NUMERICAL ELECTROMAGNETIC MODELING FOR THREE-DIMENSIONAL INSPECTION OF FERROUS METALS

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I. INTRODUCTION

The problem that we are trying to solve is: Given an excitation source, which is known to us, and a scattered field, which we can measure (albeit somewhat inaccurately, because of noise and the like), determine the spatial distribution of the electromagnetic parameters, μ, and σ, where μ is the magnetic permeability and σ the electrical conductivity. This allows us to determine the structure of a body in free space, or the structure of an internal flaw (or anomalous region) within a given body whose properties, such as size, shape and electrical parameters, are known to us. Throughout this paper we will consider only isotropic bodies, which means that the conductivity and magnetic permeability are scalar functions of positions.

Our approach to solving this problem consists of the following steps:

1. transform Maxwell's equations into the spectral domain
2. define inverse source and inverse scattering models
3. derive a system of equations from the inverse scattering model by using multifrequncy excitation
4. solve the system in a least-squares sense by using the QR-decomposition, Singular Value Decomposition, or other suitable numerical algorithm [1,2].
5. take the inverse Fourier transform via the Fast Fourier Transform (FFT) algorithm.

II. THE FIELD EQUATIONS IN THE SPECTRAL DOMAIN

Electromagnetic models are usually derived from Maxwell's field equations. The time-harmonic version of these equations, with sources acting in free-space, is given by:

$$\nabla \times \bar{E} = -j\omega\mu_0\bar{M} - j\omega\mu_0\bar{H} \tag{1}$$
$$\nabla \times \bar{H} = j\omega\varepsilon_0\bar{E} + \bar{J},$$

where $\bar{M}$ and $\bar{J}$ are the sources of the scattered field due to the ferrous body. $\bar{M}$ is the magnetization vector (magnetic moment per unit volume), and $\bar{J}$ the electric current vector. The first equation in (1) is the point form of Faraday's law when we recall that

$$\bar{B} = \mu_0(\bar{H}+\bar{M}) \tag{2}$$

We solve for the fields $\bar{E}$ and $\bar{H}$ by superposing the partial fields due to the sources acting independently. The solution for the fields in the region exterior to the sources is given by the two-dimensional Fourier transforms:

$$\bar{E}(\bar{r}) = \frac{j\omega\mu_0}{8\pi^2}\int\int_{-\infty}^{\infty} \frac{\bar{k}\times\tilde{\bar{M}}(\bar{k})e^{-j\alpha_0 z}}{\alpha_0} e^{-j(k_x x+k_y y)} dk_x dk_y$$
$$+ \frac{1}{\omega\varepsilon_0 8\pi^2}\int\int_{-\infty}^{\infty} \frac{[\bar{k}\bar{k}-k_0^2\bar{\bar{I}}]\cdot\tilde{\bar{J}}(\bar{k})e^{-j\alpha_0 z}}{\alpha_0} e^{-j(k_x x+k_y y)} dk_x dk_y$$
$$\bar{H}(\bar{r}) = \frac{-1}{8\pi^2}\int\int_{-\infty}^{\infty} \frac{\bar{k}\times\tilde{\bar{J}}(\bar{k})e^{-j\alpha_0 z}}{\alpha_0} e^{-j(k_x x+k_y y)} dk_x dk_y$$
$$+ \frac{j}{8\pi^2}\int\int_{-\infty}^{\infty} \frac{[\bar{k}\bar{k}-k_0^2\bar{\bar{I}}]\cdot\tilde{\bar{M}}(\bar{k})e^{-j\alpha_0 z}}{\alpha_0} e^{-j(k_x x+k_y y)} dk_x dk_y\ , \tag{3}$$

where

$$\bar{k} = k_x\bar{a}_x + k_y\bar{a}_y + \alpha_0\bar{a}_z\ ,\quad \alpha_0 = (k_0^2-k_x^2-k_y^2)^{1/2}\ ,\quad k_0^2 = \omega^2\mu_0\varepsilon_0 \tag{4}$$

