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Constructions of potentially eventually positive sign patterns with reducible positive part

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Potentially eventually positive (PEP) sign patterns were introduced by Berman et al. (*Electron. J. Linear Algebra* **19** (2010), 108–120), where it was noted that a matrix is PEP if its positive part is primitive, and an example was given of a 3×3 PEP sign pattern with reducible positive part. We extend these results by constructing $n \times n$ PEP sign patterns with reducible positive part, for every $n \geq 3$.

1. Introduction

A *sign pattern matrix* (or *sign pattern*) is a matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\operatorname{sgn}(A)$ is the sign pattern having entries that correspond to the signs of the entries in A. If $\mathcal{A}$ is an $n \times n$ sign pattern, the *qualitative class* of $\mathcal{A}$, denoted $Q(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n \times n}$ such that $\operatorname{sgn}(A) = \mathcal{A}$, where $\operatorname{sgn}(A) = [\operatorname{sgn}(a_{ij})]$; such a matrix A is called a *realization* of $\mathcal{A}$. Qualitative matrix problems were introduced by Samuelson [1947] in the mathematical modeling of problems from economics. Sign pattern matrices have useful applications in economics, population biology, chemistry and sociology. If P is a property of a real matrix, then a sign pattern $\mathcal{A}$ is *potentially* P (or *allows* P) if there is some $A \in Q(\mathcal{A})$ that has property P.

The *spectrum* of a square matrix A, denoted $\sigma(A)$, is the multiset of the eigenvalues of A, and the *spectral radius* of A is defined as $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. Matrix A has the *strong Perron–Frobenius property* if $\rho(A) > 0$ is a simple strictly dominant eigenvalue of A that has a positive eigenvector. A matrix $A \in \mathbb{R}^{n \times n}$ is *eventually positive* if there exists a $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k > 0$, where the inequality is entrywise. Handelman developed the following test for eventual positivity in [Handelman 1981]: a matrix A is eventually positive if and only if both A and A^T satisfy the strong Perron–Frobenius property. If there exists a k such

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that $A^k > 0$ and $A^{k+1} > 0$, then A is eventually positive [Johnson and Tarazaga 2004]. A sign pattern $\mathcal{A}$ is *potentially eventually positive* (PEP) if there exists an eventually positive realization $A \in Q(\mathcal{A})$.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, define the *positive part* of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha_{ij}^+]$ and the *negative part* of $\mathcal{A}$ to be $\mathcal{A}^- = [\alpha_{ij}^-]$, where

$$\alpha_{ij}^+ = \begin{cases} + & \text{if } \alpha_{ij} = +, \\ 0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = -, \end{cases} \qquad \alpha_{ij}^- = \begin{cases} - & \text{if } \alpha_{ij} = -, \\ 0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = +. \end{cases}$$

Clearly $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$. For a matrix $A \in \mathbb{R}^{n\times n}$, the positive part A^+ of A and negative part A^- of A are defined analogously, and $A = A^+ + A^-$.

A *digraph* $\Gamma = (V, E)$ consists of a finite, nonempty set V of vertices, together with a set $E \subseteq V \times V$ of arcs. Note that a digraph allows loops (arcs of the form (v, v)) and may have both arcs (v, w) and (w, v) but not multiple copies of the same arc. Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. The *digraph of* A, denoted $\Gamma(A)$, has vertex set $\{1, \dots, n\}$ and arc set $\{(i, j) : a_{ij} \neq 0\}$. If $\mathcal{A}$ is a sign pattern, then $\Gamma(\mathcal{A}) = \Gamma(A)$ where $A \in Q(\mathcal{A})$. A digraph Γ is *strongly connected* if for any two distinct vertices v and w of Γ, there is a path in Γ from v to w.

A square matrix A is *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are nonempty square matrices and 0 is a (possibly rectangular) block consisting entirely of zero entries, or A is the 1×1 zero matrix. If A is not reducible, then A is called *irreducible*. It is well known that for $n \geq 2$, A is irreducible if and only if $\Gamma(A)$ is strongly connected. For a strongly connected digraph Γ, the *index of imprimitivity* is the greatest common divisor of the lengths of the cycles in Γ. A strongly connected digraph is *primitive* if its index of imprimitivity is one; otherwise it is *imprimitive*. The *index of imprimitivity* of a nonnegative sign pattern $\mathcal{A}$ is the index of imprimitivity of $\Gamma(\mathcal{A})$ and $\mathcal{A} \geq 0$ is *primitive* if $\Gamma(\mathcal{A})$ is primitive, or equivalently, if the index of imprimitivity of $\mathcal{A}$ is one.

