Isomorphism of uniform algebras on the 2-torus

by

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ABSTRACT

For α a positive irrational, we consider the uniform subalgebra $\mathcal{A}_\alpha$ of $C(\mathbb{T}^2)$ consisting of those functions f satisfying $\hat{f}(m,n) = 0$ whenever $m+\alpha n < 0$. For positive irrationals $\alpha,\ \beta$, we determine when $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are isometrically isomorphic. Furthermore, we describe the group $\mathrm{Aut}(\mathcal{A}_\alpha)$ of isometric automorphisms of $\mathcal{A}_\alpha$. Finally we show how an explicit representation of $\mathrm{Aut}(\mathcal{A}_\alpha)$ can be derived from Pell's equations.

CHAPTER 1. INTRODUCTION

For α a positive irrational, let $\mathcal{A}_\alpha$ be the subalgebra of continuous functions on the two-torus whose Fourier transform vanishes at (m, n) if $m + \alpha n < 0$. These algebras were studied by Wermer and others ([1], [2]), who proved properties such as maximality and characterized the Gelfand space. One of the major themes of current work in operator algebras is classification, but none of the properties which were investigated earlier distinguished between $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$, if β is another positive irrational. In this work, we address this question. We also determine the automorphism group of $\mathcal{A}_\alpha$.

In chapter 2, we provide basic definitions, theorems, and notations that we use in this work and some known results about the algebra $\mathcal{A}_\alpha$.

In chapter 3, we show that the Gelfand space of $\mathcal{A}_\alpha$ is $\{(z, w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha = |w|\}$.

In chapter 4, we give the form of an isometric isomorphism of $\mathcal{A}_\alpha \to \mathcal{A}_\beta$ (Theorem 4.0.9). It follows that the cardinality of the isomorphism class is countable. Corollary 4.0.10 shows that there is a group invariant: let G_α be the dense subgroup of $\mathbb{R}$ consisting of $\{m+n\alpha : \ m, n \in \mathbb{Z}\}$. We show that $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are isometrically isomorphic if and only if there is a group isomorphism $G_\alpha \to G_\beta$ which maps $G_\alpha \cap \mathbb{R}^+ \to G_\beta \cap \mathbb{R}^+$.

In chapter 5, we examine the group of isometric automorphisms of $\mathcal{A}_\alpha$. If α is not a quadratic irrational, then $\mathrm{Aut}(\mathcal{A}_\alpha) \cong \mathbb{T}^2$. However, if α is a quadratic irrational, then $\mathrm{Aut}(\mathcal{A}_\alpha)$ is isomorphic to a semidirect product of $\mathbb{T}^2$ with $\mathbb{Z}$ (Theorem 5.1.9). We also give an explicit form for the automorphism group of $\mathcal{A}_\alpha$, where α is a quadratic irrational (Proposition 5.2.1, Proposition 5.2.2, and Proposition 5.2.3).

The material in chapter 2 is standard, but from chapter 3 on the material is original work.

CHAPTER 2. BACKGROUND

In this chapter, we provide basic definitions, theorems, and notations that we use for entire work. Definitions and theorems in this chapter are taken from [4], [5], and [11].

In this work, vector spaces are complex vector spaces.

2.1 Banach algebras

Definition 2.1.1. Let X be a vector space. A function $\|\cdot\| : X \to \mathbb{R}^+ \cup \{0\}$ is said to be a *norm* on X if

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$ for any $\alpha \in \mathbb{C}$ and $x \in X$;
3. $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

A vector space equipped with a norm is called a normed linear space.

Definition 2.1.2. An *algebra* is a vector space $\mathcal{A}$ equipped with a multiplication map $\cdot : \mathcal{A} \to \mathcal{A}$ such that

1. $a \cdot (b + c) = a \cdot b + a \cdot c$ for any $a, b, c \in \mathcal{A}$;
2. $(a + b) \cdot c = a \cdot c + b \cdot c$ for any $a, b, c \in \mathcal{A}$;
3. $(\alpha a) \cdot (\beta b) = (\alpha\beta)(a \cdot b)$ for any $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{A}$.

We usually write ab instead of $a \cdot b$.

Definition 2.1.3. A subspace $\mathcal{B}$ of an algebra $\mathcal{A}$ is said to be a *subalgebra* of $\mathcal{A}$ if $bb' \in \mathcal{B}$ for any $b, b' \in \mathcal{B}$.

Note that a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is also an algebra with the multiplication on $\mathcal{B}$ given by the restriction of the multiplication on $\mathcal{A}$.

Definition 2.1.4. An algebra $\mathcal{A}$ is a *normed algebra* if the norm on $\mathcal{A}$ is submultiplicative, i.e.,

$$\|ab\| \leq \|a\|\|b\|, \text{ for any } a, b \in \mathcal{A}.$$

Definition 2.1.5. A normed algebra $\mathcal{A}$ is a *unital normed algebra* if there exists an element $1 \in \mathcal{A}$ such that $1a = a = a1$ for all $a \in \mathcal{A}$ and $\|1\| = 1$. If this element 1 exists, then it is unique. 1 is called the *unit*.

Note that, a subalgebra of a unital normed algebra may not be a unital normed algebra. Also, if $\mathcal{B}$ is a unital normed subalgebra of $\mathcal{A}$, the unit of $\mathcal{B}$ may not be the same as the unit of $\mathcal{A}$.

Definition 2.1.6. Let $\mathcal{A}$ be a unital normed algebra. Then $a \in \mathcal{A}$ is invertible if there exists $b \in \mathcal{A}$ such that $ab = 1 = ba$. In this case b is unique and we write $b = a^{-1}$.

Theorem 2.1.7. *If $\mathcal{A}$ is a normed algebra, then the multiplicative map is jointly continuous.*

Definition 2.1.8. A complete normed algebra is called a *Banach algebra.*

Note that a closed subalgebra of a Banach algebra is also a Banach algebra.

Definition 2.1.9. A complete unital normed algebra is called a *unital Banach algebra.*

Definition 2.1.10. An algebra $\mathcal{A}$ is said to be *commutative* or *abelian* if $ab = ba$, for any $a, b \in \mathcal{A}$.

Definition 2.1.11. A subspace I of an algebra $\mathcal{A}$ is said to be a *left ideal* in $\mathcal{A}$ if

$$\text{for any } a \in \mathcal{A} \text{ and } b \in I, ab \in I.$$

Definition 2.1.12. A subspace I of an algebra $\mathcal{A}$ is said to be a *right ideal* in $\mathcal{A}$ if

$$\text{for any } a \in \mathcal{A} \text{ and } b \in I, ba \in I.$$

Definition 2.1.13. An *ideal* in an algebra $\mathcal{A}$ is a subspace of $\mathcal{A}$ that is both left and right ideal in $\mathcal{A}$.

Definition 2.1.14. An ideal I in an algebra $\mathcal{A}$ is a *proper ideal* if $I \neq \mathcal{A}$.

Definition 2.1.15. A proper ideal M in $\mathcal{A}$ is said to be a *maximal ideal* in $\mathcal{A}$ if for any ideal J in $\mathcal{A}$

$$\text{if } M \subseteq J \subsetneq \mathcal{A}, \text{ then } J = M.$$

Definition 2.1.16. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. Then a map $\varphi : \mathcal{A} \to \mathcal{B}$ is called an (algebra) *homomorphism* if

1. $\varphi(\alpha a) = \alpha\varphi(a)$ for any $\alpha \in \mathbb{C}$, and $a \in \mathcal{A}$;

2. $\varphi(a + b) = \varphi(a) + \varphi(b)$ for any $a, b \in \mathcal{A}$;

3. $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in \mathcal{A}$..

The third property is called *multiplicative*.

Definition 2.1.17. An algebra homomorphism is an *(algebra) isomorphism* if it is a bijection.

Definition 2.1.18. Let $\mathcal{A}$ be an algebra. An isomorphism $\phi : \mathcal{A} \to \mathcal{A}$ is called an automorphism.

Definition 2.1.19. Algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be *isomorphic* if there exists an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. We denote this by $\mathcal{A} \cong \mathcal{B}$.

Definition 2.1.20. Let $\mathcal{A}$ and $\mathcal{B}$ be normed algebras. A map $\varphi : \mathcal{A} \to \mathcal{B}$ is an *isometric isomorphism* if it is an isomorphism and

$$\|\varphi(a)\| = \|a\|, \text{ for all } a \in \mathcal{A}.$$

In this case, we say that $\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$.

Definition 2.1.21. Let $\mathcal{A}$ be an algebra. A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called *multiplicative* if

$$\varphi(ab) = \varphi(a)\varphi(b), \text{ for any } a, b \in \mathcal{A}.$$

Definition 2.1.22. Let $\mathcal{A}$ be an abelian Banach algebra. The *Gelfand space* of $\mathcal{A}$ is the set of all non-zero multiplicative linear functionals on $\mathcal{A}$. We will denote this set by $\Delta(\mathcal{A})$.

Theorem 2.1.23. *Let $\mathcal{A}$ be a unital abelian Banach algebra. Then if $\tau \in \Delta(\mathcal{A})$, then $\|\tau\| = 1$.*

Theorem 2.1.24. *Let $\mathcal{A}$ be a unital abelian algebra and $\mathfrak{M}$ a set of all maximal ideals in $\mathcal{A}$. Then $\Delta(\mathcal{A})$ is non-empty and the map $\phi : \Delta(\mathcal{A}) \to \mathfrak{M}$ defined by*

$$\phi(\tau) = \ker(\tau)$$

is a bijection from $\Omega(\mathcal{A})$ to $\mathfrak{M}$.

From Theorem 2.1.24 , $\Delta(\mathcal{A})$ is also called the maximal ideal space of $\mathcal{A}$.

2.2 C*-algebras

Definition 2.2.1. Let $\mathcal{A}$ be an algebra. An *involution* on $\mathcal{A}$ is a conjugate linear map $a \mapsto a^*$ such that

$$(a^*)^* = a \text{ and } (ab)^* = b^*a^*$$

for all $a, b \in \mathcal{A}$. $(\mathcal{A}, *)$ is called a **-algebra*.

Definition 2.2.2. A *-algebra $\mathcal{A}$ is called a *Banach *-algebra* if the norm on $\mathcal{A}$ is complete, submultiplicative and $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.

Definition 2.2.3. A *C*-algebra* is a Banach *-algebra such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

2.3 Spaces of continuous functions on compact Hausdorff spaces

Let X be a compact Hausdorff space. We will denote set of all continuous functions on X by $C(X)$. We will equip $C(X)$ with the supremum norm defined by $\|f\| = \sup_{x \in X} \|f(x)\|$. Then $C(X)$ is a unital normed algebra (with pointwise addition and pointwise multiplication of functions). Moreover, $C(X)$ is a unital abelian Banach algebra.

Theorem 2.3.1. *Let X be a compact Hausdorff space. Then the Gelfand space $\Delta(X)$ is X.*

Definition 2.3.2. Let X be a compact Hausdorff space. A subalgebra $\mathcal{A}$ of $C(X)$ is called a *uniform algebra* if

1. $\mathcal{A}$ is closed in $C(X)$;
2. $\mathcal{A}$ separates the points of X, i.e., if $x, y \in X$ and $x \neq y$, then there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$;
3. $\mathcal{A}$ contains the constant functions.

Definition 2.3.3. Let X be a compact Hausdorff space. A uniform subalgebra $\mathcal{A}$ of $C(X)$ is a *Dirichlet algebra* if the real parts of functions in $\mathcal{A}$ is dense in the set of all real-valued continuous functions in $C(X)$.

Definition 2.3.4. Let X be a compact Hausdorff space. A uniform subalgebra $\mathcal{A}$ of $C(X)$ is called *maximal* if for any closed subalgebra $\mathcal{B}$ satifying $\mathcal{A} \subset \mathcal{B} \subset C(X)$, then either $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = C(X)$.

2.4 Fourier series and double Fourier series

Definition 2.4.1. Let $f \in L^1[-\pi, \pi]$ and $n \in \mathbb{Z}$. The nth *Fourier coefficient* of f is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

and the *Fourier series* for f is the formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

Definition 2.4.2. The *Partial sums* of the Fourier series for f are $s_n(x) = \displaystyle\sum_{k=-n}^{n} \hat{f}(k) e^{ikx}, n = 1, 2, \ldots$.

Definition 2.4.3. The *Cesaro means* of the Fourier series for f are $\sigma_n = \frac{1}{n}(s_0 + s_1 + \cdots + s_{n-1}), n = 1, 2, \ldots$.

Theorem 2.4.4. *Let f be a continuous function of period 2π. Then σ_n converges to f uniformly.*

Let $\mathbb{T}^2$ denote the 2-torus $\mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle. Let $d\mu$ be normalized Lebesgue measure on $\mathbb{T}^2$.

Definition 2.4.5. If $f : \mathbb{T}^2 \to \mathbb{C}$ is in $L^2(\mathbb{T}^2)$, for $(m, n) \in \mathbb{Z}^2$, the *Fourier coefficient* $\hat{f}(m, n)$ is given by

$$\hat{f}(m,n) = \int_{\mathbb{T}^2} f(e^{is}, e^{it}) e^{-i(ms+nt)}\, d\mu$$

and the *Fourier series* for f is the formal series

$$\sum_{m,n} \hat{f}(m,n) e^{imx} e^{iny}.$$

Definition 2.4.6. *Rectangular partial sums* of the Fourier series for f are

$$S_{N,M}(x,y) = \sum_{|n| \le N, |m| \le M} \hat{f}(m,n) e^{imx} e^{iny}, M, N \in \mathbb{N}.$$

Definition 2.4.7. $\sigma_{N,M}(x,y) = \frac{1}{(N+1)(M+1)} \sum_{|k_1| \le N, |k_2| \le M} S_{k_1,k_2}(x,y)$ is called *Cesaro means* of the Fourier series for f, $M, N \in \mathbb{N}$.

Theorem 2.4.8. *Let $f \in L^2(\mathbb{T}^2)$. If f is continuous, then $\sigma_{M,N}$ converges uniformly to f.*

2.5 Disc and bidisc algebras

Let $\mathbb{D}$ be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Let $\bar{\mathbb{D}}$ be the closed unit disc $\{z \in \mathbb{C} : |z| \le 1\}$ and $\bar{\mathbb{D}}^2 = \bar{\mathbb{D}} \times \bar{\mathbb{D}}$ the bidisc.

Definition 2.5.1. The disc algebra is the space

$$\{f : \bar{\mathbb{D}} \to \mathbb{C} : f \text{ is analytic on } \mathbb{D} \text{ and continuous on } \overline{\mathbb{D}}\}.$$

We denote this algebra by $\mathcal{A}(\mathbb{D})$.

Definition 2.5.2. The bidisc algebra is the space

$$\{f : \bar{\mathbb{D}}^2 \to \mathbb{C} : f \text{ is bi-analytic on } \mathbb{D}^2 \text{ and continuous on } \overline{\mathbb{D}}^2\}.$$

We denote this algebra by $\mathcal{A}(\mathbb{D}^2)$.

Note that we can view the disc algebra as a subalgebra of $C(\mathbb{T})$ and the bidisc algebra as a subalgebra of $C(\mathbb{T}^2)$, i.e.,

$$\mathcal{A}(\mathbb{D}) = \{f \in C(\mathbb{T}) : \hat{f}(n) = 0 \text{ whenever } n < 0\}$$

and

$$\mathcal{A}(\mathbb{D}^2) = \{f \in C(\mathbb{T}^2) : \hat{f}(m,n) = 0 \text{ whenever } m < 0 \text{ or } n < 0\}.$$

Moreover, $\mathcal{A}(\mathbb{D})$ is a maximal subalgebra of $C(\mathbb{T})$. Furthermore, it is well-known that the Gelfand space $\Delta(\mathcal{A}(\mathbb{D}))$ is $\bar{\mathbb{D}}$ and the Gelfand space $\Delta(\mathcal{A}(\mathbb{D}^2))$ is $\bar{\mathbb{D}}^2$.

2.5.1 Factorization for H^1 functions

Definition 2.5.3. The space H^1 is the space $\{f : \bar{\mathbb{D}} \to \mathbb{C} : f(re^{i\theta})$ is bounded in L^1-norm$, 0 \leq r \leq 1\}$.

Definition 2.5.4. An *inner function* is a function $g : \bar{\mathbb{D}} \to \mathbb{C}$ such that g is analytic on $\mathbb{D}$, $|g(z)| \leq 1$ on $\mathbb{D}$ and $|g(z)| = 1$ almost everywhere on $\mathbb{T}$.

Definition 2.5.5. An *outer function* is a function $F : \bar{\mathbb{D}} \to \mathbb{C}$ such that F is analytic function on $\mathbb{D}$ and F has the form

$$F(z) = \lambda \exp(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} f(\theta)\, d\theta),$$

where f is a real-valued function in $L^1(\mathbb{T})$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Definition 2.5.6. An analytic function B is called a *Blaschke product* if B has the form

$$B(z) = z^n \prod_{i=1}^{\infty} (\frac{\bar{\alpha_i}}{|\alpha_i|} \frac{\alpha_i - z}{1 - \bar{\alpha_i} z})^{n_i},$$

where $n, n_1, n_2, \ldots \in \mathbb{N} \cup \{0\}$, $\alpha_i \neq 0$, for all $i \in \mathbb{N}$, $\alpha_i \neq \alpha_j$ if $i \neq j$, and $\prod_{i=1}^{\infty} |\alpha_i|^{n_i}$ is convergent.

Definition 2.5.7. S is said to be a *singular function* if S is an inner function without zeros and $S(0) > 0$.

