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PERFORMANCE OF A PHASE MODULATION SYSTEM USING M-LEVEL CODES

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INTRODUCTION

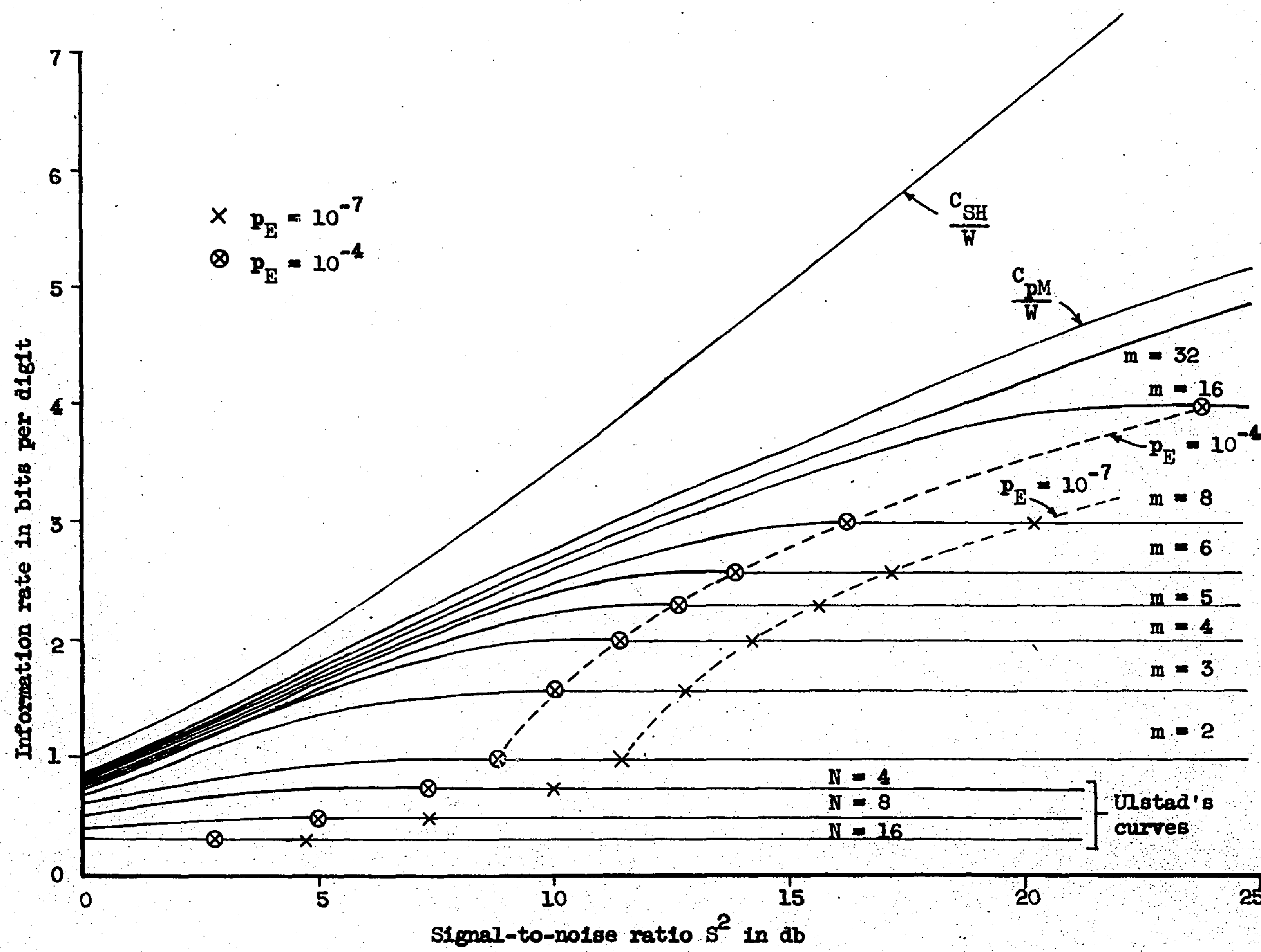
Shannon (1) in 1948 and others who followed him have provided a quantitive basis on which to judge communication systems. Shannon has shown that it is possible to communicate over a noisy channel at any rate less than or equal to that indicated by Equation 1 with an arbitrarily small probability of error if sufficiently elaborate coding schemes are used.

$$C_{SH} = W \log_2 (1 + S^2) \qquad (1)$$

In Equation 1, C_{SH} is the channel capacity in bits per second, W is the bandwidth of the channel in cycles per second, and S^2 is the channel signal-to-noise power ratio. This formula assumes that the noise is additive, gaussian, and white over the entire bandwidth W.

Ulstad (2) has shown that a communication system carrying information only in the phase of a narrowband signal, one with a bandwidth of W centered at ω_o radians per second, permits information transmission at an appreciable fraction of the theoretical maximum, C_{SH}, over a rather large range of values of S^2. Over most of the useful values of S^2 this fraction is considerably larger than the value of one half which might be expected from the sampling theorem for narrowband signals which is discussed in Appendix B. $\frac{C_{PM}}{W}$, the maximum permissible rate of information transmission by phase modulation of a narrowband signal which still permits an arbitrarily low error rate, and $\frac{C_{SH}}{W}$ are plotted in Figure 1. Note that the rates are now in bits per digit. It will be noted that the ratio of C_{PM} to C_{SH} does decrease at the larger signal-to-noise ratios and indeed it does reach the intuitive value of one

Figure 1. Information and error rates for digit-by-digit decoding



× $p_E = 10^{-7}$
⊗ $p_E = 10^{-4}$
$\frac{C_{SH}}{W}$
$\frac{C_{pM}}{W}$
m = 32
m = 16
$p_E = 10^{-4}$
$p_E = 10^{-7}$
m = 8
m = 6
m = 5
m = 4
m = 3
m = 2
N = 4
N = 8
N = 16
Ulstad's curves
Information rate in bits per digit
Signal-to-noise ratio S^2 in db
0
1
2
3
4
5
6
7
0
5
10
15
20
25

half at very large signal-to-noise ratios. This latter fact was originally shown by Blachman (3). The ratio is, however, still about 0.67 for S^2 equal to 20 db.

Figure 1 also shows the information rates possible at various signal-to-noise ratios for digit-by-digit decoding of m equally spaced phase symbols subjected to phase noise. Some error probabilities are also indicated on the curves. These curves, whose calculation is discussed in Appendix A, show that for signal-to-noise ratios in the range of 10 to 20 db more than two levels are needed to take advantage of the inherent information handling capability.

A purely phase modulated system has many practical advantages and the above results show that it has a relatively large information handling capacity. Thus in this paper a communication system using only phase modulation of a narrowband carrier is proposed. This system is not a theoretically optimum system, but it has the advantage that all the required subsystems, which are described only mathematically in the thesis, can be implemented in terms of existing technology with reasonable economy.

Consider the communication system shown in Figure 2. Here the encoder generates sequentially waveforms $x_1(t)$, $x_2(t)$, . . . , $x_M(t)$ corresponding to processed input data. All of the waveforms are assumed to be of equal time duration, of equal power content, and individually distinct. A possible encoder for nonbinary information is shown in Figure 3 where $M = 4$. Woodward (4) showed that if time stationary, white, gaussian noise is added to the output of this type of encoder no decoder can do better than to use cross-correlation processing of the noisy signal. Thus the optimum decoder

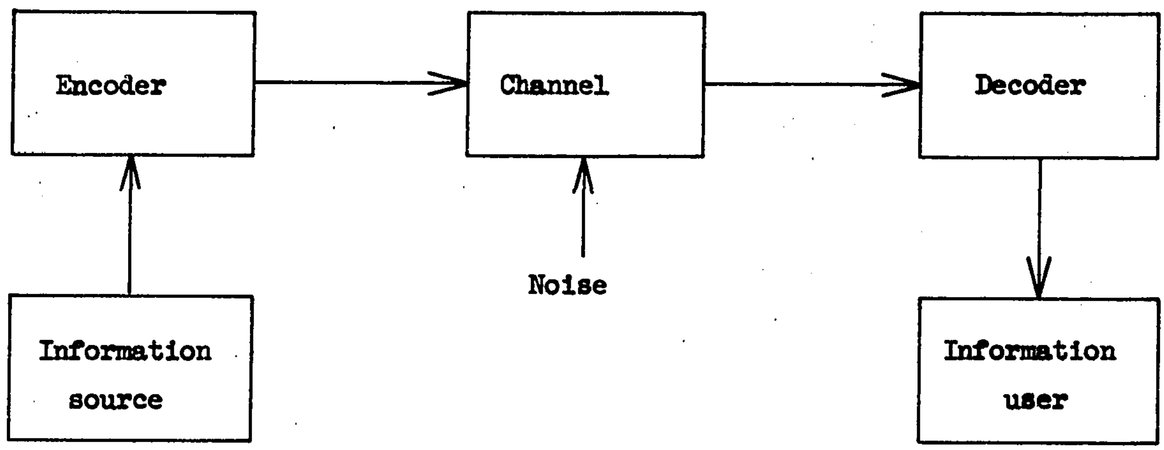


Figure 2. Block diagram of communication system

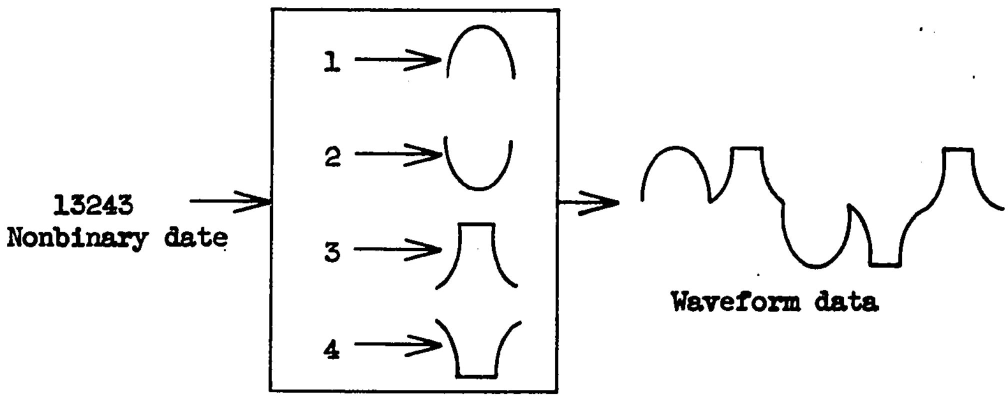


Figure 3. Example of Woodward encoder for M = 4

must store all M possible transmitted waveforms and compare the noisy signal against each. The signal with the largest cross-correlation is assumed to be the one sent by the encoder. Using the above facts, Fano (5) proposed his "Idealized Communication System" shown in Figure 4.

The communication system proposed in this paper has a basic similarity to Fano's system. It will, however, attempt to overcome some of the synchronization and decoder waveform problems associated with the prototype system.

The proposed system is very similar to the system proposed by Ulstad (2) with the exception of the coding used. This system will use nonbinary coding in forming the phase modulation codes rather than the binary codes used by Ulstad. This means that the emphasis will be placed on rates greater than one bit per digit rather than the lower rates which received Ulstad's attention. Ulstad's area of attention is indicated on Figure 1.

Sanders (6) has also described a system similar to Fano's (5) which uses digital phase modulation and a phase locked detection system. This system does not result in a well confined spectrum which may be desirable in many applications.

The heart of the system proposed in this thesis is the coding used. The codes must provide both noise immunity and word synchronization for the system to be successful.

Nonbinary coding to achieve noise immunity has been studied by many authors. Among these are Lee (7), Ulrich (8), Helstrom (9), and Shapiro and Slotnick (10). Nearly all the nonbinary coding work has assumed that blocks of N nonbinary digits are related to one another in such a way that

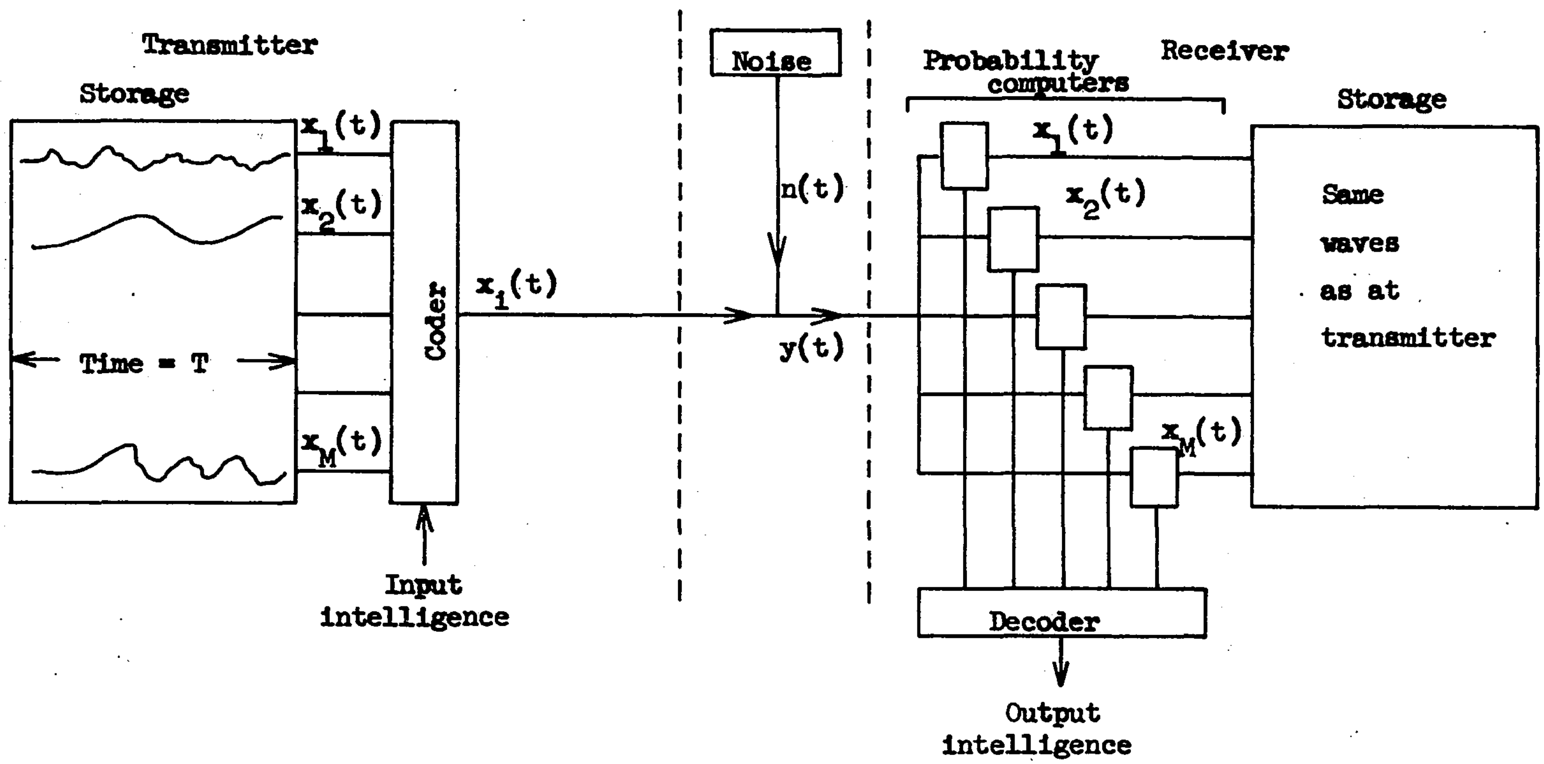


Figure 4. Fano's "Idealized Communication System"

an error in decoding one or more of the N digits can be corrected or at least detected. The digits are decoded one at a time and then the constraints are applied to detect or correct digits corrupted by noise. This type of decoding leads to a coding process in which the code words are picked by the number of digit positions in which they differ from all other allowed code words or are picked such that for any two allowed code words s and t the sum of the circular metric,

$$\rho(s_i, t_i) = \text{Min } (s_i - t_i, t_i - s_i) \pmod{m} \qquad (2)$$

over all N digits forming the words is equal to or greater than a given number. The number of symbols allowed is represented by m. The former criterion corresponds in coding terminology to using Hamming (11) distance and the second to using Lee (7) distance. Neither of these distance criterion for choosing the allowed code words is suitable for the detection scheme proposed in this paper. Thus a new coding scheme with a new distance criterion will be required.

Barker (12), Sherman (13), Gilbert (14), and Stiffler (15) have all investigated methods of word synchronization which result in a reduced information rate.

Stiffler (15) very recently proposed the use of certain codes to achieve synchronization without reducing the information rate. These codes were binary and achieved the same results as Ulstad's (2) Hadamard codes. The proposed system will use nonbinary codes to achieve synchronization in a similar manner.

After a discussion of the system and the type of detection to be used a major portion of the paper will be devoted to the construction of suitable nonbinary codes. The remainder of the paper will deal with the performance of these codes in the system and in particular with the error and information rates that they permit.

The major contribution of this thesis is the construction and performance evaluation of nonbinary codes suitable for use in a phase modulation communication system using word decoding. Codes which have greater noise immunity than those used at the same information rate for digit-by-digit decoding and which have inherent synchronization information are developed. These codes are chosen to meet a new distance criterion which is developed especially for word decoding of phase modulation information. The geometrical approach used in the code construction is believed to be unique.

DESCRIPTION OF PROPOSED COMMUNICATION SYSTEM

It is assumed that the input data to the proposed system is available at a constant rate. It is further assumed that the source and receiver have as common knowledge the code dictionary, the carrier frequency, and the transmitter bandwidth. The receivers knowledge of the code dictionary means that redundancy may be used to improve the reliability of correct message reception. The function of the channel is to transfer the input data in as short a time, in as narrow a bandwidth, and in as reliable a manner as possible to the user.

The system has several significant features. The encoding is done by a group of linear networks, the envelope amplitude is not constant with time as in a conventional phase modulation system, the detection scheme is not a matched filter problem in the normal sense, and synchronization information is carried in the received signal.

A block diagram of the proposed system is shown in Figure 5. It is assumed that the input data can be converted into code words which consist of N digits that may assume any of m values α_i, $i = 1, 2, \ldots, m$, where $-\pi \leq \alpha_i < \pi$. That is a block encoder with a block length of N is used. The code words will be given the symbol θ^j where $j = 1, 2, \ldots, M$ depending on which word is considered. These words may be thought of an N dimensional vectors with quantized component values. That is the code word θ^j may be thought of as the vector $\underline{\theta}^j = \sum_{k=1}^{N} \theta_k^j \underline{i}_k$ where θ_k^j takes on one of the values α_i, $i = 1, 2, \ldots m$, and $\underline{i}_k \cdot \underline{i}_j = 0$ if $k \neq j$. At times it will also be convenient to think of the code word θ^j as the sequence $\theta^j =$

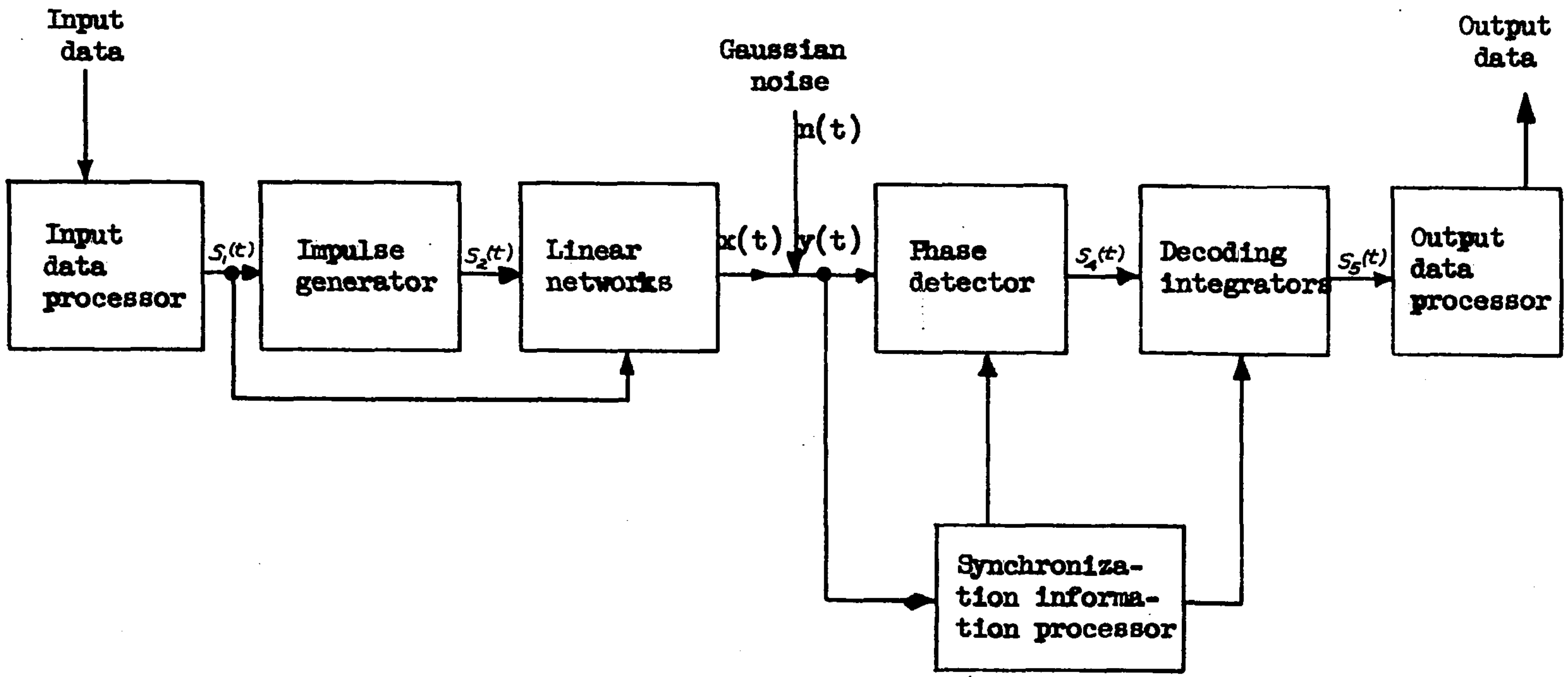


Figure 5. Block diagram of proposed communication system

$\theta_1^j, \theta_2^j, \theta_3^j, \ldots, \theta_N^j$ where the θ_k^j's are the same as above. The total number of code words is given the symbol M. M is thus the size of the code dictionary.

In Figure 5 it is the function of the input data processor to perform the conversion of the data into the allowed code words. If the data is not available at a constant rate the data processor must have some memory capacity to smooth the data rate. The signal $s_1(t)$ is a signal which takes on the one of the M values which corresponds to the code word chosen and which changes every T seconds corresponding to the rate at which new code words are sent.

Each of the code words has a unique waveform which corresponds to it and the transmitted signal, $x(t)$, consists of a continuous sequence of these waveforms. It is the function of the impulse generator and the bank of linear networks to generate these waveforms. The signal $s_1(t)$ goes to the impulse generator where a train of N unit impulses is generated. The individual impulses may be either positive or negative and are separated in time by μ seconds, where $\mu = \frac{1}{W}$. The exact form of the impulse train, whether a given impulse is positive or negative, depends on the value of $s_1(t)$. However, there will in general be a many to one correspondence between the values of $s_1(t)$ and impulse trains.

