^{n-2}\}$ colored $c_{n-5}, c_{n-6}, c_{n-7}, \ldots, c_0, c_{n-3}, c_{n-4}, c_{n-1}, c_{n-2}$ respectively

$\{sr^{i+5}, sr^{i+6}, sr^{i+7}, \ldots, sr^{i-1}, sr^i, sr^{i+1}, r^{n-5}, r^{n-4}, r^{n-3}\}$ colored $c_{n-6}, c_{n-7}, c_{n-8}, \ldots, c_0, c_{n-3}, c_{n-2}, c_{n-5}, c_{n-4}$, c_{n-1} respectively

If $\ell = i + d$ for some $0 = d$ or $6 \le d \le n-1$, then there is some positive integer m so that $d + m \equiv 0 \bmod n$ so that the elements of $sr^\ell \circ S$ are sr^{i+d}, sr^{i+d+1}, sr^{i+d+2}, $\ldots$, $sr^{i+d+m-1} = sr^{i-1}$, $sr^{i+d+m} = sr^i$, $sr^{i+d+m+1} = sr^{i+1}$, $sr^{i+d+m+2} = sr^{i+2}$, sr^{i+3}, $\ldots$, $sr^{i+d+n-5}$, $sr^{i+d+n-4}$, $r^{i-(i+d)} = r^{-d}$, $r^{i+1-(i+d)} = r^{1-d}$, $r^{i+2-(i+d)} = r^{2-d}$. These elements are colored the n distinct colors c_{n-d-1}, c_{n-d-2}, c_{n-d-3}, $\ldots$, c_0, c_{n-3}, c_{n-2}, c_{n-1}, c_{n-4}, $\ldots$, c_{n-d+4}, c_{n-d+3}, c_{n-d}, c_{n-d+1}, c_{n-d+2} respectively.

For the other direction, suppose $p^L(S) = n = p^R(S)$ and $n \ge 7$. Without loss of generality, the following argument is with respect to the left-translates. Then, the elements of S are assigned n colors without loss of generality as follows $\chi(r^\ell) = c_\ell$ for $0 \le \ell \le n-4$ and $\chi(sr^i) = c_{n-3}$, $\chi(sr^j) = c_{n-2}$, $\chi(sr^k) = c_{n-1}$. Then, the element r^{n-1} appears in left-translates with r^ℓ for all $0 \le \ell \le n-4$ (consider translates $r^{n-1} \circ S$ and $r^{n-4} \circ S$) and hence appears in left-translates with colors c_ℓ for all $0 \le \ell \le n-4$. So, one of the remaining three colors must be assigned to r^{n-1}. Also, there are only three left-translates which do not contain r^{n-1} and so potentially contain reflections which do not appear in a left-translate with r^{n-1}. These left-translates are S, $r \circ S$, $r^2 \circ S$ and the reflections in question are sr^i,sr^{i-1}, sr^{i-2}, sr^j, sr^{j-1}, sr^{j-2}, sr^k, sr^{k-1}, sr^{k-2}. So, one reflection must be in all three of these left-translates because the color that is assigned to r^{n-1} must appear in these translates. That is, one of the following must hold: $j = i-1 = k-2$, $j = i-2 = k-1$, $i = j-1 = k-2$, $i = j-2 = k-1$, $k = j-1 = i-2$, $k = j-2 = i-1$. In any case, the same relationship among i, j, k is yielded: sr^i, sr^{i+1}, sr^{i+2}. □

In general, there appears to be a pattern among subsets with $p^L(S) = n = p^R(S)$ containing both reflections and rotations. The following observations, lemma, remark, and proposition were formed in order to partially determine this pattern.

OBSERVATION 3.2.6. *Suppose $S = \{1, r^{i_1}, r^{i_2}, \dots r^{i_{\ell-1}}, sr^{j_1}, sr^{j_2}, \dots, sr^{j_{n-\ell}}\} \subset D_{2n}$ and n colors can be assigned to each left-translate of the form $r^k \circ S$ via an assignment χ so that all rotations are colored distinctly and all reflections are colored distinctly, then $p^L(S) = n$.*

Proof. Suppose χ is such an assignment of colors to the left-translates of S so that n colors are assigned to each left-translate of the form $r^k \circ S$. The remaining left-translates of S are of the form

$$s \circ S = \{s, sr^{i_1}, sr^{i_2}, \dots sr^{i_{\ell-1}}, r^{j_1}, r^{j_2}, \dots, r^{j_{n-\ell}}\}$$

and

$$sr^z \circ S = \{sr^z, sr^{i_1+z}, sr^{i_2+z}, \dots, sr^{i_{\ell-1}+z}, r^{j_1-z}, r^{j_2-z}, \dots, r^{j_{n-\ell}-z}\}$$

for all $1 \leq z \leq n-1$. Also, since χ ensures that all rotations are colored distinctly and all reflections are colored distinctly, all that must be shown is that any given rotation and any given reflection in the same left-translate of the form $s \circ S$ or $sr^z \circ S$ for all $1 \leq z \leq n-1$ are not colored using the same color. The existence of left-translates of the forms

$$r^{j_f-z} \circ S = \{r^{j_f-z}, r^{i_1+j_f-z}, r^{i_2+j_f-z}, \dots, r^{i_{\ell-1}+j_f-z}, sr^{j_1-j_f+z}, sr^{j_2-j_f+z}, \dots, sr^z, \dots, sr^{j_{n-\ell}-j_f+z}\}$$

and

$$r^{j_f-i_d-z} \circ S = \{ r^{j_f-i_d-z}, r^{j_f-i_d+i_1-z}, r^{j_f-i_d+i_2-z}, \dots, r^{j_f-i_d+i_{\ell-1}-z},$$
$$sr^{j_1-j_f+i_d+z}, sr^{j_2-j_f+i_d+z}, \dots, sr^{i_d+z}, \dots, sr^{j_{n-\ell}-j_f+i_d+z} \}$$

for all $1 \leq f \leq n - \ell$, $1 \leq d \leq \ell - 1$, and $0 \leq z \leq n - 1$ imply that $\chi(r^{j_f - z}) \neq \chi(sr^{i_d + z})$. $\square$

OBSERVATION 3.2.7. *Suppose* $S = \{1, r^{i_1}, r^{i_2}, \ldots r^{i_{\ell-1}}, sr^{j_1}, sr^{j_2}, \ldots, sr^{j_{n-\ell}}\} \subset D_{2n}$ *and* n *colors can be assigned to each right-translate of the form* $S \circ r^k$ *via an assignment* χ *so that all rotations are colored distinctly and all reflections are colored distinctly, then* $p^R(S) = n$.

Proof. Suppose χ is such an assignment of colors to the right-translates of S so that n colors are assigned to each right-translate of the form $S \circ r^k$. The remaining right-translates of S are of the form

$$S \circ s = \{s, sr^{-i_1}, sr^{-i_2}, \ldots sr^{-i_{\ell-1}}, r^{-j_1}, r^{-j_2}, \ldots, r^{-j_{n-\ell}}\}$$

and

$$S \circ sr^z = \{sr^z, sr^{z-i_1}, sr^{z-i_2}, \ldots, sr^{z-i_{\ell-1}}, r^{z-j_1}, r^{z-j_2}, \ldots, r^{z-j_{n-\ell}}\}$$

for all $1 \leq z \leq n-1$. Also, since χ ensures that all rotations are colored distinctly and all reflections are colored distinctly, all that must be shown is that any given rotation and any given reflection in the same right-translate of the form $s \circ S$ or $sr^z \circ S$ for all $1 \leq z \leq n - 1$ are not colored using the same color. The existence of right-translates of the forms

$$S \circ r^{z-j_f} = \{r^{z-j_f}, r^{z+i_1-j_f}, r^{z+i_2-j_f}, \ldots, r^{z+i_{\ell-1}-j_f}, sr^{j_1-j_f+z}, sr^{j_2-j_f+z}, \ldots, sr^z, \ldots, sr^{j_{n-\ell}-j_f+z}\}$$

and

$$S \circ r^{z-j_f-i_d} = \{ r^{z-j_f-i_d}, r^{z-j_f-i_d+i_1}, r^{z-j_f-i_d+i_2}, \ldots, r^{z-j_f-i_d+i_{\ell-1}},$$
$$sr^{j_1-j_f-i_d+z}, sr^{j_2-j_f-i_d+z}, \ldots, sr^{z-i_d}, \ldots, sr^{j_{n-\ell}-j_f-i_d+z} \}$$

for all $1 \leq f \leq n - \ell$, $1 \leq d \leq \ell - 1$, and $0 \leq z \leq n - 1$ imply that $\chi(r^{z-j_f}) \neq \chi(sr^{z-i_d})$. $\square$

DEFINITION 3.2.7. *Suppose* $S = \{1, r^{i_1}, \ldots, r^{i_{k-1}} sr^{j_1}, \ldots, sr^{j_t}\} \subset D_{2n}$ *such that* $k + t = n$. *Let colors used to color* S *be denoted by* c_i *for all* $0 \leq i \leq n - 1$. *Let* $|S|_{c_i}$ *be the number of elements belonging to* S *that are colored* c_i.

LEMMA 3.2.15. *Suppose* $p^L(S) = n$ *(also do the same argument for* $p^R(S) = n$*) where* $S = \{1, r^{i_1}, \ldots, r^{i_{k-1}} sr^{j_1}, \ldots, sr^{j_t}\} \subset D_{2n}$ *such that* $k + t = n$, *then* $|S|_{c_i} = 2$ *for all* $0 \leq i \leq n - 1$.