Theorem 2.5.8. *Let $f \in H^1$ and $f \neq 0$. Then f has the form $f = BSF$ where B is a Blaschke product, S is a singular function and F is an outter function.*

2.6 The algebra $\mathcal{A}_\alpha$

Let α be a positive irrational number. We define $\mathcal{A}_\alpha$ to be the set of continuous functions $f : \mathbb{T}^2 \to \mathbb{C}$ with the property that

$$\hat{f}(m,n) = 0 \text{ whenever } m + \alpha n < 0.$$

By the continuity of the Fourier transform, $\mathcal{A}_\alpha$ is a Banach space, and since the product of two functions in $\mathcal{A}_\alpha$ again lies in $\mathcal{A}_\alpha$, it is a Banach algebra under the norm $||f|| = \sup_{(z,w)\in\mathbb{T}^2} |f(z,w)|$.

As a norm-closed subalgebra of the C*-algebra $C(\mathbb{T}^2)$, $\mathcal{A}_\alpha$ is a commutative operator algebra. It further falls in the category of uniform algebras, as a subalgebra of $C(\mathbb{T}^2)$ it separates the points of $\mathbb{T}^2$.

For $f \in C(\mathbb{T}^2)$, let f^* denote the adjoint, i.e., $f^* = \bar{f}$, the complex conjugate. Note that $\mathcal{A}_\alpha$ is *antisymmetric*, that is, $\mathcal{A}_\alpha \cap \mathcal{A}_\alpha^* = \{\mathbb{C} \cdot 1\}$.

The *characters* of the group $\mathbb{T}^2$ will be denoted by $\chi_{m,n}$, where

$$\chi_{m,n}(z,w) = z^m w^n, \ (z,w) \in \mathbb{T}^2.$$

The characters $\chi_{m,n}$ for which $m + \alpha n \geq 0$ belong to $\mathcal{A}_\alpha$, and linear combinations of characters in $\mathcal{A}_\alpha$ are dense.

Note that $\mathcal{A}_\alpha$ is a *Dirichlet* algebra; that is $\mathcal{A} + \mathcal{A}^*$ is dense in $C(\mathbb{T}^2)$. This is clear, since $\mathcal{A} + \mathcal{A}^*$ contains all the characters of $\mathbb{T}^2$.

It is known that $\mathcal{A}_\alpha$ is a maximal subalgebra of $C(\mathbb{T}^2)$ ([2]).

CHAPTER 3. THE GELFAND SPACE OF THE $\mathcal{A}_\alpha$

In this chapter, we show that $\Delta(\mathcal{A}_\alpha) = \{(z, w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha = |w|\}$. We also show that any point in the set $\Delta(\mathcal{A}_\alpha) \cap \mathbb{D} \times \mathbb{D}$ is not in a singleton part.

3.1 Method of rational approximation

Suppose $f \in \mathcal{A}_\alpha$ has a Fourier series with only finitely many terms, so

$$f = \sum_{k=1}^{N} c_k \, \chi_{m_k,n_k}.$$

Consider the interval $I = \{t \geq 0 : m_k + tn_k \geq 0,\ 1 \leq k \leq N\}$. This is an interval containing α in its interior. Let p, q be positive integers, $\gcd\{p, q\} = 1$, such that $p/q \in I$. Furthermore, since the map $\{(m_k, n_k) : 1 \leq k \leq N\} \mapsto m_k + \alpha n_k$ is one-to-one, one can chose p, q sufficiently close to α so that the map $\{(m_k, n_k) : 1 \leq k \leq N\} \mapsto m_k + (p/q)n_k$ is one-to-one. Thus, if we define the polynomial

$$F(\zeta) = f(\zeta^q, \zeta^p)$$

then F is a polynomial in ζ with N (non-zero) terms, and f can be recovered from F.

The following observation, which we will refer to as "the method of rational approximation," extends the above observation to the Gelfand space $\Delta(\mathcal{A}_\alpha)$.

Lemma 3.1.1. *Let $(z_0, w_0) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}}$ be such that $|z_0|^\alpha = |w_0|$. Then given $\varepsilon > 0$ there is a rational approximation p/q to α so that there is ζ_0, $|\zeta_0| \leq 1$, such that*

$$(\zeta_0^q, \zeta_0^p) \text{ is within } \varepsilon \text{ of } (z_0, w_0)$$

e.g. $|\zeta_0^q - z_0| + |\zeta_0^p - w_0| < \varepsilon$.

Proof. Define ζ_0 to be a q^{th} root of z_0 with $|\zeta_0^p - w_0| \le |\zeta^p - w_0|$ as ζ runs through all the q^{th} roots of z_0. It is possible that two q^{th} roots of z_0 have the property that their p^{th} powers are equidistant from w_0, in which case choose one arbitrarily.

Observe that $||\zeta_0^p| - |w_0|| = ||z_0|^{p/q} - |z_0|^{\alpha}|$ can be made arbitrarily small by choosing p/q sufficiently close to α, and that $|\arg(\zeta_0^p) - \arg(w_0)| \le \pi/q$ which can also be made arbitrarily small by appropriate choice of p and q. □

3.2 The Gelfand space of $\mathcal{A}_\alpha$

In this section, we will show that $\Delta(\mathcal{A}_\alpha) = \{(z,w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha = |w|\}$. As the Banach algebra satisfies

$$\mathcal{A}(\mathbb{D}^2) \subset \mathcal{A}_\alpha \subset C(\mathbb{T}^2)$$

it follows that the Gelfand spaces satisfy the reverse inclusions:

$$\Delta(C(\mathbb{T}^2)) \subset \Delta(\mathcal{A}_\alpha) \subset \Delta(\mathcal{A}(\mathbb{D}^2).$$

In other words,

$$\mathbb{T}^2 \subset \Delta(\mathcal{A}_\alpha) \subset \bar{\mathbb{D}}^2$$

Lemma 3.2.1. *$\forall \epsilon > 0$, $\exists m' \in \mathbb{Z}^-$ and $n' \in \mathbb{Z}^+$ such that $0 < m' + n'\alpha < \epsilon$ and $\exists m'' \in \mathbb{Z}^+$ and $n'' \in \mathbb{Z}^-$ such that $0 < m'' + n''\alpha < \epsilon$.*

Proof. Let $\epsilon > 0$ be given. Assume that $0 < \epsilon \le 1$. Let $A = [0, \frac{\epsilon}{2})$ and $B = [\frac{\epsilon}{2}, \epsilon)$. Let $T : [0,1) \to [0,1)$ be defined by $Tx = x + \alpha \pmod 1$, $x \in [0,1)$. By ergodicity of T, $\exists k \in \mathbb{N}$ such that $m(T^{-k}(A) \cap B) > 0$. By Poincaré's Recurrence Theorm, $\exists x_0 \in T^{-k}(A) \cap B$, $\exists 0 < n_1 < n_2 < \cdots$ such that $T^{n_i} x_0 \in T^{-k}(A) \cap B$. Since $x_0 \in T^{-k}(A)$, $\exists y_0 \in A$ such that $x_0 = T^{-k} y_0$. Thus, $T^{n_i - k} y_0 = T^{n_i}(T^{-k} y_0) = T^{n_i} x_0 \in B$ for all $i \in \mathbb{N}$. Choose $i \in \mathbb{N}$ such that $n_i > k$. Then $n_i - k > 0$ and $y_0 + (n_i - k)\alpha = T^{n_i - k} y_0 \in B = [\frac{\epsilon}{2}, \epsilon)$. Since $y_0 \in A = [0, \frac{\epsilon}{2})$, we have $(n_i - k)\alpha \in [0, \epsilon)$. Let $m' = -\lfloor (n_i - k)\alpha \rfloor$, $n' = n_i - k$. Then $m' \in \mathbb{Z}^-$, $n' \in \mathbb{Z}^+$ and $m' + n'\alpha = -\lfloor (n_i - k)\alpha \rfloor + (n_i - k)\alpha \in [0, \epsilon)$. Since α is irrational, $m' + n'\alpha \ne 0$. Thus $0 < m' + n'\alpha < \epsilon$. Similarly, by inverse transformation (by replacing α by $-\alpha$), $\exists m'' \in \mathbb{Z}^+$ and $n'' \in \mathbb{Z}^-$ such that $0 < m'' + n''\alpha < \epsilon$. □

Note that, by Lemma 3.2.1, we can choose p/q to be less than α in the method of rational approximation.

Lemma 3.2.2. *$\Delta(\mathcal{A}_\alpha) \subset \{(z,w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha = |w|\}$.*

Proof. First, let $(z_0, w_0) \in \{(z,w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z| = 0 \text{ or } |w| = 0,\ (z,w) \neq (0,0)\}$. WLOG, assume that $z_0 = 0$ and $w_0 \neq 0$. Suppose that the pointwise evaluation at (z_0, w_0) defines a multiplicative linear functional Θ on $\mathcal{A}_\alpha$. Then $\Theta(z) = 0$ and $\Theta(w) = w_0$. By Lemma 3.2.1, , $\exists m' \in \mathbb{Z}^-$ and $n' \in \mathbb{Z}^+$ such that $0 < m' + n'\alpha$. Then $z^{m'} w^{n'} \in \mathcal{A}_\alpha$. Hence

$$0 \neq w_0^{n'} = \Theta(w^{n'}) = \Theta(z^{-m'} \cdot z^{m'} w^{n'}) = \Theta(z^{-m'})\Theta(z^{m'} w^{n'}) = 0,$$

a contradiction. Thus (z_0, w_0) is not in $\Delta(\mathcal{A}_\alpha)$.

Now, assume that $(z_0, w_0) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}}$ and $|z_0|^\alpha \neq |w_0|$.

Case1: $1 > |z_0|^\alpha > |w_0| > 0$.

Let $|z_0| = r$. Then $|w_0| < r^\alpha$. So $\exists\, 0 < t < 1$ such that $|w_0| = tr^\alpha$. Since $0 < t < 1$, $\exists N \in \mathbb{N}$ such that $t^{-N} > 3$. Since $0 < r < 1$, $\ln r < 0$. Then $-\frac{1}{N \ln r} > 0$. By Lemma 3.2.1, $\exists m_1 \in \mathbb{Z}^+$ and $n_1 \in \mathbb{Z}^-$ such that $0 < m_1 + n_1\alpha < -\frac{1}{N \ln r}$. Let $m = m_1 N$ and $n = n_1 N$. Then $m + n\alpha = m_1 N + n_1 N\alpha = N(m_1 + n_1\alpha) > 0$.

Claim: $t^n > 3$ and $r^{m+n\alpha} > \frac{1}{e}$.

Since $t^{-N} > 3$, $t^n = t^{n_1 N} = (t^{-N})^{-n_1} > 3^{-n_1} \geq 3$.

Since $m_1 + n_1\alpha < -\frac{1}{N \ln r}$, $\ln r^{m+n\alpha} = (m + n\alpha)\ln r = (m_1 + n_1\alpha)N\ln(r) > -1$. Thus $r^{m+n\alpha} = e^{\ln r^{m+n\alpha}} > e^{-1} = \frac{1}{e}$.

Let $f(z,w) = z^m w^n$. Then $f \in \mathcal{A}_\alpha$ and $|f(z_0, w_0)| = |z_0|^m |w_0|^n = r^m t^n r^{n\alpha} = t^n r^{m+n\alpha} > 3.\frac{1}{e} > 1$.

Case2: $1 = |z_0|^\alpha > |w_0| > 0$.

Then $|z_0| = 1$ and $\exists\, 0 < t < 1$ such that $|w_0| = t$. By Lemma 3.2.1, $\exists m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^-$ such that $0 < m + n\alpha$. Let $f(z,w) = z^m w^n$. Then $f \in \mathcal{A}_\alpha$ and $|f(z_0, w_0)| = |z_0|^m |w_0|^n = |w_0|^n = t^n > 1$.

Case3: $0 < |z_0|^\alpha < |w_0| \leq 1$.

Let $|z_0| = r$. Then $\exists t > 1$ such that $|w_0| = tr^\alpha$. Since $t > 1$, $\exists N \in \mathbb{N}$ such that $t^N > 3$. Since $0 < r < 1$, $\ln r < 0$. Then $-\frac{1}{N \ln r} > 0$ By Lemma 3.2.1, $\exists m_1 \in \mathbb{Z}^-$ and $n_1 \in \mathbb{Z}^+$ such that $0 < m_1 + n_1\alpha < -\frac{1}{N \ln r}$. Let $m = m_1 N$ and $n = n_1 N$. Then $m + n\alpha = m_1 N + n_1 N\alpha = N(m_1 + n_1\alpha) > 0$. Let $f(z,w) = z^m w^n$. Then $f \in \mathcal{A}_\alpha$ and $|f(z_0,w_0)| = |z_0|^m |w_0|^n = r^m t^n r^{n\alpha} = t^n r^{m+n\alpha} = t^{n_1 N} r^{m+n\alpha} > t^N r^{m+n\alpha} > 3.\frac{1}{e} > 1$. $\square$

Lemma 3.2.3. *Let* $f = \sum_{j=1}^{N} c_j \chi_{m_j,n_j}$, $m_j + n_j\alpha > 0$, $j = 1,2,\ldots,N$.*Let* $m,n \in \mathbb{Z}^+$ *be such that* $\frac{m}{n} < \alpha$, $nm_j + mn_j > 0$, $j = 1,2,\ldots,N$. *Let* $A = \{(z,w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha \le |w| \le |z|^{\frac{m}{n}}\}$. *Then* f *is continuous on* A.

Proof. Clearly, f is continuous at every point in $A \setminus \{(0,0)\}$. We have to show that p is continuous at the point $(0,0)$. Let $j \in \{1,2,\ldots,N\}$. Let $h(z,w) = z^{m_j} w^{n_j}$, $(z,w) \in A$. Since $nm_j + mn_j > 0$ and n is positive, $m_j + \frac{m}{n} n_j > 0$. Let $(z,w) \in A$.

Obviously, h is continuous at $(0,0)$ when $m_j \ge 0$ and $n_j \ge 0$.

Case $m_j < 0$ and $n_j > 0$.

Then $0 \le |h(z,w)| = |z|^{m_j} |w|^{n_j} \le |z|^{m_j} (|z|^{\frac{m}{n}})^{n_j} = |z|^{m_j + \frac{m}{n} n_j}$. Since $m_j + \frac{m}{n} n_j > 0$, we have $\lim\limits_{\substack{(z,w) \to (0,0) \\ (z,w)\in A}} |h(z,w)| = 0$.

Case $m_j > 0$ and $n_j < 0$.

Then $0 \le |h(z,w)| = |z|^{m_j} |w|^{n_j} \le |z|^{m_j} (|z|^\alpha)^{n_j} = |z|^{m_j + n_j\alpha}$. Since $m_j + n_j\alpha > 0$, we have $\lim\limits_{\substack{(z,w) \to (0,0) \\ (z,w)\in A}} |h(z,w)| = 0$.

Thus h is continuous at $(0,0)$ and hence f is continuous at $(0,0)$. $\square$

Lemma 3.2.4. *Let* $f = \sum_{j=1}^{N} c_j \chi_{m_j,n_j}$, $m_j + n_j\alpha \ge 0$, $j = 1,2,\ldots,N$. *Then max* $|f|$ *on* $|z|^\alpha = |w|$, $|z| \le 1$, $|w| \le 1$ *occurs on* $|z| = 1$.

Proof. Suppose that max $|f|$ on $|z|^\alpha = |w|$, $|z| \le 1$, $|w| \le 1$ occurs at (z_0,w_0). Choose $m,n \in \mathbb{Z}^+$ such that $\frac{m}{n} < \alpha$, $nm_j + mn_j \ge 0, \forall j = 1,2,\ldots,N$. Let $A = \{(z,w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha \le |w| \le |z|^{\frac{m}{n}}\}$. By Lemma 3.2.3, f is continuous on A.

Claim: For any $\varepsilon > 0$, there exists $(z',w') \in A$, $|z'| = 1$, so that $|f(z',w')| > |f(z_0,w_0)| - \varepsilon$.

Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $|f(z,w) - f(z_0,w_0)| < \frac{\varepsilon}{2}$ whenever $|z - z_0| + |w - w_0| < \delta$, $(z,w) \in A$. By Lemma 3.1.1, there is a rational approximation $\frac{m}{n} < \frac{p}{q} < \alpha$ to α so that there exists $\zeta_0, |\zeta_0| \le 1$ such that $|\zeta_0^q - z_0| + |\zeta_0^p - w_0| < \delta$. Let $F(\zeta) = f(\zeta^q, \zeta^p)$, $|\zeta| \le 1$. Then F is a polynomial in ζ. By Maximum Modulus Principle, there exists ζ' such that $|\zeta'| = 1$

and $|F(\zeta)| \leq |F(\zeta')|$, for all ζ, $|\zeta| \leq 1$. Let $z' = (\zeta')^q$ and $w' = (\zeta')^p$. Then $|z'| = 1 = |w'|$. Hence $|f(z', w')| = |F(\zeta')| \geq |F(\zeta_0)| = |f(\zeta_0^q, \zeta_0^p)| > |f(z_0, w_0)| - \varepsilon$.

Now, for each $k \in \mathbb{N}$, by Claim, there exists $(z_k, w_k) \in A$, $|z_k| = 1$ and $|f(z_k, w_k)| > |f(z_0, w_0)| - \frac{1}{k}$. Since $\{(z_k, w_k)\}_{k\in\mathbb{N}}$ is a sequence in a compact space $\mathbb{T}^2$, there exists a convergent subsequence of $\{(z_k, w_k)\}_{k\in\mathbb{N}}$ that converges to some point, say, $(z', w') \in \mathbb{T}^2$. Thus $|p(z', w')| \geq |p(z_0, w_0)|$. $\square$

Now, we will show that $\{(z, w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha = |w|\} \subset \Delta(\mathcal{A}_\alpha)$. Let $f \in \mathcal{A}_\alpha$. Let $(z_0, w_0) \in \{(z, w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} : |z|^\alpha = |w|\}$. Then there is a sequence $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathcal{A}_\alpha$ and p_n has a Fourier series with only finitely many terms, such that $p_n \to f$ uniformly as $n \to \infty$. So $|f(z_0, w_0)| = \lim\limits_{n\to\infty} |p_n(z_0, w_0)| \leq \lim\limits_{n\to\infty} \|p_n\|_\infty = \|f\|_\infty$.

