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A combinatorial model using strips and belts

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Finitely-generated modules over Special Biserial Algebras

A combinatorial model using strips and belts

Luke Kershaw

A dissertation submitted to the University of Bristol in accordance with the requirements for award of the degree of Doctor of Philosophy in the Faculty of Science.

June 2023

approx. 44,000 words

Abstract

We present a discrete model of band modules of special biserial (SB) algebras; complementing existing models for string modules. We provide efficient theoretical algorithms for calculating syzygies of band modules in terms of this discrete model, along with other functors relating to the delooping level for both string and band modules.

We first use these tools to prove some small results about the delooping levels of string and band modules for a general SB algebra; then we use these ideas specifically with radical-cube-zero SB algebras, where we show that all such algebras are either syzygy-finite or satisfy a very strict structural condition.

We also use these ideas to characterise, for a given SB algebra A, the string and band modules, $M \in \text{mod-}A$, with $\operatorname{Ext}^1_A(M, A) = 0$, which (along with the syzygy algorithms for these models) can be used when the algebra has small dimension to identify all Gorenstein-projective modules. For example, we classify the Gorenstein-projective modules for a handful of example SB algebras of dimension ≤ 20. We build on this by classifying the Gorenstein-projective band modules for any SB algebra, and then we determine some sufficient and/or necessary conditions for certain Gorenstein-homological properties, including an equivalent condition for an SB algebra to have finitely many indecomposable Gorenstein-projective modules.

Acknowledgements

First and foremost, I would like to thank my supervisor, Jeremy Rickard. I am fortunate to have benefited over many hours from his encouragement, frequent helpful examples (and counterexamples), and patience in listening to my various doomed mathematical arguments. He was also kind enough to take me on partway through my PhD at very short notice, which I will always appreciate.

I was also greatly helped by mathematical discussions with Vincent Gélinas, Birge Huisgen-Zimmermann, Henning Krause, Øyvind Solberg and Greg Stevenson during the Finitistic Dimension workshop.

I would also like to thank the people in Bristol who shaped the day-to-day of this PhD. Firstly, my deepest thanks to Jeremy's other students, Joe Allen and Charley Cummings, who took me under their respective wings as I started my research in this area; both have been great role models in how to progress through a research degree (and beyond). Also, to the various inhabitants of "the Pit" and adjacent offices, I am grateful for your consistent encouragement and timely distractions. Together you helped me find a balance between research and relaxation.

Constant throughout has been the love and support of my family. I am deeply grateful for the help and guidance you have provided through each step to get here; your unflinching belief in me has been a perpetual source of comfort. In particular, I'd like to thank my parents, Kim and Alan, and my sister, Rebecca, who have always been there when I needed them, and whose encouragement (both in mathematics and out) has been much appreciated all these years.

Finally, thank you to my partner Sarah. Your support throughout the ups and downs of this journey has been invaluable. In all honesty, I couldn't have done it without you.

Authors Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's *Regulations and Code of Practice for Research Degree Programmes* and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Chapter 1

Introduction

Special biserial algebras (SB algebras) are a class of finite-dimensional algebras that have often been used as a testing ground for new methods in representation theory. They are comparatively easy to define, and yet have been non-trivial to study; providing a rich source of examples for (and counterexamples to) various ideas.

As an illustration of this, we consider the trichotomy result for representation types by Drozd [Dro80]; that any finite-dimensional algebra is either representation-finite, representation-tame or representation-wild (in increasing order of complexity). A few years later, Wald and Waschbüsch showed that SB algebras are representation-tame [WW85]; in particular, they showed that there is a trichotomy of classes of modules of SB algebras; pin modules, string modules and band modules, where the first class is finite, the second is countable, and the third can be partitioned into a countable collection of one-parameter families.

Another important measure of complexity for a finite-dimensional algebra is its finitistic dimension; the supremum of all finite projective dimensions of modules over that algebra. There are two types of finitistic dimension that are widely considered; the big and little finitistic dimensions (where we respectively consider all modules, or just the finite-dimensional ones). It is conjectured that both the big and little finitistic dimensions are always finite for finite-dimensional algebras; these are respectively called the big and little finitistic dimension conjectures (FDCs), neither of which are known to hold in full generality. The finitistic dimension conjectures are at the top of a chain of implications of interconnected "homological conjectures", and thus they are widely studied throughout representation theory; for an overview of these implications, we direct the interested reader to [GPS21]. It is also worth noting the importance of the syzygy functor, Ω, (as defined in Definition 2.2.14) when studying the finitistic dimensions, as projective dimensions can be expressed in terms of syzygies (as discussed in Definition 2.2.20).

If we restrict our attention to SB algebras, it is known that the little FDC holds (this follows from [IT05] and [Erd+04]) but the big FDC is still open. Huisgen-Zimmermann subsequently showed that despite being representation-tame, SB algebras may not be "homologically tame" [HZ16]; in other words, there exist SB algebras which have an arbitrarily large (finite) difference between the big and little finitistic dimensions.

In [All21], Allen introduced a combinatorial approach for systematically encoding string modules and their syzygies. This approach combines "syllables" in lines to form "strips". These "strips" encode string modules in a way which makes it easier to formally study their properties (a brief review of this formalism is given in Sections 2.3 and 2.4). Allen also introduced "patches" and "patch coverings" which respectively encode indecomposable projectives and projective covers of string modules. This allowed the introduction of a syzygy algorithm for string modules using these tools.

One particular advantage of this framework is that it allowed many problems involving string modules to be studied using computer algebra software. In particular, Allen built a GAP package called SBStrips [SBS23] which implemented key parts of his model; it allows you to encode a string module as a strip, perform calculations on these strips and thus to determine properties of the string module that the strip represents. For example, for a given SB algebra you could define a strip (to represent a string module) then use the `SyzygyOfStrip` function to calculate a collection of strips to represent the syzygy of the original string module. Similarly, you can use `TransposeOfStrip` to calculate a collection of strips to represent the transpose of the original string module. By combining these functions you can calculate invariants like delooping level for a particular string module, and thus calculate the delooping level of an SB algebra (as introduced in [Gé22], see Definition 2.2.64). All of the functions implemented in SBStrips are orders of magnitude more efficient than the previous best way to perform these computations, using another GAP package called QPA [QPA22]. QPA is designed for performing calculations on a much wider class of finite-dimensional algebras, and thus can't take advantage of the additional combinatorial structure that SB algebras have. Note that QPA is used in some of the underlying parts of SBStrips, but the improved syzygy algorithm in particular means that the study of larger SB algebras is now considerably less computationally intense than when using QPA, making calculations involving bigger examples tractable.

In Chapter 2, we review the relevant background for the remainder of the thesis. We pay particular attention to the combinatorial framework introduced by Allen, and in some cases provide improvements to their results. One notable example of this is the latter half of Section 2.3, where we introduce the notion of "minimally connected overquivers". This has allowed us to implement a function, `SBAlgsFromNumVerticesAndRadLength`, in SBStrips which can generate a complete collection of SB algebras with a given number of vertices in their quiver and a given radical length and can thus be used to verify that a condition holds for all such SB algebras. Another important addition to the combinatorial framework of Allen is a description of the action of the transpose functor on string modules in terms of its effect on strips. This begins in Paragraph 2.4.28, and is based on a description in terms of "string graphs" given by Wald and Waschbüsch in [WW85]. This combinatorial description is extended to cover band modules in Chapter 3, and is vital to the results of Chapter 4.

The framework of "strips" is only useful when studying string modules of SB algebras; as discussed above there is another main class of finitely generated modules, band modules, and these require a new construction. In Chapter 3, we introduce a formalism for studying band modules called "belts"; they are also built in terms of "syllables", but instead of combining them in lines, we combine them in cycles. An informal overview of the ideas behind "belts" is given in Subsection 3.1.1 as a preview to the formal definitions being introduced in Subsection 3.1.2. Similar techniques to those for "strips" can be used to study how the syzygy functor acts on "belts", though extra care must be taken to handle the fact that the class of band modules is not generally closed under taking syzygies. The formal details of this syzygy algorithm are given in Section 3.2, followed by some small examples and a discussion of some of the immediate consequences of this algorithm. This new algorithm for "belts" means that we can now compute the syzygy of any finitely generated module of an SB algebra within the combined combinatorial framework of this thesis and Allen's; the ideas in this thesis handle the band modules, those in Allen's thesis handle the string modules, and since the pin modules are projective, their syzygy is always zero.

In Chapter 4, we give a "syllable-by-syllable" characterisation of the action of the functor $\Omega\,\mathrm{Tr} : \text{mod-}A \to \text{mod-}A^{\mathrm{op}}$ in terms of strips and belts. This functor plays a key role when considering the delooping level of a module. We then use this characterisation to better understand the delooping level of string and band modules; in particular, we give a new necessary condition for band modules to have non-zero delooping level (Proposition 4.2.1), a sufficient condition for band modules to have zero delooping level (Proposition 4.2.2), and a necessary and sufficient condition for simple non-projective modules to have non-zero delooping level (Lemma 4.2.4).

In Chapter 5, we use the tools we have developed to study string and band modules to show that a (connected) SB algebra, A, satisfying $\mathrm{rad}^3(A) = 0$ is particularly well-behaved.

> **Theorem 5.1.6.** *Suppose that A is a connected SB algebra with $\mathrm{rad}^3(A) = 0$. Then one of the following mutually exclusive conditions must hold:*
>
> - *A is syzygy-finite,*
> - *the regular module A_A is a direct sum of pin modules.*

This immediately implies the following result about the delooping level of these SB algebras, which was previously proved by Goodearl and Huisgen-Zimmermann and stated in [HZ22, Thm 4]. (*For reference, the delooping level,* $\mathrm{dell}(A)$*, of an algebra A is an invariant defined by Gélinas in [Gé22], which has implications for the big FDC. We discuss it further in Subsection 2.2.5.*)

> **Corollary 5.1.8.** *Suppose that A is an SB algebra with $\mathrm{rad}^3(A) = 0$. Then $\mathrm{dell}(A) < \infty$.*

(*Note that we no longer need the SB algebra to be connected for this result.*)

We can improve this by proving various upper bounds for $\mathrm{dell}(A)$ in this case. Section 5.2 includes various bounds, the best of which is quadratic in the number of vertices of the underlying quiver. Using SBStrips we checked the true values of the delooping level for all SB algebras on at most 4 vertices which satisfy $\mathrm{rad}^3(A) = 0$. This involved calculations for several thousand algebras, all of which satisfy a linear bound. In Example 5.2.6, we construct a family of example showing that if this linear bound holds in general, then it is sharp. This leads us to make the following conjecture of a linear bound:

> **Conjecture 5.2.8.** *Suppose that A is an SB algebra with $\mathrm{rad}^3(A) = 0$. Let n be the number of (isomorphism classes of) indecomposable projective A-modules. Then $\mathrm{dell}(A) \leq 2n - 2$.*

In Chapter 6, we begin with a classification of string and band modules, $M \in \text{mod-}A$, for a given SB algebra, A, which satisfy $\mathrm{Ext}^1_A(M, A) = 0$. This classification is done in terms of "peaks" which are pieces of a "strip" (or a "belt") consisting of two adjacent syllables. The classification assigns each "peak" a colour (red, yellow or green) and then determines whether or not $\mathrm{Ext}^1_A(M, A) = 0$ based on the colours of "peaks" present in a "strip" or "belt" representing this module.

If A is a small SB algebra, then this characterisation, along with the syzygy algorithm, is often sufficient to characterise all Gorenstein-projective $M \in \text{mod-}A$. By repeatedly applying the syzygy algorithm, we can rule out the presence of almost all combinations of peaks in a "strip" (or "belt") representing a module $M \in \text{mod-}A$ where $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \geq 1$. The remaining combinations of peaks can be checked by hand to see if they correspond to Gorenstein-projective modules.

However, when working with a larger SB algebra, this can become impractical. Fortunately, classifying Gorenstein-projective band modules can be done for general SB algebras. For example, the following result gives a single equivalent condition for a band module to be Gorenstein-projective, and appears as part of a wider collection of equivalent conditions in Theorem 6.2.13.

> **Theorem.** *Let M be a band module of A. Then M is Gorenstein-projective if and only if the Q-vertex corresponding to each simple summand of* $\mathrm{soc}(M) \oplus \mathrm{top}(M)$ *lies on a cycle of the pin graph, Φ_A.*

(*For reference, the* pin graph *Φ_A is a sub-1-regular quiver associated to the SB algebra A, which has a vertex set equal to that of the defining quiver Q of A. This was introduced in [All21], and we give the definition in Definition 2.3.39.*)

Since the pin graph is very simple to calculate, it is now very simple to verify whether or not a band module is Gorenstein-projective.

Section 6.3 then uses our improved understanding of (semi-)Gorenstein-projectives to investigate necessary and/or sufficient conditions for SB algebras to be CM-finite, CM-free and/or weakly Gorenstein (in the sense of [RZ20]). For example, our classification of Gorenstein-projective band modules is sufficient to give an equivalent condition for when an SB algebra is CM-finite (i.e. has finitely many indecomposable Gorenstein-projective modules).

> **Proposition 6.3.4.** *An SB algebra A is CM-finite if and only if it has no Gorenstein-projective band modules.*

Chapter 2

Background

We assume throughout that $\Bbbk$ is some fixed algebraically closed field of any characteristic.

To avoid confusion, we clarify that $\mathbb{N} = \{k \in \mathbb{Z} : k \geq 0\}$ and $\mathbb{Z}_+ = \{k \in \mathbb{Z} : k > 0\}$. We also write $\lfloor - \rfloor, \lceil - \rceil : \mathbb{R} \to \mathbb{Z}$ for the floor function and ceiling function respectively, and write $\mathcal{P}(X)$ for the power set of X.

We will write $g \circ f$ or gf for the composition of functions $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, and $f(x)$ for the image of an element $x \in X$. Note that this differs from [All21], which uses the reverse notation for composition where possible.

2.1 General prerequisites

2.1.1 Graphs

2.1.1. Graphs. A *graph* Γ is a triple $(\Gamma_0, \Gamma_1, \mathrm{e})$ consisting of two sets: Γ_0 (the *vertices*) and Γ_1 (the *edges*) along with an *incidence function* e; a function $\mathrm{e} : \Gamma_1 \to \mathcal{P}(\Gamma_0)$ such that for each edge $e \in \Gamma_1$ we have $1 \leq |\mathrm{e}(e)| \leq 2$.

The graph is *finite* if both Γ_0 and Γ_1 are.

2.1.2. Paths in graphs. A *path* in Γ is an alternating sequence of vertices and edges $(v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ where for each $i \in \{1, \ldots, n\}$ we have $\mathrm{e}(e_i) = \{v_{i-1}, v_i\}$. The *length* of such a path is $n \in \mathbb{N}$.

Note that some sources would call this construction a "walk", and place further restrictions to call it a "path".

2.1.3. Connectedness for graphs. In a graph Γ, two vertices v, v' are *connected* if they are the extremal vertices of some path. It is immediate that the notion of connectedness is an equivalence relation. To each equivalence class under this relation we call the corresponding full subgraph, Γ, a *connected component*. A graph Γ is called *connected* if any pair of vertices v, v' in Γ are connected (i.e. there is a unique connected component).

2.1.2 Quivers

2.1.4. Quivers. A *quiver*, Q, is a tuple (Q_0, Q_1, s, t) consisting of two sets: Q_0 (the *vertices*) and Q_1 (the *arrows*) along with *source* ($s : Q_1 \to Q_0$) and *target* functions ($t : Q_1 \to Q_0$). It is called *finite* if both Q_0 and Q_1 are finite sets, and is called *locally finite* if for every vertex $v \in Q_0$, there are finitely many arrows $\alpha \in Q_1$ whose source is v ($s(\alpha) = v$) or target is v ($t(\alpha) = v$).

The *opposite quiver* of $Q = (Q_0, Q_1, s, t)$ is $Q^{\mathrm{op}} := (Q_0, Q_1, t, s)$.

Some particularly important classes of vertices are *source* and *sink vertices*. These are the vertices $v \in Q_0$ where $|t^{-1}(v)| = 0$ (resp. $|s^{-1}(v)| = 0$). It follows immediately that source vertices of Q correspond to sink vertices of Q^{op}, and vice versa.

2.1.5. Quiver homomorphisms. For two quivers $Q = (Q_0, Q_1, s, t,)$ and $Q' = (Q'_0, Q'_1, s', t',)$ a quiver homomorphism $\phi : Q \to Q'$ is a pair of maps $\phi_k : Q_k \to Q'_k$ for $k = 0, 1$ which are compatible with the source and target maps; in other words, the following squares commute:

$$\begin{tikzcd} Q_1 \arrow[r, "s"] \arrow[d, "\phi_1"'] & Q_0 \arrow[d, "\phi_0"] \\ Q'_1 \arrow[r, "s'"'] & Q'_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Q_1 \arrow[r, "t"] \arrow[d, "\phi_1"'] & Q_0 \arrow[d, "\phi_0"] \\ Q'_1 \arrow[r, "t'"'] & Q'_0 \end{tikzcd}$$

If both ϕ_k are injective, we call ϕ an *inclusion* and call Q a *subquiver* of Q'. Dually, if both ϕ_k are surjective, we call ϕ a *projection* and call Q' a *quotient* of Q.

If $Q_0 = Q'_0$, $\phi_0 = \mathrm{id}$ and ϕ_1 is injective, then we call Q' an *augmentation* of Q, and call its additional arrows *augmented arrows*.

We now give an example of a quiver, its opposite, and a quiver homomorphism between them.

2.1.6. Example. Let Q be the quiver given by:

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta_1"] \arrow[r, bend right, "\beta_2"'] & 2 \arrow[loop above, "\gamma"] \arrow[r, bend left, "\delta_1"] & 3 \arrow[l, bend left, "\delta_2"] \end{tikzcd}$$

Then Q^{op} is given by:

$$\begin{tikzcd} 1^{\mathrm{op}} \arrow[loop left, "\alpha^{\mathrm{op}}"] & 2^{\mathrm{op}} \arrow[loop above, "\gamma^{\mathrm{op}}"] \arrow[l, bend right, "\beta_1^{\mathrm{op}}"'] \arrow[l, bend left, "\beta_2^{\mathrm{op}}"] \arrow[r, bend right, "\delta_2^{\mathrm{op}}"'] & 3^{\mathrm{op}} \arrow[l, bend right, "\delta_1^{\mathrm{op}}"'] \end{tikzcd}$$

There is a quiver homomorphism $\phi : Q \to Q^{\mathrm{op}}$ given by:

$$\phi_0(1) = 2^{\mathrm{op}} \quad \phi_0(2) = 1^{\mathrm{op}} \quad \phi_0(3) = 1^{\mathrm{op}}$$
$$\phi_1(\alpha) = \gamma^{\mathrm{op}} \quad \phi_1(\beta_1) = \beta_2^{\mathrm{op}} \quad \phi_1(\beta_2) = \beta_1^{\mathrm{op}} \quad \phi_1(\gamma) = \alpha^{\mathrm{op}} \quad \phi_1(\delta_1) = \alpha^{\mathrm{op}} \quad \phi_1(\delta_2) = \alpha^{\mathrm{op}}$$

Since neither ϕ_0 or ϕ_1 are surjective or injective, $\phi : Q \to Q^{\mathrm{op}}$ is not an inclusion or a projection.

None of the vertices in Q are source or sink vertices, as they are all the source of an arrow and the target of an arrow. This immediately means that Q^{op} has no source or sink vertices either.

2.1.7. Underlying graph. The *underlying graph* of a quiver $Q = (Q_0, Q_1, s, t)$ is the graph $(Q_0, Q_1, \alpha \mapsto \{s(\alpha), t(\alpha)\})$.

2.1.8. Connectedness for quivers. For a quiver Q, we define a *connected component* of Q to be the full subquiver of Q specified by the vertices in a connected component of the underlying graph of Q. We call Q *connected* if it has a unique connected component (i.e. the underlying graph of Q is connected).

2.1.9. Paths in quivers. Let Q be a quiver. A *path*, p, of *length* $l \geq 1$ is a sequence of arrows

$$(\alpha_1, \alpha_2, \dots, \alpha_l)$$

where for each $1 \leq k < l$ we have $t(\alpha_k) = s(\alpha_{k+1})$. We often denote such a path as $\alpha_1\alpha_2\dots\alpha_l$. We define the source and target of this path by $s(p) \coloneqq s(\alpha_1)$ and $t(p) \coloneqq t(\alpha_l)$.

A *path* of *length* zero (a *stationary path*) is associated to each vertex i of Q, denoted by e_i. We define both the source and target of e_i to be i.

We may also refer to paths in Q as Q-paths. This is particularly useful when it may be unclear which quiver a path belongs to.

2.1.10. Concatenation of paths. Let p and q be paths in Q. If $t(p) = s(q)$, then we define pq, the *concatenation* of p and q as follows:

- if p is stationary, then $pq \coloneqq q$,
- if q is stationary, then $pq \coloneqq p$, and
- if neither of p, q are stationary, they can be written as $p = \alpha_1 \dots \alpha_l$ and $q = \beta_1 \dots \beta_{l'}$ for some arrows α_i, β_i and $pq \coloneqq \alpha_1 \dots \alpha_l \beta_1 \dots \beta_{l'}$.

If $t(p) \neq s(q)$, then we leave pq undefined.

2.1.11. Regular and subregular quivers. Let $m \in \mathbb{Z}_+$. A quiver Q is called *sub-m-regular* if for each vertex $i \in Q_0$, we have inequalities $|s^{-1}(\{i\})| \leq m$ and $|t^{-1}(\{i\})| \leq m$, and is called *m-regular* if both of these inequalities are actually equalities. (Note that here s^{-1} and t^{-1} denote the preimage set).

2.1.12. Example. The quiver in Example 2.1.6 is sub-4-regular, but not 4-regular or sub-3-regular.

2.1.13. It is equivalent to define a quiver Q as 1-regular if its source and target maps are bijections $s, t : Q_1 \to Q_0$. In this case they have inverse functions $s^{-1}, t^{-1} : Q_0 \to Q_1$, which can be combined to obtain permutations of vertices ($(t \circ s^{-1}) : Q_0 \to Q_0$) and arrows ($(s^{-1} \circ t) : Q_1 \to Q_1$). Both of these permutations "move with the flow of the quiver". Repeated application of these permutations give $\mathbb{Z}$-actions on the sets of vertices and arrows respectively; to simplify notation later, for $k \in \mathbb{Z}$ we write:

$$i - k \coloneqq (t \circ s^{-1})^k(i) \qquad \text{and} \qquad \alpha - k \coloneqq (s^{-1} \circ t)^k(\alpha).$$

We can visualise this notation using the following diagrams:

$$\cdots \longleftarrow (i-2) \longleftarrow (i-1) \longleftarrow i \longleftarrow (i+1) \longleftarrow (i+2) \longleftarrow \cdots$$

$$\cdots \xleftarrow{\alpha-2} \circ \xleftarrow{\alpha-1} \circ \xleftarrow{\alpha} \circ \xleftarrow{\alpha+1} \circ \xleftarrow{\alpha+2} \cdots$$

2.1.14. In a 1-regular quiver a path is uniquely determined by its source (resp. target) and length. For a path with source i and length l, we will often use the notation ($i \xrightarrow{l} \circ$). Dually, we will occasionally denote a path with target i and length l by ($\circ \xrightarrow{l} i$). At times where the length is clear, we may denote a path with source i and target j by ($i \longrightarrow j$).

2.2 Algebraic prerequisites

2.2.1 Quiver algebras

2.2.1. Path algebra. Let Q be a quiver. We define the *path algebra* $\Bbbk Q$ by first defining its vector space structure, then defining its multiplicative structure. The set of Q-paths forms the basis of a vector space; we call this basis the *standard basis.* We define multiplication on basis vectors p and q by $p \cdot q \coloneqq pq$ (the concatenation of the paths p and q) when $t(p) = s(q)$, and $p \cdot q \coloneqq 0$ otherwise.

2.2.2. Lemma. [ASS06, Section II.2]

Let Q be a quiver and $\Bbbk Q$ be its path algebra. Then:

(a) $\Bbbk Q$ is an associative algebra,

(b) $\Bbbk Q$ has an identity element if and only if Q_0 is finite, and

(c) $\Bbbk Q$ is finite dimensional if and only if Q is finite and acyclic.

2.2.3. Arrow ideal. The *arrow ideal*, J, of a path algebra $\Bbbk Q$ is the two-sided ideal generated by the arrows, considered as elements of $\Bbbk Q$.

2.2.4. Admissible ideals and quiver algebras. Let Q be a finite quiver and J be the arrow ideal of $\Bbbk Q$. We call a two-sided ideal I of $\Bbbk Q$ an *admissible ideal* if there is an integer $m \geq 2$ such that $J^m \subseteq I \subseteq J^2$.

If I is an admissible ideal of $\Bbbk Q$, then we call the quotient algebra $\Bbbk Q/I$ a *quiver algebra.* This notion is also sometimes called a *bound quiver algebra.*

If $A = \Bbbk Q/I$ is a quiver algebra, then an *A-path* is any non-zero residue of a Q-path in A. We define *A-arrows* similarly.

2.2.5. Relations of path algebras. Let Q be a quiver. A *relation* in $\Bbbk Q$ is a $\Bbbk$-linear combination of paths of length ≥ 2 having the same source and target. In other words, a relation is an element of the form:

$$\sum_{i=1}^{m} \lambda_i w_i,$$

where the $\lambda_i \in \Bbbk$ are not all zero and the w_i are Q-paths of length at least 2 with $s(w_i) = s(w_j)$ and $t(w_i) = t(w_j)$ for all $i, j \in \{1, \ldots, m\}$.

If $m = 1$, the relation is called a *monomial relation.* If it is of the form $\lambda_1 w_1 - \lambda_2 w_2$, then it is called a *skew commutativity relation*, and if additionally $\lambda_1 = \lambda_2 = 1$, it is called a *commutativity relation.*

Part of the reason we care about relations is that if I is an admissible ideal of $\Bbbk Q$, then I is the $\Bbbk$-linear span of the relations in $\Bbbk Q$ that it contains.

2.2.6. Lemma. [ASS06, Section II.2]

Let Q be a finite quiver and I be an admissible ideal of $\Bbbk Q$. Then:

(i) the quiver algebra $\Bbbk Q/I$ is finite dimensional.

(ii) there is a finite set of relations of $\Bbbk Q$, $\{\rho_1, \ldots, \rho_m\}$, such that $I = \langle \rho_1, \ldots, \rho_m \rangle$.

Since, by item (ii), we can generate any admissible ideal by relations, for simplicity and clarity, we will always do so.

2.2.7. Residues of paths. If p is a Q-path and $A = \Bbbk Q/I$, then we call the element $p + I \in A$ the A-*residue* of p.

2.2.2 Representation theory

For this section, unless otherwise specified, A will denote an arbitrary finite-dimensional algebra over $\Bbbk$.

2.2.8. Module categories. A *(right) A-module* is a pair $(M, \cdot)$, consisting of a $\Bbbk$-vector space, M, and a binary operation $\cdot : M \times A \to M, (m, a) \mapsto ma$ satisfying the following conditions:

(a) $\cdot$ is linear in M; i.e. $\forall x, y \in M, a \in A$ we have $(x + y)a = xa + ya$,

(b) $\cdot$ is linear in A; i.e. $\forall x \in M, a, b \in A$ we have $x(a + b) = xa + xb$,

(c) $\cdot$ is compatible with multiplication in A; i.e. $\forall x \in M, a, b \in A$ we have $x(ab) = (xa)b$,

(d) multiplication by $1 \in A$ acts as the identity on M; i.e. $\forall x \in M$ we have $x1 = x$;

(e) $\cdot$ is compatible with the $\Bbbk$ vector space structures of both M and A; i.e. $\forall x \in M, a \in A, \lambda \in \Bbbk$ we have $(x\lambda)a = x(a\lambda) = (xa)\lambda$.

A *(right) A-module homomorphism* $f : (M_1, \cdot_1) \to (M_2, \cdot_2)$ is a $\Bbbk$-linear vector space map $f : M_1 \to M_2$ which satisfies $f(m \cdot_1 a) = f(m) \cdot_2 a$ for any $m \in M_1, a \in A$.

We denote the category of all A-modules by Mod-A.

A module M is called *finite-dimensional* if its dimension as a $\Bbbk$-vector space is finite. We denote the category of all finite-dimensional A-modules by mod-A.

2.2.9. Submodules and quotients. An *A-submodule* $M' \leq M$ is a vector subspace M' of M where for any $m' \in M'$ and $a \in A$ we have $m'a \in M'$. For a given submodule $M' \leq M$, the quotient vector space M/M' has a natural and well-defined A-module structure; thus we also use the notation M/M' for the *quotient* A-module.

An non-zero A-module M is called *simple* if its only submodules are 0 and M. A module is called *semi-simple* if it is a direct sum of simple modules.

The largest semi-simple submodule of M is called the *socle,* $\operatorname{soc}(M)$, of M. The smallest submodule M' of M for which the quotient M/M' is semi-simple is called the *radical,* $\operatorname{rad}(M)$, of M. The *top,* $\operatorname{top}(M)$, of M is the quotient $M/\operatorname{rad}(M)$.

Note that the socle and radical of the regular module A_A are both two-sided ideals of A.

Since $M \supsetneq \operatorname{rad}(M)$ for any non-zero module, we can consider a descending chain of the form

$$M \supseteq \operatorname{rad}(M) \supseteq \operatorname{rad}^2(M) \supseteq \dots .$$

When M is finite-dimensional, this chain must eventually stabilise at the zero module. We define the *radical length of* $M \in$ mod-A to be $\min\{n \in \mathbb{N} : \operatorname{rad}^n(M) = 0\}$.

2.2.10. Direct sums, indecomposable modules and additive closure. Note that both mod-A and Mod-A are abelian categories. This means that we can consider the direct sum of modules in these categories, denoted by $\oplus$.

A non-zero module M is *indecomposable* if $M = M_1 \oplus M_2$ implies M_1 or M_2 is zero.

For a set of A-modules, $\mathcal{M} \subseteq$ Mod-A, we define $\operatorname{Add}(\mathcal{M})$ to be the full subcategory of Mod-A whose objects are isomorphic to direct summands of coproducts of members of $\mathcal{M}$.

If $\mathcal{M} \subseteq \text{mod-}A$, we define $\text{add}(\mathcal{M})$ to be the full subcategory of $\text{Mod-}A$ whose objects are isomorphic to direct summands of finite direct sums of members of $\mathcal{M}$. Note that this is equal to $\text{Add}(\mathcal{M}) \cap \text{mod-}A$.

2.2.11. Duals and reflexivity. We denote the standard *k-dual functor* by

$$D(-) = \text{Hom}_k(-, k) : \text{mod-}A \to \text{mod-}A^{\text{op}}.$$

Similarly, we denote the standard *A-dual functor* by

$$(-)^* = \text{Hom}_A(-, A) : \text{mod-}A \to \text{mod-}A^{\text{op}}.$$

Note that both the k-dual and the A-dual are contravariant functors.

For a given $M \in \text{mod-}A$, the *evaluation morphism* for M, $\text{ev}_M : M \to M^{**}$, is defined by $\text{ev}_M(m)(f) = f(m) \in A$ for any choice of $m \in M$, $f \in M^* = \text{Hom}_A(M, A)$.

A module M is called *reflexive* if the evaluation morphism ev_M is an isomorphism.

2.2.12. Representation type. One of the main aims of representation theory is to classify the isomorphism classes of indecomposable finite-dimensional modules of a given algebra. In [Dro80], Drozd proves that for any finite-dimensional algebra, A, the representation theory of A has one of three *representation types*.

- there are finitely many isomorphism classes of indecomposable modules in $\text{mod-}A$; in this case we say that A has *finite* representation type (or is *representation finite*);
- there are infinitely many isomorphism classes of indecomposable modules in $\text{mod-}A$, and for all $n \in \mathbb{N}$, all but finitely many isomorphism classes of n-dimensional indecomposable A-modules occur in a finite number of one-parameter families; in this case we say that A has *tame* representation type;
- there is a finitely generated $k\langle x, y\rangle$-A-bimodule M which is free as a left $k\langle x, y\rangle$-module, such that the functor $(-) \otimes_{k\langle x,y\rangle} M : \text{mod-}(k\langle x, y\rangle) \to \text{mod-}A$ preserves indecomposability and reflects isomorphism class; in this case we say that A has *wild* representation type.