The study of PEP sign patterns was introduced in [Berman et al. 2010], where it was shown that if $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP, and where the first example of a PEP sign pattern with reducible positive part was given: the 3×3 pattern

$$\mathcal{B} = \begin{bmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{bmatrix}.$$

In Section 2 we extend the results of [Berman et al. 2010] by generalizing the 3×3 pattern $\mathcal{B}$ given there to a family of PEP sign patterns having reducible positive part for every order $n \geq 3$.

In Section 3 we examine the effect of the Kronecker product on PEP sign patterns and obtain another method of constructing PEP sign patterns with reducible positive part.

2. A family of sign patterns generalizing $\mathcal{B}$

The sign pattern $\mathcal{B}$ from [Berman et al. 2010] was the first PEP sign pattern with a reducible positive part. This sign pattern may be generalized by defining the $n \times n$ sign pattern

$$\mathcal{B}_n = \begin{bmatrix} + & - & \cdots & - & 0 \\ + & 0 & \cdots & 0 & - \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ + & 0 & \cdots & 0 & - \\ - & + & \cdots & + & + \end{bmatrix}.$$

The following result, which is a special case of the *Schur–Cohn criterion* (see, e.g., [Marden 1949]), will be used in the proof that $\mathcal{B}_n$ is PEP.

Lemma 2.1. *If the polynomial $f(x) = x^2 - \beta x + \alpha$ satisfies $|\beta| < 1 + \alpha < 2$, then all zeros of $f(x)$ lie strictly inside the unit circle.*

It is well known that if the characteristic polynomial of A is $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then $a_{n-k} = (-1)^k E_k(A)$, where $E_k(A)$ is the sum of the $k \times k$ principal minors of A (see, e.g., [Horn and Johnson 1985]).

Theorem 2.2. *For $n \geq 3$ the $n \times n$ sign pattern $\mathcal{B}_n$ is PEP.*

Proof. For $t > 0$, let $B_n(t)$ be the $n \times n$ matrix

$$B_n(t) = \begin{bmatrix} 1+(n-2)t & -t & \cdots & -t & 0 \\ 1+t & 0 & \cdots & 0 & -t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1+t & 0 & \cdots & 0 & -t \\ -(n-2)t - \frac{1}{2}t^2 & t & \cdots & t & 1+\frac{1}{2}t^2 \end{bmatrix}.$$

Then $B_n(t) \in Q(\mathcal{B}_n)$, and 1 is an eigenvalue of $B_n(t)$ with positive right eigenvector $\mathbb{1}$ (the all ones vector) and positive left eigenvector

$$\boldsymbol{w} = \left[\frac{2n-5}{t} \;\; 1 \;\; \cdots \;\; 1 \;\; \frac{2n-4}{t}\right]^T.$$

We show that for some choice of $t > 0$, 1 is a simple strictly dominant eigenvalue of $B_n(t)$ and hence $B_n(t)$ is eventually positive. Since $1 \in \sigma(B_n(t))$ and rank $B_n(t) \leq 3$, the characteristic polynomial $p_{B_n(t)}(x)$ of $B_n(t)$ is of the form

$$p_{B_n(t)}(x) = x^{n-3}(x-1)(x^2 - \beta x + \alpha) = x^n - (1+\beta)x^{n-1} + (\alpha+\beta)x^{n-2} - \alpha x^{n-3}.$$

Computing α and β using the sums of principal minors to evaluate the characteristic polynomial gives $\beta = \frac{1}{2}t^2 + (n-2)t + 1$ and $\alpha = (n-2)t\left(1+2t+\frac{1}{2}t^2\right)$. For $n > 3$, setting $t = 1/(2(n-2))$ gives $|\beta| < 1+\alpha < 2$, which, using Lemma 2.1, guarantees that the two nonzero eigenvalues of B_n other than 1 have modulus strictly less than 1 (recall that a 3×3 eventually positive matrix $B_3 \in Q(\mathcal{B}_3)$ was given in [Berman et al. 2010] so we have not been concerned with this case in choosing t). □

We illustrate this theorem with an example.