3.3 Parts of $\Delta(\mathcal{A}_\alpha)$

If $\mathcal{A}$ is a uniform algebra on X, we say that $\Theta_1,\ \Theta_2 \in \Delta(\mathcal{A})$, *belong to the same part of* $\Delta(\mathcal{A})$ if

$$||\Theta_1 - \Theta_2|| < 2$$

where the norm is the norm in the dual space of $\mathcal{A}$, that is

$$||\Theta_1 - \Theta_2|| = \sup\{|\Theta_1(f) - \Theta_2(f)| : f \in \mathcal{A}, ||f|| = 1\}.$$

Equivalently, $\Theta_1,\ \Theta_2$ belong to the same part of $\Delta(\mathcal{A})$ if there is a constant $c > 0$ such that Harnack's inequality is valid:

$$\frac{1}{c} u(\Theta_1) \leq u(\Theta_2) \leq c u(\Theta_1), \quad u \in \Re(\mathcal{A}),\ u > 0.$$

(See [2].)

For the algebra $\mathcal{A}_\alpha$, it is known that ([1])

1. Each point (z_0, w_0) in (the Shilov boundary) $\mathbb{T}^2$ is in a singleton part;

2. The point $(0, 0)$ is in a singleton part.

Now, we will show that any point in the set $\{(z,w) \in \Delta(\mathcal{A}_\alpha) : \ 0 < |z| < 1\}$ is not in a singleton part.

Lemma 3.3.1. *Let $(a,b) \in \mathbb{T}^2$. The map*

$$\gamma : \mathcal{A}_\alpha \to \mathcal{A}_\alpha, \ \gamma(f)(z,w) = f(az,bw)$$

is an isometric automorphism of $\mathcal{A}_\alpha$.

Proof. Clearly γ, or more precisely, the extension of γ to $C(\mathbb{T}^2)$, is an isometric automorphism of $C(\mathbb{T}^2)$. Furthermore, for any $f \in \mathcal{A}_\alpha$, a character $\chi_{m,n}$ appears as a non-zero Fourier coefficient of f if and only if it appears in $\gamma(f)$. Thus, γ maps $\mathcal{A}_\alpha$ to itself. $\square$

Definition 3.3.2. Let $\mathcal{A}$, $\mathcal{B}$ be uniform algebras with Gelfand spaces $\Delta(\mathcal{A})$, $\Delta(\mathcal{B})$ respectively. We will say that a homeomorphism $\varphi : \Delta(\mathcal{B}) \to \Delta(\mathcal{A})$ is *admissible* if $f \circ \varphi \in \mathcal{A}$ whenever $f \in \mathcal{B}$.

By Lemma 3.3.1, the homeomorphism of $\Delta(\mathcal{A}_\alpha)$ given by $(z,w) \mapsto (az,bw)$ is admissible for any $(a,b) \in \mathbb{T}^2$.

Let $(z_0,w_0), \ (z_1,w_1) \in \Delta(\mathcal{A}_\alpha)$ with $0 < |z_0|, \ |z_1| < 1$ be such that $\arg(w_0) = \arg(w_1)$. By composing with an admissible homeomorphism, we may suppose that z_0, w_0 are real and positive, and so by assumption the same is true for w_1.

Write $z_0 = r_0$ and $z_1 = r_1 e^{i\theta_1}$. We will assume z_1 belongs to the circle centered at z_0 with radius $\frac{1}{3}r_0(1-r_0)$. Then $|\sin(\theta_1)| \leq \frac{1-r_0}{3}$.

We now want to employ the method of rational approximation. Let f be a function in $\mathcal{A}_\alpha$ which has only finitely many nonzero Fourier coefficients and the real part of f is greater than 0. Let u be the real part of f. Let p/q be an approximation to α close enough so that the method can be applied to f. Let $F(\zeta) = f(\zeta^q, \zeta^p)$, $|\zeta| \leq 1$ and U the real part of F.

Now let $\zeta_0, \ \zeta_1$ in the $\zeta-$ plane corresponding to $(z_0,w_0), \ (z_1,w_1) \in \Delta_\alpha$. Let ζ_1' be the real part of ζ_1. A calculation shows that $|\zeta_1 - \zeta_1'| \leq r_1^{\frac{1}{q}} |\sin(\frac{\theta_1}{q})| \leq \frac{r_1^{\frac{1}{q}}}{q}(\frac{1-r_0}{3}) \leq \frac{1}{q}(\frac{1-r_0}{3}) \leq \frac{1}{q}(\frac{1-r_0}{2})$.

On the other hand,

$$\begin{aligned}
|\zeta_1' - \zeta_0| &\leq \max\{|r_0^{\frac{1}{q}} - (r_0 - \frac{r_0(1-r_0)}{3})^{\frac{1}{q}}|, |(r_0 + \frac{r_0(1-r_0)}{3})^{\frac{1}{q}} - r_0^{\frac{1}{q}}|\} \\
&\leq \max\{\frac{1}{q}\frac{1}{(r_0 - \frac{r_0(1-r_0)}{3})}\frac{r_0(1-r_0)}{3}, \frac{1}{q}\frac{1}{r_0}\frac{r_0(1-r_0)}{3}\} \\
&= \max\{\frac{1}{q}\frac{(1-r_0)}{(2+r_0)}, \frac{1}{q}\frac{(1-r_0)}{3}\} \\
&\leq \frac{1}{q}(\frac{1-r_0}{2}).
\end{aligned}$$

By Pythagorean theorem, we obtain

$$|\zeta_1 - \zeta_0| \leq \frac{1}{q}(\frac{1-r_0}{\sqrt{2}}).$$

Now U is continuous on $|\zeta| \leq 1 - r_0^{\frac{1}{q}}$ and harmonic in the interior. Also, $1 - r_0^{\frac{1}{q}} \geq \frac{1}{q}(1-r_0)$. By Harnack's inequality (Theorem A.0.10), we have

$$(\frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{r_0}+\frac{1}{\sqrt{2}}})U(\zeta_0) \leq U(\zeta_1) \leq (\frac{\frac{1}{r_0}+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}})U(\zeta_0).$$

Thus,

$$(\frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{r_0}+\frac{1}{\sqrt{2}}})u(\zeta_0^q, \zeta_0^p) \leq u(\zeta_1^q, \zeta_1^p) \leq (\frac{\frac{1}{r_0}+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}})u(\zeta_0^q, \zeta_0^p).$$

Observe that the set of real parts of functions of $\mathcal{A}_\alpha$ which have only finitely many Fourier coefficients is dense in the set of real parts of functions of $\mathcal{A}_\alpha$. Furthermore, by taking a sequence p_n/q_n of rationals converging to α, the method of rational approximation yields that the corresponding $(\zeta_{j,n}^{q_n}, \zeta_{j,n}^{p_n})$ converges to (z_j, w_j).

Thus we can apply Harnack's inequality to obtain that (z_1, w_1) belongs to the same part as (z_0, w_0).

CHAPTER 4. ISOMETRIC ISOMORPHISMS OF THE A_α

In this chapter, we show that an isometric isomorphism from $\mathcal{A}_\alpha$ to $\mathcal{A}_\beta$ essentially maps characters of $\mathcal{A}_\alpha$ to the charcters of $\mathcal{A}_\beta$ (Theorem 4.0.9). In order to prove this theorem, we use the method of rational approximation from chapter 3. This entails the factorization of functions in the disc algebra and the form of the Gelfand spaces of $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$.

Let $p_n,\ q_n,\ n \in \mathbb{N}$ be sequences of positive integers such that p_n/q_n converges to α. Define a sequence of measures μ_n on $C(\mathbb{T}^2)$ by

$$\mu_n(f) := \int f\, d\mu_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{iq_n\theta}, e^{ip_n\theta})\, d\theta.$$

Let μ denote normalized Lebesgue measure on $\mathbb{T}^2$.

Lemma 4.0.3. *The sequence $\{\mu_n\}$ converges in the weak $*$-topology to μ.*

Proof. First we show that for any character χ, $\lim_n \int \chi\, d\mu_n = \int \chi\, d\mu$. Indeed, if $\chi = \chi_{0,0} = 1$, then $\int \chi\, d\mu_n = 1 = \int \chi\, d\mu$, since the measures are all positive of mass 1. If $\chi = \chi_{m,n}$ with $(m,n) \neq (0,0)$, then $m + \alpha n \neq 0$, so that $m + (p_k/q_k)n \neq 0$, for k sufficiently large, hence $mq_ik + np_k \neq 0$. Then

$$\int \chi\, d\mu_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(mq_k+np_k)\theta}\, d\theta = 0 \text{ and } \int \chi\, d\mu = 0.$$

Thus the desired result holds for any $f \in C(\mathbb{T}^2)$ which is a finite linear combination of characters, which is a dense subalgebra of $C(\mathbb{T}^2)$.

Now by the weak $*$-compactness of unit ball in the dual of $C(\mathbb{T}^2)$, there is a subnet of μ_n which converges, and by the metrizability of the dual space, the subnet can be taken as a subsequence. By re-labeling, we may denote the convergent subsequence by $\{\mu_n\}$.

Next let $f \in C(\mathbb{T}^2)$, and $\varepsilon > 0$. Let f_1 be a Cesaro mean of f such that $||f - f_1|| < \varepsilon$. Then

$$\begin{aligned} |\lim_{n\to\infty} \mu_n(f) - \mu(f)| &\le |\lim_{n\to\infty} \mu_n(f - f_1)| + |\mu(f - f_1)| \\ &\le \lim_{n\to\infty} \mu_n(\varepsilon) + \varepsilon \\ &\le 2\varepsilon \end{aligned}$$

Now, the same argument shows that any subsequence of the original sequence $\{\mu_n\}$ in turn has a subsequence converging to μ. Thus $\{\mu_n\}$ converges weak $*$ to μ. □

Lemma 4.0.4. *(see [5], p. 103) Every invertible function f in $C(\mathbb{T})$ has the form $f(z) = z^n \exp(g(z))$ for $n \in \mathbb{Z}$ and some function $g \in C(\mathbb{T})$. The integer n is uniquely determined.*

Lemma 4.0.5. *Every invertible function f in $C(\mathbb{T}^2)$ has the form*

$$f = \chi_{m,n} \exp(g), \ \textit{for some } g \in C(\mathbb{T}^2).$$

Furthermore the character $\chi_{m,n}$ is uniquely determined.

Proof. Let $c = f(1,1)$. If $\frac{1}{c}f$ has the desired form, then so does f. Thus we may assume that $f(1,1) = 1$.

Given f invertible in $C(\mathbb{T}^2)$, let $f_1(z) = f(z,1)$ and $f_2(w) = f(1,w)$. Then $f_1,\ f_2 \in C(\mathbb{T})$ and are invertible, so by Lemma 4.0.4 there exist integers $m,\ n$, which are uniquely determined, and functions $g_1,\ g_2 \in C(\mathbb{T})$ so that $f_1(z) = z^m \exp(g_1(z))$ and $f_2(w) = w^n \exp(g_2(w))$.

Let $h(z,w) = z^{-m} \exp(-g_1(z)) w^{-n} \exp(-g_2(w)) f(z,w)$. Thus h satisfies

$$h(z,1) = 1 = h(1,w) \text{ for all } z,\ w \in \mathbb{T}.$$

We claim that $h = \exp(k)$ for some $k \in C(\mathbb{T}^2)$. By Corollary 2.15 of [3], this is equivalent to showing that h lies in the connected component of the constant function 1 in $C(\mathbb{T}^2)$.

Now let $w_0 \in \mathbb{T} \setminus \{1\}$, and let $t_0 \in (0, 2\pi)$ be such that $e^{it_0} = w_0$. Let $h_{w_0} \in C(\mathbb{T})$ be the function $h_{w_0}(z) = h(z, w_0)$. Observe that h_{w_0} is path homotopic to the constant 1. Define $\gamma(t)(\cdot) = h(\cdot, e^{it}),\ 0 \le t \le t_0$ and note that $\gamma(0)$ is the constant function 1, $\gamma(t_0) = h_{w_0}$, and that $\gamma(t)(1) = h(1, e^t) = 1,\ t \in [0, t_0]$. We conclude that h_{w_0} lies in the connected component of the identity of the invertibles in $C(\mathbb{T})$, hence has the form $h_{w_0} = \exp(k_{w_0})$ for some $k_{w_0} \in C(\mathbb{T})$, by Lemma 4.0.4 Now k_{w_0} is not unique, but as $h_{w_0}(1) = 1$, we have that $k_{w_0}(1) \in 2\pi i\,\mathbb{Z}$. If we specify that $k_{w_0}(1) = 0$, then k_{w_0} is unique. If we do this for each $w \in \mathbb{T}$, then the map $w \mapsto k_w$ is continuous.

Now we can define a path homotopy $F : \mathbb{T}^2 \times [0,1] \to \mathbb{C} \setminus \{0\}$ as follows: set $F(z,w,t) = \exp(tk_w(z))$. Then $F(z,w,0)$ is the constant function 1, $F(z,w,1) = h(z,w)$, and $F(1,1,t) = 1,\ t \in [0,1]$, as $k_1 = 0$. This proves the claim, hence there is a function $k \in C(\mathbb{T}^2)$ such that $h = \exp(k)$. $\square$

Lemma 4.0.6. *Let $f \in \mathcal{A}_\alpha$ and suppose $f = \chi_{m,n}\exp(g)$ for some character $\chi_{m,n}$ and some $g \in C(\mathbb{T}^2)$. Then both $\chi_{m,n}$ and $\exp(g)$ belong to $\mathcal{A}_\alpha$, and the extension of $\exp(g)$ to the Gelfand space $\Delta(\mathcal{A}_\alpha)$ does not vanish at the point $(0,0)$.*

Proof. First assume that f is a finite linear combination of characters. By the method of rational approximation, we can find integers p, q with p/q sufficiently close to α so that the function $F(\zeta) = f(\zeta^q, \zeta^p) \in \mathcal{A}(\mathbb{D})$. Since f is invertible on $\mathbb{T}^2$, it follows F is invertible on $\mathbb{T}$, so by Lemma 4.0.4 F has the form $F(\zeta) = \zeta^N \exp(G)$, with $N \geq 0$. On the other hand, setting $N_1 = mq+np$, we have $F(\zeta) = \zeta^{N_1}\exp(G_1)$, with $G_1(\zeta) = g(\zeta^q, \zeta^p)$. By the uniqueness assertion of the Lemma, $N_1 = N$, and hence $G - G_1 \in 2\pi\mathbb{Z}$. Now if B is the Blaschke factor (which is the inner factor) in the inner-outer factorization of F, then $B(\zeta) = \zeta^N(\frac{\zeta - a_1}{1-\bar{a}_1\zeta}\cdots\frac{\zeta-a_M}{1-\bar{a}_M\zeta})$ where $0 < |a_t| < 1,\ 1 \leq t \leq M$. Furthermore, each factor in the Blaschke product belongs to the disc algebra, in particular the first factor. Also, $\exp(G(\zeta)) = (\frac{\zeta-a_1}{1-\bar{a}_1\zeta}\cdots\frac{\zeta-a_M}{1-\bar{a}_M\zeta})F_0(\zeta)$, where F_0 is the outer factor. Since these are in the disc algebra, so is $\exp(G)$. Note that since N is the order of the zero of F at the origin, $\frac{F(\zeta)}{\zeta^N}$ is non-zero at the origin. So, the extension of $\exp(G)$ to the disc does not vanish at the origin.

Next we claim that both $\chi_{m,n}$ and $\exp(g) \in \mathcal{A}_\alpha$. Let $p_k,\ q_k$ be positive integers such that $\{\frac{p_k}{q_k}\}$ converges to α, and let a, b be integers such that $a + \alpha b \geq 0$. Let $F_k(\zeta) = f(\zeta^{q_k}, \zeta^{p_k}) = \zeta^{N_k}\exp(G_k(\zeta))$. Then

$$\int_{\mathbb{T}^2} \chi_{a,b}(z,w)\exp(g(z,w))\,d\mu = \lim_k \int_{\mathbb{T}} \zeta^{aq_k+bp_k}\exp(G_k(\zeta^{q_k}, \zeta^{p_k}))\,d\theta = 0,\ \zeta = e^{i\theta}$$

for k sufficiently large, as $\exp(G_k)$ is in the disc algebra.

A similar argument shows that $\chi_{m,n} \in \mathcal{A}_\alpha$.

Since the polynomial $\exp(G_k)$ is nonzero at the origin, it contains a nonzero multiple of the constant function as a Fourier coefficient, hence the same is true of $\exp(g)$.

Now for the general case, where $f = \chi_{m,n} \exp(g) \in \mathcal{A}_\alpha$, we can apply the above argument to the Cesaro means of f. Note functions in the set $\chi_{m,n} \exp(C(\mathbb{T}^2))$ constitute one of the connected components of the invertible functions in $C(\mathbb{T}^2)$, and in particular, this set is open, so that any Cesaro mean sufficiently close to f has this form. Thus considering the Cesaro means, we obtain a sequence of functions $\exp(g_n)$ converging to $\exp(g)$, such that $\exp(g_n) \in \mathcal{A}_\alpha$ for sufficiently large n. Thus $\exp(g) \in \mathcal{A}_\alpha$.

Finally, the extension of the function $\exp(g)$ to the Gelfand space $\Delta(\mathcal{A}_\alpha)$ is non-zero at the point $(0,0)$. This is due to the nature of the Cesaro approximations: the non-zero Fourier coefficients of the Cesaro approximants is a subset of the non-zero Fourier coefficients of the function $\exp(g)$. Since all of the Cesaro approximants $\exp(g_n)$ contain a non-zero multiple of the constant character, at least for sufficiently large n, the $(0,0)$ Fourier coefficient of $\exp(g)$ is non-zero. $\square$

Lemma 4.0.7. *Suppose* $g \in C(\mathbb{T}^2)$ *is such that* $\exp(g) \in \mathcal{A}_\alpha$, *and the extension of* $\exp(g)$ *to* $\Delta(\mathcal{A}_\alpha)$ *is never zero. Then* $\exp(g/2) \in \mathcal{A}_\alpha$.