If an algebra has wild representation type, then, informally speaking, classifying the isomorphism classes of its indecomposable finite-dimensional modules is at least as difficult as performing this classification for $k\langle x, y\rangle$, which is widely considered to be intractable. For a more in depth discussion of representation type, we refer to [Ben91, Section 4.4]

2.2.13. Projective and injective modules. An A-module, M, is called *projective* if for any surjection $u : U \twoheadrightarrow V$ and homomorphism $f : M \to V$, there is a homomorphism $\bar{f} : M \to U$ such that $u \circ \bar{f} = f$.

Dually, an A-module, M, is called *injective* if for any injection $v : V \hookrightarrow U$ and homomorphism $g : V \to M$, there is a homomorphism $\bar{g} : U \to M$ such that $\bar{g} \circ v = g$.

2.2.14. Projective covers and syzygies. For a given $M \in \text{mod-}A$ there is a unique smallest (in the sense of $\Bbbk$-dimension) projective module $\mathbb{P}(M)$ that surjects onto it, by some map $\pi : \mathbb{P}(M) \twoheadrightarrow M$. We call this map (and sometimes this projective module) the *projective cover* of M.

The *syzygy* of M, $\Omega(M)$ is the kernel of this map. For $k \geq 2$, the *k-th syzygy* of M, $\Omega^k(M)$ is defined inductively by $\Omega^k(M) = \Omega(\Omega^{k-1}(M))$.

2.2.15. Remark. Syzygies play a key role in many important concepts in representation theory (for example, projective dimensions and finitistic dimensions, which are both defined later). A large part of this thesis will be devoted to a formalism for calculating syzygies of particular types of modules, to aid in understanding some of these concepts.

2.2.16. Ω-periodicity and syzygy-finiteness. A module $M \in \text{mod-}A$ is called Ω*-periodic* if there exists $k \in \mathbb{Z}_+$ such that $\Omega^k(M) \cong M$.

An algebra A is called *k-syzygy-finite* if there exists $N \in \text{mod-}A$ such that $\{\Omega^{k'}(M) : k' \geq k, M \in \text{mod-}A\} \subseteq \text{add}(N)$. If there exists $k \in \mathbb{Z}_+$ such that A is k-syzygy-finite, then A is called *syzygy-finite*; otherwise it is called *syzygy-infinite*.

2.2.17. Complexes of modules. A *complex* of modules is a diagram of the form:

$$C^\bullet := (\quad \cdots \to C^{-1} \xrightarrow{d_C^{-1}} C^0 \xrightarrow{d_C^0} C^1 \xrightarrow{d_C^1} C^2 \to \cdots \quad),$$

where the *boundary morphisms* $d^i = d^i_C$ satisfy $d^i d^{i-1} = 0$ (or equivalently where $\text{im}(d^i) \leq \ker(d^{i+1})$). We will occasionally use the notation $Z^i(C^\bullet)$ (resp. $B^i(C^\bullet)$) for $\ker(d_C^{i+1})$ (resp. $\text{im}(d^i)$).

2.2.18. Homology of a complex. Fix a complex $C^\bullet$. For any $i \in Z$, the quotient module $\ker(d_C^i)/\operatorname{im}(d_C^{i-1})$ is well-defined; as such we call it the *i-th homology module of $C^\bullet$* and denote it by $H^i(C^\bullet)$. A complex $C^\bullet$ is called *acyclic* if $H^i(C^\bullet) = 0$ for all $i \in \mathbb{Z}$. (Note that $C^\bullet$ is acyclic if and only if $Z^i(C^\bullet) = B^i(C^\bullet)$ for all $i \in \mathbb{Z}$).

2.2.19. Projective/injective resolution of a module. A *projective resolution* of an A-module M is a complex of projective A-modules of the form:

$$P^\bullet \coloneqq (\quad \cdots \to P^{-2} \xrightarrow{d_P^{-2}} P^{-1} \xrightarrow{d_P^{-1}} P^0 \to 0 \to 0 \to \cdots \quad),$$

where $H^i(P^\bullet) = 0$ for $i \neq 0$ and $H^0(P^\bullet) \cong M$. A projective resolution, $P^\bullet$, is called *finite* if $P^i = 0$ for all but finitely many $i \in \mathbb{Z}$. If $P^\bullet$ is a finite projective resolution, its *length* is defined as $\operatorname{len}(P^\bullet) = \max\{i \in \mathbb{N} : P^{-i} \neq 0\}$

Dually, an *injective resolution* of an A-module M is a complex of injective A-modules of the form:

$$Q^\bullet \coloneqq (\quad \cdots \to 0 \to 0 \to Q^0 \xrightarrow{d_Q^0} Q^1 \xrightarrow{d_Q^1} Q^2 \to \cdots \quad),$$

where $H^i(Q^\bullet) = 0$ for $i \neq 0$ and $H^0(Q^\bullet) \cong M$. A injective resolution, $Q^\bullet$, is called *finite* if $Q^i = 0$ for all but finitely many $i \in \mathbb{Z}$. If $Q^\bullet$ is a finite injective resolution, its *length* is defined as $\operatorname{len}(Q^\bullet) = \max\{i \in \mathbb{N} : Q^i \neq 0\}$

2.2.20. Projective/injective dimension. Fix an A-module M. The *projective dimension,* $\operatorname{proj.dim}(M)$, of M is the minimal length of a finite projective resolution of M, if such a projective resolution exists; otherwise we define $\operatorname{proj.dim}(M) = +\infty$.

Projective dimension can also be defined equivalently using syzygies:

$$\operatorname{proj.dim}(M) \coloneqq \inf\{i \in \mathbb{N} : \Omega^i(M) \text{ is projective}\}.$$

Dually, the *injective dimension,* $\operatorname{inj.dim}(M)$, of M is the minimal length of a finite injective resolution of M, if such an injective resolution exists; otherwise we define $\operatorname{inj.dim}(M) = +\infty$.

2.2.21. Ext **functor.** Fix $i \in \mathbb{N}$ and two A-modules M and N. Let $P^\bullet$ be a projective resolution of M and consider the following complex of $\Bbbk$-vector spaces:

$$\mathrm{Hom}_A(P^\bullet, N) \coloneqq (\quad \cdots \to 0 \to 0 \to \mathrm{Hom}_A(P^0, N) \xrightarrow{\mathrm{Hom}_A(d_P^{-1}, N)} \cdots$$
$$\cdots \to \mathrm{Hom}_A(P^{-j}, N) \xrightarrow{\mathrm{Hom}_A(d_P^{-(j+1)}, N)} \mathrm{Hom}_A(P^{-(j+1)}, N) \to \cdots \quad)$$

The i-th extension group of M and N is then defined as $H^i(\mathrm{Hom}_A(P^\bullet, N))$ and denoted by $\mathrm{Ext}^i_A(M, N)$. (Note that this does not depend on the choice of projective resolution used.)

2.2.22. Transpose functor. For a given module $M \in \text{mod-}A$ we take a minimal projective presentation

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0,$$

in other words an exact sequence where $p_0 : P_0 \to M$ and $p_1 : P_1 \to \ker(p_0)$ are both projective covers. The *transpose,* $\mathrm{Tr}(M)$*,* of M is then defined to be $\mathrm{coker}(p_1^*) \in \text{mod-}A^{\mathrm{op}}$, where $p_1^* : P_0^* \to P_1^*$.

2.2.3 Gorenstein-projective modules

For this section, A will denote an arbitrary finite-dimensional algebra.

2.2.23. Totally acylic complexes. A complex $P^\bullet$ of projective A-modules is *totally acyclic* if it is acyclic, and the Hom complex $\mathrm{Hom}_A(P^\bullet, A)$ is also acyclic.

2.2.24. Gorenstein-projective modules. An A-module M is called *Gorenstein-projective* if there is a totally acyclic complex $P^\bullet$ of projective modules where $Z^0(P^\bullet) \cong M$. The complex $P^\bullet$ is called a *complete resolution* of M. The full subcategory of mod-A consisting of Gorenstein-projective modules is denoted Gproj-A.

2.2.25. Remark. Gorenstein-projective modules go under many names: they are sometimes called *maximal Cohen-Macaulay modules* (e.g. [Bel05, Def 3.2]), *modules of G-dimension zero* (e.g. [AB69]), or *totally reflexive modules* (e.g. [AM02, Section 2]).

2.2.26. Note that projective modules are always Gorenstein-projective; this can be seen by considering the totally acyclic complex:

$$\cdots \to 0 \to P \xrightarrow{\mathrm{id}} P \to 0 \to \cdots .$$

2.2.27. Gorenstein-projective modules have been well studied, and various properties are known. The next few results state the properties that will be of interest for us later on.

2.2.28. Proposition. [Che17, Lem 2.1.4]

Let $M \in \text{mod-}A$. Then M is Gorenstein-projective if and only if all of the following hold:

(G1) for all $i \in \mathbb{Z}_+$, we have $\mathrm{Ext}^i_A(M, A) = 0$;

(G2) for all $i \in \mathbb{Z}_+$, we have $\mathrm{Ext}^i_{A^{\mathrm{op}}}(M^*, A^{\mathrm{op}}) = 0$; and

(G3) M is reflexive.

2.2.29. Proposition. [Che17, Prop 2.1.7]

Let $\xi : 0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0$ be a short exact sequence of A-modules. Then we have the following statements:

(1) if N, L are Gorenstein-projective, then so is M;

(2) if ξ splits and M is Gorenstein-projective, then so are N, L;

(3) if M, L are Gorenstein-projective, then so is N;

(4) if $\mathrm{Ext}^1_A(L, A) = 0$ and N, M are Gorenstein-projective, then so is L.

2.2.30. Proposition. Let $M \in \text{mod-}A$.

- [Che17, Cor 2.1.9] if M is Gorenstein-projective, then so is $\Omega(M)$;
- [Che17, Prop 2.1.10] M is Gorenstein-projective if and only if $\mathrm{Tr}(M)$ is.

There are some particular classes of algebra where the Gorenstein-projective modules are better understood.

2.2.31. Gorenstein algebra. A finite-dimensional algebra A is *Gorenstein* if both regular modules ${}_AA$ and A_A have finite injective dimension.

In such a case, $\text{inj.dim}({}_AA) = \text{inj.dim}(A_A)$ [AR91a, Lem 6.9] and we call the common value $\mathrm{G.dim}(A)$. A finite-dimensional algebra A is called *d-Gorenstein* if $\mathrm{G.dim}(A) \leq d$.

The following result illustrates how well-behaved Gorenstein-projective modules are over Gorenstein algebras.

Theorem 2.2.32. *[Che17, Thm 2.3.3]*

Let A be a finite-dimensional algebra and let $d \geq 0$. Then A is d-Gorenstein if and only if $\mathrm{Gproj}\text{-}A = \Omega^d(\mathrm{mod}\text{-}A)$*. In this case, any module* $M \in \mathrm{mod}\text{-}A$ *with* $\mathrm{Ext}^i_A(M, A) = 0$ *for all* $i \in \mathbb{Z}_+$ *is Gorenstein-projective.*

As mentioned in Paragraph 2.2.26, all projective modules are Gorenstein-projective; however, they are usually not the only Gorenstein-projectives. The following classes of algebra place restrictions on the number of (isomorphism classes of) non-projective indecomposable Gorenstein-projectives.

2.2.33. CM-finite and CM-free. An algebra, A, is called *CM-finite* if there are only finitely many indecomposable modules in $\mathrm{Gproj}\text{-}A$ (up to isomorphism).

An algebra, A, is called *CM-free* if $\mathrm{Gproj}\text{-}A = \mathrm{proj}\text{-}A$. Clearly a CM-free algebra is automatically CM-finite.

(*The CM in these definitions stands for Cohen-Macaulay, following from one of the names discussed in Remark 2.2.25.*)

2.2.34. In the study of Gorenstein-projective modules, very few examples of modules that satisfy (G1) in Proposition 2.2.28 but not (G2) and (G3) have been found. Thus the modules satisfying (G1) have been considered for study themselves, so that the interaction between the conditions of Proposition 2.2.28 can be better understood.

2.2.35. Semi-Gorenstein-projective modules. We call a module $M \in \mathrm{mod}\text{-}A$ *semi-Gorenstein-projective* if for all $i \in \mathbb{Z}_+$, we have $\mathrm{Ext}^i_A(M, A) = 0$.

We denote the class of semi-Gorenstein-projective A-modules by ${}^\perp A$.

2.2.36. It is clear from Proposition 2.2.28 that all Gorenstein-projective modules are semi-Gorenstein-projective; thus we have an inclusion $\mathrm{Gproj}\text{-}A \subseteq {}^\perp A$. An example of a semi-Gorenstein-projective module that is not Gorenstein-projective is given in [RZ20, Section 6] so this inclusion may be strict (i.e. $\mathrm{Gproj}\text{-}A \subsetneq {}^\perp A$).

Similarly to Gorenstein-projective modules, semi-Gorenstein-projective modules behave well with short exact sequences. Compare the following to Proposition 2.2.29 above.

2.2.37. Proposition. Let $\xi : 0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0$ be a short exact sequence of A-modules. Then we have the following statements:

(1) if N, L are semi-Gorenstein-projective, then so is M;

(2) if ξ splits and M is semi-Gorenstein-projective, then so are N, L;

(3) if M, L are semi-Gorenstein-projective, then so is N;

(4) if $\operatorname{Ext}^1_A(L, A) = 0$ and N, M are semi-Gorenstein-projective, then so is L.

Proof. Parts (1), (3) and (4) follow immediately from considering the long exact sequence:

$$0 \to \operatorname{Hom}_A(L, A) \to \operatorname{Hom}_A(M, A) \to \operatorname{Hom}_A(N, A) \to \operatorname{Ext}^1_A(L, A) \to \operatorname{Ext}^1_A(M, A) \to \cdots$$

Part (2) follows immediately from the additivity of the functors $\operatorname{Ext}^i_A(-, A)$ for $i \in \mathbb{Z}_+$. $\square$

2.2.38. As discussed in Paragraph 2.2.36, we have an inclusion Gproj-$A \subseteq {}^{\perp}A$, which in some cases is strict. In [RZ20], the term *weakly Gorenstein* was introduced for when this inclusion is an equality.

2.2.39. Weakly Gorenstein algebra. An algebra A is called *weakly Gorenstein* if Gproj-$A = {}^{\perp}A$; in other words, A is weakly Gorenstein if all semi-Gorenstein-projective A-modules are Gorenstein-projective.

It follows immediately from Theorem 2.2.32 that any Gorenstein algebra is also weakly Gorenstein.

2.2.4 Special biserial algebras

2.2.40. SB algebras. A *special biserial (SB) algebra* is a quiver algebra $\Bbbk Q/I$ where

(a) Q is a finite sub-2-regular quiver,

(b) for every arrow α of Q, there is at most one arrow β with $\alpha\beta \notin I$ and at most one arrow γ with $\gamma\alpha \notin I$.

It follows immediately from the definition that A is an SB algebra if and only if A^{op} is.

2.2.41. Running example. Throughout this thesis, it will be helpful to have a concrete example of an SB algebra to work with. In these cases, we will use:

$$A \coloneqq \Bbbk \left(\begin{tikzcd} \alpha \arrow[loop left] & 1 \arrow[r, bend left, "\beta"] & 2 \arrow[l, bend left, "\gamma"] \arrow[loop right, "\delta"] \end{tikzcd} \right) / \langle \alpha^2, \beta\delta, \gamma\beta, \delta\gamma, \alpha\beta\gamma - \beta\gamma\alpha, \delta^3 \rangle.$$

The A-paths of length 2 are $\alpha\beta, \beta\gamma, \gamma\alpha, \delta^2$. The collection of all A-paths gives a basis of the regular A-module as depicted in Figure 2.1.

The graphical notation used in this diagram (and for many subsequent diagrams) is as follows:

- the vertices are basis vectors (which in this case are labelled by A-paths),
- the arrows denote connections between these vectors; an arrow labelled $\alpha' \in Q_1$ indicates that $\alpha' + I \in A = \Bbbk Q/I$ maps the basis vector at the source to a non-zero $\Bbbk$-multiple of the vector at the target,
- if a vertex is not the source of an arrow labelled $\beta' \in Q_1$, then $\beta' + I \in A = \Bbbk Q/I$ annihilates the corresponding basis vector.

Note that the vertices at the *top* of the diagram (e_1 and e_2) correspond to basis vectors of $\mathrm{top}(A_A)$, and the remaining vertices correspond to basis vectors of $\mathrm{rad}(A_A)$. Also note that the vertices at the *bottom* of the diagram ($\alpha\beta\gamma$, $\gamma\alpha\beta$ and δ^2) correspond to basis vectors of $\mathrm{soc}(A_A)$; this makes the connection to the word "socle" for the base of sculptures more clear. It is simple to see from this diagram that $\dim(A) = 12$ (as there are twelve vertices in the whole diagram), that $\dim(\mathrm{rad}(A_A)) = 10$ (as there are ten vertices not in the top row), that $\dim(\mathrm{rad}^2(A_A)) = 6$ (as there are six vertices not in the top two rows), that $\dim(\mathrm{rad}^3(A_A)) = 2$ (as there are two vertices not in the top three rows), and that $\mathrm{rad}^4(A_A) = 0$ (as there are no vertices beyond the fourth row). Thus the radical length of A_A is four.

One thing that you can see in our example is that the ideal we quotient by to form our special biserial algebra is generated by monomial relations and commutativity relations. The next proposition states that this is possible in general (up to allowing the commutativity relations to be skew).

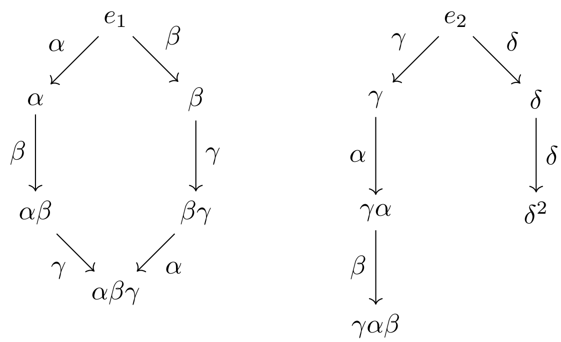


Figure 2.1: *Structure of the regular module of our running example algebra A.* The left graph shows the structure of $P_1 = e_1A$, the unique indecomposable pin module for A; the right graph shows that of $P_2 = e_2A$, the unique indecomposable projective string module for A

2.2.42. Proposition. [WW85, Prop 1.3]

Let $A = \Bbbk Q/I$ be a special biserial algebra. Then the defining admissible ideal I is generated by monomial relations and skew commutativity relations.

2.2.43. String graphs. A *string graph* for A is a quiver homomorphism $w : G \to Q$ such that:

(a) each connected component of the underlying (undirected) graph of G is linear,

(b) (i) for any subgraph $(\circ \xleftarrow{x} i \xrightarrow{y} \circ)$ of G, we have $w(x) \neq w(y) \in Q_1$,

(ii) for any subgraph $(\circ \xrightarrow{x} i \xleftarrow{y} \circ)$ of G, we have $w(x) \neq w(y) \in Q_1$,

(c) if $(\circ \xrightarrow{x_1} \circ \xrightarrow{x_2} \cdots \xrightarrow{x_l} \circ)$ is a path in G, then the A-residue of $p \coloneqq w(x_1x_2 \ldots x_l)$ is non-zero and linearly independent of all other A-paths.

We follow the convention of depicting a string graph as a labelled quiver; each vertex v (resp. arrow x) is labelled by $w(v)$ (resp. $w(x)$).

If G is connected, we call w an *indecomposable* string graph.

Note that we don't assume that G is finite; some of its connected components could be unbounded in one or both directions. We also don't assume that G is non-trivial; the empty graph is also considered a string graph.

2.2.44. String modules. Let $w : G \to Q$ be a string graph. The *string module* $\mathrm{Str}(w)$ associated to w is the A-module constructed as follows:

- its basis is the vertex set of G,
- for each arrow $v \xrightarrow{x} v'$ in G, the A-arrow $w(x)$ sends v to v'. Otherwise, each A-arrow acts as zero on these basis vectors.

Note that the connected components of G correspond to indecomposable summands of $\mathrm{Str}(w)$.

We call a string module $w : G \to Q$ *bi-infinite* if the underlying (undirected) graph of G is connected and unbounded in both directions.

A general string module is an A-module isomorphic to one of these $\mathrm{Str}(w)$. Since it is sufficient to only work with these representatives of the isomorphism classes, we shall do so.

2.2.45. Remark. Note that some sources (e.g. [HZ16]), assume (either implicitly or explicitly) that all string graphs are indecomposable. This often means that they assume that all string modules are indecomposable too.

We now briefly discuss how the class of string modules interacts with some functors that will be of interest later.

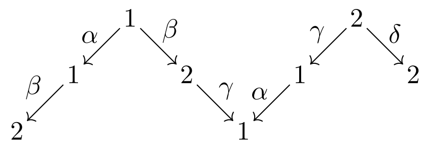
2.2.46. Lemma. [WW85, Lem 3.1(1)]
Suppose that $M \in \text{mod-}A$ is a string module with associated string graph $w : G \to Q$. Then the vector-space dual $D(M) \coloneqq \mathrm{Hom}_{\Bbbk}(M, k) \in \text{mod-}A^{\mathrm{op}}$ of M is a string module with associated string graph given by $w^{\mathrm{op}} : G^{\mathrm{op}} \to Q^{\mathrm{op}}$.

2.2.47. Example. In Figure 2.2 the first diagram is a string graph for our running example algebra A. Denote the string module that this represents by $M \in \text{mod-}A$. The second diagram is the string graph for A^{op} corresponding to $D(M) \in \text{mod-}A^{\mathrm{op}}$.

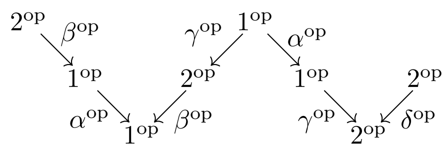
2.2.48. Proposition. Let $M \coloneqq \mathrm{Str}(w)$ be a string module over A.

(a) [WW85, Lem 3.2(2)] The transpose $\mathrm{Tr}(M)$ of M is a string module over A^{op}.

(b) [LM04, Prop 2.2] The syzygy $\Omega(M)$ of M is a string module over A.



(a) *A string graph, w, associated to a string module $M \in$ mod-A.*



(b) *A string graph associated to the string module $D(M) \in$ mod-A^{op}.*

Figure 2.2: *Calculating the vector-space dual of a string module.*

2.2.49. Band graphs. A *band graph* for A is a quiver homomorphism $w : G \to Q$ such that:

(a) the underlying (undirected) graph of G is cyclic (and in particular connected), but G is not cyclic as a quiver,

(b) (i) for any subgraph $(\circ \xleftarrow{x} i \xrightarrow{y} \circ)$ of G, we have $w(x) \neq w(y) \in Q_1$,

(ii) for any subgraph $(\circ \xrightarrow{x} i \xleftarrow{y} \circ)$ of G, we have $w(x) \neq w(y) \in Q_1$,

(c) if $(\circ \xrightarrow{x_1} \circ \xrightarrow{x_2} \cdots \xrightarrow{x_l} \circ)$ is a path in G, then the A-residue of $p \coloneqq w(x_1 x_2 \dots x_l)$ is non-zero and linearly independent of all other A-paths.

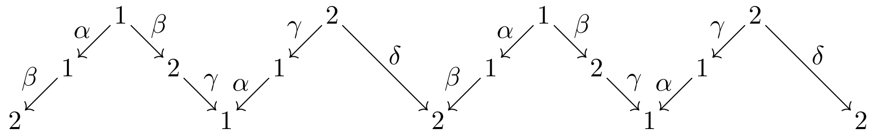
We follow the convention of depicting a band graph as a labelled quiver; each vertex v (resp. arrow x) is labelled by $w(v)$ (resp. $w(x)$).

2.2.50. Remark. We note that the definition for band graphs is identical to that of string graphs, except for in part (a). This means that all band graphs can be obtained from a special class of string graphs by identifying endpoints.

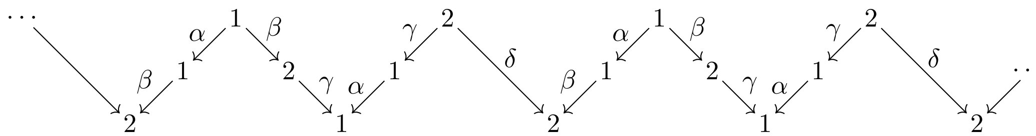
2.2.51. Powerable string graphs. A finite indecomposable string graph $w : G \to Q$ is called *powerable* if it has two sink vertices i, j, with $|t^{-1}(i)| = |t^{-1}(j)| = 1$ such that $w(i) = w(j)$ but $w(t^{-1}(i)) \neq w(t^{-1}(j))$. These sink vertices are called *endpoint vertices.*

For $m \geq 1$, the *m-th power* of a powerable string graph with endpoint vertices $i \neq j$, is the string graph obtained from m disjoint copies of w by identifying the r-th copy of j with the $(r+1)$-th copy of i. Note that this means that w^m is also powerable.

A powerable string graph u is *primitive* if it is not w^m for any powerable string graph w and integer $m \geq 2$. Every powerable string graph is a power of some primitive powerable string graph.



(a) *The second power, w^2, of w.*



(b) *The bi-infinite power, $\widehat{w}$, of w.*

Figure 2.3: *Powers of the powerable string graph, w, from Figure 2.2(a).*

If w is a powerable string graph with endpoint vertices $i \neq j$, we define the *bi-infinite power*, $\cdots www \cdots$, of w to be the string graph formed by taking a $\mathbb{Z}$-indexed disjoint union of copies of w and then identifying the r-th copy of i with the $(r+1)$-th copy of j. We will use the notation $\widehat{w}$ for the bi-infinite power of w.

2.2.52. Example. The string graph, w, depicted in Figure 2.2(a) is powerable, where its endpoint vertices are those labelled 2 at either end. We also note that w is primitive, as it can't be written as a power of any other powerable string graphs.

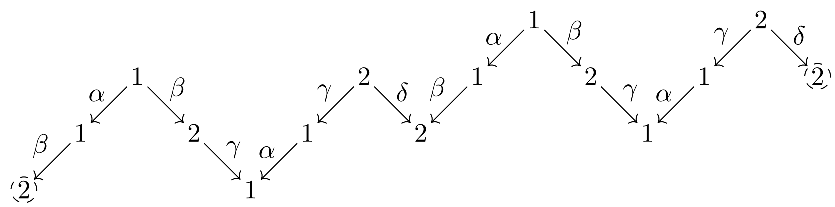
The 2nd power of w is depicted in Figure 2.3(a). The bi-infinite power of w is depicted in Figure 2.3(b). Observe that, as we expected, w^2 is also powerable.

2.2.53. Band graphs from powerable string graphs. Let w be a powerable string graph with endpoint vertices $i \neq j$. Identifying i and j yields a band graph which we denote by $\widetilde{w}$. Every band graph u can be obtained in this way from some powerable string graph w. Since every powerable string graph w can be expressed as u^m for some primitive u and $m \geq 1$, it follows that every band graph can be expressed as $\widetilde{u^m}$ for such u and m.

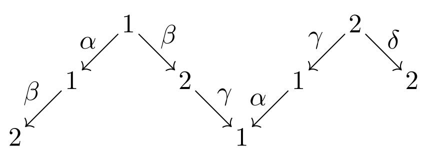
We now state an obvious lemma without proof; we give an example of this lemma in action afterwards.

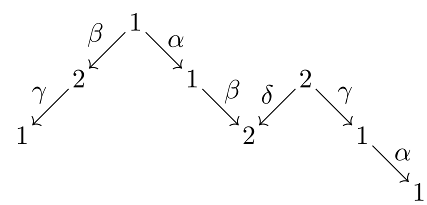
2.2.54. Lemma. Suppose $u : G \to Q, u' : G' \to Q$ are primitive string graphs and $m, m' \in \mathbb{Z}_+$. Then $\widetilde{u^m} \cong \widetilde{(u')^{m'}}$ if and only if $\widetilde{u} \cong \widetilde{u'}$ and $m = m'$.

2.2.55. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Let w be the band graph:



where the dashed circles denote vertices which are identified. We can express this band graph in the form discussed above in multiple ways. Let:





be denoted by u, u' respectively. They are clearly both primitive string graphs.

Then $\widetilde{u^2} \cong w \cong \widetilde{(u')^2}$; the first isomorphism is obvious from the diagram of w above, while the second isomorphism can be seen by reflecting the diagram of w above and shifting the point where we "cut the band graph" over to the 1s at the bottom rather than the 2s at the bottom. This second isomorphism is perhaps easier to see by noting that the "middle two peaks" of our above diagram of w are simply a reflection of our diagram for u'.

We can also see that $\widetilde{u} \cong \widetilde{u'}$ by performing a similar reflection and shifting.

Moving on from this example, we now define band modules in terms of these primitive string graphs (we follow the notation of [All21] and [HZ16]).

2.2.56. Band modules. Let v be a primitive string graph with endpoint vertices $i \neq j$. Further, let $m \geq 1$ be an integer and $\psi : \Bbbk^m \to \Bbbk^m$ be an indecomposable vector-space automorphism with companion matrix as in Figure 2.4.

For $1 \leq r \leq m$, let i_r be the m vertices of v^m corresponding to i and, (with a slight abuse of notation), write j for the sink vertex of v^m corresponding to the m-th copy of j. Each of these vertices is a basis vector of $\mathrm{Str}(v^m)$. Furthermore, there is a unique vertex idempotent e_k of A

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \lambda_2 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \lambda_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_{m-2} \\ 0 & 0 & 0 & \cdots & 1 & 0 & \lambda_{m-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \lambda_m \end{pmatrix}$$

Figure 2.4: *Companion matrix of an indecomposable automorphism associated to a band module.*

which fixes each member of this particular collection of basis vectors. Therefore, j and the i_r each span an isomorphic simple submodule of $\mathrm{Str}(v^m)$.

The *band module* $\mathrm{Bnd}(v^m, \psi)$ is the quotient module $\mathrm{Str}(v^m)/\langle j - \sum_{r=1}^{m} \lambda_r i_r \rangle$.

2.2.57. Associating band graphs to band modules. The *band graph associated to a band module* $\mathrm{Bnd}(v^m, \psi)$ is defined to be the band graph associated to the powerable string graph v^m, i.e. it is $\widetilde{v^m}$.

A priori, it is not clear that band graphs are an isomorphism invariant of band modules. This is shown as part of the next much stronger result.

We now give a restated form of [WW85, Prop 2.3], using our notation for string and band modules (which they call representations of the first and second kind respectively).

Theorem 2.2.58. *[WW85, Prop 2.3]*

The modules $\mathrm{Str}(v)$ *and* $\mathrm{Bnd}(u^m, \psi)$ *(where* v, u *are indecomposable string graphs,* u *is additionally primitive,* $m \in \mathbb{Z}_+$ *and* $\psi : \Bbbk^m \to \Bbbk^m$ *is an indecomposable automorphism) are all indecomposable. Each finitely generated indecomposable module of* A *is isomorphic to one of them, or is projective, injective and non-uniserial.*

Moreover no module of the form $\mathrm{Str}(v)$ *is isomorphic to one of the form* $\mathrm{Bnd}(u^m, \psi)$*. Furthermore,* $\mathrm{Str}(v)$ *is isomorphic to* $\mathrm{Str}(v')$ *if and only if there is a quiver isomorphism* $\sigma : G' \to G$ *such that* $v \circ \sigma = v' : G' \to Q$.

Finally, $\mathrm{Bnd}(u^m, \psi)$ *is isomorphic to* $\mathrm{Bnd}((u')^{m'}, \psi')$ *if and only if*

(i) the band graphs of $\mathrm{Bnd}(u^m, \psi)$ *and* $\mathrm{Bnd}((u')^{m'}, \psi')$ *are isomorphic,*

(ii) $m = m'$, *and*

(iii) ψ *and* ψ' *have the same eigenvalue.*

It follows from Theorem 2.2.58 and Lemma 2.2.54 that $\mathrm{Bnd}(u^m, \psi) \cong \mathrm{Bnd}((u')^{m'}, \psi')$ if and only if $\widetilde{u} \cong \widetilde{u'}$, $m = m'$ and ψ has the same eigenvalue as ψ'.