The signal $s_2(t)$, the impulse train, is the input to the bank of linear networks. The signal $s_1(t)$ also goes to this bank of networks and controls which of the networks is used. The impulse response of each of the linear networks is chosen such that each impulse controls the phase of the transmitted signal at a particular instant of time. At these instants of time, called sample points, the transmitted signal x(t) has an envelope amplitude of A and one of m possible phase values α_i depending on the polarity of the impulse and the value of $s_1(t)$.

Narrowband, white, gaussian noise n(t) is assumed to be added linearly to the transmitted signal x(t) by the channel to form the received signal y(t). That is,

$$x(t) + n(t) = y(t) \ . \tag{3}$$

The phase detector responds only to the slowly varying phase of the input signal y(t). It provides an output signal that is linearly proportional to the phase of y(t).

The synchronization information processor uses the input signal y(t) to adjust the local oscillator which provides the reference for the phase detector. It also provides sampling synchronization information to the decoding integrators. A memory capacity here will smooth out the effects of the noise in y(t).

The decoding integrators have the job of determining which of the M possible messages was sent. The phase function $s_4(t)$ is sampled at the N unique times which were called sample points earlier. That is, N samples μ seconds apart and with appropriate synchronization are taken. These N sample points form words similar to the code words. Each digit is, however, no longer quantized to m values but may have any value in the continuum from $-\pi$ to π. Thus the received word Φ may be represented as $\Phi = \sum_{k=1}^{N} \varphi_k \underline{i}_k$ where $-\pi \leq \varphi_k < \pi$ and $\underline{i}_k \cdot \underline{i}_j = 0$ if $k \neq j$ or as $\Phi = \varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_N$. The decoding integrators compare the word Φ with all the M possible code words, Θ^j, to determine which code word Φ resembles most closely in a particular sense. The type of comparison is discussed in the next section. On the basis of this comparison, a decision as to the message most probably transmitted is formed.

The signal $s_5(t)$ has M possible values corresponding to the possible transmitted messages and assumes the value corresponding to the message considered most probable. This signal provides the input to the decoding data processor which converts the signal into the input data that it represents and provides this as an output. Note that in many applications this step may be unnecessary.

There is a word synchronization problem in the formation of Φ. Namely, it must be known when the first sample point in any particular Φ is to be taken. This is particularly a problem when there are fluctuations in transmission path length and hence fluctuations in transit time between the transmitter and receiver. The essential problem in the proper operation of the receiver is one of adjusting the phase sampling times correctly within a range of μ seconds and then dividing the samples into the proper blocks of N samples. No provision for handling the latter problem is made on the block diagram. This problem will be discussed in greater detail when the coding is discussed.

Transmitter Operation

The transmitter is shown in greater detail in Figure 6. The input data processor serves the purpose mentioned earlier. Its output goes to both the impulse generator and the impulse routing logic.

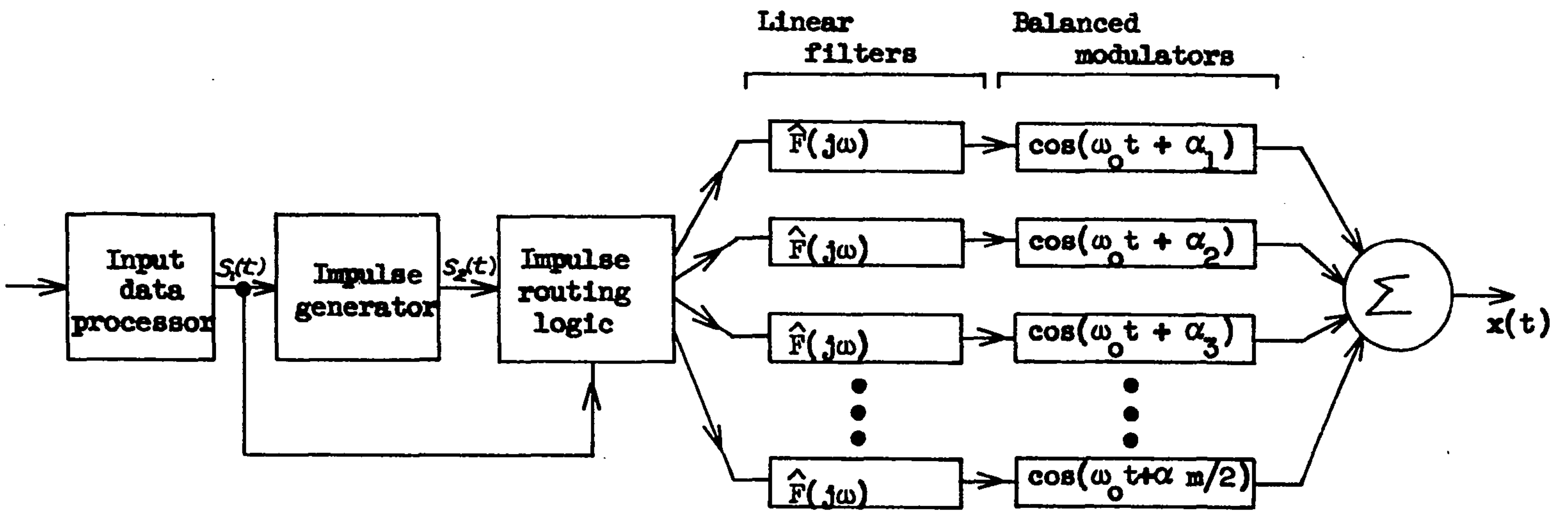


Figure 6. Transmitter for proposed communication system

Assume that a particular code word $\theta^1 = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{-\pi}{4}, \frac{-3\pi}{4}$ is to be sent, and that it is followed by the word $\theta^7 = \frac{\pi}{4}, \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{-\pi}{4}$. The corresponding impulse train is shown in Figure 7. That is, a signal $s_1(t)$ corresponding to the code word θ^1 would enter the impulse generator and its output signal $s_2(t)$ would look like the indicated portion of Figure 7.

It is shown in Appendix B that any band-limited signal which is not also time limited may be represented by the equation

$$z(t) = \sum_{k=-\infty}^{k=\infty} A_k \frac{\sin \pi W(t - \frac{k}{W})}{\pi W (t - \frac{k}{W})} \cos [\omega_o(t - \frac{k}{W}) - \lambda_k] \qquad (4)$$

where W is the bandwidth in cycles per second, ω_o the carrier frequency in radians per second, and A_k and λ_k are independent amplitude and phase values. Thus at the k^{th} sample point corresponding to $t = \frac{k}{W}$ any narrowband signal is completely determined by the values of A_k and λ_k and the sample points are spaced $\frac{1}{W}$ seconds apart.

If the transmitted signal could be built up from functions of the form

$$z_k(t) = A_k \frac{\sin \pi W (t - \frac{k}{W})}{\pi W (t - \frac{k}{W})} \cos [\omega_o (t - \frac{k}{W}) - \lambda_k] \qquad (5)$$

then in the noiseless case sampling of the received signal at $\frac{1}{W}$ intervals would yield all the information available in the waveform. It is, of course, not feasible to suggest this course of action but it should be possible to build signals by adding together functions of the form

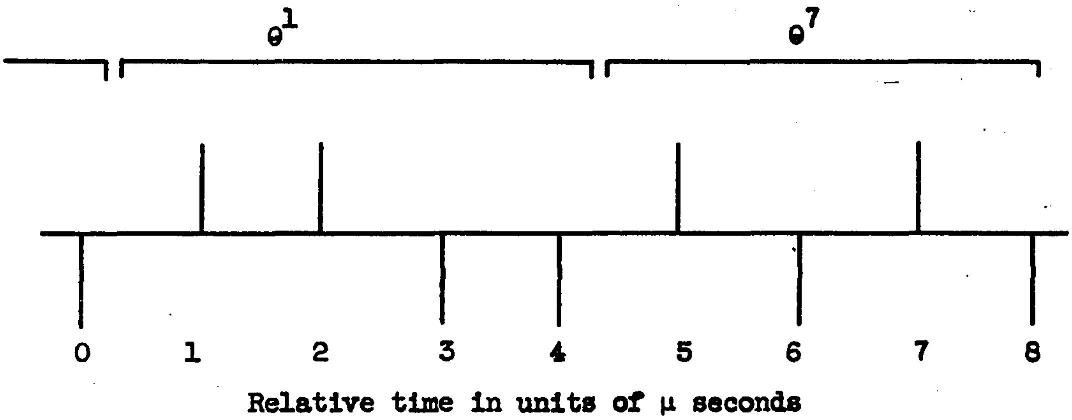


Figure 7. The signal $s_2(t)$ for the messages θ^1 and θ^7

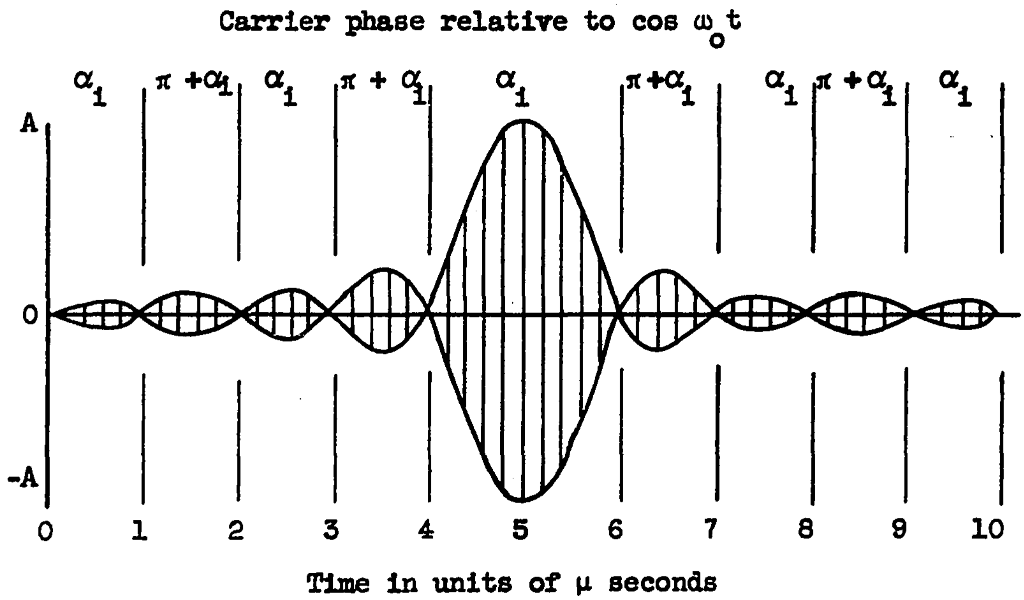


Figure 8. Sketch of the function $f^1(t)$ as given by Equation 7

$$z_k'(t) = A_k \frac{\sin \pi W(t - \frac{k+5}{W})}{\pi W(t - \frac{k+5}{W})} \cos [\omega_o(t - \frac{k+5}{W}) - \lambda_k]$$

$$[u_1 (t - \frac{k}{W}) - u_1(t - \frac{k+10}{W})] \ . \qquad (6)$$

In Equation 6 the delay is chosen arbitrarily as $\frac{5}{W}$ seconds because this results in a relatively well confined spectrum as shown in Appendix C. The function $u_1(t)$ is the unit step function. The truncation and time delay should make this time function realizable. The signal $z_k'(t)$ is now time limited and thus the resulting signal will not be limited to the frequency band W centered at ω_o. The spread in frequency caused by the truncation will, however, normally be quite small. The frequency spread is discussed in more detail in Appendix C.

A signal formed by adding together functions like $z_k'(t)$ depends only on A_k and λ_k at the k^{th} sampling point. Thus, in the noiseless case, sampling this transmitted signal at the sample points would yield the information originally put into the signal. In a noisy situation the sampled values would not agree with the original values of A_k and λ_k because of the effect of the noise but they can be processed in blocks of the same length as used in the original encoding to see which of the allowed values they most closely resemble. This method of detection is not optimal as will be shown in the next section but its ease of implementation makes it desirable.

Thus from the standpoint of confining the frequency band of the transmitted signal and simplifying the detection it is desirable to build the transmitted signal by addition of functions of the form shown in Equation 6.

In a purely phase system, such as the one under consideration, the envelope amplitude A_k will have the same value at all sample points, that is, for all values of k. Let this value be A. Note that the envelope amplitude between sample points is not equal to a constant but depends on the phase values at ten sample points. A bank of filters with impulse responses of the form

$$f^i(t) = A \frac{\sin \pi W(t - \frac{5}{W})}{\pi W(t - \frac{5}{W})} \cos(\omega_o t - \alpha_i) [u_1(t) - u_1(t - \frac{10}{W})] \quad (7)$$

can be used to form the desired signal. Here λ_k has been replaced by α_i. If a filter with the impulse response indicated in Equation 7 is driven with a negative impulse the envelope amplitude is still A but the phase angle of the carrier is $\alpha_i + 180^o$. Thus each filter of the above type is capable of yielding two of the m possible phase values α_i if m is even and the m values are evenly spaced over the 2π range. If the above is true, a total of $\frac{m}{2}$ linear filters of the form of Equation 7 must be contained in the bank.

The indicated impulse response may be realized more easily if the impulse response corresponding to the truncated $\frac{\sin x}{x}$ function

$$\hat{f}(t) = A \frac{\sin \pi W(t - \frac{5}{W})}{\pi W(t - \frac{5}{W})} [u_1(t) - u_1(t - \frac{10}{W})] \quad (8)$$

is realized and used to drive a balanced modulator operating at an angular frequency of ω_o radians per second and a phase of α_i radians relative to a reference. Figure 8 shows $f^i(t)$ and Figure 9 shows $\hat{f}(t)$.

As shown in Figure 6, the signal $s_2(t)$ is routed through the impulse routing logic by $s_1(t)$ in such a manner that the impulses go to the proper filters. Assuming that m = 4, the code word θ^1 discussed earlier illustrates the procedure. Two filters would be required, the first with an angle of $\frac{\pi}{4}$ and the second with an angle of $\frac{3\pi}{4}$. The first and fourth impulses would be routed to the first filter and the second and third impulses would be routed to the second filter. Note the correspondence between the desired phase angles and the impulse polarities.

The outputs of all the filters are summed continuously to form the transmitted signal, x(t).

If m is odd or if the allowed phase values α_i are not evenly spaced, m linear filters are required. Further, the impulse train would consist of only positive impulses separated by μ seconds and the signal $s_1(t)$ would go only to the routing logic.

Detection Theory

The encoded signal is of the form

$$x(t) = \sum_k z_k'(t) , \tag{9}$$

where $z_k'(t)$ is given by Equation 6 and λ_k is one of the m values α_i. It is desired to obtain the original phase information, the values of λ_k, from this signal after the addition of stationary, white, gaussian noise.

Woodward (4) has shown that the ideal statistical receiver is one that

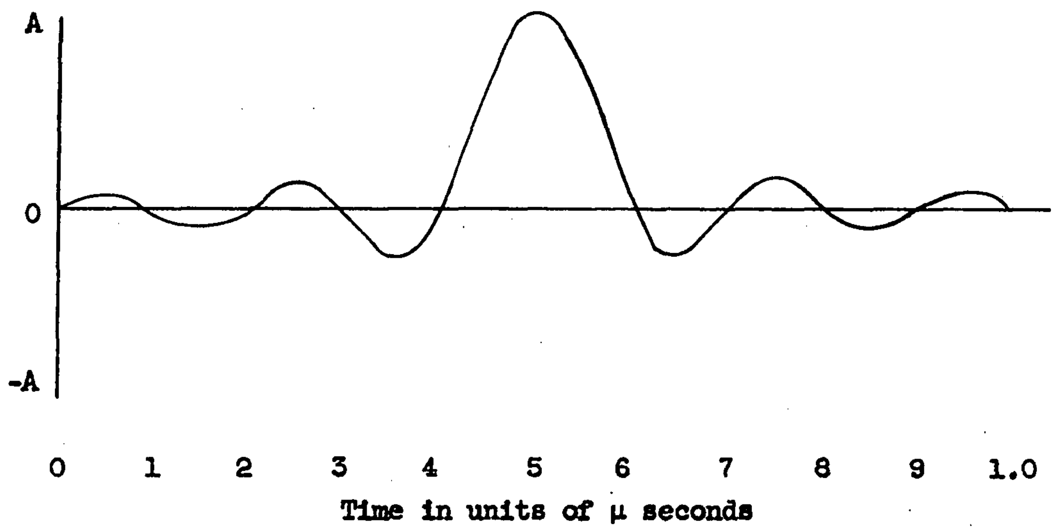


Figure 9. Sketch of $\hat{f}(t)$ as given by Equation 8

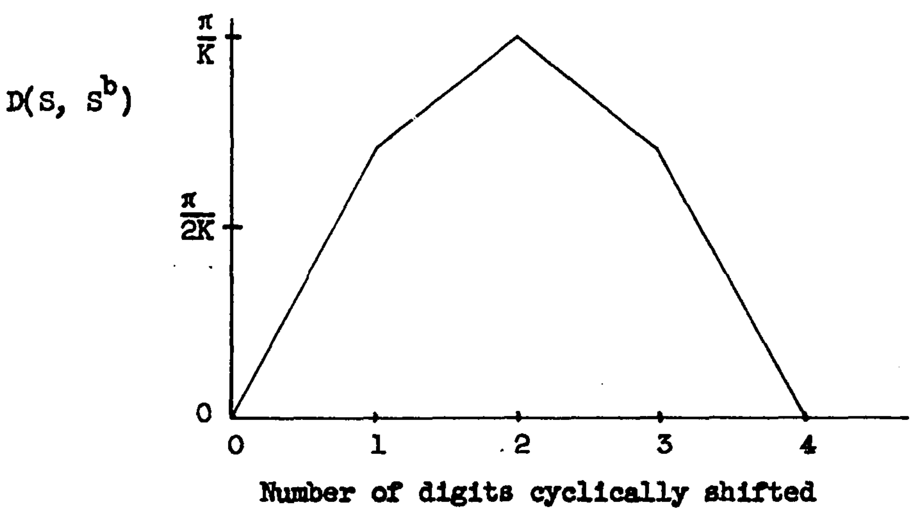


Figure 10. Plot of $D(S, S^b)$ for $S = \frac{\pi}{2K}, \frac{\pi}{2K}, 0, 0$

forms $p_y(x)$, the conditional probability that the event x has occurred given the occurrence of the effect y, or something informationally equivalent. He writes

$$p_y(x) = K\, p(x)\, p_x(y) \tag{10}$$

where K is a normalization constant dependent on y alone, $p(x)$ is the a priori probability of the event x, and $p_x(y)$ is the probability that the effect y is observed if the event x occurs. If several independent events are grouped together to form new events, $X = x_1, x_2, \ldots, x_N$, and the possible effects are now given the symbol $Y = y_1, y_2, \ldots, y_N$, Equation 10 becomes

$$p_Y(X) = K'\, p(X)\, p_X(Y) = K'\, p(X) \prod_{i=1}^{N} p_{x_i}(y_i) \tag{11}$$

The conditional probabilities $p_{x_i}(y_i)$ are called likelihood functions and for any given effect y may be thought of as only functions of x. Thus, Equation 11, for a given effect Y, is only a function of X.

In the case at hand the independent events are the sample point values, A_k and θ_k, used in forming x(t), the transmitted signal, and the effects are the sample point values, B_k and φ_k, of y(t), the received signal. The conditional probability $p_{A_k,\,\theta_k}(B_k, \varphi_k)$ is shown in Appendix E to be

$$p_{A_k,\theta_k}(B_k,\varphi_k) = C\, e^{-(B_k \cos \varphi_k - A_k \cos \theta_k)^2 - (B_k \sin \varphi_k - A_k \sin\theta_k)^2} \tag{12}$$

and thus the ideal receiver would form

$$p_{y(t)}(\theta^i) = K''p(\theta^i)e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}[(B_k \cos \varphi_k - A_k^i \cos \theta_k^i)^2 + (B_k \sin \varphi_k - A_k^i \sin \theta_k^i)^2]} \tag{13}$$

where K" is a constant, $p(\theta^i)$ is the a priori probability of the message θ^i, σ^2 is the variance of the noise, A_k^i is the envelope amplitude at the k^{th} sample point of message θ^i and θ_k^i is the phase at the k^{th} sample point of message θ^i, and B_k is the envelope amplitude and φ_k is the phase at the k^{th} sample point of the received signal y(t). Equation 13 may be written as follows

$$p_{y(t)}(\theta^i) = K''p(\theta^i)e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N} B_k^2}\, e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N} (A_k^i)^2}\, e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N} A_k^i B_k \cos(\varphi_k - \theta_k^i)} \tag{14}$$

The term

$$e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N} B_k^2}$$

depends only on y(t) and can be included in the constant for a given y(t). The envelope amplitude coefficients A_k^i have the same value A for all i and all k, and thus the term

$$e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N} (A_k^i)^2}$$

is a constant. The key term is thus

$$e^{\frac{1}{2\sigma^2} \sum_{k=1}^{N} AB_k \cos(\varphi_k - \theta_k^i)} .$$

If all the messages θ^i are equally probable a priori, the most probable transmitted message is the one that maximizes $\sum_{k=1}^{N} B_k \cos(\varphi_k - \theta_k^i)$. This term does not lend itself to easy mechanization and thus the receiver used will not be the ideal one suggested above. The amplitude information B_k will not be used and the cosine will be approximated by the first two terms of its Taylor series. Thus the decision as to the most probable message sent will be made on the basis of the minimization of the sum

$$\sum_{k=1}^{N} (\varphi_k - \theta_k^i)^2 \tag{15}$$

where $\left|\varphi_k - \theta_k^i\right| \leq \pi$. This corresponds to maximizing $p_{y(t)}(\theta^i)$. This type of detection is called maximum likelihood detection. It should be noted that only in gain stable systems could the above type of detection which depends on differences be used effectively. The linear phase detector in the receiver will provide the gain independent signal needed.

This detection scheme points out an important criterion to be used in the selection of the allowed code words. In detection the $\sum_{k=1}^{N} (\varphi_k - \theta_k^i)^2$ is formed for each of the allowed messages, θ^i; the message which yields the minimum value is assumed to be the message sent. The assumption is that the most probable noise sequence at the sample points, namely the one that yields the minimum sum, is the actual noise sequence. There is, of course, a non-zero probability that a noise sequence with a larger sum was the actual sequence and that the wrong message will be selected. That is, the signal formed by adding larger noise sample points onto the message sample points may be closer to another message θ^j than the original one θ^i in the sense that $\sum_{k=1}^{N} (\varphi_k - \theta_k^j)^2$ is smaller than $\sum_{k=1}^{N}(\varphi_k - \theta_k^i)^2$. If the allowed code words are far apart in the sense that the sum of the squares of their coordinate differences is large then the probability of incorrect decision is reduced because the probability of a phase noise sequence with coordinates large enough to make the received signal closer to a message other than the original message is greatly reduced.