Proof. As in the previous lemma, we begin by assuming that $f = \exp(g) \in \mathcal{A}_\alpha$ is expressible as a finite linear combination of characters. In that case, f extends to a function on a subset S of the closed bidisc, $S = \{(z,w) : |w| = |z|^t,\ a \le t \le b\}$ where $0 < a < \alpha < b$. Since f is nonzero on $\Delta(\mathcal{A}_\alpha)$ and uniformly continuous on S, we may assume that f is nonzero on S, possibly by replacing $[a,b]$ by a smaller interval.

Suppose that p, q are positive integers with $a < p/q < b$, and set $F(\zeta) = f(\zeta^q, \zeta^p)$, so F is in the disc algebra. If for some $|\zeta_0| < 1,\ F(\zeta_0) = 0$, then $f(z_0, w_0) = 0$, where $\zeta_0^q = z_0,\ \zeta_0^p = w_0$. But then $|w_0| = r_0^p = |z_0|^{\frac{p}{q}},\ |\zeta_0| = r_0$. Since $p/q \in (a,b)$, it follows that $(z_0, w_0) \in S$, and hence $0 = F(\zeta_0) = f(z_0, w_0)$, a contradiction.

It follows that $F(\zeta)$ is outer. Then, again by factorization, since $F(\zeta) = \exp(G)$, we obtain that $\exp(G/2)$ is in the disc algebra. ([4])

To see that $\exp(g/2) \in \mathcal{A}_\alpha$, let $\frac{p_k}{q_k}$ be a sequence of fractions converging to α. Set $F_k(\zeta) = \exp(G_k(\zeta))$. By the argument above, we have that $\exp(G_k/2)$ is in the disc algebra, at least for sufficiently large k. Then, for any $(m,n) \in \mathbb{Z}^2$ with $m + \alpha n \geq 0$, we have

$$\int_{\mathbb{T}^2} \chi_{m,n}(z,w) \exp(\frac{g}{2}(z,w))\, d\mu = \lim_k \int_{\mathbb{T}} \zeta^{mq_k+np_k} \exp(\frac{G_k}{2}(\zeta))\, d\theta = 0,$$

$\zeta = e^{i\theta}$.

For the general case, where $f = \exp(g) \in \mathcal{A}_\alpha$, apply the argument above to the Cesaro means f_n of f to get $f_n = \exp(g_n)$ with $\exp(g_n/2) \in \mathcal{A}_\alpha$. Thus $\exp(g_n/2)$ converges to $\exp(g/2)$, so that it is also in $\mathcal{A}_\alpha$. □

Proposition 4.0.8. *Suppose $\chi_{m,n}$ is a character, and $g \in C(\mathbb{T}^2)$ are such that the function $f = \chi_{m,n}\exp(g) \in \mathcal{A}_\alpha$, and f does not vanish on $\Delta(\mathcal{A}_\alpha) \setminus \{(0,0)\}$. Then $\chi_{m,n}$ and g lie in $\mathcal{A}_\alpha$. Furthermore, if $|\exp(g)| = 1$ on $\mathbb{T}^2$, then g is constant.*

Proof. Lemma 4.0.6 shows that $\chi_{m,n} \in \mathcal{A}_\alpha$, and $\exp(g) \in \mathcal{A}_\alpha$ does not vanish on $\Delta(\mathcal{A}_\alpha)$. Then, by Lemma 4.0.6 and repeated application of Lemma 4.0.7 we obtain that

$$\exp(g),\ \exp(g/2),\ \ldots, \exp(g/2^k), \dots \in \mathcal{A}_\alpha.$$

Now if t_k is any sequence of positive reals decreasing to 0, then

$$g = \lim_{k\to\infty} \frac{\exp(t_k g) - 1}{t_k}$$

where the convergence is in norm. Applying this with $t_k = \frac{1}{2^k}$, we obtain the desired result.

If $|\exp(g)| = 1$ on $\mathbb{T}^2$, then $\exp(-g) = \exp(g^*) = \exp(g)^* \in \mathcal{A}_\alpha \cap \mathcal{A}_\alpha^* = \{\mathbb{C}\cdot 1\}$, so, g is constant. □

If $\Phi : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ is an algebraic isomorphism, there is a weak* homeomorphism $\varphi : \Delta(\mathcal{A}_\beta) \to \Delta(\mathcal{A}_\alpha)$ defined by $f(\varphi(z,w)) = \Phi(f)(z,w)$. In other words, φ is admissible (Definition 3.3.2). However, if Φ is an isometric isomorphism, then more is true: φ maps the Shilov boundary of $\Delta(\mathcal{A}_\beta)$ to the Shilov boundary of $\Delta(\mathcal{A}_\alpha)$, and also maps parts of $\Delta(\mathcal{A}_\beta)$ to parts of $\Delta(\mathcal{A}_\alpha)$. Thus, φ maps $\mathbb{T}^2$ to itself, maps the singleton part $\{(0,0)\} \in \Delta(\mathcal{A}_\beta)$ to the corresponding part in $\Delta(\mathcal{A}_\alpha)$, and the set $\{(z,w) : |w| = |z|^\beta, 0 < |z| < 1\} \in \Delta(\mathcal{A}_\beta)$ to the corresponding set $\{(z,w) : |w| = |z|^\alpha, 0 < |z| < 1\} \in \Delta(\mathcal{A}_\alpha)$.

Theorem 4.0.9. *$\Phi : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ is an isometric isomorphism if and only if Φ has form $\Phi(f) = f \circ \varphi$, where $\varphi : \Delta(\mathcal{A}_\beta) \to \Delta(\mathcal{A}_\alpha)$ is of the form*

$$(z, w) \mapsto (c_1 z^{m_1} w^{n_1}, c_2 z^{m_2} w^{n_2})$$

where c_j are unimodular constants, $j = 1, 2$, and the matrix

$$A = \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \in GL(2, \mathbb{Z})$$

satisfies $m_1 + \beta n_1 > 0$ and $m_2 + \beta n_2 = \alpha(m_1 + \beta n_1)$.

Proof. Assume that $\Phi : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ is an isometric isomorphism. Then there is a homeomorphism $\varphi : \Delta(\mathcal{A}_\beta) \to \Delta(\mathcal{A}_\alpha)$ defined by $f(\varphi(z, w)) = \Phi(f)(z, w)$. Recall that the characters $\chi_{1,0},\ \chi_{0,1}$ are the coordinate functions

$$\chi_{1,0}(z, w) = z, \quad \chi_{0,1}(z, w) = w.$$

Set $f_1 = \Phi(\chi_{1,0}) = \chi_{1,0} \circ \varphi$, and similarly define f_2, with $\chi_{0,1}$ in place of $\chi_{1,0}$, so that $\varphi(z, w) = (f_1(z, w), f_2(z, w))$.

Then $f_j \in \mathcal{A}_\beta$ and since the homeomorphism φ maps $\Delta(\mathcal{A}_\beta) \to \Delta(\mathcal{A}_\alpha)$ and in particular maps the Shilov boundary of $\Delta(\mathcal{A}_\beta)$ to the Shilov boundary of $\Delta(\mathcal{A}_\alpha)$, this implies that $|f_j| = 1$ on $\mathbb{T}^2$, $j = 1, 2$. Furthermore φ maps the part $(0, 0) \in \Delta(\mathcal{A}_\beta)$ to the part $(0, 0) \in \Delta(\mathcal{A}_\alpha)$. This implies that $f_j(z, w) = 0$ if and only if $(z, w) = (0, 0)$.

Now since f_j is invertible on $\mathbb{T}^2$, by Lemma 4.0.5 it has the form $f_j = \chi_{m_j,n_j} \exp(g_j)$, $j = 1, 2$. By Proposition 4.0.8, $\exp(g_j)$ is constant, say equal to c_j, with $|c_j| = 1$. Since $f_j = c_j \chi_{m_j,n_j} \in \mathcal{A}_\beta$, we have that $m_j + \beta n_j \geq 0$. And clearly $m_j + \beta n_j > 0$, for f_j cannot be constant.

Since

$$\varphi(z, w) = (f_1(z, w), f_2(z, w)) \in \Delta(\mathcal{A}_\alpha)$$

we have that

$$|f_2(z,w)| = |f_1(z,w)|^\alpha$$

$$|\chi_{m_2,n_2}(z,w) = |\chi_{m_1,n_1}(z,w)|^\alpha$$

$$|z^{m_2}w^{n_2}| = |z^{m_1}w^{n_1}|^\alpha$$

$$|z|^{m_2}|z|^{\beta n_2} = (|z|^{m_1}|z|^{\beta n_1})^\alpha$$

since $|w| = |z|^\beta$ in $\Delta(\mathcal{A}_\beta)$. Hence $m_2 + \beta n_2 = \alpha(m_1 + \beta n_1)$.

Furthermore, since φ is invertible, the map

$$\mathbb{T}^2 \to \mathbb{T}^2,\ (z,w) \mapsto (z^{m_1}w^{n_1}, z^{m_2}w^{n_2})$$

is invertible, so that the matrix

$$A = \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \in GL(2,\mathbb{Z}).$$

Conversely, assume that Φ has form in the assumption. By composing Φ with the map in Lemma 3.3.1, we can assume that $c_1 = 1 = c_2$. First, we will show that $\varphi : \Delta(\mathcal{A}_\beta) \to \Delta(\mathcal{A}_\alpha)$ is well-defined. Let $(z,w) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}}$ be such that $|z|^\beta = |w|$. Then

$$\begin{aligned} |z^{m_1}w^{n_1}|^\alpha &= (|z|^{m_1}|z|^{\beta n_1})^\alpha \\ &= |z|^{\alpha(m_1+\beta n_1)} \\ &= |z|^{m_2+\beta n_2} \\ &= |z|^{m_2}|z|^{\beta n_2} \\ &= |z|^{m_2}|w|^{n_s} \\ &= |z^{m_2}w^{n_2}|. \end{aligned}$$

Thus φ is well-defined. Let $(m,n) \in \mathbb{Z}^2$. Since $m_2 + \beta n_2 = \alpha(m_1 + \beta n_1)$, we have $(mm_1 + nm_2) + \beta(mn_1 + nm_2) = (m + \alpha n)(m_1 + \beta n_1)$. Moreover, since $m_1 + \beta n_1 > 0$, we get the necessary and sufficient condition :

$$m + \alpha n \geq 0 \text{ if and only if } (m + \alpha n)(m_1 + \beta n_1) \geq 0.$$

Let $\chi_{m,n}$ be a character of $\mathcal{A}_\alpha$. Then

$$\begin{aligned}(\chi_{m,n}\circ\varphi)(z,w) = \chi_{m,n}(\varphi(z,w)) &= \chi_{m,n}(z^{m_1}w^{n_1}, z^{m_2}w^{n_2})\\ &= z^{mm_1+nm_2}w^{mn_1+nm_2}\\ &= \chi_{mm_1+nm_2,mn_1+nm_2}(z,w).\end{aligned}$$

Clearly, Φ is a homomorphism. Next, we will show that Φ is surjective. Let $(m,n)\in\mathbb{Z}^2$ be such that $m+\beta n\geq 0$.

Claim : $\chi_{\frac{n_2m-m_2n}{m_1n_2-m_2n_1},\frac{-n_1m+m_1n}{m_1n_2-m_2n_1}}\in\mathcal{A}_\alpha$.

Since $A = \begin{bmatrix} m_1 & n_1\\ m_2 & n_2\end{bmatrix}\in GL(2,\mathbb{Z})$, $(\frac{n_2m-m_2n}{m_1n_2-m_2n_1},\frac{-n_1m+m_1n}{m_1n_2-m_2n_1})\in\mathbb{Z}^2$. We will show that $\frac{n_2m-m_2n}{m_1n_2-m_2n_1}+\alpha\frac{-n_1m+m_1n}{m_1n_2-m_2n_1}\geq 0$. Since $m_1+\beta n_1 > 0$, it is enough to show that $(m_1+\beta n_1)(\frac{n_2m-m_2n}{m_1n_2-m_2n_1}+\alpha\frac{-n_1m+m_1n}{m_1n_2-m_2n_1})\geq 0$.

$$\begin{aligned}&(m_1+\beta n_1)(\frac{n_2m-m_2n}{m_1n_2-m_2n_1}+\alpha\frac{-n_1m+m_1n}{m_1n_2-m_2n_1})\\ &=\frac{(m_1+\beta n_1)(n_2m-m_2n)-\alpha(m_1+\beta n_1)(n_1m-m_1n)}{m_1n_2-n_1m_2}\\ &=\frac{(m_1+\beta n_1)(n_2m-m_2n)-(m_2+\beta n_2)(n_1m-m_1n)}{m_1n_2-n_1m_2}\\ &=\frac{m_1n_2m-m_1m_2n+\beta n_1n_2m-\beta n_1m_2n-n_1m_2m-\beta n_1n_2m+m_1m_2n+\beta m_1n_2n}{m_1n_2-n_1m_2}\\ &=\frac{(m_1n_2-n_1m_2)(m+\beta n)}{m_1n_2-n_1m_2}\\ &=m+\beta n\\ &\geq 0.\end{aligned}$$

Thus $\chi_{\frac{n_2m-m_2n}{m_1n_2-m_2n_1},\frac{-n_1m+m_1n}{m_1n_2-m_2n_1}}\in\mathcal{A}_\alpha$. Moreover,

$$\begin{aligned}&(\chi_{\frac{n_2m-m_2n}{m_1n_2-m_2n_1},\frac{-n_1m+m_1n}{m_1n_2-m_2n_1}}\circ\varphi)(z,w)\\ &=\chi_{\frac{n_2m-m_2n}{m_1n_2-m_2n_1},\frac{-n_1m+m_1n}{m_1n_2-m_2n_1}}(\varphi(z,w))\\ &=\chi_{\frac{n_2m-m_2n}{m_1n_2-m_2n_1},\frac{-n_1m+m_1n}{m_1n_2-m_2n_1}}(z^{m_1}w^{n_1},z^{m_2}w^{n_2})\\ &=z^{m_1(\frac{n_2m-m_2n}{m_1n_2-m_2n_1})+m_2(\frac{-n_1m+m_1n}{m_1n_2-m_2n_1})}w^{n_1(\frac{n_2m-m_2n}{m_1n_2-m_2n_1})+n_2(\frac{-n_1m+m_1n}{m_1n_2-m_2n_1})}\\ &=\chi_{m,n}(z,w).\end{aligned}$$

Thus Φ is surjective. Next, we will show that Φ is an isometry. Let $f \in \mathcal{A}_\alpha$. Since $A = \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \in GL(2, \mathbb{Z})$, the map

$$\mathbb{T}^2 \to \mathbb{T}^2,\ (z, w) \mapsto (z^{m_1} w^{n_1}, z^{m_2} w^{n_2})$$

is a bijection. Thus

$$\begin{aligned} \|\Phi(f)\| &= \sup_{(z,w)\in\mathbb{T}^2} |f(z^{m_1} w^{n_1}, z^{m_2} w^{n_2})| \\ &= \sup_{(z,w)\in\mathbb{T}^2} |f(z, w)| \\ &= \|f\|. \end{aligned}$$

Hence $\Phi : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ is an isometric isomorphism. □

If α is an irrational, let G_α denote the dense additive subgroup $\{m + \alpha n : \ m, n \in \mathbb{Z}\} \subset \mathbb{R}$.

Corollary 4.0.10. *Let $\alpha,\ \beta$ be positive irrationals. Then the algebras $\mathcal{A}_\alpha,\ \mathcal{A}_\beta$ are isometrically isomorphic if and only if the groups $G_\alpha,\ G_\beta$ are order isomorphic. That is, if and only if there is a group isomorphism from G_α to G_β which maps positive elements of G_α to positive elements of G_β.*

Proof. We use the fact that G_α and G_β are isomorphic to $\mathbb{Z}^2$, $\mathrm{Aut}(\mathbb{Z}^2) = GL(2, \mathbb{Z})$ and Theorem 4.0.9. □

CHAPTER 5. ISOMETRIC AUTOMORPHISMS OF THE A_α

Throughout, automorphisms always mean isometric automorphisms. In this chapter, we will investigate the automorphism group $\mathrm{Aut}(\mathcal{A}_\alpha)$ of the algebra $\mathcal{A}_\alpha$. By Theorem 4.0.9 from the previous section to the case $\beta = \alpha$, it follows immediately that if α is not quadratic irrational, then $\mathrm{Aut}(\mathcal{A}_\alpha) \cong \mathbb{T}^2$. Therefore, we will focus on the case when α is positive quadratic irrational. For such an α, we show that $\mathrm{Aut}(\mathcal{A}_\alpha)$ is a semidirect product of $\mathbb{T}^2$ and $\mathbb{Z}$. At the end of this chapter, we show how the solutions of Pell's equations can be employed to calculate the automorphism group $\mathrm{Aut}(\mathcal{A}_\alpha)$.

5.1 The automorphism group of $\mathcal{A}_\alpha$

First, we would like to introduce some notations that we will use throughout this chapter. For $f \in C(\mathbb{T}^2)$, $(c_1, c_2) \in \mathbb{T}^2$, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{Z})$, let $\pi((c_1,c_2)), \pi(A) : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ be defined by

$$\pi((c_1,c_2))(f)(z,w) = f(c_1 z, c_2 w), \text{ and}$$

$$\pi(A)(f) = f \circ \varphi,$$

where $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ is of the form $(z,w) \mapsto (z^a w^b, z^c w^d)$.

Note that we can view $\pi(\mathbf{c})$ and $\pi(A)$ as the restrictions of $\pi(\mathbf{c})$ and $\pi(A)$ to $\mathcal{A}_\alpha$ where $\mathbf{c} \in \mathbb{T}^2$ and $A \in GL(2,\mathbb{Z})$.

Now, for each quadratic irrational $\alpha > 0$, we want to examine the automorphism group $\mathrm{Aut}(\mathcal{A}_\alpha)$ of $\mathcal{A}_\alpha$. To find this automorphism group, we need to know some results about Pell's equations and Lie algebra.