This result also means that there is a unique isomorphism class of band modules associated to each band graph $\widetilde{u^m}$ and indecomposable automorphism $\psi : \Bbbk^m \to \Bbbk^m$, and that all band modules belong to such an isomorphism class. Thus we will occasionally use the notation $\mathrm{Bnd}(w, \psi)$ for a band module associated to a band graph, w, and compatible indecomposable automorphism, ψ, even though this is only well-defined up to isomorphism.

We now give a name to those finitely generated indecomposable modules that are not string or band modules.

2.2.59. Pin modules. We refer to the those indecomposable modules that are projective, injective and non-uniserial by the abbreviation *pin*, or as *pin modules*.

2.2.60. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Then the projective e_1A is pin, while the projective e_2A is not (it is a string module).

2.2.5 The stable module category and delooping level

In this section, A denotes an arbitrary finite-dimensional algebra over $\Bbbk$.

2.2.61. Stable module category. The *stable module* category $\underline{\mathrm{mod}}\text{-}A$ is the category whose objects are the same as those of mod-A and whose morphisms are residue classes of homomorphisms modulo those that factor through projective modules.

We will call isomorphisms in this category *stable isomorphisms*, and denote them by $\simeq$.

Note that for $M, M' \in \text{mod-}A$, we have $M \simeq M'$ if and only if there exists $N, P, P' \in \text{mod-}A$, where P, P' are projective such that $M \cong N \oplus P$ and $M' \cong N \oplus P'$.

2.2.62. Remark. The syzygy functor $\Omega : \text{mod-}A \to \text{mod-}A$ that we defined in Definition 2.2.14 is also well-defined on $\underline{\mathrm{mod}}\text{-}A$, so we will also denote it by $\Omega : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$.

2.2.63. The following invariant was introduced by Gélinas in [Gé22] (for the more general class of Noetherian semi-perfect rings), and will be helpful when we later discuss the finitistic dimension conjectures in Subsection 2.2.6.

It will also be a key point of focus in Chapter 4.

2.2.64. Delooping level of a module. The *delooping level* of $M \in \text{mod-}A$ is defined as:

$$\operatorname{dell}(M) \coloneqq \inf\{k \in \mathbb{N} \mid \Omega^k(M) \text{ is a stable retract of } \Omega^{k+1}(N) \text{ for some } N \in \underline{\text{mod}}\text{-}A\}.$$

The *delooping level* of an algebra A is defined as:

$$\operatorname{dell}(A) \coloneqq \sup\{\operatorname{dell}(S) \mid S \text{ is a simple } A\text{-module}\}.$$

Note that, in general, the delooping level of either a module or an algebra may be infinite, as shown in [KR23].

2.2.65. Lemma. Let A, B be finite-dimensional algebras and M_1 (resp. M_2) be an A-module (resp. B-module). Then:

(i) $\operatorname{dell}_{A\times B}(M_1 \oplus M_2) = \sup(\operatorname{dell}_A(M_1), \operatorname{dell}_B(M_2))$,

(ii) $\operatorname{dell}(A \times B) = \sup(\operatorname{dell}(A), \operatorname{dell}(B))$,

(iii) $\operatorname{dell}(A) = 0$ when A is self-injective.

Proof. Part (i) follows immediately from the definition of delooping level and the fact that $\Omega_{A\times B}(N_1, N_2) = (\Omega_A(N_1), \Omega_B(N_2))$. Part (ii) follows from Part (i).

Part (iii) follows from the fact that $\text{mod-}A = \Omega(\text{mod-}A)$ when A is self-injective. □

To better understand the delooping level, we define a new functor:

2.2.66. Suspension functor. We define the *suspension functor* $\Sigma \coloneqq \operatorname{Tr}\Omega\operatorname{Tr} : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A$.

This functor allows us to state an equivalent version of our definition of delooping level. The following result is part of [Gé22, Thm 1.10].

Theorem 2.2.67. *The following are equivalent for any $M \in \underline{\mathrm{mod}}\text{-}A$ and $n \in \mathbb{N}$:*

(i) $\mathrm{dell}(M) \leq n$;

(ii) $\Omega^n(M)$ *is a stable retract of* $\Omega^{n+1}(N)$ *for some* $N \in \underline{\mathrm{mod}}\text{-}A$;

(iii) $\Omega^n(M)$ *is a stable retract of* $\Omega^{n+1}\Sigma^{n+1}\Omega^n(M)$.

2.2.68. n-torsionfree modules. We call a module M *torsionfree*[1] if $\mathrm{Ext}^1_{A^{\mathrm{op}}}(\mathrm{Tr}(M), A) = 0$. Furthermore, we call a module M *n-torsionfree* if $\mathrm{Ext}^i_{A^{\mathrm{op}}}(\mathrm{Tr}(M), A) = 0$ for $1 \leq i \leq n$.

We can use the functors Ω and Σ to better understand the properties of n-torsionfree modules. The following result is part of [Gé22, Prop 1.9].

2.2.69. Proposition. Let $M \in \mathrm{mod}\text{-}A$. The following properties hold in $\underline{\mathrm{mod}}\text{-}A$.

(a) $M \simeq \Omega\Sigma(M)$ if and only if M is torsionfree.

(b) If M is n-torsionfree, then we have a chain of n stable isomorphisms:

$$M \simeq \Omega\Sigma(M) \simeq \cdots \simeq \Omega^n\Sigma^n(M).$$

(c) If $\mathrm{Tr}(M)$ is n-torsionfree, then we have a chain of n stable isomorphisms:

$$\Sigma^n\Omega^n(M) \simeq \cdots \simeq \Sigma\Omega(M) \simeq M.$$

(d) If M is Gorenstein-projective, then so is $\Sigma(M)$ and we have stable isomorphisms:

$$\Sigma\Omega(M) \simeq M \simeq \Omega\Sigma(M).$$

Another result that will be useful later is:

2.2.70. Lemma. Let $M \in \underline{\mathrm{mod}}\text{-}A$ and $n \in \mathbb{Z}_+$. Then $M \simeq \Sigma^n\Omega^n(M)$ if and only if $\mathrm{Tr}(M) \simeq \Omega^n\Sigma^n(\mathrm{Tr}(M))$.

Proof. Suppose that $M \simeq \Sigma^n\Omega^n(M)$. Then

$$\mathrm{Tr}(M) \simeq \mathrm{Tr}\,\Sigma^n\Omega^n(M) \simeq \mathrm{Tr}(\mathrm{Tr}\,\Omega^n\,\mathrm{Tr})\Omega^n(M) \simeq \Omega^n\,\mathrm{Tr}\,\Omega^n(\mathrm{Tr}^2 M) \simeq \Omega^n\Sigma^n(\mathrm{Tr}(M)),$$

as required.

[1] This notion is often referred to as *torsionless*; we follow the terminology used in [Gé22; AB69]

Now suppose that $\mathrm{Tr}(M) \simeq \Omega^n\Sigma^n(\mathrm{Tr}(M))$. Then

$$\begin{aligned} M &\simeq \mathrm{Tr}^2(M) \simeq \mathrm{Tr}(\Omega^n\Sigma^n(\mathrm{Tr}(M))) \simeq \mathrm{Tr}\,\Omega^n(\mathrm{Tr}\,\Omega^n\,\mathrm{Tr})(\mathrm{Tr}(M)) \\ &\simeq \mathrm{Tr}\,\Omega^n\,\mathrm{Tr}\,\Omega^n(\mathrm{Tr}^2(M)) \simeq \Sigma^n\Omega^n(M) \end{aligned}$$

as required. □

2.2.71. Lemma. Let A be a finite-dimensional algebra and $M \in \text{mod-}A$. Suppose that there exists $N \in \mathbb{N}$ and a finite set, $\mathcal{M}$, of indecomposable A-modules where $\Omega^n(M) \in \mathrm{add}(\mathcal{M})$ for all $n \geq N$. Then $\mathrm{dell}(M) \leq N + |\mathcal{M}|$ for all $M \in \text{mod-}A$.

Proof. For $k \in \mathbb{N}$, define $\mathcal{M}_{M,k}$ to be the set of isomorphism classes of indecomposable summands of modules in $\{\Omega^n(M) : N + k \leq n\}$. Clearly $\mathcal{M}_{M,0} \supseteq \mathcal{M}_{M,1} \supseteq \ldots \supseteq \mathcal{M}_{M,k} \supseteq \ldots$, and thus we have the following chain of inequalities:

$$|\mathcal{M}| \geq |\mathcal{M}_{M,0}| \geq |\mathcal{M}_{M,1}| \geq \ldots \geq |\mathcal{M}_{M,k}| \geq \ldots.$$

Since all entries in this chain are non-negative integers, this chain stabilises and there exists $i \in \{0, \ldots, |\mathcal{M}|\}$ such that $|\mathcal{M}_{M,i}| = |\mathcal{M}_{M,i+1}|$. Since $\mathcal{M}_{M,i} \supseteq \mathcal{M}_{M,i+1}$, this means that $\mathcal{M}_{M,i} = \mathcal{M}_{M,i+1}$. Hence $\Omega^{N+i}(M) \in \mathrm{add}(\mathcal{M}_{M,i+1})$, so $\Omega^{N+i}(M)$ is a summand of $\Omega^{N+i+1}(Y)$ for some $Y \in \text{mod-}A$. Hence $\mathrm{dell}(M) \leq N + i \leq N + |\mathcal{M}|$, as required. □

2.2.72. Corollary. Let A be a finite-dimensional algebra. Suppose that there exists $N \in \mathbb{N}$ and a finite set, $\mathcal{M}$, of indecomposable A-modules where $\Omega^n(S) \in \mathrm{add}(\mathcal{M})$ for all $n \geq N$ and all simple A-modules S. Then $\mathrm{dell}(A) \leq N + |\mathcal{M}|$.

2.2.6 The finitistic dimension conjectures

We now give the definitions of the (big and little) finitistic dimensions, and introduce the corresponding finitistic conjectures.

In this section, A denotes an arbitrary finite-dimensional algebra over $\Bbbk$.

2.2.73. Finitistic dimensions. The *big* and *little finitistic dimensions* of an algebra A are:

$$\operatorname{Fin.dim}(A) \coloneqq \sup\{\operatorname{proj.dim}(M) \,:\, M \in \text{Mod-}A, \operatorname{proj.dim}(M) < \infty\}$$
$$\operatorname{fin.dim}(A) \coloneqq \sup\{\operatorname{proj.dim}(M) \,:\, M \in \text{mod-}A, \operatorname{proj.dim}(M) < \infty\}$$

An algebra A is said to satisfy the *big* or *little finitistic dimension conjectures* when:

$$\operatorname{Fin.dim}(A) < \infty \quad \text{or} \quad \operatorname{fin.dim}(A) < \infty,$$

respectively.

We may occasionally abbreviate these conjectures to *big* and *little FDC*; or *the FDCs* when talking about both.

2.2.74. Since mod-$A \subseteq$ Mod-A, it is clear that we always have $\operatorname{fin.dim}(A) \leq \operatorname{Fin.dim}(A)$. These conjectures were originally publicised in a paper of Bass [Bas60] (though Bass attributes the questions to Rosenberg and Zalinsky).

It is also important to note that, since we can express proj.dim in terms of the syzygy functor, Ω, (as in Definition 2.2.20) the study of syzygies is very important for understanding finitistic dimensions.

2.2.75. Due to the long-standing nature of these conjectures, various techniques and connected conjectures have been built up around them.

In particular, there is a collection of connected open conjectures (often referred to collectively as the "homological conjectures"), which are implied by the (little) finitistic dimension conjecture. For a diagram showing the connections by implication between these conjectures, we refer the reader to [GPS21, Section 1].

For an overview of the history of the finitistic dimension conjectures, and the techniques that have been used to to work on them, we refer to [HZ95].

2.2.76. While the finitistic dimension conjectures remain open in general, it is known that they hold in numerous special cases, particularly for the little FDC.

We now give a short (and thus necessarily, incomplete) list of properties which imply the little

FDC. Not all of the required definitions have been given above, though relevant references are given for the interested reader.

For an algebra A, the little finitistic dimension conjecture holds if:

(a) the representation dimension of A is at most 3 [IT05] (this includes all SB algebras [Erd+04, Cor 1.3]);

(b) we have $\mathrm{rad}^3(A) = 0$ (i.e. A is radical-cube-zero) [GZH91, Thm 16];

(c) A is local (in this case $\mathrm{fin.dim}(A) = 0$ by [AR91b, proof of Prop 2.1(b)]);

(d) A is monomial [HZ91; HZ92];

(e) A is syzygy-finite, as this means that there exists a finite set of indecomposable modules $\mathcal{N}$ and an integer $s \geq 0$ such that $\{\Omega^t(M) : t \geq s, M \in \text{mod-}A\} \subseteq \mathrm{add}(\mathcal{N})$, in which case we have $\mathrm{fin.dim}(A) \leq k + \max\{\mathrm{proj.dim}(N) < \infty : N \in \mathcal{N}\}$.

2.2.77. Bounding the big finitistic dimension. Since the big FDC implies the little one, proving the big version is sufficient to prove both. One new approach to proving the big FDC for an algebra uses delooping level as a bound for Fin.dim. The following result is one half of [Gé22, Prop 1.3].

2.2.78. Proposition. Let Λ be a Noetherian semi-perfect ring. Then we have an inequality:

$$\mathrm{fin.dim}(\Lambda^{\mathrm{op}}) \leq \mathrm{dell}(\Lambda).$$

If Λ is Artinian, then we have the additional bound:

$$\mathrm{fin.dim}(\Lambda^{\mathrm{op}}) \leq \mathrm{Fin.dim}(\Lambda^{\mathrm{op}}) \leq \mathrm{dell}(\Lambda).$$

Since the delooping level can be calculated by considering the properties of finitely many (simple) modules instead of all (not necessarily finitely-generated) modules, this provides an avenue for proving that an algebra satisfies the big FDC more computationally. It is also helpful for calculating a bound on Fin.dim for algebras, A, where we know that $\mathrm{Fin.dim}(A) < \infty$, but don't have the exact value.

2.2.79. Motivation. Bounding the big finitistic dimension through the delooping level has provided the main motivation for most of the ideas in this thesis. The hope was that by improving the combinatorial understanding of Ω and $\Omega\,\mathrm{Tr}$ for modules of SB algebras, we would build enough tools to prove a general bound on $\operatorname{dell}(A)$ when A is an SB algebra. While we did not manage to prove such a bound in general, Chapter 4 contains proofs of a variety of bounds for the delooping level of special biserial algebras, A, satisfying $\operatorname{rad}^3(A) = 0$.

2.3 Permissible data and syllables

In this section we begin an overview of the methods of [All21], focusing on the encoding of special biserial algebras and the combinatorial building blocks of their methods.

While most of the ideas in this section are from [All21], there are some extensions of these ideas that are original to this thesis. For example, the ideas around "minimally connected overquivers" (see Definition 2.3.24) are original, and allow us to reduce the complexity of the combinatorial framework in some cases.
All definitions and results that are not original are cited as such.

In this section, A denotes an SB algebra over $\Bbbk$ of the form $\Bbbk Q/I$.

2.3.1 Permissible data

In [All21, Section 3.1], the concept of "permissible data" for an SB algebra was introduced. This is a collection of combinatorial data used to store the most important information about the algebra. This combinatorial data is key in later definitions and for the study of SB algebras by computer; for example it is used extensively in SBStrips. We will give a brief overview of the functionality of SBStrips in Section 2.5.

Here we review the definitions of the pieces of this collection, and follow each definition with how it applies to our running example (Paragraph 2.2.41).

2.3.1. Overquiver and vertex exchange map. [All21, Def 3.1.2]
Choose a 2-regular augmentation $\widetilde{Q}$ of Q and identify Q with its image in $\widetilde{Q}$.

We extend our notion of *A-residue* to $\widetilde{Q}$-arrows that aren't Q-arrows by defining their A-residue as $0 \in A$. This extends to $\widetilde{Q}$-paths in the natural way.

An *overquiver* is a quiver homomorphism $\mathcal{O} \twoheadrightarrow \widetilde{Q}$ satisfying the following:

(a) $\mathcal{O}$ is 1-regular and has $2|Q_0|$ vertices,

(b) the induced map $\mathcal{O}_1 \twoheadrightarrow \widetilde{Q}_1$ on arrows is a bijection,

(c) the induced map $\mathcal{O}_0 \twoheadrightarrow \widetilde{Q}_0$ on vertices is 2-to-1

(d) all $\widetilde{Q}$-paths with non-zero A-residue are images of $\mathcal{O}$-paths under this homomorphism.

We will often label the edges of $\mathcal{O}$ with the same names as their corresponding arrows in $\widetilde{Q}$, which makes the homomorphism in question unambiguous. In these cases, we will often abuse notation, and simply call $\mathcal{O}$ an *overquiver*.

The *vertex exchange map* of an overquiver is the fixpoint-free involution $\dagger : \mathcal{O}_0 \to \mathcal{O}_0$ that exchanges $\mathcal{O}$-vertices with common image in Q.

2.3.2. Example. Our running example algebra (see Paragraph 2.2.41) has a ground quiver, $Q = (\alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \circlearrowleft \delta)$, which is already 2-regular, and thus is equal to its 2-regular augmentation $\widetilde{Q}$. This algebra has a unique overquiver $\mathcal{O}$, depicted below:

$$\begin{tikzcd}
s(\beta) \arrow[r, "\beta"] \arrow[d, dash, dashed] & s(\gamma) \arrow[dl, "\gamma"] \arrow[d, dash, dashed] \\
s(\alpha) \arrow[u, bend left, "\alpha"] & s(\delta) \arrow[loop right, "\delta"]
\end{tikzcd}$$

where the dashed lines denote the $\dagger$ correspondence.

2.3.3. Existence and (degree of) uniqueness of overquivers. The fact that any SB algebra has at least one overquiver follows almost entirely from the discussions of tracks in [WW85, Section 1], and is formalised in [All21, Paragraph 3.1.4]. This formalisation shows that there are two types of choices involved in constructing an overquiver for $A = \Bbbk Q/I$:

- choosing a different 2-regular augmentation, $\widetilde{Q}$, of Q;
- choosing a different bijection $\pi_i : X_i \to Y_i$ for vertices $i \in \widetilde{Q}_0$ (there are either one or two choices here for each vertex).

(*The details of the definitions of X_i, Y_i are not important here; for a full definition and discussion we refer to [All21, Paragraph 3.1.4]. The bijections, π_i, and their implication on the construction of an overquiver will be discussed more in Lemma 2.3.21.*)

2.3.4. Definition. [All21, Def 3.1.7]

We denote the set of $\mathcal{O}$-paths with non-zero A-residue by N, and simply call these paths *non-zero*.

2.3.5. Definition. [All21, Def 3.1.8]

Some of the non-zero paths have A-residue which depends linearly on the residue of some other non-zero path. We call these *components* and write $C \subseteq N$ for the set of them.

It follows from the admissibility of I that all components must have length at least 2.

Since Proposition 2.2.42 tells us that the defining ideal of A is generated by monomial relations and skew commutativity relations, it is clear that all components of A come in pairs $p, q \in C$ where $p - \lambda q$ is a skew commutativity relation for A. This means that we can define a *component exchange map* which exchanges $p \in C$ with the unique $q \in C$ such that $p \neq q$, but the A-residues of p and q are linearly dependent. We denote this map by $\dagger : C \to C$, since it is compatible with the vertex exchange map $\dagger : \mathcal{O}_0 \to \mathcal{O}_0$ and the source and target maps of $\mathcal{O}$. In particular we have $\dagger \circ s = s \circ \dagger : C \to \mathcal{O}_0$ and $\dagger \circ t = t \circ \dagger : C \to \mathcal{O}_0$.

2.3.6. Definition. [All21, Def 3.1.11]

The *permissible data* of an SB algebra A is the tuple $(\mathcal{O}, N, C, \dagger)$, where $\mathcal{O}$ is an overquiver for A, $C \subseteq N$ are the collection of components and non-zero $\mathcal{O}$-paths, and $\dagger$ is the compatible pair of involutions $C \to C$ and $\mathcal{O}_0 \to \mathcal{O}_0$, as above.

Note that in general, there may be more than one choice of permissible data for a given SB algebra, but that once a choice of overquiver has been made, there are no further choices in this data.

2.3.7. Example. For our running example algebra, we have a single choice of overquiver, $\mathcal{O}$, and exchange map, $\dagger$ (see Example 2.3.2). The remaining parts of the permissible data are as follows:

$$C = \{\alpha\beta\gamma, \beta\gamma\alpha\} \subseteq \{e_{s(\alpha)}, e_{s(\beta)}, e_{s(\gamma)}, e_{s(\delta)}, \alpha, \beta, \gamma, \delta, \alpha\beta, \beta\gamma, \gamma\alpha, \delta^2, \alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta\} = N.$$

2.3.8. It is often convenient to encode the sets $C \subseteq N$ numerically, as a pair of sequences; one a sequence of integers, the other a binary sequence, both indexed by the vertex set, $\mathcal{O}_0$.

Assume for the following that we have a fixed tuple of permissible data $(\mathcal{O}, N, C, \dagger)$ for an SB algebra $A = \Bbbk Q/I$. There are two dual ways of defining an encoding for this choice of permissible data:

2.3.9. Definition. [All21, Def 3.1.14]

For $i \in \mathcal{O}_0$, we define:

$$a_i := \max\{\mathrm{len}(p) \,:\, s(p) = i \text{ and } p \in N\}$$

$$b_i := \begin{cases} 0 & \exists p \in C \text{ with } s(p) = i, \\ 1 & \text{otherwise.} \end{cases}$$

We then call the pair $((a_i)_{i \in \mathcal{O}_0}, (b_i)_{i \in \mathcal{O}_0})$ the *source encoding* of the permissible data $(\mathcal{O}, N, C, \dagger)$.

2.3.10. Definition. [All21, Def 3.1.15]

For $i \in \mathcal{O}_0$, we define:

$$c_i := \max\{\mathrm{len}(p) \,:\, t(p) = i \text{ and } p \in N\}$$

$$d_i := \begin{cases} 0 & \exists p \in C \text{ with } t(p) = i, \\ 1 & \text{otherwise.} \end{cases}$$

We then call the pair $((c_i)_{i \in \mathcal{O}_0}, (d_i)_{i \in \mathcal{O}_0})$ the *target encoding* of the permissible data $(\mathcal{O}, N, C, \dagger)$.

2.3.11. Example. Recall our running example algebra, with permissible data as constructed in Examples 2.3.2 and 2.3.7. The source and target encodings of this data are below:

i	a_i	b_i	c_i	d_i
$s(\alpha)$	3	0	3	0
$s(\beta)$	3	0	3	0
$s(\gamma)$	3	1	3	1
$s(\delta)$	2	1	2	1

We now note a few inequalities involving permissible data that will be useful in future arguments. Both of these lemmas follow from discussion in [All21, Remark 3.1.20].

2.3.12. Lemma. Fix $i \in \mathcal{O}_0$. Then for all $0 \leq k \leq a_i$, we have $c_{i-k} \geq k$. Dually, for all $0 \leq k \leq c_i$, we have $a_{i+k} \geq k$.

2.3.13. Lemma. Fix $i \in \mathcal{O}_0$. Then for all $k > a_i$, we have $c_{i-k} < k$. Dually, for all $k > c_i$, we have $a_{i+k} < k$.

An additional useful inequality is the following, which is a strengthening of [All21, Lem 3.1.22]. To compare, note that [All21, Lem 3.1.22] can be obtained by replacing the second summand in each inequality with 1 and then performing the obvious rearrangement.

2.3.14. Lemma. For any $i \in \mathcal{O}_0$ we have $a_i \leq a_{i-1} + b_{i-1}$ and, dually, $c_i \leq c_{i+1} + d_{i+1}$.

Proof. We prove the result for the source encoding; the result for the target encoding follows dually. We proceed by splitting into cases based on the value of b_{i-1}, and by contradiction.

Suppose that $b_{i-1} = 0$ and that $a_i > a_{i-1}$. Then $a_i \geq a_{i-1} + 1$. Thus $(i \xrightarrow{a_{i-1}+1} \circ)$ belongs to N, but the strict subpath $(i-1 \xrightarrow{a_{i-1}} \circ)$ belongs to C. This gives a contradiction, as required.

Now instead, suppose that $b_{i-1} = 1$ and that $a_i > a_{i-1}+1$. Then $a_i \geq a_{i-1}+2$. Thus $(i \xrightarrow{a_{i-1}+2} \circ)$ belongs to N, but the strict subpath $(i-1 \xrightarrow{a_{i-1}+1} \circ)$ does not. This gives a contradiction, as required. □

Another result that we can give a strengthening of is [All21, Prop 3.1.29]. To compare, note that in [All21, Prop 3.1.29] their conditions (a) and (b) correspond to our conditions (ii) and (iii), and their condition (c) follows from a combination of our conditions (ii) and (iii) and the fact that our permutation is fixed across all of the conditions. Our conditions (i), (iv) and (v) are not reflected in [All21, Prop 3.1.29] at all.

2.3.15. Proposition. Let $(\mathcal{O}, N, C, \dagger)$ be the permissible data and $((a_i)_{i\in\mathcal{O}}, (b_i)_{i\in\mathcal{O}})$, $((c_i)_{i\in\mathcal{O}}, (d_i)_{i\in\mathcal{O}})$ its encodings. Then there exists a permutation $\pi : \mathcal{O}_0 \to \mathcal{O}_0$ such that:

(i) for each connected component $\mathcal{C}$ of $\mathcal{O}$, $\mathrm{im}(\pi|_{\mathcal{C}}) = \mathcal{C}$,

(ii) for each $i \in \mathcal{O}_O$, we have $a_{\pi i} = c_i$,

(iii) for each $i \in \mathcal{O}_O$, we have $b_{\pi i} = d_i$,

(iv) for each $i \in \mathcal{O}_0$ with $b_i = 0$, we have $\pi i = i - a_i$,

(v) for each $i \in \mathcal{O}_0$ with $d_i = 0$, we have $\pi^{-1} i = i + c_i$.

Proof. We first note that in [All21, Lem 3.1.27(b)], $s(p)$ and $t(p)$ are on the same connected component of $\mathcal{O}$. Thus the argument for [All21, Prop 3.1.29(a)] gives a permutation $\pi : \mathcal{O}_0 \to \mathcal{O}_0$ satisfying conditions (i) and (ii) of the claim.

We now proceed by an inductive argument to construct a permutation satisfying conditions (i), (ii) and (iv). Let

$$
\begin{aligned}
m : \mathrm{Sym}(\mathcal{O}_0) &\longrightarrow \mathbb{N} \\
\rho &\longmapsto m(\rho) = |\{i \in \mathcal{O}_0 \mid b_i = 0 \text{ and } \rho i \neq i - a_i\}|.
\end{aligned}
$$

Clearly if $m(\sigma) = 0$ then condition (iv) is satisfied by $\pi = \sigma$.

We now assume that $m(\sigma) > 0$. Thus there exists $i_0 \in \mathcal{O}_0$ such that $b_{i_0} = 0$ and $\sigma i_0 \neq i_0 - a_{i_0}$. Now consider $\sigma' : \mathcal{O}_0 \to \mathcal{O}_0$ defined by:

$$
\sigma' i \coloneqq \begin{cases} i_0 - a_{i_0} & \text{if } i = i_0 \\ \sigma i_0 & \text{if } i = \sigma^{-1}(i_0 - a_{i_0}) \\ \sigma i & \text{otherwise} \end{cases} .
$$

It is immediately clear that since σ is a permutation, so is σ'.

Since σ restricts to a permutation on each connected component of $\mathcal{O}$ and the $\mathbb{Z}$-action on vertices [All21, Paragraph 2.1.37] restricts to an action on each connected component, it follows that σ' also restricts to a permutation on each connected component. Thus σ' still satisfies condition (i). Since $b_{i_0} = 0$, it follows from the definition of the encodings of $(\mathcal{O}, N, C, \dagger)$ that $c_{i_0 - a_{i_0}} = a_{i_0}$. Thus σ' also satisfies condition (ii).

We now show that $m(\sigma') < m(\sigma)$. Clearly $i_0 \in \{i \in \mathcal{O}_0 \mid b_i = 0 \text{ and } i\sigma \neq i - a_i\}$ but $i_0 \notin \{i \in \mathcal{O}_0 \mid b_i = 0 \text{ and } i\sigma' \neq i - a_i\}$. It is also clear that $(i_0 - a_{i_0})\sigma^{-1} \in \{i \in \mathcal{O}_0 \mid b_i = 0 \text{ and } i\sigma \neq i - a_i\}$. Therefore we have a proper inclusion $\{i \in \mathcal{O}_0 \mid b_i = 0 \text{ and } i\sigma' \neq i - a_i\} \subsetneq \{i \in \mathcal{O}_0 \mid b_i = 0 \text{ and } i\sigma \neq i - a_i\}$. Thus $m(\sigma') < m(\sigma)$ as required.

By induction, it follows that there exists a permutation satisfying conditions (i), (ii) and (iv).

We now show that any permutation $\sigma : \mathcal{O}_0 \to \mathcal{O}_0$ satisfying condition (iv) automatically satisfies conditions (iii) and (v) as well. Fix $i_0 \in \mathcal{O}_0$.

Suppose that $b_{i_0} = 0$. Since $(\, i_0 \xrightarrow{a_{i_0}} \circ \,) \in C$, this means that $d_{\sigma i_0} = d_{i - a_i} = 0$, as required.

By [All21, Prop 3.1.29(b)], $|\{i \in \mathcal{O} \mid b_i = 0\}| = |\{i \in \mathcal{O} \mid d_i = 0\}| < \infty$, and since $b_i, d_i \in \{0, 1\}$, we know that the remaining vertices $i \in \mathcal{O}_0$ with $b_i = 1$ have $d_{i\sigma} = 1$, as required.

Finally, suppose that $d_{i_0} = 0$. Since $(\circ \xrightarrow{c_{i_0}} i_0) \in C$, this means that $b_{i_0+c_{i_0}} = 0$ and that $a_{i_0+c_{i_0}} = c_{i_0}$. Hence, by (iv), we have $\sigma(i_0 + c_{i_0}) = i_0 + c_{i_0} - a_{i_0+c_{i_0}} = i_0$. Thus $\sigma^{-1}i = i + c_i$, as required.

Therefore we have shown that a permutation satisfying all five conditions exists. $\square$

We can also use the permissible data to characterise the paths corresponding to standard basis elements of $\mathrm{soc}(P)$ where P is one of the indecomposable projectives.

2.3.16. Lemma. A path $(i \xrightarrow{l} \circ)$ in the overquiver represents a basis element of the socle of the projective corresponding to the vertex i if and only if $l = a_i$ and one of the following hold:

- $b_i = 0$, or
- $b_i = 1$ and $a_i > 0$, or
- $b_i = 1$ and $a_i = a_{i^\dagger} = 0$.

Proof. Let P be the indecomposable projective corresponding to i.

We first handle the case where $b_i = 0$. In this case, P is pin, and has simple socle. The socle of P is thus represented by the (necessarily non-stationary) component paths with source i or $i^\dagger$. By the definition of our source encoding, this means that $(i \xrightarrow{l} \circ)$ represents a basis element of the socle of P if and only if $l = a_i$, as required.

We now handle the case where $b_i = 1$ and $a_i > 0$. In this case, P is a string module and P has at least one standard basis vector represented by a non-stationary $\mathcal{O}$-path (by the definition of the source encoding). This means that there is a basis vector of the socle of P represented by the maximal non-zero $\mathcal{O}$-path with source i. By the definition of our source encoding, this means that $(i \xrightarrow{l} \circ)$ represents a basis element of the socle of P if and only if $l = a_i$, as required.