Nonbinary Coding Scheme

It is desired to generate code words which can be used to represent various possible messages. A code word is actually a sequence of N digits where each digit takes on one of m values α_i, i = 1, 2, . . . , m, where $-\pi \leq \alpha_i < \pi$. These sequences are used to specify the phase of the transmitted signal at N sample points. The discussion of detection pointed out the need to choose these sequences in a particular manner to achieve the greatest noise immunity. It is desired to choose the sequences such that for any given number of code words the minimum distance separating any two of them is as large as possible. The distance metric used will resemble the Euclidean metric but differs from it due to the cyclic nature of phase values. If the sequences are put in a one-to-one correspondence with the positive integers from one to M and i and j are indices ranging over these integers, the distance D(i, j) between the sequences corresponding to i and j should be as large as possible for all i and j for which $i \neq j$. Putting the problem in another way, it is desired to find all the sequences of N digits for which $D(i, j) \geq C$ for all i and j for which $i \neq j$ and some positive real constant C. This form of the problem is more tractable.

The reason for wanting to find all the sequences that meet a given distance criterion is that the rate of transmission of information, if each of the sequences has an equal probability of being transmitted and the error rate is low, has the form

$$R = \frac{1}{N} \log_2 M. \tag{16}$$

Here M is the number of sequences or messages and R is the rate in bits per digit. Thus an increase in M corresponds to a rate increase.

Selection of words meeting a distance criterion

Because sequences meeting a certain distance criterion are to be chosen a geometrical point of view will be very useful in choosing the code words. The sequences may be thought of as points inside an N-dimensional cube of length 2π on each side. This cube will be called the phase space. The 2π restriction is imposed by the fact that only phase values in a 2π range are unique and a code suitable for phase modulation is required. Distances in this phase space are measured according to a metric which will be defined below. Each digit of one of the message sequences is a coordinate of the point corresponding to that sequence. Thus if the digits can take on any of m values the point coordinates in any dimension can have any of m values.

It will be assumed that $m = K2^g$, where K is a positive integer greater than one and $g = 0, 1, 2, \ldots$ The reason for this form will become evident later. The point coordinates will be chosen originally as

$$\alpha_i = -\pi + i\,\frac{2\pi}{m} \qquad (17)$$

where $i = 0, 1, 2, \ldots, m\text{-}1$. These values will later be shifted to aid decoding. These values can be placed in a one-to-one correspondence with the positive integers $0, 1, 2, \ldots, m\text{-}1$. The phase space may now be thought of as corresponding to a space in which the coordinate values in each dimension cyclically repeat. This space will be called the cyclic space and the coordinates in it are the integers 0 through m-1. Addition and subtraction of these integers is defined modulo m. That is $1 + (m\text{-}1) = m = 0$. The distance between two of the integers corresponding to point coordinates in this space may be defined by the ring metric.

$$\rho(i_k, j_k) = \text{Min}\,(i_k - j_k,\ j_k - i_k)\ \text{Mod}\ m \qquad (18)$$

which was first introduced into coding by Lee (7). In Equation 18 i_k and j_k are the integers corresponding to the k^{th} coordinates of the code points I and J. The total distance in cyclic space between the points I and J, corresponding to the points β^i and β^j in phase space, is

$$d(I, J) = \left[\sum_{k=1}^{N} \rho^2\,(i_k, j_k)\right]^{\frac{1}{2}} \qquad (19)$$

The actual distance, D, separating the two points in phase space can be obtained by multiplying Equation 19 by $\frac{2\pi}{m}$. That is,

$$D(\beta^i, \beta^j) = \left[\sum_{k=1}^{N} \rho^2(i_k, j_k)\right]^{\frac{1}{2}} \frac{2\pi}{m}\ . \qquad (20)$$

Note that this distance is very similar to normal Euclidean distance. The difference is introduced by the fact that the magnitude of a phase difference may never exceed π radians. This corresponds to the integer differences never exceeding $\frac{m}{2}$ which is insured by Equation 18. The metrics of Equations 19 and 20 are justified and indeed forced by the detection scheme which is being used and the fact that phase is the information carrying quantity.

The code selection problem may now be alternatively stated in terms of the foregoing models.

A. Given the dimensionality of the space and the distance required between code points what is the largest number of code points allowed, or the largest rate of transmission, and how can the points be determined?

B. Given the desired rate of transmission and minimum code point distance, what is the dimensionality of the space required and how can the appropriate points be chosen?

In either of the two forms of the problem the value of m, the number of allowed coordinate values, is allowed to vary and will ordinarily be chosen such that the rate is maximized in the first formulation and the value of N is minimized in the second.

A possible solution to the coding problem follows. This solution will be the best possible for certain values of the distance D under the assumption of equally spaced phase values. A symmetrical evenly spaced lattice is placed in the N-dimensional cube which is the phase space. The lattice spacing, D', is chosen such that it is greater than or equal to the required value of D but as small as possible under the constraint that $\frac{2\pi}{D'}$ must be an integer. This lattice will be placed in the cube such that the point ($-\pi$, $-\pi$, $-\pi$, . . .) is a point on the lattice. That is, each dimension of the cube is divided into K levels with a distance D' between each level and the first level is at $-\pi$. Note that π cannot be an allowed level because it cannot be distinguished from $-\pi$. If the above is impossible because $D > \pi$ the lattice should contain 2^N points and the levels $-\pi$ and 0 will be allowed. The lattice under discussion will be called the K-lattice where K corresponds to the K in $m = K2^g$.

If $D \leq \pi$ then all the points of the K-lattice are allowed code points because they meet the distance criterion. Note that a distance value of one in the cyclic space is sufficient because for the K-lattice $D = dD'$ and D' is greater than or equal to D. If $D > \pi$, then only some of the points of the K-lattice may be used as code points; namely, those that have a value of $d \geq \frac{D}{\pi}$ in the corresponding cyclic space. Only two levels are allowed in this case and thus the coordinate integers i_k and j_k can assume only the values 0 and 1 for any two points I and J corresponding to K-lattice points. The problem of choosing the points of the K lattice which can be used as code points now reduces to a binary coding problem. Here $\rho(i_k, j_k)$ equals zero if $i_k = j_k$ and equals one if $i_k \neq j_k$. The square of the distance d now agrees with the metric suggested by Hamming (11), and extensively used in binary coding theory. Hamming lets the distance between any two binary sequences be the sum of the number of positions at which they differ. Hamming's distance d_H may be written formally as

$$d_H (I, J) = \sum_{k=1}^{N} i_k \oplus j_k \qquad (21)$$

where $\oplus$ implies addition modulo 2 and i_k and j_k are the k^{th} digits of the binary sequences I and J. Thus the selection of K-lattice points for the case $D > \pi$ is the same as the binary problem of selecting the binary sequences which satisfy $d_H(I, J) \geq (\frac{D}{\pi})^2$ for all I and J where $I \neq J$.

Now the problem is to determine if more code points and thus a larger transmission rate are possible if the number of signal levels, possible coordinates in any dimension, is doubled. To do this the spacing between each of the original K-levels is halved. This forms a lattice called the 2K-lattice which has a lattice spacing of one half of the K-lattice spacing and which contains the K-lattice points. If N is large enough it will be possible to find additional points on this lattice which meet the distance requirement with respect to themselves and also with respect to the points of the original K-lattice already chosen. To find these points we consider the N-cube formed by the first two levels of the K-lattice, that is, the cube formed by the phase values $-\pi$ and $-\pi + \frac{2\pi}{K}$. The 2K-lattice has divided this cube into 2^N smaller cubes of length $\frac{D'}{2}$ on a side. One of these cubes has the point $(-\pi, -\pi, -\pi, \ldots)$ as a vertex. This point, which corresponds to the point $(0, 0, 0, \ldots)$ in cyclic space, will always be considered a code point. If $D' \geq D$ this point is a code point because it belongs to the K-lattice and if $D' < D$ it will be by definition one of the points of the K-lattice which are chosen as code points. This in no way limits the coding scheme.

The problem now is to determine which of the vertices of the cube formed by the 2K-lattice, which has the point $(-\pi, -\pi, -\pi, \ldots)$ as one vertex, can be considered code points. This reduced cube will be called the 2K-cube and the larger cube formed by the K-lattice which contains it will be called the K-cube. The 2K-cube is always assured of having at least one code point as it has the vertex $(-\pi, -\pi, -\pi, \ldots)$. Any other vertices of this cube which meet the distance criterion among themselves and with respect to the point $(-\pi, -\pi, -\pi, \ldots)$ will also meet the distance criterion with respect to any previously chosen code points. To show that this is true consider the following. The coordinates of the K-cube correspond to the integers 0 and 2 in the cyclic space. The coordinates of the 2K-cube correspond to the integers 0 and 1. Let a vertex of the 2K-cube be chosen and called I. The point I will have a distance d with respect to $(0, 0, 0, \ldots)$ equal to the square root of the number of coordinates of I which are 1. This is true because $\rho(0, 0) = 0$ and $\rho(1, 0) = 1$. Assume I has z coordinates which have the value 1. Then for any J which is a vertex of the K-cube $d(I, J)$ will be at least $\sqrt{z}$ because both $\rho(1, 2)$ and $\rho(1, 0)$ are equal to one and I has z coordinates which are 1. Obviously if I meets the distance criterion with respect to the allowed code points on the K-cube and any other allowed code points on the 2K-cube it will meet the criterion for any other allowed code points outside of the K cube. Thus the problem of choosing which of the 2K-cube vertices can be used as code points has been reduced to the binary coding problem of choosing those binary sequences which differ in a given number of positions from all previously chosen sequences assuming that the sequence $(0, 0, 0, \ldots)$ is always chosen first. For a vertex of the 2K-cube to meet the distance criterion with respect to $(-\pi, -\pi, -\pi, \ldots)$ in phase space its corresponding point in cyclic space must contain at least $\hat{q}$ coordinates which have the value 1, were

$$q = \frac{4D^2}{(D')^2} \qquad (22)$$

and $\hat{q}$ is the smallest integer larger than q. The value $\hat{q}$ is the smallest value that N can have if the additional phase values are to increase the number of code words. If $N < \hat{q}$ only the points originally chosen on the K-lattice are allowed. Note that if $D' = D$, N must be at least 4 before any of the 2K-lattice points other than those that correspond to K-lattice points can be used as code points.

In cyclic space to obtain all the points of the 2K-lattice which may be used as code points it is only necessary to add vectors which represent the allowed vertices of the 2K-cube onto each vector which represents an allowed K-lattice code point. The coordinates of the heads of the resulting vectors are the allowed code points. These vectors have the point (0, 0, 0, . . .) as origin. Note that the coordinates of the K-lattice points in cyclic space now have the values 0, 2, 4, . . . , 2(K-1) instead of the values 0, 1, 2, . . . , K which they had when only the K-lattice existed. The allowed code points found in cyclic space can be transferred to the corresponding phase space by using the one-to-one integer to phase value correspondence. As the 2K-cube vertex (0, 0, 0, . . .) is an allowed code word the vector $0\underline{i}_1 + 0\underline{i}_2 + \ldots + 0\underline{i}_N$ is added to all the vectors representing allowed K-lattice code points. The K-lattice code points are thus included in the 2K-lattice code points and will from now on be considered as 2K-lattice code points. All the points which can be represented by 2K levels and which are mutually separated by a distance D or more according to the metric of Equation 20 have now been obtained.

The above process is now repeated. Each lattice distance of the 2K-lattice is halved and the 4K-lattice is formed. In cyclic space the K-lattice coordinates are now 0, 4, 8, . . . , 4(K-1) and the 2K-lattice coordinates are 0, 2, 4, . . . , 4K-2. Attention will be focussed on the cube formed by the 4K-lattice which has the point ($-\pi$, $-\pi$, $-\pi$, . . .) as a vertex. As for the 2K-cube, the vertices of this cube which satisfy the distance requirement with respect to themselves and with respect to the point (0, 0, 0, . . .) in cyclic space are determined. Using the same arguments as before it can be shown that a 4K-cube vertex which satisfies the distance requirement with respect to (0, 0, 0, . . .) satisfies it with respect to all previously chosen code points. All the other allowed 4K-lattice code points can be determined by adding in cyclic space the vectors representing the allowed vertices of the 4K-cube onto the vectors representing the allowed 2K-lattice code points. The coordinates of the heads of the resulting vectors are the coordinates of the allowed 4K-lattice points in cyclic space. The corresponding points in phase space are the allowed code points. A point on the 4K-cube is an allowed code point only if $d \geq \frac{4D}{D'}$ and thus if $D = D'$, N must be at least 16 before the increased number of levels increases the number of allowed code points.

The above procedure is now repeated for an 8K-lattice and so on until no additional code points result from the introduction of more levels.

The above coding method may be viewed as follows. A N-dimensional cube with sides of length π or less is chosen as the starting point. The length of the sides is chosen as closely to the required code point separation D as an integral division of 2π will allow if $D \leq \pi$ and as the value π if

$D > \pi$. Other N-dimensional cubes that have a common vertex with the large cube but which have sides half the length of the next larger cube are successively inserted inside the large or K-cube. Vertices of these N-dimensional cubes which are mutually separated by an Euclidean distance D are chosen as code points. Note that the metric defined in Equation 20 has degenerated to the ordinary Euclidean distance because it is not necessary to worry about the ring or modularity property of phase here as all the coordinate differences are less than or equal to π, the maximum difference allowed in a phase system. First vertices of the K-cube, the largest cube, are chosen and then successively of each smaller cube until no vertex of the next smaller cube other than the vertex common to all the cubes is at a distance D from all the previously chosen points. Note that vertices of a given cube that have the desired Euclidean distance D from the common vertex and from each other have the same or a greater distance from the vertices of any of the other cubes. The rest of the allowed code points which are contained inside the K-cube may be obtained by adding the coordinates of the allowed vertices of the 4K-cube onto the coordinates of the allowed vertices of the 2K-cube and then adding the coordinates of the allowed vertices of the 8K-cube onto all the previously obtained code points inside the K-cube and continuing this process until the allowed vertices of the smallest cube have been used. As the final step all the code points can be obtained by adding the coordinates of each of the allowed points inside the K-cube onto the coordinates of each of the allowed points on the K-lattice described earlier. The nonbinary coding problem when formulated in this way reduces to a set of binary coding problems. Also a geometric interpretation which is useful in visualizing the effects of noise and lack of synchronization is available. This method of coding will prove to be very efficient for certain values of D.

The code points generated so far meet a certain distance criterion which can be directly related to an error criterion but they have several failings. One of these is that the K-lattice is not symmetrically located about the (0, 0, 0, . . .) point in phase space. This is undesirable because digit-by-digit detection of the received words to determine the closest K-lattice point will be carried out by the receiver. Another failing which is more fundamental is that all 2^N possible combinations of certain pairs of phase levels are allowed code words. These code words hinder word synchronization. These two failings can be corrected by adding certain sequences onto all the previously chosen code words.

To locate the K-lattice and its code points symmetrically inside the phase space it is only necessary to add a sequence of N digits having the value $\frac{\pi}{K}$ onto the previously determined code points. Note that the resulting digit values, α_i, must be in the range $-\pi \leq \alpha_i < \pi$. Further, note that adding the same vector onto all the code points does not change their relative separations. The proper digit values can be obtained by adding the corresponding sequence onto the code points in cyclic space and then picking the points in phase space which correspond to the resulting points in cyclic space. K-lattice symmetry will be obtained before the addition of the word synchronization sequence even though this latter sequence will spoil the symmetry of the K-lattice points because the word synchronization sequence will be subtracted from the received words before they are decoded.

The adaptation of the above codes to codes with word synchronization properties will now be discussed.

<u>Selection of words for word synchronization property</u>

For the receiver to correctly decode a word it must know the time at which reception of the word begins. In the section on detection theory it was assumed that this word synchronization was available.

If it is not available the receiver would have to assume that the message could begin at any time over a T second interval where T is the time length of the code word and would have to compare with the M allowed code words all the various received messages possible assuming various starting points. This increases the number of comparisons from M to at least WTM where W is the bandwidth of the channel. For the system under discussion the word synchronization problem has two facets. First the samplers must sample the received signal at the proper time and second the samples must be grouped into the proper blocks of N digits. This may be thought of as a problem requiring two adjustments. The received signal may be thought of as chopped up into lengths of $T = N\mu$ seconds in such a manner such that the largest error in the assumed starting time of a word is $\pm \frac{\mu}{2}$ seconds. The sampling time is then adjusted within the time range $\pm \frac{\mu}{2}$ to achieve exact synchronization. The fact that the allowed digit values are evenly spaced in the 2π continuum of allowed phase values will permit long time averages over many digits to be used in the adjustment of the sampling times. This will be discussed when the receiver is discussed. The information on which to base the choice of which N sampled values form the received words must be inherent in the coding if the need for ultra-stable time references is to be avoided. That is, the train of sampled phase values must be partitioned into blocks of length N. The codes should be such that the information on where the blocks begin is inherently contained in the received information and thus continuously available. This will remove the need for sending special timing signals and/or having ultra-stable frequency references at the receiver.

If the grouping or partitioning information is to be carried by the code words an erroneous partitioning of the received digit train should result in a word which resembles no allowed code word. Geometrically this means that the point representing the erroneously formed word must, at least on the average, be at a relatively large distance from all the allowed points. Erroneous grouping, meaning lack of word synchronization, should then have the same effect as a large noise sequence. Large noise sequences will occur only randomly but the lack of word synchronization should cause the effect to persist. Thus information on a lack of word synchronization can be obtained even though the signal is corrupted by occasional large noise bursts.

The words formed by erroneous grouping consist, in the noiseless case, of digits from two allowed code words. That is, either the first or last few digits of the word formed belong to the actual word which preceded or which will follow the word now being decoded. Thus, if the code is to supply word synchronization information none or very few code words can be allowed which can be formed by joining the last digits of one code word with the first digits of a second code word.

In the coding scheme discussed previously if $D \leq \pi$ there is no redundancy in the words formed from certain combinations of levels even though there may be a large redundancy in the words composed of other combinations of levels. If $D > \pi$ word synchronization is not such a problem because there are no combinations of levels without some redundancy. Level redundancy is defined as

$$U_L = 1 - \frac{\text{number of allowed code words formed by a combination of levels}}{\text{number of words possible with the same combination}} \tag{23}$$

Code redundancy is defined as

$$U_C = 1 - \frac{M}{m^N} \; . \tag{24}$$

Note that a value of 1 corresponds to being very redundant or using very few of the allowed combinations. In the coding scheme that has been discussed the code words formed by the points on the K-lattice have no redundancy if $D \leq \pi$. That is, the phase levels which make up the K-lattice have a level redundancy of zero. This means that a word formed from the last N-b digits of one code word and the first b digits of another is an allowed code word. Here b has the range, $1 \leq b \leq N-1$. This situation cannot be allowed because a misgrouping of a long chain of these code words would result in allowed code words and the receiver would have no word synchronization information.

For a given amount of code redundancy the optimum code from a word synchronization point of view would be one in which all the allowed digit values were equally probable and in which there were no probability constraints among the digits. Thus the code redundancy is distributed uniformly over the allowed levels. Ulstad's (2) and Stiffler's (15) binary codes have these properties.

The problem of supplying synchronization information to the nonbinary codes obtained previously will be attacked from a slightly different point of view than that of making all the levels equally probable. The solution will, however, tend to distribute the probability more evenly over the levels. The real problem is with the K-lattice points. If a method can be found to shift the K-lattice points such that allowed code words cannot be formed from the first N-b coordinates of one K-lattice point and the last b coordinates of another and if the method does not affect the distance between the code points then the word synchronization problem is effectively solved. For code points other than those on the K-lattice sufficient level redundancy is normally available to make the probability very small that one of these code words can be formed by erroneous grouping of the received digits. The effort will then be concentrated on the K-lattice points.

To provide the desired word synchronization ability a vector $\underline{S}$ will be added to all the previously obtained code points. We may think of doing the addition in cyclic space where the vector components will be added modulo m'. Because it may be necessary to introduce some code redundancy, the value m' may be larger than the value m. That is, more digit values may be allowed even though they will not increase the number of code points which meet the distance criterion. Note that the addition of the same vector to all the code points cannot change the distance between them and thus their noise immunity will not be affected.

The vector $\underline{S}$ will be chosen such that the points S^b represented by all the vectors $\underline{S}^b$ which can be formed from $\underline{S}$ by cyclic shifts of the coordinates of $\underline{S}$ by b positions where $b = 1, 2, \ldots, N-1$ have a distance greater than zero and less than $\frac{D}{2}$ from the point S which $\underline{S}$ represents. Call the original code points β^i. Now the new code points are $\theta^i = \beta^i + S = \beta_1^i + S_1, \beta_2^i + S_2, \ldots, \beta_N^i + S_N$. A received word Φ formed by erroneous grouping may be thought of as $\Phi = \gamma + S^b$ where γ is a word formed by joining together the last N-b digits of one β^i and the first b digits of another. The decoding integrators form the distance

$$D(\Phi, \theta^i) = \left[\sum_{k=1}^{N} \rho^2(\beta_k^i + s_k, \gamma_k + s_{k+b})\right]^{\frac{1}{2}} \frac{2\pi}{m'}$$

where the sum $(k + b)$ is formed modulo $N + 1$ with $N + 1 = 1$, the γ_k's are the coordinates of γ, and the s_k's are the coordinates of S. If there exists a $\beta^i = \gamma$, then for this particular θ^i, say θ^1, the distance becomes

$$\left[\sum_{k=1}^{N} \rho^2(s_k, s_{k+b})\right]^{\frac{1}{2}} \frac{2\pi}{m'} \quad .$$

The vector $\underline{S}$ was, however, chosen such that this distance is less than or equal to $\frac{D}{2}$. As all the other allowed code points have a distance of at least D from θ^1, the received word Φ fromed by erroneous grouping is closer to θ^1 than any other allowed code point. If now $\underline{S}$ is chosen so that the distance is close or equal to $\frac{D}{2}$ for all $\underline{S}^b$ faulty grouping will result in words which are on the average close to no allowed word even if only the words formed from the original K-lattice are sent.

The distance $D(S, S^b)$ should be a nondecreasing function for

$1 \leq b < \frac{N}{2}$ and a nonincreasing function for $\frac{N}{2} < b \leq N-1$. Preferably $D(S, S^b)$ would be equal to $\frac{D}{2}$ for all allowed values of b. This will not always be possible. The components of $\underline{S}$ are chosen from the phase values zero and $\frac{2\pi}{2^h K}$ where $h = 1, 2, \ldots, h_{max}$. The components of $\underline{S}$ may also be thought of as the integers in cyclic space corresponding to the phase values $-\pi$ and $-\pi + \frac{2\pi}{2^h K}$. The value $K2^{h\ max}$ is the m' discussed above. If $D' = D$ for $N = 4$, two of the permissible forms for S are $\frac{\pi}{K} \underline{i}_1 + \frac{\pi}{K} \underline{i}_2 + \frac{\pi}{2K} \underline{i}_3 + \frac{\pi}{2K} \underline{i}_4$ and $\frac{\pi}{2K} \underline{i}_1 + \frac{\pi}{2K} \underline{i}_2 + 0\ \underline{i}_3 + 0\ \underline{i}_4$. A graph of $D(S, S^b)$ for either of these synchronization vectors is shown in Figure 10. A vector of the form $\frac{\pi}{2K} \underline{i}_1 + \frac{\pi}{2K} \underline{i}_2 + 0\ \underline{i}_3 + \ldots + 0\ \underline{i}_N$ will always work if $D' = D$ but for larger values of N it is beneficial to use as many phase values as possible in forming $\underline{S}$. This will tend to make the various phase values have a more nearly equal probability and thus to distribute the code redundancy more evenly over all the levels. Thus for $N = 8$ a vector like $\underline{S} = \frac{\pi}{2K} \underline{i}_1 + \frac{\pi}{4K} \underline{i}_2 + \frac{\pi}{4K} \underline{i}_3 + 0\ \underline{i}_4 + 0\ \underline{i}_5 + \frac{\pi}{4K} \underline{i}_7 + \frac{\pi}{2K} \underline{i}_i$ might be used. Note that for this vector $D(S, S^1) = \frac{D}{4}$ while for the previous $\underline{S}$ vector $D(S, S^1) = \frac{\sqrt{2}D}{4}$. This is a disadvantage of the latter $\underline{S}$.