Pell's equation is a Diophantine equation of the form

$$x^2 - ny^2 = 1,$$

where n is a positive nonsquare integer. This equation is always solvable in integers and has the trivial solution with $x = 1$ and $y = 0$. Moreover, this equation always has nontrivial solutions. It is well-known that the set of solutions of this equation is given by

$$\begin{aligned}\{(1,0), (-1,0), (x_k, y_k), &(-x_k, y_k), (x_k, -y_k), (-x_k, -y_k) \ : \\ x_{k+1} &= x_1 x_k + n y_1 y_k, \\ y_{k+1} &= x_1 y_k + y_1 x_k, \\ k &= 1, 2, \ldots\},\end{aligned}$$

where (x_1, y_1) is the fundamental solution of $x^2 - ny^2 = 1$. The fundamental solution of $x^2 - ny^2 = 1$ is the pair (x_1, y_1), x_1 is the smallest positive interger and y_1 is the positive integer that satisfies

$$x_1^2 - ny_1^2 = 1.$$

Now, we look at the equation of the form

$$x^2 - ny^2 = -1,$$

where n is a positive nonsquare integer. This equation is called the negative Pell's equation. Note that, this equation may have no solutions in integers. There is a necessary but not sufficient condition of this equation to have integer solutions that all odd prime factors of n must be congruent to 1 modulo 4. If this negative Pell's equation has solutions, then the set of all solutions is given by

$$\begin{aligned}\{(x'_k, y'_k), &(-x'_k, y'_k), (x'_k, -y'_k), (-x'_k, -y'_k) \ : \\ x'_{k+1} &= (x_1'^2 + ny_1'^2)x'_k + 2nx'_1 y'_1 y'_k, \\ y'_{k+1} &= (x_1'^2 + ny_1'^2)y'_k + 2x'_1 y'_1 x'_k, \\ k &= 1, 2, \ldots\},\end{aligned}$$

where (x_1', y_1') is the fundamental solution of $x^2 - ny^2 = -1$. Moreover, the fundamental solution (x_1, y_1) of the Pell's equation $x^2 - ny^2 = 1$ can be obtained from the fundamental solution (x_1', y_1') of the negative Pell's equation $x^2 - ny^2 = -1$ by

$$x_1 = x_1'^2 + ny_1'^2 \text{ and } y_1 = 2x_1'y_1'.$$

Another equations that relates to our problems are the equations of the forms

$$x^2 - ny^2 = \pm 4,$$

where n is a positive nonsquare integer. The equation $x^2 - ny^2 = 4$ is always solvable over integers. The set of solutions of this equation is given by

$$\begin{aligned}
\{(2,0), (-2,0), (x_k, y_k), &(-x_k, y_k), (x_k, -y_k), (-x_k, -y_k) \ : \\
x_{k+1} &= \frac{1}{2}(x_1x_k + ny_1y_k), \\
y_{k+1} &= \frac{1}{2}(x_1y_k + y_1x_k), \\
k &= 1, 2, \ldots\},
\end{aligned}$$

where (x_1, y_1) is the fundamental solution of $x^2 - ny^2 = 4$. The equation $x^2 - ny^2 = -4$ may not be solvable over integers. But if it has solutions, then the set of all solutions is

$$\begin{aligned}
\{(x_k', y_k'), &(-x_k', y_k'), (x_k', -y_k'), (-x_k', -y_k') \ : \\
x_{k+1}' &= \frac{1}{4}((x_1'^2 + ny_1'^2)x_k' + 2nx_1'y_1'y_k'), \\
y_{k+1}' &= \frac{1}{4}((x_1'^2 + ny_1'^2)y_k' + 2x_1'y_1'x_k'), \\
k &= 1, 2, \ldots\},
\end{aligned}$$

where (x_1', y_1') is the fundamental solution of $x^2 - ny^2 = -4$. Moreover, the fundamental solution (x_1, y_1) of the equation $x^2 - ny^2 = 4$ can be obtained from the fundamental solution (x_1', y_1') of $x^2 - ny^2 = -4$ by

$$x_1 = \frac{1}{2}(x_1'^2 + ny_1'^2) \text{ and } y_1 = x_1'y_1'.$$

The following Proposition is useful.

Proposition 5.1.1. *(see [8]) Let $n > 0$ be a nonsquare integer. If $n \not\equiv 0 \ (mod\ 4)$, then any solution to the equation $x^2 - ny^2 = 4$, x and y have the same parity. Moreover, if the equation $x^2 - ny^2 = -4$ is solvable, then x and y have the same parity.*

Now, we will talk about some basic results from the special linear Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ consists of all 2×2 matrices with real entries and trace zero. The Lie bracket is given by $[A, B] = AB - BA$, for $A, B \in \mathfrak{sl}(2, \mathbb{R})$. It is well-known that (R_1, R_2, R_3) is a basis for $\mathfrak{sl}(2, \mathbb{R})$ with properties that $R_1 = [R_2, R_3], R_2 = [R_1, R_2]$, and $-R_3 = [R_1, R_3]$, where

$$R_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, R_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } R_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

By using this basis and direct calculation, it is easy to show that if $[A, B] = 0$, then A and B must be linearly dependent. In other words, if A and B commute in $\mathfrak{sl}(2, \mathbb{R})$ and $B \neq 0$, then there exists $k \in \mathbb{R}$ such that $A = kB$. We will use this result to determine the automorphism group of $\mathcal{A}_\alpha$.

Lemma 5.1.2. *Let A, B be matrices in $\mathfrak{sl}(2, \mathbb{R})$. If $B \neq 0$ and $AB = BA$, then $\exists\, k \in \mathbb{R}$ such that $A = kB$.*

Proof. Since $A, B \in \mathfrak{sl}(2, \mathbb{R})$, $\exists\, a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ so that $A = a_1R_1 + a_2R_2 + a_3R_3$ and $B = b_1R_1 + b_2R_2 + b_3R_3$. Since $AB = BA$, $[A, B] = 0$. Thus

$$\begin{aligned} 0 = [A, B] &= [a_1R_1 + a_2R_2 + a_3R_3, b_1R_1 + b_2R_2 + b_3R_3] \\ &= a_1b_1[R_1, R_1] + a_1b_2[R_1, R_2] + a_1b_3[R_1, R_3] + a_2b_1[R_2, R_1]+ \\ &\quad a_2b_2[R_2, R_2] + a_2b_3[R_2, R_3] + a_3b_1[R_3, R_1] + a_3b_2[R_3, R_2]+ \\ &\quad a_3b_3[R_3, R_3] \\ &= a_1b_2R_2 - a_1b_3R_3 - a_2b_1R_2 + a_2b_3R_1 + a_3b_1R_3 - a_3b_2R_1 \\ &= (a_2b_3 - a_3b_2)R_1 + (a_1b_2 - a_2b_1)R_2 + (a_3b_1 - a_1b_3)R_3. \end{aligned}$$

Since R_1, R_2, R_3 are linearly independent, we have

$$a_1b_2 = a_2b_1, a_2b_3 = a_3b_2, \text{ and } a_3b_1 = a_1b_3.$$

Since $B \neq 0$, there exists $i \in \{1, 2, 3\}$ such that $b_i \neq 0$. Let $k = \frac{a_i}{b_i}$. Then $A = kB$. $\square$

To find the automorphism group of $\mathcal{A}_\alpha$, when α is positive quadratic irrational, we begin by using Theorem 4.0.9 in the case $\beta = \alpha$ to get the following Corollary:

Corollary 5.1.3. *Let* $A = \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \in GL(2, \mathbb{Z})$*. Then* $\pi(A)$ *is an automorphism of* $\mathcal{A}_\alpha$ *if and only if the matrix* A *satisfies* $m_1 + \alpha n_1 > 0$ *and*

$$n_1\alpha^2 + (m_1 - n_2)\alpha - m_2 = 0.$$

Remark 5.1.4. For any matrix $A = \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \in GL(2, \mathbb{Z})$ that satisfies conditions in Corollary 5.1.3, $A \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = (m_1 + \alpha n_1) \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$, i.e., $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ is an eigenvector of A with positive eigenvalue $m_1 + \alpha n_1$.

Lemma 5.1.5. *If* α *is a positive quadratic irrational number, then there exists a non-identity matrix* $A \in SL(2, \mathbb{Z})$ *such that* $\pi(A)$ *is an automorphism of* $\mathcal{A}_\alpha$*.*

Proof. Since α is positive quadratic irrational, $\alpha = \frac{m+k\sqrt{n}}{p}$, where $m \in \mathbb{Z}$, $k \in \{-1, 1\}$, $n, p \in \mathbb{N}$, n is nonsquare, $m + k\sqrt{n} > 0$. Let (x_1, y_1) be the fundamental solution of the Pell's equation $x^2 - np^2y^2 = 1$. Let $A = \begin{bmatrix} x_1 - mpy_1 & p^2y_1 \\ (n - m^2)y_1 & x_1 + mpy_1 \end{bmatrix}$. Then $\det(A) = (x_1 - mpy_1)(x_1 + mpy_1) - ((n - m^2)y_1)(p^2y_1) = x_1^2 - m^2p^2y_1^2 - np^2y_1^2 + m^2p^2y_1^2 = x_1^2 - np^2y_1^2 = 1$. Thus A is a non-identity matrix in $SL(2, \mathbb{Z})$. Moreover,

$$\begin{aligned}
&p^2y_1\alpha^2 + (-2mpy_1)\alpha - (n - m^2)y_1 \\
&= p^2y_1(\frac{m^2 + 2mk\sqrt{n} + n}{p^2}) + (-2mpy_1)(\frac{m + k\sqrt{n}}{p}) - (n - m^2)y_1 \\
&= m^2y_1 + 2mk\sqrt{n}y_1 + ny_1 - 2m^2y_1 - 2mk\sqrt{n}y_1 - ny_1 + my_1^2 \\
&= 0.
\end{aligned}$$

Since $x_1^2 = 1 + np^2y_1^2 > np^2y_1^2$ and $x_1 > 0$, we have

$$(x_1 - mpy_1) + p^2y_1\alpha = (x_1 - mpy_1) + p^2y_1(\frac{m + k\sqrt{n}}{p}) = x_1 + k\sqrt{n}py_1 > 0.$$

By Corollary 5.1.3, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$. $\square$

Lemma 5.1.6. *Let A_1, A_2 be matrices in $GL(2,\mathbb{Z})$ and α a positive quadratic irrational number. If $\pi(A_1)$ and $\pi(A_2)$ are automorphisms of $\mathcal{A}_\alpha$, then $A_1A_2 = A_2A_1$.*

Proof. Let $A_1, A_2 \in GL(2,\mathbb{Z})$ be such that $\pi(A_1)$ and $\pi(A_2)$ are automorphisms of $\mathcal{A}_\alpha$. Let $\mathcal{H} = \{A \in GL(2,\mathbb{Z}) : A \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$, for some $\lambda > 0\}$. Then $\mathcal{H}$ is a subgroup of $GL(2,\mathbb{Z})$. Let $\phi : \mathcal{H} \to \mathbb{R}^+$ be defined by $\phi(A) = \lambda$. Then $\ker\phi = \{I\}$ Thus $\mathcal{H}$ is isomorphic to a subgroup of $\mathbb{R}^+$. Hence $\mathcal{H}$ is commutative. Since $A_1, A_2 \in \mathcal{H}$, $A_1A_2 = A_2A_1$. □

Lemma 5.1.7. *Let α be a positive irrational number. Let*

$$\mathcal{H} = \{A \in SL(2,\mathbb{Z}) : A \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \textit{ for some } \lambda > 0\}.$$

If $A \in \mathcal{H}$ with distinct eigenvalues and n is an even number, then A^n has no square root in $GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$.

Proof. Let n be even and λ an eigenvalue of A corresponding to an eigenvector $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Then $\frac{1}{\lambda}$ is the other eigenvalue of A. We write $A = P \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} P^{-1}$, where $P \in GL(2,\mathbb{R})$ and the first column of P is $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Since n is even, $n = 2k$ for some $k \in \mathbb{Z}$. Assume that A^n has a square root in $GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$. Then $P \begin{bmatrix} \lambda^k & 0 \\ 0 & -\frac{1}{\lambda^k} \end{bmatrix} P^{-1} \in GL(2,\mathbb{Z})$. Thus

$$P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1} = P \begin{bmatrix} \lambda^k & 0 \\ 0 & -\frac{1}{\lambda^k} \end{bmatrix} P^{-1}A^{-k} \in GL(2,\mathbb{Z}).$$

Note that the only matrix in $GL(2,\mathbb{Z})$ that has an eigenvector $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ with eigenvalue 1 is the identity matrix. Since $P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1} \in GL(2,\mathbb{Z})$ has an eigenvector $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ with eigenvalue 1,

$P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}$ must be the identity matrix. This is a contradiction since

$$\det(P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}) = -1.$$

Hence A^n has no square root in $GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$. $\square$

Lemma 5.1.8. *If α is a positive quadratic irrational number, then there is a matrix $A_0 \in GL(2,\mathbb{Z})$ such that for any matrix A with $\pi(A) \in \operatorname{Aut}(\mathcal{A}_\alpha)$, $\pi(A)$ is of the form $(\pi(A_0))^n$, for some $n \in \mathbb{Z}$.*

Proof. By Lemma 5.1.5, there is a non-identity matrix $B \in SL(2,\mathbb{Z})$ such that $\pi(B)$ is an automorphism of $\mathcal{A}_\alpha$. Since $B \in SL(2,\mathbb{R})$, there exist a non-zero matrix $S_1 \in \mathfrak{sl}(2,\mathbb{R})$ such that $B = \exp(S_1)$. Let $\mathcal{H} = \{A \in SL(2,\mathbb{Z}) : A \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$, for some $\lambda > 0\}$. Then $B \in \mathcal{H}$. Let $\mathcal{C} = \{t \in \mathbb{R} : \exp(tS_1) \in \mathcal{H}\}$. Then $\mathcal{C}$ is a closed additive subgroup of $\mathbb{R}$. Let $t_0 = \inf\{t > 0 : \exp(tS_1) \in \mathcal{H}\}$. Then t_0 generates the group $\mathcal{C}$. Let $A' = \exp(t_0 S_1)$. Note that A' cannot have a square root in $SL(2,\mathbb{Z})$. If A' has no square root in $GL(2,\mathbb{Z})$, then let $A_0 = A'$. If A' has a square root in $GL(2,\mathbb{Z})$, then let A_0 be the square root that has positive eigenvalue with respect to the eigenvector $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$.

Let $A \in GL(2,\mathbb{Z})$ be such that $\pi(A) \in \operatorname{Aut}(\mathcal{A}_\alpha)$. If $A \in SL(2,\mathbb{Z})$, then there exists $S_2 \in \mathfrak{sl}(2,\mathbb{R})$ such that $A = \exp(S_2)$. By Lemma 5.1.6, $BA = AB$. Thus $S_1 S_2 = S_2 S_1$. By Lemma 5.1.2, $\exists\, k \in \mathbb{R}$ so that $S_2 = kS_1$. Since $A \in \mathcal{H}$, $k \in \mathcal{C}$. Then $\exists\, j \in \mathbb{Z}$ such that $k = jt_0$ Hence $A = (A')^j$. Thus $A = A_0^n$ for some $n \in \mathbb{Z}$. If $A \notin SL(2,\mathbb{Z})$, then $A^2 \in SL(2,\mathbb{Z})$. By similar argument, we have $\exists\, n \in \mathbb{Z}$ such that $A^2 = (A')^n$. By Lemma 5.1.7, n must be odd.

Claim: A' has a square root in $GL(2,\mathbb{Z})$.

Let λ and λ' be eigenvalues of A and A' corresponding to an eigenvector $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. By Lemma 5.1.6, $AA' = A'A$. Since both A and A' are diagonalizable and they commute, A and A' are simultaneously diagonalizable. Then we can write $A = P \begin{bmatrix} \lambda & 0 \\ 0 & -\frac{1}{\lambda} \end{bmatrix} P^{-1}$ and

$A' = P \begin{bmatrix} \lambda' & 0 \\ 0 & \frac{1}{\lambda'} \end{bmatrix} P^{-1}$, where $P \in GL(2,\mathbb{R})$ and the first column of P is $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Since $A^2 = (A')^n$, we have $\lambda^2 = (\lambda')^n$. Since n is odd, $n = 2k+1$ for some $k \in \mathbb{Z}$. Thus $\lambda = (\lambda')^{k+\frac{1}{2}}$. Hence

$$\begin{aligned} P \begin{bmatrix} \sqrt{\lambda'} & 0 \\ 0 & -\frac{1}{\sqrt{\lambda'}} \end{bmatrix} P^{-1} &= P \begin{bmatrix} (\lambda')^{-k}\lambda & 0 \\ 0 & -\frac{1}{(\lambda')^{-k}\lambda} \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} \frac{1}{(\lambda')^k} & 0 \\ 0 & (\lambda')^k \end{bmatrix} P^{-1} P \begin{bmatrix} \lambda & 0 \\ 0 & -\frac{1}{\lambda} \end{bmatrix} P^{-1} \\ &= (A')^{-k} A \in GL(2,\mathbb{Z}). \end{aligned}$$

In this case, A_0 is the square root of A' that has positive eigenvalue with respect to the eigenvector $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Thus $A = A_0^n$. Hence $\pi(A) = (\pi(A_0))^n$. □

Now, if α is positive quadratic irrational, we know from Theorem 4.0.9 and Lemma 5.1.8 that $\mathrm{Aut}(\mathcal{A}_\alpha)$ is the set $\{\pi(\mathbf{c})\pi(A)^k : \mathbf{c} \in \mathbb{T}^2, k \in \mathbb{Z}\}$ for some non-identity matrix $A \in GL(2,\mathbb{Z})$.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{Z})$, let $\psi_A : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $\psi_A((c_1,c_2)) = (c_1^a c_2^b, c_1^c c_2^d)$, for $(c_1,c_2) \in \mathbb{T}^2$. Note that $\mathrm{Aut}(\mathbb{T}^2) = \{\psi_A : A \in GL(2,\mathbb{Z})\}$. For each $A \in GL(2,\mathbb{Z})$, let $\mathbb{T}^2 \rtimes_{\psi_A} \mathbb{Z}$ denote the semidirect product of $\mathbb{T}^2$ and $\mathbb{Z}$, where the group multiplication of $\mathbb{T}^2 \rtimes_{\psi_A} \mathbb{Z}$ is given by $(\mathbf{c}, m).(\mathbf{d}, n) = (\mathbf{c}\psi_A^m(\mathbf{d}), m+n)$.