Finally, we handle the case where $b_i = 1$ and $a_i = 0$. This means that P is a uniserial projective. This also means that the only non-zero $\mathcal{O}$-path with source i is the stationary one, $(i \xrightarrow{0} \circ)$, so this is the only path we need to consider. Hence, it is automatic that $l = a_i = 0$. Clearly $(i \xrightarrow{0} \circ)$ represents a basis element of the head of P; therefore it will only represent a basis element of the socle of P if P is a simple projective. This occurs exactly when $a_{i^\dagger} = 0$, as required. $\square$

2.3.17. As discussed in [All21, Paragraph 3.1.4], an overquiver is not uniquely determined by an SB algebra, $A = \Bbbk Q/I$. It depends on the choice of 2-regular augmentation, $\widetilde{Q}$, of Q, and on the choice of bijections, π_i, used in [All21, Paragraph 3.1.4]. This makes it more difficult to computationally iterate through a large collection of SB algebras; if you aren't careful, you will end up with a large number of trivially isomorphic algebras in your list.

These choices give rise to methods of constructing a new set of permissible data for A from another. Note that these constructions focus on the source encoding instead of the target encoding for an SB algebra, but this is not required; dual results work with the target encoding too.

> **2.3.18. Lemma.** Let A be an SB algebra with permissible data $(\mathcal{O}, N, C, \dagger)$ and source encoding $(a_i)_{i\in\mathcal{O}_0}, (b_i)_{i\in\mathcal{O}_0}$. Suppose that $i_1 \neq i_2 \in \mathcal{O}_0$ with $a_{i_1} = a_{i_2} = 0$. Then there exists another set of permissible data $(\mathcal{O}', N', C', \dagger')$ for A where the overquiver $\mathcal{O}'$ satisfies:
>
> $$\mathcal{O}'_0 = \{i' \mid i \in \mathcal{O}_0\}$$
> $$i' - 1 = \begin{cases} (i_2 - 1)' & \text{if } i = i_1 \\ (i_1 - 1)' & \text{if } i = i_2 \\ (i-1)' & \text{otherwise} \end{cases}$$

Proof. Let $\widetilde{Q}$ be the 2-regular augmentation of Q used in the construction of $\mathcal{O}$. Let β_1, β_2 be the arrows of $\widetilde{Q}$ corresponding to the $\mathcal{O}$-arrows $s^{-1}(i_1)$ and $s^{-1}(i_2)$ respectively.

We define $\widetilde{Q}'$ to be another 2-regular augmentation of Q as follows:

$$\widetilde{Q}'_0 \coloneqq \{v' \mid v \in \widetilde{Q}_0\} \qquad \widetilde{Q}'_1 \coloneqq \{\alpha' \mid \alpha \in \widetilde{Q}_1\}$$
$$s : \widetilde{Q}'_1 \to \widetilde{Q}'_0, \quad \alpha' \mapsto s(\alpha') \coloneqq s(\alpha)'$$
$$t : \widetilde{Q}'_1 \to \widetilde{Q}'_0, \quad \alpha' \mapsto t(\alpha') \coloneqq \begin{cases} t(\beta_2)' & \text{if } \alpha = \beta_1 \\ t(\beta_1)' & \text{if } \alpha = \beta_2 \\ t(\alpha)' & \text{otherwise} \end{cases}$$

We now define an overquiver $\mathcal{O}' \twoheadrightarrow \widetilde{Q}'$ as follows (with the homomorphism implicit from the labelling):

$$\mathcal{O}'_0 \coloneqq \{i' \mid i \in \mathcal{O}_0\} \qquad \mathcal{O}'_1 \coloneqq \{\alpha' \mid \alpha \in \mathcal{O}_1\}$$

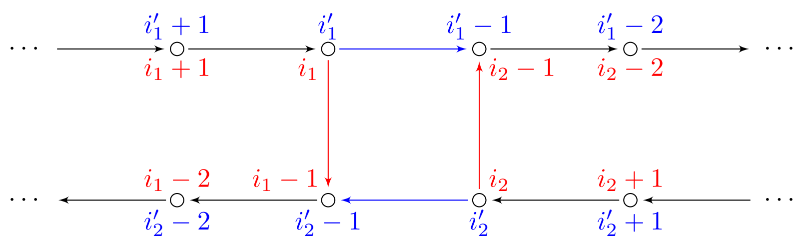
$$s : \mathcal{O}'_1 \to \mathcal{O}'_0, \quad \alpha' \mapsto s(\alpha') \coloneqq s(\alpha)'$$

$$t : \mathcal{O}'_1 \to \mathcal{O}'_0, \quad \alpha' \mapsto t(\alpha') \coloneqq \begin{cases} t(s^{-1}(i_2))' & \text{if } \alpha = s^{-1}(i_1) \\ t(s^{-1}(i_1))' & \text{if } \alpha = s^{-1}(i_2) \\ t(\alpha)' & \text{otherwise} \end{cases}$$

It is immediate from the definition that this is a well-defined overquiver, and that this overquiver satisfies the stated condition. □

2.3.19. Remark. This operation corresponds to exchanging the target vertices of "augmented arrows" in the 2-regular augmentation, $\widetilde{Q}$. Therefore, repeated application of this operation allows us to change the choice of 2-regular augmentation used in the creation of the overquiver to any of the other options.

The process of this operation can be thought of in terms of the following diagram comparing the initial overquiver to the resulting one:



In this diagram, the red labels, red lines and black lines denote the initial overquiver; the blue labels, blue lines and black lines denote the resulting overquiver (we have omitted the usual dashed lines for the † correspondence as it is not relevant here). Note that applying this operation to the resulting (blue) overquiver, gives us an overquiver isomorphic to the initial (red) one.

2.3.20. While the previous lemma handled the choice of 2-regular augmentation in constructing an overquiver, the following result handles the choice of bijections, π_i, used in [All21, Paragraph 3.1.4].

2.3.21. Lemma. Let $A = \Bbbk Q/I$ be an SB algebra with permissible data $(\mathcal{O}, N, C, \dagger)$ and source encoding $(a_i)_{i\in\mathcal{O}_0}, (b_i)_{i\in\mathcal{O}_0}$. Suppose that $i_0 \in \mathcal{O}_0$ satisfies $a_{i_0+1} \leq 1$ and $a_{i_0^\dagger+1} \leq 1$. Then there exists another set of permissible data $(\mathcal{O}', N', C', \dagger')$ for A where the overquiver $\mathcal{O}'$ satisfies:

$$\mathcal{O}'_0 = \{i' \mid i \in \mathcal{O}_0\}$$

$$i' - 1 = \begin{cases} (i_0^\dagger)' & \text{if } i = i_0 + 1 \\ (i_0)' & \text{if } i = i_0^\dagger + 1 \\ (i-1)' & \text{otherwise} \end{cases}$$

Proof. Let $\widetilde{Q}$ be the 2-regular augmentation of Q used in the construction of $\mathcal{O}$.

We now define an overquiver $\mathcal{O}' \twoheadrightarrow \widetilde{Q}$ as follows (with the homomorphism implicit from the labelling):

$$\mathcal{O}'_0 \coloneqq \{i' \mid i \in \mathcal{O}_0\} \qquad \mathcal{O}'_1 \coloneqq \{\alpha' \mid \alpha \in \mathcal{O}_1\}$$

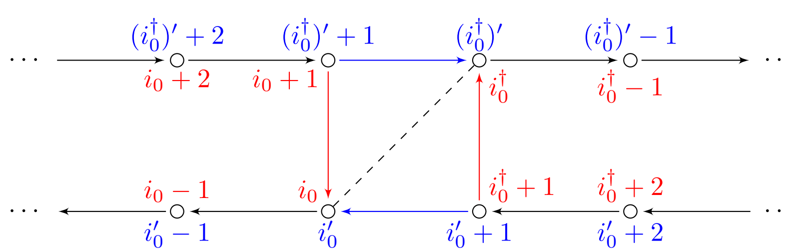
$$s : \mathcal{O}'_1 \to \mathcal{O}'_0, \quad \alpha' \mapsto s(\alpha') \coloneqq s(\alpha)'$$

$$t : \mathcal{O}'_1 \to \mathcal{O}'_0, \quad \alpha' \mapsto t(\alpha') \coloneqq \begin{cases} (i_0^\dagger)' & \text{if } \alpha = t^{-1}(i_0) \\ (i_0)' & \text{if } \alpha = t^{-1}(i_0^\dagger) \\ t(\alpha)' & \text{otherwise} \end{cases}$$

It is immediate from the definition that this is a well-defined overquiver, and that this overquiver satisfies the stated condition. $\square$

2.3.22. Remark. This operation corresponds to swapping the choice of one of the π_i bijections from [All21, Paragraph 3.1.4]. Therefore, repeated application of this operation allows us to change the selection of all these bijections to any other selection.

The process of this operation can be thought of in terms of the following diagram comparing the initial overquiver to the resulting one:



In this diagram, the red labels, red lines and black lines denote the initial overquiver; the blue labels, blue lines and black lines denote the resulting overquiver (we have omitted most of the dashed lines for the † correspondence as they are not relevant here). Note that applying this operation to the resulting (blue) overquiver, gives us an overquiver isomorphic to the initial (red) one.

2.3.23. The remainder of this section introduces and investigates a new idea of a "minimally connected" overquiver associated to an SB algebra, $A = \Bbbk Q/I$.

2.3.24. Minimally connected overquivers. Let A be an SB algebra. An overquiver $\mathcal{O}$ is called *minimally connected* if it has the maximum number of connected components, compared to all other overquivers of A.

Since the number of connected components of a quiver is bounded above by the number of its vertices, all SB algebras have at least one minimally connected overquiver. Restricting our attention to minimally connected overquivers places extra conditions on the permissible data of SB algebras and their encodings.

2.3.25. Proposition. Let A be an SB algebra. Let $(\mathcal{O}, N, C, \dagger)$ be permissible data for A where $\mathcal{O}$ is a minimally connected overquiver of A. Let $(a_i)_{i\in\mathcal{O}_0}, (b_i)_{i\in\mathcal{O}_0}$ be a source encoding of this permissible data. Then for every connected component, $\mathcal{C}$, of $\mathcal{O}$:

(i) there is at most one vertex i of $\mathcal{C}$ where $a_i = 0$, and

(ii) if $i, i^\dagger$ are both vertices of $\mathcal{C}$, then at least one of $a_{i+1}, a_{i^\dagger+1}$ is strictly greater than 1.

Proof. Suppose that condition (i) is not met; in other words, that there exists vertices $i_1, i_2 \in \mathcal{O}_0$ which lie on the same connected component $\mathcal{C}$ of $\mathcal{O}$, and which satisfy $a_{i_1} = a_{i_2} = 0$. Then applying the construction of Lemma 2.3.18 to $\mathcal{O}$ using these vertices results in an overquiver of A with strictly more connected components. This contradicts the assumption that $\mathcal{O}$ is a minimally connected overquiver, as required.

Now instead suppose that condition (ii) is not met; in other words that there exists a vertex $i \in \mathcal{O}_0$ such that $i, i^\dagger$ both lie on the same connected component $\mathcal{C}$ of $\mathcal{O}$, and which satisfy $a_{i+1} \leq 1$ and $a_{i^\dagger+1} \leq 1$. Then applying the construction of Lemma 2.3.21 to $\mathcal{O}$ using the vertex i results in an overquiver of A with strictly more connected components. This contradicts the assumption that $\mathcal{O}$ is a minimally connected overquiver, as required. □

Unfortunately, even though a minimally connected overquiver is more restricted than a general one, an SB algebra may still have more than one minimally connected overquiver.

2.3.26. Example. Let Q be the following sub-2-regular quiver:

$$\begin{tikzcd}
1 \arrow[dd, bend right, "\alpha_1"'] \arrow[dd, bend left, "\beta_1"] & & 4 \arrow[dd, bend left, "\gamma_4"] \\
& 3 \arrow[ul, "\beta_3"'] \arrow[ur, "\delta_3"] & \\
2 \arrow[ur, "\alpha_2"'] & & 5 \arrow[uu, bend left, "\gamma_5"] \arrow[ul, "\gamma_{5'}"]
\end{tikzcd}$$

and let

$$A \coloneqq \Bbbk Q/\langle \alpha\beta, \alpha\delta, \beta\alpha, \gamma\beta, \gamma\delta, \delta\gamma, \gamma_4\gamma_5 \rangle,$$

where the indices of the arrows are inserted as appropriate.

Then Q has two possible 2-regular augmentations:

$$\widetilde{Q}^1 \coloneqq \begin{tikzcd}
1 \arrow[dd, bend right, "\alpha_1"'] \arrow[dd, bend left, "\beta_1"] & & 4 \arrow[dd, bend left, "\gamma_4"] \arrow[ll, bend right=10, "\kappa_4"'] \\
& 3 \arrow[ul, "\beta_3"'] \arrow[ur, "\delta_3"] & \\
2 \arrow[ur, "\alpha_2"'] \arrow[rr, bend right=10, "\kappa_2"'] & & 5 \arrow[uu, bend left, "\gamma_5"] \arrow[ul, "\gamma_{5'}"]
\end{tikzcd}$$

$$\widetilde{Q}^2 \coloneqq \begin{tikzcd}
1 \arrow[dd, bend right, "\alpha_1"'] \arrow[dd, bend left, "\beta_1"] & & 4 \arrow[dd, bend left, "\gamma_4"] \arrow[dd, bend left=60, "\lambda_4"] \\
& 3 \arrow[ul, "\beta_3"'] \arrow[ur, "\delta_3"] & \\
2 \arrow[ur, "\alpha_2"'] \arrow[uu, bend left=60, "\lambda_2"] & & 5 \arrow[uu, bend left, "\gamma_5"] \arrow[ul, "\gamma_{5'}"]
\end{tikzcd}$$

Each of these 2-regular quivers have two corresponding choices of overquiver for A, based on the choice of π_3 (as defined in [All21, Paragraph 3.1.4]). This is the only choice we have to make with the π_i maps, as 3 is the only vertex with no non-zero paths of length 2 going through it. The two choices of overquiver corresponding to $\widetilde{Q}^1$ are:

$$\begin{tikzcd}
1 \arrow[r, "\alpha_1"] & 2 \arrow[d, "\alpha_2"] \\
4 \arrow[u, "\kappa_4"] & 3 \arrow[l, "\delta_3"]
\end{tikzcd}
\quad
\begin{tikzcd}
3' \arrow[r, "\beta_3"] & 1' \arrow[r, "\beta_1"] & 2' \arrow[d, "\kappa_2"] \\
5' \arrow[u, "\gamma_{5'}"] & 4' \arrow[l, "\gamma_4"] & 5 \arrow[l, "\gamma_5"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & 3' \arrow[r, "\beta_3"] & 1' \arrow[r, "\beta_1"] & 2' \arrow[d, "\kappa_2"] \\
4 \arrow[u, "\kappa_4"] & 3 \arrow[l, "\delta_3"] & 5' \arrow[l, "\gamma_{5'}"] & 4' \arrow[l, "\gamma_4"] & 5 \arrow[l, "\gamma_5"]
\end{tikzcd}$$

where we omit the usual dashed lines for vertex identification, and instead write i, i' for the vertices of the overquiver corresponding to i in $\widetilde{Q}^1$. The two choices of overquiver corresponding to $\widetilde{Q}^2$ are:

$$
\begin{tikzcd}
 & 1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & 3 \arrow[r, "\delta_3"] & 4 \arrow[dr, "\lambda_4"] & \\
2' \arrow[ur, "\lambda_2"] & 1' \arrow[l, "\beta_1"] & 3' \arrow[l, "\beta_3"] & 5' \arrow[l, "\gamma_{5'}"] & 4' \arrow[l, "\gamma_4"] & 5 \arrow[l, "\gamma_5"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[r, "\alpha_1"] & 2 \arrow[dr, "\alpha_2"] & \\
2' \arrow[ur, "\lambda_2"] & 1' \arrow[l, "\beta_1"] & & 3' \arrow[ll, "\beta_3"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 3 \arrow[r, "\delta_3"] & 4 \arrow[dr, "\lambda_4"] & \\
5' \arrow[ur, "\gamma_{5'}"] & & 4' \arrow[ll, "\gamma_4"] & 5 \arrow[l, "\gamma_5"]
\end{tikzcd}
$$

where we again omit the dashed lines for vertex identification, and instead i, i' are the vertices of the overquiver corresponding to i in $\widetilde{Q}^2$.

Looking at all four options for an overquiver of A, it is clear that the maximum number of connected components is 2, and that this is achieved by two non-isomorphic overquivers.

2.3.27. Remark. Choosing a minimally connected overquiver to represent a particular SB algebra is helpful when computationally iterating through a large collection of SB algebras, but it is also helpful for our intuition around this encoding of the algebra.

When considering an overquiver for an SB algebra, it is natural to think that those arrows that are on the same connected component of the overquiver have the potential for interaction. This line of thinking becomes even more natural when considering "syllables" and "descent", both of which play a key role in the combinatorial formalism, and are defined in the next section.

If you choose to use an overquiver other than a minimally connected one, then your overquiver necessarily has arrows on the same connected component which will not have any interaction in the combinatorial formalism. However, if we maximise the number of connected components, and use a minimally connected overquiver, it helps us to avoid considering arrows together that don't actually interact.

2.3.28. Remark. Note that the SBStrips function `OverquiverOfSBAlg` *does not* always generate a minimally connected overquiver. However, the SBStrips functions which are used to iterate through collections of special biserial algebras (e.g. `SBAlgsFromNumVerticesAndRadLength`) generate the algebras by constructing them from candidate overquivers which *are* minimally connected. This reduces the number of repeated SB algebras generated (up to isomorphism), but does not eliminate all repeats (since, as discussed above, some SB algebras have multiple minimally connected overquivers). The implementation of this function (and those it relies on) into SBStrips is the main contribution of the ideas of this thesis to the capabilities of computer algebra software in this area. This is particularly important when trying to test conjectures in small examples, as is done in Section 5.2.

We will discuss SBStrips further in Section 2.5.

2.3.2 Syllables

We now introduce the building blocks of Allen's theory for encoding modules of special biserial algebras, *syllables*. We will later reuse these syllables when expanding upon Allen's theory in Chapter 3. For the remainder of this section, we fix a choice of permissible data $(\mathcal{O}, N, C, \dagger)$.

We give the same definition as [All21, Def 3.2.1].

2.3.29. Syllables. [All21, Def 3.2.1]

A *syllable* is a tuple (p, ε, s) comprised of:

- a non-zero non-component path $p \in N - C$ (the *underlying path*),
- a binary bit $\varepsilon \in \{0, 1\}$ (the stability term), and
- a sign $s \in \{+1, -1\}$ (the orientation)

where $\text{len}(p) + \varepsilon > 0$.

We will generally denote syllables pictorially as $(\circ \xrightarrow{p} \circ \xrightarrow{\varepsilon} \circ)^s$ (although we will often omit the orientation s when it is unimportant or clear from context). Occasionally, the underlying path of a syllable will be specified by its length l and its source (resp. target); in these cases we will denote the syllable as $(i \xrightarrow{l} \circ \xrightarrow{\varepsilon} \circ)^s$ (resp. $(\circ \xrightarrow{l} i \xrightarrow{\varepsilon} \circ)^s$).

We have some adjectives that we use to describe syllables based on their constituent parts. Consider $(\circ \xrightarrow{p} \circ \xrightarrow{\varepsilon} \circ)^s$:

- it is *stationary* or *non-stationary* if and only if p is;
- it is *positive* or *negative* if and only if s is;
- it is *interior* if $\varepsilon = 0$ and *boundary* if $\varepsilon = 1$.

2.3.30. Compression, source and target of syllables. [All21, Def 3.2.1]
The *compression* of a syllable $\mathbf{p} \coloneqq (\, i \xrightarrow{l} \circ \xrightarrow{\varepsilon} \circ \,)$ is the $\mathcal{O}$-path $q \coloneqq (\, i \xrightarrow{l+\varepsilon} \circ \,)$. We then define the *source* and *target* of a syllable in terms of its compression; respectively as $s(\mathbf{p}) \coloneqq s(q)$ and $t(\mathbf{p}) \coloneqq t(q)$.

2.3.31. Perturbation of interior syllables. Let $\mathbf{p}$ be an interior syllable of the form $(\, \circ \xrightarrow{p} \circ \xrightarrow{0} \circ \,)^s \in \mathrm{Syll}(A)$. Then the *perturbation* of $\mathbf{p}$, is the syllable $(\, \circ \xrightarrow{p} \circ \xrightarrow{1} \circ \,)^s \in \mathrm{Syll}(A)$.

2.3.32. Opposite of interior syllables. Let $\mathbf{p}$ be an interior syllable of the form $(\, \circ \xrightarrow{p} \circ \xrightarrow{0} \circ \,)^s \in \mathrm{Syll}(A)$. Then the *opposite* of $\mathbf{p}$, denoted $\mathbf{p}^{\mathrm{op}}$, is the syllable $(\, \circ \xrightarrow{p^{\mathrm{op}}} \circ \xrightarrow{0} \circ \,)^{-s} \in \mathrm{Syll}(A^{\mathrm{op}})$.

When we later use syllables to represent string modules, the socle-quotients of pin modules will require special attention. The following are the syllables that appear in these cases.

2.3.33. Pin-boundary syllables. [All21, Def 3.2.10]
A *pin-boundary* syllable is a syllable of the form $(\, i \xrightarrow{a_i + b_i - 1} \circ \xrightarrow{1} \circ \,)$ for a vertex $i \in \mathcal{O}_0$ with $b_i = 0$.

2.3.34. When we later define Allen's method for taking syzygies by considering syllables, we will need a procedure for moving from one collection of syllables (representing a given module) to another collection of syllables (representing that modules syzygy). This will also be necessary when we expand on Allen's method in Chapter 3. The main operation in this procedure is the following partial function.

2.3.35. Descent. [All21, Def 3.2.12]
We define a partial operation on syllables called *descent* and denoted ∇. For $(\, i \xrightarrow{l} \circ \xrightarrow{\varepsilon} \circ \,)^s \in \mathrm{Syll}(A)$, we define

$$\nabla \left((\, i \xrightarrow{l} \circ \xrightarrow{\varepsilon} \circ \,)^s \right) = (\, i - (l+\varepsilon) \xrightarrow{a_i - (l+\varepsilon)} \circ \xrightarrow{b_i} \circ \,)^{-s}$$

whenever the right-hand side is a well-defined syllable.

2.3.36. Remark. Note that this is the same definition as [All21, Def 3.2.12], but there, descent would be denoted by $\mathbf{p}\nabla$, rather than $\nabla\mathbf{p}$, as we do.

A few key facts about descent are combined into the following:

2.3.37. Proposition. Fix a syllable $\mathbf{p} \coloneqq ({}_i \xrightarrow{\ l\ } \circ \xrightarrow{\varepsilon} \circ)^s$.

(i) if $\mathbf{p} \in \operatorname{supp}(\nabla)$, then $\nabla\mathbf{p}$ has orientation $-s$, and the orientation of $\mathbf{p}$ has no other effect on $\nabla\mathbf{p}$;

(ii) $\mathbf{p} \notin \operatorname{supp}(\nabla)$ if and only if $(l, \varepsilon) = (a_i + b_i - 1, 1)$;

(iii) if $\mathbf{p}$ is interior, then $\mathbf{p} \in \operatorname{supp}(\nabla)$;

(iv) if $\mathbf{p}$ is pin-boundary, then $\mathbf{p} \notin \operatorname{supp}(\nabla)$;

(v) if $\mathbf{p}, \mathbf{q}$ are syllables of the same orientation with the same compression, then $\mathbf{p} \in \operatorname{supp}(\nabla)$ if and only if $\mathbf{q} \in \operatorname{supp}(\nabla)$. Furthermore, if this is the case, $\nabla\mathbf{p} = \nabla\mathbf{q}$.

Proof. Parts (i)-(iv) are exactly [All21, Results 3.2.13-3.2.16]. Part (v) follows immediately from the definition of ∇. □

2.3.38. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41, and let $(\mathcal{O}, N, C, \dagger)$ be its previously discussed permissible data. Let $\mathbf{p} = (\circ \xrightarrow{\ \gamma\ } \circ \xrightarrow{0} \circ)$ and $\mathbf{q} = (\circ \xrightarrow{\ \delta^2\ } \circ \xrightarrow{1} \circ)$. Then the compression of $\mathbf{p}$ is $(\circ \xrightarrow{\gamma} \circ)$ and the compression of $\mathbf{q}$ is $(\circ \xrightarrow{\delta^3} \circ)$. Thus $s(\mathbf{p}) = s(\gamma) = t(\beta)$, $t(\mathbf{p}) = t(\gamma) = s(\alpha)$, $s(\mathbf{q}) = s(\delta) = t(\delta)$ and $t(\mathbf{p}) = t(\delta) = s(\delta)$.

We also know that $\mathbf{p}$ is interior, and hence belongs to $\operatorname{supp}(\nabla)$. Furthermore, $\mathbf{q}$ is boundary, not pin-boundary (as $b_{s(\delta)} = 1$) and does not belong to $\operatorname{supp}(\nabla)$.

The image of $\mathbf{p}$ under descent is $\nabla\mathbf{p} = (\circ \xrightarrow{\ \alpha\beta\ } \circ \xrightarrow{1} \circ)$, which is pin-boundary (and hence does not belong to $\operatorname{supp}(\nabla)$).

2.3.3 Pin graph

We now associate another quiver to each SB algebra, whose vertex set is equal to that of its underlying quiver. We give the same definition as in [All21, Def 5.2.14].

2.3.39. Pin graph. [All21, Def 5.2.14]

Let A be an SB algebra with underlying quiver Q. The *pin graph* Φ_A of A is the quiver with

- vertex set equal to Q_0,
- an arrow $i \to j$ if there exists a pin module P with $\mathrm{top}(P) \cong S_i$ and $\mathrm{soc}(P) \cong S_j$.

2.3.40. Some basic properties of pin graphs from the start of [All21, Section 5.2.1] include:

- A pin graph Φ_A is always sub-1-regular, and if it is 1-regular, then A is self-injective;
- A pin graph Φ_A is not necessarily connected, and it is discrete (has no arrows) if and only if A is monomial;
- The opposite of a pin graph is also a pin graph, as $\Phi_{A^{\mathrm{op}}} = (\Phi_A)^{\mathrm{op}}$.

2.3.41. The rest of this section contains new properties of pin graphs that will be helpful in proving later results.

Since a pin graph is sub-1-regular, it has two types of connected components: cyclic components and linear (acyclic) components. The next result shows that a single connected component of the overquiver can't give rise to multiple cyclic components of the pin graph of different sizes.

The details of the proof of the next result are quite complicated; the remark following the proof aims to give intuition for the argument.

2.3.42. Proposition. Let $\mathcal{C}$ be a connected component of the overquiver $\mathcal{O}$. Let i, j be vertices of $\mathcal{C}$. Suppose that the Q-vertices associated to i and j both lie on cycles of the pin graph Φ_A. Then the length of those cycles is the same.

Proof. Let us denote the Q-vertices associated to i and j by u and v respectively.

Suppose that u and v lie on distinct cycles, of lengths l_1 and l_2, on Φ_A. Suppose without loss of generality that $l_1 \leq l_2$.

Let $i_0 \coloneqq i$ and then inductively define $i_{n+1} \coloneqq i_n - a_{i_n} \in \mathcal{O}$ for all $n \in \mathbb{N}$. Similarly define j_n for all $n \in \mathbb{N}$. Then we know that $i_{n+l_1} = i_n$ and that $j_{n+l_2} = j_n$ for all $n \in \mathbb{N}$. It is clear that $i_0, \dots, i_{l_1-1}, j_0, \dots, j_{l_2-1}$ is a (not necessarily complete) collection of distinct vertices of $\mathcal{C}$, by the definition of l_1 and l_2.

Let $k_0 \in \mathbb{Z}_+$ be the minimal positive integer such that $i_0 - k_0$ is equal to one of the j_n. For simplicity of notation, we now assume, without loss of generality, that $i_0 - k_0 = j_0$, and that the indices of the remaining j_n are shifted accordingly.

We know that $k_0 \neq a_{i_0}$, else the i_n and j_n would not be distinct. Since $j_{l_2-1} - a_{j_{l_2-1}} = j_0$, we know that the $\mathcal{O}$-path ($j_{l_2-1} \xrightarrow{a_{j_{l_2-1}}} j_0$) is non-zero when considered as an A-path. Hence the strict suffix ($i_0 \xrightarrow{k_0} j_0$) is also non-zero when considered as an A-path. Therefore $k_0 < a_{i_0}$, and the $\mathcal{O}$-path ($i_0 \xrightarrow{a_{i_0}} i_1$) can be written as the product of two non-stationary paths ($i_0 \xrightarrow{k_0} j_0$) and ($j_0 \xrightarrow{a_{i_0} - k_0} i_1$).

Now, since we know that $j_0 - (a_{i_0} - k_0) = i_1$, we can apply the symmetric argument to see that $a_{i_0} - k_0 < a_{j_0}$, and the $\mathcal{O}$-path ($j_0 \xrightarrow{a_{j_0}} j_1$) can be written as the product of two non-stationary paths ($j_0 \xrightarrow{a_{i_0} - k_0} i_1$) and ($i_1 \xrightarrow{k_0 + (a_{j_0} - a_{i_0})} j_1$).

Inductively applying this argument, we see that for all $n \in \mathbb{N}$, we have:

(i) $i_n - (k_0 + \sum_{m=1}^{n}(a_{i_m} - a_{j_{m-1}})) = j_n$;

(ii) $k_0 + \sum_{m=1}^{n}(a_{i_m} - a_{j_{m-1}}) < a_{i_n}$;

(iii) $j_n - (a_{i_n} - k_0 - \sum_{m=1}^{n}(a_{i_m} - a_{j_{m-1}})) = i_{n+1}$;

(iv) $a_{i_n} + k_0 - \sum_{m=1}^{n}(a_{i_m} - a_{j_{m-1}}) < a_{j_n}$.

Now define $S \coloneqq \sum_{m=1}^{l_1}(a_{i_m} - a_{j_{m-1}})$ and consider $j_{l_1} = i_{l_1} - (k_0 + S) = i_0 - (k_0 + S)$. We handle the cases where $S < 0$, $S > 0$ and $S = 0$ separately.

Firstly, if $S < 0$, then k_0 was not chosen minimally, as $k_0 + S < k_0$. Thus we have a contradiction.

We now assume that $S > 0$.

Now we know that the non-zero $\mathcal{O}$-path ($i_0 \xrightarrow{a_{i_0}} i_1$) can be written as the product of the paths ($i_0 \xrightarrow{k_0} j_0$), ($j_0 \xrightarrow{S} j_{l_1}$) and ($j_{l_1} \xrightarrow{a_{i_0} - (k_0 + S)} i_1$). Applying the same argument as before, this means that the non-zero $\mathcal{O}$-path ($j_0 \xrightarrow{a_{j_0}} j_1$) can be written as the product of three non-stationary paths ($j_0 \xrightarrow{S} j_{l_1}$), ($j_{l_1} \xrightarrow{a_{i_0} - (k_0 + S)} i_1$) and ($i_1 \xrightarrow{k_0 + (a_{j_0} - a_{i_0})} j_1$). Repeating this argument, we see that the non-zero $\mathcal{O}$-path ($j_{l_1} \xrightarrow{a_{j_{l_1}}} j_{l_1+1}$) can be written as the product of three non-stationary paths ($j_{l_1} \xrightarrow{a_{i_0} - (k_0 + S)} i_1$), ($i_1 \xrightarrow{k_0 + (a_{j_0} - a_{i_0})} j_1$) and ($j_1 \xrightarrow{a_{j_{l_1}} - a_{j_0} + S} j_{l_1+1}$).

We can repeat this procedure to see that the non-zero $\mathcal{O}$-path ($j_{2l_1-1} \xrightarrow{a_{j_{2l_1-1}}} j_{2l_1}$) can be written as the product of three non-stationary paths ($j_{2l_1-1} \longrightarrow i_{l_1} = i_0$), ($i_0 = i_{l_1} \longrightarrow j_{l_1}$) and

($j_{l_1} \longrightarrow j_{2l_1}$) (we omit the lengths of these paths to avoid overcomplicated notation).