Example 2.3. Let $n = 5$. Following the proof of Theorem 2.2, we choose $t = \frac{1}{6}$ and define

$$B_5 = B_5\left(\frac{1}{6}\right) = \frac{1}{6}\begin{bmatrix} 9 & -1 & -1 & -1 & 0 \\ 7 & 0 & 0 & 0 & -1 \\ 7 & 0 & 0 & 0 & -1 \\ 7 & 0 & 0 & 0 & -1 \\ -\frac{37}{12} & 1 & 1 & 1 & \frac{73}{12} \end{bmatrix}.$$

Moreover, we have

$$\begin{aligned}\sigma(B_5) &= \left\{1,\ \tfrac{1}{144}\left(109 + i\sqrt{2087}\right),\ \tfrac{1}{144}\left(109 - i\sqrt{2087}\right), 0, 0\right\} \\ &\approx \{1,\ 0.7569 + 0.3172i,\ 0.7569 - 0.3172i,\ 0,\ 0\},\end{aligned}$$

and $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$ and $\begin{bmatrix}\frac{5}{6} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & 1\end{bmatrix}^T$ are right and left eigenvectors, respectively, corresponding to $\rho(B_5) = 1$. Therefore B_5 and B_5^T have the strong Perron–Frobenius property, so B_5 is eventually positive by Handelman's criterion.

In [Berman et al. 2010] it was shown that if the sign pattern $\mathcal{A}$ is PEP, then any sign pattern achieved by changing one or more zero entries of $\mathcal{A}$ to be nonzero is also PEP. Applying this to $\mathcal{B}_n$ yields a variety of additional PEP sign patterns having reducible positive part.

3. Kronecker products

The Kronecker product (sometimes called the tensor product) is a useful tool for generating larger eventually positive matrices and thus PEP sign patterns. The *Kronecker product* of $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

It is clear that if $A > 0$ and $B > 0$, then $A \otimes B > 0$. The following facts can be found in many linear algebra books; see [Reams 2006], for example. For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$, $(A \otimes B)^k = A^k \otimes B^k$. For A, C, B, D of appropriate dimensions, we have $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. There exists a permutation matrix P such that $B \otimes A = P(A \otimes B)P^T$.

Proposition 3.1. *If A and B are eventually positive matrices, then $A \otimes B$ is eventually positive.*

Proof. Assume that A and B are eventually positive matrices. Since A and B are eventually positive, there exists some $s_0, t_0 \in \mathbb{Z}$, with $s_0, t_0 > 0$, such that for all $s \geq s_0$ and $t \geq t_0$, $A^s > 0$ and $B^t > 0$. Set $k_0 = \max\{s_0, t_0\}$. Then for all $k \geq k_0$, $(A \otimes B)^k = A^k \otimes B^k > 0$. □

Corollary 3.2. *If $\mathcal{A}$ and $\mathcal{B}$ are PEP sign patterns, then $\mathcal{A} \otimes \mathcal{B}$ is PEP.*

If either A or B is a reducible matrix, then $A \otimes B$ is reducible since, without loss of generality, if

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

then

$$(P \otimes I)(A \otimes B)(P \otimes I)^T = \begin{bmatrix} A_{11} \otimes B & 0 \\ A_{21} \otimes B & A_{22} \otimes B \end{bmatrix}.$$

Thus Corollary 3.2 provides another way to construct PEP sign patterns having reducible positive part.

Example 3.3. Let

$$B = \frac{1}{100} \begin{bmatrix} 130 & -30 & 0 \\ 130 & 0 & -30 \\ -31 & 30 & 101 \end{bmatrix}.$$

In [Berman et al. 2010] it was shown that B is eventually positive, and in fact $B^k > 0$ for $k \geq 10$.

Let $A = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 0 \end{smallmatrix}\right]$. Then $A^k > 0$ for $k \geq 2$, hence A is eventually positive.

Then

$$B \otimes A = \frac{1}{100} \begin{bmatrix} 260 & 390 & -60 & -90 & 0 & 0 \\ 130 & 0 & -30 & 0 & 0 & 0 \\ 260 & 390 & 0 & 0 & -60 & -90 \\ 130 & 0 & 0 & 0 & -30 & 0 \\ -62 & -93 & 60 & 90 & 202 & 303 \\ -31 & 0 & 30 & 0 & 101 & 0 \end{bmatrix}.$$

Moreover $(B \otimes A)^{10} > 0$ and $(B \otimes A)^{11} > 0$, so $B \otimes A$ is eventually positive and $\operatorname{sgn}(B \otimes A)$ is a PEP sign pattern with reducible positive part.

Any 0 in $\operatorname{sgn}(B \otimes A)$ from Example 3.3 may be changed to $-$ to get yet another PEP sign pattern with reducible positive part.

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