Investigating the changes in teachers' pedagogical practices through the use of the

Mathematics Reasoning Heuristic (MRH) approach

by

Recai Akkus

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Education

Program of Study Committee:
Brian M. Hand, Co-Major Professor
Jennifer R. Seymour, Co-Major Professor
Lori Ann Norton-Meier
Mack Clayton Shelley
Kandauda A. Wickrama

Iowa State University

Ames, Iowa

2007

UMI Number: 3259454

UMI®

DEDICATION

To my fiancé Pınar Altun, for her endless support, motivation, inspiration, and love.

To my family for their support and encouragement.

ACKNOWLEDGMENTS

I would like to extend my appreciation and sincere thanks to my major professor, Dr. Brian M. Hand, for his support, encouragement, advice, modeling, and guidance throughout my graduate education and dissertation work. I also would like to thank Dr. Jennifer Seymour for her support, encouragement, and advice during my doctoral work. Special thanks to all the other members in my committee, Dr. Lori Ann Norton-Meier, Dr. Mack Shelley, and Dr. Kandauda Wickrama for their help during my study and for their constructive feedback on the dissertation.

I also wish to thank my colleague and lifelong friend Dr. Murat Gunel for his dedicated work and support. My great thanks and appreciation are for all the mathematics teachers and their students who participated in the study for without their participation this study would not have existed. Finally, I would like to thank the Turkish Ministry of Education for sponsoring my graduate education and Fatma and Kenan Baloglu, and Ali Aslan for their trust and their signatures on the contract, which made this journey possible.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ABSTRACT

Our changing world needs many more mathematically literate individuals. Mathematical literacy can be defined, parallel to reading and writing literacy, as not only being able to understand the fundamental notions of mathematics, develop sophisticated mathematical models and evaluate someone else's use of numbers and mathematical models but also being able to represent quantitative relations using algebraic reasoning and interpret and reflect on mathematical language patterns. In order to help students become mathematically literate, the National Council of Teachers of Mathematics (NCTM) has focused attention on students' conceptual understanding of mathematics suggesting students need to be actively involved in the learning process using their experiences and prior knowledge. Along with this view on learning, understanding of teaching has also been revised in mathematics classrooms. Teachers now need to provide students with a challenging and supportive classroom environment in which they can build new knowledge by engaging in exploration of mathematical ideas by themselves. Since the publication of Curriculum and Evaluation Standards for School Mathematics in 1989, the National Council of Teachers of Mathematics (NCTM) has paid special attention on teacher change, problem solving, and, more recently, using writing in mathematics classrooms for helping students develop thorough mathematical understanding and to becoming more mathematically literate.

This change in the views of learning and teaching has placed students in the center of learning occurring in the classroom by altering students' roles and requiring them to be actively involved in talking and writing in mathematics classrooms. The NCTM mandated that students at all levels should be able to use mathematical ideas in a

variety of situations. For this purpose, students must have the opportunity to discuss their ideas publicly, to reflect on their thoughts and problem solving processes, and to communicate their ideas using various modes of representation (graphical, pictorial, oral, written, etc.). Writing in mathematics was emphasized in *The Principles and Standards for School Mathematics* (NCTM, 2000, p. 61), which said, "Writing in mathematics can …help students consolidate their thinking…" because it requires an active involvement of learners such that they use writing as a vehicle for learning and become the center of their own learning processes by engaging in reflection on mathematical experiences.

This study focused on examining the changes in *pedagogical* practices when three high school algebra teachers shift from their traditional teaching to more student-centered practices through the use of the Mathematics Reasoning Heuristic (MRH) approach. The study also looked at the *performance* differences on the Iowa Test of Educational Development (ITED) between the students in the control classes where the teachers engaged in their traditional instructional routines and the students in the treatment classes where the teachers used the MRH approach. The goal of the MRH approach is to help teachers improve their pedagogical practices to scaffold students' understanding of mathematical concepts and their problem solving skills.

The major findings of this study are that teachers' adoption of the required pedagogical practices varied as they attempted to move away from their traditional practices and that implementing a student-oriented approach such as the MRH approach which includes embedded writing-to-learn strategies does have an impact on student performance. The student performance on the standardized test was significantly enhanced for those students in the MRH classrooms compared to students who engaged

in the more traditional approaches. The results from the analysis of the teachers'

pedagogical practices in their treatment and control classes indicate to us the importance

of pedagogical skills to promote dialogical interaction during problem solving. In

examining the results the researcher would suggest that there are two critical elements of

the MRH approach. The first is the pedagogical approach needed and the second is the

consistent use of the heuristic concept through the scaffolded writing component of the

MRH approach.

CHAPTER ONE

Introduction

General Overview

The first chapter begins with a brief review of three areas of research: problem

solving in mathematics; writing to learn; and writing as problem solving. This review

provides a backdrop for the major research questions and the context of the study. The

dissertation contains a chapter of extended literature review, one research journal article,

and a general conclusion chapter. The journal article investigates the relationships

between implementation of the mathematics reasoning heuristic (MRH) approach, which

is a student-centered, writing-embedded pedagogical tool, and students' performances on

a standardized test. This research article provides evidence of the impact of the MRH

approach on students' test scores, derived from the results of a quasi-experiment, mixed-

method study. The final chapter discusses the results from the research paper, attempts to

theorize further about the impact of the MRH approach on students' learning, suggests

implications of the study, and presents the limitations of the study.

Theoretical Background

Problem Solving in Mathematics

Problem solving in mathematics has been studied extensively by scholars (Artzt

& Armour-Thomas, 1992, 1998; Garofalo & Lester, 1985; Mayer, 1982, 1998; Pape &

Wang, 2003; Schoenfeld, 1983) since Polya (1945) first introduced his structure of a

four-phase mathematical problem solving process (understanding, planning, carrying out

the plan, and looking back). Since then, Polya's framework has been used for developing

new problem solving heuristics. For example, Schoenfeld (1983, 1985), incorporating an

information-processing perspective into Polya's structure, developed a model that explained mathematical problem solving in five phases: *analysis, design, exploration, implementation, and verification*. The analysis and design phases correspond to the understanding and planning processes, respectively, in Polya's heuristic. Schoenfeld (1985), however, included one more phase, *exploration*, that creates a bridge between the analysis and design phases. According to Schoenfeld, during the *exploration* phase, in which the problem solver uses his/her schema knowledge (a cluster of knowledge that describes the typical properties of the concept it represents[1]) the problem solver may return to the *design* stage to structure an argument about the solution, or he or she may go back to *analysis* and try to simplify or reformulate the problem, which can allow the problem solver to approach the problem differently. The last two phases of problem solving in Schoenfeld's model, implementation and verification, respectively, refer to the last two processes, carrying out the plan and looking back, in Polya's structure.

Garofalo and Lester (1985) also developed a framework based on the work of Polya (1957), Schoenfeld (1983), Sternberg (1980, 1982), and to some extent Luria (1973), which represents the cognitive and metacognitive aspects of problem solving. The framework has four categories–*orientation*, *organization*, *execution*, and *verification* –each of which is associated with metacognitive behaviors. Unlike Polya's four-phased problem solving heuristic, the categories of Garafalo and Lester's framework are broadly defined, yet are related to Polya's structure. Another distinctive feature of the framework is that at the *verification* phase the problem solver evaluates the decisions made at the earlier three stages, which is a metacognitive action as indicated by the authors.

Problem solving involves a complex set of cognitive actions requiring many connections to cognitive structure and to the context of the situation (Nesher, Hershkovitz, & Novotna, 2003). According to Polya (1981), problem solving is a process of making conjecture among the data (known) and the unknowns on one hand and the condition, which represents the connection between the unknowns and the data, on the other hand. Furthermore, problem solving is a process that can include multiple possibilities for a solution (NCTM, 2000). When solving problems, students are engaged in cognitive activities that have the potential to enhance their mathematical thinking and reasoning through examining and discussing all different possible solutions. In the course of problem solving, the learner is required to make logical argument(s) in his/her attempt to find the unknown of the problem or proving or disproving a possible answer (Polya, 1981).

Every problem solving heuristic (or structure or model or framework) involves to some extent understanding the problem, planning, carrying out the plan, and evaluation or verification. Each episode of problem solving requires, to some degree, cognitive and metacognitive behaviors inherent in mathematical problem solving. As indicated by several researchers (e.g., Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985; Mayer, 1982, Schoenfeld, 1983, 1985), some problem-solving heuristics put more emphasis on the cognitive aspect than on the metacognitive aspect of problem solving, or vice versa.

Students often struggle to solve mathematical problems, especially word problems, not because they cannot execute the algorithm but because they do not know how to analyze a problem and plan for a solution (Lorenzo, 2005). Many researchers

studying problem solving from a cognitive perspective propose that the main difficulty in problem solving may be related to students' lack of monitoring of the cognitive processes during problem solving (Artzt & Armour-Thomas, 1992; Dow & Mayer, 2004; Mayer, 1982, 1998; Silver, 1982).

Cognitive and Metacognitive Aspects of Problem Solving

The concepts of cognition and metacognition are often confused and it is difficult to observe related behaviors. However, cognition and metacognition have been of interest to both mathematics educators and psychologists, and scholars in both disciplines have contributed to the area of research on such processes through exploratory studies. Garofalo and Lester (1985), taking Flavell's (1976) description of metacognition, tried to clear up confusion by indicating the fact that there are two facets of metacognition, knowledge about cognition and monitoring cognition. Flavell described metacognition as

> …one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data.…Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective. (p. 232)

To distinguish metacognition from cognition, knowledge of cognition involves domain-specific tasks and one's abilities and resources to accomplish these tasks – doing, whereas metacognitive knowledge is about monitoring, planning, and regulating the cognitive processes or what is being done (Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985; Mayer, 1998). With Flavell's (1981) later definition, metacognition is

"knowledge or cognition that takes as its object or regulates any aspect of cognitive endeavor. Its name derives from this 'cognition about cognition' quality" (p. 37). Thus, during any given cognitive enterprise, the role of cognition is to search for paths within the knowledge structure and perform the procedures to achieve the goals. On the other hand, metacognition regulates the cognitive progress in relation to the phenomenon that is being acted on. Flavell stated, "We develop cognitive actions or strategies for *making* cognitive progress and we also develop cognitive actions or strategies for *monitoring* cognitive progress. The two might be thought of as cognitive strategies and metacognitive strategies" (p. 53). Considering these two concepts in a hierarchical relationship, the objects of metacognition are the cognitive activities such as reading a text or performing a mathematical task.

Metacognition involves not only controlling and monitoring cognitive processes but also being aware of how one learns, deciding when one does and does not understand, and evaluating cognitive progress. An effective mathematical problem-solving process requires both cognitive and metacognitive knowledge and the ability to develop cognitive and metacognitive strategies. Lester (1982, p. 59), stressing the role of metacognitive actions in problem solving, confirmed that successful problem solving in mathematics is a function of, among others, "… knowledge about one's own cognitions before, during, and after a problem-solving episode, and the ability to maintain executive control (i.e., to monitor and regulate) of the procedures being employed during problem solving." In a similar vein, Mayer (1982, 1998) and others emphasized the importance of *metaskills*, the ability to monitor and control cognitive actions: "Students need to know not only what to do, but also when to do it" (1998, p. 50).

Using Garofalo and Lester's (1985) four-category framework–*orientation*, *organization*, *execution*, and *verification*–as an example, the verification phase (e.g., the looking back phase in Polya's model) is the stage where most of the metacognitive behaviors, depending upon a given task, are most likely to occur because the problem solver returns back to the prior stages in solving the problem and evaluates the validity of the steps within each phase. This process is not only for checking the outcome of the problem but also for evaluating the progress of problem solving as accomplishing the mathematical tasks. Artzt and Armour-Thomas (1992) found that students rotated several times among different problem-solving episodes such as reading, understanding, exploring, analyzing, planning, implementing, and verifying. Verbal or nonverbal actions that signify actual processing of information demonstrate cognitive behaviors. For example, reading is classified as cognitive; understanding, analyzing, and planning as metacognitive; and exploring, implementing, and verifying as either cognitive or metacognitive depending on the type of the problem solver's action (Artzt & Armour-Thomas, 1992).

Process of Mathematical Problem Solving

Several problem-solving heuristics that have been developed try to explain to some extent the process of solving mathematical problems. Yet, most of the frameworks can be attributed to Polya's (1945) problem-solving heuristic: understanding the problem, making a plan, carrying out the plan, and looking back. The purpose of a framework is to break down the processes that problem solvers go through (Schoenfeld, 1985). For instance, to solve a problem, one must first understand what the problem is. Understanding the problem is an important ingredient of solving the problem and is

related to multiple factors (e.g., problem solver's mathematical background). Greeno (1977) suggested that understanding is an active process of constructing internal representation that is developed for the object that is understood. Greeno stated that the difference between understanding and not understanding is in the nature of the representation: "When a sentence is understood, its internal representation shows what the sentence means. The meaning corresponds to a pattern of relations among concepts that are mentioned in the sentence, and understanding is the act of constructing such a pattern" (p. 44). Such a metacognitive process is related to one's cognitive strategies such as reading the sentences of the text and extracting related information from the text by means of language, and constructing a representation of the problem using newly extracted information. Of course, the representation must include the conditions of the problem and the goals.

Greeno (1977) suggested three criteria for good understanding: 1) achievement of a coherent representation, 2) close correspondence between the internal representation and the object to be understood, and 3) connectedness of the representation to other components of the person's knowledge structure. Mayer (1982), who analyzed problem solving in two stages—representation and solution—provided a perspective from cognitive psychology that views problem solving as a series of cognitive operations that transform knowledge representations. In this view, *representation*, corresponding to understanding the problem, is translation of problem statements into an internal representation "…that includes the given state, goal state, and allowable operators. The problem space can be built upon the subject's understanding of the problem" (p. 4).

Good understanding therefore requires particular types of knowledge, for example, linguistic, factual, and schema knowledge. As stated by Greeno (1977), Lester (1982), and Mayer (1982), a problem solver reads the problem by means of linguistic knowledge to encode the information in the text for construction of internal representation that is connected to one's schema knowledge. When the problem solver reads the problem, he or she relates it to the relevant problem type in his or her schema knowledge, which later helps the problem solver to activate "the units of knowledge structure" relevant to that problem type (Pape & Wang, 2003). During problem solving, not all the units of schematic knowledge are activated; instead, only the ones related to the problem and the ones with strong connections to the input problem are activated. To summarize, understanding a problem is a process of constructing internal representation of the object to be understood (Greeno, 1977) through interpretation of the language of the problem text, which is a process of translation of the text to a set of problem solving operators in order to construct a problem space (Lester, 1982; Mayer, 1982).

Having understood the problem, the problem solver moves to the next phase in solving the problem, making a plan. Planning is inherently involved in understanding. Planning (knowledge) helps the problem solver form a symbolic construction that connects the new information to the old, already-known information (Silver, 1982) and "get a feel" for what needs to be done (Lester, 1982). During planning, students use their strategic knowledge to synthesize their factual and schema knowledge and develop models (sequence of actions that might lead to a solution) for a solution (Artzt & Armour-Thomas, 1992; Lester, 1982; Mayer, 1982; Silver, 1982). Garofalo and Lester (1985), Lester (1982), Mayer (1998), and Silver (1982) emphasized that the problem

solver uses his or her metacognitive knowledge structure to organize the information extracted from the problem text, direct the processes of understanding the problem, and create an accurate diagram for the action to be taken to achieve the goal statement in the problem. Furthermore, Silver pointed out that such metacognitive endeavors during planning might influence understanding of the problem and retrieval of information. It seems a reasonable hypothesis that understanding the problem and planning are not separate from each other; on the contrary, both processes evolve into each other.

Writing-to-learn in Mathematics

With the Writing Across the Curriculum (WAC) movement in 1970s, whose roots were based on the work of James Britton on language and writing about content areas in 1960s in UK, the emphasis in the teaching and using of writing has shifted from its mechanical features towards a process of writing and making meaning through writing. According to Britton and his colleagues, there are three functional types of (written) language: transactional, poetic, and expressive (Britton, 1970; Britton, Burgess, Martin, McLeod, & Rosen, 1975). We are not only using language to communicate or say what we want to say (instrumental function-transactional writing) but we are also using language to reflect on the meaning of our actions (self-reflection-expressive writing).

Transactional, poetic, and expressive writing are distinguished from each other in terms of the actions a writer takes and the purpose of writing. In the transactional writing, the purpose is to transfer or exchange information, ideas, and meanings. In other words, transactional writing is for the delivery of information to others and tests previously learned knowledge (Miller, 1997). The purpose of poetic writing, however, is to express one's experiences and emotions by detaching oneself from the action. On the other hand,

in the expressive writing the intent of writing is for the writer's own use to explore the current ideas and feelings or the ideas about a problem and reflect on them, and the writing promotes one's personal understanding. These three approaches to writing, especially expressive writing, have been the roots of the knowledge on understanding the process of writing and its function in learning.

Among the models that explain the writing process and its role in learning are the models by Hayes and Flower (1980), Bereiter and Scardamalia (1987), and Galbraith (1999). The common view among them is, by looking at writing from a cognitive perspective, that writing is a discovery process of what to say in the course of action and a tool for learning (Alamargot & Chanquoy, 2001). Within this common view, it is reasonable to say that the writer produces the text due to a problem (e.g., writing assignment) that will be resolved through a series of activation of knowledge structures and interaction of rhetorical goals with the problem statements (Galbraith, 1999). Another viewpoint that these models share is about the difference between expert and novice writers. This major difference is, as Galbraith stated, that "experts develop an elaborate set of goals for their text and generate ideas to satisfy these goals, whereas novices simply retrieve ideas prompted spontaneously by the topic and translate them directly into text" (p. 139).

This pervasive idea about writing was reshaped around the idea of "writing-to-learn" that suggests the role of language and writing in constituting, as well as representing, knowledge and thought (Connolly, 1989). Writing-to-learn is not only about language, which is a tool to form ideas via internal and external representations. Rather, writing is also about self-negotiation of meaning through metacognitive actions such as

planning, monitoring, reviewing, and revising—constituting knowledge in the course of dispositional dialectic (Galbraith, 1999). That is, during dispositional dialectic, the writer engages in a series of re-cyclic process of sorting (and reorganizing) the knowledge structure through what he or she knows.

Furthermore, having students write and then revise the written product gives them opportunities to constitute new meaning and generate new knowledge (Galbraith, 1999; Powell & Lopez, 1989). Writing is a tool not only for extracting what one knows about the content but also for developing/constructing thoughts but also for reflecting on this process and the ideas themselves. One of the functions of writing is to augment understanding through the cognitive and metacognitive actions that it demands. In addition, during writing, one negotiates meaning, and in negotiation one is generating knowledge and augmenting learning (Powell & Lopez, 1989). Using Vygotsky's (1962, p. 100) elegant phrase, the writer must engage in "deliberate structuring of the web of meaning." Writing, as a thinking device, helps the writers organize their thoughts, activate relevant knowledge networks aroused by the topic, and make logical connections between knowledge networks and rhetorical goals via linguistic networks in a dispositional dialectic.

Emig (1977), pointing out the unique function of writing in learning, argued that due to its demanding functional cognitive action, writing is a powerful source of thought through immediate connections, by means of lexical, syntactic, rhetorical devices, and between what has been written (product) and what is still to be said (process). Furthermore, she speculated that the process component of writing (as process-and-product) that with the fullest possible functioning of the brain, in writing process, "the

symbolic transformation of experience through the specific symbol system of verbal language is shaped into an icon (the graphic product) by the enactive hand" (p. 124). This process of writing inherently gives the writer the opportunity to integrate the information in the prose passage into his/her own knowledge structure—Emig's notion of the connective nature of writing.

People interact and share knowledge via the use of language, which is one of the communication heuristics that communities have in common. Pimm (1987, p. 7) expressed that one of the functions of language is that it allows people "access to and control over [their] thoughts." As forms of language, writing, and, Pimm stated, talking, help students to communicate with others—to make "someone else understand something or pass on some piece of information"—and to communicate with themselves —"to help organize [and reflect on] their thoughts" (pp. 23-24). One powerful role of writing is to help the writer associate concepts with language (Keys, 1999).

Galbraith (1999) argued that the writer engages in a dispositional dialectic where he or she is constantly involved in an ongoing negotiation of meaning through dialogue with oneself. In the process of writing, the quality of text (utterance), and thus dialectic, is constrained by audience, the writer's content knowledge, and linguistic knowledge. Bereiter and Scardamalia (1987) also speculated that writing empowers the use of human language and social skills attained through experience, but it is also limited by them. Through this interactive course of action, the writer communicates with his or her reader by considering the reader's understanding of the text produced.

Galbraith's model (1999), writing as a knowledge-constituting process, explains the writing process in detail. According to Galbraith, the writer's conceptual network is

activated by the input task (new sensory information). The new information is connected with the activated knowledge network units through linguistic symbols. The newly generated utterance (message) within the linguistic network, in turn, enters the knowledge network units as a new input as a result of ongoing feedback from the output of the linguistic network. However, the new information does not have to become a statement (message); instead, it can activate different units to produce new ideas on a successive cycle. The final statement (utterance) is created within the linguistic network in which the utterance is evaluated to find "the 'best fit' to what the writer has to say about the topic" (p. 146).

As a form of language, mathematics has its own symbol system, which is composed of numbers, letters, computational signs, comparative signs, the equal sign, and so on (Connolly, 1989; Morgan, 1998; Pimm, 1987; Tobias, 1989). Therefore, a "written sentence" in mathematics would be a logical combination of its own symbols (e.g., $x \in Z; x + 4 < 5$). However, the meaning of this sentence may not be explicit to someone who has not been exposed to such a language. The internal representation of this sentence might be, for example, the literal meaning of it in words (e.g., x being an element of integer numbers, x plus four less than five). Someone else who has a pictorial sense can picture this sentence on the number line representing the numbers that satisfy this condition. As can be seen, translation of any mathematical language into the "English" language might require the writer to have enough knowledge and practice in both forms of language, the ability to understand the meaning of the mathematical sentence, and lexical knowledge to translate it into words. This difference between a prose sentence and a mathematical sentence is what Emig (1977) called the distinction

between composing in words and composing in a graphic symbol system of mathematical equations. Morgan (1998) pointed out that students in mathematics classrooms are often expected to learn "the mathematical language" without making connection to their own everyday language. She suggested that writing can actually help students investigate mathematical language through exploration of the language they more commonly use.

People make/form meaning via the natural language. The unique structure of mathematical language is understood with the meaning corresponding to specific mathematical symbols. Therefore, if students have the opportunity to express their mathematical understanding in words through either orally or in written format, they are more likely to make connections between concepts (Meier & Rishel, 1998; Morgan, 1998; Pimm, 1987). Meier and Rishel (p. 90) further stated that "mathematics is embedded in language" and we use that language first to form the meaning of mathematics symbols (concepts) and then to express our understanding to the outside audience orally or in a written text.

As opposed to authors (e.g., Gries & Schneider, 1995 in Meier & Rishel) who defend first teaching rigorous proofs using a formal logic, Meier and Rishel (1998), Morgan (1998), and Ernest (1998) argued that students should build their mathematical language on their everyday language, which helps them attain the nature of mathematical symbols and special vocabulary of mathematics. This feature of mathematics language is called a "mathematical register" (Halliday, 1974; Pimm, 1987). Consequently, through constant use of mathematical concepts in classrooms by speaking or writing, students create their own comfortable mathematical register.

Mathematics Reasoning Heuristic for Mathematical Problem Solving

In light of the literature review above, incorporating writing into mathematical problem solving created the mathematics reasoning heuristic (MRH) approach for teachers to use in classrooms to promote students' problem solving abilities (Akkus & Hand, 2005). The MRH approach is a conceptual framework that focuses on the relationships among students' knowledge of mathematics, the teacher's knowledge of mathematics, interaction among students and the teacher, negotiation of ideas, writing, and the process of students' problem solving. The MRH consists of two essential components. The first is the teacher template, which is a pedagogical tool for teachers to use before and during implementation (Figure 1). According to the MRH, the teachers both need to define the big ideas of the topic, which are the essential themes of a unit, and anticipate students' prior knowledge. This planning phase for learning goals and activities is crucial in implementing the MRH approach.

Teacher Template
Preparation:
- Identify the big ideas of the unit.
- Make a concept map that relates sub-concepts to the big ideas.
- Consider students' prior knowledge
- Consider students' alternative conceptions during the lesson as they connect the prior knowledge to the big ideas
During the unit:
• **Students' knowledge of mathematics**
- Give students opportunity to discuss their ideas.
- Have students put their ideas on the board for exploration.
• **Teacher's knowledge of mathematics**
- Use your knowledge to identify students' alternative conceptions.
- Guide students to the big ideas identified earlier during the preparation.
• **Negotiation of ideas**
- Create small-group and whole-class discussion.
- Encourage students to reflect on each other's ideas.
• **Writing**
- Have students write about what they have learned in the unit to real audiences

(teacher, parents, classmates, lower grades, etc.).

Figure 1. The mathematics reasoning heuristic teacher template.

The second component of the MRH is for students and is intended to help them scaffold their problem solving abilities (Figure 2). It consists of a series of questions for students to consider when they are engaged in the problem solving process. The template allows students to clarify their thoughts, through writing, about how they will approach the problem. The MRH also gives students multiple opportunities to engage in problem solving activities by comparing and contrasting their solutions and writing explanation for different audiences. Kenyon (1989) emphasized the importance of writing on the thinking process during problem solving and the importance of the metacognitive skills that students use to reflect on their thoughts by declaring that within a "writing process, students begin to gather, formulate, and organize old and new knowledge, concepts, and strategies, and to synthesize this information as a new structure that becomes a part of their own knowledge network" (p. 77).

Student Template

- **What is my question (problem)?**
  - Specify what you are asked (What is (are) the question(s) being asked?)
  - Outline the information/data given (What information is/are given?)
- **What can I claim about the solution?**
  - Use complete sentences to explain how you will solve the problem.
  - Tell what procedures you can follow.
- **What did I do?**
  - What steps did I take to solve the problem?
  - Does my method (procedure) make sense? Why?
- **What are my reasons?**
  - Why did I choose the way I did?
  - How can I connect my findings to the information given in the problem?
  - How do I know that my method works?
- **What do others say?**
  - How do my ideas/solutions compared with others?
    - a. My classmates
    - b. Textbooks/Mathematicians

> • **Reflection**
>   - How have my ideas changed?
>   - Am I convinced with my solution? Why?

Figure 2. The mathematics reasoning heuristic student template.

Purpose

The purpose of this dissertation is to look at the impact of implementing student-centered, writing-embedded pedagogy on students' mathematics performances on a standardized test, the Iowa Test of Educational Development (ITED).

To this end, I analyzed the link between group (treatment vs. control), writing tasks, and students' performances on the ITED. This goal involves the investigation of any differences between test scores of students of the three teachers. In order to explore the effect of the implementation level of the MRH approach on students' test scores, each teacher was observed and videotaped during the teaching of their both control and treatment (MRH) classes.

Research Questions

This study investigated the following two questions:

- • Is there a difference in students' mathematical performance on a standardized test, the Iowa Test of Educational Development (ITED), between the students in the control classes where the teachers are engaged in their traditional instructional routines and the students in the treatment classes where the teachers used the MRH approach to improve their pedagogical practices and to scaffold students' problem solving skills?

- • How do the teachers change their pedagogical practices through the use of the MRH approach?

Investigation of each research question was presented in a journal article format.

Outline of the Dissertation

Chapter 1: Introduction

The research proposal was presented as the introduction chapter.

Chapter 2: Literature Review

In the literature review, relevant readings that lead this research was analyzed as outlined below:

I – Problem Solving in Mathematics

    a. Nature of mathematics (learning)

        i. Subjective and objective aspects of mathematical knowledge

    b. Problem solving in mathematics

        i. Process of problem solving

    c. Algebraic problem solving

II – Writing-to-learn

    a. Process of writing and writing models

    b. Writing as a learning tool

    c. Writing-to-learn in mathematics education

III – Writing as problem solving and the mathematics reasoning heuristic

    a. Writing as problem solving: Connections between writing and problem solving

    b. An innovative pedagogical approach to teaching mathematics: The mathematics reasoning heuristic (MRH)

        i. Development of the MRH

ii. The science writing heuristic as a deriving model for the MRH

Chapter 3: Journal Article

This paper examines the relationship among teachers' implementation of a student-centered, writing-embedded pedagogy and students' performances on a standardized test, the Iowa Test of Educational Development (ITED). In order to scrutinize the relationship, the teachers were observed and videotaped during the implementation of the MRH and during their control teaching and students' Iowa Test of Basic Skills (ITBS) (taken at grade 8) and ITED scores (taken at grade 9) were obtained. The intent is to include the journal article for publication in the middle of the full dissertation.