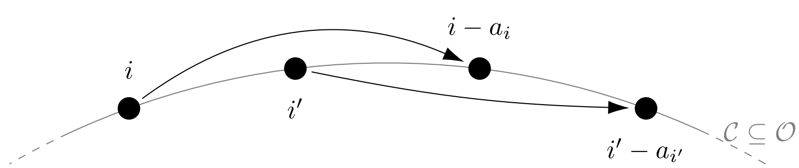
We already know that the path ($i_0 \longrightarrow j_{l_1}$) factors through j_0, it follows that the non-zero path ($i_0 \xrightarrow{a_{i_0}} i_1$) can be written as the product of the four non-stationary paths ($i_0 \longrightarrow j_0$), ($j_0 \longrightarrow j_{l_1}$), ($j_{l_1} \longrightarrow j_{2l_1}$) and ($j_{2l_1} \longrightarrow i_1$).

If we repeated this whole procedure again, we would see that the non-zero path ($i_0 \xrightarrow{a_{i_0}} i_1$) can be written as the product of five non-stationary paths. Repeating the whole procedure another $a_{i_0} - 4$ times shows that the non-zero path ($i_0 \xrightarrow{a_{i_0}} i_1$) can be written as the product of $a_{i_0} + 1$ non-stationary paths. This is clearly a contradiction.

The only case remaining is when $S = 0$. If $S = 0$, then $j_{l_1} = i_{l_1} - k_0 = i_0 - k_0 = j_0$. Thus $l_2 = l_1$, and our claim holds. $\square$

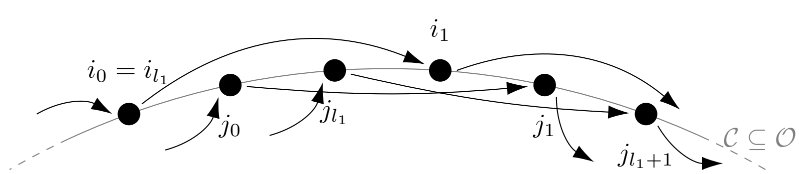
2.3.43. Remark. The key idea of the above proof is that if two vertices i, i' of the overquiver $\mathcal{O}$ satisfy $b_i = b_{i'} = 0$, and the path ($i \xrightarrow{a_i} \circ$) passes through i', then the path ($i' \xrightarrow{a_{i'}} \circ$) passes through $i - a_i$.

In other words, we have a "leap-frogging effect", illustrated in the diagram below:

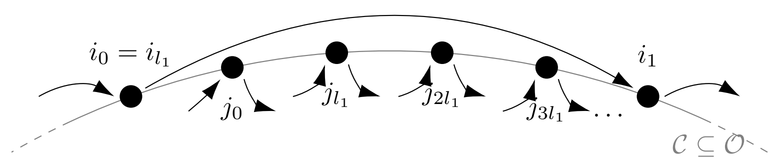


where arrows denote components of commutativity relations (in the sense of Definition 2.3.5).

If we have two different length cycles of Φ_A coming from one component of the overquiver, $\mathcal{O}$, then repeatedly applying the leap-frogging effect gives rise to a diagram like:



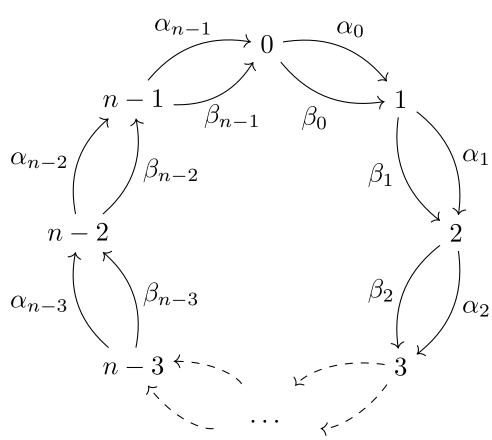
Repeating this process further results in an ever diminishing space for subsequent applications of the process to work in:



Since there are only finitely many vertices between i_0 and i_1, no such sequence of j_n can exist. Thus the length of cycles of Φ_A coming from one component of the overquiver must all be the same.

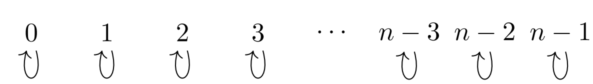
2.3.44. Note that while a single connected component of the overquiver can't give rise to multiple cyclic components of the pin graph of different sizes, it can give rise to arbitrarily many of the same size. The following example illustrates this (and for any $m \in \mathbb{Z}_+$, similar examples can be constructed to give pin graphs with an arbitrary number of connected components, all of size m).

2.3.45. Example. Let Q be the quiver on n vertices depicted below:



Let $A \coloneqq \Bbbk Q/\langle \alpha\beta, \beta\alpha, \alpha^n - \beta^n \rangle$ where the indices of the α and β are inserted into the relations as appropriate. It is clear that the overquiver of A consists of two cyclic components of length n; one corresponding to the α arrows, one corresponding to the β arrows.

The pin graph, Φ_A, of A is:



Moving on from this example, we now note that there is a relationship between the pin graph and the action of descent on syllables $\mathbf{p}$ with $b_{s(\mathbf{p})} = 0$.

2.3.46. Lemma. Let $\mathbf{p}$ be a syllable of A where $b_{s(\mathbf{p})} = 0$. Then there is an arrow in Φ_A from the Q-vertex corresponding to $s(\mathbf{p})$ to the Q-vertex corresponding to $t(\nabla \mathbf{p})$.

Proof. Since $b_{s(\mathbf{p})} = 0$, we know that $t(\nabla \mathbf{p}) = s(\mathbf{p}) - a_{s(\mathbf{p})}$. This means that the projective corresponding to the same Q-vertex as $s(\mathbf{p})$ is pin, and that the socle of this projective corresponds to the same Q-vertex as $t(\mathbf{p})$, as required. $\square$

This lemma is particularly useful when the Q-vertices corresponding to the source and target of the syllable both lie on a cycle of Φ_A.

2.3.47. Proposition. Let $\mathbf{p}$ be a syllable of A where the Q-vertices corresponding to $s(\mathbf{p})$ and $t(\mathbf{p})$ both lie on cycles of the pin graph, Φ_A. Then the length of those cycles are the same.

Furthermore, if l is the common length of the cycles, then:

$$\nabla^{2l}\mathbf{p} = \left(\ s(\mathbf{p}) \xrightarrow{\mathrm{len}(\mathbf{p})} \circ \xrightarrow{0} t(\mathbf{p})\ \right).$$

Proof. The fact that the length of the cycles is the same follows immediately from Proposition 2.3.42. Now, let l be the common length of these cycles. The fact that the source and target of $\nabla^{2l}\mathbf{p}$ are as claimed follows from Lemma 2.3.46 and the fact that Φ_A is sub-1-regular. It is also clear that the stability term of $\nabla^{2l}\mathbf{p}$ is zero; i.e. that $\nabla^{2l}\mathbf{p}$ is an interior syllable.

It remains to show that $\mathrm{len}(\mathbf{p}) = \mathrm{len}(\nabla^{2l}\mathbf{p})$. Following the notation of Proposition 2.3.42, let $i_0 \coloneqq s(\mathbf{p})$, and then inductively define $i_{n+1} \coloneqq i_n - a_{i_n} \in \mathcal{O}$ for all $n \in \mathbb{N}$. Similarly define j_n inductively with $j_0 \coloneqq t(\mathbf{p})$.

Since there is a non-zero $\mathcal{O}$-path from i_0 to j_0, we know that properties (i) - (iv) hold. Combining properties (ii) and (iv) yields the following inequalities for all $n \in \mathbb{N}$:

$$a_{i_n} + k_0 - a_{j_n} < \sum_{m=1}^{n} (a_{i_m} - a_{j_{m-1}}) < a_{i_n} - k_0.$$

Since for each $m \in \mathbb{N}$, $i_{m+l} = i_m$ and $j_{m+l} = j_m$, it follows that:

$$\sum_{m=1}^{n_0 l}(a_{i_m} - a_{j_{m-1}}) = n_0 \cdot \sum_{m=1}^{l}(a_{i_m} - a_{j_{m-1}}).$$

Combining this with the above inequalities implies that $\sum_{m=1}^{l}(a_{i_m} - a_{j_{m-1}}) = 0$. Since the i_n and j_n are periodic with period l, we can rearrange this sum to see that:

$$\sum_{m=0}^{l-1}(a_{i_m} - a_{j_m}) = 0.$$

Since for all $n \in \mathbb{N}$ we have $b_{i_n} = b_{j_n} = 0$, applying the definition of descent, ∇, repeatedly gives:

$$\nabla^{2n}\mathbf{p} = (\ i_n \xrightarrow{\operatorname{len}(\mathbf{p}) - \sum_{m=0}^{n-1}(a_{i_m} - a_{j_m})} \circ \xrightarrow{0} j_n\ ).$$

Since we have shown that $\sum_{m=0}^{l-1}(a_{i_m} - a_{j_m}) = 0$, and we know that $i_l = i_0 = s(\mathbf{p})$ and $j_l = j_0 = t(\mathbf{p})$, the result follows. $\square$

2.4 Patches and strips

In this section we complete our overview of the methods of [All21], focusing on how the syllables of the last section can be used to represent;

- indecomposable projective A-modules (using (2×2)-grids called *patches*); and
- string modules of A (using rows called *strips*).

While most of the ideas in this section are from [All21], there are some extensions of these ideas that are original to this thesis. For example, we include a bound on the number of string modules of a given "size", beginning in Paragraph 2.4.19. We also include a description of the action of the transpose functor, Tr, on string modules in terms of its effect on strips, beginning in Paragraph 2.4.28.

All definitions and results that are not original are cited as such.

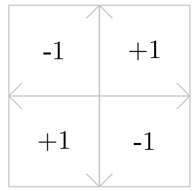


Figure 2.5: *Orientations of syllables in patches.*

2.4.1 Patches

We first introduce our set of patches for an SB algebra $A = \Bbbk Q/I$. Throughout, we have a fixed set of permissible data $(\mathcal{O}, N, C, \dagger)$ for A, and we let $(a_i)_{i\in\mathcal{O}}, (b_i)_{i\in\mathcal{O}}, (c_i)_{i\in\mathcal{O}}, (d_i)_{i\in\mathcal{O}}$ be the sequences given by the source and target encodings of this permissible data.

Each patch will be a (2×2)-grid populated with syllables with orientations as in Figure 2.5. Since the orientations of syllables are specified by their position, we will not indicate the orientations of these syllables going forward.

Following the framework of [All21, Section 4.1.1], we split our construction of patches into five cases, and give an example of each case at the end:

2.4.1. Blank patch. There is a single patch with all of its cells blank, which we call the *blank patch*.

2.4.2. Patches with no pin-boundary syllable in the top row. For every pair of syllables $(\mathbf{p}, \mathbf{p}')$, neither of which is pin-boundary, and which satisfy $s(\mathbf{p})^\dagger = s(\mathbf{p}')$, we construct a patch. On one side it has $\nabla\mathbf{p}$ below $\mathbf{p}$, on the other it has $\nabla\mathbf{p}'$ below $\mathbf{p}'$.

From this patch, we create additional *amended patches* if either of $\mathbf{p}, \mathbf{p}'$ is stationary. In either case, we create a copy of the original patch, replacing one of the stationary syllables with the blank syllable and leaving the other cells unchanged. Note that we don't perform this replacement simultaneously if both of $\mathbf{p}, \mathbf{p}'$ are stationary, and only create two amended patches in this case.

2.4.3. Patches with one pin-boundary syllable in the top row. For every pin-boundary syllable $\mathbf{p}$ and non-pin-boundary syllable $\mathbf{p}'$ satisfying $s(\mathbf{p})^\dagger = s(\mathbf{p}')$, we define $\mathbf{q}'$ to be the perturbation of $\nabla\mathbf{p}'$. We then construct two patches. The first has a blank cell below $\mathbf{p}$ on one side, and $\mathbf{q}'$ below $\mathbf{p}'$ on the other. The second is its reflection.

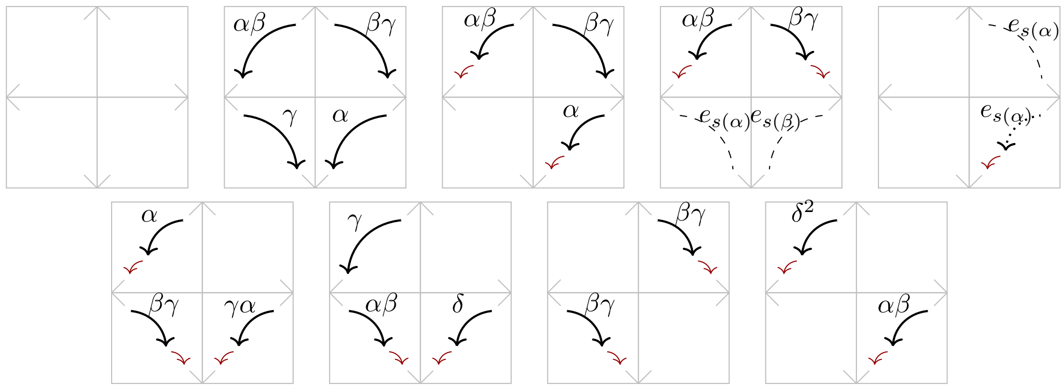


Figure 2.6: *Examples of patches for our running example algebra..*

For each of these patches, we create additional *amended patches* if $\mathbf{p}'$ is stationary. Similarly to the previous case, we create a copy of these patches where $\mathbf{p}'$ is replaced with the blank syllable.

2.4.4. Patches with two pin-boundary syllables in the top row. For each pair of pin-boundary syllables $\mathbf{p}, \mathbf{p}'$ satisfying $s(\mathbf{p})^\dagger = s(\mathbf{p}')$, we define the *virtual syllables* $\mathbf{q} = (\, t(\mathbf{p}) \xrightarrow{0} \circ \xrightarrow{0} \circ \,)$ and $\mathbf{q}' = (\, t(\mathbf{p}') \xrightarrow{0} \circ \xrightarrow{0} \circ \,)$. We then create a patch with $\mathbf{q}$ below $\mathbf{p}$ on one side and $\mathbf{q}'$ below $\mathbf{p}'$ on the other side.

2.4.5. Virtual patches. For each vertex $i \in \mathcal{O}_0$ with $d_i = 0$, we construct two *virtual patches*. The first has a virtual syllable $(\, i \xrightarrow{0} \circ \xrightarrow{0} \circ \,)$ in the top-left cell with the corresponding stationary syllable $(\, i \xrightarrow{0} \circ \xrightarrow{1} \circ \,)$ in the bottom-left cell, and the remaining cells blank. The second is its reflection.

2.4.6. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Then the top row of Figure 2.6 depicts a single (unamended) patch in each of the above cases, while the bottom row depicts a selection of amended patches.

2.4.7. Remark. For more detail on patches, and further examples, we refer the reader to [All21, Section 4.1.1].

2.4.8. Properties of patches. The remainder of this section is a brief rundown of the technical results from [All21, Section 4.1.2], as we will need them later.

2.4.9. Lemma. [All21, Lem 4.1.15]

The only patches containing virtual syllables are virtual patches (having one such, which lies in the top row) and patches with two pin-boundary syllables in the top row (having two such, in the bottom row).

2.4.10. Lemma. [All21, Lem 4.1.16]

Let X be a patch with entries $\mathbf{p}$ and $\mathbf{p}'$ in the top row, and with entry $\mathbf{q}$ under $\mathbf{p}$.

(a) If neither $\mathbf{p}$ nor $\mathbf{p}'$ is blank, then $s(\mathbf{p})^\dagger = s(\mathbf{p}')$.

(b) If $\mathbf{p}$ is non-blank, then $\mathbf{q} = \nabla \mathbf{p}$ unless

  (i) both $\mathbf{p}, \mathbf{p}'$ are pin-boundary (in which case $\mathbf{q}$ is a virtual syllable), or

  (ii) $\mathbf{p}'$ is pin-boundary and $\mathbf{p}$ is not (in which case $\mathbf{q}$ is the perturbed version of $\nabla \mathbf{p}$), or

  (iii) $\mathbf{p}$ is a virtual syllable (in which case $\mathbf{q}$ is the corresponding stationary syllable).

(c) If $\mathbf{p}$ is blank, then $\mathbf{q}$ is either blank or is $\nabla \mathbf{e}_i$ for some stationary syllable $\mathbf{e}_i$.

(d) If $\mathbf{p} = (\, i \xrightarrow{l} \circ \xrightarrow{0} \circ \,)$, then $s(\mathbf{q}) = t(\mathbf{p})$.

2.4.11. Lemma. [All21, Lem 4.1.17]

No two patches have the same top row.

2.4.12. Projective associated to a patch. [All21, Paragraph 4.1.18]

To each non-virtual patch, X, we can associate either an indecomposable projective or the zero module.

If the patch is blank, then it is associated to the zero module. If the patch is non-blank (and non-virtual), then it has at least one non-blank syllable $\mathbf{p}$ in its top row. Then $s(\mathbf{p})$ corresponds to a Q-vertex, i; if there is another non-blank syllable $\mathbf{p}'$ in the top row, then $s(\mathbf{p})^\dagger = s(\mathbf{p}')$, and thus they correspond to the same Q-vertex. Hence it is well-defined to associate X to the projective $P_i = e_i A$.

2.4.2 Strips

We now introduce our encoding of string modules in terms of syllables, following the definitions given in [All21, Section 4.2.1].

2.4.13. Peaks and valleys. [All21, Def 4.2.9]

Let $(\mathbf{p}, \mathbf{q})$ be a pair of (non-virtual) syllables for A.

(a) We say $\mathbf{p}$ and $\mathbf{q}$ are *peak compatible* if: both are blank (this is called a *blank peak*), or exactly one is blank (an *implied peak*), or neither is blank, $s(\mathbf{p})^\dagger = s(\mathbf{q})$ and the orientations of $\mathbf{p}$ and $\mathbf{q}$ are respectively -1 and $+1$ (an *interior peak*).

(b) We say $\mathbf{p}$ and $\mathbf{q}$ are *valley compatible* if: both are blank (this is called a *blank valley*), or one is blank and the other is boundary (a *boundary valley*), or both are interior and $t(\mathbf{p})^\dagger = t(\mathbf{q})$ and the orientations of $\mathbf{p}$ and $\mathbf{q}$ are respectively -1 and $+1$ (an *interior valley*).

A *peak* is a pair of peak compatible syllables, often denoted as $\boxed{\mathbf{p} \uparrow \mathbf{q}}$, leaving the orientations implicit. Similarly, a *valley* is a pair of valley compatible syllables, denoted as $\boxed{\mathbf{p} \downarrow \mathbf{q}}$. This notation makes defining their *reflections* as $\boxed{\mathbf{q} \uparrow \mathbf{p}}$, and $\boxed{\mathbf{q} \downarrow \mathbf{p}}$ obvious.

2.4.14. Strips. [All21, Def 4.2.11]

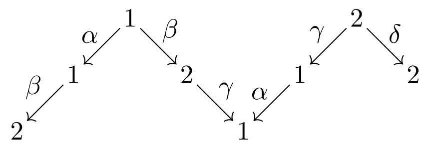
A *strip* w is a concave, not necessarily bounded juxtaposition of (non-virtual) syllables alternately forming peaks and valleys (here concave means that no blank syllable is between two non-blank syllables).

Formally we consider the juxtaposition to be a single row of cells with columns indexed by $\mathbb{Z}$. Thus, a strip w is a function $w : \mathbb{Z} \to \mathrm{Syll}(A)$. The *entry* of cell k is $w(k)$.

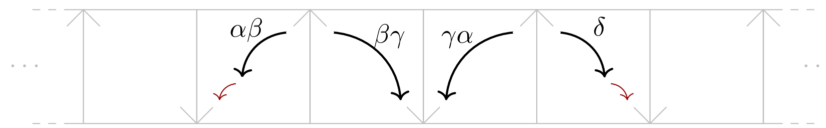
The *support*, $\mathrm{supp}(w)$, of a strip w is the interval subset of $\mathbb{Z}$ where the entries are non-blank. The *interior*, $\mathrm{int}(w)$, of a strip w is the interval subset of $\mathbb{Z}$ where the entries are interior syllables. The *interior width*, $\mathrm{intwid}(w)$, of a strip w is defined as $\mathrm{intwid}(w) \coloneqq |\,\mathrm{int}(w)|$.

The *reflection* of a strip $w : \mathbb{Z} \to \mathrm{Syll}(A)$ is obtained by precomposing with the reflection $k \mapsto -k$ of $\mathbb{Z}$ and postcomposing with the orientation involution $\mathbf{p}^s \mapsto \mathbf{p}^{-s}$ of $\mathrm{Syll}(A)$.

2.4.15. Any strip represents a string graph (and hence a string module), as formalised by the following:



(a) *An indecomposable string graph associated to a string module of our running example algebra.*



(b) *A strip corresponding to the above string graph.*

Figure 2.7: *Representing an indecomposable string graph with a strip.*

2.4.16. Proposition. [All21, Prop 4.2.17]

Any strip w represents a well-defined indecomposable string graph, hence an indecomposable string module, and moreover w and its reflection both represent the same string graph.

Conversely, any indecomposable string graph can be represented by a strip.

While we won't go into an explicit discussion of the association of a string graph to a strip (we refer the reader to the above reference for that), we will give a small example which is indicative and leads to the correct intuition.

2.4.17. Example. In Figure 2.7, the first diagram is a string graph for our running example algebra A (the same string graph as in Figure 2.2(a)). The second diagram is a strip representing this string graph.

2.4.18. Rounding off a strip. [All21, Paragraph 4.2.3]

However, Proposition 2.4.16 doesn't completely classify when two strips represent the same string graph. We also have to handle "rounding off".

When a strip contains an implied peak, we can replace the blank syllable in that peak with the unique stationary syllable that is still peak compatible. This process (which we call *rounding off*) does not change the string graph (and hence string module) represented by a strip.

We will almost always work with rounded off strips, as they are more convenient for performing syzygy calculations.

2.4.19. One of the advantages of this formalism for representing string graphs, is that it allows us to completely enumerate all string graphs of a given size (where "size" here means interior width). The following few results provide bounds on how many strips (or string modules) there are of a given size.

2.4.20. Proposition. Let $d \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. Then:

$$\left|\left\{\mathrm{Str}(w) \,:\, \mathrm{intwid}(w) = 2d, \mathrm{rad}^{r+1}(\mathrm{Str}(w)) = 0\right\}\right| \leq n \cdot (r+1)^2 \cdot r^{2d}$$

where n is the number of simple modules of A.

Proof. For any strip, we can uniquely encode it by the following data:

- the vertex of $\mathcal{O}$ that is the source of the left boundary syllable,
- the ordered sequence of lengths of underlying paths of the interior syllables,
- the ordered pair of lengths of underlying paths of the two boundary syllables.

There are clearly at most $|\mathcal{O}| = 2n$ options for the first piece of data. For the second piece of data, there is a ordered sequence of $2d$ choices, each of which has at most r choices (the length can be any value in $\{1, \ldots, r\}$). Thus there are at most r^{2d} choices for the second piece of data. For the third piece of data, there is a ordered pair of 2 choices, each of which has at most $r+1$ choices (the length can be any value in $\{0, 1, \ldots, r\}$).

Thus it follows that there are at most $2n \cdot (r+1)^2 \cdot r^{2d}$ choices of encoding.

To show the required bound holds, it is now sufficient to show that each isomorphism class of a string module in our set has exactly two choices to encode it. Recall from Proposition 2.4.16 that a strip and its reflection both represent the same string module. Hence it is sufficient to show that no strip is equal to its own reflection.

Suppose that w is a strip which is equal to its own reflection. This means that there are two adjacent non-blank syllables of w which have identical underlying path and stability term, and opposite orientation. However, no non-blank syllable is peak or valley compatible in this way. Hence we have a contradiction, as required. $\square$

2.4.21. Corollary. Let $d \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. Let n denote the number of simple modules of A. Then, if $r > 1$:

$$\left|\left\{\mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{intwid}(w) \leq 2d, \mathrm{rad}^{r+1}(\mathrm{Str}(w)) = 0\right\}\right| \leq n \cdot (r+1) \cdot \frac{r^{2d}-1}{r-1}$$

while if $r = 1$,

$$\left|\left\{\mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{intwid}(w) \leq 2d, \mathrm{rad}^{r+1}(\mathrm{Str}(w)) = 0\right\}\right| \leq n \cdot (r+1)^2 \cdot (d+1).$$

Proof. First we fix notation $N = |\{\mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{intwid}(w) \leq 2d, \mathrm{rad}^{r+1}(\mathrm{Str}(w)) = 0\}|$.

Assume that $r > 1$. Since $\mathrm{intwid}(w) \in 2\mathbb{N}$ for all strips w, applying Proposition 2.4.20 gives:

$$N \leq n \cdot (r+1)^2 \cdot \sum_{k=0}^{d} r^{2k} = n \cdot (r+1)^2 \cdot \frac{r^{2d}-1}{r^2-1} = n \cdot (r+1) \cdot \frac{r^{2d}-1}{r-1},$$

as required.

Now instead suppose that $r = 1$. Since $\mathrm{intwid}(w) \in 2\mathbb{N}$ for all strips w, applying Proposition 2.4.20 gives:

$$N \leq n \cdot (r+1)^2 \cdot \sum_{k=0}^{d} r^{2k} = n \cdot (r+1)^2 \cdot \sum_{k=0}^{d} 1 = n \cdot (r+1)^2 \cdot (d+1),$$

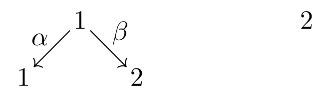
as required. $\square$

2.4.22. Example. It is easy to verify that for any member of the family of SB algebras (indexed by $n \in \mathbb{Z}_+$) given in Example 2.3.45, each of the bounds in Proposition 2.4.20 and Corollary 2.4.21 is sharp when $r < n$.

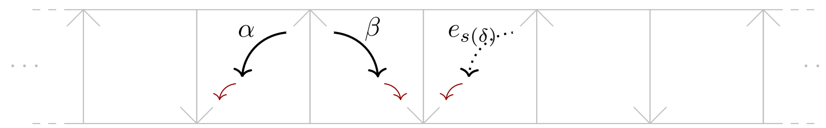
While we mostly deal directly with string modules (which by our definition are indecomposable), it is sometimes helpful to be able to encode direct sums of string modules in a similar way to strips. We can do this by extending our definition of valleys.

2.4.23. Pseudo-valleys. Let $(\mathbf{p}, \mathbf{q})$ be a pair of (non-virtual) syllables for A. We say $\mathbf{p}$ and $\mathbf{q}$ are *pseudo-valley compatible* if: they are valley compatible or both are boundary.

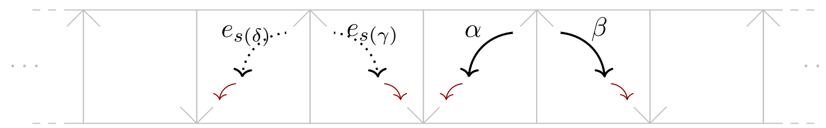
A *pseudo-valley* is a pair of pseudo-valley compatible syllables, denoted similarly to a valley. Its reflection is also defined analogously.



(a) *A string graph associated to a string module of our running example algebra.*



(b) *A flattened family of strips corresponding to the above string graph.* Note that this flattened family of strips contains an implied peak, so could be rounded off without changing the string graph represented.



(c) *Another flattened family of strips corresponding to the above string graph.* Note that this flattened family of strips has no implied peaks; also note that the components of the string graph are depicted in a different order to the above.

Figure 2.8: *Representing a string graph with a flattened family of strip.*

2.4.24. Flattened families of strips. A *flattened family of strips* is defined identically to a strip (see Definition 2.4.14) apart from replacing all instances of "valley" with "pseudo-valley".

2.4.25. Remark. A flattened family of strips can be thought of as a family of strips arranged boundary to boundary. Note in particular that we still require the collection of non-blank syllables to be concave.

2.4.26. Example. In Figure 2.8, the first diagram is a string graph for our running example algebra A. The second and third diagram depict two different flattened families of strips representing this string graph.

2.4.27. Remark. Observe from Example 2.4.26 that a string graph which is not indecomposable can be represented as a flattened family of strips in multiple ways which can't be reached from each other by reflections and rounding off. Hence there is no equivalent result to Proposition 2.4.16 for flattened families of strips.

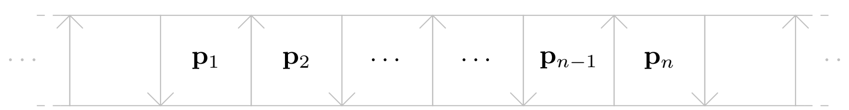
2.4.28. The remainder of this section focuses on a characterisation of the action of the transpose functor $\mathrm{Tr} : \text{mod-}A \to \text{mod-}A^{\mathrm{op}}$ on string modules, in terms of its effect on strips. Propositions 2.4.30–2.4.32 handle all non-projective string modules between them, relying on the construction of $\operatorname{Tr} D$ from [WW85]. Thus we first give a remark explaining how the notation of [WW85] translates into our notation of strips and syllables.

2.4.29. Remark. In [WW85], they call string modules "representations of the first kind", and represent them in terms of "V-sequences". These "V-sequences" are analogous to string graphs, and can therefore be uniquely identified with a strip (up to reflections and rounding off).

The construction for $\operatorname{Tr} D$ takes a string graph which is thought of as "W-shaped" as an input, which is exactly the way we would think of a dual of a string module represented by a strip (see Lemma 2.2.46 and Example 2.2.47).

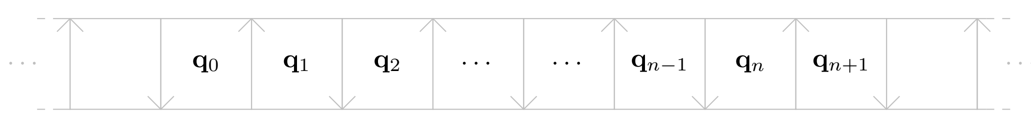
The other key notion used in the construction is whether a "V-sequence" is "extendable". This corresponds exactly with the cases where the underlying path of a syllable does not have maximal length, i.e. the syllable belongs to $\operatorname{supp}(\nabla)$.

2.4.30. Proposition. Suppose that $M \in \text{mod-}A$ is a string module with $\operatorname{intwid}(M) > 0$, represented by a strip of the form:



We denote the compression of each $\mathbf{p}_i$ by p_i. We also denote the target of each p_i by t_i (note that t_i may not be the same as the target of the underlying path of $\mathbf{p}_i$ when the syllable is boundary).

Then $\operatorname{Tr}(M) \in \text{mod-}A^{\text{op}}$ is a string module represented by a strip of the form:



where $\mathbf{q}_3 = \mathbf{p}_3^{\text{op}}, \ldots, \mathbf{q}_{n-2} = \mathbf{p}_{n-2}^{\text{op}}$, for $i \in \{0, 1, 2\}$ the $\mathbf{q}_i$ are defined as:

$$\mathbf{q}_0 \coloneqq \begin{cases} \left(\, (t_1^{\text{op}})^\dagger \xrightarrow{c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1} \circ \xrightarrow{1} \circ \, \right) & \text{if } \mathbf{p}_1 \in \operatorname{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_1 \notin \operatorname{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_1 \coloneqq \begin{cases} (\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)} \circ \xrightarrow{0} \circ\ ) & \text{if } \mathbf{p}_1 \in \mathrm{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_1 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_2 \coloneqq \begin{cases} \mathbf{p}_2^{\mathrm{op}} & \text{if } \mathbf{p}_1 \in \mathrm{supp}(\nabla_A) \\ (\ t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2)-1} \circ \xrightarrow{1} \circ\ ) & \text{if } \mathbf{p}_1 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

and for $i \in \{n-1, n, n+1\}$, the $\mathbf{q}_i$ are defined symmetrically as:

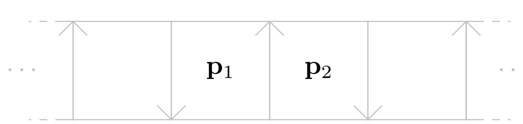
$$\mathbf{q}_{n+1} \coloneqq \begin{cases} (\ (t_n^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_n)^\dagger} + d_{(t_n)^\dagger} - 1} \circ \xrightarrow{1} \circ\ ) & \text{if } \mathbf{p}_n \in \mathrm{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_n \notin \mathrm{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_n \coloneqq \begin{cases} (\ t_n^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_n)} \circ \xrightarrow{0} \circ\ ) & \text{if } \mathbf{p}_n \in \mathrm{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_n \notin \mathrm{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_{n-1} \coloneqq \begin{cases} \mathbf{p}_{n-1}^{\mathrm{op}} & \text{if } \mathbf{p}_n \in \mathrm{supp}(\nabla_A) \\ (\ t_{n-1}^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_{n-1})-1} \circ \xrightarrow{1} \circ\ ) & \text{if } \mathbf{p}_n \notin \mathrm{supp}(\nabla_A) \end{cases}$$

Proof. This result follows immediately from [WW85, Section 3] when considered in the language of syllables introduced in [All21]. This selection of strips corresponds to the case where $m \geq 3$ in their argument (note that m is always odd). In this case the $(-)_1$ and $(-)_2$ constructions are always non-empty, and will never interact with each other. $\square$

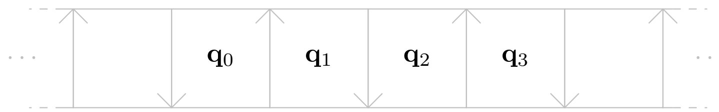
2.4.31. Proposition. Suppose that $M \in \text{mod-}A$ is a non-projective string module with $\mathrm{intwid}(M) = 0$, represented by a strip of the form:



where $\mathbf{p}_1 \in \mathrm{supp}(\nabla_A)$.