Receiver Operation

It is the function of the decoder to decide on the basis of the received signal y(t) which of the M possible messages was sent. The accuracy of the decision will depend to a great extent on the relative amount of noise introduced by the channel. The receiver knows the constraint or block length N of the code words and the bandwidth W of the channel. The decoder makes a decision every $\frac{N}{W} = N\mu$ seconds of which message was most probably received on the basis of the N sample points of the received signal y(t) that are available in that time. The receiver must have certain synchronization properties. It must be able to provide a signal of proper frequency and phase as a reference for the phase detector and it must be able to determine the starting time of the waveform associated with each code word. The later problem involves both the synchronization of the sampling times and the proper grouping of the sample values to form words. The grouping problem is called the word synchronization problem and was discussed under the coding problem.

The receiver is shown in Figure 11. Its form will be slightly different from that indicated in the overall system discussion. The changes are made to save on the number of decoding integrators required.

The signal y(t) goes to two linear phase detectors. The upper phase detector has its reference signal provided by a local oscillator of appropriate phase and frequency. The phase of the local oscillator is controlled partially by the portion of the receiver enclosed by dashed lines and partially by a feedback signal $s_{11}(t)$. Phase synchronization will be discussed later. The linear phase detector has the transfer characteristic shown in Figure 12. The input phase angles are transformed to directly proportional voltages or currents.

The output of the upper phase detector, the signal $s_4(t)$, is thus directly proportional to the phase function of the received signal. This signal is sampled every μ seconds to form the signal $s_5(t)$. As stated above the time between sample points is known information but the exact time of the start of each word in the received information must be provided to the

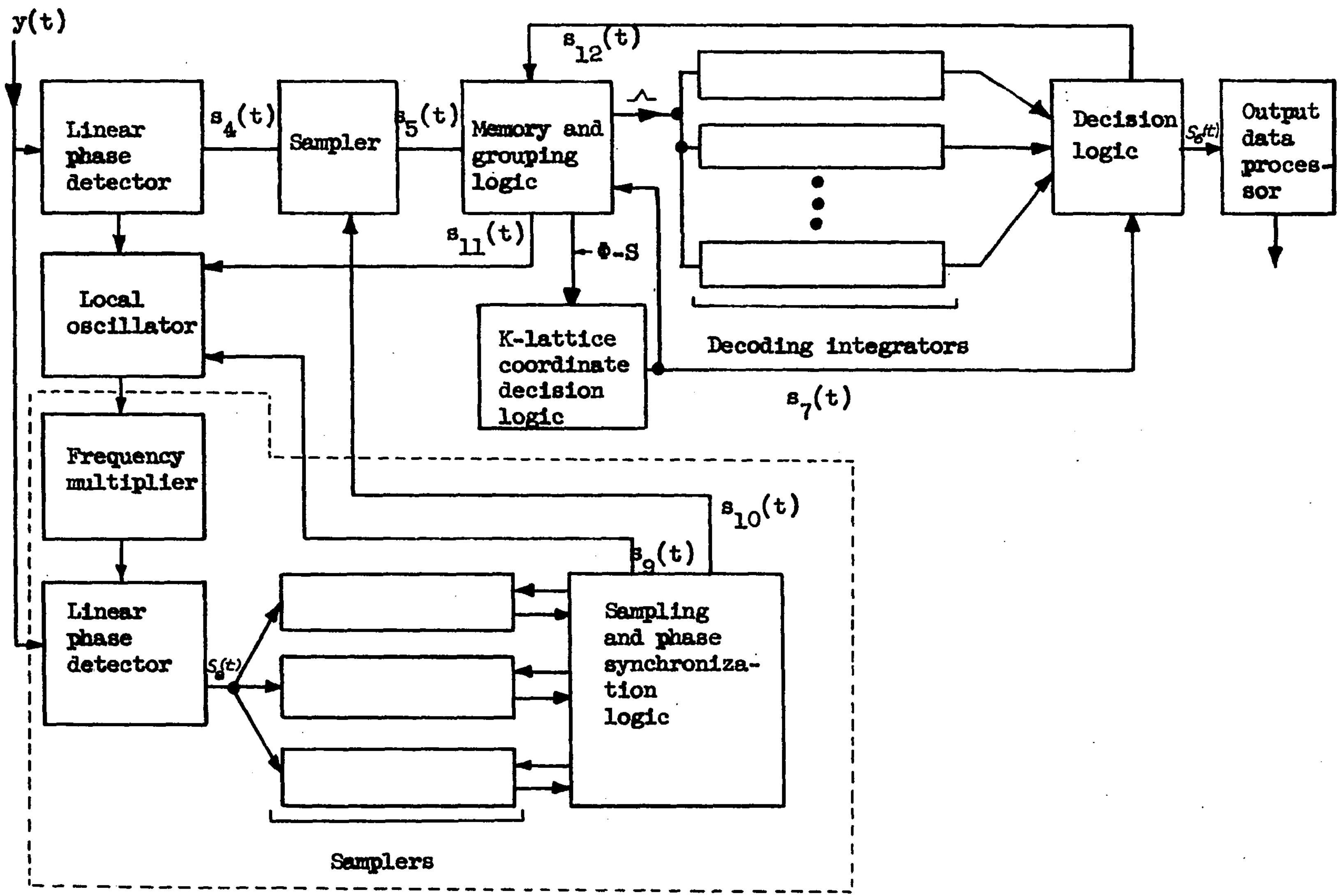


Figure 11. Receiver for proposed communication system

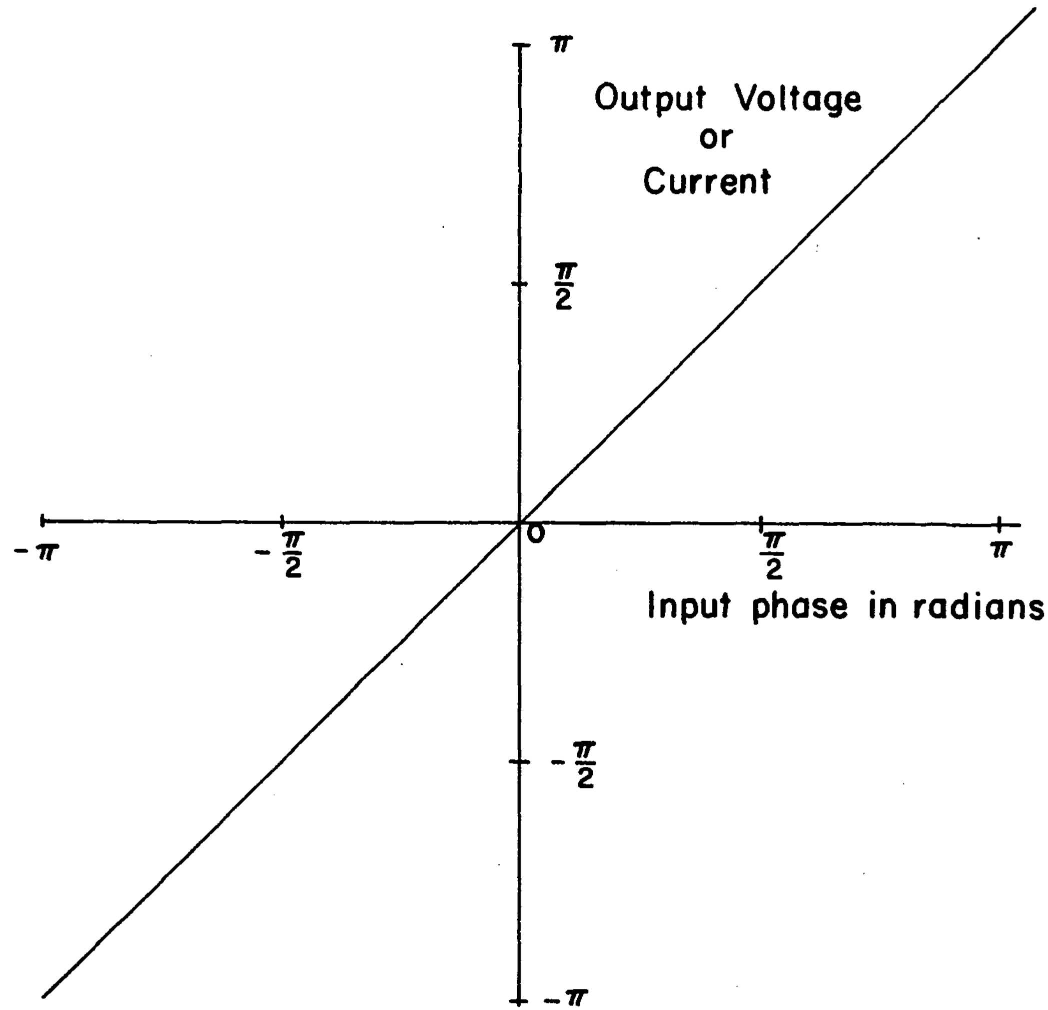


Figure 12. Phase detector transfer function

decoder. This word synchronization information must either be supplied by synchronizing waveforms or by properties of the codes used. The later choice will be used here. Also, sampling synchronization is necessary. That is, the samples must be taken from the received signal at times which correspond to the original sample points of the transmitted signal. Sampling synchronization information is supplied to the sampler from the portion of the receiver enclosed by the dashed lines.

The signal $s_5(t)$ consisting of a train of digits which are sampled values of $s_4(t)$ goes to the block labelled memory and grouping logic. Here the train of digits is partitioned into groups of N in three different ways. One partition is placed at what the receiver feels is the correct position, another is placed one digit ahead of this partition and a third is placed one digit behind it. Thus, three words are formed for each received word. These words will be called Φ, Φ', and Φ'', respectively and will be used to obtain and hold word synchronization. Only the word Φ will be used in determining the word sent.

In the memory the sequence $\underline{S}$ is subtracted from each of the three words Φ, Φ', and Φ'', and the resulting words Φ - S, Φ' - S, and Φ'' - S, are supplied one at a time to the decoding portion of the receiver. Note that the decoder must act on three words during the time that one is received, however, the decision as to which word was sent is based only on Φ - S. The decoder consists of essentially three parts: K-lattice coordinate decision logic, decoding integrators, and code word decision logic. This assumes that all the points of the K-lattice are code words. If this is not true, only the decoding integrators and code word decision logic are necessary. If only K-lattice points are allowed code points then digit-by-digit decoding is appropriate and only the K-lattice coordinate decision logic is necessary for decoding. In this case decoding integrators would, however, still be needed for word synchronization.

If all the K-lattice points are code words a noise coordinate of more than $\frac{\pi}{K}$ in any dimension will cause a decoding error regardless of the noise coordinates in the other dimensions. This being true it is possible to determine with which of the K-lattice points the received signal should be associated and to achieve a savings in the number of decoding integrators required. That is, it is possible to determine the closest K-lattice point called the K-point by digit-by-digit decoding of the received word and to use this information to decrease the number of decoding integrators required without increasing the probability of a decoding error. The subtraction of the sequence S restores symmetry so that the K-lattice levels are located symmetrically in the phase space. Thus a type of threshold detection is adequate for the block called K-lattice coordinate decision logic. The situation in any two dimensions is shown in Figure 13. The received signal must correspond to a code word inside the dashed square. Note that one and only one K-lattice point lies inside this square unless one or more of the signal coordinates are exactly between two K-lattice coordinates. If this is true, the smaller K-lattice coordinate is chosen.

As stated above, the synchronizing sequence is subtracted from each of the three words formed. The subtraction is done such that the resulting coordinates are in the range $-\pi$ to π. The K-point is then determined. To

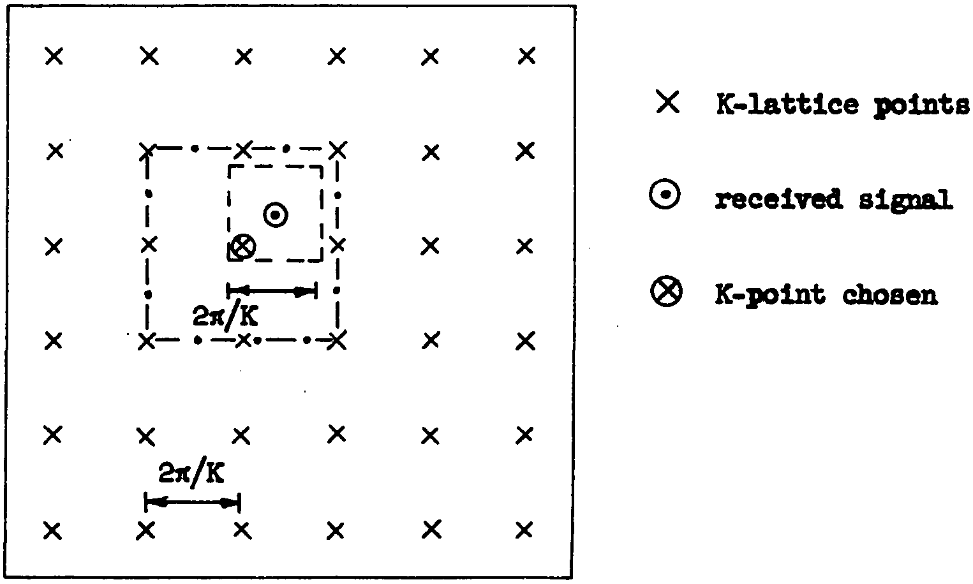


Figure 13. Two-dimensional illustration of K-point choice

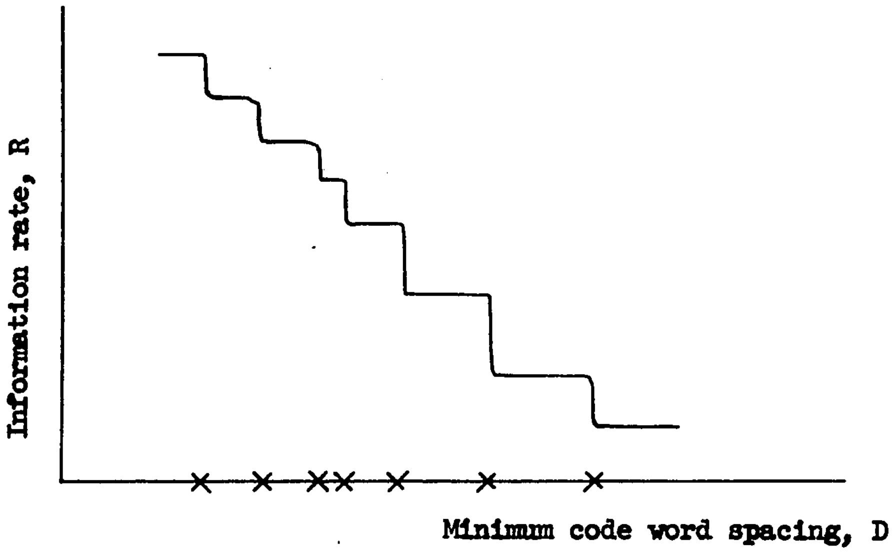


Figure 14. Qualitative picture of R versus D

determine which of the possible code points associated with this K-point corresponds most closely to the received word the distance of the received word from each of the code points contained in an N-dimensional cube centered at the K-point with sides of length $\frac{4\pi}{K}$ must be determined. The received word must be decoded as one of the code words contained inside this cube. There are only $1 + 2^N Z$ allowed code points inside this cube. Thus, only $1 + 2^N Z$ decoding integrators are required if the K-point information is properly used. For the situation of Figure 13 the cube enclosing the possible received words is formed by dash-dot lines. The variable Z is the total number of points associated with each K-lattice point in the original coding scheme minus one, that is, the number of code points <u>inside</u> what was called a K-cube. The number of decoding integrators is thus less than the $M = K^N(Z + 1)$ required if one is used for each allowed code word.

The things that have been referred to as decoding integrators really form

$$D^2(\Phi - S, \beta^i) = \sum_{k=1}^{N} r^2(\varphi_k - s_k, \beta_k^i) \tag{25}$$

where

$$r(\alpha, \beta) = |\alpha - \beta| \text{ or } \big||\alpha - \beta| - 2\pi\big|$$

whichever is smaller. The form of r is dictated by the fact that the difference between any two phase coordinates may never exceed π in magnitude. The β^i are the code points before the addition of S. Thus, the decoding integrator corresponding to the code word β^i forms the difference between each received coordinate and the corresponding coordinate of the i^{th} code word, squares this difference, and forms the sum over all N coordinates.

If the K-point coordinates are used to bias a set of decoding integrators such that they represent the decoding integrators corresponding to the code words possible with that K-point then only $1 + 2^N Z$ decoding integrators are needed.

To avoid modularity problems and simplify the decoding integrators the following scheme may be used. The decoding integrators correspond to the code words that would be possible if the point (0, 0, 0, . . .) in the center of phase space was the K point. The word, $\hat{\ }$, sent to these decoding integrators is formed by adding to the sequence Φ - S a sequence corresponding to the vector from its K-point to the point (0, 0, 0, . . .). Thus the point corresponding to Φ-S in phase space is moved so that it has the same relative position with respect to the point (0, 0, 0, . . .) as it had with respect to its K-point. The decoding integrators may now form the squares of the distances from the received word to the possible code words as simply the sum of the squares of the coordinate differences because all coordinate differences have a magnitude less than π.

Each decoding integrator sends its sum to the code word decision logic after every N digits. Here a decision is made as to the most probable transmitted word by choosing the word corresponding to the decoding integrator with the smallest sum. The actual word will, of course, depend on the K-point and this information is supplied by the signal $s_7(t)$ to the decision logic as shown in Figure 11. The decision as to the most probable word sent depends only on the word formed by the first partition. The other two words, however, go through the same process and the minimum decoding integrator sums that they yield are obtained and compared with the minimum sum yielded by the word corresponding to the assumed correct partitioning. If on the average over several received words one of these two words yields a smaller sum than the word with the assumed correct partitioning it is then an indication that the word synchronization is out of phase by at least one digit and the starting point for the N samples should be advanced or retarded by a digit. The exact use to be made of this word synchronization information will not be discussed. The point of interest is that the information is available. The signal $s_{12}(t)$ carries the word synchronization information back to the partitioning logic.

The signal $s_6(t)$ assumes the one of its M values that corresponds to the code word most probably transmitted and is the input to the output data processor. The output data processor transforms the data into the desired form and finally yields the output data.

As stated earlier the portion of the receiver enclosed in dashed lines provides synchronization for the sampler. The output of the same local oscillator used as a reference for the linear phase detector used to generate $s_4(t)$ is also fed to a frequency multiplier. Here the frequency of the signal is multiplied m times where m is the number of allowed sample point phase values in the transmitted signal. This signal is used as the reference for a second linear phase detector. This linear phase detector is also supplied with the signal $y(t)$. The signal $s_8(t)$ corresponds to the phase of $y(t)$ with respect to the multiple frequency signal. The signal $s_8(t)$ is sampled every μ seconds by three samplers which sample at times differing by $p\mu$ seconds where $p \leq 1/2$. If there was no noise and the frequency of the local oscillator agreed with the carrier frequency of the received signal and the sampling was done at the proper time all the sampled values of $s_8(t)$ would be identical. This is true because in the coding only symmetrically spaced values of phase were used and thus all the allowed phases differed by $\lambda(\frac{2\pi}{m})$ from any one arbitrarily chosen as a reference where $\lambda = 1, 2, \ldots, m$. With respect to a reference which has a frequency m times as large as the original reference the sampled phase values should be identical because multiplication of the reference frequency by m causes a multiplication of the phase values by m and thus the resulting phases differ by $\lambda(2\pi)$ where λ is an integer. If the sampling is not done at the proper times the phase values obtained will not all agree because these phase values will not in general be evenly spaced with respect to the original reference frequency. Thus by comparing the output of the three samplers it should be possible to adjust the sampling time so that on the average all the sampled values agree. Noise will, of course, cause fluctuations from the single value even when the sampling times are correct. Again, however, the effect of noise should average out in the sense that large deviations from the expected value should occur only randomly and most of the values should be near the average value. If the sampling times are not correct large deviations should be common and indeed the values should be more or less uniformly distributed over the 2π range. After original synchronization the value of p would be chosen so that $p\mu$ is the largest allowed variation in sampling time synchronization. Note that the above method of obtaining sampling time synchronization is independent of the phase of the local oscillator. In Figure 11 the signal $s_{10}(t)$ carries the sampling time synchronization information to the sampler used in forming Φ.

The decision making logic will also contain long term information on the sampling synchronization as a lack of sampling synchronization will tend to make the minimum distance between the received words and all of the allowed code words large. That is, the received words will on the average not agree well with any of the allowed code words for any partitioning.

Information to provide phase synchronization between the transmitter reference and the local oscillator is also available. If the sampler is properly synchronized the sampled values of $s_8(t)$ should be spread about the average value zero. If they do not have an average value of zero the phase of the local oscillator can be adjusted to make this true. Note that this method only allows adjustment over a phase range of $\pm \frac{\pi}{m}$. Additional phase synchronization information is available from the received words. After subtraction of the sequence S the phase values $+\alpha$ and $-\alpha$ where $0 \leq \alpha \leq \pi$ should be equally probable as coordinate values of the resulting words. This is inherent from the coding scheme used and the symmetry of the noise. Thus, the long term average of the coordinates of Φ - S should be zero and this fact can be used to adjust the local oscillator phase. The range of adjustment is not limited here. Again, the exact use made of the above criteria to maintain phase synchronization will not be discussed. The important point is that a means of providing phase synchronization is available. The signals $s_9(t)$ and $s_{11}(t)$ in Figure 11 carry the phase synchronization information to the local oscillator.

PERFORMANCE OF PROPOSED SYSTEM

Using the proposed codes a certain information rate, R, will correspond to assumed values of D and N. It is assumed that m is chosen to maximize R. The error probability of the code will depend on D, N, and s^2.