Proof. Since $p^L(S) = n$ ($p^R(S) = n$), n elements must be colored distinctly. Thus, the coloring of n elements must be determined. Suppose it is not the case that these n elements are distributed evenly among color classes. Then, there is some color class that contains only one element. However, this element can not appear in every translate, so this is impossible. Therefore, $|S|_{c_i} = 2$ for all $0 \leq i \leq n-1$. □

REMARK 3.2.7. *If $S \subset D_{2n}$ contains both reflections and rotations, it can always be assumed there are more rotations than reflections.*

Proof. Suppose $S = \{1, r^{j_1}, r^{j_2}, \ldots, r^{j_{m-1}}, sr^{i_1}, sr^{i_2}, \ldots, sr^{i_{m+k}}\}$ where $k, m \geq 1$. Clearly, this translate contains more reflections than rotations. A left-translate of S is

$$sr^{i_1} \circ S = \{1, r^{i_2-i_1}, \ldots, r^{i_{m+k}-i_1}, sr^{i_1}, sr^{i_1+j_1}, sr^{i_1+j_2}, \ldots, sr^{i_1+j_{m-1}}\}$$

which has more rotations than reflections. A right-translate of S is

$$S \circ sr^{i_1} = \{1, r^{i_1-i_2}, \ldots, r^{i_1-i_{m+k}}, sr^{i_1}, sr^{i_1-j_1}, sr^{i_1-j_2}, \ldots, sr^{i_1-j_{m-1}}\}$$

which contains more rotations than reflections. □

Corollary 3.2.3. *If $S \subset D_{2n}$ such that $|S| = n$ and contains both reflections and rotations, then it can be assumed that the number of reflections is less than or equal to $\frac{n}{2}$.*

PROPOSITION 3.2.8. *Suppose $S \subset D_{2n}$ with $n \geq 2$ and S contains k rotations and t reflections so that $|S| = n = k + t$ and $k \neq t$. If $S = \{1, r^{i_1}, \ldots, r^{i_{k-1}}, sr^{n-i_k+j}, \ldots, sr^{n-i_{k+(t-1)}+j}\}$ for any $0 \leq j \leq n-1$ and any $0 \leq i_1, \ldots, i_s, \ldots i_{k-1} \leq n-1$ so that $i_s \neq i_d$ for any $k \leq d \leq k+(t-1)$, then $p(S) \leq |S| = p^L(S) = p^R(S)$.*

Proof. First, since $k \neq t$, it can be assumed that $k > t$ by Remark 3.2.7. Next, if $S = \{1, r^{i_1}, \ldots, r^{i_{k-1}}, sr^{n-i_k+j}, \ldots, sr^{n-i_{k+(t-1)}+j}\}$ for any $0 \leq j \leq n-1$ and any $0 \leq i_1, \ldots, i_{k-1} \leq n-1$, note that n left-translates are of the form

$$r^i \circ S = \{r^i, r^{i+i_1}, \ldots, r^{i+i_{k-1}}, sr^{n-i_k+j-i}, \ldots, sr^{n-i_{k+(t-1)}+j-i}\}.$$

So, the rotations in these left-translates can be written uniquely as r^{i+i_s} for any $0 \leq i \leq n-1$ and $1 \leq s \leq k-1$ and the reflections can be written uniquely as sr^{n-i_d+j-i} for any $k \leq d \leq k+(t-1)$ and $0 \leq i \leq n-1$. Thus, an S^L−polychromatic coloring is given by $\chi(r^{i+i_s}) = c_{i+i_s \bmod n}$ for any $0 \leq i \leq n-1$ and $1 \leq s \leq k-1$ and $\chi(sr^{n-i_d+j-i}) = c_{n-(n-i_d+j-i)+j \bmod n} = c_{i_d+i \bmod n}$ for any $k \leq d \leq k+(t-1)$ and $0 \leq i \leq n-1$. Thus, $r^i \circ S$ consists of elements that are distinctly colored c_i, c_{i+i_1}, …, $c_{i+i_{k-1}}$, c_{i_k+i}, …, $c_{i_{k+(t-1)}+i}$. Therefore, by Observation 3.2.6, $p^L(S) = n$. Next, note that n right-translates are of the form

$$S \circ r^i = \{r^i, r^{i_1+i}, \ldots, r^{i_{k-1}+i}, sr^{n-i_k+j+i}, \ldots, sr^{n-i_{k+(t-1)}+j+i}\}.$$

So, the rotations in these right-translates can be written uniquely as r^{i_s+i} for any $0 \leq i \leq n-1$ and $1 \leq s \leq k-1$ and the reflections can be written uniquely as sr^{n-i_d+j+i} for any $k \leq d \leq k+(t-1)$ and $0 \leq i \leq n-1$. Thus, an S^R−polychromatic coloring is given by $\chi'(r^{i_s+i}) = c_{i_s+i \bmod n}$ for all $0 \leq i \leq n-1$ and $1 \leq s \leq k-1$ and $\chi'(sr^{n-i_d+j+i}) = c_{n-(n-i_d+j+i)+j+2i \bmod n} = c_{i_d+i \bmod n}$ for any $k \leq d \leq k+(t-1)$ and $0 \leq i \leq n-1$. Thus, $S \circ r^i$ consists of elements that are distinctly colored c_i, c_{i_1+i}, …, $c_{i_{k-1}+i}$, c_{i_k+i}, …, $c_{i_{k+(t-1)}+i}$. Therefore, by Observation 3.2.7, $p^R(S) = n$. Hence, by Theorem 3.2.1 $p(S) \leq n$.

□

There are many subsets of the dihedral group that are left unclassified and future work on this group is enticing.

3.2.2 The Quaternion Group and the Dicyclic Group

Consider the small *quaternion* group denoted by Q_8 and defined by $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ with products $\cdot$ computed as follows [10]:

$$1 \cdot a = a \cdot 1 = a, \text{ for all } a \in Q_8$$

$$(-1) \cdot (-1) = 1, \ (-1) \cdot a = a \cdot (-1) = -a, \text{ for all } a \in Q_8$$

$$i \cdot i = j \cdot j = k \cdot k = -1$$

$$i \cdot j = k,\ j \cdot i = -k$$

$$j \cdot k = i,\ k \cdot j = -i$$

$$k \cdot i = j,\ i \cdot k = -j$$

Note that because of these relationships, if $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ where $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, and $c \notin \langle a \rangle, \langle b \rangle$, then $-b \cdot a = a \cdot b$, $-a \cdot b = b \cdot a$, $-c \cdot a = a \cdot c$, $-a \cdot c = c \cdot a$, $-b \cdot c = c \cdot b$, $-c \cdot b = b \cdot c$. Also, $a \cdot b, b \cdot a \in \{\pm c\}$, $a \cdot c, c \cdot a \in \{\pm b\}$, $b \cdot c, c \cdot b \in \{\pm a\}$.

REMARK 3.2.8. *By Theorem 3.2.3, the polychromatic number of any two element subset of Q_8 has been determined.*

PROPOSITION 3.2.9. $p_{Q_8}(\{d, e, f\}) = 2$ *for any* $d \neq e \neq f \in Q_8$.

Proof. First, note that any three element subset of Q_8, $\{d, e, f\}$ has a translate of the form $\{1, -de, -df\}$. Also, note that the order of any element in $Q_8 \setminus \{\pm 1\}$ is four. For subsets of Q_8 of the form $\{1, a, a^2\}$ or $\{1, a, a^3\}$, by Theorem 3.2.2, $p_{Q_8}(\{1, a, a^2\}) = p_{\mathbb{Z}_4}(\{0, 1, 2\}) = 2$, $p_{Q_8}(\{1, a, a^3\}) = p_{\mathbb{Z}_4}(\{0, 1, 3\}) = 2$, and $p_{Q_8}(\{1, -1, a\}) = p_{\mathbb{Z}_4}(\{0, 2, 1\}) = 2$. Next, suppose $b \notin \langle a \rangle$ and $a \notin \langle b \rangle$ such that $S = \{1, a, b\}$. Then, $Q_8 \setminus \{\pm 1\} = \{\pm a, \pm b, \pm c\}$. Suppose, there is an $S-$polychromatic coloring with three colors so that without loss of generality $\chi(1) = c_0$, $\chi(a) = c_1$, and $\chi(b) = c_2$. However, then the assignment $\chi(c)$ can not be made because c appears in $\{c, c \cdot a, c \cdot b\}$ and $\{c, a \cdot c, b \cdot c\}$, so $\chi(c) = \chi(1) = c_0$, but the translates $\{-b, a \cdot b, 1\}$ and $\{-b, b \cdot a, 1\}$ also exist and imply $\chi(c) \neq \chi(1)$. Consider the assignment $\chi(\pm 1) = c_0$, $\chi(\pm a) = c_1$, $\chi(\pm b) = c_0$, and $\chi(\pm c) = c_1$. The left-translates are $\{-1, -a, -b\}$, $\{a, -1, a \cdot b\}$, $\{-a, 1, b \cdot a\}$, $\{b, b \cdot a, -1\}$, $\{-b, a \cdot b, 1\}$, $\{c, c \cdot a, c \cdot b\}$, $\{-c, a \cdot c, b \cdot c\}$ and the right-translates are $\{-1, -a, -b\}$, $\{a, -1, b \cdot a\}$, $\{-a, 1, a \cdot b\}$, $\{b, a \cdot b, -1\}$, $\{-b, b \cdot a, 1\}$, $\{c, a \cdot c, b \cdot c\}$, $\{-c, c \cdot a, c \cdot b\}$. Together with the knowledge that $c \cdot b, b \cdot c \in \{\pm a\}$, $a \cdot c, c \cdot a \in \{\pm b\}$, $a \cdot b, b \cdot a \in \{\pm c\}$, χ is an $S-$polychromatic coloring in two colors. □

Corollary 3.2.4. $ex(Q_8, \{d, e, f\}) \geq 4$ *for any* $d \neq e \neq f \in Q_8$.