We denote the compression of each $\mathbf{p}_i$ by p_i. We also denote the target of each p_i by t_i (note that t_i may not be the same as the target of the underlying path of $\mathbf{p}_i$ when the syllable is boundary).

Then $\mathrm{Tr}(M) \in \text{mod-}A^{\mathrm{op}}$ is a string module represented by a strip of the form:



where $\mathbf{q}_i$ are defined as follows:

$$\mathbf{q}_0 \coloneqq \left(\ (t_1^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1} \circ \xrightarrow{1} \circ\ \right)$$

$$\mathbf{q}_1 \coloneqq \begin{cases} \left(\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)} \circ \xrightarrow{0} \circ\ \right) & \text{if } \mathbf{p}_2 \in \mathrm{supp}(\nabla_A) \\ \left(\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1) - 1} \circ \xrightarrow{1} \circ\ \right) & \text{if } \mathbf{p}_2 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

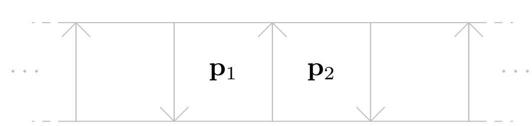
$$\mathbf{q}_2 \coloneqq \begin{cases} \left(\ t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2)} \circ \xrightarrow{0} \circ\ \right) & \text{if } \mathbf{p}_2 \in \mathrm{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_2 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_3 \coloneqq \begin{cases} \left(\ (t_2^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_2)^\dagger} + d_{(t_2)^\dagger} - 1} \circ \xrightarrow{1} \circ\ \right) & \text{if } \mathbf{p}_2 \in \mathrm{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_2 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

Proof. This result follows immediately from [WW85, Section 3] when considered in the language of syllables introduced in [All21]. This selection of strips corresponds to the case where $m = 1$ in their argument and the $(-)_1$ construction increases the size of the string graph. In this case the $(-)_1$ construction is always non-empty, and can thus be computed first. $\square$

Note that if you are considering a strip for M with $\mathrm{intwid}(M) = 0$, and that strip has $\mathbf{p}_2 \in \mathrm{supp}(\nabla_A)$, you can reflect the strip to apply this result (as reflected strips always represent the same string modules).

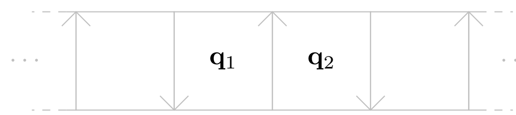
2.4.32. Proposition. Suppose that $M \in \text{mod-}A$ is a non-projective string module with $\mathrm{intwid}(M) = 0$, represented by a strip of the form:



where $\mathbf{p}_1, \mathbf{p}_2 \notin \mathrm{supp}(\nabla_A)$.

We denote the compression of each $\mathbf{p}_i$ by p_i. We also denote the target of each p_i by t_i (note that t_i may not be the same as the target of the underlying path of $\mathbf{p}_i$ when the syllable is boundary).

Then $\mathrm{Tr}(M) \in \text{mod-}A^{\mathrm{op}}$ is a string module represented by a strip of the form:



where $\mathbf{q}_i$ are defined as follows:

$$\mathbf{q}_1 \coloneqq (\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)-1} \circ \xrightarrow{1} \circ\ ), \qquad \mathbf{q}_2 \coloneqq (\ t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2)-1} \circ \xrightarrow{1} \circ\ ).$$

Proof. Note that since M is non-projective, it is necessary for $b_{s(\mathbf{p}_i)} = 0$ for $i = 1, 2$.

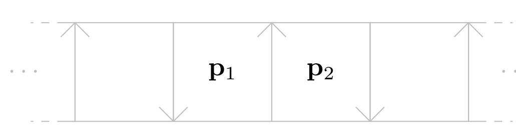
This result follows immediately from [WW85, Section 3] when considered in the language of syllables introduced in [All21]. This selection of strips corresponds to the case where $m = 1$ in their argument and both the $(-)_1$ and $(-)_2$ constructions reduce the size of the string graph. $\square$

The fact that these propositions cover all non-projective string modules follows from the fact that projective string modules M always have $\mathrm{intwid}(M) = 0$.

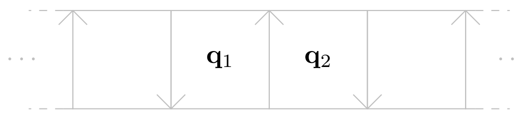
This characterisation also makes the following corollary much easier to notice.

2.4.33. Corollary. Suppose that $M \in \text{mod-}A$ is a non-projective string module. Then M is a socle-quotient of a pin A-module if and only if $\mathrm{Tr}(M)$ is a socle-quotient of a pin A^{op}-module.

Proof. First suppose that M is a socle-quotient of a pin A-module. Then M is represented by a strip of the form required for Proposition 2.4.32. In particular, M is represented by a strip of the form:



where $\mathbf{p}_i = (\ s_i \xrightarrow{a_{s_i}-1} \circ \xrightarrow{1} t_i\ )$ for $i = 1, 2$ and $b_{s_i} = 0$ for $i = 1, 2$. Thus applying Proposition 2.4.32, we know that $\mathrm{Tr}(M) \in \text{mod-}A^{\mathrm{op}}$ is a string module represented by a strip of the form:



where $\mathbf{q}_i = (\ (t_i)^{\mathrm{op}} \xrightarrow{a_{s_i}-1} \circ \xrightarrow{1} (s_i)^{\mathrm{op}}\ )$ for $i = 1, 2$. Since $a_{s_i} = c_{t_i} (= a_{(t_i)^{\mathrm{op}}})$ and $0 = d_{t_i} (= b_{(t_i)^{\mathrm{op}}})$ for $i = 1, 2$, this means that $\mathrm{Tr}(M)$ is a socle-quotient of a pin A^{op}-module.

For the converse, suppose that $\mathrm{Tr}(M)$ is a socle-quotient of a pin A^{op}-module. Then by applying the first part, we know that $\mathrm{Tr}^2(M)$ is a socle-quotient of a pin A-module. Since $M \simeq \mathrm{Tr}^2(M) \in \underline{\mathrm{mod}}\text{-}A$, the result follows. $\square$

2.4.3 The syzygy fabric

We now briefly review the results of [All21, Sections 4.2.2-4.2.3]. For proofs and more examples, we point the reader to this reference.

We first note that there is a correspondence between peaks and patches.

2.4.34. Proposition. [All21, Prop 4.2.18]

Any peak $\mathbf{p}$ $\mathbf{q}$ is the top row of exactly one patch.

This gives us a way to associate a line of patches to a strip.

2.4.35. Corollary. [All21, Cor 4.2.19]

Any strip is the top row of a well-defined line of patches.

This line of patches corresponds to a projective cover of the associated string module in an obvious way.

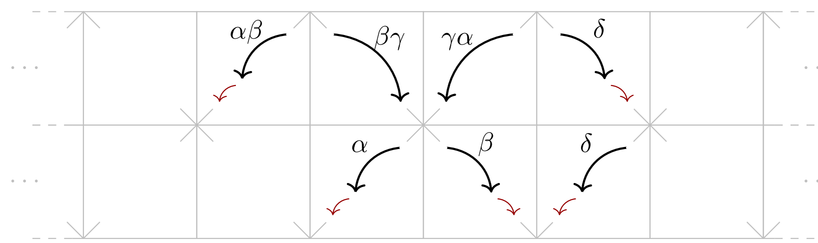
2.4.36. Proposition. [All21, Prop 4.2.20]

If the strip w is the top row of a line of patches, then $\mathbb{P}(\mathrm{Str}(w))$ is the direct sum of the associated projectives (using the association given in Paragraph 2.4.12).

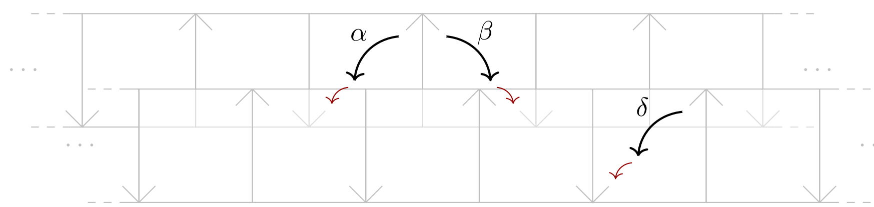
This gives a canonical way to assign a line of patches to a strip.

2.4.37. Patch covers. [All21, Def 4.2.22]

We define the *patch cover* $\mathbb{P}(w)$ of a strip w to be the corresponding line of patches, given in Corollary 2.4.35. (Note that this line is "bi-infinite" and features blank patches on either end as required.)



(a) *An example of a patch cover.*



(b) *Strips corresponding to summands of a flattened family of strips.* These strips represent summands of the string module corresponding to the bottom row of (a).

Figure 2.9: *Example of a patch cover and summands of syzygies using (flattened families of) strips.*

We can use these patch covers to calculate syzygies of string modules in terms of strips (and flattened families of them). The following is exactly equivalent to [All21, Prop 4.2.24], but uses the language of "flattened families of strips" (see Definition 2.4.24) instead of "segments".

2.4.38. Proposition. Suppose that the strip w does not represent the socle-quotient of a pin module.

(a) The bottom row of $\mathbb{P}(w)$ is a valid flattened family of strips, w'.

(b) That flattened family of strips, w', represents the syzygy, $\Omega(\mathrm{Str}(w))$.

2.4.39. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Then Figure 2.9(a) depicts an example of a patch cover. Figure 2.9(b) then depicts the two strips corresponding to the parts of the flattened family of strips in the bottom row of (a). By Proposition 2.4.41, these are the summands of the syzygy of the string modules corresponding to the top row of (a).

2.4.40. Since the above result only applies to strips that don't represent socle-quotients of pin modules, we need a separate result to handle calculating the syzygies of socle-quotients of pin modules in this formalism.

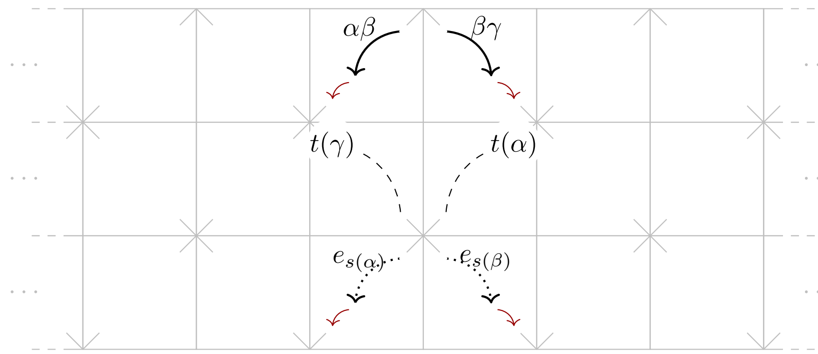


Figure 2.10: *Example of a syzygy of a socle-quotient of a pin module.*

2.4.41. Proposition. [All21, Prop 4.2.28]

If the strip w represents the socle-quotient $P/\operatorname{soc}(P)$ of a pin module P, then the bottom row of $\mathbb{P}(w)$ is blank except for two virtual syllables.

Moreover, the bottom row of $\mathbb{P}(w)$ is the top row of a new line of (virtual and blank) patches whose bottom row is a strip representing the simple module $\operatorname{soc}(P) = \Omega(P/\operatorname{soc}(P))$.

2.4.42. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Then Figure 2.10 depicts an example of the two lines of patches constructed when applying Proposition 2.4.41. In fact, since there is a unique pin module of A (up to isomorphism), this is the only arrangement of patches resulting from Proposition 2.4.41 for A (up to reflection).

2.4.43. Syzygy fabric for a strip. [All21, Def 4.2.29]

Let w be a strip representing a string module. By iteratively applying the above algorithm we can construct an array which contains strips representing all the syzygies of a given string module. We call this array the *syzygy fabric* associated to w.

Each row of this array is a strip indexed by the vertices of a rooted tree $\mathcal{T}$. The columns of the array are indexed by $\mathbb{Z}$ (corresponding to the indices of syllables in each strip).

The row corresponding to the root of $\mathcal{T}$ contains w. For a row corresponding to $t \in \mathcal{T}_0$, containing a strip v, the child vertices of t in $\mathcal{T}$ correspond to the strips contained in the flattened family of strips at the bottom row of $\mathbb{P}(v)$.

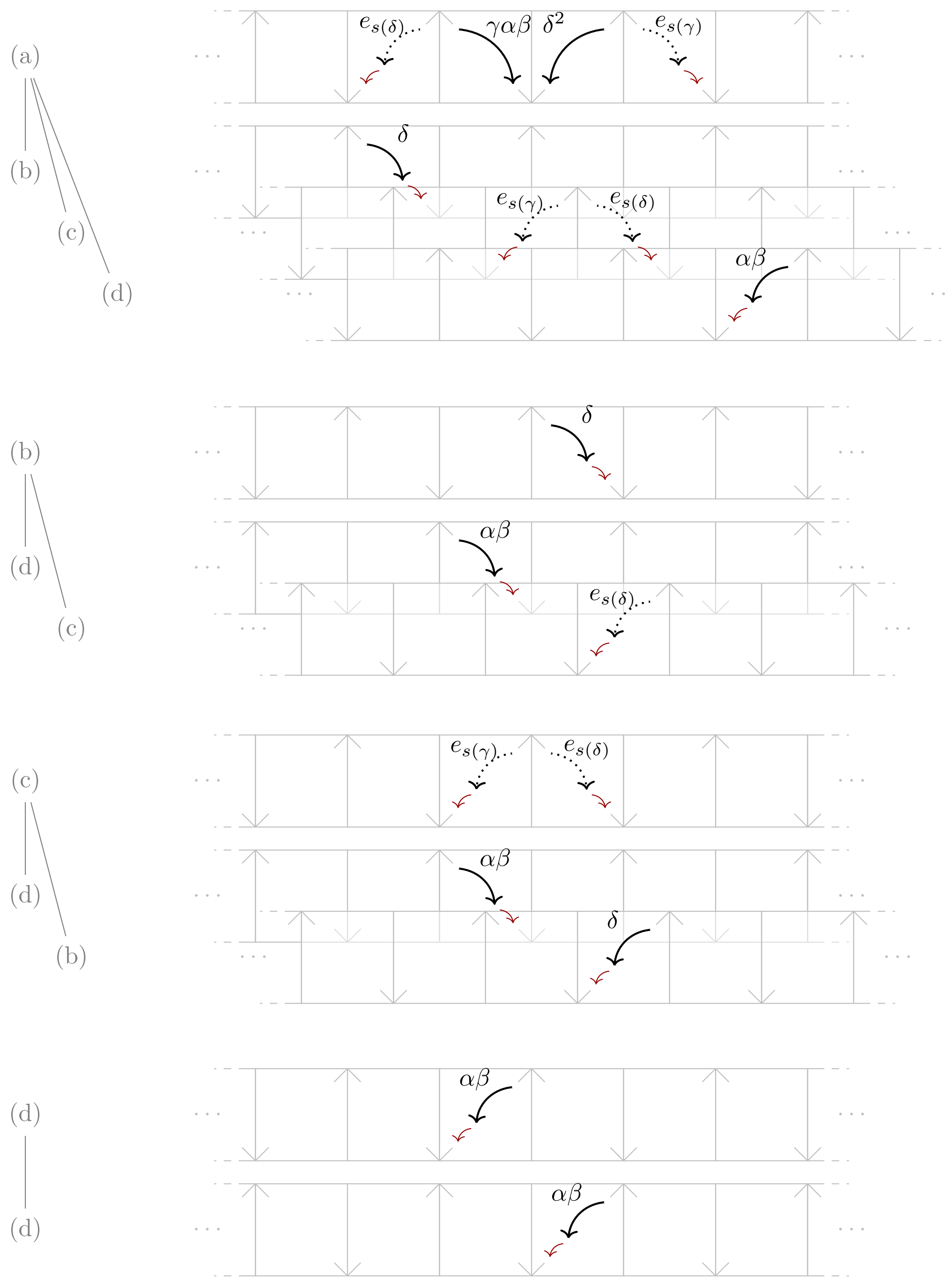
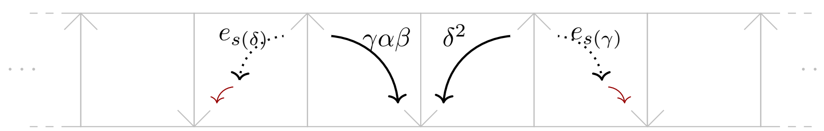


Figure 2.11: *Pieces of the syzygy fabric associated to an injective string module.* Note that some strips which are labelled the same are not identical; however, they can be made so by a combination of rounding off and reflecting. This means that they represent the same string modules.

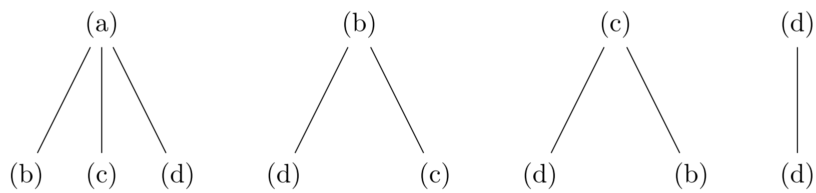
2.4.44. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41, and w the strip:



Note that w represents the indecomposable injective string module associated to the vertex $2 \in Q_0$.

We now describe the tree $\mathcal{T}$ that we use to index the rows of the syzygy fabric. We define $\mathcal{T}$ to be the limit of an increasing sequence of rooted trees $\mathcal{T}_{(0)} \subseteq \mathcal{T}_{(1)} \subseteq \mathcal{T}_{(2)} \subseteq \cdots$, with each $\mathcal{T}_{(k)}$ containing all vertices reachable from the root of $\mathcal{T}$ by paths of length at most k. Thus our initial tree $\mathcal{T}_{(0)}$ consists of a single vertex, our root, which we label (a), and contains no arrows.

If we have constructed the k-th tree $\mathcal{T}_{(k)}$, then we construct $\mathcal{T}_{(k+1)}$ by adding child vertices to the leaves of $\mathcal{T}_{(k)}$ based on their label in accordance with the following diagrams:



The rows corresponding to each of the vertices of $\mathcal{T}$ are populated by the strips present in Figure 2.11. Iteratively connecting one row to its children rows yields our complete syzygy fabric for w.

2.5 SBStrips

Here we provide a brief overview of the functionality in SBStrips. For more detail, we refer to the documentation (available on the SBStrips site) which also contains a full worked example going through the various methods available.

Strips are the principal objects of the SBStripspackage. You can construct a particular strip using the `Stripify` method, but often you will want to be working with a canonical type of strip; in these cases there is often a helpful method for constructing them, for example `SimpleStripsOfSBAlg`, `IndecProjectiveStripsOfSBAlg` and `IndecInjectiveStripsOfSBAlg`. Many common operations on strips have been implemented; we have already mentioned `SyzygyOfStrip` and `TransposeOfStrip`, but there is also `VectorSpaceDualOfStrip` and `NthSyzygyOfStrip`, which do what you would expect from the name. Other helpful methods include:

- `ModuleOfStrip` - this takes in a strip and returns a module in the form used by QPA, allowing further calculations in a more general context.
- `IsFiniteSyzygyTypeStripByNthSyzygy` - this takes in a strip and an integer, N, and returns `true` if the strips appearing in the N-th syzygy of the input strip have all appeared among earlier syzygies, and `false` otherwise.
- `DeloopingLevelOfStripIfAtMostN` - this takes in a strip and an integer, N, and returns the delooping level of the strip if it is at most N, and `fail` if the delooping level is greater than N.

Another method from SBStrips that is worth mentioning is `DeloopingLevelOfSBAlgIfAtMostN` which takes in an SB algebra and an integer, N, and returns the delooping level of the algebra if it is at most N, and `fail` if the delooping level is greater than N.

All of the previous methods were implemented by Allen. Using the ideas of minimally connected overquivers (see Definition 2.3.24), we have added some more helper methods for generating complete collections of SB algebras with particular properties.

The most commonly used will probably be `SBAlgsFromNumVerticesAndRadLength`, which takes in two positive integers, N and R, and produces a collection of all SB algebras on N vertices with radical length R. This is useful for trying to verify conjectures for all SB algebras up to a given size; we use it for exactly this purpose in Section 5.2. For instance, our running example algebra (as defined in Paragraph 2.2.41) would appear in the collection generated by `SBAlgsFromNumVerticesAndRadLength(2,4)` as its underlying quiver has 2 vertices and it has radical length 4.

If you require more control over the structure of the SB algebras that are generated, then you can use `SBAlgsFromCyclesAndRadLength` which takes as an input, a list of positive integers, L, and another positive integer, R. It then generates a collection of all SB algebras of radical length at most R which have a minimally connected overquiver, $\mathcal{O}$, where the multiset formed by the number of vertices of each connected component of $\mathcal{O}$ match with the list L. Note that since any overquiver has an even number of vertices, it is necessary that the list L sums to an even number. For instance, our running example algebra (as defined in Paragraph 2.2.41) would appear in the collection generated by `SBAlgsFromCyclesAndRadLength([3,1],4)` as it has a minimally connected overquiver where the two connected components have 3 and 1 vertices respectively, and it has radical length 4.

There are also various methods for working with permissible data, overquivers, and syllables; for these we refer to the documentation, as working with them directly is not necessary for the vast majority of computations.

Chapter 3

Belts and syzygies of band modules

In this chapter, we extend the approach to syzygy calculations for string modules of special biserial algebras (introduced by Allen [All21]), to syzygy calculations for band modules. This approach involves representing band modules as finite words made up of syllables; these words only contain interior syllables, the start and end of the word are thought of as identified, and they are called *belts*.

We first establish how to represent a band module as a belt and its projective cover as an associated set of patches. We then demonstrate and verify our method is compatible with syzygies of band modules, and that these syzygies can be recorded into an array in a similar way to the syzygy fabric for strips when calculating syzygies of string modules.

3.1 Belts

3.1.1 Preview of belts

Before we dive into the formal definition of belts, we will explore our method for encoding band modules and their syzygies using several worked examples. The aim is to build intuition first; the details of definitions and precise properties will follow afterwards. Recall our running example algebra:

$$A \coloneqq \Bbbk \left(\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & 2 \arrow[l, bend left, "\gamma"] \arrow[loop right, "\delta"] \end{tikzcd} \right) / \langle \alpha^2, \beta\delta, \gamma\beta, \delta\gamma, \alpha\beta\gamma - \beta\gamma\alpha, \delta^3 \rangle,$$

whose uniserial A-modules can be identified with the strict prefixes of $\alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta\gamma$ and δ^3.

3.1.1. Overview of the idea. Similarly to the method of representing string graphs in terms of rows of syllables, we will represent a band graph as a cyclic row of interior syllables. As with strips, the cells of a belt alternate between orientations, positive (pointing to the right from top to bottom) and negative (pointing to the left from top to bottom). Thus adjacent pairs of cells alternately form peaks (a negative syllable then a positive) and valleys (a positive syllable then a negative).

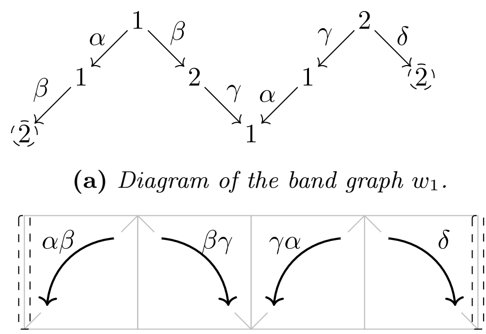
Again, like strips, the content of a peak in one row uniquely determines the contents of a valley in a row underneath it. The possibility of branching (one row having several rows underneath it) remains in our handling of belts. However, we also have to contend with the possibility of the row beneath remaining cyclic (and thus containing a belt) or splitting into a collection of acyclic (linear) rows (and thus containing a collection of strips).

While with strips it is strictly necessary to operate "peakwise" (due to the perturbations caused by pin-boundary syllables), for belts it is possible to operate "cellwise". This is due to the fact that in a patch with two interior syllables in its top row, the lower row is exactly the image of the upper row under descent. However, while it is possible to proceed "cellwise", there are several points in our arguments where it is useful to continue working "peakwise", so we will proceed on that basis for the time being.

3.1.2. Peaks and valleys. Consider the band graph w_1, shown in Figure 3.1(a). It has four linear subgraphs that start at source vertices and end at sink vertices. The corresponding restrictions are $(2 \xleftarrow{\beta} 1 \xleftarrow{\alpha} 1)$, $(1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 1)$, $(1 \xleftarrow{\alpha} 1 \xleftarrow{\gamma} 2)$ and $(2 \xrightarrow{\delta} 2)$. The first and last of these end at the same vertex, as illustrated by the dashed circles in the diagram. We represent w_1 using the belt in Figure 3.1(b).

Let us now review this figure. Firstly, it comprises a finite ($\mathbb{Z}/4\mathbb{Z}$-indexed) row of cells which alternate between positive orientation and negative orientation . Therefore they alternately form potential peaks and potential valleys .

Next, notice that the left edge of the leftmost cell is identified with the right edge of the rightmost



(a) *Diagram of the band graph w_1.*

(b) *Diagram of a belt representing w_1.*

Figure 3.1: *First example of representing a band graph with a belt.* Note that the circled vertices in the band graph are identified.

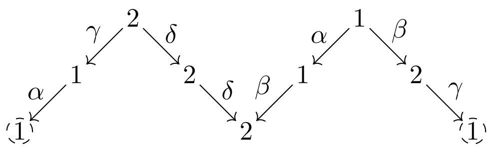
cell. Thus the leftmost cell and the rightmost cell also form a potential valley.

As with strips representing string graphs, the entries in the cells are syllables that uniquely identify the subgraphs of the band graph that we highlighted earlier. Unlike strips representing (finite) string graphs, all of our syllables are interior. This is due to the fact that in a band graph, all source vertices have outdegree 2 and all sink vertices have indegree 2.

We now notice that the leftmost syllable is valley compatible with the rightmost syllable. So in fact the entries in these extremal cells don't just form a potential valley, they form a bona fide valley. The fact these syllables should be valley compatible follows from condition (b)(ii) in the definition of a band graph.

Again, like strips representing string graphs, the underlying path of a syllable represents a path in A, but it is not itself an A-path. The underlying paths of syllables are paths in the overquiver $\mathcal{O}$, not in the ground quiver Q. This distinction is sometimes important, particularly when dealing with stationary paths. Fortunately, since all of the syllables in a belt are interior, none of the underlying paths are stationary.

3.1.3. Rotation of cyclic rows. We now consider the band graph w_2, shown in two forms in Figure 3.2(a). The first diagram shows w_2 as a string graph with two sink vertices identified, the second shows w_2 as a string graph with two source vertices identified. The fact that these diagrams represent the same band graph can be seen by "rotating" the point where we identify vertices to the next source/sink vertex. We represent w_2 using a belt, which is illustrated in two different ways in Figure 3.2(b).



(a) *Two diagrams of the band graph w_2.*

(b) *Diagram of two belts representing w_2.*

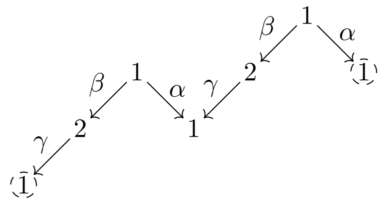
Figure 3.2: *Example of 'rotation' of band graph and associated belt.*

Let us now review this figure. As with Figure 3.1, there is an obvious correspondence between the maximal linear subgraphs of the band graph and the syllables in the belt. The "rotation" of the band graph corresponds to shift of the syllables in the belt, where the endpoint wraps around.

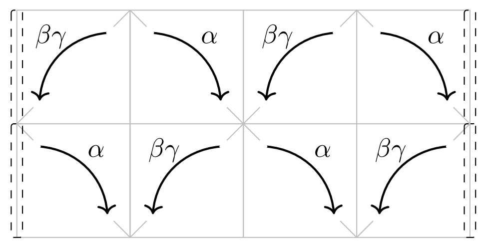
3.1.4. Moving to the next row. Unlike string modules, the class of band modules is not generally closed under taking syzygies. Fortunately, like string modules, there is a local rule for calculating the belt (or strip) representing the syzygy of a band module. To demonstrate how this works when the syzygy of a band module is (or isn't) a band module, we will give an example in each case.

First, consult Figure 3.3. In Figure 3.3(a), we see a diagram of the band graph w_3. There could be infinitely many band modules associated to this band graph (if $\Bbbk$ is infinite), but all of them have a similar structure. In particular, the syzygies of each of these representative band modules are all band modules represented by the same band graph (we will discuss this fact further in Subsection 3.2.2).

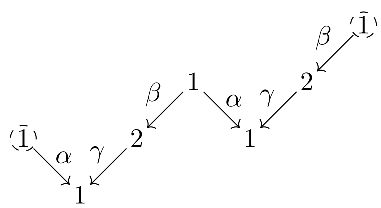
To better understand the structure of these syzygies, we look at Figure 3.3(b). The first row is a belt representing w_3, and the second row is a belt which we claim represents the band graph of these syzygies, and where the entries are determined by our local rule. This second belt has the same number of cells (so its intwid is the same), and the wrapping columns are aligned with the row above. In fact, this second belt is actually a rotation of the first; this may be easier to see by observing the band graph it represents in Figure 3.3(c). This means that the class of band modules with underlying band graph w_3 is closed under taking syzygies.



(a) *Diagram of the band graph w_3.*

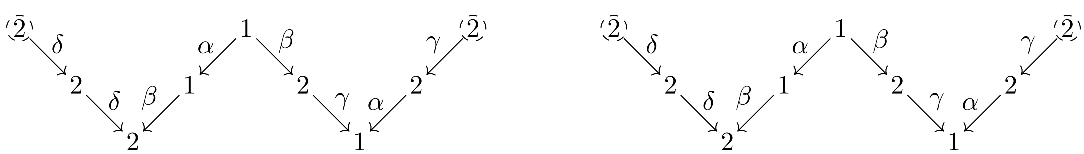


(b) *Calculating syzygies of band modules associated to w_3.*

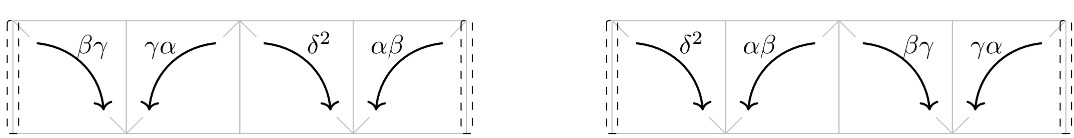


(c) *Another diagram of the band graph w_3.* This helps to illustrate that the bottom row of (b) represents the same band graph as the upper row.