Theorem 5.1.9. *Let α be a positive irrational. If α is not a quadratic irrational, then from Theorem 4.0.9 and Corollary 5.1.3,* $\mathrm{Aut}(\mathcal{A}_\alpha) \cong \mathbb{T}^2$. *If α is positve quadratic irrational, then* $\mathrm{Aut}(\mathcal{A}_\alpha) \cong \mathbb{T}^2 \rtimes_{\psi_A} \mathbb{Z}$ *for some non-identity matrix $A \in GL(2,\mathbb{Z})$.*

Proof. Since α is positive quadratic irrational, $\mathrm{Aut}(\mathcal{A}_\alpha) = \{\pi(\mathbf{c})\pi(A)^k : \mathbf{c} \in \mathbb{T}^2, k \in \mathbb{Z}\}$ for some non-identity matrix $A \in GL(2,\mathbb{Z})$. Let N_α denote the subgroup generated by $\{\pi(\mathbf{c}) : \mathbf{c} \in \mathbb{T}^2\}$ and $< \pi(A) >$ denote the subgroup generated by $\pi(A)$. First, we will show that $N_\alpha \trianglelefteq \mathrm{Aut}(\mathcal{A}_\alpha)$.

Let $k \in \mathbb{Z}$ and $(c_1, c_2) \in \mathbb{T}^2$. Let $f \in A_\alpha$. Let $A^k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned}
\pi(A)^k \pi((c_1, c_2))\pi(A)^{-k}(f)(z, w) &= \pi((c_1, c_2))\pi(A)^{-k}(f)(z^a w^b, z^c w^d) \\
&= \pi(A)^{-k}(f)(c_1 z^a w^b, c_2 z^c w^d) \\
&= f(c_1^{\frac{d}{ad-bc}} c_2^{-\frac{b}{ad-bc}} z^{\frac{ad-bc}{ad-bc}} w^{\frac{bd-bd}{ad-bc}}, c_1^{-\frac{c}{ad-bc}} c_2^{\frac{a}{ad-bc}} z^{\frac{-ac+ac}{ad-bc}} w^{\frac{-bc+ad}{ad-bc}}) \\
&= f(c_1^{\frac{d}{ad-bc}} c_2^{-\frac{b}{ad-bc}} z, c_1^{-\frac{c}{ad-bc}} c_2^{\frac{a}{ad-bc}} w) \\
&= \pi((c_1^{\frac{d}{ad-bc}} c_2^{-\frac{b}{ad-bc}}, c_1^{-\frac{c}{ad-bc}} c_2^{\frac{a}{ad-bc}}))(f)(z, w).
\end{aligned}$$

Thus $\pi(A)^k \pi((c_1, c_2))\pi(A)^{-k} \in N_\alpha$. Hence $N_\alpha \trianglelefteq \mathrm{Aut}(\mathcal{A}_\alpha)$. Now we have that $\mathrm{Aut}(\mathcal{A}_\alpha) = N_\alpha < \pi(A) >$ and $N_\alpha \cap < \pi(A) >= \{1\}$. Note that

$$\begin{aligned}
\pi(\mathbf{c})\pi(A)^{k_1}\pi(\mathbf{d})\pi(A)^{k_2} &= \pi(\mathbf{c})(\pi(A)^{k_1}\pi(\mathbf{d})\pi(A)^{-k_1})\pi(A)^{k_1+k_2} \\
&= \pi(\mathbf{c})\pi(\psi_A^{-k_1}(\mathbf{d}))\pi(A)^{k_1+k_2},
\end{aligned}$$

for $\mathbf{c}, \mathbf{d} \in \mathbb{T}^2, k_1, k_2 \in \mathbb{Z}$. Thus $\mathrm{Aut}(\mathcal{A}_\alpha) \cong \mathbb{T}^2 \rtimes_{\psi_A} \mathbb{Z}$. □

Theorem 5.1.10. *Let α, β be positive quadratic irrationals. If $\mathrm{Aut}(\mathcal{A}_\alpha) \cong \mathbb{T}^2 \rtimes_{\psi_A} \mathbb{Z}$ and $\mathrm{Aut}(\mathcal{A}_\beta) \cong \mathbb{T}^2 \rtimes_{\psi_B} \mathbb{Z}$, then $\mathrm{Aut}(\mathcal{A}_\alpha) \cong \mathrm{Aut}(\mathcal{A}_\beta)$ if and only if $B = C^{-1}AC$ or $B^{-1} = C^{-1}AC$ for some $C \in GL(2, \mathbb{Z})$.*

Proof. First, we will show that there is no group homomorphism from $\mathbb{T}^2$ onto $\mathbb{Z}$. Assume that there is a group epimorphism $\phi : \mathbb{T}^2 \to \mathbb{Z}$. Then $\exists\, \mathbf{c} \in \mathbb{T}^2$ such that $\phi(\mathbf{c}) = 1$. Since $\mathbf{c} \in \mathbb{T}^2$, $\exists\, \mathbf{d} \in \mathbb{T}^2$ so that $\mathbf{d}^2 = \mathbf{c}$. Thus $1 = \phi(\mathbf{c}) = \phi(\mathbf{d}.\mathbf{d}) = \phi(\mathbf{d}) + \phi(\mathbf{d}) = 2\phi(\mathbf{d})$ Hence $\phi(\mathbf{d}) = \frac{1}{2}$, a contradiction. Therefore, there is no such an epimorphism. By Theorem A.0.11, $\mathbb{T}^2 \rtimes_{\psi_A} \mathbb{Z} \cong \mathbb{T}^2 \rtimes_{\psi_B} \mathbb{Z}$ if and only if ψ_A is conjugate to ψ_B or ψ_B^{-1}. Thus $\mathbb{T}^2 \rtimes_{\psi_A} \mathbb{Z} \cong \mathbb{T}^2 \rtimes_{\psi_B} \mathbb{Z}$ if and only if A is conjugate to B or B^{-1}, i.e., $\exists\, C \in GL(2, \mathbb{Z})$ such that $B = C^{-1}AC$ or $B^{-1} = C^{-1}AC$. □

5.2 Computation of the automorphism group of $\mathcal{A}_\alpha$

In this section, we will find an explicit formula for any matrix $A \in GL(2, \mathbb{Z})$ such that $\pi(A) \in \mathrm{Aut}(\mathcal{A}_\alpha)$. First, notice that for any positive quadratic irrational number α, we can write α in one of following forms:

(1) $\alpha = \sqrt{\frac{p}{q}}$, $p, q \in \mathbb{N}$, and $\gcd(p,q) = 1$,

(2) $\alpha = \frac{r}{s} + k\sqrt{\frac{p}{q}}$, $r \in \mathbb{Z}, p, q, s \in \mathbb{N}$, $k \in \{-1, 1\}$, $\gcd(r,s) = 1$, and $\gcd(p,q) = 1$.

Note that if $\sqrt{\frac{p}{q}}$ is irrational, then pq is nonsquare.

Proposition 5.2.1. *Let α be a positive irrational number of the form $\sqrt{\frac{p}{q}}$, where $p, q \in \mathbb{N}$, and $\gcd(p,q) = 1$.*

(1) If the equation $x^2 - pqy^2 = -1$ is not solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - pqy^2 = 1$.

(2) If the equation $x^2 - pqy^2 = -1$ is solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1', y_1') is the fundamental solution of the negative Pell's equation $x^2 - pqy^2 = -1$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{Z})$ such that $a + b\sqrt{\frac{p}{q}} > 0$ and

$$b\frac{p}{q} + (a-d)\sqrt{\frac{p}{q}} - c = 0.$$

Then $a = d$ and $c = \frac{p}{q}b$. So A is of the form $\begin{bmatrix} a & b \\ \frac{p}{q}b & a \end{bmatrix}$. Since $A \in GL(2,\mathbb{Z})$ and $\gcd(p,q) = 1$, $q|b$. So $\exists\, j \in \mathbb{Z}$ such that $b = qj$. Thus

$$A = \begin{bmatrix} a & qj \\ pj & a \end{bmatrix}.$$

If $\det(A) = 1$, then $a^2 - pqj^2 = 1$. So $a^2 = 1 + pqj^2 > pqj^2 = \frac{p}{q}b^2$. Thus $|a| > |b|\sqrt{\frac{p}{q}}$. Since $a + b\sqrt{\frac{p}{q}} > 0$, a must be positive. In this case, we have to find integers a, j such that $a > 0$, $a^2 - pqj^2 = 1$.

If $\det(A) = -1$, then $a^2 - pqj^2 = -1$. So $\frac{p}{q}b^2 = pqj^2 = 1 + a^2 > a^2$. Thus $|b|\sqrt{\frac{p}{q}} > |a|$. Since $a + b\sqrt{\frac{p}{q}} > 0$, b must be positive, i.e., j must be positive. In this case, we have to find integers

a, j such that $j > 0$, $a^2 - pqj^2 = -1$.

Case (1) : The equation $x^2 - pqy^2 = -1$ is not solvable over the integers.

Then $\det(A)$ must be 1. Now, A is of the form $\begin{bmatrix} a & qj \\ pj & a \end{bmatrix}$, where $a, j \in \mathbb{Z}$ such that $a > 0$ and $a^2 - pqj^2 = 1$. Let (x_1, y_1) be the positive fundamental solution of the Pell's equation $x^2 - pqy^2 = 1$. Then all nonnegative solutions (x, y) are given by the set

$$\{(1,0), (x_n, y_n) : x_{n+1} = x_1x_n + pqy_1y_n, y_{n+1} = x_1y_n + y_1x_n, n = 1, 2, \ldots\}.$$

Next, we will show by induction that for any $n \in \mathbb{N}$,

$$\begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix}^n = \begin{bmatrix} x_n & qy_n \\ py_n & x_n \end{bmatrix}.$$

Basis step: Trivial.

Inductive step: Let $n \in \mathbb{N}$. Assume that $\begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix}^n = \begin{bmatrix} x_n & qy_n \\ py_n & x_n \end{bmatrix}$. Then

$$\begin{aligned} \begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix}^{n+1} &= \begin{bmatrix} x_n & qy_n \\ py_n & x_n \end{bmatrix} \begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix} \\ &= \begin{bmatrix} x_1x_n + pqy_1y_n & qy_1x_n + qx_1y_n \\ px_1y_n + py_1x_n & pqy_1y_n + x_1x_n \end{bmatrix} \\ &= \begin{bmatrix} x_{n+1} & qy_{n+1} \\ py_{n+1} & x_{n+1} \end{bmatrix}. \end{aligned}$$

Thus, for any $n \in \mathbb{N}$,

$$\begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix}^{-n} = \begin{bmatrix} x_n & q(-y_n) \\ p(-y_n) & x_n \end{bmatrix}.$$

Note that if $a = 1$ and $j = 0$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $A = \begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix}^n$, for some $n \in \mathbb{Z}$.

Case (2) : The equation $x^2 - pqy^2 = -1$ is solvable over the integers.

Then A must be of the form $\begin{bmatrix} a & qj \\ pj & a \end{bmatrix}$ where $a, j \in \mathbb{Z}$,

$$a > 0 \text{ and } a^2 - pqj^2 = 1$$

or

$$j > 0 \text{ and } a^2 - pqj^2 = -1.$$

Let (x_1', y_1') be the fundamental solution of the negative Pell's equation $x^2 - pqy^2 = -1$. Then $(x_1, y_1) = (x_1'^2 + pqy_1'^2, 2x_1'y_1')$ is the fundamental solution of $x^2 - pqy^2 = 1$. Also, all positive solutions (x, y) of $x^2 - pqy^2 = -1$ are given by the set

$$\begin{aligned} \{(x_n', y_n') : & x_{n+1}' = (x_1'^2 + pqy_1'^2)x_n' + 2pqx_1'y_1'y_n', \\ & y_{n+1}' = (x_1'^2 + pqy_1'^2)y_n' + 2x_1'y_1'x_n', n = 1, 2, \ldots\} \end{aligned}$$

and all nonnegative solutions (x, y) of $x^2 - pqy^2 = 1$ are given by the set

$$\{(1, 0), (x_n, y_n) : x_{n+1} = x_1x_n + pqy_1y_n, y_{n+1} = x_1y_n + y_1x_n, n = 1, 2, \ldots\}.$$

We will show that, for any $n \in \mathbb{N}$,

$$\begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{2n-1} = \begin{bmatrix} x_n' & qy_n' \\ py_n' & x_n' \end{bmatrix}.$$

Let $n \in \mathbb{N}$. Assume that $\begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{2n-1} = \begin{bmatrix} x_n' & qy_n' \\ py_n' & x_n' \end{bmatrix}$. Then

$$\begin{aligned} \begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{2(n+1)-1} &= \begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{2n-1} \begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{2} \\ &= \begin{bmatrix} x_n' & qy_n' \\ py_n' & x_n' \end{bmatrix} \begin{bmatrix} x_1'^2 + pqy_1'^2 & 2qx_1'y_1' \\ 2px_1'y_1' & x_1'^2 + pqy_1'^2 \end{bmatrix} \\ &= \begin{bmatrix} (x_1'^2 + pqy_1'^2)x_n' + 2pqx_1'y_1'y_n & 2qx_1'x_n'y_1' + qy_n'(x_1'^2 + pqy_1'^2) \\ py_n'(x_1'^2 + pqy_1'^2) + 2px_1'x_n'y_1' & 2pqx_1'y_1'y_n + (x_1'^2 + pqy_1'^2)x_n' \end{bmatrix} \\ &= \begin{bmatrix} x_{n+1}' & qy_{n+1}' \\ py_{n+1}' & x_{n+1}' \end{bmatrix}. \end{aligned}$$

Hence, for any $n \in \mathbb{N}$,

$$\begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{2n-1} = \begin{bmatrix} x_n' & qy_n' \\ py_n' & x_n' \end{bmatrix}.$$

Thus, for any $n \in \mathbb{N}$,

$$\begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{-(2n-1)} = \begin{bmatrix} -x_n' & qy_n' \\ py_n' & -x_n' \end{bmatrix}.$$

We also have,

$$\begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{2n} = \begin{bmatrix} x_1'^2 + pqy_1'^2 & 2qx_1'y_1' \\ 2px_1'y_1' & x_1'^2 + pqy_1'^2 \end{bmatrix}^{n} = \begin{bmatrix} x_1 & qy_1 \\ py_1 & x_1 \end{bmatrix}^{n} = \begin{bmatrix} x_n & qy_n \\ py_n & x_n \end{bmatrix}$$

and

$$\begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{-2n} = \begin{bmatrix} x_n & q(-y_n) \\ p(-y_n) & x_n \end{bmatrix}, n \in \mathbb{N}.$$

If $a = 1$ and $j = 0$, then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $A = \begin{bmatrix} x_1' & qy_1' \\ py_1' & x_1' \end{bmatrix}^{n}$, for some $n \in \mathbb{Z}$. $\square$

Proposition 5.2.2. *Let α be a positive irrational number of the form $\frac{r}{s} + k\sqrt{\frac{p}{q}}$, where $r \in \mathbb{Z}, p, q, s \in \mathbb{N}$, $k \in \{-1, 1\}$, s is odd,* $\gcd(r,s) = 1$ *and* $\gcd(p,q) = 1$. *Let $d_1 = \gcd(ps^2 - qr^2, qs)$.*
(1) If the equation $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$ is not solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} -\frac{qrs}{d_1}y_1 + x_1 & \frac{qs^2}{d_1}y_1 \\ (\frac{ps^2-qr^2}{d_1})y_1 & \frac{qrs}{d_1}y_1 + x_1 \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1, y_1) is the fundamental solution of $x^2 - \frac{pqs^4}{d_1^2}y^2 = 1$.
(2) If the equation $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$ is solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} -\frac{qrs}{d_1}ky_1' + x_1' & \frac{qs^2}{d_1}ky_1' \\ (\frac{ps^2-qr^2}{d_1})ky_1' & \frac{qrs}{d_1}ky_1' + x_1' \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1', y_1') is the fundamental solution of $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$.