Research setting and participants

This study was conducted in a high school with three algebra teachers, two males and one female. Pseudonyms (Mike, John, and Amy) will be used for the male and female teachers, respectively. The school is located in a rural area of Iowa with a low percentage of minorities (Hispanic or African-American). The teachers have a total of ten classroom sections for an Algebra I course. The data were collected at two different times. Mike was the first teacher implementing the MRH during the 2004-2005 school year. He had three classes, one control (21 students) and two MRHs (44 students). After Mike left the school district for the following school year due to his personal reasons, Amy was placed in his position and she started to teach the Algebra I course. Therefore, the study continued with Amy and John in the 2005-2006 school year. Amy taught three classes, two of which were treatment groups (25 MRH and 24 control students). John taught four Algebra I classes, two of which were treatment groups (45 MRH and 43

control students). Therefore, of the ten classes there were four control and six treatment classrooms.

Mike had 20 years of mathematics teaching experience in different schools in Iowa, the last five years of which had been at the high school featured in this study. Mike also taught an Applied Mathematics course. During his career he has taught different levels of high school mathematics. He also coached football as an extracurricular activity in the school. John had 33 years of teaching experience, 28 years of which have been at his current school. He was the head of the mathematics department in the school and taught Probability and Statistics and Algebra I courses for ninth grade students. He also had a master's degree in English and Guidance Counseling. Amy had been teaching for five years and this was her first year in the school district. She taught Algebra I (ninth grade) and Applied Math II (tenth grade) courses.

Research framework

In this study, a mixed-research approach (qualitative and quantitative) was used to provide empirical evidence and to add more robustness to the results of the study for generalization of its findings. Fraser and Tobin (1992), Reichardt and Rallis (1994), and Smith and Heshusius (1986) argued that even though the qualitative and quantitative research traditions differ, using a mixed-method may enrich the practice of research by informing each other and overcoming each one's weaknesses. Therefore, it is reasonable to use combined research traditions in this study in order to explore the relationship between the implementation of the MRH and students' performances on the tests.

Qualitative research design

A qualitative research approach was appropriate to use in order to explore the characteristics of implementation and identify teacher's level of implementation. The practice of each teacher was analyzed individually using an interpretative case study design, and the analysis informed the implementation level. To this end, each teacher was observed and videotaped during their implementation and interviewed to add richness to the researcher's characterizing of their level of teaching in both MRH and control classrooms. To document the level of teaching, we adapted an observation protocol called the Reformed Teaching Observation Protocol (RTOP) used to measure "reformed" teaching in mathematics and science classrooms (Sawada, Piburn, Falconer, Turley, Benford, & Bloom, 2000). The focus of the observation was to identify how teachers promoted dialogical interaction by means of questioning, encouraging students to participate in the process of problem solving in the classroom, and requiring students to find different ways of solving a problem and connecting them to other problems that have been studied.

Level of Teaching

The RTOP is an observational instrument that can be used to assess the degree to which mathematics or science instruction is reformed. The instrument draws on the recommendations and standards for the teaching of mathematics and science that have been promulgated by NCTM (1989, 1991, 1995, 2000) and National Science Education Standards (NRC, 1995). The instrument consists of 25 items, with each rated on a scale from 0 (not observed) to 4 (very descriptive). The RTOP Cronbach' Alpha is 0.954 for math and science classes (Sawada et al., 2000). However, we modified the RTOP and chose 14 items (Cronbach' Alpha was 0.976) according to the following criteria.

First, creating dialogical interaction is important for implementing the MRH. In other words, types of questions asked by teacher and students, teacher's response to students' answer and questions, and the direction of communication (e.g., from teacher to student) are of the importance for creating dialogical interaction. The questions teachers ask in classrooms can either promote or limit classroom conversation. Second, focus of learning is crucial to reflecting an important step away from traditional mathematics classroom practice. The main point of this criterion is "allowing students to take the responsibility of thinking process and problem solving process and moderating the conversation." Teacher domination of classroom discussion affects not only students' sharing ideas and reflection on other students' ideas but also the participation of students in classroom discussion. Finally, teachers are expected to allow students to discover their own problem solving methods either as groups or individually rather than to provide an explanation of their own method. Moreover, students should be encouraged to find different ways of solving a problem and provide justification for their solution methods. The comparison of the fourteen RTOP items and the MRH implementation criteria can be seen in the Appendix.

Quantitative research design

A quasi-experimental design will be used as the quantitative research method. Quasi-experimental designs are often used in educational research due to the lack of randomization, the limited control of variables, and/or sometimes the lack of a control group (Campbell & Stanley, 1996; McMillan & Schumacher, 1997; Merriam, 1988). Such a design is confounded by the fact that control and treatment groups are not

random; yet, pre- and post-tests were administered in this study to both groups as assuming they were random.

The results of the qualitative analysis were used in the quantitative analysis as independent variables: teacher's level of teaching. The outcome variable was students' ITED mathematics scores. To account for possible differences prior to the study, students' previous ITBS mathematics scores were used as a covariate in the analysis.

Statistical Analysis

For social sciences, especially for education, it is hard to attribute any differences in groups to one single variable (e.g., treatment effect). Therefore, analysis of covariance (ANCOVA) was chosen as the statistical method to examine the effects of the variables on students' performance and the possible interaction of those variables as well as to control for students' prior knowledge differences (Agresti & Finlay, 1997). Possible group differences were reported as "effect sizes" (Cohen *d* index), which is widely used in social science, because it enabled us to measure "the difference between two means expressed in standard deviation units" (Sheskin, 2000, p. 835).

There are three advantages of reporting effect sizes (Wilkinson & Task Force on Statistical Inference, 1999). First, reporting effect size makes meta-analyses possible for a given report. Second, effect size reporting allows a researcher to determine more appropriate study expectations in future studies. Third, reporting and interpreting effect sizes facilitates assessment and comparison of a study's results across existing related studies.

Chapter 4: General Conclusion, Implications, and Limitations

  The last chapter of the dissertation presents a general conclusion, implications, and limitations based on the literature review. Emerging themes of the literature on mathematical problem solving, writing-to-learn, and the impact of the level of teaching on students' learning outcomes are outlined. Second, findings of the study, their implications to practice, and their relation to current literature are discussed. Then, surrounding limitations of the study are delineated.

CHAPTER TWO

Literature Review

This literature review will address the literature to provide a theoretical

framework for the research questions. This chapter is organized in sections. In the first

part of the review, the theoretical underpinnings of the mathematics reasoning heuristic

(MRH) are discussed. Particular attention is given to problem solving in mathematics,

writing-to-learn, and the relationship between the writing process and the problem-

solving process. The literature review finally describes the MRH approach as a means to

assist teachers and students in their journey to develop robust mathematical problem

solving capabilities through writing.

Problem Solving in Mathematics Education

The National Council of Teachers of Mathematics (NCTM) places a strong

emphasis on problem solving in the Principles and Standards for School Mathematics

(NCTM, 2000). The NCTM states that problem solving is a crucial part of mathematics

learning; therefore, problem solving should be fully integrated into mathematics

programs. Because problem solving requires high cognitive and metacognitive actions,

students develop thorough understanding of the mathematical concepts in the problem

when engaged in the problem solving process. Thus, challenging students with well-

chosen problems that support their learning of mathematics, that enhance their

understanding of the nature of mathematical knowledge, and that require them to work

collaboratively in order to reach the solutions is essential in math classrooms. In this way,

students not only become aware of the nature of mathematics but also develop an

understanding of the subjective and objective aspects of mathematical knowledge. The

NCTM states that students should be able to construct their mathematical knowledge through problem solving and monitor and reflect on the process of problem solving.

Consequently, this section of the literature review focuses not only on mathematical problem solving, but also on the nature of mathematical knowledge and learning, as well as the construction of mathematical knowledge by individuals and the mathematical community including teachers and students. Taking a social constructivist perspective, I indicate how mathematical knowledge is constructed through negotiation within the mathematical community and the implications in classrooms. I later discuss the major problem-solving heuristics on which the MRH approach is based, the cognitive and metacognitive aspects of problem solving, and algebraic problem solving. Finally, the implications of problem solving in this study are discussed.

The Nature of Mathematics

Mathematics, by its nature, is perceived differently by different communities. For example, for mathematicians, mathematics is a process of creation and exploration which they are constantly studying to expand their repertoire of mathematical knowledge by engaging in mathematical activities (i.e., proof, refutation, counterargument) (Khait, 2005); whereas, for ordinary people or students, or even for some mathematics teachers, mathematics is "a collection of preestablished facts, rules, and techniques essentially having to do with numbers and (at best) geometric shapes" (Borasi, 1992, p. 158). When math teachers see school mathematics as nothing but a set of rules, their belief shapes their "instructional decisions about curriculum choices, teaching strategies, and classroom organization and management" (p. 157). Such beliefs isolate human factors from the creation of knowledge.

Borasi (1992) and Ernest (1998) argued that the construction of mathematical knowledge, as with any other knowledge, inherently involves the human mind. Both authors emphasized the fact that mathematical knowledge is socially and historically constructed; that is, it is "the result of one of the forms of human knowing" (Ernest, 1998, p. 47-48) and mathematics is "a *humanistic* discipline" (Borasi, 1992, p. 159, original emphasis). The humanistic view of mathematics states that culturally constructed knowledge is based on culturally situated, shared foundation, human agreement. Therefore, as Ernest pointed out, mathematical objects are the consequences of communication among mathematicians over time, based on the previously created set of knowledge. As in science (see Norris, 1992), mathematical knowledge passes through a series of iterative processes before its acceptance by the mathematical community. As I will discuss in the following section, when an individual mathematician or a group of mathematicians propose a mathematical knowledge claim via publication or conference, it has been made public for an extensive review process by other mathematicians in the field. This process mandates the owner(s) of the proposal to defend their arguments and provide logical justification for any counterarguments raised by the community. In short, since the mathematical knowledge is inherently conversational and dialectical (Ernest, 1998), it is based on a shared and agreed upon foundation.

This dialogical aspect of the mathematical knowledge is based on a coherent set of argumentation patterns supported with logical reasoning within problem solving. Through problem solving, one is engaged in setting a knowledge structure using prior schematic knowledge and the problem context. Therefore, the nature of mathematics can then be simply described as problem solving, *which is an endeavor of human minds.*

Borasi (1992) suggested that since the nature of mathematics is humanistic, students should be encouraged to appreciate its humanistic aspect. She further points out that school mathematics should strive to make students aware of the unique characteristics of mathematics and mathematical modes of thoughts. Moreover, students should also be encouraged to see that mathematics and mathematical thinking is not apart from their everyday activities. In order for students to understand the nature of mathematics and its humanistic aspects, they should be engaged in activities in which they can work together as communities of learners and thinkers who create original mathematical knowledge. Within such an environment, "mathematical communication becomes an essential way of sharing guesses and ideas, providing and using feedback constructively, and ultimately building the consensus that sanctions new knowledge" (Borasi, 1992, p. 170).

But, school mathematics, as a discipline, typically hides the dialogical aspect of mathematics under its "monological appearance" (Ernest, 1998, p. 173). Mathematics is seen as a concrete body of knowledge created by people called mathematicians and it includes no personal view or subjective knowledge. As a result, the dialogical nature of mathematics comes as a surprise to some students. Most students think that such a superhuman knowledge structure must be taught by teachers rather than created by the students themselves. Cobb, Yackel, and Wood (1993) argued that students sometimes expect the teacher to simply give instructions for them to follow. Such expectations indicate that the student believes his or her only role is to follow the procedural instructions. Students in classrooms where such situations often occur conceive a passive role in learning of mathematics and see the teacher as the active donor of the mathematical knowledge, rather than *a* source of knowledge among many sources. And

students' conceptions of mathematics play an important role. As Teppo (1998) stated, students' success in learning mathematics is significantly affected by their understanding (or misunderstanding) of the nature of mathematics.

Misunderstandings about mathematics might come from teachers' teaching styles, beliefs about the nature of mathematics learning and teaching, and behaviors in the classroom (Hart, 1989). In mathematics classrooms, students are typically expected to follow the teacher's instructions and to work on routine problems individually. They are not encouraged to create their own problem solving methods; rather, they are expected to memorize the teacher-transmitted knowledge without attempting to understand the reasons behind a method. This learning and teaching style continues during learners' school career; that is, "learners work on textual or symbolically presented teacher-set tasks" (Ernest, 1998, p. 223). Doing mathematics in school is associated with *solving word problems* whose solution method is shown by the teacher before students attempt a solution. Barnes and Todd (1995) discussed that the structure of schooling (lectures, lessons, textbooks, exams, quizzes, questioning, etc.) implies "that knowledge is made up primarily of information (*facts*) to be memorized [and]… that all that is needed is to produce the *right* answer, …[which] misleads some students about what kind of learning is required of them" (p. 14, original emphasis). Such an approach exemplifies the misunderstanding of the nature of *learning*.

Taking into account the humanistic nature of mathematics and mathematical knowledge, Borasi (1992) suggested that we should rethink learning and teaching mathematics. From a constructivist point of view, learning is associated with personal meaning making based on the already existing knowledge structure and the social

interaction with environment. Therefore, learning mathematics should no longer be considered an accumulation of teacher-set knowledge by means of listening, memorizing, and practicing. Rather, learning mathematics should be considered an active process of personal construction of mathematical understanding through use of natural language in a socially situated conversation by students. Thus, as Borasi (p. 176) pointed out, "…in the context of schooling, learning mathematics should include activities such as articulating and sharing results and interpretations, … examining collectively the soundness of arguments and explanations, and trying to reach consensus." Students should also be aware that mathematics, as a humanistic inquiry approach, involves uncertainty and, to some degree, confusion.

Consequently, teaching mathematics should be restructured around how students best learn mathematics. Borasi explained the whole notion of teaching and learning in the following:

> Good mathematics teaching should be conceived not as the "clear and efficient" transmission of established mathematical results but as the creation of a community of learners engaged collaboratively in the construction of mathematical knowledge in order to increase their understanding of the world, to solve specific problems, and to come to appreciate and expand their mathematical ability. This, in turn, will involve the development of a "rich" classroom environment, which can stimulate students to engage in humanistic inquiries about mathematics and provide the necessary support for pursuing such inquiries (Borasi, 1992, p. 181).

Indeed, teaching should provide opportunities for students to participate in dialogical interaction in which they can negotiate their meanings of mathematical concepts. Watson and Mason (2005) pointed to the fact that mathematics is a constructive activity, and, thus, learners should be actively involved in the process of the construction of mathematical objects, relations, questions, problems, and meanings.

Another key point that affects students' understanding of the nature of mathematics is that the "same" words have "different" meanings in mathematics and everyday practice. For example, the word "difference" has distinct uses. While, in one case, it refers to comparison of two objects in terms of their physical characteristics (i.e., a square table vs. a round table), in mathematics, it refers to the "quantity" between two numbers. Thus, a small child, when asked what is the difference between 10 and 5, might say "there are two numbers [digits] in 10 and one in 5." As Gee (1999) stated, the meanings of words are associated with different contexts and form a pattern specific to those contexts. In our example, the child has experienced the meaning of the word "difference" within various situations. His or her parents might have said, "What is the difference between this teddy bear and that one?" In a series of experiences with the word "difference," Gee argues, the child created a pattern associated with the word. Sierpinska (1998) argued that since the child practices the concepts in social settings, he or she has to discriminate the different uses of the concepts in different contexts when he or she is at school.

During everyday interactions, children are often exposed to informal ways of dealing with mathematics. Unfortunately, however, school mathematics seems isolated from both everyday spontaneous use of mathematics and the activity of mathematicians;

whereas, the everyday use of mathematics, to some extent, is similar to activity of professional mathematicians (Borasi, 1992). Yet, by focusing on the humanistic notion of mathematics and of the inquiry approach, students can be encouraged to see that what they do in learning mathematics is not different than what "real" mathematicians do. This will also help them "[internalize] the set of beliefs and values that belong to mathematics as a 'culture'" (Borasi, 1992, p. 170). Artzt and Thomas-Armour (1992) have found that when students work in small groups to solve mathematical problems, they approach problems like expert problem solvers (Artzt & Yaloz-Femia, 1999).

Subjective and Objective Aspects of Mathematical Knowledge

The central strand of the nature of knowledge is that knowledge is a socially constructed endeavor within a community through negotiated meaning of experiences, not static or stable, but inconstant (Ernest, 1998; Connolly, 1989). As Cobb, Yackel, and Wood (1993) and Ernest (1998) pointed out, we, as people, construct our knowledge, contextually and historically, based on our interaction with other people (or their artifacts). Therefore, knowledge is socially and historically constructed within a community through the very act of communication via different discourse tools (e.g., oral, written, etc.). Yet, even though knowledge is constructed by individuals, the collective objective knowledge lies on the shared language, experiences, and understanding, which are embedded in a social structure of community (Atweh, Bleicher, & Cooper, 1998; Cobb, Boufi, McClain, & Whiteneck, 1997; Ernest, 1998; Krummheuer, 2000). Thus, the objective knowledge and subjective knowledge are interrelated. Ernest attributed to Harding (1991) that the "concept of 'strong objectivity' [is] based on the recognition that all knowledge is thus socially constituted and that knowledge increases

in objectivity when its social roots and presuppositions are laid bare and acknowledged" (p. 147).

Yet, the question "how is objective knowledge warranted by the community?" emerges. Ernest offered an answer to the question about how mathematical knowledge is accepted within the mathematical community. Considering the fact that knowledge is socially and historically constituted, negotiation and renegotiation is the center of such a knowledge construction process. Ernest claimed that once knowledge has been externally represented, it becomes a candidate of objective knowledge. In other words, the proposed knowledge is present in public for a possible revision. Sfard (2000) also argued that mathematical objects (theorems, proofs, examples, etc.) become accessible to the mathematical community by means of communication, which might be publication or a speech at a conference. Yet, the dialectical feature of mathematical knowledge is what determines the objectivity of the knowledge. This warranting process is based on the previous knowledge structure in the community and "dialectical, socially situated interpersonal 'conversations'" (Ernest, 1998, p. 136). The relationship between the "past" and "present" knowledge structures in the mathematical community is in a cyclic revolution such that mathematicians join or participate in an already-existing mathematical discourse where they study mathematical objects created by previous mathematicians before them; thus in turn, they *recreate* the past mathematical knowledge and develop their own knowledge based on the existing structure. This cycle of development of mathematical knowledge helps not only to preserve the existing knowledge but also to expand it further. However, in order to be a part of mathematical

knowledge, the existing knowledge and the new knowledge emerging from it has to be negotiated and warranted within the mathematical community.

Mathematical knowledge, by its nature, needs proof to persuade the mathematical community, so publications are necessary for the knowledge claims to be warranted. Once the mathematical knowledge claim is published, the owner of the claim has the opportunity to defend his or her claim against the mathematical community. This process is, either written form or orally or both, through "formal dialectical conversational exchanges" (Ernest, 1998, p. 148). The participants of the mathematical community, using their subjective (personal) mathematical knowledge, make the decision on the acceptance or rejection of the proposed claim to be warranted. From a constructivist view, the formal dialectical conversational exchanges are the "genesis and warranting of objective mathematical knowledge" (p. 149).

However, Ernest (1998) also argued that the members of the community might suggest revisions in the proposal by providing a refutation, a counterexample, or a counterargument. After the modified proposal is submitted, depending on the structure of extension or modification, it is either accepted with minor or major revisions, or accepted with no revision, or fully rejected. These revisions might be the mathematical content of the proposal and might also be its sublanguage or examples. Ernest focused on the very dialectical conversational nature of the warranting process. He further argued that the objectivity of the mathematical knowledge relies on the dialectical process explained above and on the social acceptance of the knowledge. Valid mathematical knowledge is the knowledge that has survived in this dialectical process. I believe that it would be "false" to consider the objectivity discussed above as an *absolute objectivity* since the

acceptance of the mathematical knowledge involves the perspectives of the individuals in the community, who have subjective perspectives on any particular topic given. Moreover, Ernest pointed out that the positioning and relationships of mathematicians within the social institution of mathematics influence the acceptance of new mathematical knowledge since there might be personal interests or institutionalized effects on the decisions. Yet, to a large extent, decision-making depends on a logical argumentation structure in a conversational manner.

In relation to the concept of conversation or "conversational," even though there might be different definitions or interpretations for this concept, Ernest's definition appears to be a comprehensive treatment of most, if not all, of the aspects of the concept. The intended meaning of conversation is as follows:

> A conversation is a sequence of linguistic utterances or *texts* in a common language (or languages) made by a number of speakers or *authors*, who take it *in turn* to "speak" (contribute) and who respond with *further relevant contributions* to the conversation (Ernest, 1998, p. 163, emphasis added).

Thus conversation, as defined above, plays an important role in the construction of mathematical knowledge because the interlocutors involved in the act of conversation negotiate the explicated utterances through their perspectives. This negotiation is the result of the dialectical, persuasive reasoning and the social exchange aspects of the conversation (Ernest, 1998; Russell, 1983; Sfard, 2000). As Ernest (1998, p. 166) discussed, during conversation, genesis, acquisition, or justification of objective mathematical knowledge is in relation to the personal (subjective) characteristic of knowledge such that "without conversation and its feedback mechanisms, the individual

appropriation of collective knowledge cannot be conducted or validated. Likewise, the social construction and acceptance of objective mathematical knowledge cannot take place."

Cobb, Boufi, McClain, and Whitenack (1997) also argued that individuals actively construct their mathematical understandings as they participate in a collective discourse. Similarly, Krummheuer (2000) emphasized the importance of social interaction in the construction of mathematical knowledge and stated that participants of conversation (defined as above) make sense of mathematical objects created within the community by means of their individual interpretations. Ernest (1998) attributed to Mead (1964a, 1964b) that individuals take the conversation of the group into account and debate with themselves accordingly. Ernest further argued that "socially situated conversation between persons" plays an important role in the construction of the knowledge of individuals (p. 211). Thus, individuals make meaning through prolonged participation in many socially situated conversations in different contexts with different people. Personal meaning making is an individual act; therefore, individuals engage in thinking even after a long collective or public conversation. The individual private thinking is now free of collective public conversation, yet it is based on the previously socially situated conversation.

The individual appreciation and the construction of the collective mathematical knowledge,[2] and their mutual use are irrevocably interwoven because students are presented sets of the mathematical knowledge created previously, on the one hand, and yet they have to participate in the *recreation* of the mathematical knowledge to interiorize the collective knowledge (Ernest, 1998; Schwarz, Neuman, Gil, & Ilya, 2003). However,

Schwarz et al. (2003) also showed that individual students partly internalized the collectively constructed arguments. This study suggests that the "appropriation" from a socially constructed knowledge is constrained by (a) the individual's own perspective on the topic and the individual's interpretation of what has been discussed, (b) the social relationships between group members (i.e., peer effect), and (c) the counter-examples created in the argumentative activities (Yackel, 2002). Analyzing the roles of the teacher in collective argumentation, Yackel suggested that argumentation is crucial to students' learning of mathematical concepts both as a collective and an individual act. The teacher plays an important role in initiating such an argument, supporting students' arguments as they interact, and supplying supports (data, warrant, and backing) that are omitted or left implicit in arguments (Yackel, 2002).

In addition to the local construction of mathematical knowledge, we should also analyze how different "locals" or people (e.g., school mathematics or academic mathematics) are related to each other. For example, school mathematics is part of, in fact a site of, academic mathematics, mathematics constructed by mathematicians, mostly, at universities. In other words, school mathematics uses the product of the academic mathematics. Consider a mathematical knowledge claim made by a mathematician. This knowledge passes through the processes I mentioned above to be accepted as an objective knowledge. That is, individual or subjective knowledge becomes public or objective knowledge. This new mathematical knowledge created in the academic domain of mathematics is selected and recontextualized in the domain of schooling by teachers. Then, newly constructed objective knowledge is internalized and personalized by individual students by means of conversation and interaction with others

(mostly teachers) (Ernest, 1998). However, the process is not finished yet, since the individual students will reproduce the knowledge in order to be certified as being in possession of that knowledge. At the end of the cycle, there is a possibility that *an* individual from the school context pursues an advanced mathematics degree and becomes a member of the academic mathematics community and thus contributes to the field of mathematics using the acquired knowledge.

Implications of socially constructed individual knowledge

One of the intents of schooling is to transfer the sets of mathematical knowledge to students through classroom activities and to teach students to communicate the knowledge back to the teacher. However, as students enter into formalized educational settings, they are presented, somewhat, specific institutionally sanctioned mathematical knowledge. Therefore, which mathematical knowledge is presented and how it is represented is determined by the social context (e.g., school, classroom). Ernest (1998) argued that the classroom (or school) discourse (discourse of teaching) dominates and controls the discourse of learning mathematics as well as students' production of text. In the classroom, mathematical knowledge passes through a series of iterations of transformation as a result of student participation, as it does in the community of professional mathematicians—the iterative process of proofs and refutations (Borasi, 1992). In other words, the certification of students' personal knowledge of mathematics is analogous to the justification of objective knowledge in the domain of research mathematics. Yet, the teacher, as the authority of the knowledge in the classroom, mostly determines whether students' construction of mathematical knowledge is "acceptable." Learning of mathematics in schools is partly based on mathematical conversations

structured by teachers based on their own mathematical knowledge and on institutionally determined texts.

The role of the teacher in such a context, from a social constructivist point of view, is to provide opportunities for students to discuss their alternative ideas to the problems, while individual responses may not be conventional but may be valid and indicative of students' own thinking (Marshall, 2004; NCTM, 2000), so that the class as a whole community can determine which opinion(s) or solution(s) should be accepted. Yackel (2002) pointed out that the teacher's understanding of mathematical concepts and of students' mathematical perspective is important for the teacher to step into the classroom conversation to push the argumentation forward. She further suggested that by understanding "how students make sense of mathematical ideas" and "what [they] are and are not capable of making sense of," teachers can better help students build mathematical knowledge on their own (p. 439).

On the other hand, the study by Fried and Amit (2003) showed that traditional teachers sometimes emphasize the public aspect of the knowledge, which leaves the students with little room for their own exploration of mathematical ideas. They analyzed two teachers' classrooms in terms of their use of notebooks in the classroom. They found that the teachers wanted "their students' notebook to look like their own" (Fried & Amit, 2003, p. 100). They argued that such an approach constrains students' ability to reflect on mathematical concepts. Even though the authors did not mention it specifically, I argue that when students are not involved in the *recreation* of the mathematical knowledge, they have very little opportunity to internalize much, if any, of the knowledge presented by the teacher. As stated previously, the objective and subjective aspects of knowledge

by its nature are intertwined; therefore, individuals have to participate in the *construction*

or *reconstruction* of the knowledge in order to appropriate the collectively constructed

knowledge.

Classroom discussion and the negotiation process play an important role in

individual appropriation of collective knowledge. Such opportunities can be made

possible by having students argue for their perspectives publicly so that they can engage

in mathematical conversations as mathematicians do in warranting their mathematical

knowledge. The individual's knowledge of mathematics and that of others are blended in

a deliberately structured mathematical conversation, which has a life as long as individual

contribution and shared participation continues. Ernest (1998, p. 221) argued that for

school mathematics, such "continual participation in dialogue … is necessary for the

personal appropriation and internalization of mathematical knowledge," which is a

transition of "individual's personal knowledge of mathematics to be regarded as an

interiorization of collective knowledge."

Cobb, Yackel, and Wood (1993) stated that these social interactions in the

classroom can create contradictions and conflicts in children, and, in the process of

resolving these conflicts, students in turn reorganize their mathematical ways of knowing.

The relationship between individual students' learning and mutually constructed social

norms is reflexive. They inform each other such that classroom social norms are the

results of negotiation of individual perspectives (teacher's own role, students' roles, and

the nature of mathematical activity), and in turn, the individual interpretations are

constrained by the social norms collectively created by the individuals. This negotiation

process is needed to communicate effectively; that is, individual interpretations of social

situations are compatible for the purposes. This compatibility refers to "taken-as-shared" meaning (p. 26). In sum, Layzer (1989, p. 129) argued that conventional methods of teaching do not provide "a meaningful context for what is being learned" and suggested that "[mathematics] should be learned and taught not as a collection of facts, formulas, and rules but as a living language, or, more precisely, a family of living languages."

Problem Solving in Mathematics

As described in the preceding section, the nature of mathematics is simply problem solving, which is one of the major goals in mathematics education. There has been a great deal of interest in mathematics education on mathematical problem solving (Cai, Mamona-Downs, & Weber, 2005) since Polya (1945) first introduced his four-stage problem-solving heuristic, which has been the cornerstone of the heuristics thereafter (Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985; Mayer, 1992; Schoenfeld, 1983, 1985). Polya's heuristic has influenced a wide variety of mathematics communities. For example, the National Council of Teachers of Mathematics (NCTM) has placed problem solving at the center of school mathematics and its role in learning and doing mathematics. The NCTM (2000) stated that the effectiveness of school mathematics in work, school, and life "lies at the heart of problem solving" (NCTM, 2000, p. 334). Through problem solving, students can create mathematical thinking skills that will serve for them in and out of school. Similarly, Hiebert and Wearne (2003) argued that problem solving leads to deep understanding. Through problem solving, students are engaged in organizing and reorganizing their knowledge of mathematical concepts, and therefore, developing and enhancing mathematical understanding. In order

to support such understanding, students should be challenged with well-chosen problems (Kahan & Wyberg, 2003; NCTM, 2000; Schoenfeld, 1985).