Figure 3.3: *First example of representing syzygies of band modules with belts.*

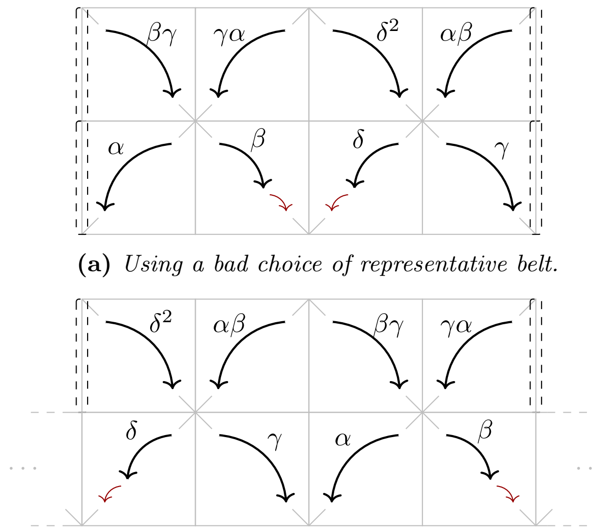


(a) *Two diagrams of the band graph w_2.*



(b) *Diagram of two belts representing w_2.*

Figure 3.4: *Two 'rotations' of a band graph and associated belt.* The right-hand belt is more useful for calculations, since the wrapping splits a peak whose source is a non-pin vertex.



(a) *Using a bad choice of representative belt.*

(b) *Using a good choice of representative belt.*

Figure 3.5: *Unfurling the syzygy fabric.* The choice in (b) is better as we have boundary syllables where the wrapping would be. This means that we no longer have to consider the wrapping, and can consider the second row as a strip in the usual way.

Now, consult Figure 3.4. In Figure 3.4(a), we see a previous diagram of a band graph from earlier, w_2, alongside a new diagram of w_2. Figure 3.4(b) contains diagrams of the corresponding belts to represent w_2. We will see that the right-hand choice of belt is more useful for calculations involving syzygies. This is due to the fact that the choice of wrapping point splits a peak with non-pin source vertex. Observe Figure 3.5 to see why this is helpful.

Since our local rule puts boundary syllables beneath syllables with non-pin source, by using this choice of wrapping point, we ensure that two of our boundary syllables in the second row appear in the leftmost cell and the rightmost cell. This means that any wrapping in these positions would have no effect, and we can safely consider this row as a strip in the usual way.

By choosing a representative belt in this way, it is easier to see (and work with) the "unfurling of the syzygy fabric" as we switch from a cyclic row (that is $\mathbb{Z}/4\mathbb{Z}$-indexed) to an acyclic one (that is $\mathbb{Z}$-indexed).

This calculation also shows that no matter which band module we choose for the band graph w_2, its syzygy is always a string module represented by the same string graph. In particular, this means that these syzygies are all isomorphic.

3.1.5. Branching. As with string modules, sometimes the syzygy of a band module has a non-trivial direct sum decomposition. Whereas in our examples so far, applying our local rule to belts has resulted in rows with at most two boundary syllables, it is possible to have any even number of boundary syllables, depending on the structure of the original belt.

Consider Figure 3.6. In Figure 3.6(a), we see a diagram of a band graph with three source vertices which are non-pin (note the identification between the left and right ends). The fact that the vertices we are identifying are non-pin source vertices means that the choice of belt corresponding to this diagram is an example of a "good" representative of the band graph in the sense of Paragraph 3.1.4. If we take this choice of belt (see the top row of Figure 3.6(b)) and apply our local rule, we end up with a (flattened) row with six boundary syllables (see the bottom row of Figure 3.6(b)).

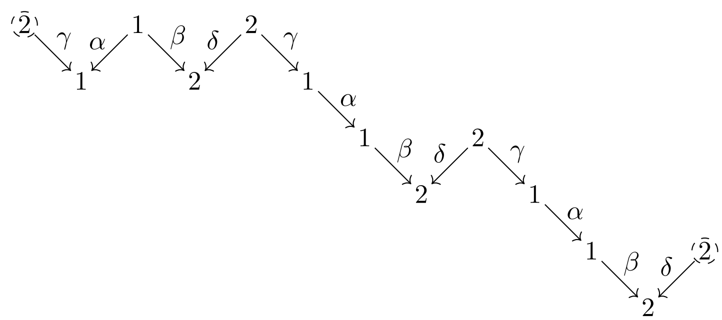
As this bottom row represents a collection of strips, it is best to avoid this flattening. We do this by assigning each of these strips its own row as in Figure 3.6(c). This means that we must generally use the vertices of a rooted tree to index our rows (whether they are cyclic or acyclic). Thus we can use the branching of the tree to represent the direct sum decomposition of the syzygies of the (family of) band module(s). One thing to note is that this branching only occurs when we are moving to an acyclic row; in other words, there is a non-trivial direct sum decomposition of the syzygy only when the syzygy is the sum of a family of string modules (this will be formalised in Section 3.2).

3.1.2 Belts and band modules

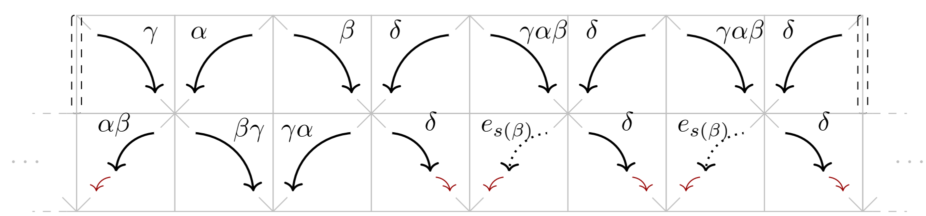
We first refer the reader back to Definition 2.4.13, as the definitions of peaks and valleys are instrumental in what follows.

3.1.6. Belts. A *belt* w is a cyclic juxtaposition of (non-virtual, non-blank) interior syllables alternately forming peaks and valleys.

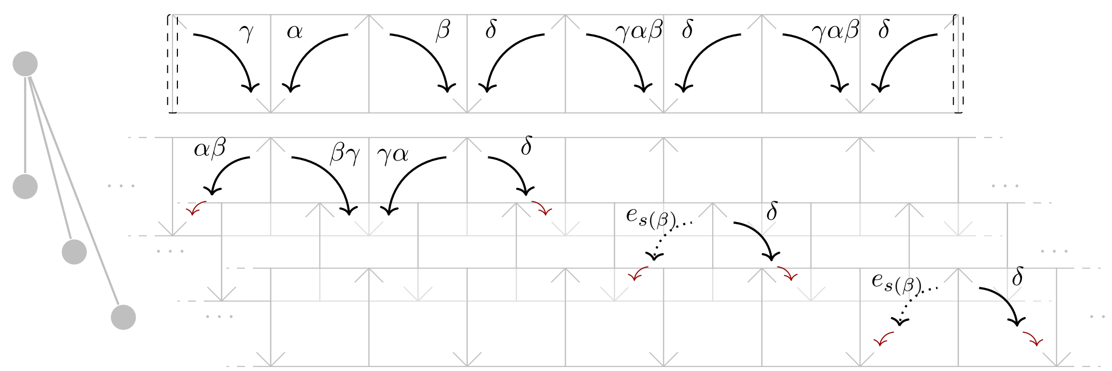
Formally we consider the juxtaposition to be a single row of cells with columns indexed by $\mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$. Thus, a belt w is a function $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$. The *entry* of cell k is $w(k)$. The *(interior) width* of w is defined as $\mathrm{intwid}(w) := n$.



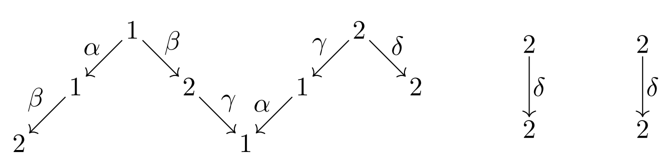
(a) *Diagram of the band graph w_4.*



(b) *Calculating (flattened) syzygies of band modules associated to w_4.*



(c) *Calculating (branched) syzygies of band modules associated to w_4.*



(d) *Diagram of the string graphs associated to syzygies.*

Figure 3.6: *Branching.* Handling syzygies of band modules with non-trivial direct sum decompositions.

3.1.7. Since the syllables in a belt alternately from peaks and valleys, they can be considered as a union of peaks (or dually, a union of valleys). Thus $\mathrm{intwid}(w) \in 2\mathbb{Z}_+$ for all belts, w.

3.1.8. Support of a belt. Since a belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ has no blank syllables, to match the notation for strips, we define its support as $\mathrm{supp}(w) := \mathbb{Z}/n\mathbb{Z}$.

3.1.9. Neighbours. We define peak/valley neighbours similarly to strips. For any $k \in \mathrm{supp}(w)$, one out of $\{w(k), w(k+1)\}$ and $\{w(k-1), w(k)\}$ is a peak and one is a valley. The *peak neighbour* of k is the cell forming a peak with k; the *valley neighbour* is defined similarly.

3.1.10. Reflections and rotations. Similar to strips, we can define the *reflection* of $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ by precomposing with the reflection $k \mapsto -k$ on $\mathbb{Z}/n\mathbb{Z}$ and postcomposing with the orientation involution $\mathbf{p}^s \mapsto \mathbf{p}^{-s}$ on $\mathrm{Syll}(A)$.

We can also define the *rotation by l steps* of a belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ by precomposing with the rotation $k \mapsto k + l$ on $\mathbb{Z}/n\mathbb{Z}$. This can be thought of as an analogue of "translating" a strip.

These are both clearly well-defined operations, since they carry peaks to peaks and valleys to valleys.

3.1.11. Running example. Several examples of belts for our running example algebra were given in Figures 3.1(b), 3.2(b), 3.3(b), 3.4(b) and 3.6(b).

3.1.12. Any belt represents a band graph in a similar way to strips representing string graphs. However, unlike string graphs and string modules, there is not a one-to-one correspondence between band graphs and band modules. This is due to the choice of indecomposable automorphism $\psi : \Bbbk^m \to \Bbbk^m$ in the definition of band modules (Definition 2.2.56). The following proposition formalises the identification between belts and band graphs, following a similar style of argument to [All21, Prop 4.2.17].

3.1.13. Proposition. Any belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ represents a well-defined band graph, and moreover w and its reflection both represent the same band graph. Furthermore, w and its rotations represent the same band graph. Thus for any belt, there is at least one band module represented by it.

Conversely, any band graph can be represented by a belt. Thus for any band module, there is at least one belt to represent it.

Proof. We first handle the forward implication. When $\mathbf{p}, \mathbf{q}$ are syllables in w, we denote their underlying paths (or equivalently their compressions) by p, q. Due to the definition of syllables (Definition 2.3.29), these $\mathcal{O}$-paths, p, q, belong to $N - C$ (i.e. they are $\mathcal{O}$-paths which correspond to A-paths which non-zero and are not components). Since $\mathbf{p}, \mathbf{q}$ are interior, p, q are non-stationary. Thus p, q represent Q-paths whose A-residue is linearly independent of all other A-paths.

Each p, q can be viewed as a linear subgraph $(\circ \to \circ \to \cdots \to \circ)$ of $\mathcal{O}$, where each arrow (resp. vertex) canonically represents an arrow (resp. vertex) of Q, the ground quiver of A. This viewpoint gives rise to quiver homomorphisms $p \to Q$ and $q \to Q$. As belts do not contain blank syllables, these homomorphisms always have non-empty domain. In particular, p, q are non-stationary $\mathcal{O}$-paths, so these linear subgraphs have at least one arrow and at least two vertices.

When $\boxed{\mathbf{p} \uparrow \mathbf{q}}$ is a peak, then the sources of p, q are distinct $\mathcal{O}$-vertices but represent the same Q-vertex. This means that these sources have the same image in the quiver homomorphism. Thus, if we consider the disjoint union of the quiver homomorphisms $p \to Q$ and $q \to Q$, we can identify the sources of p, q to obtain another well-defined quiver homomorphism. If we continue to assume that $\boxed{\mathbf{p} \uparrow \mathbf{q}}$ is a peak then the first arrows of p, q are distinct, thus the arrows of Q that they represent must be distinct. Therefore, when we perform this identification of source vertices, pairs of arrows in the domain with common sources are mapped to distinct arrows of Q by this homomorphism.

The previous paragraph is still true if we perform the following replacements simultaneously: "peak" by "valley"; "peak-neighbour(ing)" by "valley-neighbour(ing)"; "sources" by "targets" and; "source vertex" by "sink vertex".

By identifying all of these linear quivers at their source vertices or sink vertices, we obtain a quiver G whose underlying graph is cyclic. This repeated identification preserves the property that pairs of arrows in G whose common source (resp. target) is a source vertex (resp. sink vertex) are mapped to distinct arrows of Q. Since we only identify source vertices with source vertices and sink vertices with sink vertices, we also preserve the property that any maximal connected linear subquiver in G represents an A-path that is non-zero and not a component (in the sense of Definition 2.3.5). In other words, a maximal connected linear subquiver in G represents an A-path that is linearly dependent of all other A-paths. Thus an arbitrary connected linear subquiver in G also represents a non-zero non-component A-path too. In other words, any path in G represents a non-zero non-component A-path, meaning condition (c) in Definition 2.2.49 holds. Hence all conditions in the definition of band graphs hold, so the first claim is proven.

The fact that reflections and rotations of belts preserve the band graph follows immediately, since this construction is not affected by reflecting or rotating w.

We now turn our attention to the converse assertion. Let $v : G \to Q$ be a band graph.

To each maximal connected linear subquiver $\Gamma := (\circ \xrightarrow{x_1} \circ \xrightarrow{x_2} \cdots \xrightarrow{x_l} \circ)$ of G, we associate an A-path $p := v(x_1 x_2 \dots x_l)$ which is linearly independent of all other A-paths. We now associate an interior syllable $\mathrm{syll}(\Gamma) := (\circ \xrightarrow{p} \circ \xrightarrow{0} \circ)$.

Choose one such maximal connected linear subquiver Γ_0 of G. Let Γ_1 be the maximal connected linear subquiver of G sharing a sink vertex with Γ_0 (by the definition of band graphs we know such a Γ_1 exists). Let Γ_2 be the maximal connected linear subquiver of G sharing a source vertex with Γ_1 (by the definition of band graphs we know such a Γ_2 exists). Let Γ_3 be the maximal connected linear subquiver of G sharing a sink vertex with Γ_2. Continue this process iteratively until we obtain a Γ_{n-1} which shares a source vertex with Γ_0 (we know this process will terminate since G is a finite quiver).

For any $k \in \mathbb{Z}$ with $0 \leq k \leq n-1$, define $w(k + n\mathbb{Z}) := \mathrm{syll}(\Gamma_k)^{(-1)^k}$. (Note that the exponent $(-1)^k$ is the orientation of the syllable.) This clearly gives a well-defined function $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$. We now show that w is also a well-defined belt. To do this, all that remains is to show the peak/valley compatibility of adjacent syllables.

Potential peaks come from neighbouring subquivers $\Gamma_k, \Gamma_{k'}$ whose common vertex is a source vertex of G. Because the two arrows incident to that source vertex have different images in v, we know that $\mathrm{syll}(\Gamma_k)$ and $\mathrm{syll}(\Gamma_{k'})$ are distinct and have their sources exchanged by $\dagger : \mathcal{O}_0 \to \mathcal{O}_0$. It follows that each of these potential peaks is actually peak compatible.

A dual argument shows that all potential valleys are actually valley compatible.

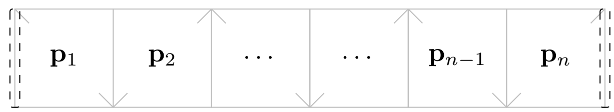
We can now conclude that w is a well-defined belt, as claimed.

The fact that any band module has at least one belt representing it then follows immediately by constructing the belt associated to its band graph. $\square$

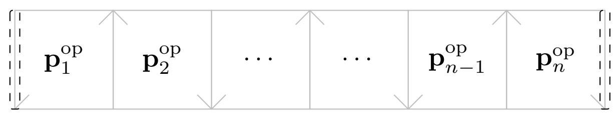
Due to this identification between band graphs and belts, we will often use one in the place of the other in various constructions. For example, we often use the notation $\mathrm{Bnd}(w, \psi)$ for a band module associated to a belt, w, and a compatible indecomposable automorphism $\psi : \mathbb{k}^m \to \mathbb{k}^m$.

The next proposition follows immediately from the discussion in [WW85, Section 3].

3.1.14. Proposition. Let $M \in \text{mod-}A$ be a band module represented by a belt of the form:



Then $D(M) \in \text{mod-}A^{\text{op}}$ is a band module represented by a belt of the form:



where for $\mathbf{p}_i = (\circ \xrightarrow{p_i} \circ \xrightarrow{0} \circ)^o \in \text{Syll}(A)$, we define $\mathbf{p}_i^{\text{op}} := (\circ \xrightarrow{p_i^{\text{op}}} \circ \xrightarrow{0} \circ)^{-o} \in \text{Syll}(A^{\text{op}})$.

Furthermore $\text{Tr}(M) \simeq D(M)$ in $\underline{\text{mod}}\text{-}A^{\text{op}}$, so can be represented by the same belt.

3.2 Syzygies of band modules

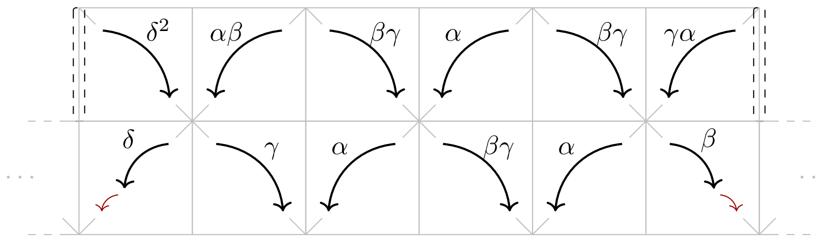
3.2.1 Patch covers of belts

In a similar way to the construction of [All21, Section 4.2.2], we relate belts and patches in a way that mimics the projective covers of band modules.

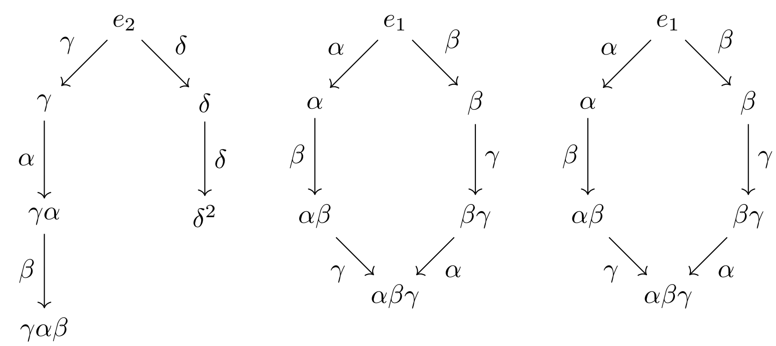
Firstly, we refer back to Proposition 2.4.34, which tells us that there is a correspondence between peaks and patches. As with strips (see Corollary 2.4.35), this gives us a way to associate an arrangement of patches to belts.

3.2.1. Corollary. Any belt is the top row of a well-defined finite arrangement of patches.

Proof. We can partition a belt into a finite arrangement of peaks (possibly with one peak across the wrapping). Each of these peaks is the top of some patch by the previous proposition, so we obtain a finite set of patches. ☐



(a) *Arrangement of patches from a belt.*



(b) *Corresponding projective module.* Note that the first indecomposable projective corresponds to the patch wrapping between the first and last columns.

Figure 3.7: *An example of a patch cover of a belt.* The top row of (a) is a belt representing the band graph w. This determines the arrangement of patches in the rest of (a). This arrangement then corresponds to the projective module in (b).

We can then use this association to construct a projective cover of a band module. This is analogous to Proposition 2.4.36.

> **3.2.2. Proposition.** If the belt w is the top row of a finite arrangement of patches, then $\mathbb{P}(\mathrm{Bnd}(w, \psi))$ is the direct sum of the associated projectives, for any choice of indecomposable automorphism ψ.

Proof. We know from Proposition 2.4.34 that to any peak of w we may associate a non-blank, non-virtual patch. Since each such patch corresponds to an indecomposable projective P_i, each peak corresponds to such a projective.

We know from the construction in Proposition 3.1.13 that the peaks of w correspond to the source vertices in the corresponding band graph. These source vertices are a basis of $\mathrm{Bnd}(w, \psi)/\operatorname{rad}(\mathrm{Bnd}(w, \psi))$, and each is fixed by exactly one primitive idempotent e_j. Clearly $i = j$ for each peak of w and the result follows. $\square$

3.2.3. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41, and let

$$w \coloneqq 2 \xrightarrow{\delta} 2 \xrightarrow{\delta} 2 \xleftarrow{\beta} 1 \xleftarrow{\alpha} 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 1 \xleftarrow{\alpha} 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 1 \xleftarrow{\alpha} 1 \xleftarrow{\gamma} 2$$

be a band graph (with circled vertices identified as usual). Then $\mathbb{P}(\mathrm{Bnd}(w, \psi))$ is given in Figure 3.7(b). This projective module is represented by the arrangement of patches in Figure 3.7(a), where the top row is a belt which represents w.

3.2.4. Patch covers of belts. In analogue to the same construction for strips, we define the *patch cover* $\mathbb{P}(w)$ of a belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ to be the corresponding arrangement of patches given in Corollary 3.2.1. Unlike with patch covers of strips, this arrangement is always finite (it consists of $\frac{n}{2}$ patches), and never includes blank patches.

3.2.2 Syzygy algorithm for belts

We have now established that any belt w has a patch cover $\mathbb{P}(w)$. Unlike with patch covers of strips, the bottom row of a patch cover of a belt will never contain virtual syllables (as w will never contain boundary syllables, let alone pin-boundary syllables). However, with belts we also have the complication of dealing with whether or not the syzygy of the corresponding band module is also a band module, or is in fact a string module instead. A characterisation of when each case occurs is given in the following proposition, which was originally stated in [HZ16, Prop 2.2], referencing a (seemingly unpublished) Ph.D. thesis [Gal] for the proof. However, since this thesis can't be found in the literature, we refer to the first instance where the proofs can be found.

3.2.5. Proposition. [All21, Prop 5.1.15]
Let v be a primitive string graph and let $\widehat{v}$ be the corresponding bi-infinite power. Moreover, let $m \in \mathbb{Z}_+$ and ψ and an indecomposable automorphism of $\Bbbk^m$ with companion matrix as in Figure 2.4. Then the following statements are equivalent:

(a) The syzygy $\Omega^1(\mathrm{Bnd}(v^m, \psi))$ is a band module,

(b) The syzygy $\Omega^1(\mathrm{Str}(\widehat{v}))$ is an indecomposable, infinite-dimensional string module,

(c) None of the indecomposable direct summands of $\mathbb{P}(\mathrm{Str}(v))$ is a string module; that is, they are all pin.

If these conditions fail to be satisfied, then $\Omega^1(\mathrm{Bnd}(v^m, \psi))$ is a string module.

We are now in a position to formalise the algorithm for calculating syzygies of band modules in terms of belts, as previously discussed informally in Subsection 3.1.1. We need to split the handling of this algorithm into two cases (based on whether or not the syzygy is a band module), and each case has a simple example following it.

The proofs of these results follow the structure of [All21, Prop 4.2.24], which proves the analogous result for strips.

3.2.6. Proposition. Suppose that M is a band module which is represented by a belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ where all syllables $\mathbf{p}$ of w have $b_{s(\mathbf{p})} = 0$ and $\mathbf{p}_0 = w(0 + n\mathbb{Z})$ is a negatively oriented syllable. Then $\Omega(M)$ is a band module represented by the belt $\nabla \circ w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$.

Proof. The fact that $\Omega(M)$ is a band module follows immediately from Proposition 3.2.5.

We split the remainder of the proof into two parts; first we show that $\nabla \circ w$ is indeed a belt, then we show that it represents $\Omega(M)$.

Firstly, to show that $\nabla \circ w$ is a belt, it is sufficient to show that all its syllables are interior, and that adjacent syllables are peak compatible or valley compatible as appropriate. The fact that all the syllables of $\nabla \circ w$ are interior follows immediately from the condition that $b_{s(\mathbf{p})} = 0$ for all syllables $\mathbf{p}$ of w.

Consider a pair of adjacent syllables $\boxed{\mathbf{q}}\boxed{\mathbf{q}'}$. To show that $\mathbf{q}, \mathbf{q}'$ are peak compatible (as required), it suffices to show that $s(\mathbf{q}) = s(\mathbf{q}')^{\dagger}$. This immediately follows from the fact that $\mathbf{q} = \nabla\mathbf{p}$ and $\mathbf{q}' = \nabla\mathbf{p}'$ for some valley compatible $\mathbf{p}, \mathbf{p}'$.

Now consider a pair of adjacent syllables $\boxed{\mathbf{q}}\boxed{\mathbf{q}'}$. To show that $\mathbf{q}, \mathbf{q}'$ are valley compatible (as required), it suffices to show that $\mathbf{q}, \mathbf{q}'$ are interior, and that $t(\mathbf{q}) = t(\mathbf{q}')^{\dagger}$. This immediately follows from the fact that $\mathbf{q} = \nabla\mathbf{p}$ and $\mathbf{q}' = \nabla\mathbf{p}'$ for some peak compatible $\mathbf{p}, \mathbf{p}'$ where $b_{s(\mathbf{p})} = b_{s(\mathbf{p}')} = 0$.

Now, to show that $\nabla \circ w$ represents $\Omega(M)$, we show that there is a basis of the projective module $\mathbb{P}(w)$ admitting a partition into a top and bottom part. The basis vectors of the bottom part correspond to the basis of a band module represented by $\nabla \circ w$, modulo which the basis vectors of the top part give rise to a basis of M. The standard inclusion and projection maps of this basis of $\mathbb{P}(w)$ therefore yield a short exact sequence

$$0 \to \mathrm{Bnd}(\nabla \circ w, \psi') \to \mathbb{P}(\mathrm{Bnd}(w, \psi)) \to \mathrm{Bnd}(w, \psi) \to 0,$$

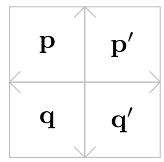


Figure 3.8: *Notation for syllables in the patch X.*

so $\Omega(M)$ is a band module represented by a belt of the form $\nabla \circ w$ as required.

Before we begin constructing our basis, we let $m \in \mathbb{Z}_+$ be the largest divisor of n such that $w(i + \frac{n}{m}) = w(i)$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Note that this corresponds naturally to the m that appears in Lemma 2.2.54 when associating a positive integer to any band graph.

Now, as with the syzygy algorithm for strips, for an initial basis of $\mathbb{P}(\mathrm{Bnd}(w, \psi))$, we take the disjoint union of the standard bases of its direct summands. We then alter this basis on a patch by patch basis.

Let X be a patch in $\mathbb{P}(w)$. Write P_X for the indecomposable projective A-module associated to X. Let $\mathbf{p}, \mathbf{p}', \mathbf{q}, \mathbf{q}'$ denote the syllables of X as in Figure 3.8. We denote the underlying paths (or equivalently compressions) of $\mathbf{p}, \mathbf{p}'$ as p, p'. By the definition of interior syllables, both p and p' are non-stationary paths in $\mathcal{O}$. The paths comparable with p, p' in the prefix order represent basis vectors of P_X. We divide these basis vectors into two parts: an upper part and a lower part.

In the upper part, we place all basis vectors represented by strict prefixes of p and p'. In the lower part, we place all basis vectors represented by paths with p or p' as a strict prefix. It just remains to determine where to put the vectors represented by p and p'. How we handle these vectors depends on where the patch under consideration fits into the patch cover of the belt.

We note that since all valleys of the belt are interior, $\mathbf{p}$ and $\mathbf{p}'$ necessarily appear in valleys $\boxed{\mathbf{u} \mid \mathbf{p}}$ and $\boxed{\mathbf{p}' \mid \mathbf{u}'}$ next to other interior syllables, $\mathbf{u}$ and $\mathbf{u}'$, whose compressions we denote u and u'. Thus we can consider how to handle the remaining basis vectors by considering them valley-by-valley.

The first case we handle is the valley consisting of $\mathbf{p}_{n-1}$ and $\mathbf{p}_0$, as we must treat it differently to the others. Here we define $x = p_{n-1} - \sum_{r=1}^{m} \lambda_r p_{\frac{n}{m}(r-1)}$, replace $\{p_{n-1}, p_0\}$ in the basis of $\mathbb{P}(\mathrm{Bnd}(w, \psi))$ by $\{x, p_0\}$, and then place p_0 in the upper part and x in the lower part.

We now handle the remaining valleys, consisting of $\mathbf{p}_{2k-1}$ and $\mathbf{p}_{2k}$ for $k \in \{1, \ldots, \frac{n}{2} - 1\}$. In these cases, we replace $\{p_{2k-1}, p_{2k}\}$ in the basis of $\mathbb{P}(\mathrm{Bnd}(w, \psi))$ by $\{p_{2k-1} - p_{2k}, p_{2k}\}$, and then place p_{2k} in the upper part and $p_{2k-1} - p_{2k}$ in the lower part.

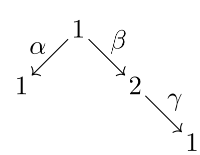
Performing these changes for all valleys yields a well-defined, bi-partitioned basis for $\mathbb{P}(\mathrm{Bnd}(w, \psi))$.

The basis vectors of the lower part $\Bbbk$-span an A-submodule of $\mathbb{P}(\mathrm{Bnd}(w, \psi))$. Note that the basis vectors represented by x and the differences $p_{2k-1} - p_{2k}$ together $\Bbbk$-span the top of this submodule. Also, each $\alpha \in Q_1$ annihilates at least one component of each of these differences and any vector not so annihilated is mapped to a linear multiple of another vector in the lower part. The rest of the lower basis vectors (those represented by paths with p, p' as a strict prefix) are annihilated by all A-arrows except (at most) one. It follows that the lower part is the standard basis for a band module represented by the belt $\nabla \circ w$ (up to rescaling). Thus the lower basis vectors are the basis for a submodule isomorphic to $\mathrm{Bnd}(\nabla \circ w, \psi')$, for some indecomposable automorphism ψ', as required.

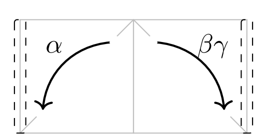
Taking a quotient by all of the lower basis vectors except x, the underlying paths of neighbouring interior syllables are identified. This gives us the standard basis for the string module in the definition of $\mathrm{Bnd}(w, \psi)$. Adding x to the basis of the submodule we quotient by thus matches the definition of $\mathrm{Bnd}(w, \psi)$ exactly. The result follows. $\square$

To aid in understanding, we now give a (relatively simple) example.

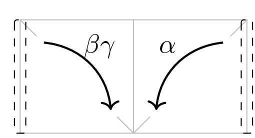
3.2.7. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Consider the band module, $M \in$ mod-A, associated to the primitive string graph



and the indecomposable automorphism $\mathbb{1} = \psi : \Bbbk^1 \to \Bbbk^1$ (i.e. $m = 1$ and $\lambda_1 = 1$). Then M is represented by the belt



Applying our algorithm from Proposition 3.2.6 shows that $\Omega(M) \in$ mod-A is a band module represented by the belt



3.2.8. Proposition. Suppose that M is a band module which is represented by a belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ where $\mathbf{p}_0 = w(0 + n\mathbb{Z})$ is a positively oriented syllable with $b_{s(\mathbf{p}_0)} = 1$. Then $\Omega(M)$ is a string module, represented by a flattened family of strips $s : \mathbb{Z} \to \mathrm{Syll}(A)$ of the form:

$$\begin{aligned} s : \mathbb{Z} &\longrightarrow \mathrm{Syll}(A) \\ k &\longmapsto s(k) = \begin{cases} \nabla\Big(w(k + n\mathbb{Z})\Big) & \text{if } 0 \leq k < n \\ (\quad) & \text{otherwise} \end{cases} \end{aligned}$$

where $(\quad)$ denotes the blank syllable.

Proof. By the peak compatibility condition in the definition of belts, we know that $\mathbf{p}_{n-1} = w((n-1) + n\mathbb{Z})$ is a negatively oriented syllable with $b_{s(\mathbf{p}_{n-1})} = 1$. This means that the reflection of w also meets the conditions required for this proposition. This symmetry will be helpful later in the proof.

The fact that $\Omega(M)$ is a string module follows immediately from Proposition 3.2.5.

We split the remainder of the proof into two parts; first we show that $s : \mathbb{Z} \to \mathrm{Syll}(A)$ is indeed a flattened family of strips, then we show that it represents $\Omega(M)$.