The question then arises as to which subsets of the quaternion group have $p(S) = |S|$. Lemma 3.2.1 suggests that the only subsets of Q_8 that have $p(S) = |S|$ are subsets of sizes two and four. Since subsets of size two have already been considered, subsets of size four is the next topic of discussion.

REMARK 3.2.9. *Note that any four element subset of Q_8 $\{a, b, c, d\}$ contains either no elements of order two, one element of order two, or two elements of order two. If it is the first case, then $a \neq b \neq c \neq d \in Q_8 \setminus \{\pm 1\}$ and so at least two elements must belong to the same subgroup of order four, but since each element must have order four, the set is of the form without loss of generality $\{a, b, c, -a\}$ if $b \notin \langle c \rangle$ and $c \notin \langle b \rangle$ and $\{a, b, -b, -a\}$ otherwise. If it is the second case, then without loss of generality $d \in \{\pm 1\}$ and $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ and so the set is of the form $\{1, a, b, c\}$ if $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, $c \notin \langle a \rangle, \langle b \rangle$ or $\{1, a, -a, c\}$ if $a \notin \langle c \rangle$ and $c \notin \langle a \rangle$. Also, $\{-1, a, b, c\}$ is an option for the form of this set, however a translate of $\{1, a, b, c\}$ is $\{-1, -a, -b, -c\}$. If it is the latter most case, then without loss of generality $c, d \in \{\pm 1\}$ and $a \neq b \in Q_8 \setminus \{\pm 1\}$ and so the set is of the form $\{1, -1, a, b\}$ if $a \notin \langle b \rangle$ and $b \notin \langle a \rangle$ or $\{-1, 1, a, -a\}$.*

PROPOSITION 3.2.10. *Let $S \subset Q_8$ such that $|S| = 4$. Then, $p(S) = |S|$ if and only if S is a translate of $\{1, a, -1, -a\}$ where $a \in Q_8 \setminus \{\pm 1\}$ or of $\{1, a, b, c\}$ where $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ and $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, and $c \notin \langle a \rangle, \langle b \rangle$ or of $\{a, b, -b, -a\}$ where $a \neq b \in Q_8$ and $a \notin \langle b \rangle$, $b \notin \langle a \rangle$.*

Proof. Let $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ where $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, and $c \notin \langle a \rangle, \langle b \rangle$. Consider the coloring given by $\chi(1) = c_0$, $\chi(-1) = c_2$, $\chi(a) = c_1$, $\chi(-a) = c_3$, $\chi(b) = c_0$, $\chi(-b) = c_1$, $\chi(c) = c_2$, $\chi(-c) = c_3$. The left- and right-translates are the same three distinct subsets $\{1, -1, a, -a\}$, $\{b, -b, a{\cdot}b, b{\cdot}a\}$, and $\{c, -c, a{\cdot}c, c{\cdot}a\}$. Since $a{\cdot}b, b{\cdot}a \in \{\pm c\}$ and $a{\cdot}c, c{\cdot}a \in \{\pm b\}$, all translates contain four distinct colors. Next, let $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ and $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, and $c \notin \langle a \rangle, \langle b \rangle$. Consider the coloring given by $\chi(1) = c_3 = \chi(-1)$, $\chi(a) = c_2 = \chi(-a)$, $\chi(b) = c_1 = \chi(-b)$, $\chi(c) = c_0 = \chi(-c)$. Note also that $b \cdot c \neq c \cdot b \in \{\pm a\}$, $a \cdot c \neq c \cdot a \in \{\pm b\}$, and $b \cdot a \neq a \cdot b \in \{\pm c\}$. The left- and right-translates are $\{1, a, b, c\}$, $\{-1, -a, -b, -c\}$, $\{a, -1, a \cdot b, a \cdot c\}$, $\{a, -1, b \cdot a, c \cdot a\}$, $\{-a, 1, b{\cdot}a, c{\cdot}a\}$, $\{-a, 1, a{\cdot}b, a{\cdot}c\}$, $\{b, b{\cdot}a, -1, b{\cdot}c\}$, $\{b, a{\cdot}b{-}1, c{\cdot}b\}$, $\{-b, a{\cdot}b, 1, c{\cdot}b\}$, $\{-b, b{\cdot}a, 1, b{\cdot}c\}$,

$\{c, c \cdot a, c \cdot b, -1\}$, $\{c, a \cdot c, b \cdot c, -1\}$, $\{-c, a \cdot c, b \cdot c, 1\}$, and $\{-c, c \cdot a, c \cdot b, 1\}$. Thus, all translates four distinct colors. Finally, let $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ where $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, and $c \notin \langle a \rangle, \langle b \rangle$. Consider the coloring given by $\chi(1) = c_3$, $\chi(-1) = c_0$, $\chi(a) = c_0$, $\chi(-a) = c_3$, $\chi(b) = c_1$, $\chi(-b) = c_2$, $\chi(c) \neq \chi(-c) \in \{c_1, c_2\}$. Note that $a \cdot b \neq b \cdot a \in \{\pm c\}$. The left- and right-translate are $\{a, b, -b, -a\}$, $\{-1, a \cdot b, b \cdot a, 1\}$, $\{c \cdot a, c \cdot b, b \cdot c, a \cdot c\}$. Thus, all translates contain four distinct colors.

By Remark 3.2.9, any such subset's form is given. Suppose there is an $S-$polychromatic coloring χ consisting of four colors in any case. If $\{a, b, c, -a\}$ is the set in question with $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ and its translates are considered, a left-translate is $\{-b, c \cdot b, -1, b\}$. Without loss of generality assign the colors c_0, c_1, c_2, and c_3 respectively. However, a right-translate is $\{-b, b \cdot c, -1, b\}$ and so $\chi(b \cdot c) = c_1$. Note that $c \cdot b \neq b \cdot c \in \{\pm a\}$. Yet, the existence of translate $\{a, b, c, -a\}$ ensures that there is a contradiction. Next, consider $\{1, -1, a, b\}$ with $a \neq b \in Q_8 \setminus \{\pm 1\}$ and $a \notin \langle b \rangle$ or $b \notin \langle a \rangle$. Without loss of generality assign the colors c_0, c_1, c_2, and c_3 to 1, -1, a, b respectively. The existence of left-translates $\{-1, 1, -a, -b\}$ and $\{a, -a, -1, a \cdot b\}$ ensure $\chi(-a) = c_3$. These translates along with $\{-a, a, 1, b \cdot a\}$ ensure $\chi(-b) = c_2$, $\chi(a \cdot b) = c_0$, and $\chi(b \cdot a) = c_1$. However, because $\{-b, b, a \cdot b, 1\}$ is also a translate a contradiction is reached. Finally, consider $\{1, a, -a, c\}$ with $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ where $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, and $c \notin \langle a \rangle, \langle b \rangle$. Without loss of generality assign the colors c_0, c_1, c_2, and c_3 to 1, a, $-a$, c respectively. The translates $\{-1, -a, a, -c\}$ and $\{a, -1, 1, a \cdot c\}$ ensure $\chi(-1) = c_3$ but, the translate $\{c, a \cdot c, c \cdot a, -1\}$ gives a contradiction. In all other instances the subset has such an $S-$polychromatic coloring because it is of the desired form. □

Corollary 3.2.5. *Let $S \subset Q_8$ such that $|S| = 4$. Then, $ex(Q_8, S) \geq 6$ if and only if S is a translate of $\{1, a, -1, -a\}$ where $a \in Q_8 \setminus \{\pm 1\}$ or of $\{1, a, b, c\}$ where $a \neq b \neq c \in Q_8 \setminus \{\pm 1\}$ and $a \notin \langle b \rangle, \langle c \rangle$, $b \notin \langle a \rangle, \langle c \rangle$, and $c \notin \langle a \rangle, \langle b \rangle$ or of $\{a, b, -b, -a\}$ where $a \neq b \in Q_8$ and $a \notin \langle b \rangle$, $b \notin \langle a \rangle$.*

The quaternion group is one of the smaller self-contained finite nonabelian groups that one learns as an undergraduate student in abstract algebra, however it is actually one group in a larger family of finite nonabelian groups. Consider the dicyclic group denoted Dic_n for each integer $n > 1$

is given by the presentation:

$$Dic_n = \langle a, x | a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle.$$

Thus, every element of Dic_n can be uniquely written as $a^k x^j$ where $0 \leq k < 2n$ and $j = 0$ or 1. The multiplication rules are given by:

(i) $a^k a^m = a^{k+m}$;

(ii) $a^k a^m x = a^{k+m} x$;

(iii) $a^k x a^m = a^{k-m} x$;

(iv) $a^k x a^m x = a^{k-m+n}$

where superscripts of generator a are taken modulo $2n$. It follows that Dic_n has order $4n$. Also, when $n = 2$, the dicyclic group is the quaterion group Q_8 (simply take $a = i$ and $b = j$) and when n is a power of two, is called the generalized quaterion group.

By the above result, it is known which translates of Dic_n of cardinality two have $p(S) = |S|$, however what is the answer for $|S| = 3$ and also is it true that $p^L(S) = p^R(S)$? Work towards partial answers to these questions is what follows.

Without loss of generality, in the results and proofs that follow the identity 1 is always assumed to be an element of S.