Accordingly, Heibert and Wearne (2003) suggested that to promote students' deep understanding we should pose problems that require them to struggle for solutions. They further argued that mathematics should be "problematic" enough for students "to wrestle with what is mathematically challenging" (p. 6). The scholars who plead for learning and teaching mathematics through problem solving claim that problem solving is the perfect tool for students to deepen their understanding of mathematics and mathematical thinking skills and that important mathematical ideas can be embedded in problem solving tasks (Goldenberg & Walter, 2003; Hiebert & Wearne, 2003; Levasseur & Cuoco, 2003). Kahan and Wyberg (2003), Rasmussen, Yackel, and King (2003), and others have stated that teaching and learning mathematics through problem solving gives both teachers and students opportunities to make sense of mathematical ideas by appreciating the individual and collective aspects of mathematics—that is, the conversational nature of mathematical knowledge (Ernest, 1998).

Even though problem solving can be used for conceptual understanding of mathematical ideas, how this understanding occurs indeed depends on what the problem solver does while solving a problem. How does problem solving contribute to understanding? I think the answer to this question lies in the process of problem solving. So, what does the problem solver do during problem solving that leads to understanding of mathematical concepts? The following section scrutinizes the problem-solving process using different problem-solving heuristics. However, I will first discuss the major

problem-solving heuristics (or models or frameworks) that I found important for the development of the mathematics reasoning heuristic (MRH).

Problem-solving heuristics and problem-solving process

Polya's problem-solving heuristic constitutes the basis of the other heuristics created thereafter; therefore, I will first briefly discuss his four-stage heuristic before discussing the others. The heuristic consists of *understanding the problem, devising a plan, carrying out the plan, and looking back*. As discussed by Polya, and later by many others, understanding is the heart of solving the problem, and indeed of all types of learning (Lester, 1982). One first needs to understand the problem in order to solve it. Understanding is closely related to one's internal representation of the problem context (Greeno, 1977). Polya (1945, 1981) argued that to understand a problem, the problem solver must be aware of and correctly identify the data, the unknown, and the condition(s) of the problem.[3] In other words, he or she constructs a representation of the patterns of the relationships within the problem context (Greeno, 1977; Lester, 1982). During the course of understanding, one tries to extract the information (data, unknown, relation between them, etc.) from the problem statements. This stage leads the person to the *planning* phase where he or she actually finds the relationship between the data and the unknown, constructs a model for the conditions of the problem, and devises a plan for the solution. The problem solver also searches for his or her knowledge structure in order to relate the given problem to a familiar problem, methods that have worked for similar problems, or any information (definitions and theorems) that can be used for the solution of the problem. Based on the method generated for the problem, the problem solver *carries out the plan*. The execution of the plan includes verifying that each step for the

solution (algorithm) is correct. Finally, Polya suggested the problem solver *examine* the solution regarding the accuracy of the result and relate the method used in the problem and the result obtained to other problems studied before (*looking back* and checking the results and arguments and making connections to other problems).

This problem solving process has been explained by others to a certain degree by considering different cognitive and metacognitive processes involved during the solution. The definition of metacognition that all the following authors agreed on is drawn from Flavell (1976):

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data. … Metacognition refers, among others, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete [problem solving] goal or objective. (p. 232).

This definition is inclusive enough to explain the relationship between what is cognitive and what is metacognitive; that is, "cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done" (Garofalo & Lester, 1985, p. 164).

The first problem-solving heuristic is Schoenfeld's (1983, 1985), which was derived from Polya's four-stage heuristic. Schoenfeld's heuristic begins with *analysis*[4] of the problem, where the problem solver "reads" and "examines" the problem statements and identifies what is given, what is being asked, and the conditions of the problem. During analysis a person gets the feel for the problem. That is, the problem solver decides

if there is enough information given to solve the problem or if there is a need for extra information not explicitly stated in the problem but that might help to meet the subgoals. For example, if one is trying to find the cost of carpeting a room, he or she first needs to figure out the area of the room (even though it is not stated in the problem text) as a subgoal in order to get to finding the price. As in Polya's heuristic, the stages in Schoenfeld's heuristic overlap (they are not really even separate at all). *Design*, even though it is given as a separate stage, is actually a part of analysis and indeed "pervades the entire solution process" (Schoenfeld, 1985, p. 108). During the design stage, the problem solver is engaged in the construction of global argumentation about the solution and in outlining the solution process.

The bridge between the analysis and design stages is the *exploration* stage. The role of exploration is to provide feedback for the problem solver to examine the status of the solution to determine whether the subgoals are met, there is a more accessible (solvable) related problem, or there is a need for new information or reorganization of the givens and the conditions of the problem. Exploration helps the problem solver recast the problem based on what is needed for the solution. For example, one might feel that he or she does not understand the problem, then he or she goes back to *analysis* and, say, reexamines the given and the conditions of the problem; or, one might believe that he or she has made a progress, then he or she returns to *design* to restructure the plan of the solution.

After this cyclic process, the problem solver *carries out the plan* step by step by verifying the steps. Schoenfeld (1985, p. 111, original emphasis) said, "*Implementation… should (usually) be the last step in the actual problem solving*." On the other hand, the

verification stage is crucial because one can approach the problem differently or discover connections to other problems as a result of this looking-back process with a wider perspective and find mistakes or insights within specific areas at a local level.

Knowledge categories for problem solving

Besides the *heuristic* knowledge (strategies and techniques that help individuals to progress on unfamiliar or nonstandard problems), Schoenfeld (1985) also identified three different categories of knowledge and behavior that he thinks are crucial for understanding the human problem-solving behavior. These levels include (a) *resources* (knowledge that can be used by individuals to bear on a specific problem), (b) *control* (knowledge that guides the person's resource and heuristic knowledge), and (c) *belief systems* (the person's views about self, the environment, the topic, and mathematics that determine one's behavior). These levels are in fact related to one's solution attempts described above.

Schoenfeld argued that the problem solver searches for his or her repertoire of facts, procedures, and skills that help him or her attempt to solve the problem. These *resources*, depending on the degree to which the person knows or how certain about that knowledge he or she is, guide the problem solver throughout the solution. However, how and when the resources are used is related to one's decision making, which requires metacognitive skills (*control*). In other words, as Schoenfeld emphasized, *resources* (facts, procedures, and skills) and *control* (making plans, selecting subgoals, monitoring process of solution, etc.) are inherently interwoven. The key ingredient of effective control "lies in [the periodic] monitoring the state of a solution as it evolves and taking appropriate action in the light of new information" (p. 134). As this suggests, problem

solving is deeply metacognitive, for the problem solver is engaged in constructing logical argument about the solution of the problem, and he or she has to consider the cognitive actions progressing within the solution.

Another problem-solving framework that I will discuss was proposed by Garofalo and Lester (1985). Garofalo and Lester approached problem solving from a cognitive and metacognitive perspective. They argued that metacognitive knowledge (knowledge about one's cognition) plays an important role in problem solving and mathematical performance. By drawing on various literatures on metacognition, they expressed that successful mathematical performance depends not only on content knowledge but also on consciousness, regulation, and control of that knowledge. Their "cognitive-metacognitive framework," based on the work of Polya (1945), Schoenfeld (1983, 1984), Sternberg (1980, 1982), and, to some degree, Luria (1973), consists of four categories: *orientation* (one's assessing and understanding of a problem); *organization* (planning goals and subgoals and choice of actions); *execution* (monitoring and regulating behavior to carry out plans); and *verification* (evaluation of actions made in prior stages).

Garofalo and Lester's strong emphasis on metacognition pervades the four categories. They argued that the actions taken during these stages are connected to and pass through metacognitive components. Therefore, at each stage the problem solver evaluates the decisions made and the actions executed previously. The four stages are strongly related to Polya's four phases but are explicitly connected to metacognitive behaviors. Schoenfeld's categories, except for exploration, are also incorporated: reading and analysis taken together as *orientation*, planning as *organization*, implementation as *execution*, and verification as *verification*. Garofalo and Lester called for further research

to scrutinize the relationship between cognitive and metacognitive behaviors and their impact on problem solving.

This call was addressed by Artzt and Armour-Thomas (1992), who approached problem solving from a cognitive-metacognitive perspective. The Artzt and Armour-Thomas model was based on the work of Garofalo and Lester (1985), Polya (1945), and Schoenfeld (1983) with respect to problem solving, and on the work of Brown (1978), Flavell (1981), and Jacobs and Paris (1987) about metacognition. Unlike previous problem-solving models, the Artzt and Armour-Thomas model classified the problem-solving episodes observed in the small-group setting according to the type and level of processes used by the problem solver. The framework consists of the following sequence of episodes: *read, understand, analyze, explore, plan, implement, verify*, and *watch and listen*.

METACOGNITIVE BEHAVIOR

understand plan analyze	implement explore	verify

read	implement explore	verify

COGNITIVE BEHAVIOR
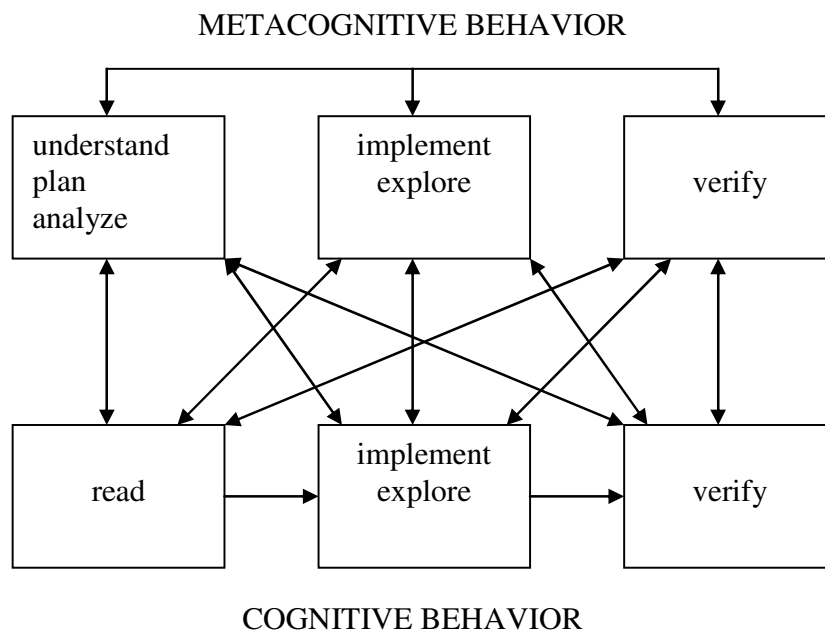
Figure 3. The Artzt and Armour-Thomas cognitive-metacognitive model.

Acknowledging that problem-solving behaviors can be cognitive or metacognitive depending on the purpose, they proposed *reading* as cognitive, *understanding, analyzing*, and *planning* as metacognitive, and *exploring, implementing*, and *verifying* as cognitive and metacognitive (see Figure 3). The figure illustrates that the problem solver makes various movements among episodes during solving a problem.

One of the significant findings of the work of Artzt and Armour-Thomas is that their model delineated the role of metacognitive processes in successful problem solving. They argued that the most successful problem solving groups had the highest percentages of metacognitive behaviors. This speaks to the argument that monitoring and regulating cognitive actions lie at the heart of problem solving (Garofalo & Lester, 1985; Schoenfeld, 1985). Another key contribution of the Artzt and Armour-Thomas work, supporting Schoenfeld's findings, is that students show changing problem-solving patterns by returning several times to different problem-solving episodes as needed. Furthermore, studying problem solving from a metacognitive perspective using Garofalo and Lester's framework, Stillman and Galbraith (1998) also found similar results: students used graphs and diagrams or some sort of pictorial aids during the representation of the problem in order to *organize* the data given and *clarify* the relationships between pieces of information in the problem text. They also argued that few students were *aware* of such metacognitive behaviors.

Process of problem solving

Each of the models explained above and many other problem-solving heuristics seek to explain the process of problem solving from different perspectives. In light of the frameworks that I have discussed here and by using Mayer's (1992) analysis of

mathematical problem solving, I will now discuss the process of problem solving, the types of knowledge needed, and cognitive and metacognitive behaviors used during this process.

The most common characteristic of the frameworks discussed above is the importance of understanding in order to solve a problem. As Greeno (1977, p.44) stated, "understanding is the act of constructing … a pattern of relations among concepts … [and]… its internal representation." Similarly, Mayer (1992) argued that the first step to solving a problem is "converting the words (and pictures) of the problem into an internal mental representation" (p. 459). This *problem representation* includes *translation* of the problem statements (sentences, phrases, pictures, etc.) into an internal mental representation that shows one's understanding of the meaning of the sentences (Greeno, 1977). Moreover, Greeno (1977), Hayes and Simon (1977), and Mayer (1992) argued that the translation is a cognitive process through one's linguistic and semantic knowledge structure. Translation and representation, the key ingredients of understanding, lie at the heart of constructing the pattern of relationships among the concepts in the problem context.

Greeno specified three criteria for developing good understanding: 1. achievement of a *coherent representation*; 2. subtle *correspondence* between the internal representation and the object to be understood; 3. *connectedness* of the representation to general concepts and procedures in the person's knowledge structure. In other words, the problem solver generates a cognitive network and new relationships among the elements of the problem—*coherence*—and eventually creates a solution pattern that satisfies the conditions of the problem in a relatively direct way—*correspondence*—in which the

problem solver connects and adds the newly generated knowledge to his or her existing knowledge structure (e.g., schematic knowledge, problem space) in order to apply it to new problem situations—*connectedness*. Such understanding relies on one's background knowledge of language, the concepts in the problem, and the schema.

Furthermore, Goldin (1987, p. 61), Greeno (1977), Janvier (1987), and Mayer (1992) discussed that one's *linguistic knowledge* plays an important role in understanding the problem because problem solvers "interpret the grammar of sentences and associate words and phrases with other words and phrases." This interpretation allows individuals to comprehend the relationships in the problem text that connect the words in a meaningful way in which their representation closely corresponds to the object that is understood. In this case, not only is the natural linguistic knowledge of importance but mathematical language is also crucial to create a representation of the problem situation. For example, to solve a problem about the amount of carpet used for a rectangular room one should know the meaning of "carpeting" as well as of "area," what they refer to and how they are used in everyday and mathematical language. Mayer also argued that the problem solver needs to have *semantic knowledge* (facts and resources about the world, such as 1 meter equals 100 centimeters) and *schematic knowledge* (knowledge of problem types) for the representation of the problem. These knowledge structures include, for example, understanding *relational statements* (which express a numerical relation between two variables), metric systems or geometric shapes, and the fact that area problems for rectangular shapes are based on the formula area=length *x* width. As the problem solver reads the problem and engages in the representation of the relationships in the context, he or she activates various knowledge structures relevant to

the problem as needed (Pape & Wang, 2003). However, constructing this representation is related to the linguistic forms of the problem and the problem solvers' existing schematic knowledge structures (Cummins, Kintsch, Reusser, & Weimer, 1988, in Mayer, 1992).

Mayer (1992) argued that schematic knowledge helps to distinguish relevant from irrelevant information and to activate necessary knowledge. During problem representation, the problem solver puts all the information together into a coherent structure. If students do not have appropriate schematic knowledge, they struggle to represent the problem. As Greeno (1977), Hayes and Simon (1977), and Schoenfeld (1985) expressed, one has to decode the problem text using linguistic, semantic, and schematic knowledge structures and "reconstruct" the relationships using the same knowledge networks. This representation is a process of translating the problem text to a set of problem solving operators using the knowledge structures mentioned above. Once a representation is established, the amount of searching for information is reduced (Lester, 1982; Stillman & Galbraith, 1998); thus, the problem solver focuses on the solution of the problem.

In the frameworks described in the preceding paragraphs, the entire *problem representation* was broken into two or three subprocesses (e.g., reading, understanding, planning). As discussed by several authors (e.g., Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985; Mayer, 1992), metacognitive knowledge plays an important role not only in constructing such a representation but also in the *problem solution* phase of the process. Problem solution (e.g., implementation and verification) involves the

"planning and monitoring" and "execution" subprocesses intertwined with each other (Mayer, 1992).

Since problem solvers move between different problem-solving episodes (Artzt & Thomas-Armour, 1992; Artzt & Yaloz-Femia, 1999; Garofalo & Lester, 1985), planning is already a part of problem representation. During planning, the problem solver outlines appropriate and necessary problem solving operators using his or her strategic knowledge. For example, for solving a linear equation, a problem solver might devise a plan such as rearranging the equation so that the unknowns are on one side and the numbers on the other. Mayer (1992) argued that this strategy might depend on the nature of the problem and the characteristics of the problem solver, and different strategies may lead the problem solver to different problem solution patterns. For instance, consider the following equation, $9 (x + 40) = 5 (x + 40)$ (Mayer, 1992, p. 475). Besides a typical strategy for solving this algebraic equation such as isolation, one might think, "when would multiplying the same number by 9 and 5 be equal? The answer is when it is *zero*; thus, $x + 40$ must equal to *zero*. Therefore, x is –40."

This solution pattern, for example, is related to the problem solver's schematic and semantic knowledge. He or she may have known that the equation cannot be simplified by dividing both sides by $x + 40$ because $x + 40$ may be zero, which conflicts with "division by zero." As pointed out by many scholars, problems solvers must know how and when to apply these strategies. This metacognitive characteristic of problem solving indeed pervades the entire solution process. The problem solver plans, monitors, and controls the problem solution by reflecting on the algorithms, and the regulation of the problem solution allows the problem solver to constantly check the conditions of the

problem. Accordingly, Mayer (1998) argued that metacognition helps the problem solver

cope with and orchestrate the cognitive and motivational components of problem solving.

Throughout the course of the problem-solving endeavor the solver often sets and

implements subgoals until he or she gets to the main goal by monitoring the process.

Expert Problem Solver

Problem Text	Problem Representation	Problem Solution

Novice Problem Solver
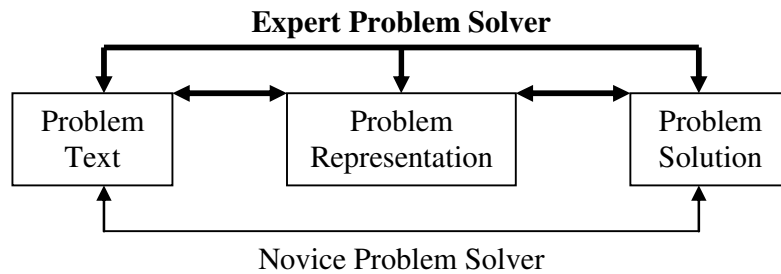
Figure 4. Difference between expert and novice problem solvers

However, the problem-solving process reveals differentiation between experts and

novices in terms of cognitive and metacognitive behaviors (Artzt & Thomas-Armour,

1992; Artzt & Yaloz-Femia, 1999; Mayer, 1998). For example, expert problem solvers

create a well-structured internal representation of the problem given by means of their

semantic, schematic, and strategic knowledge networks. They spend more time on a

representation that allows them to understand the relationship among the concepts in the

problem and to set subgoals to achieve the main goals (Lester, Garofalo, & Kroll, 1989;

Schoenfeld, 1985; Sternberg, 1999). On the other hand, novices often directly begin to

solve the problem without forming a representation that scaffolds their solution processes

(Pape & Wang, 2003; Sternberg, 1999). As depicted in Figure 4, novice problem solvers

bypass the problem representation phase or spend less time on constructing mental

representation of the problem. Pape and Wang further argued that the missing problem-

solving behavior of novices is that they do not record or transform their reading for an

image; rather, they repeatedly read and carry out the computations without referring to the text of the problem. Furthermore, Sternberg stated that students often do not monitor their problem solving process and evaluate whether their answers even make sense and that this leads to mistakes. Moreover, another difference between expert and novice problem solvers is that experts transfer what they learn from one problem to another problem situation in order to form a schema or expand their schemas (Steele & Johanning, 2004); whereas, novices see the problem solution as separate and isolated from other problem types and even similar problems.

Algebraic Problem Solving

   Rationale for focus on algebra to study the MRH. Algebraic thinking is crucial for developing thorough mathematical habits of mind (NCTM, 1989, 2000). Algebra has been the stumbling-block for many high school students, who have difficulty in making the transition from arithmetic to algebra (Brenner et al., 1997) due to the lack of expertise in problem representation, which involves understanding word problems. Research in high school mathematics has been limited to problem-solving tasks prepared by researchers (Nesher, Hershkovitz, & Novotna, 2003); few studies have looked at students' performances on state-mandated standardized tests, such as Iowa Test of Basic Skills. Nesher et al. stated that teachers often fail to pass on their expert knowledge in solving algebra word problems to students because they focus on teaching skills for applying pre-established problem-solving methods rather than asking students to develop their own.

   The nature of problem solving described above does by no means differ among different branches of mathematics, such as algebra. In other words, the problem-solving

process given above was not characterized by considering a specific domain in mathematics. This general process can, however, be applied to a particular branch, by still keeping the very nature of problem solving. The deduction helps researchers and teachers understand the applications of the general in the particular, and, in turn, helps them make generalizations about problem solving and, on the whole, about learning mathematics. Realizing the importance of algebraic thinking and reasoning to advanced mathematics and other academic courses, educators and researchers have agreed on the inclusion of algebra in early grades and called for studies experimenting in learning and teaching algebra in schools (Greenes & Findell, 1999; Steele & Johanning, 2004).

Since algebra requires an abstract level of thinking, most students have difficulties connecting the studies of arithmetic that they learned during elementary and middle school to algebra (Nesher et al., 2003; Romberg, Carpenter, & Fennema, 1993). Brenner et al. (1997, p. 664) argued that "prealgebra courses focus mainly on symbol manipulation skills, such as how to solve equations but do not emphasize the underlying problem representation skills, such as understanding what a word problem means." Even though algebraic thinking is a part of every level of school mathematics, it has been considered by teachers as a separate and isolated school subject because of its language. Swafford and Langrall (2000) defined algebraic thinking as "the ability to operate on an *unknown quantity as if the quantity was known* [emphasis added], in contrast to arithmetic reasoning which involves operations on known quantities" (p. 2). This unknown quantity is represented with symbols in different forms (e.g., x, y, $f(x)=3x + 1$), which creates an unfamiliar language system for novice students (Layzer, 1989; Nesher et al., 2003).

Greenes and Findell (1999) have proposed that the big ideas of algebraic thinking are deductive and inductive reasoning, representation, equality, variable, function, and proportion. These "concepts" are interrelated to each other and based on arithmetical relationships. For example, inductive and deductive reasoning are related to recognizing number patterns and forming the relationship among those cases using symbols. It is important for developing algebraic thinking to be able to move between these concepts.

For example, to determine the function between two sets of numbers in a word problem, a learner must simultaneously use his or her understanding of the concept of representation, variable, and function. Brenner et al. (1997) and Nesher et al. (2003) have supported the interrelated nature of these concepts, pointing out that symbol manipulation *and* problem representation (which involves translating the text of the problem into algebraic representation) are important for successful algebraic story problem solving.

Students find story problems difficult at both the arithmetic and algebraic levels according to Koedinger and Nathan (2004). Referencing other studies they conducted, Koedinger and Nathan also stated that there is a common belief among mathematics teachers and educators that story problems are difficult for students. Koedinger and Nathan also argued that this difficulty is related to the phases of problem solving—that errors in the stages account for word problems difficulties. However, studying high school students, Koedinger and Nathan also investigated the symbolic, situation, and verbal facilitation hypotheses and found that symbolic problems can be more difficult for students than story problems and that students do not always use equations to solve story problems. They also found that students have difficulty in equation solving due to errors in the comprehension phase as well as in the solution process. Similarly, Brenner et al.

(1997) found that students have difficulty transitioning from arithmetic to algebra. This may be related to the contradiction between the language forms of the problem text and the cognitive map of the problem solver.

To this end, Layzer (1989) suggested that prealgebra students have to learn the language of algebra (the syntax of arithmetic) like the syntax of English, and transform arithmetic statements. Mayer (1992) suggested that students have difficulty because the linguistic forms of the problem text do not match with their existing forms of concepts and language (Nesher et al., 2003). Therefore, this mismatch during the translation phase causes the problem solver to be unsuccessful in solving the problem. For example, consider the following statement Mayer (1992, p. 461) used: "There are 6 times as many students as professors at this university." He said that one-third of the college students made mistakes in this problem by producing the relational statement "6S=P." This mistake results from difficulty in comprehending relational statements; the representation of word problems is not quite a literal translation of the sentences in the problem. Thus, the problem solver has to consider the relationships among the objects of the problem. Kieran (1993) stated that representing such functional relations in words problems is often the struggle that middle-school students face. Creating a coherent representation of the statements is to some degree related to one's knowledge of problem type—schemas. Mayer affirmed, "the failure to solve word problems may be caused by lack of appropriate schemas rather than poor arithmetic or logical skills" (p. 468).

As stated above, expert problem solvers develop a schema or expand their existing schemas during problem solving. Steele and Johanning (2004) attributing to Marshall (1995), among others, suggested that schematic knowledge is important to

algebraic problem solving. A schema helps problem solvers create a web of related concepts around which they organize subsequent experiences. Moreover, a schema allows individuals to connect a particular experience (e.g., consistent pattern of numbers) to a more general cognitive structure. Steele and Johanning stated that mathematical problem solving and algebraic thinking can be connected to schemas via the notion of generalization. They argued that during problem solving schemas are activated and eventually generalized by their usage in a wider scale of related experiences. As Simon (2006) argued, learners assimilate their experiences into their existing knowledge structure by making logical connections among the concepts. This is the notion of problem solving that Greeno (1977) emphasized—*connectedness*.

Writing-To-Learn

Writing to learn has grown in importance in education and educational research since James Britton and his colleagues' work on language in the 1960s and Janet Emig's landmark article "Writing as a Mode of Learning." With the writing across the curriculum movement in 1970's, the emphasis on teaching and learning writing has shifted from a focus on mechanical aspects to an emphasis on using writing as a learning tool for construction of meaning. Britton and his colleagues proposed three functional types of (written) language: transactional, poetic, and expressive (Britton, 1970; Britton, Burgess, Martin, McLeod, & Rosen, 1975). According to Britton and his colleagues, we not only use language to say what we want to say (instrumental function— transactional writing) or to create beautiful objects (poetic writing) but we also use language to explore ideas and reflect on the meaning of our actions (self-reflection— expressive writing). Arguing that writing can be used as a learning tool across disciplines, Zinsser (1988, p.

56) emphasized the importance of using different types of writing in the classroom because the varying types of writing are all "equally valid and useful." He suggested that the role of writing should not be underestimated, especially, when one is exploring what he or she means: "Meaning, in fact, doesn't exist until a writer goes looking for it" (Zinsser, 1988, p. 57).

Distinguished from other types of writing, expressive writing, which is related to what Vygotsky called "inner speech," has been viewed as the root of understanding the process of writing and its function in learning (Bazerman et al., 2005). In expressive writing the intent is for the writer to explore his or her current ideas and feelings about a problem and reflect on them; in turn, exploration and reflection promotes one's personal understanding (Miller, 1997).

Similarly, Emig (1977, p. 122), who saw writing as "a unique mode of learning," indicated that due to its demanding functional cognitive action, writing is a powerful source of thought. Immediate connections between what has been written (product) and what is still to be said (process) are made by means of lexical, syntactic, and rhetorical devices and create new meaning. Building on Vygotsky's ideas about writing, learning, and inner speech, Emig proposed that writing is a tool for originating and constructing a unique verbal structure that is graphically recorded (symbolization of sound in written signs). In a similar vein, Galbraith (1999) defined writing as generating new ideas and new meanings via an ongoing dialogue between written product (text) and one's thoughts. He argued the writers engage in ongoing negotiation of meaning in a dialectical disposition through self-dialogue.