Information Rates for the Proposed Codes

While it is possible to specify the minimum distance, D, between any two code points as any desired value, only certain distances take full advantage of the information handling capacity of the proposed coding scheme. If $D \leq \pi$ these distances are those than can be obtained by dividing 2π by an integer and multiplying by 1, $\sqrt{2}$, or $\sqrt{3}$. This is reasonable in view of Equation 20. For the K-lattice the distance between points is $d\frac{2\pi}{K}$ where K is an integer and d has the form $\sqrt{n}$ where n is an integer. Tables 7 and 8 in Appendix D show the number of binary sequences of length N that differ in $n = d_H$ places. They show the futility of choosing a K value larger than the value of the first integer less than or equal to $\frac{2\pi}{D}$. That is, they show the futility, if $D \leq \pi$, of choosing the K-lattice so that not all of the K-lattice points are code points. Thus, the allowed code points of the K-lattice may be chosen with distances of $\frac{2\pi}{K}$, $\sqrt{2}\,\frac{2\pi}{K}$, and $\sqrt{3}\,\frac{2\pi}{K}$. Furthermore, for reasonable values of N the total number of allowed code points with a minimum spacing between two of these desired values will be almost the same as the number allowed with the larger spacing. A graph of information rate versus distance tends to look like Figure 14. It is of course desirable to operate at the largest distance permitted for a given rate as this will mean a lower error rate. Figure 14 shows that the distance may be increased to one of the values specified above with little or no sacrifice in rate. The desired distances correspond to the points marked with x's in Figure 14. Tables 7 and 8 also show that for block lengths of the form $N = 2^v$ where $v = 1, 2, 3, \ldots,$ the distance $\sqrt{3}\,\frac{2\pi}{K}$ is not a desirable code point separation distance because there are as many sequences that differ in 4 places as there are that differ in 3. Unless N is large the same number of code points will result for distances of $\sqrt{3}\,\frac{2\pi}{K}$ and $\frac{2\pi}{K/2}$. If $D > \pi$ and thus $K = 2$, all distances of the form $\sqrt{n}\,\pi$ are good distances in the sense described above.

Due to the above considerations the effort will be concentrated on determining the rates possible if the code points have a minimum separation equal to one of the so-called desirable distances $\frac{2\pi}{K}$ or $\sqrt{2}\,\frac{2\pi}{K}$ for various values of $N = 2^v$. The N values were chosen as 2^v to facilitate the error rate calculations. Some rates for D values of the form $\sqrt{n}\,\pi$ will also be determined. Table 1 shows the rates possible for various values of D and N. The rates indicated were computed from Equation 16 and thus contain no equivocation term. If the error rate for a given D and N is not low the actual rate will be less than that indicated. The values of M for the various values of D and N were determined by use of the theorems and tables of Plotkin (16) given in Appendix D. Appendix D also gives an example of a rate calculation.

Table 1 shows that it is possible to obtain the same rate with several different distances and block lengths. Which of these is best in the sense of least error probability per symbol will depend on the signal-to-noise

Table 1. Information rates for proposed codes

Minimum code point separation D	Information rate in bits per digit						
	Number of digits per word						
	N = 1	N = 2	N = 4	N = 8	N = 16	N = 32	N = 64
$2\sqrt{2}\,\pi$	-	-	-	0.125	0.313	0.531	0.765
2π	-	-	0.25	0.5	0.75	1.0	1.25
$\sqrt{2}\,\pi$	-	0.50	0.75	1	1.25	1.5	1.75
π	1	1	1.25	1.5	1.75	2.0	2.25
$\sqrt{2}\,\frac{\pi}{2}$	-	1.5	1.75	2.0	2.25	2.5	2.75
$2\,\frac{\pi}{3}$	1.58	1.58	1.84	2.09	2.34	2.58	2.84
$\frac{\pi}{2}$	2.0	2.0	2.25	2.5	2.75	3.0	3.25
$\sqrt{2}\,\frac{\pi}{3}$	-	2.09	2.34	2.58	2.84	3.09	3.34
$2\,\frac{\pi}{5}$	2.32	2.32	2.57	2.82	3.07	3.32	3.57
$\sqrt{2}\,\frac{\pi}{4}$	-	2.5	2.75	3.0	3.25	3.5	3.75
$\frac{\pi}{3}$	2.58	2.58	2.84	3.09	3.34	3.58	3.84
$\sqrt{2}\,\frac{\pi}{5}$	-	2.82	3.07	3.32	3.57	3.82	4.07
$\frac{\pi}{4}$	3.0	3.0	3.25	3.50	3.75	4.0	4.25
$\sqrt{2}\,\frac{\pi}{6}$	-	3.09	3.34	3.58	3.84	4.09	4.34
$\frac{\pi}{5}$	3.32	3.32	3.57	3.82	4.07	4.32	4.57

ratio. The next part of the paper will be concerned with determining the error probability versus distance for various values of N and S^2.

Error Rates for Proposed Codes

For an error to result in the decoding process it is necessary for noise, or lack of synchronization which has the same effect as noise, to cause the received message to lie in the decoding volume of a code word other than the one sent. This means that the received word is closer, using the metric of Equation 20, to a code word which was not sent than it is to the one sent.

If we think of the code points as vectors in the phase space with components in each of the directions equal to the coordinate of the point in that direction we may think of the effect of noise as the addition of a noise vector onto the code point vector. The resulting vector is the received word. If the resulting vector lies in the N-dimensional polyhedron formed by planes which bisect the lines joining all adjacent code points to the code point which was sent the decoder will choose the correct message and no error will occur. If the noise vector is too big, that is, if the square root of the sum of the squares of its coordinates is too large, it will cause the resultant vector to lie outside this decoding volume and an error will occur. Note that the noise coordinates must have values between plus and minus π. Thus a determination of the error probability for given values of D, N, and S^2 becomes a study of the probability that the noise vector will exceed certain magnitudes in certain directions.

To simplify the problem the decoding volume will be assumed to be an

N-dimensional sphere of radius $\frac{D}{2}$ centered at the code point. This sphere certainly lies inside the actual decoding volume as the code point is at a distance of at least $\frac{D}{2}$ from any of the planes bounding the actual decoding volume. Thus, the answers obtained will be pessimistic. Note that as the number of code points increases for a given N the sphere will become a closer approximation to the actual decoding volume.

With the above simplification we are interested in the probability that

$$\sum_{k=1}^{N} (\gamma_k)^2 < \left(\frac{D}{2}\right)^2$$

where γ_k is the noise component in the k^{th} coordinate direction. The probability density of γ_k, the noise component at the k^{th} sampling point, is derived in Appendix E. To determine the probability that

$$\sum_{k=1}^{N} (\gamma_k)^2 < \left(\frac{D}{2}\right)^2$$

it is necessary to perform an N-fold convolution of the probability density of $(\gamma_k)^2$ to obtain the probability density of the sum. This density is then integrated over the range 0 to $(\frac{D}{2})^2$ to determine the probability that the sum is less than $(\frac{D}{2})^2$. The probability of error is one minus the value of the above integral or the integral of the sum density from $(\frac{D}{2})^2$ to $N\pi^2$. The probability density of $(\gamma_k)^2$ as determined in Appendix F is

$$p\left[(\gamma_k)^2 = \delta\right] = \frac{e^{-S^2}}{2\pi\sqrt{\delta}} + \frac{1}{2}\sqrt{\frac{S^2}{\pi}}\,\frac{\cos\sqrt{\delta}}{\sqrt{\delta}}\left[1 + \operatorname{erf} S\cos\sqrt{\delta}\right] e^{-S^2\sin^2\sqrt{\delta}} \tag{26}$$

As the convolutions did not lend themselves to an analytical solution they were performed numerically on a digital computer. Note that the probability density of $(\gamma_k)^2$ has a singularity at $(\gamma_k)^2$ equal to zero. This makes the function very unsuitable for numerical operations. To overcome this difficulty the first convolution to obtain the probability density of $(\gamma_1)^2 + (\gamma_2)^2$ was performed in the manner described in Appendix G. The resulting probability density was finite at the origin and was more amenable to numerical operations. This probability density was convolved with itself numerically to yield the probability density of $\sum_{k=1}^{4}(\gamma_k)^2$. The resultant probability density was convolved with itself to yield the probability density of $\sum_{k=1}^{8}(\gamma_k)^2$. This process was continued until a sufficiently large value of N was reached. Note that the possible values of N using this method are 2, 4, 8, 16, . . . While this is a restriction the results will demonstrate the error trends of the codes under consideration.

Figures 15, 16, 17 are the curves of error probability per digit versus square of the decoding sphere radius for various values of N and S^2. These curves were obtained by numerical integration of the probability densities determined above and division by N. The word probability of error can be obtained by adding the distance between the value one and the zero ordinate of the curve onto the curve. This corresponds to multiplying by N. Note that for a given signal-to-noise power ratio the curves exhibit a threshold effect. As N increases the error probability does not decrease and approach the $N = 1$ curve until a certain decoding sphere radius is reached. Thus, for a large N, $\frac{D}{2}$ must be greater than the threshold if

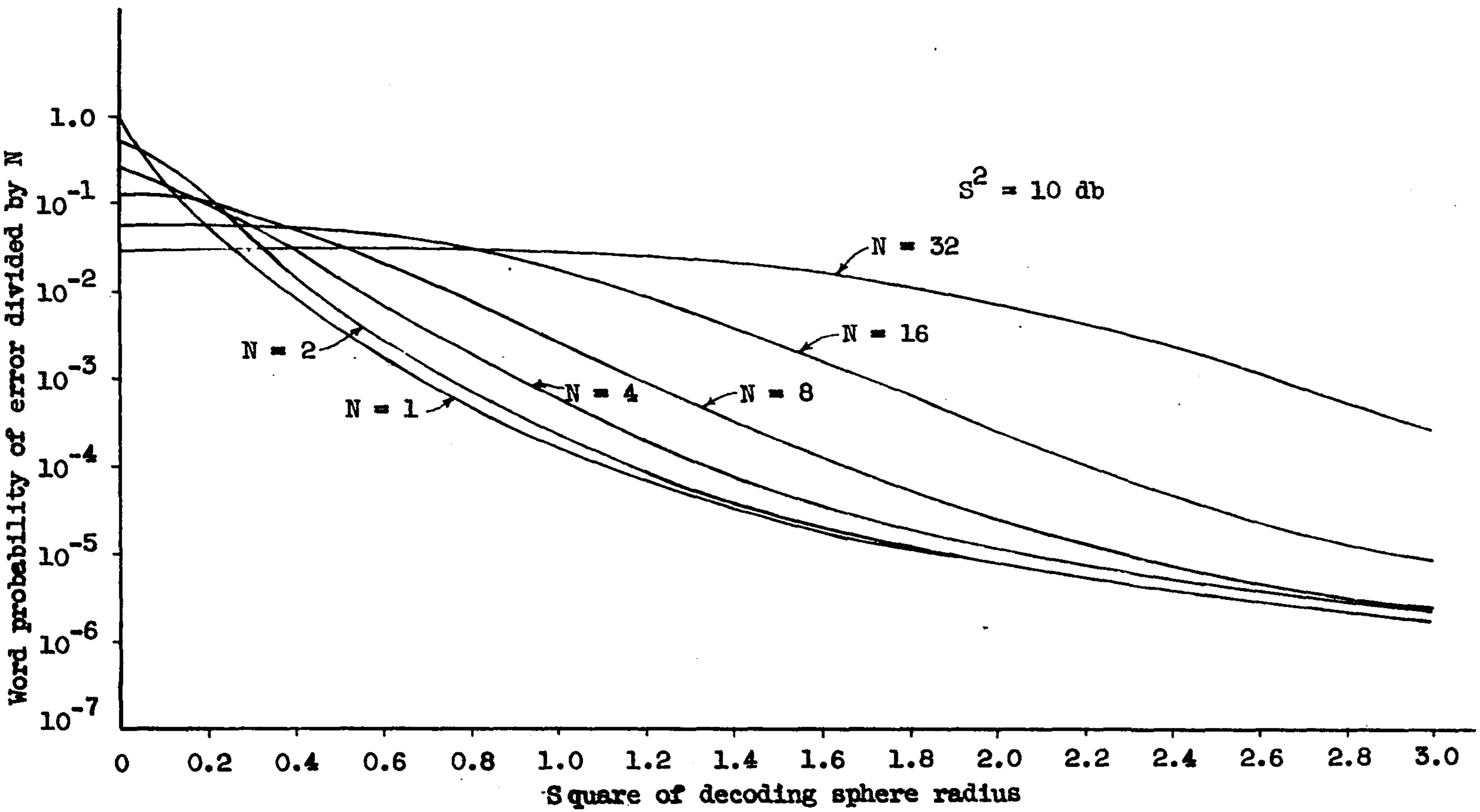


Figure 15. Error curves for $S^2 = 10$ db

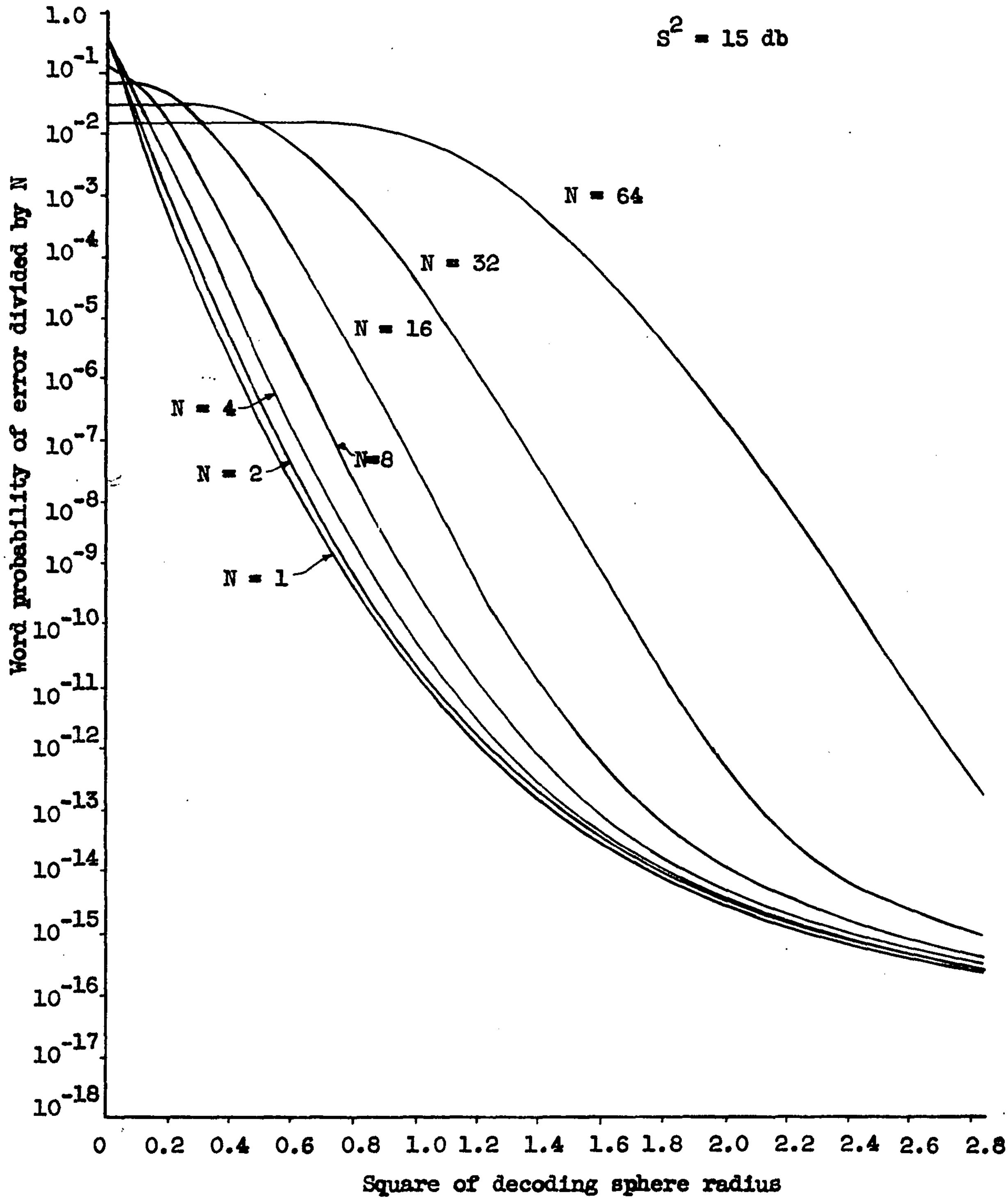
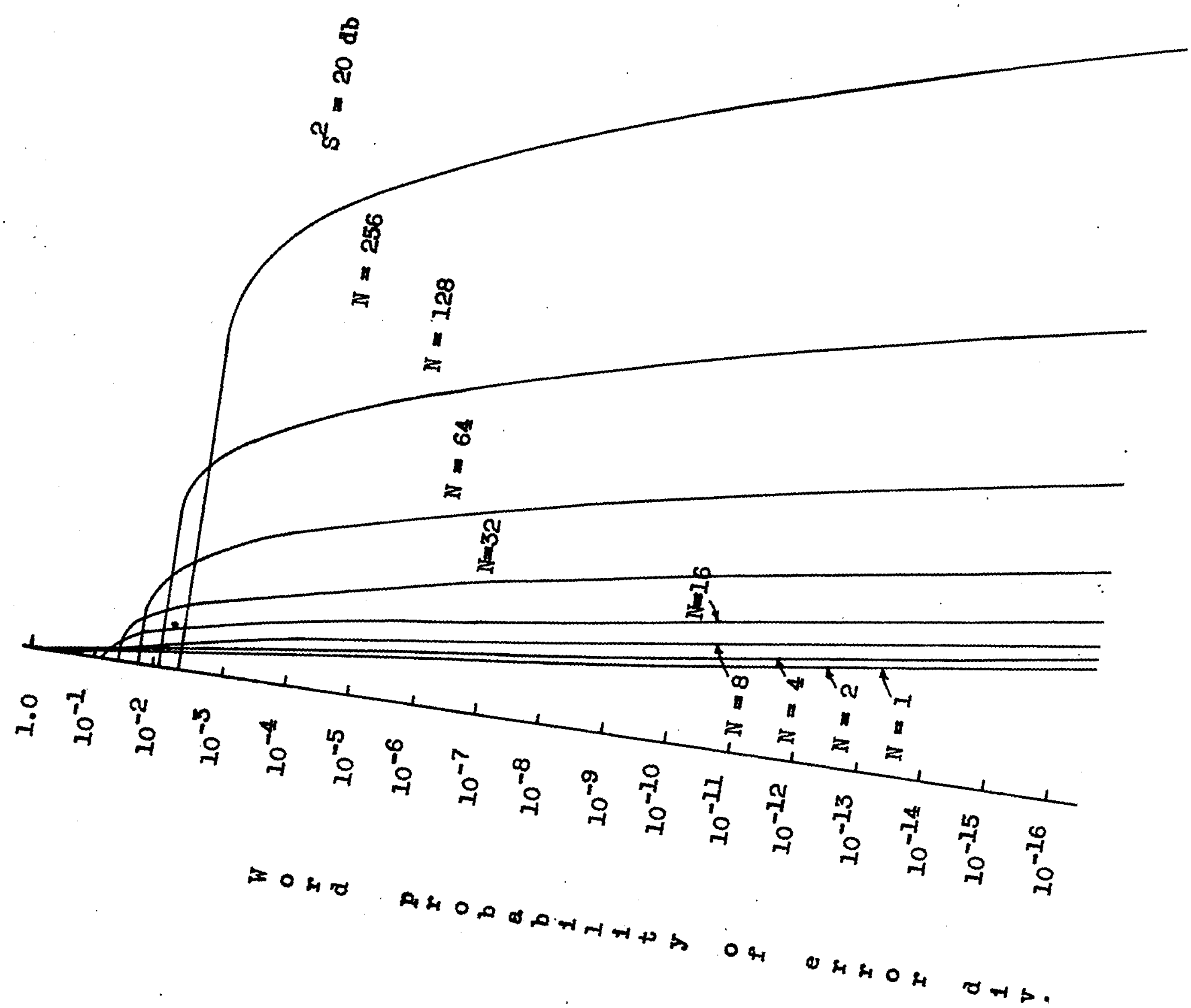
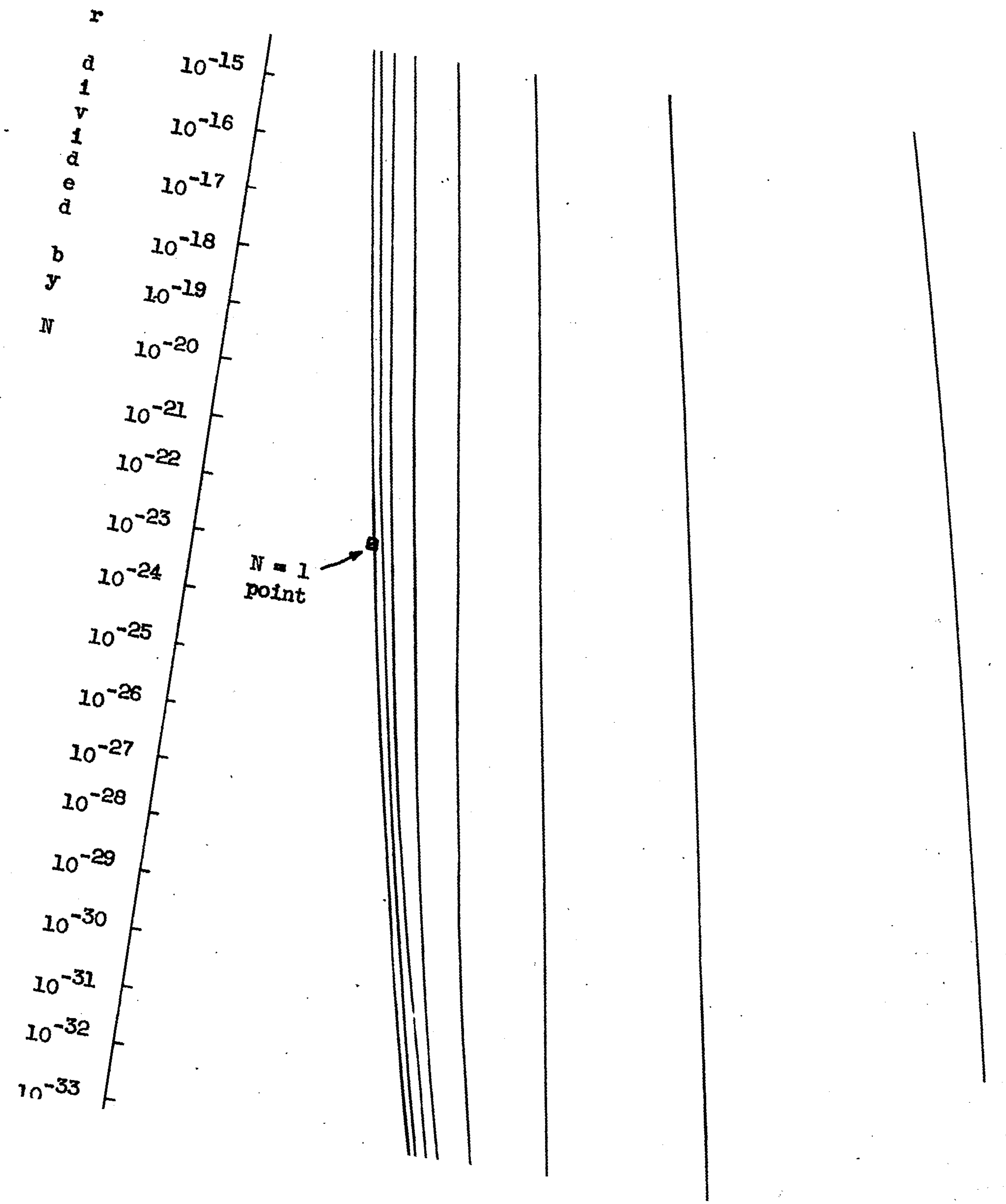


Figure 16. Error curves for $S^2 = 15$ db



S² = 20 db
N = 256
N = 128
N = 64
N=32
N=16
N = 8
N = 4
N = 2
N = 1
1.0
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
10⁻⁸
10⁻⁹
10⁻¹⁰
10⁻¹¹
10⁻¹²
10⁻¹³
10⁻¹⁴
10⁻¹⁵
10⁻¹⁶
Word probability of error div.