LEMMA 3.2.16. *If $S = \{1, a^{k_1}, \ldots, a^{k_t}\}$, then $p^L_{Dic_n}(S) = p^R_{Dic_n} = p_{\mathbb{Z}_{2n}}(S)$.*

Proof. All left-translates are of the form $\{a^k, a^{k+k_1}, \ldots, a^{k+k_t}\}$ and $\{a^k x, a^{k-k_1} x, \ldots, a^{k-k_t} x\}$ for all $0 \leq k \leq 2n-1$. Note that both collections are disjoint from each other. Coloring the former collection of translates is equivalent to coloring translates $\{k, k+k_1, \ldots, k+k_t\} = k + \{0, k_1, \ldots, k_t\}$ in $\mathbb{Z}_{2n}$. Coloring the latter collection of translates is equivalent to coloring $\{k, k-k_1, \ldots, k-k_t\}$ in $\mathbb{Z}_{2n}$. Note that subtracting k and then multiplying -1 into $\{k, k-k_1, \ldots, k-k_t\}$ yields $\{0, k_1, \ldots, k_t\}$. So, $p_{\mathbb{Z}_{2n}}(\{k, k-k_1, \ldots, k-k_t\}) = p_{\mathbb{Z}_{2n}}(\{0, k_1, \ldots, k_t\})$ and so if the first collections of translates can be colored with r colors then so can the second collection of translates. That is to say

$p^L_{Dic_n}(\{a^k, a^{k+k_1}, \ldots, a^{k+k_t}\}) = r \implies p^L_{Dic_n}(\{a^kx, a^{k-k_1}x, \ldots, a^{k-k_t}x\}) = r$ for all $0 \leq k \leq 2n-1$. Next, note that the right-translates are of the form $\{a^k, a^{k_1+k}, \ldots, a^{k_t+k}\}$ and $\{a^kx, a^{k_1+k}x, \ldots, a^{k_t+k}x\}$ for all $0 \leq k \leq 2n-1$. Notice once again these collections are disjoint from each other. Clearly, coloring either collection of right-translates in Dic_n is equivalent to coloring the translates of $\{k, k_1+k, \ldots, k_t+k\} = k + \{0, k_1, \ldots, k_t\}$ in $\mathbb{Z}_n$. □

PROPOSITION 3.2.11. *If $S = \{1, a^{k_1}, a^{k_2}, \ldots, a^{k_{r-1}}, a^{\ell_1}x, a^{\ell_2}x, \ldots, a^{\ell_t}x\}$, then $p_{Dic_n}(S) \geq 2$.*

Proof. Note that the left-translates are $a^i \star S = \{a^i, a^{i+k_1}, a^{i+k_2}, \ldots, a^{i+k_{r-1}}, a^{i+\ell_1}x, a^{i+\ell_2}x, \ldots, a^{i+\ell_t}x\}$ and $a^ix \star S = \{a^ix, a^{i-k_1}x, a^{i-k_2}x, \ldots, a^{i-k_{r-1}}x, a^{i-\ell_1+n}, a^{i-\ell_2+n}, \ldots, a^{i-\ell_t+n}\}$ for all $0 \leq i \leq 2n-1$. The right-translates are $S \star a^i = \{a^i, a^{k_1+i}, a^{k_2+i}, \ldots, a^{k_{r-1}+i}, a^{\ell_1-i}x, a^{\ell_2-i}x, \ldots, a^{\ell_t-i}x\}$ and $S \star a^ix = \{a^ix, a^{k_1+i}x, a^{k_2+i}x, \ldots, a^{k_{r-1}+i}x, a^{\ell_1-i+n}, a^{\ell_2-i+n}, \ldots, a^{\ell_t-i+n}\}$ for all $0 \leq i \leq 2n-1$. Assign one color to elements of the form a^i and another distinct color to elements of the form a^ix. □

LEMMA 3.2.17. *Suppose $S = \{1, a^ix, a^jx\}$, then $p^L_{Dic_n}(S) = p^R_{Dic_n}(S)$.*

Proof. Suppose $S = \{1, a^ix, a^jx\}$ for any $0 \leq i, j \leq 2n-1$. Note that any left-translate of S can be written in the forms $\{a^r, a^{i+r}x, a^{j+r}x\}$ or $\{a^rx, a^{r-i+n}, a^{r-j+n}\}$ and any right-translate of S can be written in the form $\{a^{-r}, a^{i+r}x, a^{j+r}x\}$ or $\{a^rx, a^{i-r+n}, a^{j-r+n}\}$ for all $0 \leq r \leq 2n-1$. Consider $\phi : Dic_n \to Dic_n$ such that $\phi(a^m) = a^{-m}$ and $\phi(a^mx) = a^mx$ for all $0 \leq m \leq 2n-1$. Note that $\chi(a^{r-i+n}) = a^{-r+i-n} = a^{-r+i+n}$ since $n \equiv -n \bmod 2n$. So, the following correspondence between left- and right-translates results:

$$\phi(a^r \star S) = \{\phi(a^r), \phi(a^{i+r}x), \phi(a^{j+r}x)\} \mapsto S \star a^{-r} = \{a^{-r}, a^{i+r}x, a^{j+r}x\}$$

and

$$\phi(a^rx \star S) = \{\phi(a^rx), \phi(a^{r-i+n}), \phi(a^{r-j+n})\} \mapsto S \star a^rx = \{a^rx, a^{i-r+n}, a^{j-r+n}\}.$$

□

LEMMA 3.2.18. *$p^L(S) = p^R(S)$ for any $S \subset Dic_n$ such that $|S| = 3$.*

Proof. Any subset of Dic_n is of the form $\{a^k, a^\ell, a^m\}$, $\{a^k, a^\ell x, a^m x\}$, $\{a^k, a^\ell, a^m x\}$, or $\{a^k x, a^\ell x, a^m x\}$. Note that a left-translate of $\{a^k, a^\ell, a^m\}$ is $a^{2n-k} \star \{a^k, a^\ell, a^m\} = \{1, a^{\ell-k}, a^{m-k}\}$, a left-translate of $\{a^k, a^\ell x, a^m x\}$ is $a^{2n-k} \star \{a^k, a^\ell x, a^m x\} = \{1, a^{\ell-k}x, a^{m-k}x\}$, a left-translate of $\{a^k, a^\ell, a^m x\}$ is $a^{m+n}x \star \{a^k, a^\ell, a^m x\} = \{a^{m+n-k}x, a^{m+n-\ell}x, 1\}$, and a left-translate of $\{a^k x, a^\ell x, a^m x\}$ is $a^{k+n}x \star \{a^k x, a^\ell x, a^m x\} = \{1, a^{k-\ell}, a^{k-m}\}$. By Lemma 3.2.16 and Lemma 3.2.17, any subset of the form $\{1, a^i x, a^j x\}$ and $\{1, a^i, a^j\}$ have a equivalence between their left- and right-polychromatic numbers. □

PROPOSITION 3.2.12. *If* $S = \{1, a^i x, a^j x\} \subset Dic_n$ *such that* $3|n$ *and* $i \not\equiv j \bmod 3$ *or* $S = \{1, a^i, a^j x\} \subset Dic_n$, $3|n$, $0 \leq i, j \leq 2n-1$, *and one of the following holds*

1. $i \not\equiv j \bmod 3$
2. $i \equiv j \equiv 1 \bmod 3$
3. $i \equiv j \equiv 2 \bmod 3$

then $p^L(S) = p^R(S) = 3$.

Proof. Suppose $S = \{1, a^i x, a^j x\}$ and $3|n$. By Lemma 3.2.17, $p^L_{Dic_n}(S) = p^R_{Dic_n}(S)$. So, without loss of generality the following argument will be given with respect to left-translates.

Case 1: Without loss of generality, $i \equiv 1 \bmod 3$ and $j \equiv 2 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r) = \chi(a^r x) = c_{r \bmod 3} \text{ for all } 0 \leq r \leq 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}x, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{r \bmod 3}$, $c_{r+1 \bmod 3}$, and $c_{r+2 \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i+n}, a^{r-j+n}\}$ also consist of elements colored $c_{r \bmod 3}$, $c_{r+2 \bmod 3}$, and $c_{r+1 \bmod 3}$ respectively.

Case 2: Without loss of generality, $i \equiv 0 \bmod 3$, $j \equiv 1 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r) = c_{r \bmod 3} \text{ and } \chi(a^r x) = c_{r+1 \bmod 3} \text{ for all } 0 \leq r \leq 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}x, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{r \bmod 3}$, $c_{r+1 \bmod 3}$, and $c_{r+2 \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i+n}, a^{r-j+n}\}$

also consist of elements colored $c_{r+1 \bmod 3}$, $c_{r \bmod 3}$, and $c_{r+2 \bmod 3}$ respectively.

Case 3: Without loss of generality, $i \equiv 0 \bmod 3$, $j \equiv 2 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r) = c_{3-r \bmod 3} \text{ and } \chi(a^r x) = c_{4-r \bmod 3} \text{ for all } 0 \leq r \leq 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}x, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{3-r \bmod 3}$, $c_{4-r \bmod 3}$, and $c_{5-r \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i+n}, a^{r-j+n}\}$ also consist of elements colored $c_{4-r \bmod 3}$, $c_{3-r \bmod 3}$, and $c_{5-r \bmod 3}$ respectively.

The arguments are repeatable if i and j switch congruence classes.

Suppose $S = \{1, a^i, a^j x\}$ and $3|n$.

Case 1: Without loss of generality, $i \equiv 2 \bmod 3$ and $j \equiv 1 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r x) = c_{r \bmod 3} \text{ for all } 0 \leq r \leq 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{r \bmod 3}$, $c_{r+2 \bmod 3}$, and $c_{r+1 \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i}x, a^{r-j+n}\}$ also consist of elements colored $c_{r \bmod 3}$, $c_{r+1 \bmod 3}$, and $c_{r+2 \bmod 3}$ respectively.