Upon taking two-dimensional Fourier transforms of the measured fields, $\bar{E}$ and $\bar{H}$, over the plane z = 0, we get the algebraic system:

$$\tilde{\bar{E}}(k_x,k_y) = \frac{j\omega\mu_0}{2}\tilde{\bar{\bar{R}}}(\bar{k})\cdot\tilde{\bar{M}}(\bar{k}) + \frac{1}{2\omega\varepsilon_0}\tilde{\bar{\bar{S}}}(\bar{k})\cdot\tilde{\bar{J}}(\bar{k})$$
$$\tilde{\bar{H}}(k_x,k_y) = \frac{j}{2}\tilde{\bar{\bar{S}}}(\bar{k})\cdot\tilde{\bar{M}}(\bar{k}) - \frac{1}{2}\tilde{\bar{\bar{R}}}(\bar{k})\cdot\tilde{\bar{J}}(\bar{k}) \tag{5}$$

in which

$$\tilde{\bar{\bar{R}}}(\bar{k}) = \frac{1}{\alpha_0}\begin{bmatrix} 0 & -\alpha_0 & k_y \\ \alpha_0 & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}, \quad \tilde{\bar{\bar{S}}}(\bar{k}) = \frac{1}{\alpha_0}\begin{bmatrix} k_x^2-k_0^2 & k_xk_y & \alpha_0k_x \\ k_yk_x & k_y^2-k_0^2 & \alpha_0k_y \\ \alpha_0k_x & \alpha_0k_y & \alpha_0^2-k_0^2 \end{bmatrix}$$

This is the basic system from which we derive our model equations. Before defining the inverse source and scattering models, however, we should point out the following facts about (5):

1. because $\bar{k}\cdot\tilde{\bar{E}} = 0$ and $\bar{k}\cdot\tilde{\bar{H}} = 0$, there are only two independent field components in the region exterior to the sources.
2. $\tilde{\bar{E}}(k_x,k_y)$, $\tilde{\bar{H}}(k_x,k_y)$ are two-dimensional transforms of measured data
3. $\tilde{\bar{M}}(\bar{k})$ and $\tilde{\bar{J}}(\bar{k})$, the three-dimensional transforms of the sources, are known only on the sphere $\bar{k}\cdot\bar{k} = k_0^2$
4. $DET(\tilde{\bar{\bar{R}}}(\bar{k}))=0$, $DET(\tilde{\bar{\bar{S}}}(\bar{k}))=0$.

III. INVERSE SOURCE MODEL

By using the inverse source model we seek to determine the Fourier transforms of the source densities, $\bar{J}$ and $\bar{M}$, and then invert the transforms to compute the spatial distributions of the sources. Because of the facts outlined at the end of the last section, this program is generally impossible. For example, the source transforms are known only over a sphere in the spectral domain, and it does no good to change the frequency (thereby changing the radius of this sphere) because at each frequency we introduce a new source field distribution. In addition the matrices $\tilde{\bar{\bar{R}}}$ and $\tilde{\bar{\bar{S}}}$ in (5) are not invertible. This means that in order to determine the source densities, one needs additional information, such as the fact that the source densities may be solenoidal. The source model cannot distinguish between the electric and magnetic sources, using measurements made outside the source, unless additional information is available, such as $\bar{J}$=0, or $\bar{M}$=0, or that only certain components of $\bar{J}$ or $\bar{M}$ are nonzero. For these reasons we will not pursue the inverse source model, but will go on to the inverse scattering model.

IV. INVERSE SCATTERING MODEL

In this model we write the sources in terms of the electric and magnetic field and the electrical conductivity and magnetic permeability. Then the problem is solved iteratively by computing the actual fields within the anomalous region, as in a classical direct scattering problem. This approach, though rigorously correct, is time consuming, so approximations are usually made to allow the computations to proceed quickly.

Consider a system in which a current sheet excites a plane-parallel ferrous plate containing a localized anomalous region. The exciting current flows in the x-direction and is uniform in the x- and y-directions. The sensors are located at the plate z=0. The fields within the unflawed plate are independent of x and y:

$$\bar{E}_0(\bar{r},\omega) = E_0(z,\omega)\bar{a}_x \quad , \quad \bar{H}_0(\bar{r},\omega) = H_0(z,\omega)\bar{a}_y \quad . \tag{6}$$

In the presence of the flaw, we use the same fields and write for the source densities:

$$\begin{aligned} \bar{J}_a(x,y,z,\omega) &= E_0(z,\omega)\sigma_a(x,y,z)\bar{a}_x \\ \bar{M}_a(x,y,z,\omega) &= H_0(z,\omega)\mu_a(x,y,z)\bar{a}_y \quad , \end{aligned} \tag{7}$$

where the material parameters $\sigma_a(x,y,z,)$, $\mu_a(x,y,z)$ of the anomalous region are the unknowns. The approximation that allows us to write (7) follows from the fact that the anomalous region is small, and therefore does not greatly perturb the field that existed at the same point in the unflawed plate. This approximation allows us to decouple the direct scattering problem from the inverse problem.