r divided by N
10-15
10-16
10-17
10-18
10-19
10-20
10-21
10-22
10-23
10-24
10-25
10-26
10-27
10-28
10-29
10-30
10-31
10-32
10-33
N = 1 point

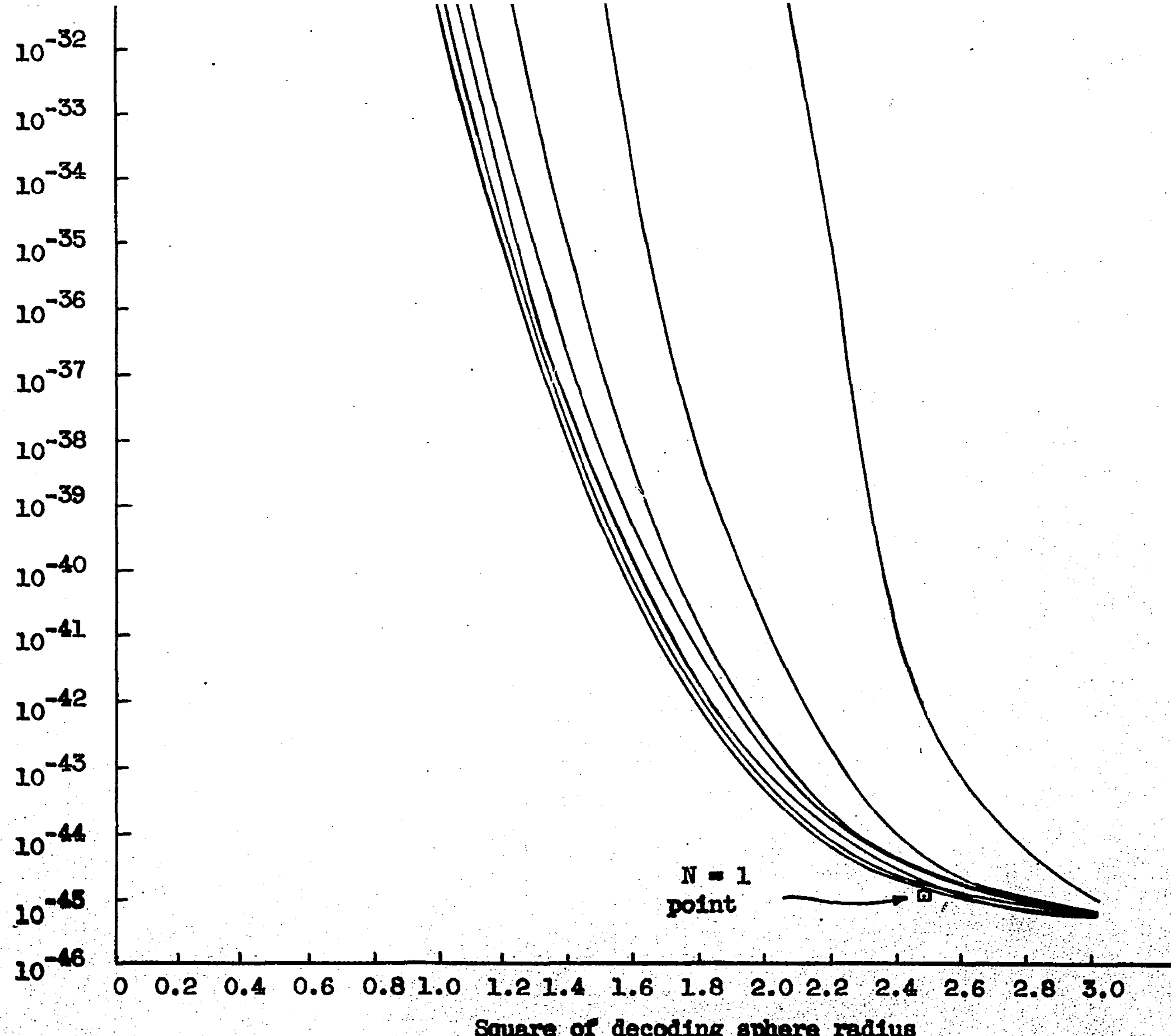


Figure 17. Error curves for S^2 = 20 db

small per digit error rates are to be obtained and the coding is to be effective. The information rate also increases with N as shown in Table 1. If the coding is to be effective for a given N and D the information rate must increase more than the error probability. This means that increasing the value of N will not decrease the error probability at high information rates. This is shown by the error tables which are obtained next.

Tables 2, 3, and 4 show the error rate per digit for various rates and block lengths for three values of signal-to-noise power ratio. These were obtained by using Table 1 and Figures 15, 16, and 17. The smallest error rate per symbol is underlined for each information rate. Note that the optimum block lengths for a given rate tend to be relatively small which is ideal from an equipment complexity viewpoint. The value of 1 for N means digit-by-digit decoding. Note that the proposed coding scheme allows certain rates which cannot be obtained by digit-by-digit decoding. These rates are marked with an asterisk. Their digit-by-digit decoding error rate is shown for comparison purposes even though it is not obtainable. Note that some times at high information rates it may be necessary to accept a higher error rate than that attainable digit-by-digit decoding to obtain the message synchronization properties of the proposed codes. Table 5 shows the improvement in error rate over digit-by-digit decoding for three values of S^2. Table 5 shows that the largest gains occur in middle range of rates for any given value of S^2. There are probably larger gains available in the lower range also if values of $\frac{D}{2}$ greater than $\frac{\pi}{2}$ are used. Unfortunately, the computer time required to obtain the error rates

Table 2. Error probability per digit versus information rate for S^2 = 10 db

Information Rate $R \frac{\text{bits}}{\text{digit}}$	Error Probability Per Digit					
	Number of Digits Per Word					
	N = 1	N = 2	N = 4	N = 8	N = 16	N = 32
1.0	3.9×10^{-6}	4.0×10^{-6}				
1.25*	1.8×10^{-5}		4.9×10^{-6}			
1.5 *	7.4×10^{-5}	8.8×10^{-5}		6.5×10^{-6}		
1.58	1.1×10^{-4}	1.8×10^{-4}				
1.75*	3.5×10^{-4}		1.8×10^{-4}		8.1×10^{-5}	
1.84*	6×10^{-4}		4.1×10^{-4}			
2.00	1.6×10^{-3}	2.7×10^{-3}		8.3×10^{-4}		1.9×10^{-3}
2.09*	2.6×10^{-3}	4.4×10^{-3}		1.9×10^{-3}		
2.25*	6.6×10^{-3}		6.5×10^{-3}			
2.32	8.6×10^{-3}	1.4×10^{-2}	1.0×10^{-2}			
2.5 *	1.8×10^{-2}	2.9×10^{-2}		2.2×10^{-2}		
2.58	2.6×10^{-2}	3.9×10^{-2}				

Table 3. Error probability per digit versus information rate for S^2 = 15 db

Information Rate $R \frac{\text{bits}}{\text{digit}}$	Error Probability Per Digit						
	Number of Digits Per Word						
	N = 1	N = 2	N = 4	N = 8	N = 16	N = 32	N = 64
1.0	7.5×10^{-16}	7.6×10^{-16}					
1.25 *	1.0×10^{-14}		7.7×10^{-16}				
1.5 *	8.0×10^{-13}	1.4×10^{-12}		8×10^{-16}			
1.58	4.0×10^{-12}	7.5×10^{-12}					
1.75 *	1.1×10^{-10}		2.4×10^{-12}		1.3×10^{-15}		
1.84 *	7.3×10^{-10}		1.5×10^{-11}				
2.0	1.7×10^{-8}	3.1×10^{-8}		9.1×10^{-12}		4.4×10^{-15}	
2.09 *	6.3×10^{-8}	1.5×10^{-7}		8×10^{-11}			
2.25 *	6.0×10^{-7}		1.2×10^{-7}		4.3×10^{-10}		1.3×10^{-10}
2.32	1.6×10^{-6}	5.9×10^{-6}	6.2×10^{-7}		6.1×10^{-9}		
2.5 *	2.0×10^{-5}	5.9×10^{-5}		1.6×10^{-6}		9.4×10^{-7}	
2.58	6.7×10^{-5}	1.5×10^{-4}	2.5×10^{-5}	8.2×10^{-6}		1.0×10^{-5}	
2.75 *	4.1×10^{-4}		2.4×10^{-4}		9.8×10^{-5}		
2.84	8.4×10^{-4}	1.3×10^{-3}	5.8×10^{-4}	2.7×10^{-4}	3.7×10^{-4}		
3.0	3.3×10^{-3}	4.5×10^{-3}		1.9×10^{-3}		6.1×10^{-3}	
3.08 *	5.4×10^{-3}	7.5×10^{-3}	4.2×10^{-3}	3.8×10^{-3}	7.7×10^{-3}	1.1×10^{-2}	
3.17	8.4×10^{-3}	1.2×10^{-2}					
3.25 *	1.3×10^{-2}		1.2×10^{-2}		1.6×10^{-2}		
3.34	1.4×10^{-2}	2.3×10^{-2}	1.9×10^{-2}	1.7×10^{-2}	2.3×10^{-2}		
3.46	2.7×10^{-2}	4×10^{-2}					
3.5 *	3.0×10^{-2}			3.6×10^{-2}		3.0×10^{-2}	
3.58	3.9×10^{-2}		4.1×10^{-2}	4.7×10^{-2}			

Table 4. Error probability versus information rate for S^2 = 20 db

Information Rate R $\frac{\text{bits}}{\text{digit}}$	Error Probability Per Digit							
	Number of Digits Per Word							
	N = 1	N = 2	N = 4	N = 8	N = 16	N = 32	N = 64	N = 128
1.0	10^{-45}	1.1×10^{-45}						
1.25*	3×10^{-41}		1.2×10^{-45}					
1.5*	2.8×10^{-36}	3.3×10^{-36}		1.4×10^{-45}				
1.58	1.2×10^{-33}	1.6×10^{-33}						
1.75*	6.7×10^{-30}		5.5×10^{-36}		1.6×10^{-45}			
1.84*	2.4×10^{-27}		2×10^{-33}					
2.0	1.5×10^{-23}	2.6×10^{-23}		10^{-35}		2×10^{-45}		
2.09*	1.5×10^{-21}	2.8×10^{-21}		6.2×10^{-33}				
2.25*	5.2×10^{-18}		7.5×10^{-23}		10^{-34}		3.2×10^{-45}	
2.32	1.2×10^{-16}	2.1×10^{-16}	1.1×10^{-20}		1.3×10^{-29}			
2.5*	9×10^{-14}	2.6×10^{-13}		1.1×10^{-21}		2.7×10^{-32}		5×10^{-43}
2.59	2×10^{-12}	4.7×10^{-12}	2.2×10^{-15}	2.1×10^{-19}				
2.75*	1.8×10^{-10}		1.8×10^{-12}		6.2×10^{-19}		2.8×10^{-23}	
2.84	1.6×10^{-9}	4.1×10^{-9}	3.5×10^{-11}	4.7×10^{-14}	1.8×10^{-16}		1.9×10^{-18}	
3.0	7×10^{-8}	2×10^{-7}		7.6×10^{-11}		8.1×10^{-14}		
3.09*	3.2×10^{-7}	10^{-6}	3×10^{-8}	1.2×10^{-9}	3.3×10^{-11}	1.6×10^{-11}		4.7×10^{-6}
3.17	9×10^{-7}	3.4×10^{-6}						
3.25*	4×10^{-6}		1.3×10^{-6}		3×10^{-8}		2.4×10^{-8}	
3.34	1.3×10^{-5}	3.3×10^{-5}	6×10^{-6}	7.1×10^{-7}	3.5×10^{-7}	3.1×10^{-7}	6.6×10^{-6}	
3.46	6×10^{-5}	1.7×10^{-4}	2.2×10^{-6}					
3.5*	10^{-4}	2.7×10^{-4}		2.6×10^{-5}		4.8×10^{-5}		
3.58	2.5×10^{-4}	6×10^{-4}	1.6×10^{-4}	9.4×10^{-5}	6.9×10^{-5}	2.6×10^{-4}	1.6×10^{-3}	
3.7	10^{-3}	1.6×10^{-3}	7.4×10^{-4}	2.9×10^{-4}				
3.75*	1.3×10^{-3}		1.1×10^{-3}		1×10^{-3}		9×10^{-2}	
3.82	2.2×10^{-3}	3.9×10^{-3}	2.2×10^{-3}	1.5×10^{-3}	2.6×10^{-3}			
3.91	4×10^{-3}	6.4×10^{-3}						
4.0	6×10^{-3}	1.1×10^{-2}		6.8×10^{-3}		1.1×10^{-2}		
4.09*	8×10^{-3}			1.1×10^{-2}		2.2×10^{-2}		

Table 5. Improvement of digit error probability over digit-by-digit decoding

Information Rate $\frac{\text{bits}}{\text{digit}}$	Improvement Ratio of Digit Error Probability Over Digit-by-Digit Decoding		
	S^2 = 10 db	S^2 = 15 db	S^2 = 20 db
1.0	1.0	1.0×10^0	1.0
1.25	2.8	1.4×10^1	2.5×10^4
1.5	11.4	1.0×10^3	2×10^9
1.58	1.0	1.0×10^0	1.0
1.75	4.3	8.8×10^4	4.2×10^{15}
1.84	1.5	4.9×10^1	1.2×10^6
2.0	1.9	3.9×10^6	7.5×10^{21}
2.09	1.4	7.9×10^2	2.4×10^{12}
2.25	1.0	4.7×10^3	1.6×10^{27}
2.32	1.0	2.6×10^2	9.2×10^{12}
2.5	0.84	2.1×10^1	1.8×10^{29}
2.59	1.0	8.2×10^0	9.5×10^6
2.75		4.2×10^0	6.4×10^{12}
2.84		3.1×10^0	8.4×10^8
3.0		1.7×10^0	8.6×10^5
3.09		1.4×10^0	2×10^4
3.17		1.0×10^0	1.0
3.25		1.0×10^0	1.7×10^2
3.34		1.0×10^0	4.2×10^1
3.46		1.0×10^0	2.3
3.5		1.0×10^0	4.0
3.58		1.0×10^0	3.6
3.7			3.4
3.75			1.3
3.84			1.5
3.91			1.0
4.0			1.0

for the large distances becomes excessive and thus they were not obtained and no comparison is possible. At the high signal-to-noise ratios these gains at low information rates are not of importance but at smaller values of S^2 they may become significant.

In a practical communication system error rates of interest probably lie between 10^{-2} and 10^{-12} and thus these are the main areas of interest in Tables 2, 3, and 4. These tables show the trade off possible between information rate, error rate, signal-to-noise ratio, and coding block length.

SUMMARY AND CONCLUSIONS

This thesis has studied a communication system in which information is carried only in the phase of the transmitted signal. The system has a confined frequency spectrum and a relatively simple receiver. The receiver does not require the generation of the transmitter waveforms at the receiver even though a form of cross-correlation detection is used. Nonbinary codes which are suitable for this system and which permit information transmission at rates larger than one bit per digit have been developed. The code construction is based on a geometrical approach which permits an easy visualization of the effects of noise and lack of synchronization. This thesis thus completes for all information rates the original study of Ulstad (2) which was limited to rates less than one bit per digit. It further adds to that study by proposing some nonbinary codes for rates less than a bit per digit. These codes should not have the encoder output power problem of Ulstad's binary codes -- namely that the envelope amplitude deviated considerably from the sample point value between the sample points. The envelope amplitude deviations will still be present but the large deviations will be less probable.

The rate curves of Figure 1 are to the author's best knowledge original. They show that increasing the number of permissible digit values at the transmitter increases the information rate. Figure 1 and Table 6 show that the error rate at a given signal-to-noise ratio also increases as the number of allowed digit values increases. Table 1 shows that the larger information rates can also be obtained with the coding and decoding proposed in this thesis. Table 5 shows that at the same values of S^2 the encoding-decoding procedure proposed here results in lower error rates than those possible at the same information rate with digit-by-digit decoding. Tables 2, 3, and 4 show that certain information rates not possible with digit-by-digit decoding are possible in the system proposed here. These tables also show the exchange possible between information rate, error rate, signal-to-noise ratio, and word length. Note that word length is related to receiver complexity. Greater design flexibility is now possible because of the additional parameter -- word length -- and the additional information rates. Of equal importance are the synchronization properties which the proposed system possesses.

The gains in reducing the decoding error probability are not as large as had been hoped for at the rates near the channel phase modulation capacity, C_{PM}. Figures 15, 16, and 17, which show the error probability per digit for various word lengths, provide an explanation for the above fact. The digit error rate curves show a threshold effect at long word lengths. Thus, for a given word length the error rate increases more than the information rate until a certain code point separation is reached. As the word length increases the threshold moves to large values of code point separation. As either small code point separations for a given value of word length or longer word lengths for a given code point separation are necessary to achieve rates near the channel capacity the proposed coding is inefficient at these rates.

It should be pointed out that the coding discussed in this thesis was used only in forming the words to be transmitted. Further coding, that is relationships among the code words transmitted, is possible and would further reduce the error rates but at some cost in information rate. The codes of Ulrich (8) could, for example, be used for this purpose.

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APPENDIX A: CALCULATION OF PHASE MODULATION ERROR AND INFORMATION RATES FOR DIGIT-BY-DIGIT-DECODING

This appendix is concerned with the information rates and error probabilities that result from digit-by-digit decoding of a phase modulated signal subjected to additive, white, gaussian noise. The transmitted phase values are assumed to be evenly spaced over the range from $-\pi$ to π. The gaussian noise will cause the phase values to be subjected to a noise called phase noise with the result that the received phase values have the conditioned probability density given by Equation E-23. An error results if the phase noise causes the received phase value to lie outside of the range $\alpha - \frac{\pi}{m}$ to $\alpha + \frac{\pi}{m}$ where α is the transmitted phase value and m is the number of phase values permitted at the transmitter.

If it is assumed for convenience that the phase value transmitted is zero, Equation E-23 becomes

$$q(\varphi) = \frac{e^{-S^2}}{2\pi} + \frac{1}{2}\sqrt{\frac{S^2}{\pi}} \cos \varphi \, e^{-S^2 \sin^2 \varphi} [1 + \operatorname{erf} S \cos \varphi] \qquad \text{(A-1)}$$

The probability of correct reception, P_A, involves the integral of $q(\varphi)$ from $-\frac{\pi}{m}$ to $\frac{\pi}{m}$. Thus

$$P_A = \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} q(\varphi)\, d\varphi = 2\int_0^{\frac{\pi}{m}} q(\varphi)\, d\varphi \qquad \text{(A-2)}$$

as $q(\varphi)$ is symmetrical about zero.

If $\frac{\pi}{m} \leq \frac{\pi}{2}$ and the substitution of $y = S \sin \varphi$ is made

$$P_A = \frac{e^{-S^2}}{m} + \frac{1}{\sqrt{\pi}} \int_0^{S \sin \frac{\pi}{m}} e^{-y^2} dy + \frac{2}{\pi} \int_0^{S \sin \frac{\pi}{m}} e^{-y^2} \int_0^{\sqrt{S^2-y^2}} e^{-z^2} dz\, dy \tag{A-3}$$

The portion

$$I = \frac{2}{\pi} \int_0^{S \sin \frac{\pi}{m}} e^{-y^2} \int_0^{\sqrt{S^2-y^2}} e^{-z^2} dz\, dy \tag{A-4}$$

is the troublesome part of P_A. Breaking this integral up into the integral over the two regions shown in Figure 18 yields

$$I = \frac{2}{\pi} \int_0^{S \sin \frac{\pi}{m}} e^{-y^2} \int_0^{y \cot \frac{\pi}{m}} e^{-z^2} dz\, dy + \int_0^{\frac{\pi}{m}} \int_0^{S} e^{-r^2} r dr d\beta$$

$$= \frac{2}{\pi} I^* + \frac{1}{m} - \frac{e^{-S^2}}{m} \tag{A-5}$$

In Figure 18 the angle α is used in place of $\frac{\pi}{m}$.

Now in the integral I^*, e^{-z^2} is replaced by its Taylor's series to yield

$$I^* = \int_0^{S \sin \frac{\pi}{m}} e^{-y^2} dy \int_0^{y \cot \frac{\pi}{m}} \sum_{i=0}^{\infty} \frac{(-1)^i z^{2i}}{i\,!} dz \tag{A-6}$$

Interchanging the order of summation and integration and carrying out the integration on z yields

$$I^* = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{(\cot \frac{\pi}{m})^{2i+1}}{2i+1} \int_0^{S \sin \frac{\pi}{m}} e^{-y^2} y^{2i+1} dy \qquad \text{(A-7)}$$

Using repeated integration by parts yields

$$\int_0^{S \sin \frac{\pi}{m}} e^{-y^2} y^{2i+1} dy = \frac{i!}{2} - \frac{e^{-(S \sin \frac{\pi}{m})^2}}{2} \sum_{j=0}^{i} (S \sin \frac{\pi}{m})^{2j} \frac{i!}{j!} \qquad \text{(A-8)}$$

Thus

$$I^* = \sum_{i=0}^{\infty} \left\{ (-1)^i \frac{(\cot \frac{\pi}{m})^{2i+1}}{2i+1} \left[\frac{1}{2} - \frac{e^{-(S \sin \frac{\pi}{m})^2}}{2} \sum_{j=0}^{i} \frac{(S \sin \frac{\pi}{m})^{2j}}{j} \right] \right\} \qquad \text{(A-9)}$$

The value of I^* may be obtained through the use of a digital computer to any desired accuracy. If $\frac{\pi}{m}$ is in the range from $\frac{\pi}{2}$ to $\frac{\pi}{4}$ the convergence is better than for smaller angles. To get the value of I^* for smaller values of $\frac{\pi}{m}$ the following scheme is employed. In Figure 19 the integral over region 1 is equal to the same kernel integrated over region 2. This is true because the kernel depends only on the distance from the origin. The integral over regions 2 and 3 can be obtained through the use of erf tables. Thus the integral over region 3 which is the I^* of interest is

$$I^* = \int_0^{S \sin \alpha} e^{-y^2} dy \int_0^{S \cos \alpha} e^{-z^2} dz - [\text{integral over region 1}] \qquad \text{(A-10)}$$

and the angle involved in the integral over region 1 is greater than or equal to $\frac{\pi}{4}$. Thus

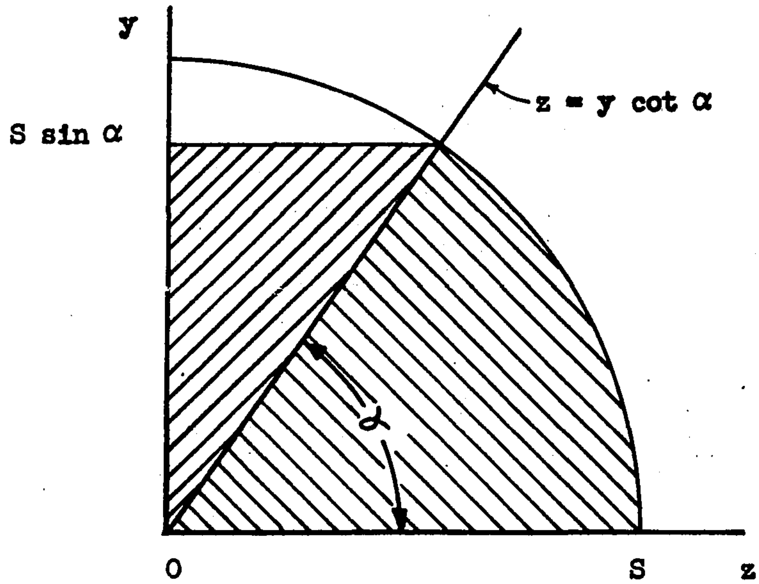


Figure 18. Integration regions for Equation A-5

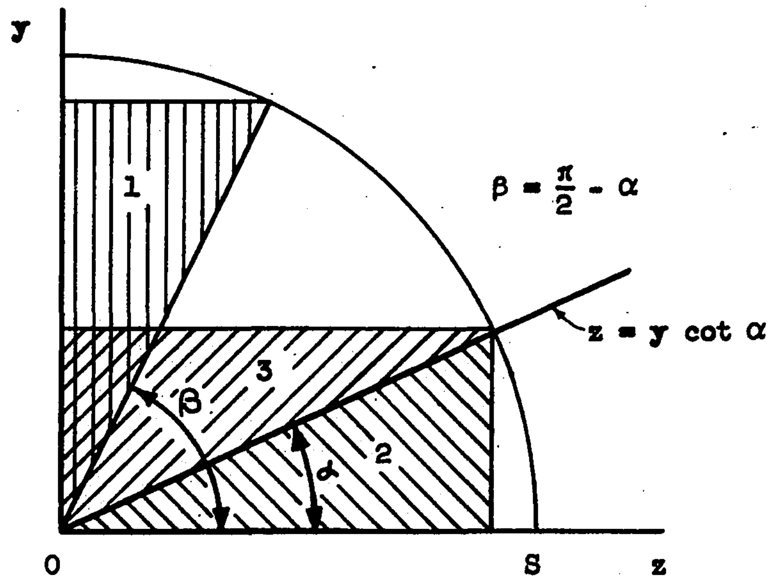


Figure 19. Integration regions used to form I^* for $\alpha < \frac{\pi}{4}$

$$P_A = \frac{1}{m} + \frac{1}{2}\,\text{erf}\,(S \sin \frac{\pi}{m}) + \frac{2}{\pi} I^* \tag{A-11}$$

and in the special case of m = 2

$$P_A = \frac{1}{2} + \frac{1}{2}\,\text{erf}\,S \tag{A-12}$$

If m = 4, another special case,

$$P_A = \frac{1}{4} + \frac{1}{2}\,\text{erf}\,\frac{S}{\sqrt{2}} + \frac{1}{4}\,[\text{erf}\,\frac{S}{\sqrt{2}}]^2 \tag{A-13}$$

Of course P_E, the probability of error, is

$$P_E = 1 - P_A \tag{A-14}$$

The value of P_E for various values of m and S^2 are tabulated in Table 6.