Case 2: Suppose $i, j \equiv 1 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r) = c_{r \bmod 3} \text{ and } \chi(a^r x) = c_{r+1 \bmod 3} \text{ for all } 0 \leq r \leq 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{r \bmod 3}$, $c_{r+1 \bmod 3}$, and $c_{r+2 \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i}x, a^{r-j+n}\}$ also consist of elements colored $c_{r+1 \bmod 3}$, $c_{r \bmod 3}$, and $c_{r+2 \bmod 3}$ respectively.

Case 3: Suppose $i, j \equiv 2 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r) = c_{3-r \bmod 3} \text{ and } \chi(a^r x) = c_{4-r \bmod 3} \text{ for all } 0 \leq r \leq 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{3-r \bmod 3}$, $c_{4-r \bmod 3}$, and $c_{5-r \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i}x, a^{r-j+n}\}$ also consist of elements colored $c_{4-r \bmod 3}$, $c_{3-r \bmod 3}$, and $c_{5-r \bmod 3}$ respectively.

Case 4: Suppose $i \equiv 1 \bmod 3$, $j \equiv 0 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r) = c_{r \bmod 3} \text{ and } \chi(a^r x) = c_{r+2 \bmod 3} \text{ for all } 0 \leq r \leq 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{r \bmod 3}$, $c_{r+1 \bmod 3}$, and $c_{r+2 \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i}x, a^{r-j+n}\}$ also consist of elements colored $c_{r+2 \bmod 3}$, $c_{r+1 \bmod 3}$, and $c_{r \bmod 3}$ respectively.

Case 5: Suppose $i \equiv 2 \bmod 3$, $j \equiv 0 \bmod 3$. Consider the coloring χ so that:

$$\chi(a^r) = c_{3-r \bmod 3} \text{ and } \chi(a^r x) = c_{5-r \bmod 3} \text{ for all } 0 \le r \le 2n-1$$

Then each translate of the form $\{a^r, a^{r+i}, a^{r+j}x\}$ consists of three distinct colors as the elements are colored $c_{3-r \bmod 3}$, $c_{4-r \bmod 3}$, and $c_{5-r \bmod 3}$ respectively. Translates of the form $\{a^r x, a^{r-i}x, a^{r-j+n}\}$ also consist of elements colored $c_{5-r \bmod 3}$, $c_{4-r \bmod 3}$, and $c_{3-r \bmod 3}$ respectively. □

This result does not capture all subsets of Dic_n of size three with polychromatic number equal to three.

3.2.3 A Note on the Symmetric Group of Degree n, S_n

Let Ω be any nonempty set and let S_Ω be the set of all bijections from Ω to itself. The set S_Ω is a finite nonabelian group under function composition: $\circ$. This group is called the *symmetric group on the set* Ω. Note that the elements of S_Ω are the *permutations* of Ω, not the elements of Ω itself. When $\Omega = \{1, 2, 3, \ldots, n\}$, the symmetric group on Ω is denoted S_n and is called the *symmetric group of degree n*. A *cycle* is a string of integers which represents the element of S_n which cyclically permutes these integers (and fixes all other integers). The cycle $(a_1 a_2 \ldots a_m)$ is the permutation which sends a_i to a_{i+1}, $1 \le i \le m-1$ and sends a_m to a_1. For example, (213) maps 2 to 1, 1 to 3, and 3 to 2. In general, for each $\sigma \in S_n$ the numbers from 1 to n can be rearranged and collected into k cycles of the form

$$(a_1 a_2 \ldots a_{m_1})(a_{m_1+1} a_{m_2+2} \ldots a_{m_2}) \ldots (a_{m_{k-1}+1} a_{m_{k-1}+2} \ldots a_{m_k}).$$

This notation is read as for any $x \in \{1, 2, 3, \ldots, n\}$, find x in the above expression. If x is not followed immediately by a right parenthesis, then $\sigma(x)$ is the integer appearing immediately to the right of x. If x is followed by a right parenthesis, then $\sigma(x)$ is the number which is at the start of the cycle ending with x. The product of all the cycles is called the *cycle decomposition* of σ [10]. For example, let $n = 13$ and $\sigma \in S_{13}$ so that $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 3$, $\sigma(4) = 1$, $\sigma(5) = 11$, $\sigma(6) = 9$, $\sigma(7) = 5$, $\sigma(8) = 10$, $\sigma(9) = 6$, $\sigma(10) = 4$, $\sigma(11) = 7$, $\sigma(12) = 8$, $\sigma(13) = 2$. The cycle decomposition is

$$\sigma = (1\ \ 12\ \ 8\ \ 10\ \ 4)(2\ \ 13)(3)(5\ \ 11\ \ 7)(6\ \ 9).$$

The symmetric group of degree n can be generated by two elements namely any $n-$cycle of the form $(a_1 a_2 \ldots a_n)$ and any $2-$cycle consisting of adjacent elements in said $n-$cycle i.e. $(a_i a_{i+1})$. In much the same fashion as results on the dihedral group were obtained, for further exploration of the polychromatic number of subsets of S_n, it is useful to note that grids can be used to represent translates.

DEFINITION 3.2.8. *Consider the 2 by n grid*

$(2,0)$	$\cdots$	$(2,i)$	$(2,i+1)$	$\cdots$	$(2,n-1)$
$(1,0)$	$\cdots$	$(1,i)$	$(1,i+1)$	$\cdots$	$(1,n-1)$

Figure 3.10 $2 \times n$ *grid*

where $0 \leq i \leq n-1$ correspond to the columns in the grid. Note that when $i = n-1$, $i+1 = 0$. An ar-tile *is a shape of the following form within said* 2 *by n grid.*

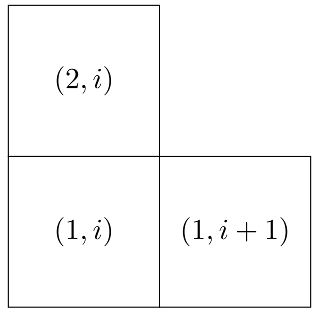

Figure 3.11 An ar-tile

OBSERVATION 3.2.8. *All left-translates of the subset of* S_n *containing* (1), $(a_i a_{i+1})$, *and* $(a_1 a_2 \ldots a_n)$ *are ar-tiles in multiple 2 by n grids.*

Proof. Any element of S_n can be written in cycle decomposition form

$$\overline{a} = (a_{1,1} a_{1,2} \ldots a_{1,k_1})(a_{2,1} a_{2,2} \ldots a_{2,k_2}) \cdots (a_{r,1}, a_{r,2} \ldots a_{r,k_r})$$

where each cycle is of length k_i for $1 \leq i \leq r$ are not necessarily distinct values. The left-translate that one obtains from composing this element on the left of $S = \{(1), (a_i a_{i+1}), (a_1 a_2 \ldots a_n)\}$ is $\overline{a} \circ S = \{\overline{a} \circ (a_i a_{i+1}, \overline{a}, \overline{a} \circ (a_1 a_2 \ldots a_n)\}$. Consider the 2 by n grid with some elements of S_n in its entries as follows

$(a_i a_{i+1})$	$(a_1 a_2 \ldots a_n)(a_i a_{i+1})$	$(a_1 a_2 \ldots a_n)^2 (a_i a_{i+1})$	$\cdots$	$(a_1 a_2 \ldots a_n)^{n-1} (a_i a_{i+1})$
(1)	$(a_1 a_2 \ldots a_n)$	$(a_1 a_2 \ldots a_n)^2$	$\cdots$	$(a_1 a_2 \ldots a_n)^{n-1}$

Figure 3.12 Some elements of S_n arranged in a $2 \times n$ grid

Next, apply $\overline{a}$ via composition to the left-side of every element in the previous grid. The following grid is obtained.

$\overline{a} \circ (a_i a_{i+1})$	$\overline{a} \circ (a_1 a_2 \ldots a_n)(a_i a_{i+1})$	$\overline{a} \circ (a_1 a_2 \ldots a_n)^2 (a_i a_{i+1})$	$\cdots$	$\overline{a} \circ (a_1 a_2 \ldots a_n)^{n-1} (a_i a_{i+1})$
$\overline{a}$	$\overline{a} \circ (a_1 a_2 \ldots a_n)$	$\overline{a} \circ (a_1 a_2 \ldots a_n)^2$	$\cdots$	$\overline{a} \circ (a_1 a_2 \ldots a_n)^{n-1}$

Figure 3.13 Composition applied to previous $2 \times n$ grid

$\overline{a} \circ S$ clearly is an ar-tile in this grid. □

REMARK 3.2.10. *Note that elements can be repeated in the grids described above as* $\overline{a} \circ (a_i a_{i+1}) \circ S = \{\overline{a}, \overline{a} \circ (a_i a_{i+1}), \overline{a} \circ (a_i a_{i+1}) \circ (a_1 a_2 \ldots a_n)\}$ *is also a left-translate that appears as an ar-tile in the following grid:*

$\overline{a}$	$\overline{a} \circ (a_i a_{i+1})(a_1 a_2 \ldots a_n)(a_i a_{i+1})$	$\cdots$	$\overline{a} \circ (a_i a_{i+1})(a_1 a_2 \ldots a_n)^{n-1} (a_i a_{i+1})$
$\overline{a} \circ (a_i a_{i+1})$	$\overline{a} \circ (a_i a_{i+1})(a_1 a_2 \ldots a_n)$	$\cdots$	$\overline{a} \circ (a_i a_{i+1})(a_1 a_2 \ldots a_n)^{n-1}$

Figure 3.14 Another $2 \times n$ grid containing some of the elements of S_n

Because of Observation 3.2.8 and Remark 3.2.10, work on S_n seems feasible but also challenging as when translates can be represented within grids where grid elements repeat, extra care must be taken when assigning colors. Similarly, there are many larger translates on S_n to consider and find the polychromatic number of as well as the Turán number, but as far as this work is concerned these problems are completely open.