Next we partition the plate into N_z layers in the z-direction and then expand $E_0(z,\omega)\sigma_a(x,y,z)$ and $H_0(z,\omega)\mu_a(x,y,z)$ in pulse functions with respect to this partition. Then, upon taking the three-dimensional Fourier transform of the source functions, and evaluating them over the appropriate sphere in k-space, we get:

$$\tilde{J}_a(k_x,k_y,\alpha_0) = 2\sum_{i=1}^{N_z} e^{j\alpha_0 z_i} \frac{\sin(\alpha_0\delta/2)}{\alpha_0} E_0(z_i\omega)\tilde{\sigma}_i(k_x,k_y) \tag{8}$$

$$\tilde{M}_a(k_x,k_y,\alpha_0) = 2\sum_{i=1}^{N_z} e^{j\alpha_0 z_i} \frac{\sin(\alpha_0\delta/2)}{\alpha_0} H_0(z_i,\omega)\tilde{\mu}_i(k_x,k_y)$$

where z_i is the midpoint of the ith layer and δ is the z-length of a layer.

The surviving equations in the spectral domain are found from (5) to be:

$$\tilde{\tilde{H}}(k_x,k_y\omega) = \frac{j}{2}\begin{bmatrix} k_xk_y \\ k_y^2-k_0^2 \\ \alpha_0 k_y \end{bmatrix}\frac{\tilde{M}_a}{\alpha_0} - \frac{1}{2}\begin{bmatrix} 0 \\ \alpha_0 \\ -k_y \end{bmatrix}\frac{\tilde{J}_a}{\alpha_0} \tag{9}$$

$$\tilde{\tilde{E}}(k_x,k_y,\omega) = \frac{j\omega\mu_0}{2}\begin{bmatrix} -\alpha_0 \\ 0 \\ k_x \end{bmatrix}\frac{\tilde{M}_a}{\alpha_0} + \frac{1}{2\omega\varepsilon_0}\begin{bmatrix} k_y^2-k_0^2 \\ k_xk_y \\ \alpha_0 k_x \end{bmatrix}\frac{\tilde{J}_a}{\alpha_0} \quad .$$

Consider the y-component of the $\tilde{\tilde{H}}$-equation; it becomes

$$\tilde{H}_y(k_x,k_y,\omega) = \frac{j}{2}(k_y^2-k_0^2)\frac{\tilde{M}_a}{\alpha_0} - \frac{1}{2}\tilde{J}_a \; , \tag{10}$$

which couples the two sources. This equation has the advantage, however, in not having the coefficients of $\tilde{M}_a$ or $\tilde{J}_a$ vanish for any values of $\bar{k}$ (except for $k_y=k_0$, which is not an "essential zero"). On the other hand, we can uncouple the sources by taking the x-component of the $\tilde{\tilde{H}}$-equation and the y-component of the $\tilde{\tilde{E}}$-equation:

$$\tilde{H}_x(k_x,k_y,\omega) = \frac{j}{2}\frac{k_xk_y}{\alpha_0}\tilde{M}_a \tag{11}$$

$$\omega\varepsilon_0\tilde{E}_y(k_x,k_y,\omega) = (-\alpha_0\tilde{H}_x+k_x\tilde{H}_z) = \frac{k_xk_y}{2\alpha_0}\tilde{J}_a \quad ,$$

where we have used the second of Maxwell's equations, (1), to rewrite $\tilde{E}_y$ in terms of $\tilde{H}_x$ and $\tilde{H}_z$. We do this because we assume that it is easier to measure the magnetic field rather than the electric field at the low frequencies of interest here. In (11) we see that there is an essential singularity at $k_x=0$ and $k_y=0$. We will come back to this point shortly.

Upon substituting (8) into either (10) or (11), we get the following generic forms:

$$B_1(k_x,k_y,\omega) = \sum_{i=1}^{N_z} e^{j\alpha_0 z_i}\left\{\frac{j(k_y^2-k_0^2)}{\alpha_0}H_0(z_i,\omega)\tilde{\mu}_i - E_0(z_i,\omega)\tilde{\sigma}_i\right\}, \tag{12}$$

$$B_2(k_x,k_y,\omega) = \sum_{i=1}^{N_z} e^{j\alpha_0 z_i} H_0(z_i,\omega)\tilde{\mu}_i$$

$$B_3(k_x,k_y,\omega) = \sum_{i=1}^{N_z} e^{j\alpha_0 z_i} E_0(z_i,\omega)\tilde{\sigma}_i \quad .$$