The maximum information rate possible for narrowband phase modulation, the channel capacity, is given by the formula

$$\frac{C}{W} = \log_2 m + P_A \log_2 P_A + P_{E_1} \log \frac{P_{E_1}}{2} + P_{E_2} \log \frac{P_{E_2}}{2}$$

$$+ \ldots + \begin{cases} \dfrac{P_{E_{\frac{m-1}{2}}}}{2} \log_2 \dfrac{P_{E_{\frac{m-1}{2}}}}{2} & \text{m odd} \\[2ex] \dfrac{P_{E_{\frac{m}{2}}}}{2} \log_2 \dfrac{P_{E_{\frac{m}{2}}}}{2} & \text{m even} \end{cases} \tag{A-15}$$

where $\frac{C}{W}$ is the rate in bits per digit, C is the channel capacity in bits per second, W is the bandwidth in cycles per second, P_{E_1} is the probability that noise causes the transmitted digit to be interpreted as one of its two nearest neighbors, P_{E_2} is the probability that the transmitted digit is interpreted as one of the two next nearest neighbors and so on. As the probabilities are functions of S^2, the signal-to-noise power ratio, the channel capacity per unit bandwidth is also a function of S^2. The quantity $\frac{C}{W}$ for various values of m is plotted versus S^2 in Figure 1. The various probabilities were calculated using formula A-11 with $\frac{\pi}{m}$ replaced by the appropriate angles. Thus P_{E_1} was obtained as P_A evaluated for the angle $\frac{\pi}{m}$ minus P_A evaluated for the angle $\frac{3\pi}{m}$.

Table 6. P_E versus m with S^2 as a parameter

Number of Digit Values, m	Error Probability for Digit-by-Digit Decoding				
	S^2 = 0 db	S^2 = 5 db	S^2 = 10 db	S^2 = 15 db	S^2 = 20 db
2	7.9×10^{-2}	5.9×10^{-3}	3.9×10^{-6}	7.5×10^{-16}	1.0×10^{-45}
3	1.8×10^{-1}	2.7×10^{-2}	1.1×10^{-4}	4.0×10^{-12}	1.2×10^{-33}
4	2.9×10^{-1}	7.4×10^{-2}	1.6×10^{-3}	1.7×10^{-8}	1.5×10^{-23}
5	3.9×10^{-1}	1.4×10^{-1}	8.6×10^{-3}	2.8×10^{-6}	1.2×10^{-16}
6	4.6×10^{-1}	2.1×10^{-1}	2.6×10^{-2}	6.7×10^{-5}	1.6×10^{-9}
8	5.7×10^{-1}	3.4×10^{-1}	8.7×10^{-2}	2.3×10^{-3}	7.0×10^{-8}
10	6.5×10^{-1}	4.4×10^{-1}	1.7×10^{-1}	1.4×10^{-2}	1.3×10^{-5}
12	7.1×10^{-1}	5.1×10^{-1}	2.5×10^{-1}	3.9×10^{-2}	2.5×10^{-4}
16	7.8×10^{-1}	6.2×10^{-1}	3.8×10^{-1}	1.2×10^{-1}	5.8×10^{-3}
24	8.5×10^{-1}	7.4×10^{-1}	5.6×10^{-1}	3.0×10^{-1}	6.5×10^{-2}
32	8.9×10^{-1}	8.1×10^{-1}	6.6×10^{-1}	4.3×10^{-1}	1.7×10^{-1}

APPENDIX B: COMPLEX NARROWBAND SAMPLING THEOREM

This appendix is a summary of the theory of uniform sampling of narrowband waveforms. It follows very closely the work of Goldman (17). A signal $s_r(t)$ has a spectrum which is $2\pi W$ radians per second wide and which is centered at ω_o radians per second. There is, of course, a symmetrical spectrum centered at $-\omega_o$ radians per second. To express $s_r(t)$ in terms of values at sample points it is necessary to form a Fourier series expansion of the corresponding frequency function and this requires a continuous frequency spectrum. As W is assumed small with respect to $2\omega_o$ this is not true in the present case. Therefore, a new complex function

$$s_c(t) = s_r(t) - j\, s_i(t) \tag{B-1}$$

is defined such that the negative frequency spectrum is removed. The real part of $s_c(t)$ is the original function $s_r(t)$ and $s_i(t)$ is chosen such that it removes the negative frequency spectrum. Goldman shows that $s_i(t)$ is really the Hilbert transform of $s_r(t)$.

The complex signal $s_c(t)$ and its spectrum $S_c(j\omega)$ which runs from $\omega_o - \frac{2\pi W}{2}$ to $\omega_o + \frac{2\pi W}{2}$ radians per second are assumed to be Fourier transformable. Thus

$$s_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_c(j\omega)\, e^{j\omega t}\, dt \tag{B-2}$$

And

$$S_c\,(j\omega) = \int_{-\infty}^{\infty} s_c(t)\, e^{-j\omega t}\, dt \tag{B-3}$$

Now $S_c(j\omega)$ is expanded in a Fourier series in the frequency range $\omega_o - \pi W$ to $\omega_o + \pi W$ to obtain

$$S_c(j\omega) = \sum_{k=-\infty}^{\infty} C_k \, e^{-j\frac{k\omega}{W}} \tag{B-4}$$

where

$$C_k = \frac{1}{2\pi W} \int_{\omega_o - \pi W}^{\omega_o + \pi W} S_c(j\omega) \, e^{j\frac{k\omega}{W}} \, d\omega \tag{B-5}$$

Substituting $t = \frac{k}{W}$ in Equation B-2 yields

$$C_k = \frac{1}{W} \; s_c(\tfrac{k}{W}) \tag{B-6}$$

Thus substituting Equation B-6 into Equation B-4 and the resulting equation into Equation B-2 yields

$$s_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} \frac{1}{W} \, s_c(\tfrac{k}{W}) \, e^{-j\frac{k\omega}{W}} \right] e^{j\omega t} \, dt \tag{B-7}$$

Performing the indicated integration and taking the real part of the result yields an $s_r(t)$ of the form

$$s_r(t) = \sum_{k=-\infty}^{\infty} A_k \frac{\sin \pi W (t - \frac{k}{W})}{\pi W (t - \frac{k}{W})} \cos [\omega_o (t - \tfrac{k}{W}) - \lambda_k] \tag{B-8}$$

Equation B-8 shows that $s_r(t)$ is completely specified by the two numbers A_k and λ_k at the k^{th} sample point and that these sample points are spaced $\frac{1}{W}$ seconds apart. These sample values contain all the information that can be put in the narrowband signal. The value A_k determines the amplitude of the $\frac{\sin x}{x}$ envelope function centered at the k^{th} sample point and λ_k

determines the carrier phase.

Note that the envelope function of a given sample point is zero at all other sample points.

APPENDIX C: FREQUENCY SPECTRUM OF TRANSMITTED SIGNAL

The transmitted signal, $x(t)$, has the form

$$x(t) = \sum_{k=0}^{K} f_k(t - k\mu) \tag{C-1}$$

where

$$f_k(t) = \frac{\sin \pi W(t - 5\mu)}{\pi W(t - 5\mu)} [u_1(t) - u_1(t - 10\mu)] [\cos \omega_o t \cos \lambda_k + \sin \omega_o t \sin \lambda_k] \tag{C-2}$$

In the above equations W is the nominal channel bandwidth in cycles per second, $\mu = \frac{1}{W}$ seconds, $u_1(t)$ is the unit step function, ω_o is the carrier frequency in radians per second, and λ_k is the phase of the carrier at the k^{th} sample point. In Equation C-1 K will approach infinity if the transmitter is assumed to send messages continuously after it is turned on.

It is desired to examine the frequency spectrum of the transmitted signal. To do this the Fourier transform of $x(t)$ is formed as

$$X(j\omega) = \int_{-\infty}^{\infty} \sum_{k=0}^{K} f_k (t - k\mu) \; e^{-j\omega t} \, dt \tag{C-3}$$

Now interchanging the order of integration and summation and letting

$$\hat{t} = t - 5\mu - k\mu \tag{C-4}$$

yields

$$X(j\omega) = \sum_{k=0}^{K} \cos\lambda_k \; e^{-j\omega k\mu} \int_{-\infty}^{\infty} (e^{-j\omega 5\mu}) \frac{\sin\pi W\hat{t}}{\pi W\hat{t}} \, [u_1(\hat{t}+5\mu)-u_1(\hat{t}-5\mu)]$$

$$[\cos\omega_o(\hat{t}+5\mu)] \; e^{-j\omega\hat{t}} \, d\hat{t} + \sum_{k=0}^{K} \sin\lambda_k \; e^{-j\omega k\mu} \int_{-\infty}^{\infty} (e^{-j\omega 5\mu}) \frac{\sin\pi W\hat{t}}{\pi W\hat{t}}$$

$$[u_1(\hat{t}+5\mu) - u_1(\hat{t}-5\mu)] \; [\sin\omega_o(\hat{t}+5\mu)] \; e^{-j\omega\hat{t}} \, d\hat{t} \qquad \text{(C-5)}$$

If the integrals of Equation C-5 are called respectively $F_1(j\omega)$ and $F_2(j\omega)$ the equation becomes

$$X(j\omega) = \sum_{k=0}^{K} [\cos\lambda_k e^{-j\omega k\mu} \; F_1(j\omega)] + \sum_{k=0}^{K} [\sin\lambda_k e^{-j\omega k\mu} \; F_2(j\omega)] \qquad \text{(C-6)}$$

The power density spectrum of $x(t)$ is defined as

$$G(\omega) = \frac{1}{K\mu} X(j\omega) \; X(-j\omega) \qquad \text{(C-7)}$$

if K is finite and this becomes

$$\begin{aligned} G(\omega) = \frac{1}{K\mu}\Big\{ & F_1(j\omega)F_1(-j\omega) \sum_{n=0}^{K} \sum_{k=0}^{K} \cos\lambda_k \cos\lambda_n \; e^{-j\omega\mu(k-n)} \\ & + F_2(j\omega)F_1(-j\omega) \sum_{n=0}^{K} \sum_{k=0}^{K} \sin\lambda_k \cos\lambda_n \; e^{-j\omega\mu(k-n)} \\ & + F_2(-j\omega) \; F_1(j\omega) \sum_{n=0}^{K} \sum_{k=0}^{K} \sin\lambda_n \cos\lambda_k \; e^{-j\omega\mu(k-n)} \\ & + F_2(j\omega) \; F_2(-j\omega) \sum_{n=0}^{K} \sum_{k=0}^{K} \sin\lambda_n \sin\lambda_k \; e^{-j\omega\mu(k-n)} \Big\} \end{aligned} \qquad \text{(C-8)}$$

If we now take the ensemble average of $G(\omega)$, that is, average Equation C-8 over all the possible x(t)'s, the average power spectrum will be obtained. In the various x(t)'s the λ_k's may have the values α_i, $i = 1, 2, \ldots, m$. The codes are such that α and $-\alpha$ where $0 \leq \alpha \leq \pi$ are equally probable values for the λ_k's. Also if m is odd, all the α_i are equally probable and if m is even α and $(\alpha + \pi)$ are equally probable where $0 \leq \alpha \leq \pi$. It is also assumed that there are no probability constraints in the codes. That is, it is assumed that in a particular x(t) the probabilities of the various values of λ_k at any value of k do not depend on the value of λ_k at any other value of k. The actual codes will normally, as mentioned in the section on code synchronization properties, have some probability constraints among the digits. These constraints will vary from code to code and thus computational convenience makes the above assumption necessary. Thus,

$$\widetilde{\sum_{n=0}^{K} \sum_{k=0}^{K} \cos\lambda_k \cos\lambda_n \, e^{j\omega\mu(k-n)}} = K\,\widetilde{\cos^2\lambda_k} \qquad \text{(C-9)}$$

$$\widetilde{\sum_{n=0}^{K} \sum_{k=0}^{K} \sin\lambda_k \cos\lambda_n \, e^{j\omega\mu(k-n)}} = 0 \qquad \text{(C-10)}$$

$$\widetilde{\sum_{n=0}^{K} \sum_{k=0}^{K} \sin\lambda_n \sin\lambda_k \, e^{j\omega\mu(k-n)}} = K\,\widetilde{\sin^2\lambda_k} \qquad \text{(C-11)}$$

where $\widetilde{\cos^2\lambda_k}$ and $\widetilde{\sin^2\lambda_k}$ have the same value for all k. In the above the wavy lines indicate ensemble averages.

$F_1(j\omega)$ may be written in the form

$$F_1(j\omega) = \frac{e^{-j\omega\mu 5}}{2} \int_{-\infty}^{\infty} \left[e^{j(\omega_o \hat{t} + 5\mu\omega_o)} + e^{-j(\omega_o \hat{t} + 5\mu\omega_o)} \right] \frac{\sin \pi W \hat{t}}{\pi W \hat{t}}$$

$$[u_1(\hat{t} + 5\mu) - u_1(\hat{t} - 5\mu)] \, e^{-j\omega\hat{t}} \, d\hat{t} \qquad \text{(C-12)}$$

Using the fact that

$$\int_{-\infty}^{\infty} f(t) e^{-j\omega_o t} \, e^{-j\omega t} dt = F\,[j(\omega - \omega_o)] \qquad \text{(C-13)}$$

it is seen that $F_1(j\omega)$ will consist of a spectrum centered at $+\omega_o$ and a similar one centered at $-\omega_o$. If $\frac{W}{2} << \omega_o$ these spectrums will not overlap appreciably and the effort can be concentrated on the one centered at ω_o.

Let

$$I_1(j\omega) = \frac{e^{-j\omega 5\mu}}{2} \int_{-\infty}^{\infty} \left[e^{j(\omega_o \hat{t} + 5\mu\omega_o)} \right] \frac{\sin \pi W \hat{t}}{\pi W \hat{t}} \, [u_1(\hat{t}+5\mu) - u_1(\hat{t}-5\mu)] \, e^{-j\omega\hat{t}} d\hat{t} \qquad \text{(C-14)}$$

$$I_2(j\omega) = \frac{e^{-j\omega 5\mu}}{2} \int_{-\infty}^{\infty} \left[e^{j(\omega_o \hat{t} + 5\mu\omega_o)} \right] \frac{\sin \pi W \hat{t}}{\pi W \hat{t}} [u_1(\hat{t}+5\mu) - u_1(\hat{t}-5\mu)] \, e^{-j\omega\hat{t}} \, d\hat{t} \qquad \text{(C-15)}$$

Then if $G^*(\omega)$ is the portion of the power spectrum centered at ω_o

$$\widetilde{G^*(\omega)} = \frac{\widetilde{\cos^2 \lambda_k}}{\mu} \, I_1(j\omega) \, I_1(-j\omega) + \frac{\widetilde{\sin^2 \lambda_k}}{\mu} \, I_2(j\omega) \, I_2(-j\omega) \qquad \text{(C-16)}$$

$$= \frac{I_2(j\omega) \, I_2(-j\omega)}{\mu}$$

as

$$I_1(j\omega) \, I_1(-j\omega) = I_2(j\omega) \, I_2(-j\omega) \qquad \text{(C-17)}$$

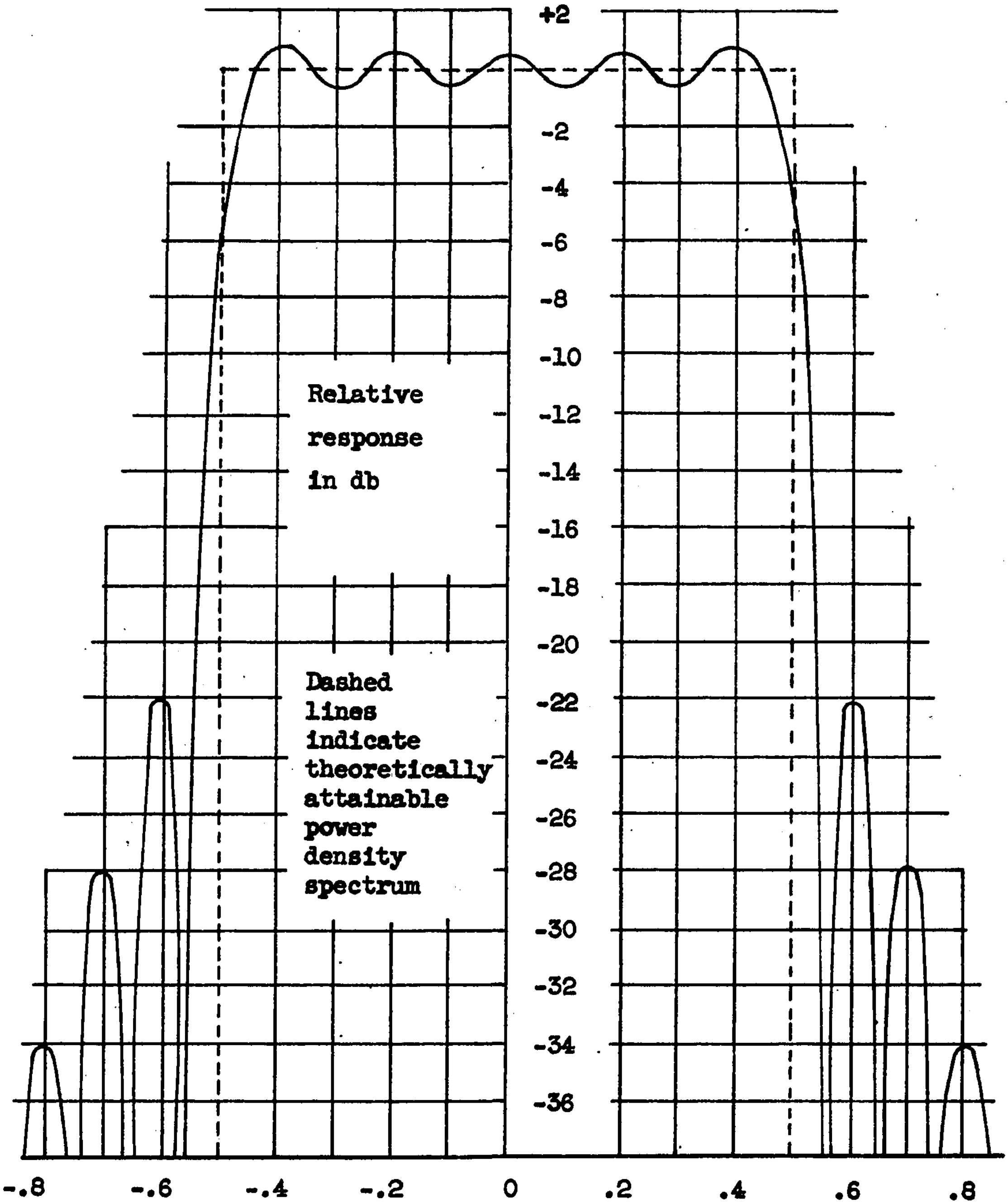
and $\widetilde{\cos^2 \lambda_k} + \widetilde{\sin^2 \lambda_k} = 1$

Note that the middle two terms of Equation C-8 would cancel even if the ensemble average of $\sin \lambda_k \cos \lambda_n$ were not zero because of the j term in Equation C-18. Also if K approaches infinity Equation C-7 becomes

$$G(\omega) = \lim_{k \to \infty} \frac{1}{K\mu} X(j\omega)\, X(-j\omega) \qquad \text{(C-17)}$$

and $G(\omega)$ for positive values of ω will approach $G^*(\omega)$. This is true because for any value of (k-n) there will be many terms in the sum which will yield that value and the average over all the terms which yield a particular value of (k-n) will equal the ensemble average for a particular k and n which have this value of (k-n). The latter is true because the x(t) generation process is assumed stationary.

Equation C-16 is the same spectrum arrived at by Ulstad and his Figure 13 is repeated here as Figure 20. The fact that the actual codes do contain some probability constraints will affect the spectrum somewhat. This actual spectrum will, of course, depend on the particular constraints. The dashed curve in Figure 20 corresponds to the theoretical spectrum of a band limited signal of bandwidth W. Note that the assumption that the negative spectrum may be neglected in the determination of the spectrum near ω_o is justified by Figure 20.