CHAPTER 4. CONCLUSION AND FUTURE WORK

The characterization and proof of subsets of size three which have polychromatic number equal to three given in Theorem 2.0.3 led to the result Theorem 2.0.9 for subsets of $\mathbb{Z}_n$ of a cardinality which is odd and prime. In [6], as has been mentioned, Lemma 2.0.10 gives a few partial results in determining the polychromatic number of subsets of the integers modulo n of size four. The polychromatic number of some subsets of composite cardinalities and whose cardinalities are greater than or equal to four have been determined in Proposition 2.0.1, Proposition 2.0.3, and Proposition 2.0.4 for example. Nevertheless, a complete characterization of such subsets is still open.

Given the results explored in small subsets of the integers modulo n, it is now possible to extend to products of these groups such as $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$ as was done in Section 2.0.2. However, there are still many subsets of many finite abelian groups left to determine. Furthermore, is it possible to further apply the Fundamental Theorem of Finitely Generated Abelian Groups to extend results to all finite abelian groups?

Also, given Theorem 2.0.4 in [6], there is clearly a connection between tiling by translation and determining the polychromatic number of a given abelian group. In [9], multiple terms and interesting results related to tiling by translation which is simply referred to as "tiling" are introduced. Therefore, it is highly possible that the results on tiling by translation can aid in the quest to determine the polychromatic number of any subset of any given abelian group.

The dihedral group D_{2n} was explored extensively with respect to subsets whose left- and right-translates can be colored with the same number of colors, however, is it very probable that there are more subsets of $S \subset D_{2n}$ with $p^L(S) = p^R(S)$ left undetermined. Furthermore, for any finite nonabelian group, is it true that $p^L(S) = p^R(S)$? Similarly, what can be said of the value of $p(S)$ for such subsets? There were many subsets of D_{2n} with $p^L(S) = p^R(S) = |S|$ with $|S| \geq 4$ determined in results such as Theorem 3.2.5, Proposition 3.2.4, Lemma 3.2.12, Lemma 3.2.13, Lemma 3.2.14, Proposition 3.2.5, and Proposition 3.2.6. However, there are still more to determine.

The Dicyclic Group Dic_n was explored a little and only some of the subsets of size three with $p^L(S) = p^R(S) = 3$ were determined in Proposition 3.2.12. Finishing this characterization would be desirable. Similarly, working more extensively with this group and determining other subsets with $p^L(S) = p^R(S) = |S|$ such that $|S| \geq 4$ is an interesting problem. Moreover, determining $p(S)$ for subsets of size three and greater is advantageous. After working with finite nonabelian groups with two generators like the dihedral group and the Dicyclic Group, is it possible to use the previous methods and extend the results to other finite groups generated by two elements? Along the same lines, what can be said of the Symmetric group of degree n? Is it possible to use Cayley's Theorem to extend results to all groups?

It is also important to note that while determining the polychromatic number of finite groups is an interesting problem in its own right and helps in improving the lower bound on the Turán number, there does not appear to be a way to naturally extend this work to improve on the upper bound of the Turán number. Thus, finding and implementing a method to decrease the upper bound on the Turán number is a completely open and worthwhile problem.

Finally, this work is very abstract. However, that should not suggest there are no real world applications. A few such real world applications to Turán-type problems are mentioned in [15]. With this in mind, how can this work on finite groups relate to the real world?

CHAPTER 5. EXAMPLES

In this section, selected examples of some of the colorings employed in the previous chapters are given.

EXAMPLE 5.0.1. *In Lemma 2.0.1, $S = \{0, 1\} \subset \mathbb{Z}_n$.*

Let $n = 4$. The translates are $\{0,1\}$, $\{1,2\}$, $\{2,3\}$, and $\{3,0\}$. Consider the $S-$polychromatic coloring $\chi(i) = c_{i \bmod 2}$ for all $0 \leq i \leq 3$. Then, $\chi(0) = c_0$, $\chi(1) = c_0$, $\chi(2) = c_0$, and $\chi(3) = c_1$.

EXAMPLE 5.0.2. *In Lemma 2.0.4, $S = \{0, a, b\}$ such that $n \equiv 0 \bmod 3^{j+1}$, $a = 3^j m_a$, $b = 3^j m_b$, $m_a \equiv 1 \bmod 3$, $m_b \equiv 2 \bmod 3$, and $j \geq 0$.*

Let $n = 9$, $a = 1$, $b = 2$. The translates are $\{0,1,2\}$, $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, $\{4,5,6\}$, $\{5,6,7\}$, $\{6,7,8\}$, $\{7,8,0\}$, and $\{8,0,1\}$. Consider the $S-$polychromatic coloring $\chi(t \cdot 3 + r) = c_{t \bmod 3}$ for all $0 \leq t \leq 8$ and $0 \leq r \leq 0 \implies r = 0$. Then, $\chi(0) = c_0$, $\chi(1) = c_1$, $\chi(2) = c_2$, $\chi(3) = c_0$, $\chi(4) = c_1$, $\chi(5) = c_2$, $\chi(6) = c_0$, $\chi(7) = c_1$, and $\chi(8) = c_2$.

EXAMPLE 5.0.3. *In Theorem 2.0.5, $S = \{0, 1, \ell\}$ in $\mathbb{Z}_n$ such that $n \equiv r \bmod \ell - 2$ where $r = 0, 1, 2,$ or 3 and ℓ is odd.*

Let $n = 9$, $\ell = 5$, $r = 0$. The translates are $\{0,1,5\}$, $\{1,2,6\}$, $\{2,3,7\}$, $\{3,4,8\}$, $\{4,5,0\}$, $\{5,6,1\}$, $\{6,7,8\}$, $\{7,8,3\}$, and $\{8,0,4\}$. Consider the $S-$polychromatic coloring

$$\chi_0(x) = \begin{cases} c_0 & \text{if } x = i(3) \\ c_0 & \text{if } x = i(3)+1 \\ c_1 & \text{if } x = i(3)+2 \end{cases}$$

for all $0 \leq i \leq 2$. Then, $\chi_0(0) = c_0$, $\chi_0(1) = c_0$, $\chi_0(2) = c_1$, $\chi_0(3) = c_0$, $\chi_0(4) = c_0$, $\chi_0(5) = c_1$, $\chi_0(6) = c_0$, $\chi_0(7) = c_0$, $\chi_0(8) = c_1$.

EXAMPLE 5.0.4. *In Theorem 2.0.6, $S = \{0, 1, \ell\}$ in $\mathbb{Z}_n$ such that $n \equiv r \bmod \ell - 2$ and $r \geq 4$ and ℓ is odd.*

Let $n = 19$, $\ell = 7$, $r = 4$. The translates are $\{0,1,7\}$, $\{1,2,8\}$, $\{2,3,9\}$, $\{3,4,10\}$, $\{4,5,11\}$, $\{5,6,12\}$, $\{6,7,13\}$, $\{7,8,14\}$, $\{8,9,15\}$, $\{9,10,16\}$, $\{10,11,17\}$, $\{11,12,18\}$, $\{12,13,0\}$, $\{13,14,1\}$, $\{14,15,2\}$, $\{15,16,3\}$, $\{16,17,4\}$, $\{17,18,5\}$, *and* $\{18,0,6\}$. *Consider the* $2-$*polychromatic interval precoloring*

$$\chi(x) = \begin{cases} c_0 & \text{if } x = i(5) \\ c_0 & \text{if } x = i(5) + 2k + 1 \text{ where } 0 \leq k \leq 1 \\ c_1 & \text{if } x = i(5) + 2k \text{ where } 1 \leq k \leq 2 \end{cases}$$

where $0 \leq i \leq 1$. *Then,* $\chi(0) = c_0$, $\chi(1) = c_0$, $\chi(2) = c_1$, $\chi(3) = c_0$, $\chi(4) = c_1$, $\chi(5) = c_0$, $\chi(6) = c_0$, $\chi(7) = c_1$, $\chi(8) = c_0$, $\chi(9) = c_1$. *Consequently,* $\chi(10) = c_0$, $\chi(11) = c_0$, $\chi(12) = c_1$, $\chi(13) = c_1$, $\chi(14) = c_0$, $\chi(15) = c_1$, $\chi(16) = c_0$, $\chi(17) = c_1$, $\chi(18) = c_1$.

EXAMPLE 5.0.5. *In Lemma 2.0.7, set* $S = \{0,1,3\}$ *in* $\mathbb{Z}_n$ *such that either* $n \geq 9$ *or* $n = 5$ *with* n *odd.*

Let $n = 11$. The translates are $\{0,1,3\}$, $\{1,2,4\}$, $\{2,3,5\}$, $\{3,4,6\}$, $\{4,5,7\}$, $\{5,6,8\}$, $\{6,7,9\}$, $\{7,8,10\}$, $\{8,9,0\}$, $\{9,10,1\}$, *and* $\{10,0,2\}$. *Consider the* $S-$*polychromatic coloring* $\chi(0) = c_0$, $\chi(1) = c_1$, $\chi(2) = c_1$, $\chi(3) = c_0$, $\chi(4) = c_0$, $\chi(5) = c_0$, $\chi(6) = c_1$, $\chi(7) = c_1$, $\chi(8) = c_1$, $\chi(9) = c_0$, $\chi(10) = c_0$.

EXAMPLE 5.0.6. *In Theorem 2.0.7,* $S = \{0,1,\ell\}$ *in* $\mathbb{Z}_n$ *such that* $n \neq 7$ *and* $\ell \neq 3,5$.