Proof. First, note that $\frac{qrs}{d_1}, \frac{qs^2}{d_1}, \frac{ps^2-qr^2}{d_1} \in \mathbb{Z}$. We will show that $\frac{pqs^4}{d_1^2}$ is a nonsquare integer. Since $d_1|(ps^2 - qr^2)$ and $d_1|qs^2$, we have $d_1^2|(pqs^4 - q^2r^2s^2)$. We also have that $d_1^2|q^2r^2s^2$. Thus

$d_1^2|pqs^4$. Since α is irrational, $\sqrt{\frac{p}{q}}$ is irrational. Thus pq is nonsquare. Hence $\frac{pqs^4}{d_1^2}$ must be nonsquare. Now, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{Z})$ be such that $a + b\alpha > 0$ and

$$b\alpha^2 + (a-d)\alpha - c = 0.$$

Then $a - d = -2\frac{r}{s}b$ and $c = (\frac{ps^2-qr^2}{qs^2})b$. So A is of the form

$$\begin{bmatrix} a & b \\ (\frac{ps^2-qr^2}{qs^2})b & a + 2\frac{r}{s}b \end{bmatrix}.$$

Since $A \in GL(2,\mathbb{Z})$ and $\gcd(2r, s) = 1$, we have $s|b$. Then there exists $j \in \mathbb{Z}$ such that $b = sj$. Now,

$$A = \begin{bmatrix} a & sj \\ (\frac{ps^2-qr^2}{qs})j & a + 2rj \end{bmatrix}.$$

Since $\gcd(\frac{ps^2-qr^2}{d_1}, \frac{qs}{d_1}) = 1$, $\frac{qs}{d_1}|j$. So $j = \frac{qs}{d_1}l$ for some $l \in \mathbb{Z}$. Thus

$$A = \begin{bmatrix} a & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & a + 2\frac{qrs}{d_1}l \end{bmatrix}.$$

If $\det(A) = 1$, then $a^2 + 2\frac{qrs}{d_1}la - (\frac{ps^2-qr^2}{d_1})\frac{qs^2}{d_1}l^2 = 1$. By the quadratic formula,

$$a = -\frac{qrs}{d_1}l \pm \sqrt{\frac{pqs^4}{d_1^2}l^2 + 1}.$$

If $a = -\frac{qrs}{d_1}l + \sqrt{\frac{pqs^4}{d_1^2}l^2+1}$, then

$$\begin{aligned} a + b(\frac{r}{s} + k\sqrt{\frac{p}{q}}) &= -\frac{qrs}{d_1}l + \sqrt{\frac{pqs^4}{d_1^2}l^2 + 1} + \frac{qs^2}{d_1}l(\frac{r}{s} + k\sqrt{\frac{p}{q}}) \\ &= \sqrt{\frac{pqs^4}{d_1^2}l^2 + 1} + k\frac{s^2\sqrt{pq}}{d_1}l > 0. \end{aligned}$$

If $a = -\frac{qrs}{d_1}l - \sqrt{\frac{pqs^4}{d_1^2}l^2+1}$, then

$$\begin{aligned} a + b(\frac{r}{s} + k\sqrt{\frac{p}{q}}) &= -\frac{qrs}{d_1}l - \sqrt{\frac{pqs^4}{d_1^2}l^2 + 1} + \frac{qs^2}{d_1}l(\frac{r}{s} + k\sqrt{\frac{p}{q}}) \\ &= -\sqrt{\frac{pqs^4}{d_1^2}l^2 + 1} + k\frac{s^2\sqrt{pq}}{d_1}l < 0. \end{aligned}$$

Hence $a=-\frac{qrs}{d_1}l+\sqrt{\frac{pqs^4}{d_1^2}l^2+1}$. Now A is of the form

$$\begin{bmatrix} -\frac{qrs}{d_1}l+\sqrt{\frac{pqs^4}{d_1^2}l^2+1} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l+\sqrt{\frac{pqs^4}{d_1^2}l^2+1} \end{bmatrix}.$$

Since $A \in GL(2,\mathbb{Z})$, $\sqrt{\frac{pqs^4}{d_1^2}l^2+1}$ must be an integer. Thus

$$A=\begin{bmatrix} -\frac{qrs}{d_1}l+x & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l+x \end{bmatrix},$$

where $x,l \in \mathbb{Z}, x>0$ and $x^2-\frac{pqs^4}{d_1^2}l^2=1$. In this case, we have to find integers x,l such that $x>0$ and $x^2-\frac{pqs^4}{d_1^2}l^2=1$.

If $\det(A)=-1$, then $a^2+2\frac{qrs}{d_1}la-(\frac{ps^2-qr^2}{d_1})\frac{qs^2}{d_1}l^2=-1$. So

$$a=-\frac{qrs}{d_1}l \pm \sqrt{\frac{pqs^4}{d_1^2}l^2-1}.$$

If $a=-\frac{qrs}{d_1}l+\sqrt{\frac{pqs^4}{d_1^2}l^2-1}$, then

$$\begin{aligned} a+b(\frac{r}{s}+k\sqrt{\frac{p}{q}}) &= -\frac{qrs}{d_1}l+\sqrt{\frac{pqs^4}{d_1^2}l^2-1}+\frac{qs^2}{d_1}l(\frac{r}{s}+k\sqrt{\frac{p}{q}}) \\ &= \sqrt{\frac{pqs^4}{d_1^2}l^2-1}+k\frac{s^2\sqrt{pq}}{d_1}l \begin{cases} <0 & \text{if } k=1 \text{ and } l<0 \\ >0 & \text{if } k=1 \text{ and } l>0 \\ >0 & \text{if } k=-1 \text{ and } l<0 \\ <0 & \text{if } k=-1 \text{ and } l>0 \end{cases}. \end{aligned}$$

If $a=-\frac{qrs}{d_1}l-\sqrt{\frac{pqs^4}{d_1^2}l^2-1}$, then

$$\begin{aligned} a+b(\frac{r}{s}+k\sqrt{\frac{p}{q}}) &= -\frac{qrs}{d_1}l-\sqrt{\frac{pqs^4}{d_1^2}l^2-1}+\frac{qs^2}{d_1}l(\frac{r}{s}+k\sqrt{\frac{p}{q}}) \\ &= -\sqrt{\frac{pqs^4}{d_1^2}l^2-1}+k\frac{s^2\sqrt{pq}}{d_1}l \begin{cases} <0 & \text{if } k=1 \text{ and } l<0 \\ >0 & \text{if } k=1 \text{ and } l>0 \\ >0 & \text{if } k=-1 \text{ and } l<0 \\ <0 & \text{if } k=-1 \text{ and } l>0 \end{cases}. \end{aligned}$$

Since $A \in GL(2,\mathbb{Z})$, $\sqrt{\frac{pqs^4}{d_1^2}l^2 - 1}$ must be an integer. If $k = 1$, then

$$A = \begin{bmatrix} -\frac{qrs}{d_1}l + x & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l + x \end{bmatrix},$$

where $x, l \in \mathbb{Z}, l > 0$ and $x^2 - \frac{pqs^4}{d_1^2}l^2 = -1$. In this case, we have to find integers x, l such that $l > 0$ and $x^2 - \frac{pqs^4}{d_1^2}l^2 = -1$. If $k = -1$, then

$$A = \begin{bmatrix} -\frac{qrs}{d_1}l + x & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l + x \end{bmatrix},$$

where $x, l \in \mathbb{Z}, l < 0$ and $x^2 - \frac{pqs^4}{d_1^2}l^2 = -1$. In this case, we have to find integers x, l such that $l < 0$ and $x^2 - \frac{pqs^4}{d_1^2}l^2 = -1$.

Case (1) : The equation $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$ is not solvable over the integers.

Then $\det(A)$ must be 1. Thus $A = \begin{bmatrix} -\frac{qrs}{d_1}l + x & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l + x \end{bmatrix}$, for some $x, l \in \mathbb{Z}, x > 0$ and $x^2 - \frac{pqs^4}{d_1^2}l^2 = 1$. Let (x_1, y_1) be the fundamental solution of the Pell's equation $x^2 - \frac{pqs^4}{d_1^2}y^2 = 1$. Then all nonnegative solutions (x, y) are given by the set

$$\begin{aligned} \{(1,0), (x_n, y_n) : & x_{n+1} = x_1 x_n + \frac{pqs^4}{d_1^2} y_1 y_n, \\ & y_{n+1} = x_1 y_n + y_1 x_n, n = 1, 2, \ldots\}. \end{aligned}$$

By induction, for $n \in \mathbb{N}$,

$$\begin{bmatrix} -\frac{qrs}{d_1}y_1 + x_1 & \frac{qs^2}{d_1}y_1 \\ (\frac{ps^2-qr^2}{d_1})y_1 & \frac{qrs}{d_1}y_1 + x_1 \end{bmatrix}^n = \begin{bmatrix} -\frac{qrs}{d_1}y_n + x_n & \frac{qs^2}{d_1}y_n \\ (\frac{ps^2-qr^2}{d_1})y_n & \frac{qrs}{d_1}y_n + x_n \end{bmatrix}.$$

Thus, for each $n \in \mathbb{N}$,

$$\begin{bmatrix} -\frac{qrs}{d_1}y_1 + x_1 & \frac{qs^2}{d_1}y_1 \\ (\frac{ps^2-qr^2}{d_1})y_1 & \frac{qrs}{d_1}y_1 + x_1 \end{bmatrix}^{-n} = \begin{bmatrix} -\frac{qrs}{d_1}(-y_n) + x_n & \frac{qs^2}{d_1}(-y_n) \\ (\frac{ps^2-qr^2}{d_1})(-y_n) & \frac{qrs}{d_1}(-y_n) + x_n \end{bmatrix}.$$

Hence $A = \begin{bmatrix} -\frac{qrs}{d_1}y_1 + x_1 & \frac{qs^2}{d_1}y_1 \\ (\frac{ps^2-qr^2}{d_1})y_1 & \frac{qrs}{d_1}y_1 + x_1 \end{bmatrix}^n$, for some $n \in \mathbb{Z}$.

Case (2) : The equation $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$ is solvable over the integers.

If $k=1$, then A must be of the form $\begin{bmatrix} -\frac{qrs}{d_1}l+x & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l+x \end{bmatrix}$ where $x,l \in \mathbb{Z}$,

$$x>0, \text{ and } x^2-\frac{pqs^4}{d_1^2}l^2=1$$

or

$$l>0, \text{ and } x^2-\frac{pqs^4}{d_1^2}l^2=-1.$$

If $k=-1$, then A must be of the form $\begin{bmatrix} -\frac{qrs}{d_1}l+x & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l+x \end{bmatrix}$ where $x,l \in \mathbb{Z}$,

$$x>0, \text{ and } x^2-\frac{pqs^4}{d_1^2}l^2=1$$

or

$$l<0, \text{ and } x^2-\frac{pqs^4}{d_1^2}l^2=-1.$$

Let (x_1', y_1') be the fundamental solution of the negative Pell's equation $x^2-\frac{pqs^4}{d_1^2}y^2=-1$. Then $(x_1,y_1)=(x_1'^2+\frac{pqs^4}{d_1^2}y_1'^2, 2x_1'y_1')$ is the fundamental solution of $x^2-\frac{pqs^4}{d_1^2}y^2=1$. Also, all positive solutions (x,y) of $x^2-\frac{pqs^4}{d_1^2}y^2=-1$ are given by the set

$$\begin{aligned}\{(x_n', y_n') :& x_{n+1}'=(x_1'^2+\frac{pqs^4}{d_1^2}y_1'^2)x_n'+2\frac{pqs^4}{d_1^2}x_1'y_1'y_n', \\ & y_{n+1}'=(x_1'^2+\frac{pqs^4}{d_1^2}y_1'^2)y_n'+2x_1'y_1'x_n', n=1,2,\ldots\}\end{aligned}$$

and all nonnegative solutions (x,y) of $x^2-\frac{pqs^4}{d_1^2}y^2=1$ are given by the set

$$\begin{aligned}\{(1,0),(x_n,y_n) :& x_{n+1}=x_1x_n+\frac{pqs^4}{d_1^2}y_1y_n, \\ & y_{n+1}=x_1y_n+y_1x_n, n=1,2,\ldots\}.\end{aligned}$$

Thus $A=\begin{bmatrix} -\frac{qrs}{d_1}ky_1'+x_1' & \frac{qs^2}{d_1}ky_1' \\ (\frac{ps^2-qr^2}{d_1})ky_1' & \frac{qrs}{d_1}ky_1'+x_1' \end{bmatrix}^n$, for some $n \in \mathbb{Z}$. □

Proposition 5.2.3. *Let α be a positive irrational number of the form $\frac{r}{s}+k\sqrt{\frac{p}{q}}$, where $r \in \mathbb{Z}, p,q,s \in \mathbb{N}$, $k \in \{-1,1\}$, s is even,* $\gcd(r,s)=1$ *and* $\gcd(p,q)=1$. *Let* $d_1=\gcd(ps^2-$

$qr^2, 2qs)$.

(1) If $d_1 \mid qs$ and the equation $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$ is not solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} -\frac{qrs}{d_1}y_1 + x_1 & \frac{qs^2}{d_1}y_1 \\ (\frac{ps^2-qr^2}{d_1})y_1 & \frac{qrs}{d_1}y_1 + x_1 \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1, y_1) is the fundamental solution of $x^2 - \frac{pqs^4}{d_1^2}y^2 = 1$.

(2) If $d_1 \mid qs$ and the equation $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$ is solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} -\frac{qrs}{d_1}ky_1' + x_1' & \frac{qs^2}{d_1}ky_1' \\ (\frac{ps^2-qr^2}{d_1})ky_1' & \frac{qrs}{d_1}ky_1' + x_1' \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1', y_1') is the fundamental solution of $x^2 - \frac{pqs^4}{d_1^2}y^2 = -1$.

(3) If $d_1 \nmid qs$ and the equation $x^2 - \frac{4pqs^4}{d_1^2}l^2 = -4$ is not solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} -\frac{qrs}{d_1}y_1 + \frac{x_1}{2} & \frac{qs^2}{d_1}y_1 \\ (\frac{ps^2-qr^2}{d_1})y_1 & \frac{qrs}{d_1}y_1 + \frac{x_1}{2} \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1, y_1) is the fundamental solution of $x^2 - \frac{4pqs^4}{d_1^2}l^2 = 4$.

(4) If $d_1 \nmid qs$ and the equation $x^2 - \frac{4pqs^4}{d_1^2}l^2 = -4$ is solvable over the integers, then, for any $A \in GL(2,\mathbb{Z})$, $\pi(A)$ is an automorphism of $\mathcal{A}_\alpha$ if and only if $A = \begin{bmatrix} -\frac{qrs}{d_1}ky_1' + \frac{x_1'}{2} & \frac{qs^2}{d_1}ky_1' \\ (\frac{ps^2-qr^2}{d_1})ky_1' & \frac{qrs}{d_1}ky_1' + \frac{x_1'}{2} \end{bmatrix}^n$, for some $n \in \mathbb{Z}$, where (x_1', y_1') is the fundamental solution of $x^2 - \frac{4pqs^4}{d_1^2}l^2 = -4$.

Proof. First, note that if $d_1 \mid qs$, then $\frac{qrs}{d_1}, \frac{qs^2}{d_1}, \frac{ps^2-qr^2}{d_1} \in \mathbb{Z}$. In this case, we will show that $\frac{pqs^4}{d_1^2}$ is a nonsquare integer. Since $d_1|(ps^2-qr^2)$ and $d_1|qs^2$, we have $d_1^2|(pqs^4-q^2r^2s^2)$. We also have that $d_1^2|q^2r^2s^2$. Thus $d_1^2|pqs^4$. Since α is irrational, $\sqrt{\frac{p}{q}}$ is irrational. Thus pq is nonsquare. Hence $\frac{pqs^4}{d_1^2}$ must be nonsquare. Now, if $d_1 \nmid qs$, then $\frac{ps^2-qr^2}{d_1}, \frac{qs^2}{d_1} \in \mathbb{Z}$. In this case, we will show that $\frac{4pqs^4}{d_1^2}$ is a nonsquare integer. Since $d_1|2qs$, $d_1|4qs^2$. Since $d_1|(ps^2 - qr^2)$ and $d_1|4qs^2$, we have $d_1^2|(4pqs^4 - 4q^2r^2s^2)$. We also have that $d_1^2|4q^2r^2s^2$. Thus $d_1^2|4pqs^4$. Since α is irrational, $\sqrt{\frac{p}{q}}$ is irrational. Thus pq is nonsquare. Hence $\frac{4pqs^4}{d_1^2}$ must be nonsquare.

Now, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{Z})$ be such that $a + b\alpha > 0$ and

$$b\alpha^2 + (a-d)\alpha - c = 0.$$

Then $a-d=-2\frac{r}{s}b$ and $c=(\frac{ps^2-qr^2}{qs^2})b$. So A is of the form

$$\begin{bmatrix} a & b \\ (\frac{ps^2-qr^2}{qs^2})b & a+2\frac{r}{s}b \end{bmatrix}.$$

Since s is even, $\frac{s}{2}\in\mathbb{N}$. Since $A\in GL(2,\mathbb{Z})$ and $\gcd(r,\frac{s}{2})=1$, we have $\frac{s}{2}|b$. Then there exists $j\in\mathbb{Z}$ such that $b=\frac{s}{2}j$. Now,

$$A=\begin{bmatrix} a & \frac{s}{2}j \\ (\frac{ps^2-qr^2}{2qs})j & a+rj \end{bmatrix}.$$

Since $\gcd(\frac{ps^2-qr^2}{d_1},\frac{2qs}{d_1})=1$, $\frac{2qs}{d_1}|j$. So $j=\frac{2qs}{d_1}l$ for some $l\in\mathbb{Z}$. Thus

$$A=\begin{bmatrix} a & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & a+2\frac{qrs}{d_1}l \end{bmatrix}.$$

Case : $d_1|qs$.

By the same argument in the proof of Proposition 5.2.2,

$$A=\begin{bmatrix} -\frac{qrs}{d_1}y_1+x_1 & \frac{qs^2}{d_1}y_1 \\ (\frac{ps^2-qr^2}{d_1})y_1 & \frac{qrs}{d_1}y_1+x_1 \end{bmatrix}^n,$$

for some $n\in\mathbb{Z}$, where (x_1,y_1) is the fundamental solution of $x^2-\frac{pqs^4}{d_1^2}y^2=1$ if the equation $x^2-\frac{pqs^4}{d_1^2}y^2=-1$ is not solvable over the integers and

$$A=\begin{bmatrix} -\frac{qrs}{d_1}ky_1'+x_1' & \frac{qs^2}{d_1}ky_1' \\ (\frac{ps^2-qr^2}{d_1})ky_1' & \frac{qrs}{d_1}ky_1'+x_1' \end{bmatrix}^n,$$

for some $n\in\mathbb{Z}$, where (x_1',y_1') is the fundamental solution of $x^2-\frac{pqs^4}{d_1^2}y^2=-1$ if the equation $x^2-\frac{pqs^4}{d_1^2}y^2=-1$ is solvable over the integers.

Case : $d_1\nmid qs$.