In addition, one of the functions of writing is to augment understanding through the cognitive and metacognitive actions that writing demands. During writing one negotiates meaning, and in negotiation, one is constructing knowledge and augmenting learning (Powell & Lopez, 1989). As Zinsser (1988, p. 49) suggested, "Writing is a tool that enables people in every discipline to wrestle with facts and ideas." One way writers create new meaning is through the interactive process between the writer's written product and his or her own thoughts. Writing is not only a text production, a reflection of thought, but also a learning tool that provides the writer a "conversational partner" for a constant social interaction (Bereiter & Scardamalia, 1987; Galbraith & Torrance, 1999). During this process the writer reflects on his or her own meanings, which leads to new learning. Accordingly, Galbraith and Torrance argued that generating new ideas is a part of this writing process. From the point of view of cognitive theory, the writer engages in modifying existing knowledge while writing and this process helps the writer access the knowledge network (Pittard, 1999). The activation of the necessary knowledge structure is related to the connection between the individual's knowledge and the topic. In this sense, Galbraith and Torrance argued, writers generate the content knowledge in the course of writing process. However, different writing models explain the writing process and the generation of ideas differently. The commonalities among three of these models are detailed in the next section, then each of the models are explored in detail in separate sections, focusing on how each of the models' contribute to the assumptions of the MRH.

Process of Writing and Writing Models

Among the models that explain the writing process and its role in learning are the models of Hayes and Flower (1980), Bereiter and Scardamalia (1987), and Galbraith

(1999). The common view among these models is that writing is a discovery process of what to say in the course of action and a heuristic for learning; that is, writing facilitates learning (Alamargot & Chanquoy, 2001; Langer & Applebee, 1987). The models also agree that in the process (of writing) the writer produces the text due to a problem (e.g., writing assignment) that will be resolved through a series of activations of knowledge structures and interaction of rhetorical goals with the problem statements (Galbraith, 1999).

Another viewpoint that these models share is about the difference between expert and novice writers. The major difference between the two types of writers is that "experts develop an elaborate set of goals for their text and generate ideas to satisfy these goals, whereas novices simply retrieve ideas prompted spontaneously by the topic and translate them directly into text" (Galbraith, 1999, p. 139). In other words, writing processes are not the same for all writers. For example, inexperienced writers generally use knowledge-telling processes as a means to reduce the cognitive load, while expert writers develop complex sets of writing strategies, which requires reconstruction and transformation of knowledge structure (Van der Hoeven, 1999). This difference between the two groups of writers is also related to the writers' linguistic skills as inexperienced writers have difficulties generating new ideas due to this lack of linguistic knowledge (Bereiter & Scardamalia, 1987; Van der Hoeven, 1999). Studying differences in writing performance, Van der Hoeven found that even though inexperienced writers' writing process could not be simply described as knowledge telling, there is a gradual transition from knowledge telling to knowledge transforming, with a continuous two-way interaction between developing knowledge and developing text.

In the following section I will outline how these common aspects of the writing models inform the assumptions of the mathematics reasoning heuristic (MRH), the tool created by the author that will be the focus of this paper.

As stated above, the three basic processes proposed by Hayes and Flower (1980) are, to some extent, common to all the writing models; that is, planning, translating plans into text, and revising. These three processes operate upon two kinds of information: (1) a representation of the task environment consisting of writing assignment and the text produced and (2) the knowledge stored in the long-term memory consisting of topic knowledge, linguistic knowledge, audience, and writing plan (Galbraith & Torrance, 1999). Of course, the working progress of these knowledge structures and how they are processed depends on the writer's experience in writing and the strength of the connections among those knowledge structures. These three processes (planning, translating, and revising) will be elaborated on within each of the other two models in the following sections.

Model 1: Knowledge-Telling vs. Knowledge-Transforming

Among the attempts to explain the complex process of writing, Bereiter and Scardamalia (1987) connected the writing activity to two distinguishable processes. First, the writing process can be "explainable within a 'psychology of the natural,'" which takes the position that human language competence and skill is the product of social experience (p. 5). They called this way of writing "knowledge-telling." The second way of explaining the writing process is through psychology of the problematic; that is, the person has to reprocess the knowledge learned through social interaction beyond normal linguistic abilities. Such a writing model is called "knowledge-transforming."

Knowledge-telling and knowledge-transforming will be detailed in sections after a discussion of their interrelationships.

Bereiter and Scardamalia (1987), using Vygotsky's idea of language efficacy in children, gave a temporal progress for children's writing development: from conversation to knowledge-telling to knowledge-transforming. This progress is related to the experiences of the child in writing as well as his/her age. Bereiter and Scardamalia argued that the knowledge generated in each level of progress and how it is generated varies depending on the availability of information and the relevancy of the topic to the knowledge structure in the long-term memory. Therefore, it is legitimate to say that since novice writers have less experience in language activities, they have to pay more attention to the rules of language during writing, which reduces their knowledge generating process. In contrast, expert writers do not have to struggle with basic language skills (e.g., correctly using grammatical rules); therefore, they develop specific writing strategies by reprocessing and transforming the knowledge available (Bereiter & Scardamalia, 1987; Van der Hoeven, 1999).

Knowledge-telling model

In the knowledge-telling model, the writing process starts with a writing assignment or a self-chosen writing task. Knowledge telling is a "think-say" process; that is, knowledge is activated and retrieved by the topic and discourse cues to be translated into text. In the knowledge-telling process, the writer tends to retrieve and write down all the information he or she possesses activated by the text already written. Bereiter and Scardamalia (1987, p. 22) expressed that from the perspective of the knowledge-telling model, the writer produces text so that someone else can understand:

Writing is a matter of conveying a selection of this knowledge [one already has] to someone else. It is absolutely about asking questions like, "Will my sentence make sense to others?" "Is it clear enough for someone who doesn't know anything about the subject?"

In another related approach to discussing writing to learn across disciplines, Zinsser (1988) generalized two kinds of writing: explanatory writing, which is similar to knowledge telling with the purpose of explanatory writing as informing people and "it has no deeper content that the writer will discover in the act of writing" (p. 56). The other type of writing, exploratory writing, enables writers to discover what they want to say. Similarly, Keys (1999) argued that in the knowledge-telling model, no generation of new knowledge takes place because the writer only uses already established connections between knowledge structures (topic knowledge, semantic knowledge, and discourse knowledge).

The diagram in Figure 5 shows that the knowledge-telling process begins with the mental representation of the topic created from cues extracted from the writing assignment by means of the topic identifier and the genre identifier. For example, Bereiter and Scardamalia gave the example of an assignment to write an essay on "whether boys and girls should play in the same sports team" (p. 7). In this example, they suggested that *amateur sports* and *sexual equality* might be topic identifiers and the structure of the sentence might be a cue for a genre identifier, an *opinion* or *argument essay*. Therefore, the information retrieved by means of the identifiers is more likely used in the composition depending on the relevance to the topic and the structure of the composition. The writer, in this process, seeks cues in the long-term memory for the

appropriateness of the information with respect to the topic, genre, and the text already produced before he or she actually writes it down.
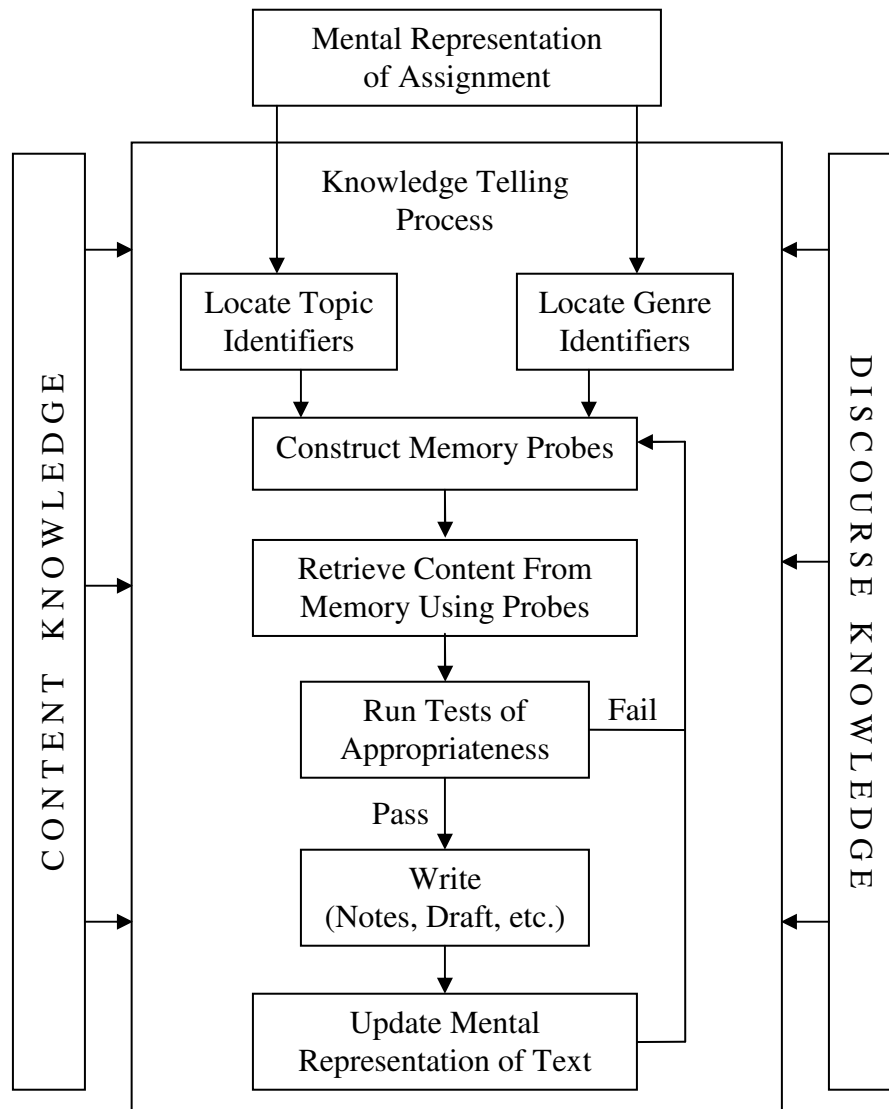
Figure 5. Structure of the knowledge-telling model. Adapted from Bereiter and Scardamalia (1987, p.8)

Knowledge-transforming model

As Bereiter and Scardamalia described, knowledge-transforming goes beyond knowledge-telling because of the cognitive and metacognitive demands that a writer

encounters during the process of writing. In the knowledge-transforming process, the writer takes into account rethinking and restating what he or she wants to say, which helps develop new knowledge. There is a cyclic process between the content problem space and rhetorical problem space that forces the writer to clarify what to say. This course of action leads to two distinct but inseparable processes—knowledge processing and text processing—and with these two processes together, the writer makes modifications in the text and in his or her knowledge/thoughts. The distinctive feature of the model is that knowledge transforming is a process of formulating and solving problems within the interaction between continuously developing text and knowledge. Furthermore, in the knowledge-transforming model, writers restructure their knowledge of the subject through translation of ideas mediated by active problem solving. The reformulated knowledge leads to a deeper understanding.

This writing process is sketched in Figure 6. According to the model in the figure, the problem analysis of the writing assignment and composition planning take place in two kinds of problem spaces. The content problem space consists of knowledge states that can be characterized as opinions, moral decisions, inferences about matters of fact, causal explanations, and so on. It is the kind of space in which writers work out "what to say." The other type of space is the rhetorical problem space in which writers deal with the problems of "how to say." Writers create a mental representation of "what to say" from their content space, and once the mental representations are transferred into text, which is what is being said, the representations are converted back to the content space for clarification of "what do I mean?" (p. 303). Bereiter and Scardamalia argued that, for an expert writer, these two problem spaces operate together to feed into each other with

their products; that is, the experienced writer "…carries on a two-way process of information transfer, which results in the joint evolution of the composition and the writer's understanding of what he or she is trying to say…" (p. 304). This two-way interactional process in the act of writing leads to growth in both knowledge and text.

Mental Representation of Assignment

Problem Analysis And Goal Setting

Content Knowledge

Discourse Knowledge

Content Problem **S**pace (What do I mean?)

Rhetorical **P**roblem **S**pace (What do I say?)

Problem Translation

Problem Translation

Knowledge Telling Process
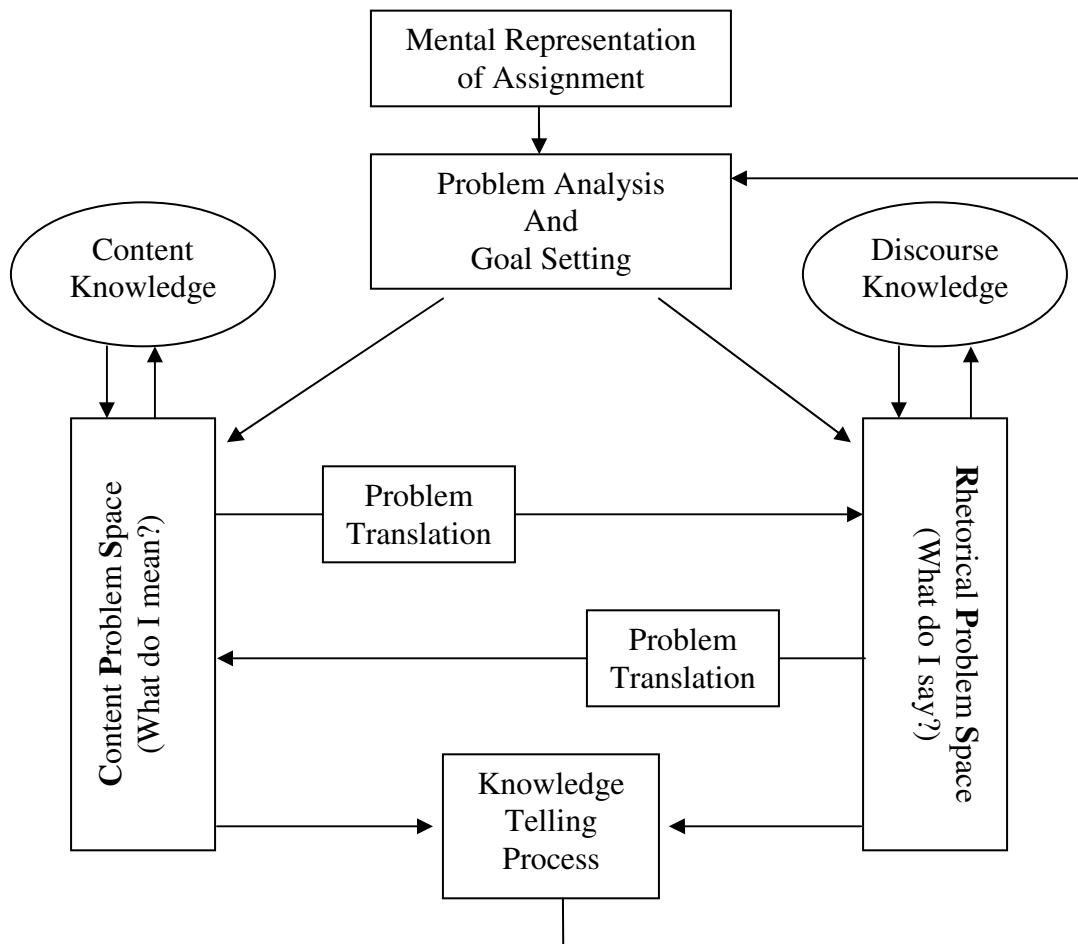
Figure 6. Structure of knowledge-transforming model. Adapted from Bereiter and Scardamalia (1987, p. 12)

As can be seen, both the knowledge telling and knowledge transforming models describe cognitive actions during writing and not the text itself. Both models also share the common idea that writers are eventually obligated to make themselves understood. Bereiter and Scardamalia highlighted that the knowledge-telling model is depicted in the

knowledge-transforming model. They pointed out that "the crucial difference between knowledge telling and knowledge transforming lies in the problem-formulating and problem-solving activities associated with the latter" (p. 299). Specifically, as explained above, there is a reflective process in the knowledge-transforming model encompassing the two problem spaces. This process is a cyclic process in which the problem spaces provide each other with inputs and outputs, and the interaction between the content problem space and the rhetorical problem space creates a reflective process in composition. Moreover, van der Hoeven (1999) confirmed that the frequency of generating activities throughout the entire writing process shows differences in knowledge telling and knowledge transforming: There is a constant curve for the knowledge-telling strategy whereas there is a decreasing curve for the knowledge-transforming strategy. Another alternative viewpoint for the cognitive actions during writing was acclaimed by Galbraith (1999), who perceives writing as a knowledge-constituting process.

Model 2: Writing as a Knowledge-Constituting Process

As an alternative account of the writing process, Galbraith (1999) proposed a dual process model, one process (roughly) equivalent to the knowledge-transforming model of Bereiter and Scardamalia (1987) and with an additional component Galbraith refers to as the knowledge-constituting process that features successive generation of ideas produced in the act of writing. Galbraith suggested that the production of new ideas during writing occurs in these two distinct processes. Moreover, he argued that in the knowledge-transforming process, writers evaluate and reorganize their existing ideas in episodic memory to satisfy the rhetorical goals. On the other hand, in the knowledge-constituting

process, writers produce a dispositional dialectic between their implicit disposition and the emerging text in order to capture and develop understanding. In other words, because writers have the disposition (or propensity), or habits of mind, to engage their own thoughts through writing, they create a conversation with themselves, an inner dialectic— what Vygotsky has successfully communicated using the terms "inner speech" or, more explicitly, "inner dialogue."
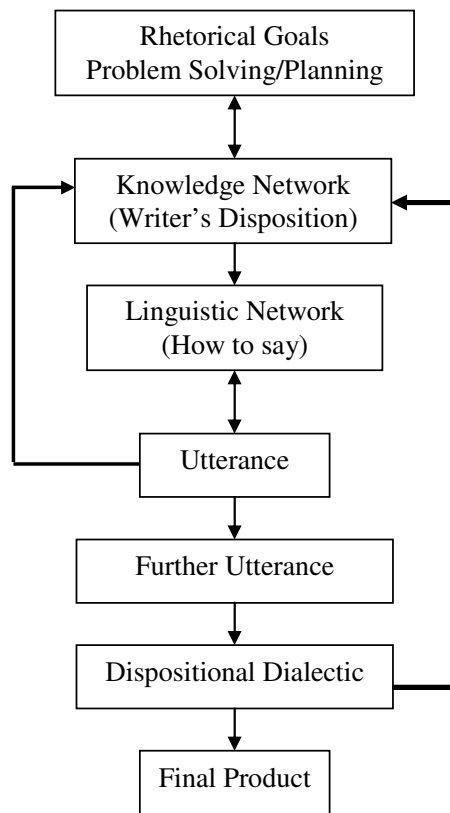
Figure 7. Simplified version of the knowledge-constituting model.

In Figure 7, I provide a simplified version of the knowledge-constituting model, which should be considered my own understanding of the model proposed by Galbraith. In the simplified figure, which is an offspring of Galbraith's original diagram in Figure 8, the writer activates his or her knowledge structure with the input coming from

topic/writing assignment. This activated knowledge structure essentially consists of connected knowledge units corresponding to individual ideas, which is used by the writer's implicit disposition and linguistic network to produce a single statement. The roles of the disposition and linguistic network are both to form a message and to articulate the message in words. The newly generated utterance becomes a new input into the writer's knowledge structure for evaluation by explicit planning and problem-solving processes, and further utterances can be produced as a result of successive cycles between the writer's disposition and linguistic network for "the best fit" to what the writer wants to say about the topic (p. 146). Galbraith further articulated that this cyclic process is actually a dispositional dialectic, which "is essentially a consequence of the tension between the amount of potential content within the writer's disposition and the amount which can be expressed in a single utterance" (pp. 148-149).

However, the dispositional dialectic is not produced quite as simply as described above. The articulation of thoughts in a dispositional dialectic depends on and is constrained by many other factors such as input, content, and linguistic knowledge. First of all, as can be seen in Figure 8, the writer's knowledge network consists of conceptually linked relationships, and the ideas are "synthesized by constraint satisfaction within this network, rather than being directly retrieved" (Galbraith, 1999, p. 143). According to the diagram in Figure 8, the writer's rhetorical goals, which include planning and problem solving, activate the knowledge network in regards to the topic. Activation of the units within the network depends on the strength of the connections among the units and the relevance of the ideas to the topic. In other words, not all ideas

are activated during this process: only ideas with strong connections to the topic and to

the previously activated ideas are activated.

B
Linguistic Propositions

A
Linguistic
C **Network**
Feedback

D

F

E

Producing
different
ideas on
successive
cycles

Knowledge Network
(Writer's Disposition)

TOPIC + TASK SPECS
(Planning and Problem Solving)
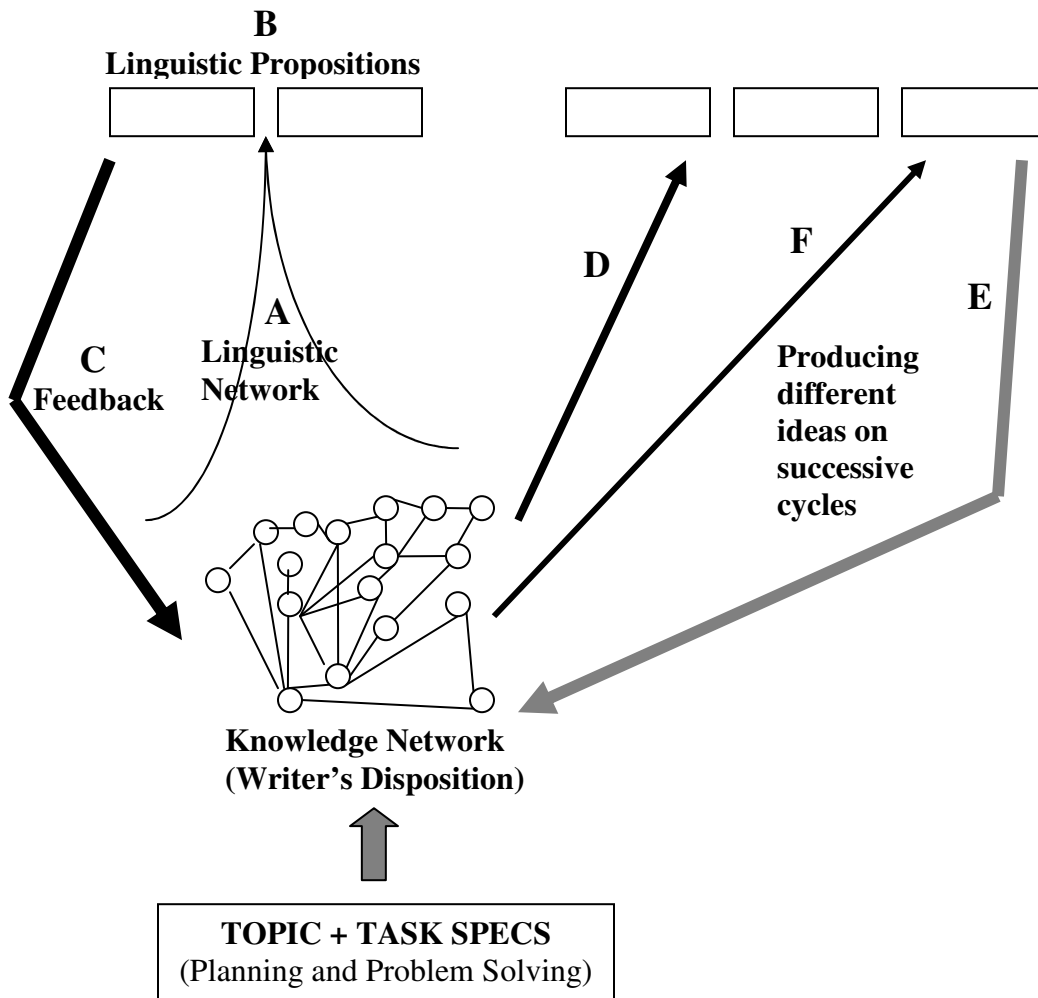
Figure 8. An illustration of the main features of the knowledge-constituting model.
Adapted from Galbraith (1999, p. 144).

Galbraith explained that this activation process is in a cycle within the units. Once

a stable message is reached, it is passed on to the linguistic network for production as a

single sentence. Generation of this single utterance is constrained by topic knowledge

(input), semantic knowledge (linguistic network), and discourse knowledge (writer's

disposition). As Galbraith suggests, there is a considerable resolution process between the

writer's disposition and linguistic network such that a single utterance is a partial

statement created by the linguistic network as a consequence of the message formulated

within the writer's disposition. Ideas with weak connections in the message are "lost out"

during this conflict resolution.

The newly generated utterance is added to the writer's disposition for evaluation

by planning and problem-solving processes. Even though further utterances can be

produced as a result of this input to the knowledge network by simply retrieving or

reorganizing existing ideas (as in knowledge telling or knowledge transforming),

Galbraith argued that one of the main features of the model is that not until the message

is transferred to a single utterance from the disposition to linguistic network does

feedback occur. Consequently, this input becomes a part of the knowledge structure in

the writer's disposition, which influences the whole disposition itself. Therefore, the

dynamic within the network will change; that is, the pattern of activation produced by

constraint satisfaction leads to the production of a new idea. As Galbraith stated, "[this

activation] enables the network to produce different ideas on *successive cycles* [emphasis

added] without requiring a change in the input from TOPIC AND TASK SPECS" (p.

147, original emphasis).

Galbraith further argued that through these successive cycles not only is a

particular utterance that represents the dominant knowledge units in the network

produced due to constraint satisfaction but also this feedback has the potential for the

writer's disposition to activate the previously suppressed content. As Galbraith (1999, p.

147) suggested, "Overall, then, the disposition's response to its input will be constituted

over a series of cycles, and will be represented by the set of interdependent utterances as

a whole, rather than by one particular utterance." Feedback also impacts the overall synthesis of an idea, which takes place between the writer's disposition, where the formation of the message occurs, and the linguistic network, where the articulation of the message in words is created. Moreover, the cyclic process also occurs within the disposition and the linguistic network as well as between them. The cycle within the disposition works for creating single or multiple *stable messages*, and the cycle within the linguistic network produces *stable utterance(s)* for those messages. During this dual process, the "lost out" ideas may be reactivated, which may produce conflicting ideas about the topic because of previously suppressed messages. In sum, the whole sequence of the composition is a result of dispositional dialectic between the writer's disposition and the produced text.

Conditions for the dispositional dialectic

As explained in the preceding paragraphs, according to the knowledge-constituting model, content can be generated through a retrieval process from episodic memory or it can be produced as a result of continuous dialogical interaction between the writer's implicit disposition and the previously generated text. However, the generation of content depends on some conditions relative to the length of the dispositional dialectic. Galbraith analyzed these conditions in two groups of factors: factors related to differences in writers' knowledge and factors related to writers' strategy for translation.

In terms of the knowledge-constituting model, the dispositional dialectic is closely related to the relationship between the potential content knowledge within the writer's disposition and the amount that can be transferred into a single utterance. This relationship can be affected by three factors: the complexity of the writer's disposition,

the range of units activated within the writer's disposition, and linguistic knowledge. Writers who have more a complex dispositional structure produce more cycles before they actually activate the units within the disposition. In addition, the strength of the connections between individual units also has a role on the extent of the dispositional dialectic. This varies though within individuals when they face different topics. Topics that activate weaker connections will result in shorter dialectic. Galbraith argued that since the dispositional dialectic occurs between the writer's disposition and linguistic network, the writer's linguistic skills are also of importance in order to express the message of the disposition in a single utterance. Linguistic competence may result in producing more content in a single utterance, therefore leading to a longer dialectic.

Galbraith suggested that the second factor that influences the length of the dispositional dialectic is the strategies that the writer uses for translation. These strategies are in relation to activation of the units in the disposition, how to express the message in language, and the writer's goals. The writer may engage in different planning stages before the act of writing, and depending on the planning activity, the range of the units activated within the writer's disposition, which is related to the strength of the connections between the units, may cause a longer dispositional dialectic. The writer may also directly translate the knowledge into text retrieved from episodic memory. Furthermore, how the content is expressed in language is important. The type of writing (prose or note) either elicits a longer dispositional dialectic because of the cycles produced for generation of a single utterance or reduces the activation of new ideas because of the labeled phrases. Finally, Galbraith argued that if the successive utterances are directed to the writer's rhetorical goals, this interrupts the dialectic; on the other hand,

if the utterances are sent into the dispositional goals, they get feedback from the

preceding utterances based on the dispositional goals and, hence, produce a longer

dialectic.

In short, Galbraith offers the following:

[T]he knowledge-constituting model enables us to describe different kinds of

translation in terms of the extent to which they involve the dispositional dialectic,

and to specify that the conditions under which the dispositional dialectic will be at

a maximum correspond to the synthetically planned, dispositionally-driven

articulation of thought in continuous prose (p. 150).

To summarize the importance of the knowledge-constituting model in the

generation of new ideas and text production, the dispositional dialectic enables the writer

to negotiate his or her own meaning through the successive feedback from the preceding

utterances to the writer's disposition. This process is two-fold. First, the preceding ideas

produce successive ideas in relation to one another; this pushes the writer to go beyond

his existing knowledge. Second, the ideas produced in the disposition and translated into

text become the sources of new knowledge generation in the writer's disposition. The

knowledge-constituting model claims that knowledge production is a consequence of the

dispositional dialectic that occurs between the writer's disposition and the linguistic

network by means of simultaneous activation of the units with the strongest connections

to the topic without pre-planning and without rhetorical evaluation.