Let $n = 7$, $\ell = 4$. The translates are $\{0,1,4\}$, $\{1,2,5\}$, $\{2,3,6\}$, $\{3,4,0\}$, $\{4,5,1\}$, $\{5,6,2\}$, *and* $\{6,0,3\}$. *Consider the* $S-$*polychromatic coloring* $\chi(0) = c_0$, $\chi(1) = c_1$, $\chi(2) = c_0$, $\chi(3) = c_0$, $\chi(4) = c_1$, $\chi(5) = c_0$, $\chi(6) = c_1$.

EXAMPLE 5.0.7. *In Lemma 2.0.9,* $S = \{0,a,b\}$ *such that* $n \equiv 0 \bmod 7$ *and without loss of generality* $b = 3a$ *or* $b = 5a$ *with* $|\langle a \rangle| = 7$ *can not occur simultaneously.*

Let $n = 12$, $a = 3$, $b = 4$. The translates are $\{0,3,4\}$, $\{1,2,5\}$, $\{2,5,6\}$, $\{3,6,7\}$, $\{4,7,8\}$, $\{5,8,9\}$, $\{6,9,10\}$, $\{7,10,11\}$, $\{8,11,0\}$, $\{9,0,1\}$, $\{10,1,2\}$, *and* $\{11,2,3\}$. *The following grid can also be formed*

0	4	8
3	7	11
6	10	2
9	1	5

Figure 5.1 Example of 4 by 3 grid

where any translate $\{i, i+3, i+4\}$ *is an ell - tile in this grid of the form*

i	$i+4$
$i+3$	

Figure 5.2 The ell - tiles of the above grid

Since this is a 4×3 *matrix, consider the ell-tile 2-coloring given by*

$$\chi((i,j)) = \begin{cases} c_{i+j \bmod 2} & \text{if } (i,j) \neq (\ell,3) \text{ for any } \ell \\ c_0 & \text{if } (i,j) = (\ell,3) \text{ for odd values of } \ell \\ c_1 & \text{if } (i,j) = (\ell,3) \text{ for even values of } \ell. \end{cases}$$

More explicitly, $\chi((1,1)) = c_0$, $\chi((1,2)) = c_1$, $\chi((1,3)) = c_0$, $\chi((2,1)) = c_1$, $\chi((2,2)) = c_0$, $\chi((2,3)) = c_1$, $\chi((3,1)) = c_0$, $\chi((3,2)) = c_1$, $\chi((3,3)) = c_0$, $\chi((4,1)) = c_0$, $\chi((4,2)) = c_0$, $\chi((4,3)) = c_1$.

EXAMPLE 5.0.8. *In Proposition 2.0.3, for any* $a \in \mathbb{Z}_n$, $S = \{a, a+1, a+2, \ldots, a+m-1\}$ *and* $n \equiv 0 \bmod m$.

Let $a = 0$, $m = 10$, $n = 20$. *The translates are* $\{0,1,2,\ldots,9\}$, $\{1,2,3,\ldots,10\}$, $\{2,3,4,\ldots,11\}$, $\{3,4,5,\ldots,12\}$, $\{4,5,6,\ldots,13\}$, $\{5,6,7,\ldots,14\}$, $\{6,7,8,\ldots,15\}$, $\{7,8,9,\ldots,16\}$, $\{8,9,10,\ldots,17\}$, $\{9,10,11,\ldots,18\}$, $\{10,11,12,\ldots,19\}$, $\{11,12,13,\ldots,0\}$, $\{12,13,14,\ldots,1\}$, $\{13,14,15,\ldots,2\}$, $\{14,15,16,\ldots,3\}$, $\{15,16,17,\ldots,4\}$, $\{16,17,18,\ldots,5\}$, $\{17,18,19,\ldots,6\}$, $\{18,19,0,\ldots,7\}$, *and*

$\{19, 0, 1, \ldots, 8\}$. *Consider the* $S-$*polychromatic coloring* $\chi(0) = c_0$, $\chi(1) = c_1$, $\chi(2) = c_2$, $\chi(3) = c_3$, $\chi(4) = c_4$, $\chi(5) = c_5$, $\chi(6) = c_6$, $\chi(7) = c_7$, $\chi(8) = c_8$, $\chi(9) = c_9$, $\chi(10) = c_0$, $\chi(11) = c_1$, $\chi(12) = c_2$, $\chi(13) = c_3$, $\chi(14) = c_4$, $\chi(15) = c_5$, $\chi(16) = c_6$, $\chi(17) = c_7$, $\chi(18) = c_8$, $\chi(19) = c_9$.

EXAMPLE 5.0.9. *In Theorem 2.0.9,* $S = \{a, a + \ell^j m_1, a + \ell^j m_2, \ldots, a + \ell^j m_{\ell-1}\}$ *where* $\ell^{j+1} | n$, $j \geq 0$, *and* $0 \not\equiv m_1 \not\equiv m_2 \not\equiv \ldots \not\equiv m_{\ell-1} \bmod \ell$ *where* ℓ *is an odd prime and* $a \in \mathbb{Z}_n$.

Let $a = 0$, $\ell = 5$, $j = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 4$, $n = 25$. *The translates are* $\{0, 5, 10, 15, 20\}$, $\{1, 6, 11, 16, 21\}$, $\{2, 7, 12, 17, 22\}$, $\{3, 8, 13, 18, 23\}$, $\{4, 9, 14, 19, 24\}$, $\{5, 10, 15, 20, 0\}$, $\{6, 11, 16, 21, 1\}$, $\{7, 12, 17, 22, 2\}$, $\{8, 13, 18, 23, 3\}$, $\{9, 14, 19, 24, 4\}$, $\{10, 15, 20, 0, 5\}$, $\{11, 16, 21, 1, 6\}$, $\{12, 17, 22, 2, 7\}$, $\{13, 18, 23, 3, 8\}$, $\{14, 19, 24, 4, 9\}$, $\{15, 20, 0, 5, 10\}$, $\{16, 21, 1, 6, 11\}$, $\{17, 22, 2, 7, 12\}$, $\{18, 23, 3, 8, 13\}$, $\{19, 24, 4, 9, 14\}$, $\{20, 0, 5, 10, 15\}$, $\{21, 1, 6, 11, 16\}$, $\{22, 2, 7, 12, 17\}$, $\{23, 3, 8, 13, 18\}$, *and* $\{24, 4, 9, 14, 19\}$. *Consider the* $S-$*polychromatic coloring* $\chi(0) = c_0$, $\chi(1) = c_0$, $\chi(2) = c_0$, $\chi(3) = c_0$, $\chi(4) = c_0$, $\chi(5) = c_1$, $\chi(6) = c_1$, $\chi(7) = c_1$, $\chi(8) = c_1$, $\chi(9) = c_1$, $\chi(10) = c_2$, $\chi(11) = c_2$, $\chi(12) = c_2$, $\chi(13) = c_2$, $\chi(14) = c_2$, $\chi(15) = c_3$, $\chi(16) = c_3$, $\chi(17) = c_3$, $\chi(18) = c_3$, $\chi(19) = c_3$, $\chi(20) = c_4$, $\chi(21) = c_4$, $\chi(22) = c_4$, $\chi(23) = c_4$, $\chi(24) = c_4$.

EXAMPLE 5.0.10. *In Proposition 3.2.3, there is a* $3 - 2 \times 2$ *frame coloring of a* $2 \times M$ *grid if and only if* $3 | M$.

Let $M = 3$. *Then, consider the following* $3 - 2 \times 2$ *frame coloring of a* 2×3 *grid*

c_2	c_1	c_0
c_1	c_0	c_2

Figure 5.3 An example of a grid and a $3 - 2 \times 2$ *frame coloring*

EXAMPLE 5.0.11. *In Lemma 3.2.9, if M is even, there exists a $4-2\times 2$ box coloring of a $2\times M$ grid.*

Let $M=4$. Then, consider the following $4-2\times 2$ box coloring of a 2×4 grid

c_3	c_1	c_3	c_1
c_2	c_0	c_2	c_0

Figure 5.4 An example of a grid and a $4-2\times 2$ box coloring

EXAMPLE 5.0.12. *In Lemma 3.2.12, $|S|=2n-1$.*

Set $n=3$ and $S=\{1,r,r^2,sr,sr^2\}$. The left-translates are $\{r,r^2,1,s,sr\}$, $\{r^2,1,r,sr^2,s\}$, $\{s,sr,sr^2,r,r^2\}$, $\{sr,sr^2,s,1,r\}$, and $\{sr^2,s,sr,r^2,1\}$. The right-translates are $\{r,r^2,1,sr^2,s\}$, $\{r^2,1,r,s,sr\}$, $\{s,sr^2,sr,r^2,r\}$, $\{sr,s,sr^2,1,r^2\}$, and $\{sr^2,sr,s,r,1\}$. Consider the $S-$polychromatic coloring given by the following assignment $\chi(1)=c_0$, $\chi(r)=c_1$, $\chi(r^2)=c_2$, $\chi(s)=c_0$, $\chi(sr)=c_1$, $\chi(sr^2)=c_2$.