Recall that $A=\begin{bmatrix} a & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & a+2\frac{qrs}{d_1}l \end{bmatrix}$, for some $j\in\mathbb{Z}$

If $\det(A)=1$, then $a^2+2\frac{qrs}{d_1}la-(\frac{ps^2-qr^2}{d_1})\frac{qs^2}{d_1}l^2=1$. By the quadratic formula,

$$a=-\frac{qrs}{d_1}l\pm\frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2}.$$

If $a = -\frac{qrs}{d_1}l + \frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2}$, then

$$a + b(\frac{r}{s} + k\sqrt{\frac{p}{q}}) = -\frac{qrs}{d_1}l + \frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2 + 4}}{2} + \frac{qs^2}{d_1}l(\frac{r}{s} + k\sqrt{\frac{p}{q}})$$
$$= \sqrt{\frac{pqs^4}{d_1^2}l^2 + 1} + k\frac{s^2\sqrt{pq}}{d_1}l > 0.$$

If $a = -\frac{qrs}{d_1}l - \frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2}$, then

$$a + b(\frac{r}{s} + k\sqrt{\frac{p}{q}}) = -\frac{qrs}{d_1}l - \frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2 + 4}}{2} + \frac{qs^2}{d_1}l(\frac{r}{s} + k\sqrt{\frac{p}{q}})$$
$$= -\sqrt{\frac{pqs^4}{d_1^2}l^2 + 1} + k\frac{s^2\sqrt{pq}}{d_1}l < 0.$$

Hence $a = -\frac{qrs}{d_1}l + \frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2}$. Now A is of the form

$$\begin{bmatrix} -\frac{qrs}{d_1}l + \frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l + \frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2} \end{bmatrix}.$$

Since $d_1 \nmid qs$ and $d_1 \mid 2qs$, d_1 must be even. Since $\gcd(r, s) = 1$ and s is even, r must be odd. Since $d_1 = \gcd(ps^2 - qr^2, 2qs)$ and d_1 is even, q must be even. Since $\gcd(p, q) = 1$ and q is even, p must be odd. Now we write $s = 2^{m_1}n_1, q = 2^{m_2}n_2$, where $m_1, m_2 \in \mathbb{N}$ and n_1, n_2 are odd integers. Since $d_1 \nmid qs$ and $d_1 \mid 2qs$, $d_1 = 2^{m_1+m_2+1}n_3$ for some odd integer n_3 and $n_3 \mid n_1n_2$. Since n_1, n_2, n_3 are odd and $n_3 \mid n_1n_2$, $\frac{n_1n_2}{n_3}$ is an odd integer. Now we have

$$\frac{qrs}{d_1} = \frac{2^{m_2}n_2r2^{m_1}n_1}{2^{m_1+m_2+1}n_3} = \frac{r(\frac{n_1n_2}{n_3})}{2}.$$

Thus A is of the form

$$\begin{bmatrix} \frac{-r(\frac{n_1n_2}{n_3})l+\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{r(\frac{n_1n_2}{n_3})l+\sqrt{\frac{4pqs^4}{d_1^2}l^2+4}}{2} \end{bmatrix}.$$

Since $A \in GL(2, \mathbb{Z})$, $\sqrt{\frac{4pqs^4}{d_1^2}l^2 + 4}$ must be an integer. Thus

$$A = \begin{bmatrix} \frac{-r(\frac{n_1n_2}{n_3})l+x}{2} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{r(\frac{n_1n_2}{n_3})l+x}{2} \end{bmatrix}.$$

where $x, l \in \mathbb{Z}, x > 0$ and $x^2 - \frac{4pqs^4}{d_1^2} l^2 = 4$.

Since $\frac{-r(\frac{n_1 n_2}{n_3})l + x}{2}$ and $\frac{r(\frac{n_1 n_2}{n_3})l + x}{2}$ must be integers and $r, \frac{n_1 n_2}{n_3}$ are odd, x and l must have the same parity. We want to show that $\frac{4pqs^4}{d_1^2} \not\equiv 0 \pmod 4$, i.e., we have to show that $d_1^2 \nmid pqs^4$. Since $d_1 \mid ps^2 - qr^2$ and $d_1 = 2^{m_1+m_2+1} n_3$, we have $2^{m_1+m_2+1} \mid ps^2 - qr^2$. Let $m = \min\{2m_1, m_2\}$. Then

$$ps^2 - qr^2 = 2^m \left(p\frac{s^2}{2^m} - \frac{q}{2^m} r\right).$$

Since $m = \min\{2m_1, m_2\}$, $p\frac{s^2}{2^m}$ or $\frac{q}{2^m} r$ must be odd. Since $m \le m_2 < m_1 + m_2 + 1$, $p\frac{s^2}{2^m} - \frac{q}{2^m} r$ must be even. That is $p\frac{s^2}{2^m}$ and $\frac{q}{2^m} r$ must be odd. Thus $m = m_2 = 2m_1$. Now we have

$$pqs^4 = p2^{m_2} n_2 2^{4m_1} n_1^4 = 2^{m_2+4m_1} pn_2 n_1^4$$

and

$$d_1^2 = 2^{2m_1+2m_2+2} n_3^2.$$

Since $m_2 + 4m_1 = 2m_2 + 2m_1 + 1 < 2m_1 + 2m_2 + 2$, $d_1^2 \nmid pqs^4$. Thus $\frac{4pqs^4}{d_1^2} \not\equiv 0 \pmod 4$. By Proposition 5.1.1, x and l that satisfy the equation $x^2 - \frac{4pqs^4}{d_1^2} l^2 = 4$ have the same parity. Thus, in this case, we have to find integers x, l such that $x > 0$ and $x^2 - \frac{4pqs^4}{d_1^2} l^2 = 4$.

If $\det(A) = -1$, then $a^2 + 2\frac{qrs}{d_1} la - (\frac{ps^2 - qr^2}{d_1})\frac{qs^2}{d_1} l^2 = -1$. So

$$a = -\frac{qrs}{d_1} l \pm \frac{\sqrt{\frac{4pqs^4}{d_1^2} l^2 - 4}}{2}.$$

If $a = -\frac{qrs}{d_1} l + \frac{\sqrt{\frac{4pqs^4}{d_1^2} l^2 - 4}}{2}$, then

$$\begin{aligned} a + b\left(\frac{r}{s} + k\sqrt{\frac{p}{q}}\right) &= -\frac{qrs}{d_1} l + \frac{\sqrt{\frac{4pqs^4}{d_1^2} l^2 - 4}}{2} + \frac{qs^2}{d_1} l\left(\frac{r}{s} + k\sqrt{\frac{p}{q}}\right) \\ &= \sqrt{\frac{pqs^4}{d_1^2} l^2 - 1} + k\frac{s^2\sqrt{pq}}{d_1} l \begin{cases} < 0 & \text{if } k = 1 \text{ and } l < 0 \\ > 0 & \text{if } k = 1 \text{ and } l > 0 \\ > 0 & \text{if } k = -1 \text{ and } l < 0 \\ < 0 & \text{if } k = -1 \text{ and } l > 0 \end{cases}. \end{aligned}$$

If $a=-\frac{qrs}{d_1}l-\frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2-4}}{2}$, then

$$
\begin{aligned}
a+b(\frac{r}{s}+k\sqrt{\frac{p}{q}}) &= -\frac{qrs}{d_1}l-\frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2-4}}{2}+\frac{qs^2}{d_1}l(\frac{r}{s}+k\sqrt{\frac{p}{q}}) \\
&= -\sqrt{\frac{pqs^4}{d_1^2}l^2-1}+k\frac{s^2\sqrt{pq}}{d_1}l\begin{cases} <0 & \text{if } k=1 \text{ and } l<0 \\ >0 & \text{if } k=1 \text{ and } l>0 \\ >0 & \text{if } k=-1 \text{ and } l<0 \\ <0 & \text{if } k=-1 \text{ and } l>0 \end{cases}.
\end{aligned}
$$

Thus, if $k=1$, then A is of the form

$$
\begin{bmatrix} -\frac{qrs}{d_1}l\pm\frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2-4}}{2} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l\pm\frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2-4}}{2} \end{bmatrix}, l>0.
$$

If $k=-1$, then A is of the form

$$
\begin{bmatrix} -\frac{qrs}{d_1}l\pm\frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2-4}}{2} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l\pm\frac{\sqrt{\frac{4pqs^4}{d_1^2}l^2-4}}{2} \end{bmatrix}, l<0.
$$

Since $A\in GL(2,\mathbb{Z})$ and $\frac{qrs}{d_1}=\frac{r(\frac{n_1n_2}{n_3})}{2}$, $\sqrt{\frac{4pqs^4}{d_1^2}l^2-4}$ must be an integer. If $k=1$, then

$$
A=\begin{bmatrix} -\frac{qrs}{d_1}l+\frac{x}{2} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l+\frac{x}{2} \end{bmatrix},
$$

where $x,l\in\mathbb{Z}, l>0$ and $x^2-\frac{4pqs^4}{d_1^2}l^2=-4$.

If $k=-1$, then

$$
A=\begin{bmatrix} -\frac{qrs}{d_1}l+\frac{x}{2} & \frac{qs^2}{d_1}l \\ (\frac{ps^2-qr^2}{d_1})l & \frac{qrs}{d_1}l+\frac{x}{2} \end{bmatrix},
$$

where $x,l\in\mathbb{Z}, l<0$ and $x^2-\frac{4pqs^4}{d_1^2}l^2=-4$.

By the same argument as above, if $x,l\in\mathbb{Z}$ satisfying $x^2-\frac{4pqs^4}{d_1^2}l^2=-4$, $-\frac{qrs}{d_1}l+\frac{x}{2}$ and $\frac{qrs}{d_1}l+\frac{x}{2}$ are integers. Thus, in this case, we have to find integers x,l such that $kl>0$ and

$x^2 - \frac{4pqs^4}{d_1^2} l^2 = -4.$

By the same argument in the proof of Proposition 5.2.2, we have

$$A = \begin{bmatrix} -\frac{qrs}{d_1} y_1 + \frac{x_1}{2} & \frac{qs^2}{d_1} y_1 \\ (\frac{ps^2 - qr^2}{d_1}) y_1 & \frac{qrs}{d_1} y_1 + \frac{x_1}{2} \end{bmatrix}^n,$$

for some $n \in \mathbb{Z}$, where (x_1, y_1) is the fundamental solution of $x^2 - \frac{4pqs^4}{d_1^2} l^2 = 4$ if the equation $x^2 - \frac{4pqs^4}{d_1^2} l^2 = -4$ is not solvable over the integers and

$$A = \begin{bmatrix} -\frac{qrs}{d_1} k y_1' + \frac{x_1'}{2} & \frac{qs^2}{d_1} k y_1' \\ (\frac{ps^2 - qr^2}{d_1}) k y_1' & \frac{qrs}{d_1} k y_1' + \frac{x_1'}{2} \end{bmatrix}^n,$$

for some $n \in \mathbb{Z}$, where (x_1', y_1') is the fundamental solution of $x^2 - \frac{4pqs^4}{d_1^2} l^2 = -4$ if the equation $x^2 - \frac{4pqs^4}{d_1^2} l^2 = -4$ is solvable over the integers. $\square$

Example 5.2.4. $\alpha = \sqrt{5}$.

The fundamental solution of the equation $x^2 - 5y^2 = -1$ is $(x_1', y_1') = (2, 1)$. Thus any matrix A in $GL(2, \mathbb{Z})$ that $\pi(A)$ is automorphisms of $\mathcal{A}_{\sqrt{5}}$ is of the form $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^n$, $n \in \mathbb{Z}$, i.e., $\pi(A)$ is an automorphism of $A_{\sqrt{5}}$ if and only if $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^n$ for some $n \in \mathbb{Z}$.

Example 5.2.5. $\alpha = \sqrt{7}$.

Since $7 \not\equiv 1 (\mod 4)$, the equation $x^2 - 7y^2 = -1$ is not solvable over integers. So we look at the fundamental solution of the equation $x^2 - 7y^2 = 1$. Since $(x_1, y_1) = (8, 3)$ is the fundamental solution of $x^2 - 7y^2 = 1$, any matrix A in $GL(2, \mathbb{Z})$ that $\pi(A)$ is automorphisms of $\mathcal{A}_{\sqrt{7}}$ is of the form $A = \begin{bmatrix} 8 & 3 \\ 21 & 8 \end{bmatrix}^n$, $n \in \mathbb{Z}$.

Example 5.2.6. $\alpha = \frac{1+\sqrt{7}}{3}$.

Since $(x_1, y_1) = (8, 3)$ is the fundamental solution of $x^2 - 7y^2 = 1$, by Proposition 5.2.2, any matrix A in $GL(2, \mathbb{Z})$ that $\pi(A)$ is automorphisms of $\mathcal{A}_{\frac{1+\sqrt{7}}{3}}$ is of the form $A = \begin{bmatrix} 5 & 9 \\ 6 & 11 \end{bmatrix}^n$, $n \in \mathbb{Z}$.

Example 5.2.7. $\alpha = \frac{1+\sqrt{5}}{2}$.

The fundamental solution of the equation $x^2 - 5y^2 = -4$ is $(x_1', y_1') = (1, 1)$. By Proposition 5.2.3, any matrix A in $GL(2, \mathbb{Z})$ that $\pi(A)$ is automorphisms of $\mathcal{A}_{\frac{1+\sqrt{5}}{2}}$ is of the form
$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$, $n \in \mathbb{Z}$

Note that $\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$ is not conjugate to $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ nor $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$ since they don't have the same trace. By Theorem 5.1.10, $\mathcal{A}_{\sqrt{5}}$ is not isomorphic to $\mathcal{A}_{\frac{1+\sqrt{5}}{2}}$.

Example 5.2.8. Let $A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$. Then $A \in GL(2, \mathbb{Z})$. Moreover, $-1 + (\frac{1+\sqrt{7}}{3})1 = \frac{-2+\sqrt{7}}{3} > 0$, and

$$\begin{aligned} 3 + (\frac{1+\sqrt{7}}{3})(-2) &= \frac{7 - 2\sqrt{7}}{3} \\ &= \sqrt{7}(\frac{-2+\sqrt{7}}{3}) \\ &= \sqrt{7}(-1 + (\frac{1+\sqrt{7}}{3})(1)). \end{aligned}$$

Thus, by Theorem 4.0.9, $\mathcal{A}_{\sqrt{7}}$ is isomorphic to $\mathcal{A}_{\frac{1+\sqrt{7}}{3}}$.

APPENDIX . ADDITIONAL THEOREMS

Theorem A.0.9. (Poincaré's Recurrence Theorem) *(see [14]) Let* $(X, \mathfrak{B}, m)$ *be a probability space. Let* $T : X \to X$ *be such that* $m(E) = m(T^{-1}(E))$ *for any* $E \in \mathfrak{B}$*. Let* $F \in \mathfrak{B}$ *be such that* $m(F) > 0$*. Then there is a sequence* $n_1 < n_2 < \ldots$*,* $n_1, n_2, \ldots \in \mathbb{N}$ *such that* $T^{n_i}(x) \in F$ *for almost all points* $x \in F$*, for all* $i \in \mathbb{N}$*.*

Theorem A.0.10. (Harnack's Inequality) *(see [12]) Let* $u : \bar{B}(a, R) \to \mathbb{R}$ *be a continuous function. If* u *is harmonic in* $B(a, R)$*, and* $u \geq 0$*, then*

$$\frac{R-r}{R+r} u(a) \leq u(a + re^{i\theta}) \leq \frac{R+r}{R-r} u(a),$$

where $0 \leq r < R, \theta \in \mathbb{R}$*.*

Theorem A.0.11. *(see [9], p.364) Let* G *be a group. For each* $\varphi \in \operatorname{Aut}(G)$*, let* $G \rtimes_\varphi \mathbb{Z}$ *denote the semidirect product of* G *and* $\mathbb{Z}$*, where the group multiplication of* $G \rtimes_\varphi \mathbb{Z}$ *is given by* $(x, m).(y, n) = (x\varphi^m(y), m + n)$*. Let* $\Phi : \operatorname{Aut}(G) \to \operatorname{Aut}(G)/\operatorname{Inn}(G)$ *be the canonical map. Suppose that there is no group epimorphism from* G *onto* $\mathbb{Z}$*. Let* φ *and* $\psi \in \operatorname{Aut}(G)$*. Then* $G \rtimes_\varphi \mathbb{Z}$ *and* $G \rtimes_\psi \mathbb{Z}$ *are isomorphic if and only if* $\Phi(\varphi)$ *is conjugate to* $\Phi(\psi)$ *or* $\Phi(\psi)^{-1}$ *in* $\operatorname{Aut}(G)/\operatorname{Inn}(G)$*.*

BIBLIOGRAPHY

[1] J. Wermer, *Banach Algebras and Analytic Functions*, Advances in Math **1**, fasc. 1 (1961), 51 - 102.

[2] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, 1969.

[3] R. Douglas, *Banach Algebra Techniques in Operator Theory, 2^{nd} edition*, Springer, Graduate Texts in Mathematics 179, 1998..

[4] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, 1962.

[5] G. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, 1990.

[6] E. L. Stout, *The Theory of Uniform Algebras*, Bogden & Quigley, 1971.

[7] E. J. Barbeau, *Pell's Equation*, Springer-Verlag, 2002.

[8] J. P. Robertson, *Solving the generalized Pell equation* $x^2 - Dy^2 = N$, www.jpr2718.org/pell.pdf, July 2004.

[9] G. Arzhantseva, J.-F. Lafont, A. Minasyan, *Isomorphism versus commensurability for a class of finitely presented groups*, J. Group Theory **17** (2014), 361-378.

[10] O. Karpenkov, *Geometry of Continued Fractions*, Springer-Verlag, 2013.

[11] E. Kaniuth, *A Course in Commutative Banach Algebras*, Springer-Verlag, 2009.

[12] J. B. Conway, *Functions of One Complex Variable*, Springer-Verlag, 1978.

[13] A. Zygmund, *Trigonometric Series*, Vol II, Cambridge University Press, 1977.

[14] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, 1982.