In all these cases, there are $2N_z$ unknowns, $\tilde{\mu}_i(k_x,k_y)$, $\tilde{\sigma}_i(k_x,k_y)$, for each spectral-pair (k_x,k_y). Hence, in order to get a linear system whose solution will be these unknowns, we need to evaluate (12) at a number of different frequencies. Hence, the inverse scattering model involves multifrequency excitation. The number of frequencies ought to be larger than $2N_z$ in order to provide an overdetermined system for a least-squares solution of the first equation in (12). We can use N_z frequencies if we measure the two independent components, H_x, H_z, of $\bar{H}$ and solve the second and third equations in (12).

V. NUMERICAL METHODS

If we choose N_f frequencies then any of the equations in (12) becomes an N_f-by-N_z (or $2N_z$) system of linear equations. We solve this system in a least-squares manner [1, 2]. Because the system is usually ill-conditioned, we must regularize it by using a Levenberg-Marquardt parameter [1] or some other suitable regularization technique, such as constrained least-squares [2]. The two methods of solving least-squares problems that are most attractive use either the QR-decomposition or the singular value decomposition (SVD). These methods are thoroughly described in references [1, 2]. The solutions of the equations are the two-dimensional Fourier transforms of the conductivity and permeability at the ith layer of the body. After solving the equations, we take the inverse Fourier transform, using the fast Fourier transform algorithm (FFT), to obtain the spatial distribution of the material parameters. We use analytic continuation to continue the transform solutions through the "essential zeros" that were described in the last section.

Hence, we can summarize the inverse scattering model algorithm as:

Measure the appropriate components of the $\bar{H}$-field at N_f frequencies, $\omega_1,\ldots\ldots,\omega_{N_f}$, and then compute the 2D FFT of these components. Then, for each (k_x,k_y) we have a system of N_f equations in $2N_z$ unknowns, $\tilde{\mu}_i(k_x,k_y)$, $\tilde{\sigma}_i(k_x,k_y)$, $i=1,\ldots,N_z$. Let $N_f \geq N_z$ (in fact, it's best to have a strongly overdetermined system, with N_f several times larger than N_z), and solve the resulting system in a least-squares sense, using a stabilizing method to reduce effects of noise. This gives $\tilde{\mu}_i(k_x,k_y)$, $\tilde{\sigma}_i(k_x,k_y)$, which are, respectively, the two-dimensional Fourier transforms of the magnetic permeability at the ith level of the anomalous region and the electrical conductivity. Use analytic continuation, when necessary, to continue the solution through "essential zeros". Upon taking the inverse Fourier transform, we get $\mu_i(x,y)$, $\sigma_i(x,y)$, which is the three-dimensional distribution of the electromagnetic parameters throughout the anomalous region.

VI. COMMENTS AND CONCLUSIONS

The approach that we have outlined in this paper is computationally intensive and requires efficient algorithms and computer hardware for execution. Such items are rapidly becoming part of the scientific and engineering scene. We have not performed any numerical experiments with this model, yet, but we have performed some on a similar model for nonferrous tubes. The results there were quite encouraging, and we hope to have similar results with the ferrous model soon. An additional feature of the present model is that it should allow the distinction between electrical conduction current and magnetic permeability effects in ferrous metals. This is desirable in, for instance, those problems in which a ferrous metal has been stressed, but not cracked, so that the magnetic permeability, but not electrical conductivity, has been changed from its nominal value. Our inversion technique, in this case, should therefore inform us of such a condition.

VII. REFERENCES

1. Charles L. Lawson and Richard J. Hanson, Solving Least Squares Problems, Prentice-Hall, Inc., Englewood Cliffs, 1974.
2. Gene H. Golub and Charles F. Van Loan, Matrix Computations, Johns Hopkins University Press, Baltimore, 1983.

DISCUSSION

W. Lord (Colorado State University): The implication of your work is that you can determine the permeability and conductivity at any point in the material. What value of permeability would that be?

H.A. Sabbagh: It would be the departure from the nominal permeability, which in our program was a relative permeability of 70.

W. Lord: What does that correspond to in terms of the behavior of the material? Is that the initial permeability?

S. Marinov (Dresser Atlas): Yes, it would be the initial permeability. It's a linear model; we are in the Rayleigh area.

H.A. Sabbagh: That's exactly right. We are assuming a simple model and the initial permeability would be the reasonable thing we would use there.