EXAMPLE 5.0.13. *In Lemma 3.2.13, let $S=\{1,r,r^2,r^3,sr^3\}$ with $i=3$ and $n=4$. The left-translates are $\{r,r^2,r^3,1,sr^2\}$, $\{r^2,r^3,1,r,sr\}$, $\{r^3,1,r,r^2,s\}$, $\{s,sr,sr^2,sr^3,r^3\}$, $\{sr,sr^2,sr^3,s,r^2\}$, $\{sr^2,sr^3,s,sr,r\}$, and $\{sr^3,s,sr,sr^2,1\}$. The right-translates are $\{r,r^2,r^3,1,s\}$, $\{r^2,r^3,1,r,sr\}$, $\{r^3,1,r,r^2,sr^2\}$, $\{s,sr^3,sr^2,sr,r\}$, $\{sr,s,sr^3,sr^2,r^2\}$, $\{sr^2,sr,s,sr^3,r^3\}$, and $\{sr^3,sr^2,sr,s,1\}$. Consider the $S-$polychromatic coloring given by the following assignment $\chi(1)=c_0$, $\chi(r)=c_1$, $\chi(r^2)=c_2$, $\chi(r^3)=c_3$, $\chi(s)=c_0$, $\chi(sr)=c_1$, $\chi(sr^2)=c_2$, $\chi(sr^3)=c_3$.*

EXAMPLE 5.0.14. *In Lemma 3.2.14, let $n=3, j=1, k=2$. Then, $S=\{1,r^2,sr^2\}$. The left-translates are $\{r,1,sr\}$, $\{r^2,r,s\}$, $\{s,sr^2,r^2\}$, $\{sr,s,r\}$, and $\{sr^2,sr,1\}$. The right-translates are $\{r,1,s\}$, $\{r^2,r,sr\}$, $\{s,sr,r\}$, $\{sr,sr^2,r^2\}$, and $\{sr^2,s,1\}$. Consider the S^L-polychromatic coloring given by the following assignment $\chi(1)=c_0$, $\chi(r)=c_2$, $\chi(r^2)=c_1$, $\chi(s)=c_0$, $\chi(sr)=c_1$, $\chi(sr^2)=c_2$.*

EXAMPLE 5.0.15. *In Proposition 3.2.5, let $n=4$, $i=3$ and $S=\{1,r,r^2,sr,sr^3\}$. Then, the left-translates are $\{r,r^2,r^3,s,sr^2\}$, $\{r^2,r^3,1,sr^3,sr\}$, $\{r^3,1,r,sr^2,s\}$, $\{s,sr,sr^2,r,r^3\}$, $\{sr,sr^2,sr^3,1,r^2\}$,*

$\{sr^2, sr^3, s, r^3, r\}$, *and* $\{sr^3, s, sr, r^2, 1\}$. *The right-translates are* $\{r, r^2, r^3, sr^2, s\}$, $\{r^2, r^3, 1, sr^3, sr\}$, $\{r^3, 1, r, s, sr^2\}$, $\{s, sr^3, sr^2, r^3, r\}$, $\{sr, s, sr^3, 1, r^2\}$, $\{sr^2, sr, s, r, r^3\}$, *and* $\{sr^3, sr^2, sr, r^2, 1\}$. *Consider the* S^L-*polychromatic coloring given by the following assignment* $\chi(1) = c_0$, $\chi(r) = c_2$, $\chi(r^2) = c_1$, $\chi(r^3) = c_3$, $\chi(s) = c_0$, $\chi(sr) = c_2$, $\chi(sr^2) = c_1$, $\chi(sr^3) = c_3$.

EXAMPLE 5.0.16. *In Proposition 3.2.6, let* $n = 5$, $j = 1$, $i = 4$, *and* $S = \{1, r, r^2, sr^3, sr^4\}$. *Then, left-translates are* $\{r, r^2, r^3, sr^2, sr^3\}$, $\{r^2, r^3, r^4, sr, sr^2\}$, $\{r^3, r^4, 1, s, sr\}$, $\{r^4, 1, r, sr^4, s\}$, $\{s, sr, sr^2, r^3, r^4\}$, $\{sr, sr^2, sr^3, r^2, r^3\}$, $\{sr^2, sr^3, sr^4, r, r^2\}$, $\{sr^3, sr^4, s, 1, r\}$, *and* $\{sr^4, s, sr, r^4, 1\}$. *The right-translates are* $\{r, r^2, r^3, sr^4, s\}$, $\{r^2, r^3, r^4, s, sr\}$, $\{r^3, r^4, 1, sr, sr^2\}$, $\{r^4, 1, r, sr^2, sr^3\}$, $\{s, sr^4, sr^3, r^2, r\}$, $\{sr, s, sr^4, r^3, r^2\}$, $\{sr^2, sr, s, r^4, r^3\}$, $\{sr^3, sr^2, sr, 1, r^4\}$, *and* $\{sr^4, sr^3, sr^2, r, 1\}$. *Consider the* S^L-*polychromatic coloring given by the following assignment* $\chi(1) = c_0$, $\chi(r) = c_1$, $\chi(r^2) = c_2$, $\chi(r^3) = c_4$, $\chi(r^4) = c_3$, $\chi(s) = c_2$, $\chi(sr) = c_1$, $\chi(sr^2) = c_0$, $\chi(sr^3) = c_3$, $\chi(sr^4) = c_4$.

EXAMPLE 5.0.17. *In Proposition 3.2.7, let* $n = 5$, $i = 2$, *and* $S = \{1, r, sr^2, sr^3, sr^4\}$. *Then, left-translates are* $\{r, r^2, sr, sr^2, sr^3\}$, $\{r^2, r^3, s, sr, sr^2\}$, $\{r^3, r^4, sr^4, s, sr\}$, $\{r^4, 1, sr^3, sr^4, s\}$, $\{s, sr, r^2, r^3, r^4\}$, $\{sr, sr^2, r, r^2, r^3\}$, $\{sr^2, sr^3, 1, r, r^2\}$, $\{sr^3, sr^4, r^4, 1, r\}$, *and* $\{sr^4, s, r^3, r^4, 1\}$. *The right-translates are* $\{r, r^2, sr^3, sr^4, s\}$, $\{r^2, r^3, sr^4, s, sr\}$, $\{r^3, r^4, s, sr, sr^2\}$, $\{r^4, 1, sr, sr^2, sr^3\}$, $\{s, sr^4, r^3, r^2, r\}$, $\{sr, s, r^4, r^3, r^2\}$, $\{sr^2, sr, 1, r^4, r^3\}$, $\{sr^3, sr^2, r, 1, r^4\}$, *and* $\{sr^4, sr^3, r^2, r, 1\}$. *Consider the* S^L-*polychromatic coloring given by the following assignment* $\chi(1) = c_0$, $\chi(r) = c_1$, $\chi(r^2) = c_4$, $\chi(r^3) = c_3$, $\chi(r^4) = c_2$, $\chi(s) = c_1$, $\chi(sr) = c_0$, $\chi(sr^2) = c_2$, $\chi(sr^3) = c_3$, $\chi(sr^4) = c_4$.

EXAMPLE 5.0.18. *In Proposition 3.2.8, let* $n = 4$, $k = 3$, $t = 1$, $j = 2$, $i_1 = 2$, $i_2 = 3$, $i_3 = 1$. *Then, the left-translates are* $\{r, r^3 1, s\}$, $\{r^2, 1, r, sr^3\}$, $\{r^3, r, r^2, sr^2\}$, $\{s, sr^2, sr^3 r\}$, $\{sr, sr^3, s, 1\}$, $\{sr^2, s, sr, r^3\}$, *and* $\{sr^3, sr, sr^2, r^2\}$. *The right-translates are* $\{r, r^3, 1, sr^2\}$, $\{r^2, 1, r, sr^3\}$, $\{r^3, r, r^2, s\}$, $\{s, sr^2, sr, r^3\}$, $\{sr, sr^3, sr^2, 1\}$, $\{sr^2, s, sr^3, r\}$, *and* $\{sr^3, sr, s, r^2\}$. *Consider the* S^L-*polychromatic coloring given by the following assignment* $\chi(1) = c_0$, $\chi(r) = c_1$, $\chi(r^2) = c_2$, $\chi(r^3) = c_3$, $\chi(s) = c_2$, $\chi(sr) = c_1$, $\chi(sr^2) = c_0$, $\chi(sr^3) = c_3$.

EXAMPLE 5.0.19. *In Proposition 3.2.12, let* $n = 3$, $i = 1$, $j = 4$. *Then* $S = \{1, a, a^4 x\}$. *The left-translates are* $\{a, a^2, a^5 x\}$, $\{a^2, a^3, x\}$, $\{a^3, a^4, ax\}$, $\{a^4, a^5, a^2 x\}$, $\{a^5, 1, a^3 x\}$, $\{x, a^5 x, a^5\}$,

$\{ax, x, 1\}$, $\{a^2x, ax, a\}$, $\{a^3x, a^2x, a^2\}$, $\{a^4x, a^3x, a^3\}$, *and* $\{a^5x, a^4x, a^4\}$. *The right-translates are* $\{a^5, a^2, a^5x\}$, $\{a^4, a^3, x\}$, $\{a^3, a^4, ax\}$, $\{a^2, a^5, a^2x\}$, $\{a, 1, a^3x\}$, $\{x, a^4, a\}$, $\{ax, a^3, 1\}$, $\{a^2x, a^2, a^5\}$, $\{a^3x, a, a^4\}$, $\{a^4x, 1, a^3\}$, *and* $\{a^5x, a^5, a^2\}$. *Consider the* S^L-*polychromatic coloring given by the following assignment* $\chi(1) = c_0$, $\chi(a) = c_1$, $\chi(a^2) = c_2$, $\chi(a^3) = c_0$, $\chi(a^4) = c_1$, $\chi(a^5) = c_2$, $\chi(a^6) = c_0$, $\chi(x) = c_1$, $\chi(ax) = c_2$, $\chi(a^2x) = c_0$, $\chi(a^3x) = c_1$, $\chi(a^4x) = c_2$, $\chi(a^5x) = c_0$, $\chi(a^6x) = c_1$.

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