Writing as a Learning Tool

The concept of writing as a powerful tool for learning is now widely accepted.

Writing has been considered a powerful learning tool not only because it provides writers

with instant feedback about his or her thoughts but also because it helps writers negotiate their own meaning in a dialogue: dialogue between the writer's disposition and the emerging text (Galbraith, 1999). Galbraith argued that this dispositional dialectic is a knowledge constituting process through cognitive and metacognitive actions such as planning, monitoring, reviewing, and revising.

Many scholars (Galbraith, 1999; Hand & Prain, 1996; Powell & Lopez, 1989; Pugalee, 2004) assert that writing is a tool not only for extracting what one knows about a subject but also for generating new ideas in the course of negotiation of meaning and, therefore, for augmenting learning. This generation of new ideas is, though, connected to the activation of knowledge structure by the topic. In this sense, writing is closely and mutually related to the structure of the knowledge network. That is, since knowledge is conceptually stored within a web of meanings (Simon, 1995), the quality and length of writing is related to this knowledge network structure. In turn, writing, as a thinking device, helps the writers organize their thoughts, activate relevant knowledge networks, and enhance/expand their knowledge network by reorganizing the connections between units of information in the network. Moreover, during writing the writer must engage in a "deliberate structuring of the web of meaning" (Vygotsky, 1986, p. 182). The relationship between writing and knowledge generation is mutual because emerging text eventually becomes a part of the knowledge network. Emig (1977) reinforced the connective nature of writing by arguing that the process of writing inherently gives the writer the opportunity to integrate the information in the prose passage into his/her own knowledge structure.

As stated in the preceding paragraphs, writing has multiple roles in the construction and communication of knowledge, ideas, or thoughts. First of all, the writer is in a constant conversation with himself or herself during the very act of writing as if talking to an "imaginary reader." Of course, depending on the type of writing (summary, reflection, journal, or diary) and the audience (self, peers, parents, or teacher), the flow of the conversation on the paper takes its own shape. However, since the audience is not present at the time of writing, writers have to create the situation and represent it to themselves. Ernest (1998, p. 163) suggested while not an immediate conversation, "writing is much more than symbolized speech and has a life and history of its own." Furthermore, Vygotsky (1986, p. 183) argued that writing adds awareness to speech and augments the "intellectuality of the child's actions." This growth of one's intellectuality (e.g., knowledge) is related to active participation in the process of finding what to say and how to say it (Ernest, 1998). In writing, this process is the conversation (dialogical interaction) between the writer and the imaginary reader (Ernest, 1998), question and answer (Vygotsky, 1986), the disposition and the emerging text (Galbraith, 1999).

Summary

I reiterate that writing is not simply a display of what one has learned. As pointed out by many scholars (Emig, 1977; Keys, 1999; Stehney, 1990), writing is itself a unique mode of learning. Writing helps the writer crystallize and develop his or her own concept web (knowledge network structure) of the subject matter by reflecting on one's own thought process. As the proponents of writing to learn have argued, the writer is actively engaged in thinking during the very act of writing. As one writes, he or she generates successive ideas emerging from the preceding written ideas. Simply, one utterance

triggers an idea that sparks another "brilliant" or "controversial" idea. This process

continues through the writing activity. Not surprisingly, the "live" interaction between

one's thinking and the emerging text leads the person to the very heart of the negotiation

of ideas, which in turn causes changes/improvements in the person's knowledge

structure. As Zinsser (1988, p. 166) stated, after his observation of Countryman's

calculus classroom during a writing activity, "writing and thinking and learning had

merged into one process."

Implications of writing-as-a-learning tool

We need to understand that learning is not a separated/isolated act from the

context, nor is it a communal act. Two facets of this concept require consideration. First,

learning is an individual act of creating a web of knowledge structure. But, this does not

mean that social interaction has nothing to do with learning. Rather, the second facet of

learning for the individual person is in the interaction with his or her environment

(including people on the one extreme side, and the chair that he or she sits on to the other

side). Taking this view into account, providing such an environment for students where

they can experience learning by being active in the process should be, and is, a crucial

part of the education system. Barnes (1992) explored different aspects of learning in the

school classroom and argued that students should take part in the formulation of

knowledge. Of course, students' contributions in the production/creation of knowledge

are made possible by, as the proponents of the constructivist view claim, valuing the

ideas students have formed outside of the school and bring into the classroom.

In a broader sense, language (any form of language—oral, written, pictorial, etc.)

is the only tool that humankind has for "representing or symbolizing [their] experience of

the world" (Britton, 1970, p. 135) and for communicating with others—to make "someone else understand something or pass on some piece of information"—and with themselves—"to help organize [and reflect on] their thoughts" (Pimm, 1987, pp. 23-24; Norris & Phillips, 2003). However, there are *usually* two modes of communication used in the classroom, spoken and written (Ernest, 1998; Pimm, 1987); and Barnes (1992) and Pimm (1987) argued that talking and writing (as two forms of language) are both used as communication tools and learning tools. Similarly, the advocates of writing to learn have argued that talking and writing are jointly beneficial for students' learning because they both provide a powerful mirror to reflect on their thoughts.

In order for talking and writing and their connection to learning to make sense to students, the instruction should be oriented away from traditional didactic approaches (Countryman, 1992). As pointed out above, constructivist learning theories (specifically social constructivism) view learning as an individual construction of meaning as a consequence of the social interactions of the learner. Therefore, from this point of view, students should be given opportunities where they can be exposed to different perspectives from their peers and they can individually work on the ideas discussed in the classroom. Writing becomes crucial to creating such an environment where the individual students can reorganize their formerly created ideas with the ideas that have emerged from the classroom discussion.

In summary, writing alone is not fully useful unless it is supported with a talk, conversation, or discussion in the classroom. Taking the views of Gee (1999) and Pimm (1987) on language, talking and writing are considered the two critical elements for generation of thoughts because when students have the opportunity to discuss their ideas

before a writing activity, they negotiate their own meaning publicly and they then

renegotiate meaning during individual writing. As Galbraith (1999) argued, the act of

writing provides writers a unique structure in which they can engage in the dispositional

dialectic. Overall, discussion before writing scaffolds this dialectic.

Writing to Learn in Mathematics Education

Our changing world needs many more mathematically literate individuals.

Mathematical literacy can be defined, parallel to reading and writing literacy, as not only

being able to understand the fundamental notions of mathematics, develop sophisticated

mathematical models and evaluate someone else's use of numbers and mathematical

models but also being able to represent quantitative relations using algebraic reasoning

and interpret and reflect on mathematical language patterns (Jablonka, 2003; NCTM,

2000). Writing is one of the tools via which students can build mathematical literacy.

Writing-to-learn has gained favor among mathematics educators for at least the

last three decades. Starting with the writing across the curriculum movement, many

mathematics educators have begun to use writing in their classrooms in a variety of ways,

such as journaling, explaining a mathematical problem solution, or reflecting on the

mathematics being studied (Cooley, 2002; Meier & Rishel, 1998; Morgan, 1998; Powell

& Lopez, 1989). The Mathematics Association of America has included "writing to learn

in mathematics" into MAA Notes and Reports series in college mathematics (Meier &

Rishel, 1998). Other similar interest in writing in K-12 mathematics classrooms exists,

but what lacks is the attention on the roles of writing in improving content knowledge

and communicational skills given by the earlier advocates of "writing across the

curriculum" (Connolly, 1989).

Accordingly, with the 2000 *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM) has focused attention on students' conceptual understanding of mathematics suggesting students need to be actively involved in the learning process using their experiences and prior knowledge. This view has been supported with constructivist learning theories that essentially claim that the learner constructs his or her own understanding and meaning based on the interaction with his or her environment. Therefore, students are now considered active creators of mathematics rather than passive consumers (Morgan, 1998). Along with this view on learning, understanding of teaching has also been revised in mathematics classrooms. Teachers now need to provide students with a challenging and supportive classroom environment in which they can build new knowledge by engaging in exploration of mathematical ideas by themselves.

This change in the views of learning and teaching has placed students in the center of learning occurring in the classroom by altering students' roles and requiring them to be actively involved in talking and writing in mathematics classrooms (Countryman, 1992; Morgan, 1998). The NCTM mandated that students at all levels should be able to use mathematical ideas in a variety of situations. For this purpose, students must have the opportunity to discuss their ideas publicly, to reflect on their thoughts and problem solving processes, and to communicate their ideas using various modes of representation (graphical, pictorial, oral, written, etc.). Writing in mathematics was emphasized in The Principles and Standards for School Mathematics (NCTM, 2000, p. 61), which said, "Writing in mathematics can …help students consolidate their thinking…" because it requires an active involvement of learners such that they use

writing as a vehicle for learning and become the center of their own learning processes by engaging in reflection on mathematical experiences (Powell & Lopez, 1989).

Advocates of "writing-to-learn" mathematics focus on the investigative feature of writing, claiming that writing helps students learn mathematical ideas by reflecting on problem solving processes, and they state that writing helps develop understanding of mathematical language (Kenyon, 1989; Morgan, 1998). Indeed, Birken (1989) stated that writing allows students to explore the structures of mathematical language via the natural language in which they are most comfortable of expressing their ideas. In the act of writing and talking, students are in a conversation with themselves as well as with their peers such that they negotiate their meanings of mathematical concepts.

Given that language has a powerful role in the production of knowledge, writing, as a unique form of using language in generation of ideas, is a powerful tool for construction of mathematical knowledge and for discovery of mathematical ideas. Connolly (1989) stated that writing, as a learning tool in mathematics and science, is mostly used for developing students' conceptual understanding by enhancing their use of the languages of mathematics and science fields. Recent studies on writing, especially in science, have, however, focused on characteristics of writing in the construction of ideas (Klein, 2004). Klein found that setting content goals, applying moderately sophisticated writing strategies, and extensive use of content sources were crucial for learning. He further suggested that the purpose and the audience of the writing activity should be discussed with students so that they know that writing is used for learning, not for improving their communicational skills or for testing their knowledge, per se.

Burton and Morgan (2000) mentioned that students at many levels have difficulties in writing, and they may indeed struggle to explain their mathematical ideas or problem solutions. Phrases such as "I know how to do it, but I don't know how to explain it" are common for student's talking and writing in mathematics classrooms. One of the reasons for difficulties may have been that students (and teachers as well) see problem solving (mathematics) and writing (in terms of explaining it) as two separate entities, which cause them not to connect writing to the stages of problem solving. Burton and Morgan focused on the "natural language"[5] in which mathematical language is embedded. They argued that natural language is used in the construction of mathematical ideas. Since natural language is established on the social needs within the culture and is an expressive tool with which we reflect on our experience and thinking, learners (of mathematics) attach the mathematical objects to their natural language (Bereiter & Scardamalia, 1987; Burton & Morgan, 2000; Connolly, 1989; Morgan, 1998). Furthermore, Connolly put it:

> our natural language, operating as the "metadiscourse" of all our other symbol systems, from math through money, from dance to drawing, enables us to distance ourselves from, for example, our own mathematical problem solving, and reflect on our procedure, thereby making knowledge of it (p. 9).

However, writing mathematically (e.g., learning conventions of mathematical writing) needs practice in both natural and mathematical languages. Indeed, Ernest (1987) argued that the acquisition of competence in mathematical language (figural and symbolic linguistic forms) is crucial for the learning of mathematics. In order to enhance learning mathematics, Burton and Morgan (2000), Kenney (1992), Keys (1999), and

Morgan (1998) suggested that a writer's knowledge about his or her natural language and about its nature fosters (empowers) his or her writing in the mathematical genre. In other words, (knowing of) the natural language serves for students' learning of mathematical language and writing mathematically. As Birken (1989, p. 41) stated, "Writing allows students to explore the constructs of a foreign language (mathematics) using a language in which most are fluent." In considering writing and mathematics as two discourses used to communicate through writing mathematically, students need knowledge and practice in both discourses not separately but rather jointly by embedding one in another. Furthermore, Ernest (1987), Morgan (1998), and Pimm (1987) argued that students need knowledge of the symbolization of mathematics and an understanding that mathematics—like written English has its own grammar (consisting of mathematical objects, symbols, and verbs)—both of which are conventions of mathematical objects.

In relation to learning the conventions of mathematics, by drawing from different literature on writing and writing-in-mathematics, Burton and Morgan (2000, p. 431) suggested that students should have the opportunity "to practice in high-status forms of discourse." They cited that knowledge of the characteristics of the forms of discourses (natural and mathematical languages in particular) is crucial to enhancing students' learning mathematics through writing. Similarly, Adler (1999) suggested that students should "practice being users of educated discourses" (p. 51). Writing enables the writer to internalize "mathematical concepts and thus their appropriation into the learner's own words and thought processes" (John-Steiner, 1989, p. 287).

Implications of writing-to-learn in mathematics

Mathematics has long been considered a "concrete" set of pre-established knowledge consisting of facts, rules, numbers, and right or wrong answers all of its own (Borasi, 1992). Such a perception among students and teachers puts mathematics in a position where it is seen as an isolated course subject and there is no solid connection to other studies, for example, to social studies. This compartmentalized view is even true for the branches of mathematics: geometry is seen as a different area of mathematics from algebra, and students cannot make absolute connections between the branches. This perception is somewhat related to students and teachers' understanding of the overall nature of mathematics.

Borasi (1992) and Rose (1989) identified several traditional perceptions of mathematics that students have: mathematics is the discipline of certainty; the nature of mathematics is unclear; mathematics is as a body of knowledge consisting of its own symbols and right or wrong answers; and teachers have the authority of the mathematical knowledge in the classroom and teachers must transfer the set of concrete knowledge to students. Rose suggested that using writing in mathematics classrooms can create changes in students' traditional beliefs about teaching and learning mathematics toward an attitude that reflects collective construction of mathematical knowledge through active experiencing of "doing mathematics." As mentioned in the nature of mathematics section, because the uses of the "same" words (i.e., *difference*) in everyday language and in mathematics show differentiation, students often struggle to make the connection between everyday language and mathematical language. Writing helps students build their mathematical language on their everyday language, which they use to make meaning of their experiences.

Accordingly, Countryman (1992) used different writing-to-learn strategies in her mathematics classrooms with the belief that learning mathematics is doing mathematics. Indeed, doing mathematics is not simply executing an algorithm by just plugging numbers in right places in the formula. Rather, students must experience the mathematics they are studying by being actively involved in the construction of mathematical knowledge. In other words, students play in the sand to get the necessary ingredients to make a castle, rather than waiting for someone (e.g., teacher) to supply the materials for them. In this process, students fully engage in experiencing the work they are doing.

This hands-on and minds-on experience gives students opportunities to explore what they know and do not know, to claim what is true and not true, to argue for and against an idea or view, to justify and reason their claims and arguments, to question others and themselves, to raise questions about new ideas, to negotiate their own meaning of the phenomena, and, in turn, to construct and reconstruct their knowledge network structure. Countryman (1992) and others, in my opinion, have quite rightly argued that writing, with its cognitive and metacognitive demands, is an ideal activity that supports such an experience.

Despite the many benefits of writing to learn, the practice has not been easily welcomed by mathematics teachers or students at secondary and college levels. Drawing from the literature on writing to learn mathematics, Rose (1989) gives different writing tasks (summaries, reports, essays, word problems, free writing, letters, journals, etc.) to deal with/address the issues stated above. But even with all these choices for activities, Birken (1989), Countryman (1992), and Tobias (1989) identified a strong resistance or reluctance to use writing in math classrooms in which neither teachers nor students

believe that writing should be part of a mathematics course. Tobias (pp. 48-49) suggested that mathematics teachers, at least, can partition their grading accordingly: "one-third credit for a right answer, one-third credit for finding variety of ways to get to that answer, and one-third credit for an essay … in which the student would be asked to reflect on 'what makes this problem mathematically interesting.'"

Similarly, Kenyon (1989) suggested that students can write about their problem solving process and explain it to an audience. He asserted that the steps of writing and problem solving are similar; a writer and a problem solver engages in the same stages. As Hayes and Flower (1980), who see writing as problem solving, Kenyon argued that writing *is* problem solving. Pointing out the cognitive and metacognitive processes during writing, he said that writing is a means for students to engage in self-reflection on their problem-solving process, which in turn becomes more effective.

Writing as Problem Solving and Mathematics Reasoning Heuristic

This section of the literature review discusses the connections between writing and problem solving and explains the mathematics reasoning heuristic (MRH) as a pedagogical approach to teaching mathematics. The assumptions of the MRH approach have been outlined in the previous literature review sections. In short, the MRH approach assumes that supporting students' ability to develop a dispositional dialectic for writing in the context of mathematical problem solving will enhance their mathematical understanding and abilities. In the first part of this section, the ideas from the problem solving and writing literature reviews are outlined and the parallel nature of the writing process and the problem-solving process is clearly delineated. Based on this parallel nature and the previously discussed research on writing-to-learn, in the last part of this

section the MRH approach will be introduced as a pedagogical tool for teaching mathematics that guides teachers and students through the problem-solving process. Eventually, the entire process of the MRH approach becomes an internalized problem-solving heuristic for students.

Connections between Writing and Problem Solving

To recap, current research into the use of writing-to-learn strategies within classrooms has indicated that writing is an effective learning tool. As Emig (1977) suggested, writing is a unique mode of learning because of its connective nature. Building on Britton's (1970) work on language and writing, current writing models that explain the writing process and its function in learning (Bereiter & Scardamalia, 1987, Hayes & Flower, 1980; Galbraith, 1999) consider writing as a process of developing one's ideas, a tool for communicating them, and a heuristic for learning. Furthermore, Hayes and Flower stated that writing is a problem-solving activity—a matter of finding the *solution to rhetorical problems*. This view was based on the findings from research on problem solving in general that could be applied to different areas (Galbraith & Torrance, 1999).

After Britton's work and the writing across the curriculum movement, the use of writing in mathematics classrooms has increased, incorporating a range of different purposes (Gopen & Smith, 1989; Morgan, 1998; Stonewater, 2002). Even though many studies on using writing in mathematics already exist, few studies have been conducted specifically exploring the connection between the writing and mathematical problem-solving processes. For example, Bell and Bell (1985) discussed that the writing and problem-solving processes are similar: the two processes share the same thought

processes since Polya's four stages of problem solving (understanding, devising a plan, carrying out the plan, and looking back) are very similar to the stages of prewriting, planning, drafting, and revising and rewriting (Steele, 2005). Kenyon (1989) and Connolly and Vilardi (1989) also mentioned that writing can be used as a problem solving tool and as a tool for communicating students' mathematical knowledge, yet they did not outline how the writing process is similar to the problem-solving process. Therefore, the purpose of this section is to provide a theoretical framework for connecting the writing-process and problem-solving process using the preceding literature review.

I will first outline the skeleton of the rest of the discussion. Based on Galbraith's writing model and Artzt and Armour-Thomas' problem solving framework, the discussion will be centered around the following phases: input, knowledge network, planning and strategic network, sub-product, feedback, further product, verification, and final product. Note that these stages are not linear or hierarchical; rather, they are cyclic. I have used these two models because they are the most comprehensive of all existing models in their fields.

Both writing and problem-solving processes begin with an input (writing assignment or mathematical problem). The input activates the knowledge network units related to the topic (see Figure 9). The network includes one's knowledge about the topic or the problem. For example, if one is reading a mathematics problem, the problem statements activate his or her schemas (knowledge about the types of problems) and semantics (knowledge about facts). Once activated, the person moves into a how-to-process, the process in which he or she tries to find a way to use the network units or

schema knowledge in order to reach the subgoals. In this stage, the writer makes a plan

for creating an utterance using his or her linguistic knowledge, and the problem solver

develops an algorithm using his or her strategic knowledge. The planning phase is also a

part of the problem representation for the problem solver. The sentence is sent back to
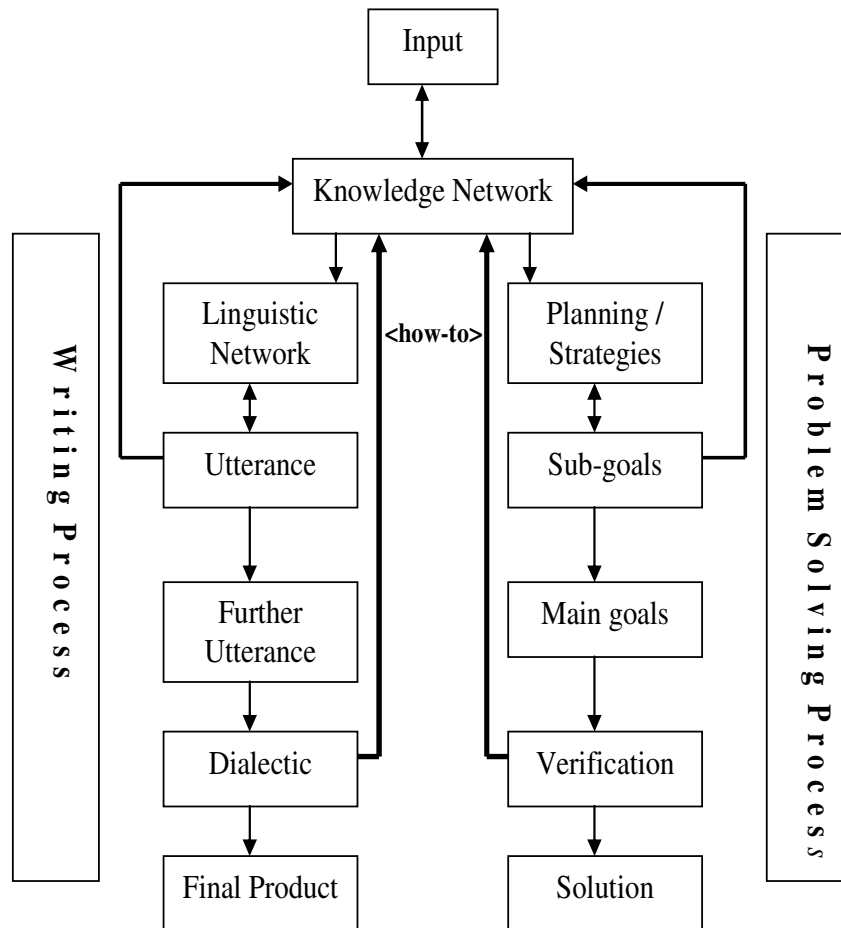
Figure 9. Parallel structure of problem-solving and writing processes.

the knowledge network for feedback to check if the sentence corresponds to what the

writer wants to say; and the problem solver controls his or her algorithms and checks to

see if the sub-solution satisfies the conditions of the problem. This verification and

feedback process continues until the sub-goals of the writer and problem solver are met.

After this local cycle process, the writer creates a further utterance that best fits to his or

her goals; and the problem solver uses the results of the local cycle to reach the main goals. For example, if the problem asks to find the cost of carpeting a room, first finding the area of the room would be a sub-goal. Thus, the problem solver uses this new information to find the cost integrating the information given back into the problem (e.g., unit price of carpet).

According to Galbraith (1999), the writer engages in a dispositional dialectic between his or her knowledge network and the produced text. Therefore, the writing process involves both local and global feedback processes. Similarly, as pointed out by Schoenfeld (1985), the person is engaged in making connection between different knowledge structures (resources, heuristics, belief system, etc.) during problem solving. For example, he or she interacts with his or her internal knowledge network in order to retrieve related information that helps to solve the problem (Vygotsky, 1978). Furthermore, the problem solver (Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985) and the writer (Van den Bergh & Rijlaarsdam, 1999) both go back and reread the assignment; however, the same reading activity serves a different purpose than the first time the student reads the assignment. Now reading the assignment is a check if the person is still on the right track in solving the problem and the writing process.

Lester (1982) argued that the problem solver interprets the language of the instructions and constructs the problem space during the representation of the problem. The representation process is for understanding the problem as well as for sanctioning the information. Greeno (1977) disputed that the problem solver creates new relationships among the components of the problem, attaining a structure that satisfies the conditions of the problem and which is eventually added in to the knowledge network of the

problem solver. This ongoing generation of problem structure helps the problem solver

activate different knowledge networks that make the connections stronger, and,

eventually, the newly established knowledge structure becomes a part of the larger

problem space.

Similarly, Galbraith (1999) articulated that during the composition process the

writer engages in a dispositional dialectic where he or she activates the knowledge

structures related to the topic and constructs an utterance using his or her linguistic

network. The newly generated utterance becomes a new input into the writer's

knowledge structure and is evaluated to determine whether it fits into what the writer

wants to say about the topic. As depicted in Figure 9, both the problem solver and the

writer engage in an internal negotiation process through both problem solving or writing,

which is what Galbraith called dispositional dialectic.

Studies exploring the relationship between writing and problem solving have

shown that writing helps students organize and monitor their problem solving behaviors

and develop their schemata knowledge (Pugalee, 2001; Steele, 2005). Steele found that

when students use writing during problem solving they use all the aspects of their

schemata knowledge and develop their mathematical structure and algebraic thinking.

Pugalee, likewise, provided evidence of metacognitive behaviors in students'

mathematical writing. These studies also show that the demanding and connective notion

of writing scaffolds students' problem solving activities and, therefore, enhances their

mathematical knowledge structure.

The difference between experts and novices also shows similarities in writing and

problem solving. For example, Galbraith (1999) stated that the writing process differs for

expert and novice writers such that "experts develop an elaborated set of goals for their text and generate ideas to satisfy these goals, whereas novices simply retrieve ideas prompted spontaneously by the topic and translate them directly into text" (p. 139). Pape and Wang (2003, p. 419) described novice and expert problem solvers similarly:

> Expert and successful problem solvers transform the problem text to form a mental model of a cognitive representation of the problem that corresponds to the problem elements and their relationships… [whereas] less successful problem solvers may not form this mental model. Rather, they often directly translate the problem elements to a solution without an image of the problem to facilitate their solution processes.

Therefore, such differences between experts and novices create differences in cognitive and metacognitive behaviors as well. Considering the fact that metacognition plays an important role in problem solving, it is crucial to point out the dual effects of writing: providing a cognitive tool for the writer to reflect on his or her thoughts and thinking process (Galbraith, 1999; Halliday, Yore, & Alvermann, 1994) and enhancing the students' problem solving abilities by acting as a monitoring tool students during their problem solving process (Artzt & Armour-Thomas, 1992; Dow & Mayer, 2004; Garofalo & Lester, 1985; Kenyon, 1989).

As articulated by Pape and Wang, cognitive (e.g., organizing and transforming, seeking information, and rehearsing and memorizing) and metacognitive (e.g., self-evaluating, goal setting and planning, keeping records and monitoring, and reviewing) strategies are very important in problem solving. Writing is a unique tool for such purposes (Kenyon, 1989; Pugalee, 2001, 2004).

Greeno (1977) said that understanding is a process of constructing internal representation of the object to be understood through interpretation of the language of the problem text. One powerful function of writing in mathematics is that it inherently pushes the translation between the problem text and problem solving operators to the forefront to help the problem solver be aware of how he or she can associate mathematical concepts with language (Galbraith, 1999; Keys, 1999; Pugalee, 2004). As Galbraith said, the writer engages in a dispositional dialectic during writing where he or she negotiates the meaning through self-dialogue (Vygotsky, 1986).

The Mathematics Reasoning Heuristic Approach

Kenyon (1989) argued that asking students to write about their problem-solving process allows them to clarify their thoughts about how they will approach the problem. Furthermore, he said, "As students write down, reflect on, and react to their thoughts and ideas, they enhance the executive problem-solving abilities, and the problem-solving process becomes more effective" (p. 77). Moreover, teachers need to develop conceptual understanding of the subject matter (Simon, 1995) in order to effectively implement writing-to-learn strategies (Hand & Prain, 2002) so that students develop mathematical habits of mind for solving problems (Driscoll, 1999; Levasseur & Cuoco, 2003). In other words, a mathematical dispositional dialectic is necessary for expert writing in the context of mathematical problem solving.

Development of the MRH

The intent of the MRH is to provide a framework for teachers to combine different aspects of mathematics teaching and learning such as students' knowledge of mathematics, teachers' knowledge of mathematics (Simon, 1995), negotiation of problem

solving methods, and embedding writing into mathematics instruction (Ernest, 1998; Galbraith, 1999; Hand & Keys, 1999; Morgan, 1998). Another intent of the MRH is to guide students' problem-solving behaviors and enhance students' problem-solving skills via the use of writing (Connolly & Vilardi, 1989; Kenyon, 1989; Morgan, 1998). Therefore, using the structure of the science writing heuristic (SWH) developed by Hand and Keys (1999), two templates were created for the teacher and students (Akkus & Hand, 2005).