Departure from $f_o = \frac{\omega_o}{2\pi}$ in units of W cps

Figure 20. Average power density spectrum

APPENDIX D: BINARY CODING THEOREMS

A great deal of work has been done in the field of binary coding. Most of the work has been concerned with finding codes which were easily decoded, that is, codes which have a very systematic relationship among a group of bits. The most common of these codes have been the parity check codes suggested by Hamming (11) and studied in great detail by Slepian (18). These codes are normally constructed to detect and/or correct any number of errors up to some maximum which normally depends primarily on the constraint length of the code. The fact that the decoding requires a systematic relationship among the bits tends to reduce slightly the number of code words possible for a given word length.

Plotkin (16) has studied nonsystematic codes and obtained some limits and theorems concerning the number of words of a given length which meet a certain distance criterion. Others have also looked at this problem but they have not in general suggested, as has Plotkin, means of constructing the codes which meet their coding bounds. The nonbinary codes discussed in the body of the paper require binary codes for their formulation. As there is no need for these codes to be systematic, due to the type of detection used, Plotkin's work will be reviewed here. With this information it is possible to determine the information rates for the nonbinary codes discussed in the paper.

Plotkin's work concerns choosing binary sequences of length N which differ from each of the other chosen sequences in at least d_H places. Thus he uses Hamming distance and obtains many of the same results as

Hamming. The symbol $A(N,d_H)$ is used for the number of allowed sequences of length N and mutual distance d_H. This is similar to Hamming's $B(N,d_H)$ except that $B(N,d_H)$ must be a power of 2. Thus $A(N,d_H) \geq B(N,d_H)$.

The following results are due to Hamming.

$$A(N, 1) = 2^N \tag{D-1}$$

$$A(N, 2) = 2^{N-1} \tag{D-2}$$

$$A(N + 1, 2K) = A(N, 2K - 1) \qquad K \text{ an integer} \tag{D-3}$$

$$A(N, 2K - 1) \leq \frac{2^N}{1 + C(N, 1) + \ldots + C(N, K - 1)}$$

$$C(N, h) = \frac{N!}{h!\,(N - h)!} \tag{D-4}$$

Plotkin has proven the following theorems.

Theorem 1: If $2d_H > N$, then $A(N,d_H) \leq 2m \leq \frac{2d_H}{2d_H-N}$, m an integer.

Corollary: $A(4m - 1, 2m) \leq 4m$ and $A(4m - 2, 2m) \leq 2m$.

Theorem 2: $A(N,d) \leq 2A(N-1, d_H)$.

Corollary: $A(4m, 2m) \leq 8m$.

If $A(4m,2m) = 8m$, then $A(4m-1,2m) = 4m$ and $A(4m-2,2m) = 2m$.

Theorem 3: If 4m-1 is prime, then $A(4m,2m) = 8m$.

Paley (20) has shown that this is also true if $4m-1 = 2^K(p^h + 1)$ for p an odd prime and h, K integers.

Theorem 4: $A(2N,2d) \geq A(N,2d)\,A(N,d)$.

Theorem 5: If $A(4m, 2m) = 8m$ holds for $m = x$ then it holds for $m = 2x$.

Using the above results Plotkin obtained the value of $A(N,d_H)$ shown in Table 7. Plotkin also has a second table showing the values of $B(N,d_H)$, Hamming's bound, which is reproduced here as Table 8.

These tables and theorems make it possible to calculate at least the minimum information rates for the suggested nonbinary codes.

Assume $D = \frac{\pi}{2}$ and $N = 16$. The number of messages is

$$M = 4^{16}\ A(16,4)\ A(16,16) \geq 4^{16}\ 2^{11}\ 2 = 2^{44} \tag{D-5}$$

The 4^{16} term is for the K-lattice points as they are all included, the A(16,4) is the number of allowed 2K-lattice points associated with each allowed K-lattice point, and A(16,16) is the number of allowed 4K-lattice points associated with each allowed 2K-lattice point. As $D' = D$, d must be 2 for the 2K-lattice points and 4 for the 4K-lattice points and thus d_H must be 4 and 16 respectively. The total number of allowed code points <u>inside</u> the K-cube is 2^{12}-1 in this case.

The information rate for the code of the above example is

$$R = \frac{1}{N} \log_2 M \geq \frac{1}{16} \log_2 2^{44} = 2.75 \text{ bits/digit} \tag{D-6}$$

All of Plotkin's theorems which bound $A(N,d_H)$ from below are constructive in nature. Thus it is possible to construct the codes which give the information rages indicated in Table 1 of the main paper. The construction methods are indicated in Plotkin's (16) paper and will not be discussed here.

Table 7. Plotkin's table of $A(N,d_H)$

d_H	N=1	2	3	N=4	5	6	7	N=8	9	10	11	N=12	13	14	15	N=16
								$A(N,d_H)$								
8								2	2	2	2	4	4	8	16	32
7							2	2	2	2	4	4	8	16	32	
6						2	2	2	4	6	12	24				
5					2	2	2	4	6	12	24					
4				2	2	4	8	16								2^{11}
3			2	2	4	8	16								2^{11}	2^{11}
2		2	4	8	16	32	64	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}
1	2	4	8	16	32	64	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}

Table 8. Hamming's bound on the number of code words

d_H	N=1	2	3	N=4	5	6	7	N=8	9	10	11	N=12	13	14	15	N=16
								$B(N,d_H)$								
8								2	2	2	2	4	4	8	16	32
7							2	2	2	2	4	4	8	16	32	
6						2	2	2	4	4	8	16				
5					2	2	2	4	4	8	16					
4				2	2	4	8	16	16	32	64	2^7	2^8	2^9	2^{10}	2^{11}
3			2	2	4	8	16	16	32	64	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
2		2	4	8	16	32	64	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}
1	2	4	8	16	32	64	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}

APPENDIX E: CONDITIONAL PROBABILITY OF RECEIVER PHASE SAMPLE VALUES

The transmitted signal x(t) has the form

$$x(t) = \sum_k A \frac{\sin \pi W(t - \frac{k}{W} - \frac{5}{W})}{\pi W(t - \frac{k}{W} - \frac{5}{W})} \cos [\omega_o(t - \frac{k}{W} - \frac{5}{W}) - \lambda_k]$$

$$[u_1(t - \frac{k}{W}) - u_1(t - \frac{k}{W} - \frac{10}{W})] \qquad \text{(E-1)}$$

In this equation W is the nominal channel bandwidth in cycles per second, ω_o is the carrier frequency in radians per second, $u_1(t)$ is the unit step function, and λ_k is one of m allowed phase values α_i, i = 1, 2, . . ., m.

In the channel this signal is corrupted by additive, white, gaussian, band-limited noise. Bennett (19) shows that this narrowband noise may be expressed in the form

$$n(t) = u(t) \cos \omega_o t + v(t) \sin \omega_o t \qquad \text{(E-2)}$$

where u(t) and v(t) are slowly time varying functions of the form

$$u(t) = \sum_{i=1}^{I} \sqrt{2G(\omega_i)\Delta f} \cos [(\omega_i - \omega_o) t + \gamma_i] \qquad \text{(E-3)}$$

$$v(t) = \sum_{i=1}^{I} \sqrt{2G(\omega_i)\Delta f} \sin [(\omega_i - \omega_o)t + \gamma_i] \qquad \text{(E-4)}$$

and ω_o is the center frequency of the narrow band. In Equations E-3 and E-4 $G(\omega_i)$ is the power spectral density in watts per cycles per second of the white noise at the frequency ω_i after being passed through the narrow band filter, the channel in this case. This power spectrum was originally approximated by I sinusoids of uniform and independently distributed phase to arrive at Equation E-2. In the above equations the ω_i are frequencies in the band centered at ω_o, Δf is the spacing between frequencies, and the ζ_i's are uniformly and independently distributed phase variables.

As I approaches infinity and Δf approaches zero the central limit theorem may be used to show that u(t) and v(t) have gaussian distributions. Bennett (19) shows that u(t) and v(t) are independent random variables. Thus the joint probability density of u(t) and v(t) for any given instant of time is

$$p(u,v) = p(u)\ p(v) = \frac{e^{-\frac{u^2 + v^2}{2\sigma_N^2}}}{2\pi\ \sigma_N^2} \qquad \text{(E-5)}$$

where $\sigma_N^2 = \sum_{i=1}^{I} G(\omega_i)\ \Delta f$ is the total noise power in the channel.

The received signal y(t) has the form

$$y(t) = x(t) + n(t) \qquad \text{(E-6)}$$

The signal y(t) will be sampled at times corresponding to the sample points used in forming x(t). Near a sampling point

$$x(t) \simeq A \cos \omega_o t \cos \lambda_k + A \sin \omega_o t \sin \lambda_k \qquad \text{(E-7)}$$

and thus

$$y(t) \simeq A \cos \omega_o t \cos \lambda_k + A \sin \omega_o t \sin \lambda_k + n(t) \qquad \text{(E-8)}$$

near a sampling point. The probability densities associated with the ensemble of functions given by Equation E-8 are of interest.

Near a sampling point y(t) has the forms

$$y(t) = [A \cos \lambda_k + u(t)] \cos \omega_o t + [A \sin \lambda_k + v(t)] \sin \omega_o t \quad \text{(E-9)}$$

and

$$y(t) = r(t) \cos [\omega_o t - \varphi(t)] \quad \text{(E-10)}$$

where

$$r(t) = \sqrt{[A \cos \lambda_k + u(t)]^2 + [A \sin \lambda_k + v(t)]^2} \quad \text{(E-11)}$$

and

$$\varphi(t) = \tan^{-1} \frac{A \sin \lambda_k + v(t)}{A \cos \lambda_k + u(t)} \quad \text{(E-12)}$$

If $u'(t) = A \cos \lambda_k + u(t)$ (E-13)

and $v'(t) = A \sin \lambda_k + v(t)$ (E-14)

the probability densities are

$$p(u') = \frac{e^{-\frac{(u' - A \cos \lambda_k)^2}{2 \sigma_N^2}}}{\sqrt{2\pi} \; \sigma_N} \quad \text{(E-15)}$$

and

$$p(v') = \frac{e^{-\frac{(v' - A \sin \lambda_k)^2}{2 \sigma_N^2}}}{\sqrt{2\pi} \; \sigma_N} \quad \text{(E-16)}$$

and $u'(t)$ and $v'(t)$ are still independent.

The probability density of interest is that of $\varphi(t)$. To get at this it is necessary to let

$$q(r, \varphi) dr d\varphi = p(u', v') \, du' \, dv' \quad \text{(E-17)}$$

that is, equate the probabilities. Thus,

$$q(r, \varphi) = \frac{e^{-\frac{A^2}{2\sigma_N^2}} \, r \, e^{-\frac{(r^2 - A\, r \cos\varphi \cos\lambda_k - A\, r \sin\varphi \sin\lambda_k)}{2\,\sigma_N^2}}}{2\pi\,\sigma_N^2} \tag{E-18}$$

To obtain the probability density of φ it is necessary to integrate Equation E-18 over all possible values of r. Thus

$$q(\varphi) = \frac{e^{-\frac{A^2}{2\sigma_N^2}}}{2\pi\sigma_N^2} \int_0^\infty r\, e^{-\frac{r^2}{2\sigma_N^2}} \, e^{\frac{A\, r \cos(\varphi - \lambda_k)}{2\sigma_N^2}} \, dr$$

$$= \frac{e^{-\frac{A^2}{2\sigma_N^2}}}{2\pi} + \frac{1}{2}\sqrt{\frac{A^2}{2\sigma_N^2}} \, \frac{\cos(\varphi - \lambda_k)}{\sqrt{\pi}} \, e^{-\frac{A^2 \sin^2(\varphi - \lambda_k)}{2\sigma_N^2}} \left[1 + \mathrm{erf} \frac{A\cos(\varphi - \lambda_k)}{\sqrt{2}\ \sigma_N}\right] \tag{E-19}$$

for $-\pi \leq \varphi < \pi$

where

$$\mathrm{erf}\ (x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2}\ dr \tag{E-20}$$

The probability density given by Equation D-19 is really the conditional probability density of φ, that is, the probability density that φ is the phase sample value given that λ_k was the sample point value of the transmitted signal.

If

$$S^2 = \frac{A^2}{2\sigma_N^2} \tag{E-21}$$

and

$$\lambda_k = \theta_k^j = \alpha_i \tag{E-22}$$

where θ_k^j is the k^{th} coordinate of the j^{th} code word and α_i is one of the m allowed transmitter sample point phase values, then

$$q_{\theta_k^j = \alpha_i}(\varphi_k) = \frac{e^{-S^2}}{2\pi} + \frac{1}{2}\sqrt{\frac{S^2}{\pi}}\ \cos(\varphi_k - \alpha_i) e^{-S^2 \sin^2(\varphi_k - \alpha_i)} [1 + \text{erf}\ S \cos(\varphi_k - \alpha_i)] \tag{E-23}$$

for $-\pi \leq \varphi < \pi$.

Equation E-23 is the desired conditional probability of the sampled phase values at the receiver.

APPENDIX F: PROBABILITY DENSITY OF THE SQUARE OF PHASE NOISE

If

$$\lambda = \partial^2 \tag{F-1}$$

then

$$P(x < \lambda < x + dx) = P(y < \partial < y + dy) + P(-y < \partial < -y - dy) \tag{F-2}$$

for $\lambda \geq 0$ where the P's are probabilities. In terms of probability densities Equation F-2 is

$$p(\lambda = x)\, dx = p\,(\partial = y) dy + p\,(\partial = -y)\, dy \tag{F-3}$$

for all $x \geq 0$. Therefore

$$p(\lambda = x) = \begin{cases} \left. \dfrac{p(\partial = y) + p(\partial = -y)}{\frac{dx}{dy}} \right|_{y = \sqrt{x}} = \dfrac{p(\partial = \sqrt{x}) + p(\partial = \sqrt{x})}{2\sqrt{x}} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{F-4}$$

In Appendix E the conditional probability density of the k^{th} sample point phase value in a received word given the k^{th} coordinate of the transmitted word is derived. This is really the probability density of the k^{th} coordinate of any noise word and thus if the noise sequences are

$$\Gamma = \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_N \tag{F-5}$$

the probability density of γ_k is

$$p(\gamma_k = f) = \frac{e^{-S^2}}{2\pi} + \frac{1}{2}\sqrt{\frac{S^2}{\pi}}\ \cos f\ e^{-S^2 \sin^2 f}[1 + \text{erf } S \cos f] \tag{F-6}$$

for $-\pi \leq \gamma < \pi$.

Using Equation F-4 the probability density of $(\gamma_k)^2$ is

$$p\,[(\gamma_k)^2 = \delta] = \begin{cases} \dfrac{e^{-S^2}}{2\pi\sqrt{\delta}} + \dfrac{S\cos\sqrt{\delta}}{2\sqrt{\pi}\,\sqrt{\delta}}\,[1 + \operatorname{erf} S\cos\sqrt{\delta}\,]\; e^{-S^2 \sin^2\sqrt{\delta}} & \text{for } 0 \leq \delta < \pi^2 \\ 0 & \text{for } \delta < 0 \text{ and } \delta > \pi^2 \end{cases} \qquad \text{(F-7)}$$

APPENDIX G: FIRST CONVOLUTION OF THE PROBABILITY DENSITY OF THE SQUARE OF PHASE NOISE

It is necessary in the error rate calculations to obtain the probability density of $(\gamma_1)^2 + (\gamma_2)^2$ where γ_1 and γ_2 are the first two coordinates of a noise word composed of N coordinates. The probability density of $(\gamma_k)^2$, the square of the noise component at any sample point, is derived in Appendix F and given by Equation F-7. The probability density of the sum of two independent random variables is the convolution of their probability densities. Therefore

$$p\,[(\gamma_1)^2 + (\gamma_2)^2 = t] = \int_{-\infty}^{\infty} p[(\gamma_1)^2 = \delta]\; p\,[(\gamma_2)^2 = t - \delta]\,d\delta$$

$$= \begin{cases} 2\displaystyle\int_0^{t/2} p\,[(\gamma_1)^2 = \delta]\,p\,[(\gamma_2)^2 = t - \delta]\,d\delta & t \leq \pi^2 \\ 2\displaystyle\int_{t-\pi^2}^{t/2} p[(\gamma_1)^2 = \delta]\,p[(\gamma_2)^2 = t - \delta]\,d\delta & \pi^2 < t \leq 2\pi^2 \end{cases} \qquad \text{(G-1)}$$

For $t \leq \pi^2$, substitution of Equation F-7 into Equation G-1 yields

$$p[(\gamma_1)^2+(\gamma_2)^2 = t] = 2\int_0^{t/2} \frac{e^{-2S^2}}{4\pi^2\sqrt{\delta}\sqrt{t-\delta}}\,d\delta$$

$$+ 2\int_0^{t/2} \frac{e^{-S^2}}{4\pi}\sqrt{\frac{S^2}{\pi}}\frac{\cos\sqrt{\delta}}{\sqrt{\delta}\sqrt{t-\delta}}[1 + \mathrm{erf}(S\cos\sqrt{\delta})]e^{-S^2\sin^2\sqrt{\delta}}\,d\delta$$

$$+ 2\int_0^{t/2} \frac{e^{-S^2}}{4\pi}\sqrt{\frac{S^2}{\pi}}\frac{\cos\sqrt{t-\delta}}{\sqrt{\delta}\sqrt{t-\delta}}[1 + \mathrm{erf}(S\cos\sqrt{t-\delta})]e^{-S^2\sin^2\sqrt{t-\delta}}\,d\delta$$

$$+ 2\int_0^{t/2} \frac{S^2}{4\pi}\frac{\cos\sqrt{\delta}}{\sqrt{\delta}}\frac{\cos\sqrt{t-\delta}}{\sqrt{t-\delta}}[1 + \mathrm{erf}(S\cos\sqrt{\delta})]\,[1 + \mathrm{erf}\,S\cos\sqrt{t-\delta}]$$

$$e^{-S^2(\sin^2\sqrt{\delta} + \sin^2\sqrt{t-\delta})}\,d\delta \qquad \text{(G-2)}$$

Equation G-2 does not lend itself to an analytical solution and the singularity of $p[(\gamma_k)^2]$ at $(\gamma_k)^2 = 0$ also makes a numerical solution of the equation as it stands virtually impossible. Therefore, it will be necessary to change the form of Equation G-2 to one more suited to a numerical solution.

If the substitution

$$y = \sqrt{\delta} \qquad \text{(G-3)}$$

is made, the probability has the form

$$p[(\gamma_1)^2 + (\gamma_2)^2 = t] = \frac{e^{-2S^2}}{2\pi^2} \int_0^{t/2} \frac{2\,dy}{\sqrt{t - y^2}}$$

$$+ \frac{e^{-S^2}}{2\pi}\sqrt{\frac{S^2}{\pi}} \int_0^{t/2} \frac{2\cos\sqrt{t-y^2}}{\sqrt{t-y^2}}[1 + \mathrm{erf}(S\cos\sqrt{t-y^2})]e^{-S^2\sin^2\sqrt{t-y^2}}dy$$

$$+ \frac{e^{-S^2}}{2\pi}\sqrt{\frac{S^2}{\pi}} \int_0^{t/2} \frac{2\cos y}{\sqrt{t-y^2}}[1 + \mathrm{erf}(S\cos y)]\, e^{-S^2\sin^2 y}\,dy$$

$$+ \frac{S^2}{\pi} \int_0^{t/2} \frac{\cos y \cos\sqrt{t-y^2}}{\sqrt{t-y^2}}[1 + \mathrm{erf}(S\cos y)]$$

$$[1 + \mathrm{erf}(S\cos\sqrt{t-y^2})]\, e^{-S^2(\sin^2 y + \sin^2\sqrt{t-y^2})}dy$$

(G-4)

The further substitution

$$z = \sin^{-1}\frac{y}{t} \qquad \text{(G-5)}$$

yields

$$p[(\gamma_1)^2 + (\gamma_2)^2 = t] = \frac{e^{-2S^2}}{\pi^2} \int_0^{\pi/4} dz + \frac{Se^{-S^2}}{\sqrt{\pi}\ \pi} \int_0^{\pi/4} \cos[\sqrt{t} \cos z]$$

$$1 + \text{erf}\, [S \cos(\sqrt{t} \cos z)] \quad e^{-S^2 \sin^2(\sqrt{t} \cos z)} dz$$

$$+ \frac{Se^{-S^2}}{\sqrt{\pi}\ \pi} \int_0^{\pi/4} \cos(\sqrt{t} \sin z) \left\{1 + \text{erf}\, [S \cos(\sqrt{t} \sin z)]\right\}$$

$$e^{-S^2 \sin^2(\sqrt{t} \sin z)}\, dz$$

$$+ \frac{S^2}{\pi} \int_0^{\pi/4} \cos(\sqrt{t} \cos z) \cos(\sqrt{t} \sin z) \left\{1 + \text{erf}\right.$$

$$\left.[S \cos(\sqrt{t} \sin z)]\right\} \left\{1 + \text{erf}\, [S \cos(\sqrt{t} \cos z)]\right.$$

$$e^{-S^2 [\sin^2(\sqrt{t} \sin z) + \sin^2(\sqrt{t} \cos z)]}\, dz \qquad \text{(G-6)}$$

Equation G-6 allows the calculation of the probability density of $(\gamma_1)^2 + (\gamma_2)^2$ at any desired value in the range 0 to π^2. The integrals of Equation G-6 do not lend themselves to an analytical solution but they are well suited to a numerical solution on a digital computer as they are definite and have well behaved integrands. Equation G-6 was used to obtain the values of the convolved probability density function for the convolution of the probability density of the square of phase noise with itself. After this convolution the resulting function was of such a form that convolution of the function with itself could be carried out by straightforward numberical methods. All the subsequent convolutions required to form

$$p[\sum_{1}^{N} (\gamma_k)^2]$$

for N of the form 2^V, V = 0, 1, 2, . . ., were also carried out numerically.

The determination of P_E, the probability of a detection error, from the probability density functions involves an inherent accuracy problem. The density functions have a large range and also have very large higher order derivatives. These properties make accurate numerical integration very difficult. The error probability, P_E, is

$$P_E = 1 - P_A = 1 - \int_0^{\delta} p[\sum_{k=1}^{N} (\gamma_k)^2] \, d\theta \tag{G-7}$$

where δ is the square of the decoding sphere radius. Calculation of P_E by use of Equation G-7 requires the integral to be formed very accurately because the P_E values of interest are in the range from 10^{-2} to 10^{-12}. For the reasons mentioned above, this degree of accuracy cannot be obtained, particularly in a computer whose inherent accuracy per operation is one part in 10^8. The error probability, P_E, can also be written as

$$P_E = \int_{\delta}^{N\pi^2} p[\sum_{k=1}^{N} (\gamma_k)^2] \, d\theta \tag{G-8}$$

because

$$p[\sum_{k=1}^{N} (\gamma_k)^2]$$

is zero for argument values larger than $N\pi^2$. Fortunately for the values of signal-to-noise power ratio which are of primary interest, the integral,

$$\int_{4}^{N\pi^2} p[\sum_{k=1}^{N} (\gamma_k)^2] d\theta ,$$

is very small with respect to the values of P_E which are of interest. This integral can be bounded analytically to show that it is small. Thus, P_E can be formed as

$$P_E \simeq \int_{\delta}^{4} p [\sum_{k=1}^{N} (\gamma_k)^2] d\theta \qquad \text{(G-9)}$$

and the probability densities need only be formed for arguments in the range from zero to four. The integration of Equation G-9 only has to be performed with two or three digit accuracy to yield sufficiently accurate values of P_E.