The science writing heuristic (SWH) as a deriving model

The MRH has been influenced by the basic ideas of the science writing heuristic (SWH), adapting the main structure of the model. The SWH was constructed as a heuristic that links writing, reading, and science laboratory activities (Hand & Keys, 1999). The main focus of the SWH is to promote students' participation in setting their investigative laboratory work, framing their own questions, proposing methods to address these questions, and carrying out appropriate investigations. The SWH also scaffolds students' scientific thinking about the relationships between questions, evidence, and claims. It consists of two templates: the teacher template and the student template.

The teacher template provides important phases of negotiation of science knowledge created in the classroom. The teacher needs to build upon his or her students' background knowledge by creating a discussion that allows students clarify their own understanding of the concepts studied. Therefore, the SWH activities are organized so that students go through various forms of negotiation (i.e., individual writing, sharing, comparing to other resources, individual reflection).

The student template is designed to scaffold student understanding of scientific concepts while they are engaging in laboratory activities. The template allows students to think about their claims, interpret the data, and provide supporting evidence for their claims. The SWH is also based on the assumption that science laboratory activities and writing genres in school should reflect some of the characteristics of scientists' writing (Omar, 2004).

Akkus and Hand (2005), as an attempt to develop a teaching/learning approach embedding writing into problem solving, constructed the mathematics reasoning heuristic (MRH), which they consider to be an approach that links teacher promotion of negotiation of mathematical ideas, writing, and problem-solving activities. The framework for designing the MRH includes the shift to constructivist theory, teachers' conceptual understanding of mathematics, embedding writing into mathematics, and promoting problem solving.

Teacher template

The teacher template (Figure 10) provides important phases of knowledge construction in mathematics classroom that teachers need to be aware of before and during instruction. For example, the template emphasizes the importance of planning the learning goals and the major concepts of a unit. As Simon (1995) argued, a teacher's knowledge of mathematics and students' knowledge of mathematics interact in the course of learning in the classroom environment. Prior to engaging students in a learning process, the teacher has initial understanding of students' mathematical knowledge and their learning of mathematics (Simon, 1995, 1997).

According to the MRH, it is crucial that teachers are engaged in the planning phase that provides them the opportunity to determine the major concepts that they want their students to learn, identify what they know and do not know about the topic, and, importantly, align their teaching according to the notion of how students learn. As Simon emphasized, if we store knowledge as a conceptual frame, we then need to help students construct a mathematically acceptable framework. To this end, the MRH suggests that teachers consider their mathematical knowledge and students' knowledge of mathematics, which they represent during classroom interaction, and create small- and large-group discussion opportunities for students to negotiate their meanings of mathematical activities and problem solution methods.

Teacher Template

Preparation:
- Identify the big ideas of the unit.
   - Make a concept map that relates sub-concepts to the big ideas.
 	
  - Consider students' prior knowledge
 	
  - Consider students' alternative conceptions during the lesson as they connect the prior knowledge to the big ideas
 	
During the unit:
    • **Students' knowledge of mathematics**
- Give students opportunity to discuss their ideas.
- Have students put their ideas on the board for exploration.
    • **Teacher's knowledge of mathematics**
- Use your knowledge to identify students' alternative conceptions.
- Guide students to the big ideas identified earlier during the preparation.
    • **Negotiation of ideas**
- Create small-group and whole-class discussion.
- Encourage students to reflect on each other's ideas.
    • **Writing**
- Have students write about what they have learned in the unit to real audiences (teacher, parents, classmates, lower grades, etc.).

Figure 10. The mathematics reasoning heuristic teacher template.

In addition, the purpose of such classroom activities is to promote students' mathematical thinking and talking. Adler (1999) argued that students and the teacher all, to some degree, engage in classroom discourses in varying ways. She suggested that students should have opportunities "to practice being users of educated discourses"[6] (p. 51). To foster classroom discourse, teachers need to be continuously aware of students' mathematical ideas and questions. Simon (1997) argued that the dynamic of the classroom discourse depends upon the relationship between the teacher's mathematical knowledge and his or her knowledge of students' mathematical knowledge (of related concepts). However, it is important to note that the mathematics knowledge in a classroom is, in turn, based on this classroom dynamic (Ernest, 1998). The MRH facilitates both public and private aspects of knowledge construction via classroom discussions and individual writing activities.

Furthermore, the teacher template focuses on the importance of writing-to-learn strategies as one of the main foci of learning mathematics. For example, the template encourages teachers to ask students to explain the mathematics they have learned to different audiences (e.g., parents, younger students, peers, teacher, etc.). Writing-to-learn concepts suggest that the role of language and writing is to scaffold students' process of constituting thoughts and knowledge (Galbraith, 1999), as well as their representation of that knowledge externally (Connolly, 1989). Therefore, teachers are asked to provide students with the opportunities to discuss their ideas publicly before they actually write their individual understanding.

Student template

The second component of the MRH, the student template (Figure 11), is constructed to promote students' problem solving. It is a series of questions for students to consider when they are engaged in the problem solving process. The student template resembles Polya's (1945) four-stage problem-solving heuristic (understanding, planning, carrying out the plan, and looking back) or Schoenfeld's (1985) phases of problem solving (analysis, design, exploration, and implementation). However, one distinct feature of the template from all the other problem-solving frameworks is indeed that it asks students to *compare* their solutions with their peers and to *reflect on* their problem solution after classroom discussion.

Student Template

- **What is my question (problem)?**
- Specify what you are asked (What is (are) the question(s) being asked?).
- Outline the information/data given (What information is/are given?).
 - **What can I claim about the solution?**
- Use complete sentences to explain how you will solve the problem.
- Tell what procedures you can follow.
 - **What did I do?**
- What steps did I take to solve the problem?
- Does my method (procedure) make sense? Why?
 - **What are my reasons?**
- Why did I choose the way I did?
- How can I connect my findings to the information given in the problem?
- How do I know that my method works?
 - **What do others say?**
- How do my ideas/solutions compared with others?
    a. My classmates
    b. Textbooks/Mathematicians
 - **Reflection**
- How have my ideas changed?
- Am I convinced with my solution? Why?

Figure 11. The mathematics reasoning heuristic student template.

The template allows students to clarify their thoughts, through writing, about how they will approach the problem (Kenyon, 1989). This course of action may be completed

individually or in a group depending on the teacher's decision. When a teacher chooses group discussion, students would discuss the question (the question might be a word problem or any mathematical task chosen by the teacher) and negotiate possible ways of solving it within each group. Then each group shares their methods with the whole class. The student template also encourages students to think about their metacognitive actions and reflect on their thoughts (Pugalee, 2001, 2004). Kenyon (1989) emphasized the importance of writing on the thinking process during problem solving and the importance of the metacognitive skills that students use to reflect on their thoughts declaring that within a "writing process, students begin to gather, formulate, and organize old and new knowledge, concepts, and strategies, and to synthesize this information as a new structure that becomes a part of their own knowledge network" (p. 77). Pugalee (2004) showed that students who wrote about their problem solving process scored significantly higher than students who provided verbal description. This also indicates the importance of asking students to write about their problem solving process, which is why the MRH encourages students to present their understanding of mathematics to different audiences (e.g., parents, peers, and younger students).

Students using the template to structure their writing can engage in a self-negotiation of meaning through cognitive and metacognitive actions such as planning, monitoring, reviewing, and revising. Therefore, the process of text production during problem solving is an internal discussion about schema knowledge, problem-solving strategies, and rhetorical knowledge, which, Galbraith (1999) argued, are all interconnected. This internal discussion is intended to support and benefit from external classroom discussions of students' reasoning about the mathematics. Once students and

the teacher are aware of students' informal mathematical ideas, they are better able to negotiate the connections to the formalized mathematical language. The MRH encourages teachers to embed mathematical language into mathematical activities rather than giving isolated definitions of mathematics concepts. Sternberg (1999) emphasized the importance of asking students to identify the nature of the problem to be solved to help them improve their mathematical reasoning abilities. This also gives students opportunities to transfer the method learned during problem solving to other situations.

Summary

The literature review of the dissertation traces a wide range of the literature on mathematical problem solving, the nature of mathematics, the construction of mathematical knowledge, writing-to-learn strategies in mathematics, and the relationship between the processes of mathematical problem solving and writing. The studies on problem solving point out the importance of integrating problem solving into mathematics programs to promote students' conceptual understanding of the mathematics they study (Hiebert & Wearne, 2003; NCTM, 2000). Specifically, the NCTM states that the effectiveness of school mathematics in work, school, and life "lies at the heart of problem solving" (p. 334).

Given the significance of problem solving in learning mathematics, the literature review also shows that teaching and learning mathematics through problem solving gives both teachers and students opportunities to make sense of mathematical ideas by appreciating the individual and collective aspects of mathematics—that is, the conversational nature of mathematical knowledge (Ernest, 1998; Kahan & Wyberg, 2003; Rasmussen, Yackel, & King, 2003). Moreover, Hiebert and Wearne (2003) have argued

that students should be challenged with mathematical problems that make them struggle in finding solutions and that require them to work collaboratively to solve the problems.

Based on the literature review on the nature of mathematics and social construction of mathematical knowledge and taking a constructivist perspective to learning, I conclude that learning mathematics should be considered as an active process of personal construction of mathematical understanding through the use of natural language in a socially situated conversation by students. However, considering that learning can be, and is, an individual act and that conversation can occur within an individual (without an interlocutor), I highlight the importance of writing in learning (mathematics); that is, writing as a "unique mode of learning" allows the writer to generate new ideas and new meanings via an ongoing dialogue between the written product (text) and one's thoughts (Bereiter & Scardamalia, 1987; Emig, 1977; Galbraith, 1999; Galbraith & Torrance, 1999; Vygotsky, 1986). Furthermore, Galbraith (1999) and Powell and Lopez (1989) have argued that during writing one engages in a dispositional dialectic where he or she negotiates meaning, and in negotiation, one is constructing knowledge and augmenting learning.

Building on the work in the areas of problem solving and writing and on the bases of the SWH approach by Hand and Keys (1999), I propose a framework—the mathematics reasoning heuristic (MRH) approach—for teachers and students to combine different aspects of mathematics teaching and learning such as teachers' knowledge of mathematics, students' knowledge of mathematics, and students' learning of mathematics. These aspects come together with the MRH approach in a dialogical interaction and writing. The intent of the MRH approach is to guide students' problem

solving behaviors and their problem solving skills via the use of writing (Connolly &

Vilardi, 1989; Kenyon, 1989; Morgan, 1998).

CHAPTER THREE

Journal Article

Abstract

This study examines the pedagogical differences when three teachers change their practices from traditional teaching to student-centered practices. The student-centered approach adopted—the Mathematics Reasoning Heuristic (MRH) approach—was developed by the researcher. A professional development program was implemented over one semester where the MRH approach was initially modeled for teachers before the three teachers implementing the approach. The level of student-centered teaching in the MRH classes varied as a result of the feedback provided by the researcher, while the teachers' traditional-style teaching level stayed the same. The patterns of the change for all three teachers, with different scales, were similar; that is, their MRH teaching stayed at the level of their control teaching for a while and began to move up depending on the pedagogical area on which they put emphasis. Their questioning was the first move to change, yet it was not aligned with other areas at the beginning. However, variations existed in the differences between the teachers' control and MRH instructions.

Introduction

From a social constructivist point of view, dialogical interaction is crucial to students' learning because learning requires active involvement in the construction of mathematical knowledge. However, teachers are generally uncomfortable with using such an approach. Teachers need to change their pedagogical practices in order to promote such mathematical learning in classrooms (Nelson, 1997). Promoting dialogic learning within classrooms is challenging for teachers and requires ongoing support. This

study focused on working with three teachers to enhance dialogical interaction within mathematics classrooms through the use of the Mathematics Reasoning Heuristic (MRH) approach, which is designed on the premise that supporting students' ability to develop a dispositional dialectic for writing in the context of mathematical problem solving will enhance their mathematical understanding. That is, when students create a dispositional dialectic during writing, they sort the context of the problem through what they know in a cyclic process.

The results from the study indicated that students in the treatment classrooms gained enhanced mathematical understanding. Students' Iowa Test of Educational Development (ITED) performance in the treatment classes where the three teachers implemented the MRH approach was significantly higher than that of students in the control classes where the teachers engaged in typical instructional routines. Given this statistical difference and the results from the complementary qualitative analysis of the teachers using the MRH approach, the researcher would suggest that the teachers' ability to support dialogic interaction for mathematical problem solving improved across time. Finally, during the process of learning to support dialogic interaction, the teachers' questioning skills were a forerunner to improving other pedagogical skills important for promoting dialogic interaction, such as giving students voice, promoting student problem solving, and listening attentively to students' ideas.

Dialogic Learning

Social constructivist approaches stress the importance of dialogical interaction between students (Borasi, 1992; Cobb, Yackel, & Wood, 1993; Ernest, 1998). As such, learning is seen as a process of negotiation of meaning between individuals through a

series of "dialogues." Borasi and Ernest describe the conversational aspect of

mathematical knowledge as a human endeavor, with Ernest further defining conversation

as "a sequence of linguistic utterances or *texts* in a common language (or languages)

made by a number of speakers or *authors*, who take it *in turn* to 'speak' (*contribute*) and

who respond with *further relevant contributions* to the conversation" (Ernest, 1998, p.

163, emphasis added). Thus, conversation plays an important role in the construction of

mathematical knowledge because it initiates all participants into negotiating their

explicated utterances through their perspectives. This negotiation is the result of the

dialectical, persuasive reasoning, and social exchange aspects of the conversation (Ernest,

1998; Russell, 1983; Sfard, 2000).

  Alro and Skovsmose (2002), Cobb, Yackel, and Wood (1992), Ernest (1998), and

Frijters, ten Dam, and Rijlaarsdam (in press) argued that participants involved in a

conversation collaboratively construct knowledge by recognition of different perspectives

or through an extended exchange of points of view. Furthermore, speakers are challenged

to mutually understand each other in order to communicate that what they say is

understood as what they intend to mean (Alro & Skovsmose, 2002; Cobb et al., 1993).

Thus, the dialogue is inter-related to the dynamic of mutual negotiation of meanings.

  The central strand of the nature of knowledge is that knowledge is socially

constructed within a community through negotiated meaning of experiences, not static or

stable, but inconstant (Connolly, 1989; Ernest, 1998). Cobb et al. (1992) and Ernest

(1998) have pointed out that children partly construct their knowledge as a form of

collaborative meaning making based on their interaction with others. Seeing mathematics

learning as an active problem-solving process, Cobb et al. (1993) stated that social

interactions in the classroom can create contradictions and conflicts in children, and in the process of resolving these conflicts, students in turn reorganize their mathematical experiences and mathematical ways of knowing. Similarly, von Glaserfeld (1993) asserted that through talk (dialogical interaction) students find inconsistencies in their thoughts that lead them to change or reorganize their conceptual relationships.

The role of the teacher in such a context, from a social constructivist point of view, is to generate disturbances for students about their conceptual structures (von Glaserfeld, 1993) and to provide opportunities for students to discuss their alternative ideas for solving problems (Marshall, 2004; NCTM, 2000) so that the class as a community can determine which opinion(s) or solution(s) should be accepted. While individual responses may not always be conventional, they are valid and indicative of students' own thinking. Teacher questioning is crucial to creating such a classroom climate where students are actively involved in doing mathematics through problem solving (Grouws, 2003). Questioning can engage students in a dialogical interaction where they can organize and reorganize their knowledge of mathematical concepts and, therefore, develop and enhance their mathematical understanding (Cobb et al., 1993). On the other hand, the problem solving activity can be an individual act where students are engaged in a self-dialogue. In this case, teacher questioning is still important because it helps an individual construct the conceptual relationships by creating conflicts. Yackel (2002) pointed out that the teacher's understanding of mathematical concepts and of students' mathematical perspective is important for the teacher to step into the classroom conversation to push the argumentation forward.

Social constructivism accords a crucial role to dialogue and argumentation in both collective and individual aspects of knowledge because "without conversation and its feedback mechanisms, the individual appropriation of collective knowledge cannot be conducted or validated" (Ernest, 1998, p. 166). Ernest further pointed out that mathematical knowledge claims are constructed through a series of "formal dialectical conversational exchanges" by individuals using their subjective (personal) mathematical knowledge and perspectives (p. 149). In a similar vein, Cobb, Boufi, McClain, and Whitenack (1997, p. 264) argued that "children actively construct their mathematical understandings as they participate in classroom social processes." Individuals, as they are talking, at the same time, are reflecting on their own speech and what has been said thus far. Cobb and his colleagues also pointed out that while participating in such social practices individuals restructure their mathematical activities and experiences based on the reflection they have made through irrevocable dialogue. The value of talk in students' learning is crucial because talk provides opportunities for students to modify their existing concepts and conceptual relationships according to the inconsistencies and perturbations they are experiencing at the moment (von Glaserfeld, 1993).

A study by Schwarz, Neuman, Gil, and Ilya (2003) showed that individual students partly internalized the collectively constructed arguments. This study suggests that the "appropriation" of socially constructed knowledge is constrained by (a) the individual's own perspective of the topic and the individual's interpretation of what has been discussed, (b) the social relationships between group members (i.e., peer effect), and (c) the counter-examples created in the argumentative activities (Yackel, 2002). Analyzing the roles of the teacher in collective argumentation, Yackel argued that

argumentation is crucial to students' learning of mathematical concepts both as a collective and an individual act. Yet, the teacher plays an important role in initiating such an argument, supporting students as they interact, and supplying supports (data, warrant, and backing) that are omitted or left implicit in arguments (Yackel, 2002). That said, change is difficult and it is often hard for teachers to give up old habits in favor of new, student-centered techniques; thus, in order for teachers to transform their teaching to support argument within their classrooms, they need support and guidance (Borko, Davinroy, Bliem, & Cumbo, 2000; Weissglass, 1994).

Professional Development

Early studies of teacher change (Carpenter, Fennema, Peterson, & Carey, 1988; Lampert, 1987; NCTM, 1991) identified a set of changes in belief that teachers go through as they shifted their teaching to a practice based on a student-centered learning: (a) coming to see students as learners who are actively engaged in problem solving rather than as vessels to be filled; (b) coming to see that teaching could be and should be aligned with students' learning and thinking; (c) the need for changing the locus of authority from teachers to classroom interaction and negotiation; and (d) coming to see that students can use mathematical reasoning to construct mathematical knowledge (Nelson, 1997).

Scholars have asserted that teachers need ongoing support (e.g., feedback) as they make these pedagogical shifts (Borko et al., 2000; Franke et al., 1997; Goldsmith & Shifter, 1997; Weissglass, 1994). Considering the required changes stated above, teachers' promotion of students' active involvement in problem-solving process is crucial because, as Cobb et al. (1993) have stated, learning is about students' reflection on their

experiences and reorganization of their interpretive framework. Therefore, the locus of teacher change should target such a goal in teachers because, as Franke et al. (1997), Cooney and Shealy (1997), and Weissglass (1994) have argued, teachers' beliefs influence their practices in classrooms. Furthermore, Cooney (1994) pointed out that teacher change cannot be considered separate from teachers' everyday activities of teaching and Borko et al. expressed that the change takes time and effort and can be very rewarding. Similarly, McCaffrey, Hamilton, Stecher, Klein, Bugliari, and Robyn (2001) suggested that such changes require acceptance and adoption by teachers. The MRH is an approach that supports such teacher changes in order to enhance students' problem solving skills through dialogical interaction in classrooms and individual writing activities in the context of mathematical problem solving.

Writing as Problem Solving and the Mathematics Reasoning Heuristic Approach for Dialogic Learning

Building on Britton's (1970) work on language and writing, current writing models that explain the writing process and its function in learning (Bereiter & Scardamalia, 1987, Hayes & Flower, 1980; Galbraith, 1999) consider writing as a process of developing one's ideas, a tool for communicating them, and a heuristic for learning. Furthermore, Hayes and Flower stated that writing is a problem-solving activity—a matter of finding the *solution to rhetorical problems*. This view was based on the findings from research on problem solving in general that could be applied to different areas (Galbraith & Torrance, 1999).

Studies exploring the relationship between writing and mathematical problem solving have shown that writing helps students organize and monitor their problem

solving behaviors and develop their schemata knowledge (Pugalee, 2001; Steele, 2005). Steele found that when students use writing during problem solving, they use all the aspects of their schemata knowledge and develop their mathematical structure and algebraic thinking. Pugalee, likewise, provided evidence of metacognitive behaviors in students' mathematical writing. These studies also show that the demanding and connective notion of writing scaffolds students' problem solving activities and, therefore, enhances their mathematical knowledge structure. One powerful function of writing in mathematics is that it inherently pushes the translation between the problem text and problem solving operators to the forefront to help the problem solver be aware of how he or she can associate mathematical concepts with language (Galbraith, 1999; Greeno, 1977; Keys, 1999; Pugalee, 2004). As Galbraith said, the writer engages in a dispositional dialectic during writing where he or she negotiates the meaning through self-dialogue.

Kenyon (1989, p. 77) expressed, "As students write down, *reflect on, and react to their thoughts and ideas* [emphasis added], they enhance the executive problem-solving abilities, and the problem-solving process becomes more effective." Writing helps students develop a mathematical dispositional dialectic that is necessary for becoming an expert in the context of mathematical problem solving. Kenyon further argued that asking students to write about their problem-solving process allows them to clarify their thoughts about how they will approach the problem. Implementing writing-to-learn strategies effectively requires teachers to develop conceptual understanding of the subject matter so that they can help students develop mathematical habits of mind for solving problems (Driscoll, 1999; Hand & Prain, 2002; Levasseur & Cuoco, 2003; Simon, 1995).

Akkus and Hand (2005), as an attempt to develop a teaching/learning approach with embedded writing into problem solving, constructed the mathematics reasoning heuristic (MRH) approach, which they consider to be an approach that links promotion of negotiation of mathematical ideas, writing, and problem-solving activities. The framework for designing the MRH approach includes shifting to constructivist theory, promoting teachers focus on conceptual understanding of mathematics, embedding writing into mathematics, and the promotion of problem solving.

Development of the mathematics reasoning heuristic approach

The intent of the MRH approach is to provide a framework for teachers to combine different aspects of mathematics teaching and learning such as students' knowledge of mathematics, teachers' knowledge of mathematics (Simon, 1995), negotiation of problem solving methods, and embedding writing into mathematics instruction (Ernest, 1998; Hand & Keys, 1999; Galbraith, 1999; Morgan, 1998). Another intent of the MRH approach is to guide students' problem-solving behaviors and enhance students' problem-solving skills through the use of writing (Connolly & Vilardi, 1989; Kenyon, 1989; Morgan, 1998).

The science writing heuristic (SWH) as a deriving model

The MRH approach has been influenced by the basic ideas of the science writing heuristic (SWH) approach by adapting the main structure of the model. The SWH approach was constructed as a heuristic that links writing, reading, and science laboratory activities (Hand & Keys, 1999). The main focus of the SWH approach is to promote students' participation in setting their investigative laboratory work, framing their own questions, proposing methods to address these questions, and carrying out appropriate

investigations. The SWH approach also scaffolds students' scientific thinking about the relationships between questions, claims, and evidence. Therefore, using the structure of the SWH approach developed by Hand and Keys, two MRH templates were created for the teacher and students (Akkus & Hand, 2005).

Teacher template

The teacher template (Figure 12) provides important phases of knowledge construction in mathematics classrooms that teachers need to be aware of before and during instruction. For example, the template emphasizes the importance of planning the learning goals and the major concepts of a unit. As Simon (1995) has argued, a teacher's knowledge of mathematics and students' knowledge of mathematics interact in the course of learning in the classroom environment. Prior to engaging students in a learning process, the teacher has initial understanding of students' mathematical knowledge and their learning of mathematics (Simon, 1995, 1997). This knowledge is used to guide the teacher in the teaching/learning environment.

Using the MRH approach, teachers engage in a planning phase that requires them to determine the major concepts that they want their students to learn, to identify what they (teachers) know and do not know about the topic, and, importantly, to align their teaching according to the concepts of how students learn. As Simon (1995) emphasized, if we store knowledge as a conceptual frame, we then need to help students construct a mathematically acceptable framework. To this end, the MRH approach suggests that teachers consider their mathematical knowledge and students' knowledge of mathematics, which they represent during classroom interaction, and create small- and

large-group discussion opportunities for students to negotiate their meanings of

mathematical activities and problem solution methods.

Teacher Template

Preparation:
 - Identify the big ideas of the unit.
   - Make a concept map that relates sub-concepts to the big ideas.
 	
   - Consider students' prior knowledge
 	
   - Consider students' alternative conceptions during the lesson as they
     connect the prior knowledge to the big ideas

During the unit:
 • **Students' knowledge of mathematics**
 - Give students opportunity to discuss their ideas.
 - Have students put their ideas on the board for exploration.
 • **Teacher's knowledge of mathematics**
 - Use your knowledge to identify students' alternative conceptions.
   - Guide students to the big ideas identified earlier during the preparation.
 • **Negotiation of ideas**
 - Create small-group and whole-class discussion.
 - Encourage students to reflect on each other's ideas.
 • **Writing**
 - Have students write about what they have learned in the unit to real audiences
   (teacher, parents, classmates, lower grades, etc.).

Figure 12. The mathematics reasoning heuristic teacher template.

In addition, the purpose of such classroom activities is to promote students'

mathematical thinking and talking. Adler (1999) argued that students and the teacher all,

to some degree, engage in classroom discourses in varying ways. She suggested that

students should have opportunities "to practice being users of educated discourses"[7] (p.

51). In order to foster classroom discourse, teachers need to be continuously aware of

students' mathematical ideas and questions. Simon (1997) argued that the dynamic of the

classroom discourse depends upon the relationship between the teacher's mathematical

knowledge and his or her knowledge of students' mathematical knowledge (of related

concepts). However, it is important to note that the mathematics knowledge in a

classroom is, in turn, based on this classroom dynamic (Ernest, 1998). The MRH approach facilitates both public and private aspects of knowledge construction via classroom discussions and individual writing activities.

Furthermore, the teacher template focuses on the importance of writing-to-learn strategies as one of the main foci of learning mathematics. For example, the template encourages teachers to ask students to explain the mathematics they have learned to different audiences (e.g., parents, younger students, peers, teacher, etc.). Writing-to-learn concepts suggest that the role of language and writing is to scaffold students' process of constituting thoughts and knowledge (Galbraith, 1999), as well as their representation of that knowledge externally (Connolly, 1989). Therefore, teachers are asked to provide students with the opportunities to discuss their ideas publicly before they actually write their individual understanding.

Student template

The second component of the MRH approach, the student template (Figure 13), is constructed to promote students' problem solving. The template consists of a series of questions for students to consider when they are engaged in the problem solving process. The student template resembles Polya's (1945) four-stage problem-solving heuristic (understanding, planning, carrying out the plan, and looking back) or Schoenfeld's (1985) phases of problem solving (analysis, design, exploration, and implementation). However, one distinct feature of the template from other problem-solving frameworks is indeed that it specifically asks students to *compare* their solutions with their peers and to *reflect on* their problem solution after a classroom discussion.

Student Template
• **What is my question (problem)?**
- Specify what you are asked (What is (are) the question(s) being asked?).
- Outline the information/data given (What information is/are given?).
• **What can I claim about the solution?**
- Use complete sentences to explain how you will solve the problem.
- Tell what procedures you can follow.
• **What did I do?**
- What steps did I take to solve the problem?
- Does my method (procedure) make sense? Why?
• **What are my reasons?**
- Why did I choose the way I did?
- How can I connect my findings to the information given in the problem?
- How do I know that my method works?
• **What do others say?**
- How do my ideas/solutions compared with others?
a. My classmates
b. Textbooks/Mathematicians
• **Reflection**
- How have my ideas changed?
- Am I convinced with my solution? Why?

Figure 13. The mathematics reasoning heuristic student template.

The template allows students to clarify their thoughts, through writing, about how they will approach the problem (Kenyon, 1989). This course of action may be completed individually or in a group depending on the teacher's decision. When the teacher chooses group discussion, students discuss the question (the question might be a word problem or any mathematical task chosen by the teacher) and negotiate possible ways of solving it within each group. Then each group shares their methods with the whole class. A critically important criterion is that the teacher needs to create opportunities for students to share their methods, hear alternative methods, and then compare advantages and disadvantages of the methods (Hiebert & Wearne, 2003). The student template also encourages students to think about their metacognitive actions and reflect on their thoughts (Pugalee, 2001, 2004). Kenyon (1989) emphasized the importance of writing on

the thinking process during problem solving and the importance of the metacognitive skills that students use to reflect on their thoughts declaring that within a "writing process, students begin to gather, formulate, and organize old and new knowledge, concepts, and strategies, and to synthesize this information as a new structure that becomes a part of their own knowledge network" (p. 77). The Pugalee's study (2004) showed that students who wrote about their problem solving process scored significantly higher than students who provided verbal description. This also indicates the importance of asking students to write about their problem solving process, which is why the MRH approach encourages students to present their understanding of mathematics in text to different audiences (e.g., parents, peers, and younger students).

Students using the template to structure their writing can engage in a self-negotiation of meaning through cognitive and metacognitive actions such as planning, monitoring, reviewing, and revising. Therefore, the process of text production during problem solving is an internal discussion about schema knowledge, problem-solving strategies, and rhetorical knowledge, which, Galbraith (1999) has argued, are all interconnected. This internal discussion is intended to support and benefit from external classroom discussions of students' reasoning about the mathematics they learn. Once students and the teacher are aware of the students' informal mathematical ideas, they are better able to negotiate the connections to the formalized mathematical language. The MRH approach encourages teachers to embed mathematical language into mathematical activities rather than giving isolated definitions of mathematics concepts. Sternberg (1999) emphasized the importance of asking students to identify the nature of the problem to be solved in order to help them improve their mathematical reasoning

abilities. This also gives students opportunities to transfer the method learned during problem solving to other situations.

This study was guided by the following two research questions:

1. Is there a difference in students' mathematical performance on a standardized test, the Iowa Test of Educational Development (ITED), between the students in the control classes where the teachers are engaged in their traditional instructional routines and the students in the treatment classes where the teachers used the MRH approach to improve their pedagogical practices and to scaffold students' problem solving skills?

2. How do the teachers change their pedagogical practices through the use of the MRH approach?

Methods

Research Setting and Participants

The researcher initially started working with a grade-nine algebra teacher (Mike) at a high school located in a rural area of Midwest. During the initial phase of his collaboration, Mike and the researcher worked closely together to implement the fundamentals of the MRH approach and on numerous occasions engaged in team-teaching. This way the researcher had an opportunity to model the new approach. As Mike gained more experience and felt more confident with the approach, he decided to try teaching on his own for the rest of the semester. During the second semester, the head of the mathematics department in the school (John), who was also teaching algebra sections, became interested in the project and agreed to participate in the following school year. Later, Mike moved to a different school district and was replaced with a new

teacher (Amy) who agreed to participate for the first semester of the second year. Consequently, in this study we used the data from Mike's first semester in the first year and John and Amy's first semester in the second year. The reason I used the first semester data was because the Iowa Test of Basic Skills (ITBS) and Iowa Test of Educational Development (ITED) were administered in this school in February, at the beginning of the second semester. This enabled the same conditions to apply for each teacher in terms of implementation time prior to collection of standardized test scores. All three teachers taught the same units of polynomials and fractions. The number of students in each teacher's control and treatment classes is provided in Table 1.

Mike. Mike had 20 years of mathematics teaching experience in different schools in Iowa, the last five years of which had been at the high school featured in this study. Mike also taught an Applied Mathematics course. During his career he has taught all different levels of high school mathematics as well as coaching football as an extracurricular activity in the school. He had three algebra classes participating in this study.

John. John had 33 years of teaching experience, 28 years of which were at his current school district, where he started as a middle school mathematics and English teacher. He had worked in the middle school for 18 years before moving to the high school. He was the head of the mathematics department in the school and taught Probability and Statistics and Algebra I courses for ninth grade students. He also had master's degrees in English and Guidance Counseling. In his years at the current high school, he taught algebra, general mathematics, consumer mathematics, applied mathematics, and probability and statistics. He was also employed by the Area Education

Agency (the local professional development organization) as a facilitator for staff development programs. John coached basketball and tennis at the school. He had four classes participating in this study.

Amy. Amy had been teaching for five years; this was her first year in this school district. She taught Algebra I (ninth grade) and Applied Math II (tenth grade) courses. She received her bachelor's degree in education with a double major in mathematics and mathematics education. She also had a coaching endorsement and coached cross country and girl's basketball. Amy had three classes participating in this study.

Table 1. Distribution of students according to teacher and group.

Teacher	Group	Number of students	Male	Female	High Achv[**]	Med Achv	Low Achv
Mike	Control	21	11	10	1[***]	12	3
	Treatment (MRH)	44[*]	19	25	3	25	7
John	Control	43[*]	26	17	5	26	8
	Treatment (MRH)	45[*]	35	10	6	29	6
Amy	Control	24	13	11	1	12	5
	Treatment (MRH)	25[*]	8	17	2	20	2

[*] Teachers had two classes in this group.
[**] Achv (achievement) refers to proficiency level provided by the testing center using students' ITBS scores, relative to each other.
[***] The number of students for proficiency level was obtained according to the data reported.

Participating School District

The school district that participated in this study is located in a moderate-sized federally-designated rural poverty town in Midwest. The school district is classified as a 3-A district, with 106 teachers in the middle and high school buildings. Total student enrollment was 2,276. The distribution of students' ethnicities was: 96% white, 1.60% African American, 0.37% Asian, 1.85% Hispanic, and 0.18% Native American. The number of minority students per classroom was, at most, two for each of the three participating teachers. Based on the decision made by the school administration, only

those students who are considered proficient in mathematics based on the results obtained in the 7th grade are admitted to take algebra in the 8th grade. The students who did not qualify to take algebra I in the 8th grade constituted 60% of the students, and this group participated in this study (see Table 1 for the proficiency level distribution across teachers).

Interventions

The year in which each teacher participated was different; Mike participated in fall of 2004, while John and Amy participated in fall of 2005. The researcher first met Mike in September 2004 to introduce the project. He explained the details of the project and the major components of the MRH approach. After the first meeting, the researcher visited Mike weekly to model or to co-teach. Mike raised his questions or concerns about the implementation, and the researcher addressed them at the weekly planning meeting, where the focus was on the mathematics topics that they were planning to teach for data collection. The intent of the planning meetings was to get Mike to identify the main ideas (or concepts) of the units and make a concept map related to the concepts identified. When the researcher would co-teach or model, he arrived during Mike's daily planning hour and briefly discussed the major points of the lesson (Frenke et al., 1997). The MRH students were provided with the student template as a poster to hang on the wall and as a bookmark for their individual use, and were encouraged to use the MRH student template in their studies. This orientation period lasted for two months, with ten visits total. During the following three months, Mike then taught two mathematics units (polynomials and fractions) using the MRH approach based on the planning. During Mike's implementation, the researcher observed the lessons and debriefed with Mike afterwards.

The purpose of the debriefings was to make Mike aware of his own actions during teaching, to highlight the pedagogical areas that needed improvement, and to provide suggestions for the future. There also was a lot of email correspondence because Mike had been told he could send an email about any concern at any time.

The same support was provided for John and Amy; they also had the chance to work together with or without the researcher being present. However, there were differences in the intervention for John and Amy compared to Mike, one of which was that the researcher met them before school started and had them read related articles. The other difference was that with two participating teachers more discussions occurred and more ideas were raised than during Mike's planning meetings. There were seven visits with John and Amy for the first two months, and eighteen for the last three months during their implementation. Similarly, debriefing sessions and email corresponding occurred with John and Amy, individually or together.

The intervention provided for the teachers was based on the Iowa Professional Development Model, which requires on-going support. The main commitment between the researcher, the teachers, and their school administration was to provide the support necessary for the teachers and to help the researcher with the data collection process.

Data Collection

There were two main data sources. The quantitative data included the ITBS and ITED scores that were used to examine performance differences between the control and treatment classes. The qualitative data included observations through both on-site observations and videotape recordings in both MRH and control classrooms and interviews that shed light on the level of teacher implementation.

All data were collected by the researcher, who had been involved in data collection and analysis from the beginning of the study. Data collection occurred during the last three months of the first semester of the school year. Mike was observed, videotaped, and interviewed during the first semester of the 2004-2005 school year, and John and Amy were observed, videotaped, and interviewed during the first semester of the 2005-2006 school year.

Observations and Videotaping. For each teacher, two types of observations occurred in both their treatment and control classes—on-site observations and videotaped observations. Of particular interest for both types of observations were teacher-student dialogical interaction, teacher questioning and its promotion for interaction, and the focus of learning and problem solving.

During the on-site observations, the observer was physically present in the classroom at least once and often twice a week for three months, following the teachers, taking field notes on teacher-student interactions, and filling in the observation protocol (the Reformed Teaching Observation Protocol [RTOP]). At the end of each visit, the observer had a short debriefing session with the teachers, where constructive feedback was provided after the teachers' identification/self evaluation of strengths, weaknesses, and difficulties with implementation. Such debriefings targeted several areas of interest such as increasing the level of confidence and trust between the teachers and researcher, promoting the teachers' awareness of certain behaviors observed, highlighting the pedagogical areas that needed improvement, and suggesting some strategies that the teachers might use to improve their implementation of the required student-oriented approaches in the future. Every time the observer had a debriefing session, detailed field

notes were kept and transferred into electronic format for further analysis. The number of

visits, observations, and videotape recordings for each teacher are provided in Table 3

below. The videotapes were independently analyzed and assigned an RTOP-score by the

three researchers and a retired teacher in terms of the level of implementation.

Interviews. The teacher interviews occurred once at the end of the semester. The

interviews were semi-structured, for 45 minutes to an hour, targeting the teachers'

perceptions of learning and teaching, their role during problem solving, and use of

writing. The researcher asked the teachers to compare their teaching experience before

and after using the MRH approach and posed follow-up questions based on the teachers'

responses. Each interview was audiotaped and transcribed.

Student scores. For each student participating in the study, the ITBS and ITED

scores were obtained in such areas as mathematics concepts and estimation, mathematics

problem solving and data interpretation, and writing. The ITBS (Grades 3-8) and ITED

(Grades 9-12) are district-wide, annually-administered standardized tests that are used in

the participating school district. The tests are administered around the same time every

year (in February) across the district. The ITBS and ITED cover a wide variety of subject

areas, such as vocabulary, reading comprehension, spelling, capitalization, mathematics

concepts and estimation, mathematics problem solving and data interpretation, social

studies, reference materials, and science.

Analyses

The analyses consisted of a series of passes through various aspects of the data. In

terms of quantitative data, an analysis of covariance (ANCOVA) was estimated to control

for other variables that might impact students' mathematics achievement (Agresti &

Finlay, 1997). To ensure the accuracy of the data that were collected, both frequency

distributions and descriptive statistics were obtained using the SPSS Frequencies

procedure (Mertler & Vannatta, 2002). The SPSS Casewise Diagnostic procedure was

employed to examine whether outliers possibly could affect the results of the study

(Levine & Roos, 2002). Students' ITED mathematics scores of the year of the

implementation were used as the dependent variable, with teacher and group as the

independent variables, and students' ITBS scores from the previous year as the covariate.

 In terms of qualitative data, the classroom and videotape observations and the

field notes were analyzed in terms of the level of teaching (i.e., level of dialogic teaching)

in both control and MRH classrooms. The findings from both types of observations were

triangulated with the interviews, email correspondence, and field notes, which provided a

general trend of teachers' pedagogical implementation that confirmed our classification

of the level of their teaching.

 Level of teaching. To determine the level of teaching, we adapted an observation

protocol called the Reformed Teaching Observation Protocol (RTOP) used to measure

the degree of "reformed" teaching in mathematics and science classrooms (Sawada et al.,

2000). The instrument draws on the recommendations and standards for teaching

mathematics and science that have been promulgated by NCTM (1989, 1991, 1995, and

2000) and the National Science Education Standards (NRC, 1995). The instrument

consists of 25 items, each rated from 0 (not observed) to 4 (very descriptive). The RTOP

Chronbach's Alpha is .954 for both mathematics and science classes (Sawada et al.,

2000). However, we modified the RTOP and chose the 14 items in Table 3 (Chronbach's

Alpha was .976). The items were then categorized according to the relevancy to each

other, and four sub-categories are constructed: *Student Voice, Teacher Role, Problem Solving and Reasoning, and Questioning*. The values of Cronbach's alpha for the sub-categories (Student Voice, Teacher Role, and Problem Solving and Reasoning) were .908, .882, and .952, respectively. There was only one item related to questioning so Cronbach's alpha was not reported. The RTOP sub-categories identified are also similar to the categories of classroom practice documented by Franke, Fennema, and Carpenter (1997): providing opportunity for children to solve mathematical problems in their own ways; listening to children's mathematical ideas; and using children's mathematical thoughts in making instructional decisions through questioning. The RTOP sub-categories were defined as followed: a) *Student Voice*: The amount of opportunities for students to share their ideas; b) *Teacher Role*: Allowing students to take responsibility of thinking process and problem solving process and moderating the conversation; c) *Problem Solving and Reasoning*: Allowing and encouraging students to discover their own problem solving methods either as groups or individually rather than providing a method, and asking for justification for their solution methods; and d) *Questioning*: The type and purpose of questions asked by the teacher. There was only one item related to questioning in the original RTOP instrument; therefore, questioning stands alone.

The researcher provided an RTOP score for the on-site observations. However, not *all* the observations were RTOP-scored and the number of videotape recordings was not equal for all teachers. The number of observations and RTOP scores for each teacher is shown below in Table 2. The corresponding *on-site* RTOP scores were replaced with their *videotape* RTOP scores negotiated by the four raters. In other words, the corresponding videotape RTOP scores were used in the analysis. Thus, for all teachers

combined, 11 MRH and 5 control, final RTOP scores were provided, and the analyses were done accordingly.

Table 2. Number of observations and RTOP scores provided for teachers.

		Total Number of Observations	On-site RTOP Scoring	Videotape RTOP Scoring	Final RTOP Scoring
Mike	Control	5	5	5	5 (0[*])
	MRH	20	11	9	11 (2)
John	Control	5	5	5	5 (0)
	MRH	18	11	8	11 (3)
Amy	Control	5	5	5	5 (0)
	MRH	18	11	7	11 (4)

[*] Represents the number of final RTOP scores based on the on-site observation.

The percentage of agreement (or the inter-rater reliability) between any pairs of observers for teachers' level of teaching ranged from 90% to 95%. If there were any disagreement about a score, all observers watched the videotape and made a decision based on a discussion revolving around the problematic part of teaching. Such discussions resulted in 100% agreement by providing rationales for the scores. The final score of the observers was also compared to the researcher's score for that particular lesson based on his on-site observations. There was a 90% match between the group scores and the on-site observation scores.

Table 3. Reformed Teaching Observation Protocol (RTOP)—modified[*].

		Never Occurred				Very Descriptive
Student Voice	-**The** instructional strategies and activities respected students' prior knowledge and the preconceptions inherent therein.	0	1	2	3	4
	-**The** focus and direction of the lesson was often determined by ideas originating with students.	0	1	2	3	4
	-**Students** were involved in the communication of their ideas to others using a variety of means and media.	0	1	2	3	4
	-**There** was a high proportion of student talk and a significant amount of it occurred between and among students.	0	1	2	3	4
	-**Student** questions and comments often determined the focus and direction of classroom discourse.	0	1	2	3	4
	OVERALL (Total/5)					
Teacher Role	-**The** teacher acted as a resource person, working to support and enhance student investigations.	0	1	2	3	4
	-**The** metaphor "teacher as listener" was very characteristic of this classroom.	0	1	2	3	4
	OVERALL (Total/2)					
Math Problem Solving and Reasoning	-**This** lesson encouraged students to seek and value alternative modes of investigation or of problem solving.	0	1	2	3	4
	-**Students** were actively engaged in thought-provoking activity that often involved the critical assessment of procedures.	0	1	2	3	4
	-**Students** were reflective about their learning.	0	1	2	3	4
	-**Intellectual** rigor, constructive criticism, and the challenging of ideas were valued.	0	1	2	3	4
	-**Active** participation of students was encouraged and valued.	0	1	2	3	4
	-**Students** were encouraged to generate conjectures, alternative solution strategies, and/or different ways of interpreting evidence.	0	1	2	3	4
	OVERALL (Total/6)					
Q.ing	- **The** teacher's questions triggered divergent modes of thinking.	0	1	2	3	4
	OVERALL (Total/1)					

[*] The items were not changed but rearranged.

Results

Overall Results

The results from the study indicated that students in treatment classrooms did gain

mathematical understanding. To back up this claim, we first provide the details of

students' ITED performance in three teachers' classrooms: both the classes where the

teachers implemented the MRH approach and their control classes where the teachers

engaged in their typical instructional routines. The complementary qualitative analysis

unpacks the evidence we gathered to document our second claim that there was

improvement across time in the three teachers' ability to support dialogic interaction for

mathematical problem solving. Finally, in order to support our description of how

teachers shifted their pedagogical practices over the time of the study, we illustrate our

evidence that teacher questioning skills were a forerunner to improving other pedagogical

skills important for promoting dialogic interaction, such as giving students voice,

promoting student problem solving, and listening attentively to students' ideas.

Treatment Improved ITED Scores Significantly over Control

Statistical results are reported for all the three teachers. Before estimating an

ANCOVA model, possible violations of key assumptions were investigated. Normal

probability plots of model residuals, along with the Kolmogorov-Smirnov test, were used

to examine the normality assumption for the estimated model. The linearity assumption

was addressed by plotting standardized residual values against the predicted values.

Using the SPSS Casewise Diagnostic procedure, the outliers (below and above 3 standard

deviations) were removed from the data file (Levine & Roos, 2002). Examination of the

Normal Q-Q Plots obtained through the SPSS Explore procedure enabled the researchers

to monitor the patterns of lines in which the resemblance to linearity was judged. The results of tests of the linearity and normality assumptions and transformation procedures were not reported unless one or both were violated. Finally, the homogeneity assumption was examined by using Levene's test for equal variances within each ANCOVA analysis. Levene's test results were provided with each set of analysis of results.

To decide the potential covariate for the ANCOVA model, students' ITBS scores for the previous year in different subject areas such as language, reading, and mathematics were analyzed. The ITBS Mathematics (Mathematics: Problem solving and Data interpretation) was the only subject that was significantly different between the control and MRH groups for each teacher (e.g., F (1, 47) = 4.545, p = .038) or among the teachers for each group (i.e., F (2, 64) = 3.365, p = .041). Therefore, the ITBS Mathematics scores were chosen as the covariate for the ANCOVA model.

An overall ANCOVA model was estimated using the ITED mathematics test of the same year as the dependent variable, with the teacher and group as the independent variables, and the ITBS Mathematics Problem Solving and Data Interpretation test of the previous year as the covariate. The model yielded a significant main effect of group in favor of the MRH (F (1, 163) = 5.381, p = .022, η^2 = .032) (see Table 4 for adjusted means and standard errors). Moreover, even though there was no significant interaction effect (F (2, 163) = .350, p = .706), the analyses of pairwise comparisons indicated that John's MRH classes (M = 284.554, SD = 18.721) significantly outperformed Mike's control class (M = 273.645, SD = 14.700), t (52) = 2.288, p< .05, Amy's control class (M = 268.555, SD = 14.361), t (54) = 3.518, p< .05, and John's own control classes (M = 276.154, SD = 18.585), t (75) = 1.976, p< .05. Corresponding Cohen's d effect sizes

were, respectively, .618 SD, .916 SD, and .450 SD (see Table 5). Even though the

difference was not significant, Cohen's *d* effect size difference between John and Amy's

control classes was .437 SD. Moreover, Mike and Amy's MRH classes had higher mean

scores than their control classes (significantly higher in Amy's case). Mean Square Error

was 345.281, and adjusted R^2 was .316 for this model. Finally, Levene's test of equality

of error variance showed non-significant results ($F (5, 164) = 1.154, p = .334$), which

confirms that the error variance of the dependent variable is equal across groups.

Table 4. Descriptive statistics for ITED Mathematics scores

Group	Teacher	*n*	Adj. Mean	Std. Error
	Mike	16	273.645	3.675
Control	John	39	276.154	2.976
	Amy	18	268.555	3.385
	Mike	35	279.463	3.145
MRH	John	38	284.554	3.037
	Amy	24	279.113	3.800

Table 5. Pairwise comparisons in Cohen's *d* effect size

		t (d.f.)	Cohen's *d*
	Mike (Control)	2.288* (52)	.618
	Amy (Control)	3.518* (54)	.916
John (MRH)	John (Control)	1.976* (75)	.450
	Mike (MRH)	1.165 (71)	.273
	Amy (MRH)	1.119 (60)	.291
John (Control)	Mike (Control)	.531 (60)	.143
	Amy (Control)	1.686 (55)	.437
Mike (MRH)	Mike (Control)	1.203 (49)	.332
Amy (MRH)	Amy (Control)	2.075* (40)	.623

*$p < .05$

Teachers' Pedagogical Practices Improved Across Time

Each teacher's level of implementation was identified using the RTOP

instrument. Each lesson observed and/or videotaped was attributed an RTOP score

ranging from "0" to "4," with 4 being the highest level of teaching (see Figure 14). The

mean scores of all individual lesson RTOP scores were assigned to the teachers as their

final RTOP scores. Mike and Amy's control teachings were rated .22 and .26,

respectively. Their MRH teaching levels were 1.60 and .88, respectively. John's control

and MRH teaching levels were 1.40 and 2.10, respectively. Overall, while the teachers'

MRH teaching improved, their control teaching level tended to score the same throughout

the study. John's routine control teaching level was as high as Mike's MRH teaching and

higher than Amy's MRH teaching. Moreover, Mike and Amy occasionally regressed in

their implementation of the MRH approach, whereas John continued to improve after the

5th observation. Mike and Amy's control teachings were the same over the course of the

study. These observations can be seen in the left-hand side of Figure 14, where they were

purposefully represented together in the same figure in order to present the comparisons

of pedagogical practices between the teachers. Furthermore, the detailed analysis of the

teachers' MRH teachings provides evidence that there was an improvement in the

teachers' practices, with "questioning" being the critical component (the right-hand side

of Figure 14). For example, even though the RTOP scores for John's forth and fifth

observations were low (between 1 and 1.5), his questioning was high (3); and then he

improved all the pedagogical areas together for the next time. Below is the description of

each teacher's level of teaching.

Case 1 (Mike)

The left-hand side of Figure 14 provides Mike's observation scores for his MRH

and control classes. As can be seen in Figure 14a, his MRH teaching went up while his

control teaching remained the same throughout the data collection. The MRH approach

provided Mike with a scaffold that he could use during his teaching, more specifically during problem-solving activities. Mike's control teaching was traditional teaching where a teacher tells students what to do all the time, provides them with problem solving methods, and has students practice those methods. The detailed analysis of Mike's MRH classroom practices showed that he put a lot of emphasis on questioning; however, the maximum RTOP score for questioning was 2.5, which occurred towards the end of the study. There were fluctuations in Mike's practice between the 4th and 8th observations. Those were the times when he focused on his questioning yet was unable to pull up the other pedagogical areas.

Case 2 (John)

John's classroom environment was very different from the other two teachers. He put a great deal of importance on the relationship with his students. He was always friendly and "*a big believer of relationship. [I] need to connect with the kids on a first level, non-math level, uhm, on a human level*" (Interview, John, 01/11/2006). His students were active in the classroom: "*a noisy class never bothers me,*" said John during the interview. John always began his lessons with an activity he called "The Event of the Day," which asked, for example, "What happened at this time 10 years ago?" In both MRH and control classes, he always made problems a detective game and connected to a real-life situation in a meaningful way. As can be seen in Figure 14c, John's control teaching was also higher compared to Mike and Amy's control teachings. The detailed analysis of John's MRH teaching practices also showed fluctuations yet consistently rose over time (the right-hand side of Figure 14). His questioning level was always higher compared to other pedagogical areas (student voice, teacher role, and problem solving)

and he was able to pull them up after the 4th and 5th observations where there were larger gaps between the questioning and other areas. As can be seen in Figure 14d, John began to more improve his questioning during the 9th, 10th, and 11th observations along with other areas of his pedagogical practice, although these areas did not increase at the same rate.

Case 3 (Amy)

Amy was new at this school and was reluctant to try something different than her regular way of teaching. Even though she made progress in her 6th, 7th, and 8th videotaped observations, this improvement was not consistent and she often fell back to her traditional way of teaching. Her regular questioning pattern was asking for factual information and mostly yes-no questions without following up with "why" question. She often missed the opportunity to create student-centered problem solving; instead there was a rush to get the "right" answer. Any student-student interaction was filtered by the teacher. Amy's MRH teaching was parallel to her control teaching except for the 6th, 7th, and 8th observations (the left-bottom corner of Figure 14). Amy showed an improvement in her practice of the MRH during these three observations where the first two were examples of high questioning without connecting to the other pedagogical areas and the third observation was a case of pulling these areas together with questioning even though the overall teaching was not high (the right-bottom corner of Figure 14).

Treatment and Control Comparison

RTOP scores for sub-categories in MRH

a. Mike

b. Mike
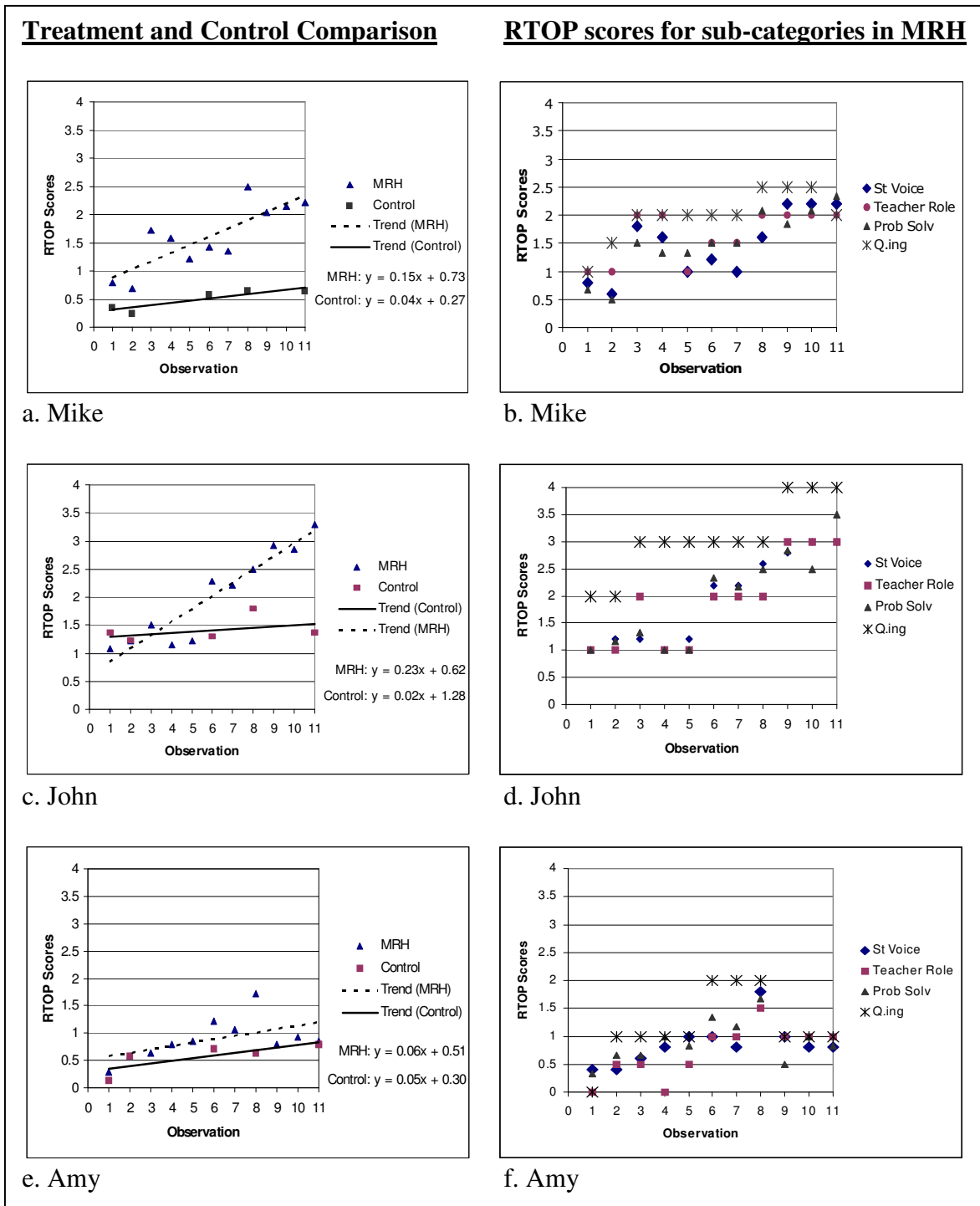

c. John

d. John

e. Amy

f. Amy

Figure 14. Teachers' levels of teaching and RTOP scores for sub-categories in MRH.

Patterns in the Teachers' Pedagogical Shift to Supporting Dialogic Interaction

As a result of intensive qualitative analysis, the researchers have four assertions for how the process of shifting pedagogical practices unfolded for these teachers. The assertions are in **bold**, while supportive evidence is in *italics* and the researchers' comments are in normal type.

Assertion 1. The MRH teaching strategies improved as a result of the feedback.

Throughout the contact with the researcher, the teachers' MRH teaching improved while their control teaching remained the same throughout the study. As can be seen in Figures 14a, 14c, and 14e, their MRH teaching resembled their control teaching at the beginning. Yet, their improvement occurred at different times and at different rates and amounts. For example, Mike showed a big gain during the third and forth observations and his level of MRH teaching dropped a little bit during the next two observations. On the other hand, John and Amy's MRH teaching was stable (according to their control teaching) throughout the first half of the study. While Amy had a small improvement and then regressed to her control teaching level, John continuously improved his teaching with the MRH approach for the rest of the study. Mike, similarly, had a gain for the rest of the study; however, it was smaller compared to John's. The following is the part of an example of feedback provided for the teachers after debriefing sessions:

> *In general, I realized that your questioning is improving. Here are some remarks:*
> *give students chance to write their ideas on the board – in this way they will have*
> *feeling of ownership (For example, when Natasha said "divide, or split the L*
> *shape," she could have gone to the board and showed what exactly she had*

meant). Therefore, you would have had the opportunity to reflect on students'

responses. Instead of asking too general/vague questions, try to be specific and

clear for your question. (For example: What properties of addition,

multiplication, and division did you use to solve this problem? How and why did

you choose to use them?) **Missed opportunity:** *Creating a discussion on*

perimeter. When a student said that they needed to find the perimeter to find the

cost of the carpet, a group discussion could have been created by asking, "What

do you guys think? Let's make groups of three or four to decide what we need to

do." (E-mail feedback for Mike, 11/06/2004)

Such feedback was given to the teachers after debriefing sessions following the

classroom observations. The focus was to highlight weaknesses and strengths in their

implementations. The reactions of the teachers to such feedbacks were positive and they

were willing to put them in practice in the next lesson. However, Amy showed reluctance

to changing her implementation, with various reasons that are mentioned in the next

assertions.

Assertion 2. Promoting negotiation for problem solving was a challenge for the

    teachers.

  Since the teachers were used to their traditional teaching, where they first

introduced the method for solving a problem and then had students practice it, moving to

ensure they were "*getting everybody's contribution was a challenge*" (Interview, John,

01/11/2006). They had to force themselves to step back and let the kids discover different

ways of solving problems. Mike, for example, said, *"I think I need to let them discover*

on their own if it is right or wrong, let them go their own path" (Interview, Mike, 01/

17/2005). On the other hand, the teachers struggled with how much they needed to let the students go down their own path. This challenge was addressed differently by each teacher. For instance, Amy was extremely frustrated when her students did not give the right answer in a lesson. During the debriefing session afterwards, she stated, *"See, they don't get it! Sometimes, I need to directly tell them the answer. We have so much to cover"* (Debrief, Amy, 12/05/2005). Mike had the same opinion that

> *I need to guide them along. Otherwise, we can end up not being very efficient. …If they are getting way off, I need to come in and steer them back to the direction. I can do that by giving them some facts or some questions to consider.* (Interview, Mike, 01/17/2005)

On the other hand, John took an opposite perspective on this. He admitted that he did not give students enough time to discover their own methods and *"I always think that big part of figuring out is for me to instruct them.… But now with this project, [I am] trying to step back and see how it [letting kids discover] works"* (Interview, John, 01/11/2006). When he compared his control teaching and MRH teaching, he noticed the changes in his teaching:

> *I always find myself thinking "ohh, Reggie would be proud of me. Or Reggie would slap my hand." So, I tried it, and it went so well, because I had to shut up. We went back and forth, back and forth, arguing…. I know in my heart that this class [MRH] learned better than the other period [control] where I was more the instructor.* (Interview, John, 01/11/2006)

Even though Mike did not mention his thought-process during teaching, analyses of the videos revealed that he often went back and forth between his traditional teaching and the

new way of teaching. For example, he often stopped telling students how to do the problem and said, *"Well, uhm, okay, I guess I'll let you guys think on this, okay? Think about it, either on your own or with your partner, for a moment. And we'll come back together"* (Videotape, Mike, No. 5). In a lesson where Mike followed students' ideas, even though he was good at asking high-level questions, he eventually ended up telling students what to do: *"The 'square' part tells you that you need to find the area. Go ahead and find the area"* (Videotape, Mike, No. 7). Mike, like Amy, was also constrained by the curriculum.

> *When we did student-led learning, we ended up with, uhm, pushing them for 15 minutes or so, and ended up not getting the concepts out of that we wanted them to. And to get through the curriculum, and get through the things that we want to teach each year, we've got to be on a, at least, certain speed, or certain direction on each day.* (Interview, Mike, 01/17/2005)

On the other hand, John pointed out the importance of spending time on a single problem, by saying *"We don't get as much our curriculum covered because it takes longer, so, but the students are learning in greater depth. Perhaps, this is more important"* (Interview, John, 01/11/2006).

For Mike and Amy, behavioral problems were also a challenge in this student-oriented learning environment as Mike said, *"Some students don't know their boundaries in student-led learning"* (Interview, Mike, 01/17/2005) whereas, for John, *"A noisy class never bothers me.…That's why I do those warm-up activities [e.g., The Event of the Day] before we get started. To get to know students on a human level."* (Interview, John, 01/11/2006). Indeed, even though John had more students with behavioral problems, he

was able to get them involved in solving problems by assigning them roles in their groups. On the other hand, Mike and Amy used "punishment" (e.g., suspension or the principal's office) to control such behaviors, which turned students off to the lesson. Interestingly, the on-site and videotape observations revealed that students with behavioral problems in Mike and Amy's classes in fact did make significant contributions to the discussion about solving problems.

Assertion 3. While the teachers were more active in their questioning strategies, they were unable to move to a consistent pattern of dialogical interaction.

From Figures 14b, 14d, and 14f, it can be seen that all the teachers, at some point, had a high level of questioning yet were unable to incorporate that with problem solving and dialogical interaction. For example, the major area that John needed to work on was his domination of classroom interaction; that is, even though he asked thought-provoking questions, the interaction was only with a particular student. This notion of interaction meant that student voice was lost in his class. Throughout the debriefing sessions, he realized that *"So, you just want me to shut up and not tell the answer?"* (Fieldnotes, John, 11/13/2006). John's level of questioning also improved his role during problem solving by giving students more opportunities to discover their own methods. He indicated that

> *In math there is always a right answer, and I tell the answer. But now with this project, [I am] trying to step back. …The first time we did, it wasn't as good as it should be. Because they'd never done it before. But they get better at listening. I discovered ways of looking at stories I've never thought of. One time, about a*

problem, one girl came up an idea, which I never thought of, because I've never

given them think time to tell me how they do it. (Interview, John, 01/11/2006)

Similarly, Mike's classroom observations showed that he often lost track of students' ideas since he focused on his questioning and had a teacher-only-one-student interaction pattern. During the debriefing sessions the researcher focused on questioning and the effects of higher-order questioning in other pedagogical areas. Mike worked on his questioning and on not telling students the answer if the correct one did not emerge. To this end, he started to call on different students, which required other students to be engaged in problem solving and listening to their peers, and he often rephrased his questions to make them explicit to the students (e.g., *Well, what I'm trying to ask is …).* Thus, he managed to incorporate his questioning with problem solving later in his implementation. For example, he spent more than half of a lesson trying to come up with different ways of solving an equation. The students were fully engaged, working together to come up with, as they said at the end of the lesson, *"different routes to get to the town if the high-way is closed"* (Videotape, Mike, No. 10). On the other hand, Mike solved four different problems in 15 minutes in his control class, without any discussion or getting students' ideas. This type of teaching was his regular control teaching throughout the project. This view also came out during the interview, as he defined learning: *"It [learning] goes like hand in hand. I have x amount of knowledge and skills that I can pass on to the students"* (Interview, Mike, 01/17/2005). Even though he acknowledged students' own knowledge and ways of knowing, this idea was consistent neither with his pedagogical practices nor with his definition of the teacher's role in the classroom, as he indicated during the interview: *"I think it [the role of the teacher] depends on what type*

of learning atmosphere we want for the day. Sometimes, I can be more of a facilitator, and other times, if it is information type, I need to lecture and give it to them directly" (Interview, Mike, 01/17/2005).

The same perspective on teaching was also held by Amy. Her implementation of the MRH approach was not as high as Mike or John's. Yet, there were instances where she tried harder to get students involved in discussion, as can be seen in her eighth observation (Figure 14f). From the observation, it appeared as though Amy was struggling with managing classroom discipline. As such she tended to go back to her traditional teaching, where she lectured most of the time. She often indicated during the debriefing session and the interview that *"these students are not used to this kind of learning, so lecturing works better for them"* (Debrief, Amy, 12/05/2005).

Assertion 4. The teachers implemented writing in different ways.

The teachers used writing in a variety of ways such as filling in the MRH student template while solving a problem, writing an explanation of a problem solution to an audience, and writing a letter about a topic. While Mike and Amy struggled to implement many writing activities in their classes—for example, Amy implemented only two writing activities during the first unit—John, on the other hand, was very active trying to get writing activities going, as he said, *"Writing is always natural to me. But I never used that much in my lesson plan, writing in this way, so simple. 'Explain the concepts so that a fifth grade student can understand it. Or imagine somebody is absent.'"* (Interview, John, 01/11/2006). This notion of writing, *"not just writing, but writing to an audience,"* was appealing to John, as he said, *"before they only wrote to me."* All three teachers used writing prompts such as "Write a letter to a 5th-grade student about multiplying

fractions." or "Explain to your parents what you have learned this week about factoring polynomials." Mike, for example, asked his students to write a letter for 5th-grade students about ratios and how they could use them in shopping. He actually took the letters to a fifth grade teacher and asked her to distribute to her students and ask them to read and evaluate the letters. Similarly, John had his MRH students write an explanation of the solution of a problem they solved in class to 7th-grade students, while his control students were asked to "*write an explanation about this problem so that I can understand that you know how to solve it.*" John was excited when he was talking about this activity during a debriefing session:

> *You know what, Reggie? The seventh-grade writing came out wonderful. They drew pictures, and all other stuff. Even they came up with different ways of solving the problem, which we didn't in the class. They thought of what they knew back in seventh grade.* (Debrief, John, 12/15/2005)

On the other hand, Amy was using writing as an assessment tool rather than a learning tool. While John spent time in class discussing what students wrote, Amy and Mike did not provide feedback on writing assignments. In fact, there was only one instance where Amy spent some time on discussing students' writings: when John and she collaborated for a writing activity during the second unit. However, although Amy did not provide feedback on writing assignments, she was more insistent on having students complete the MRH student template as part of the problem solving process.

Discussion

This study focuses on examining the changes in *pedagogical* practices when three teachers shift from their traditional teaching to more student-centered practices. The

study also looked at the *performance* differences on the Iowa Test of Educational Development (ITED) between the students in the control classes where the teachers engaged in their traditional instructional routines and the students in the treatment classes where the teachers used the MRH approach to improve their pedagogical practices and to scaffold students' problem solving skills. The study particularly focused on changes in the teachers' pedagogical practices when using the MRH approach. The major findings of this study are that implementing a student-oriented approach such as the MRH approach which includes embedded writing-to-learn strategies does have an impact on student performance and that teachers' adoption of the required pedagogical practices varied as they attempted to move away from their traditional practices.

The student performance on the standardized test was significantly enhanced for students in the MRH classrooms than students who engaged in the more traditional approaches—the statistically significant main effect of group was in favor of the MRH (F (1, 163) = 5.381, p = .022, η^2 = .032). Moreover, the performance differences in effect sizes in the ITED scores between John's MRH classes and the other two teachers' and his own control classes—Mike' control (.618 standard deviation units), Amy's control (.916 standard deviation units), and John's control (.450 standard deviation units)—indicate the benefit of using student-oriented approaches such as the MRH approach with embedded writing-to-learn strategies in problem solving. I argue that there are two reasons why these differences occurred. First, the results from the analysis of the teachers' pedagogical practices in their treatment and control classes show us the importance of pedagogical skills to promote dialogical interaction during problem solving. Such a result supports the earlier work of Cobb, Yackel, and Wood (1993) and Simon (1995, 1997)

who have argued for implementing a constructivist approach to learning, where the teacher's role is crucial to promoting social aspects of learning by creating dialogical interaction. Specifically, Lester et al. (1994) and Simon (1997) stated that the teacher, by avoiding being the authority of knowledge in the classroom contrary to the view of appropriate teacher behavior in the past, can promote students' negotiation of mathematical ideas and problem solution methods. This concept of dialogical interaction and negotiation of mathematical ideas and solutions gives students the ownership of problem solving. The students in the MRH classes were involved in the process of construction of mathematical relations, questions, problems, and meanings rather than, as Ernest (1998) says, working on textual or symbolically presented teacher-set tasks. This study suggests that teachers should change their pedagogical practices from an algorithmic view of problem solving to a negotiation view of problem solving. Similarly, Cobb, Boufi, et al. (1997) and Watson and Mason (2005) also argue that the importance of social constructive nature of mathematical objects is that individuals actively construct their mathematical understandings as they participate in a socially situated discourse.

In examining the results the researcher would suggest that there are two critical elements of the MRH approach. The first is the pedagogical approach needed and the second is the consistent use of the heuristic concept through the scaffolded writing component of the MRH approach. Given that John's RTOP scores and his willingness to use writing activities in his classroom are consistently higher than Mike and Amy's, I would suggest that he is able to build both of these critical components, that is, the pedagogy and the heuristic writing. However, when examining Mike and Amy's results, I would suggest that each had a different emphasis in his/her classroom. Mike's RTOP

scores were higher than Amy's which indicates that he was more prepared to engage with the required pedagogical practices. However, the classroom observation data does provide evidence that Amy was more insistent on having students complete the MRH student template as part of the problem solving process. While she struggled with the pedagogical changes her students were still able to engage in the heuristic problem solving process. This data does begin to indicate the importance of building both of these aspects together as a means to benefit student performance.

In examining the changing pedagogical practices, the researcher would suggest that developing appropriate questioning skills is a necessary skill in promoting dialogical interaction. The break-down for the RTOP scores in the teachers' MRH classes (the right-hand side of Figure 14) shows that questioning was the skill that moved the most and was the one that brought the others (student voice, problem solving and reasoning, and teacher role) up to match it. This structure was parallel across the three teachers and had a match with the fluctuations in their MRH implementations shown on the left-hand side of Figure 14. That is, as the teachers improved their questioning skills, they were better able to implement the MRH approach. This suggests that in helping teachers to implement the MRH approach we need to focus attention on their questioning skills. Moreover, for better results, teachers should align their questioning with dialogical interaction by providing students with more opportunities to discuss their problem solving methods in public.

The structure of the MRH approach is two fold. The teacher component requires teachers to make connections among different aspects of learning mathematics in the classroom. For example, they are to consider the importance of student-student dialogue

as well as of self-dialogue (e.g., through writing) in problem solving. Students in the MRH classes are engaged in such a learning experience in the classroom where they can discuss their ideas with their peers, and through individual writing they are engaged in self-problem solving using the MRH student template. Therefore, it is possible to see two different sources of effect on students' learning when using the MRH approach. First, teachers using the MRH approach are required to change their pedagogical practices in a way to promote student dialogical interaction through questioning. Second, using the MRH student template students are engaged in self-dialogue through writing—the heuristic function of the MRH approach. In other words, students use the template as a thinking device rather than as a worksheet. For this reason, the MRH approach should be considered as a whole rather than as two different parts for teachers and students in order to observe its impact on the improvement of students' learning and problem-solving skills.

CHAPTER FOUR

General Conclusions, Implications, and Limitations

General Overview

The last chapter of the dissertation first addresses the general conclusions

emerging from the literature on mathematical problem solving and writing-to-learn in

mathematics and makes the connection to the themes of the results of the study. Second,

implications based on the findings of the study and their relationships to the literature are

discussed. Finally, surrounding limitations of the study are delineated.

General Conclusions

Literature Review

The literature review of the dissertation traces a wide range of the literature on

mathematical problem solving, the nature of mathematics, the construction of

mathematical knowledge, writing-to-learn strategies in mathematics, and the relationship

between the processes of mathematical problem solving and writing. The studies on

problem solving point out the importance of integrating problem solving into

mathematics programs to promote students' conceptual understanding of the mathematics

they study (Hiebert & Wearne, 2003; NCTM, 2000). Specifically, the NCTM states that

the effectiveness of school mathematics in work, school, and life "lies at the heart of

problem solving" (p. 334).

Given the significance of problem solving in learning mathematics, the literature

review also shows that teaching and learning mathematics through problem solving gives

both teachers and students opportunities to make sense of mathematical ideas by

appreciating the individual and collective aspects of mathematics—that is, the

conversational nature of mathematical knowledge (Ernest, 1998; Kahan & Wyberg, 2003; Rasmussen, Yackel, & King, 2003). Moreover, Hiebert and Wearne (2003) have argued that students should be challenged with mathematical problems that make them struggle in finding solutions and that require them to work collaboratively to solve the problems.

Based on the literature review on the nature of mathematics and social construction of mathematical knowledge (Borasi, 1992; Cobb, Boufi, McClain, & Whiteneck, 1997; Cobb, Yackel, & Wood, 1992, 1993; Ernest, 1998; Krummheuer, 2000) and taking a constructivist perspective to learning, I conclude that learning is associated with personal meaning making based on the already existing knowledge structure and the social interaction with the environment. Therefore, learning mathematics should be considered an active process of personal construction of mathematical understanding through the use of natural language in a socially situated conversation by students. Further, Borasi and Ernest point out the humanistic and conversational aspects of learning mathematics.

However, considering that learning can be, and is, an individual act and that conversation can occur within an individual (without an interlocutor), I highlight the importance of writing in learning (mathematics); that is, writing as a "unique mode of learning" allows the writer to generate new ideas and new meanings via an ongoing dialogue between the written product (text) and one's thoughts (Bereiter & Scardamalia, 1987; Emig, 1977; Galbraith, 1999; Galbraith & Torrance, 1999; Vygotsky, 1986). Furthermore, Galbraith (1999) and Powell and Lopez (1989) have argued that during writing one engages in a dispositional dialectic where he or she negotiates meaning, and in negotiation, one is constructing knowledge and augmenting learning.

Using the problem-solving model of Artzt and Armour-Thomas (1992) and Galbraith's (1999) writing as knowledge-constituting model, I have examined the parallel structures of the problem-solving process and writing process. In both processes, one has to move between different episodes in order to solve a problem or write a text. For example, both writing and problem-solving processes begin with an input (writing topic or mathematical problem), which activates the knowledge network units related to the topic. Both problem solver and writer engage in an internal negotiation process through both problem solving and writing, which is what Galbraith called dispositional dialectic. Even though there are studies exploring the relationship between writing and problem solving, these studies have not outlined the parallel structures (Pugalee, 2001; Steele, 2005). Pugalee and Steele's work provide evidence of metacognitive behaviors in students' mathematical writing. These studies also show that the demanding and connective notion of writing scaffolds students' problem solving activities and, therefore, enhances their mathematical knowledge structure.

Building on the work in the areas of problem solving and writing and on the bases of the SWH approach by Hand and Keys (1999), I propose a framework—the mathematics reasoning heuristic (MRH) approach—for teachers and students to combine different aspects of mathematics teaching and learning such as teachers' knowledge of mathematics, students' knowledge of mathematics, and students' learning of mathematics. These aspects come together with the MRH approach in a dialogical interaction and writing. The intent of the MRH approach is, besides supporting teachers' pedagogical practices, to guide students' problem solving behaviors and their problem

solving skills through the use of writing (Connolly & Vilardi, 1989; Kenyon, 1989; Morgan, 1998).

The Study

The purpose of the study is to examine the changes in *pedagogical* practices when three teachers change their practices from traditional teaching to student-centered practices and to explore the *performance* differences in the ITED scores between the students in the control classes where the teachers engaged in traditional instructional practices and the students in the treatment classes where the teachers used the MRH approach to improve their pedagogical practices and to scaffold students' problem solving skills. The major findings of the study are that implementing a student-centered approach such as the MRH approach with embedded writing-to-learn strategies in mathematics classrooms does have an impact on students' performance and that teachers improve pedagogical practices at different times during an attempt to shift their teaching practices to practices based on a student-centered learning.

Performance on the ITED mathematics test significantly favored those students who were involved in the MRH approach compared to students involved in the teachers' traditional approach. Moreover, significant mean score differences between John's MRH classes and the other two teachers' and John's own control classes indicate the benefit of using the MRH approach with embedded writing-to-learn strategies in the problem solving process. The first reason for this difference is the teachers' ability to promote dialogical interaction during problem solving. The results from the analysis of the teachers' pedagogical practices in their treatment and control classes illustrate the importance of pedagogical skills in promoting dialogical interaction. In implementing a

constructivist approach to learning, the teacher's role is crucial to promoting the social aspects of learning by creating dialogical interaction (Cobb, Yackel, & Wood, 1993; Simon, 1995, 1997). Simon (1997) further argued that the teacher, by avoiding being the authority of knowledge in the classroom, can promote students' negotiation of mathematical ideas and problem solution methods. This concept of dialogical interaction and negotiation of mathematical ideas and solutions gives students ownership of problem solving. The students in the MRH classes are involved in the process of the construction of mathematical relations, questions, problems, and meanings rather than, as Ernest (1998) says, working on textual or symbolically presented teacher-set tasks.

The results obtained from observational data indicate the importance of questioning in creating dialogical interaction. For example, even though John's questioning was high during the third, fourth, and fifth observations, he was unable to use it to improve other areas of his pedagogy. On the other hand, throughout the debriefing sessions, John used his questioning not only to get richer responses but also to challenge students' ideas and create conflicts in students. He was then able to have students discuss with each other in order to resolve these conflicts. Von Glaserfeld (1993) argues that through dialogical interaction students have the opportunities to modify their existing concepts and conceptual relationships according to inconsistencies they are experiencing at the moment. Thus, questioning is a crucial part of pedagogical practices because it creates perturbations requiring students to negotiate their meanings and mathematical experiences among themselves and thus providing opportunities for them to resolve these conflicts.

The second reason for the difference comes from the heuristic/writing function of the MRH approach. The MRH approach provides students with a heuristic as a problem-solving device. In order to have an impact on students' standardized test scores, the pedagogical practices and the heuristic/writing function of the MRH approach should be combined. While John was able to make a greater connection between these two components, Amy and Mike tried various, sometimes inconsistent, combinations of the pedagogy and heuristic/writing functions of the MRH approach during their practices. However, even though Mike's implementation rated by the RTOP was greater than Amy's, their MRH class means were equal. This would indicate that the heuristic and writing functions of the MRH approach may also play a critical role on students' standardized test scores, given that Amy's pedagogical practices were the weakest, yet her class means were the same as Mike's. These two functions make problem solving a valuable tool for students to deepen their understanding of mathematical ideas (Goldenberg & Walter, 2003; Heibert & Warne, 2003) and mathematics as "a living language, or, more precisely, a family of living languages" (Layzer, 1989, p. 129).

The results from this study indicate that the MRH approach should be considered as a whole interweaving the teacher and student templates together by integrating writing into problem solving. In terms of implementation, the studies conducted in science areas using the SWH approach (Akkus, Gunel, & Hand, in press; Gunel, 2006) also showed that students' test scores were directly correlated with the level of implementation. Low-level teachers tended to use the SWH approach as a template without incorporating the major pedagogical practices suggested by the SWH approach.

Implications

The results of the study suggested a number of implications for students during problem solving and for mathematics teachers who wish to shift their teaching practices to more student-oriented approaches where they can integrate writing into mathematical problem solving. First, the MRH approach provides a scaffold for teachers to understand the fundamental aspects of learning and teaching mathematics in a student-centered manner because it requires teachers to consider various components of classroom learning such as students' knowledge of mathematics, teacher's knowledge of mathematics, and teacher's knowledge of students' learning. To implement such an approach, teachers should identify the fundamental concepts of a unit and relate them to how students learn. The teacher template is a tool for teachers to consider different aspects of learning through the phases of negotiation. Having said the MRH approach provides a scaffold for teachers, teachers can also use the RTOP instrument to evaluate their own or a colleague's teaching practice. Even though this study suggests the importance of long-term support for teachers, there is still a need for research that examines teachers' pedagogical changes over a longer period of time documenting the effect of these changes on students' standardized test scores.

Second, the MRH approach provides a problem solving heuristic for students during problem solving. The student template is not just a template to fill in; rather, it is a device to scaffold students' mathematical thinking and problem solving. One of the major connotations that can be drawn from the results is that this study suggests that students should be provided opportunities to negotiate problem solving rather than engage in the more expected role of memorizing and practicing teacher-generated problem solving

methods. In accordance with students' learning through negotiation, teachers should shift their view of problem solving from an algorithmic view to a negotiation view of problem solving. The performance difference between students in the control and MRH classes indicates that when students have the opportunities to negotiate their problem solving either in public or private through writing, they construct their own mathematical understanding. However, this study did not examine students' use of the student template. In other words, further research needs to be conducted to examine students' problem solving episodes in order to explore what aspects of cognitive and metacognitive behaviors occur when students use the MRH student template during problem solving.

Third, the results of the study suggest that the MRH approach should be implemented by combining the heuristic and writing functions of the approach with the necessary pedagogical skills in order to promote students' mathematical learning. However, one can argue that the effect of the approach may only come from a teacher's implementation level (rated by the RTOP), and, therefore, writing has nothing to do with the students' scores. Another argument might be that writing is the only key for students' success on the standardized test. In these cases, further experimental studies are needed that focus on only (a) the teacher component of the approach, (b) the student component (the heuristic and writing functions), and (c) the combined version as suggested by this study to delineate effects of each model component of the MRH approach.

General Remarks

This study looked at the changes in teachers' pedagogical practices through the use of the MRH approach as a way to promote students' problem solving via dialogical interaction and writing. The study was able to show the need for analyzing the

relationship between different areas of teacher pedagogical practices and helping teachers to use their questioning skills to promote a better classroom negotiation. Moreover, the study suggests that teachers should change their practices from an algorithmic view of problem solving to a negotiation view of problem solving.

Limitations

There are several limitations surrounding the study. First, because of the nature of the study and the teacher volunteers, random sampling was not possible; therefore, one of the statistical assumptions was violated. In addition, even though the town that the school is located in can be considered typical for Iowa, the characteristics of the school district and town may be different from other school districts and towns in general. Thus, generalization of the findings from the study is limited to the sample of the study in the population in Iowa, not to a population in general. The assignment of students to classes was completed by the school district and assumed to be random. Indeed, random sampling for research purposes in educational settings is rarely possible due to several organizational and cultural restrictions.

Another limitation is that even though standardized tests have been widely used in educational research for measuring learning outcomes, they lack measures of the complexity of the learning process. Yet, standardized tests are stronger indicators than teacher generated tests. The ITED, in particular, measures not only content areas with factual knowledge, but it also tests the mathematical problem solving and data interpretation.

The other limitation of the study is the time of the intervention for the teachers and their teaching experience in the school district featured in the study. For example, the

data of the study were collected at two different times, the 2004-2005 and 2005-2006 school years. Moreover, while John has been in this school district for 28 years, Mike and Amy have been for 5 and 1 years, respectively. This might have effected Amy's implementation. Further, the number of videotapes for the control and treatment classes was not equal. Therefore, the ability to draw a general conclusion about the relationships between teachers' pedagogical shifts and students' test scores is constrained.

The final limitation is the researcher's biases. Since the interaction between the researcher and the teachers occurred at different times and in different conditions, the relationships between the teachers and researcher may have influenced the researcher's view and interpretation of the data. However, the researcher attempted to reduce this influence by using multiple data collection methods, triangulation of the data, and independent observers. Moreover, one would argue that without the researcher's subjective perspective, it is not possible to draw such conclusions. Subjectivity is part of research.

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FOOTNOTES

[1] The schema (knowledge) is defined in Silver (1982) as "a cluster of knowledge that describes the properties of the concept it represents" (p. 16).

[2] The collective knowledge can be a mathematical knowledge introduced by the teacher (e.g., $A = l \times w$), or it can be created by the participating students in the classroom.

[3] The condition of the problem is the relationship between the data and the unknown (e.g., in a problem finding the sides of a rectangle whose circumference is known with the largest area, the known circumference and the area are the two conditions). In other words, it is the *constraint* or *limit* of the problem.

[4] Schoenfeld's 1983 edition of the heuristic includes "reading" as the first stage of problem solving; however, the 1985 version considers reading within analysis.

[5] "Ordinary or 'natural' language is the medium in which we think and speak about the world of ordinary experience. The words and phrases of a natural language derive their meanings, directly or indirectly, from shared experience" (Layzer, 1989, pp. 124-125).

[6] New ways of using language (e.g., in algebra "let x be any number") (Adler, 1999, p. 51).

[7] New ways of using language (e.g., in algebra "let x be any number") (Adler, 1999, p. 51).