Firstly, to show that s is a flattened family of strips, it is sufficient to show that adjacent syllables are peak compatible or pseudo-valley compatible as appropriate.

Consider a pair of adjacent syllables $\boxed{\mathbf{q}}\boxed{\mathbf{q}'}$. To show that $\mathbf{q}, \mathbf{q}'$ are peak compatible (as required), it suffices to show that $s(\mathbf{q}) = s(\mathbf{q}')^{\dagger}$. This immediately follows from the fact that $\mathbf{q} = \nabla\mathbf{p}$ and $\mathbf{q}' = \nabla\mathbf{p}'$ for some valley compatible $\mathbf{p}, \mathbf{p}'$.

Now consider a pair of adjacent syllables $\boxed{\mathbf{q}}\boxed{\mathbf{q}'}$. To show that $\mathbf{q}, \mathbf{q}'$ are pseudo-valley compatible (as required), we must handle several cases differently.

If both $\mathbf{q}, \mathbf{q}'$ are blank, then they are valley compatible (not just pseudo-valley compatible).

We now handle the case where exactly one of the $\mathbf{q}, \mathbf{q}'$ is blank. This only occurs "at the edge" of the row; i.e. when considering the potential valleys in cells -1 and 0 or in cells $n-1$ and n. By symmetry, we will only consider the "left edge"; the potential valley in cells -1 and 0 of s. Clearly, $s(-1)$ is a blank syllable, by definition of s. Since $b_{s(\mathbf{p}_0)} = 1$, we know that $s(0) = \nabla(\mathbf{p}_0)$ is a boundary syllable. Thus $s(-1)$ and $s(0)$ are valley compatible (not just pseudo-valley compatible).

Next, we handle the case where $\mathbf{q}$ is interior and $\mathbf{q}'$ is non-blank. For this case, we must show that $\mathbf{q}'$ is also interior, meaning the pair is valley compatible (not just pseudo-valley compatible). Since neither is blank, $\mathbf{q} = \nabla(\mathbf{p})$ and $\mathbf{q}' = \nabla(\mathbf{p}')$ for some peak compatible pair $\mathbf{p}$ and $\mathbf{p}'$. Since $\mathbf{q}$ is an interior syllable, it follows that $b_{s(\mathbf{p})} = 0$. Now, as $\mathbf{p}$ and $\mathbf{p}'$ are peak compatible, we know that $b_{s(\mathbf{p}')} = 0$, and thus $\mathbf{q}'$ is also interior, as required.

Finally, we consider the case where $\mathbf{q}$ is a boundary syllable and $\mathbf{q}'$ is non-blank. Here, we must show that $\mathbf{q}'$ is also a boundary syllable, as this is the only remaining option for the pair to be pseudo-valley compatible (although not valley compatible). As in the previous case, since neither is blank, $\mathbf{q} = \nabla(\mathbf{p})$ and $\mathbf{q}' = \nabla(\mathbf{p}')$ for some peak compatible pair $\mathbf{p}$ and $\mathbf{p}'$. Since $\mathbf{q}$ is a boundary syllable, we know that $b_{s(\mathbf{p})} = 1$. Now, as $\mathbf{p}$ and $\mathbf{p}'$ are peak compatible, we know that $b_{s(\mathbf{p}')} = 1$, and thus $\mathbf{q}'$ is also boundary, as required.

Now, to show that s represents $\Omega(M)$, we use the same method as Proposition 3.2.6. We split a basis of $\mathbb{P}(w)$ into an upper and lower part, representing M and $\Omega(M)$ respectively.

As with the previous syzygy algorithms, we take an initial basis of $\mathbb{P}(\mathrm{Bnd}(w, \psi))$, as the disjoint union of the standard bases of its direct summands. We then handle this basis working with each patch separately, in exactly the same way as in Proposition 3.2.6.

The resulting lower part of the basis $\Bbbk$-spans an A-submodule of $\mathbb{P}(\mathrm{Bnd}(w, \psi))$. Again, note that the basis vectors represented by x and the differences $p_{2k-1} - p_{2k}$ together $\Bbbk$-span the top of this submodule. As before, each $\alpha \in Q_1$ annihilates at least one component of each of these differences and any vector not so annihilated is mapped to a linear multiple of another vector in the lower part. It follows that the lower part is a basis for the string module represented by the flattened family of strips s.

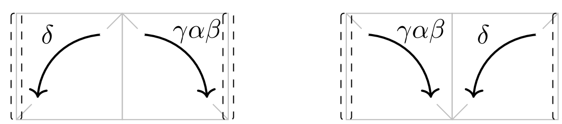
It remains to show that the quotient by this submodule is isomorphic to $\mathrm{Bnd}(w, \psi)$. The logic for this is identical to that in Proposition 3.2.6, but for completeness we include it here. Taking a quotient by all of the lower basis vectors except x, the underlying paths of neighbouring interior syllables are identified. This gives us the standard basis for the string module in the definition of $\mathrm{Bnd}(w, \psi)$. Adding x to the basis of the submodule we quotient by thus matches the definition of $\mathrm{Bnd}(w, \psi)$ exactly. The result follows. $\square$

To aid in understanding, we again give a (relatively simple) example.

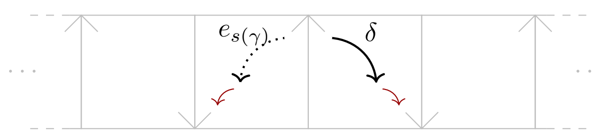
3.2.9. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Consider the band module, $M \in \text{mod-}A$, associated to the primitive string graph

$$\begin{tikzcd} & 2 \arrow[dl, "\delta"'] \arrow[dr, "\gamma"] & & & \\ 2 & & 1 \arrow[dr, "\alpha"] & & \\ & & & 1 \arrow[dr, "\beta"] & \\ & & & & 2 \end{tikzcd}$$

and the indecomposable automorphism $\mathbb{1} = \psi : \mathbb{k}^1 \to \mathbb{k}^1$ (i.e. $m = 1$ and $\lambda_1 = 1$). Then M is represented by either of the following belts



Applying our algorithm from Proposition 3.2.8 to the right belt, shows that $\Omega(M) \in \text{mod-}A$ is a string module represented by the flattened family of strips below (which is in fact an actual strip):



3.2.10. Comparing the syzygy algorithm from Proposition 3.2.8 to the one for strips immediately gives the following:

3.2.11. Corollary. Suppose that M is a band module, and $\widehat{M}$ the corresponding bi-infinite string module. If $\Omega(M)$ is a direct sum of string modules then $\Omega(\widehat{M}) = \oplus_{i \in \mathbb{Z}} \Omega(M)$.

Proposition 3.2.6 tells us about the belts that represent band modules which are syzygies. However, it is not clear that all band modules represented by such a belt are themselves syzygies. To prove this, we first note a useful lemma.

3.2.12. Lemma. Any valley $\boxed{\mathbf{q} \,|\, \mathbf{q}'}$ where $\mathbf{q}, \mathbf{q}' \in \operatorname{im}(\nabla)$ are interior syllables is the bottom row of exactly one patch with interior syllables in both positions of the top row.

The projective associated to any such patch is a pin module.

Proof. Since $\mathbf{q} \in \operatorname{im}(\nabla)$, there exists $\mathbf{p} = (s \xrightarrow{l} \circ \xrightarrow{\varepsilon} \circ) \in \operatorname{Syll}(A)$ such that $b_s = 0$ and $\mathbf{q} = (s-(l+\varepsilon) \xrightarrow{a_s-(l+\varepsilon)} \circ \xrightarrow{0} \circ)$. By the definition of syllables and descent, $\mathbf{p}_0 = (s \xrightarrow{l+\varepsilon} \circ \xrightarrow{0} \circ) \in \operatorname{Syll}(A)$ has $\nabla \mathbf{p}_0 = \nabla \mathbf{p} = \mathbf{q}$. Thus we assume without loss of generality that $\mathbf{p}$ is an interior syllable.

Symmetrically, we know that there exists an interior syllable, $\mathbf{p}' = (s' \xrightarrow{l'} \circ \xrightarrow{0} \circ)$, such that $\nabla \mathbf{p}' = \mathbf{q}'$.

We now claim that

$$\begin{array}{|c|c|} \hline \mathbf{p} & \mathbf{p}' \\ \hline \mathbf{q} & \mathbf{q}' \\ \hline \end{array}$$

is a valid patch, and that it is the unique patch meeting the conditions of the lemma.

Firstly, to show that the patch is valid, all that remains is to show that $\begin{array}{|c|c|} \hline \mathbf{p}' & \mathbf{q}' \\ \hline \end{array}$ is a valid peak, since patches where both top syllables are interior are entirely determined by descent. This simply requires showing that $s^\dagger = s'$. For the syllables $\mathbf{p}, \mathbf{q}, \mathbf{p}', \mathbf{q}'$, let their compressions (or equivalently underlying paths, since they are all interior) be denoted by p, q, p', q' respectively. By construction, composing these $\mathcal{O}$-paths gives $pq = (s \xrightarrow{a_s} t(q))$ and $p'q' = (s' \xrightarrow{a_{s'}} t(q'))$. Since $\mathbf{q}$ and $\mathbf{q}'$ are valley compatible and interior, it follows that $t(q)^\dagger = t(q')$. Now, as $b_s = b_{s'} = 0$, it follows that pq and $p'q'$ are components that are swapped under the component exchange map $\dagger : C \to C$. It follows that $s^\dagger = s'$, as required.

Now, to show that this is the unique patch meeting the conditions of the lemma, it is enough to show that $\mathbf{p}$ is the unique interior syllable which maps to $\mathbf{q}$ under ∇ (showing the same for $\mathbf{p}'$ and $\mathbf{q}'$ would then follow similarly). The reason that this is sufficient is a fact mentioned above; patches where both top syllables are interior are entirely determined by descent.

So, suppose for contradiction that $\mathbf{p}_0, \mathbf{p}_1 \in \operatorname{Syll}(A)$ are interior syllables satisfying $\nabla \mathbf{p}_0 = \mathbf{q} = \nabla \mathbf{p}_1$. We denote the compressions of these syllables as p_0, p_1, q in the obvious way. Then (as previously discussed above) both $p_0 q$ and $p_1 q$ are components ending at $t(q)$. Thus $p_0 q = p_1 q$, as distinct components can't have the same target (or source). The result follows. $\square$

3.2.13. Proposition. Suppose that M is a band module which is represented by a belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$. Then the following conditions are equivalent:

(i) there exists $Y \in \text{mod-}A$ such that $\Omega(Y) \simeq M$ in $\underline{\text{mod}}\text{-}A$,

(ii) there exists a band module $Y \in \text{mod-}A$ such that $\Omega(Y) \simeq M$ in $\underline{\text{mod}}\text{-}A$,

(iii) $w(k + n\mathbb{Z}) \in \mathrm{im}(\nabla)$ for all $k \in \mathbb{Z}$.

Proof. Clearly (ii) $\implies$ (i). The reverse implication follows from Theorem 2.2.58 and the fact that syzygies of string modules are string modules by Proposition 2.2.48. The implication (ii) $\implies$ (iii) follows from the description of syzygies of band modules represented by belts in Proposition 3.2.6. Thus it remains to show that (iii) $\implies$ (ii).

We now assume that (iii) holds. By Lemma 3.2.12, for each valley of w, there exists a unique peak formed of interior syllables where that peak is the top row of a patch with bottom row our valley. The peak compatibility of the valleys of w immediately gives us the valley compatibility of the peaks in the top row of these patches. Thus the top row of these patches give a well-defined belt w', where when we apply our syzygy algorithm to w', we obtain w.

Consider the families of band modules represented by the belts w' and w; call the former B' and the latter B. There are three key facts about B' and B that we need to consider:

(a) The syzygy of any module in B' is a module in B.

(b) The projective cover of any module in B' is the direct sum of pin modules (and hence injective).

Since $D : \text{mod-}A \to \text{mod-}A^{\text{op}}$ is exact, and the cosyzygy functor $\Omega^{-1} : \text{mod-}A \to \text{mod-}A$ can be written as $D\Omega D = \Omega^{-1}$, we have dual properties to the above

(c) The cosyzygy of any module in B is a module in B'.

(d) The injective hull of any module in B is the direct sum of pin modules (and hence projective).

Combining these properties, we see that for any module $M \in B$ there exists a short exact sequence:

$$0 \to M \to Q \to N \to 0,$$

where Q is projective and injective, and where $N \in B'$. Hence $\Omega(N) \simeq M$, as required to satisfy (ii). □

This immediately gives the following:

> **3.2.14. Corollary.** Suppose that a belt w represents a band module $M \in \text{mod-}A$ where $M \simeq \Omega(N)$ for some band module $N \in \text{mod-}A$. Then for any other band module $M' \in \text{mod-}A$ represented by the same belt, w, there exists a band module $N' \in \text{mod-}A$ such that $M' \simeq \Omega(N')$.

3.2.15. Remark. This corollary is very important when considering band modules which are syzygies, particularly when calculating the delooping level of band modules, which we will investigate further in Chapter 4. It allows us to consider these things entirely at the level of belts, rather than handling the extra parameter involved in defining a band module.

We can also use the ideas of Proposition 3.2.13 to obtain a uniqueness result for band modules which are syzygies.

> **3.2.16. Corollary.** Suppose that M is a band module. If there is a band module $N \in \text{mod-}A$ such that $\Omega(N) \simeq M$ in $\underline{\text{mod}}\text{-}A$, then this choice of N is unique up to isomorphism.

Proof. We keep the same notation from Proposition 3.2.13.

By applying properties (a)-(d), we see that for any module $N' \in B'$ there exists a short exact sequence:

$$0 \to N' \to Q \to M' \to 0,$$

where Q is projective and injective, and where $M' \in B$. Hence $N' \simeq \Omega^{-1}(\Omega(N'))$. Since this did not depend on $N' \in B'$, we know that the composition of maps $\Omega^{-1}|_B \circ \Omega|_{B'}$ is the identity map on B'. This means that $\Omega|_{B'} : B' \to B$ must be injective.

Therefore any band module $N \in \text{mod-}A$ with $\Omega(N) \simeq M$ must be unique, as required. □

3.2.17. Remark. Note that this result does not hold for string modules, as can be seen immediately for our running example algebra (introduced in Paragraph 2.2.41). The simple module, S_2, corresponding to vertex 2 can be considered as a submodule of the projective $P_2 = e_2A$ in two obviously different ways; by mapping to the basis element corresponding to the path $\gamma\alpha\beta$ or the basis element corresponding to δ^2. Hence S_2 can clearly be expressed as a syzygy of a string module in (at least) two different ways.

Chapter 4

Delooping level in special biserial algebras

In this chapter, we investigate delooping levels of string and band modules of special biserial algebras. This involves using the constructions discussed so far for syzygies of string and band modules in terms of their strips and belts (Propositions 2.4.38, 2.4.41, 3.2.6 and 3.2.8), alongside the algorithms for calculating the transpose of these modules in terms of the same (Propositions 2.4.30–2.4.32 and 3.1.14).

During discussions with Gélinas about delooping level, he mentioned that characterising the functors $\Omega\,\mathrm{Tr}$ and Ω^n for an algebra, A, and its opposite, A^{op}, is often enough to identify all the A-modules with delooping level at most n (as $\Omega^{n+1}\Sigma^{n+1}\Omega^n = \Omega^n(\Omega\,\mathrm{Tr})\Omega^n(\Omega\,\mathrm{Tr})\Omega^n$ is the main functor in one of the equivalent definitions of delooping level, see Theorem 2.2.67). Gélinas has already completed such a characterisation for radical-square-zero algebras in an unpublished set of notes.

The first section of this chapter focuses on giving a characterisation of $\Omega\,\mathrm{Tr}$ for special biserial algebras, within the formalism of strips and belts.

The second section then applies this understanding of $\Omega\,\mathrm{Tr}$ to obtain results about the delooping levels of band and string modules. In particular, we give a new necessary condition for band modules to have non-zero delooping level (Proposition 4.2.1), a sufficient condition for band modules to have zero delooping level (Proposition 4.2.2), and a necessary and sufficient condition for simple non-projective modules to have non-zero delooping level (Lemma 4.2.4).

The original aim of this chapter was to build enough understanding of the combinatorial formalism surrounding strips and belts to prove that SB algebras always have finite delooping level, as conjectured by Huisgen-Zimmermann [HZ22, Section 4]. Unfortunately, we were not able to reach this goal; however Chapter 5 uses the results of this chapter to prove that if an SB algebra, A, satisfies $\mathrm{rad}^3(A) = 0$, then it must have finite delooping level.

4.1 Syzygy-transpose of string and band modules

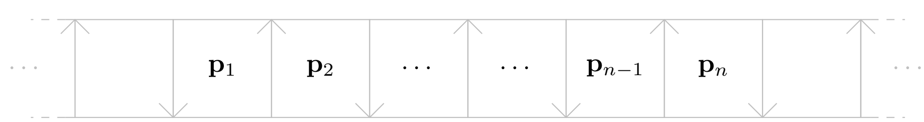
4.1.1. Twisting. The following operation on syllables is denoted $\bowtie_A = \bowtie$ and called *twisting*. For $(s \xrightarrow{l} \circ \xrightarrow{\varepsilon} t)^o \in \mathrm{Syll}(A)$, we define

$$\bowtie\left((s \xrightarrow{l} \circ \xrightarrow{\varepsilon} t)^o\right) \coloneqq (s^{\mathrm{op}} \xrightarrow{\max(0, c_t - (l+\varepsilon))} \circ \xrightarrow{1-(\varepsilon-1)(d_t-1)} \circ)^o \in \mathrm{Syll}(A^{\mathrm{op}}).$$

4.1.2. Remark. Note that the twisting operation, $\bowtie$, preserves the orientation of syllables, and that $\bowtie(s \xrightarrow{l} \circ \xrightarrow{\varepsilon} t)^o$ is a boundary syllable if and only if $d_t = 1$ or $\varepsilon = 1$.

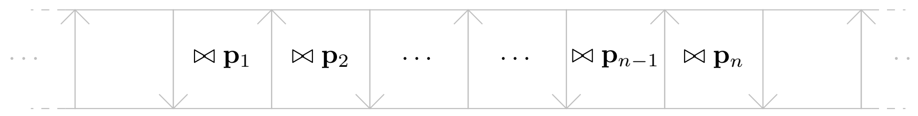
This twisting operation completely determines the behaviour of the functor $\Omega\,\mathrm{Tr} : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A^{\mathrm{op}}$. We handle string modules (represented by strips) first.

4.1.3. Proposition. Let M be a non-projective string module of A represented by a strip of the form:



where $\mathbf{p}_1, \mathbf{p}_n$ are boundary syllables, and $\mathbf{p}_2, \ldots, \mathbf{p}_{n-1}$ are interior syllables. (Note that $\mathbf{p}_1, \mathbf{p}_n$ may be stationary syllables.)

Then $\Omega(\mathrm{Tr}(M))$ is a string module of A^{op} represented by a flattened family of strips of the form:



Proof. To handle all of the possible non-projective string modules of A, we will need to apply Propositions 2.4.30–2.4.32. Thus we will split into the three cases that each of them handle:

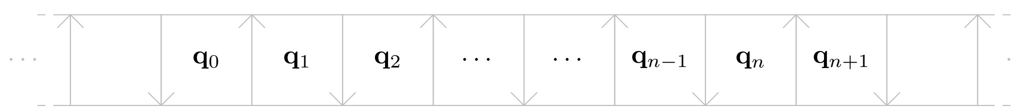
(i) $\mathrm{intwid}(M) > 0$ (i.e. $n > 2$),

(ii) $\mathrm{intwid}(M) = 0$ (i.e. $n = 2$) and $\mathbf{p}_1 \in \mathrm{supp}(\nabla_A)$,

(iii) $\mathrm{intwid}(M) = 0$ (i.e. $n = 2$) and $\mathbf{p}_1, \mathbf{p}_2 \notin \mathrm{supp}(\nabla_A)$.

Case (i): $\mathrm{intwid}(M) > 0$ (i.e. $n > 2$)

Let us denote the compression of each $\mathbf{p}_i$ by p_i, and the source and target of p_i by s_i and t_i respectively. Let $o_i \in \{1, -1\}$ be the orientation of the syllable $\mathbf{q}_i$ for each $i \in \{1, \ldots, n\}$. By Proposition 2.4.30, we know that $\mathrm{Tr}(M)$ is a string module represented by a strip of the form:



where $\mathbf{q}_3 = \mathbf{p}_3^{\mathrm{op}}, \ldots, \mathbf{q}_{n-2} = \mathbf{p}_{n-2}^{\mathrm{op}}$, for $i \in \{0, 1, 2\}$ the $\mathbf{q}_i$ are defined as:

$$\mathbf{q}_0 \coloneqq \begin{cases} \left(\, (t_1^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1} \circ \xrightarrow{1} \circ \, \right) & \text{if } \mathbf{p}_1 \in \mathrm{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_1 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_1 \coloneqq \begin{cases} \left(\, t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)} \circ \xrightarrow{0} \circ \, \right) & \text{if } \mathbf{p}_1 \in \mathrm{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_1 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_2 \coloneqq \begin{cases} \mathbf{p}_2^{\mathrm{op}} & \text{if } \mathbf{p}_1 \in \mathrm{supp}(\nabla_A) \\ \left(\, t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2) - 1} \circ \xrightarrow{1} \circ \, \right) & \text{if } \mathbf{p}_1 \notin \mathrm{supp}(\nabla_A) \end{cases}$$

and for $i \in \{n-1, n, n+1\}$, the $\mathbf{q}_i$ are defined symmetrically (see statement of Proposition 2.4.30 for explicit definitions).

We now note that when we are taking syzygies we compute "peak-by-peak", that $\mathbf{q}_2$ and $\mathbf{q}_{n-1}$ are never pin-boundary syllables, and that their compressions q_2 and q_{n-1} are independent of which case their definitions fall into. Thus when computing the syzygy of $\mathrm{Tr}(M)$, columns 2 to $n-1$ can be computed by applying descent to the interior syllables $\mathbf{p}_i^{\mathrm{op}}$. Now:

$$\begin{aligned}
\nabla_{A^{\mathrm{op}}} \mathbf{p}_i^{\mathrm{op}} &= \nabla_{A^{\mathrm{op}}} \left(t_i^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_i)} \circ \xrightarrow{0} \circ \right)^{-o_i} \\
&= \left(t_i^{\mathrm{op}} - \mathrm{len}(p_i) \xrightarrow{c_{t_i} - \mathrm{len}(p_i)} \circ \xrightarrow{d_{t_i}} \circ \right)^{-o_i} \\
&= \left(s_i \xrightarrow{c_{t_i} - \mathrm{len}(p_i)} \circ \xrightarrow{d_{t_i}} \circ \right)^{o_i} \\
&= \bowtie_A \mathbf{p}_i
\end{aligned}$$

as required.

Thus it remains to verify that strip representing $\Omega(\mathrm{Tr}(M))$ agrees with the required form at both ends of the strip. Since both ends are symmetrical, and the syzygy algorithm preserves this symmetry, we will handle the left end and the right will follow by symmetry.

We now split into three subcases based on whether or not $\mathbf{p}_1$ belongs to $\mathrm{supp}(\nabla_A)$ and the what value of $d_{t_1} \in \{0, 1\}$ is.

First, suppose that $\mathbf{p}_1 \notin \mathrm{supp}(\nabla_A)$. Thus $\mathbf{q}_0$ and $\mathbf{q}_1$ are both blank syllables, so the patch defined by them is the blank patch. In this case, the module defined by the strip is preserved under adding the appropriate stationary syllable to the end (i.e., rounding off the strip). Since $\mathbf{p}_1 = \left(s_1 \xrightarrow{a_{s_1} + b_{s_1} - 1} \circ \xrightarrow{1} t_1 \right)$ we know that:

$$\begin{aligned}
\bowtie \mathbf{p}_1 &= \left(s_1^{\mathrm{op}} \xrightarrow{\max(0,\, c_{t_1} - ((a_{s_1} + b_{s_1} - 1) + 1))} \circ \xrightarrow{1 - (1-1)(d_{t_1} - 1)} \circ \right) \\
&= \left(s_1^{\mathrm{op}} \xrightarrow{\max(0,\, c_{t_1} - (a_{s_1} + b_{s_1}))} \circ \xrightarrow{1} \circ \right) \\
&= \left(s_1^{\mathrm{op}} \xrightarrow{0} \circ \xrightarrow{1} \circ \right)
\end{aligned}$$

which is the stationary syllable required.

Now suppose instead that $\mathbf{p}_1 \in \mathrm{supp}(\nabla_A)$ and $d_{t_1} = 0$. Then $\mathbf{q}_0$ is a pin-boundary syllable, so the syllable below it in the syzygy fabric is blank. This also implies that the syllable below $\mathbf{q}_1$ is given by the perturbed syllable:

$$\left(s_1^{\mathrm{op}} \xrightarrow{c_{t_1} - \mathrm{len}(p_1)} \circ \xrightarrow{1} \circ \right) = \bowtie \mathbf{p}_1$$

as required.

Now suppose instead that $\mathbf{p}_1 \in \operatorname{supp}(\nabla_A)$ and $d_{t_1} = 1$. Then $\mathbf{q}_0 \notin \operatorname{supp}(\nabla_A)$ so the syllable below it in the syzygy fabric is blank. Since $\mathbf{q}_0$ is not pin-boundary, the syllable below $\mathbf{q}_1$ is given by:

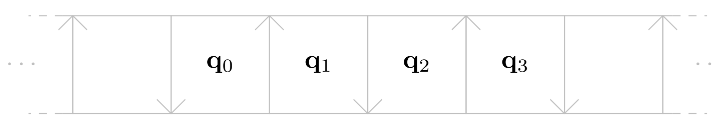
$$\nabla_{A^{\mathrm{op}}}\mathbf{q}_1 = (\ s_1^{\mathrm{op}} \xrightarrow{c_{t_1} - \operatorname{len}(p_1)} \circ \xrightarrow{1} \circ\ ) \mathrel{=\!\!\bowtie} \mathbf{p}_1$$

as required.

Thus we have shown that if $M \in \text{mod-}A$ belongs to this case, then $\Omega(\operatorname{Tr}(M))$ is represented by a strip of the required form.

Case (ii): $\operatorname{intwid}(M) = 0$ (i.e. $n = 2$) and $\mathbf{p}_1 \in \operatorname{supp}(\nabla_A)$

Let us denote the compression of each $\mathbf{p}_i$ by p_i, and the source and target of p_i by s_i and t_i respectively. By Proposition 2.4.31, we know that $\operatorname{Tr}(M)$ is a string module represented by a strip of the form:



where $\mathbf{q}_i$ are defined as follows:

$$\mathbf{q}_0 \coloneqq (\ (t_1^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1} \circ \xrightarrow{1} \circ\ )$$

$$\mathbf{q}_1 \coloneqq \begin{cases} (\ t_1^{\mathrm{op}} \xrightarrow{\operatorname{len}(p_1)} \circ \xrightarrow{0} \circ\ ) & \text{if } \mathbf{p}_2 \in \operatorname{supp}(\nabla_A) \\ (\ t_1^{\mathrm{op}} \xrightarrow{\operatorname{len}(p_1) - 1} \circ \xrightarrow{1} \circ\ ) & \text{if } \mathbf{p}_2 \notin \operatorname{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_2 \coloneqq \begin{cases} (\ t_2^{\mathrm{op}} \xrightarrow{\operatorname{len}(p_2)} \circ \xrightarrow{0} \circ\ ) & \text{if } \mathbf{p}_2 \in \operatorname{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_2 \notin \operatorname{supp}(\nabla_A) \end{cases}$$

$$\mathbf{q}_3 \coloneqq \begin{cases} (\ (t_2^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_2)^\dagger} + d_{(t_2)^\dagger} - 1} \circ \xrightarrow{1} \circ\ ) & \text{if } \mathbf{p}_2 \in \operatorname{supp}(\nabla_A) \\ (\quad) & \text{if } \mathbf{p}_2 \notin \operatorname{supp}(\nabla_A) \end{cases}$$

We first note that the compression q_1 of $\mathbf{q}_1$ is independent of which case their definition fall into. Thus the lower row of the patch defined by $\mathbf{q}_0, \mathbf{q}_1$ is independent of which case the definition of $\mathbf{q}_1$

falls into.

Therefore (as with the subcases for $\mathbf{p}_1 \in \mathrm{supp}(\nabla_A)$ in case (i)), the syllable below $\mathbf{q}_0$ in the syzygy fabric is always blank, and the syllable below $\mathbf{q}_1$ always takes the form:

$$(\; s_1^{\mathrm{op}} \xrightarrow{c_{t_1} - \mathrm{len}(p_1)} \circ \xrightarrow{1} \circ \;) = \bowtie \mathbf{p}_1$$

as required.

It now remains to show that the syllables below $\mathbf{q}_2, \mathbf{q}_3$ take the form required (possibly after rounding off).

We first consider the case where $\mathbf{p}_2 \in \mathrm{supp}(\nabla_A)$. This case can then be reflected and the same logic applied as for $\mathbf{p}_1$ above. Hence the syllable below $\mathbf{q}_3$ is blank and the syllable below $\mathbf{q}_2$ always takes the form:

$$(\; s_2^{\mathrm{op}} \xrightarrow{c_{t_2} - \mathrm{len}(p_2)} \circ \xrightarrow{1} \circ \;) = \bowtie \mathbf{p}_2$$

as required.

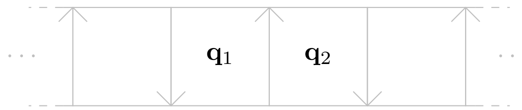
Now, instead consider the case where $\mathbf{p}_2 \notin \mathrm{supp}(\nabla_A)$. In this case, both $\mathbf{q}_2$ and $\mathbf{q}_3$ are blank syllables, and thus the patch that they define is the blank patch. Then (as with the subcase for $\mathbf{p}_1 \notin \mathrm{supp}(\nabla_A)$, we can round off the strip with the appropriate stationary syllable. As before, the stationary syllable required is:

$$\begin{aligned}
\bowtie \mathbf{p}_2 &= (\; s_2^{\mathrm{op}} \xrightarrow{\max(0,\, c_{t_2} - ((a_{s_2} + b_{s_2} - 1) + 1))} \circ \xrightarrow{1 - (1-1)(d_{t_2} - 1)} \circ \;) \\
&= (\; s_2^{\mathrm{op}} \xrightarrow{\max(0,\, c_{t_2} - (a_{s_2} + b_{s_2}))} \circ \xrightarrow{1} \circ \;) \\
&= (\; s_2^{\mathrm{op}} \xrightarrow{0} \circ \xrightarrow{1} \circ \;).
\end{aligned}$$

Thus we have shown that if $M \in \text{mod-}A$ belongs to this case, then $\Omega(\mathrm{Tr}(M))$ is represented by a strip of the required form.

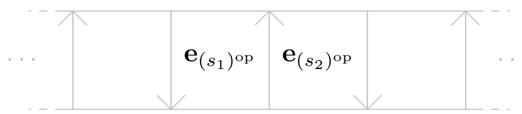
Case (iii): $\mathrm{intwid}(M) = 0$ (i.e. $n = 2$) and $\mathbf{p}_1, \mathbf{p}_2 \notin \mathrm{supp}(\nabla_A)$

Since $\mathbf{p}_1, \mathbf{p}_2 \notin \mathrm{supp}(\nabla)_A$, they can both be written in the form $\mathbf{p}_i = (\; s_i \xrightarrow{a_{s_i} - 1} \circ \xrightarrow{1} t_i \;)$ (noting that $b_{s_i} = 0$ for $i = 1, 2$ since M is non-projective). By Proposition 2.4.32, we know that $\mathrm{Tr}(M)$ is a string module represented by a strip of the form:



where $\mathbf{q}_1 := (\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)} \circ \xrightarrow{1} (s_1)^{\mathrm{op}}\ )$ and $\mathbf{q}_2 := (\ t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2)} \circ \xrightarrow{1} (s_2)^{\mathrm{op}}\ )$.

Hence, by applying the syzygy algorithm for strips (and skipping the row with virtual syllables), we observe that $\Omega(\mathrm{Tr}(M)) \in \text{mod-}A^{\mathrm{op}}$ is a string module represented by a strip of the form:



where as usual $\mathbf{e}_{(s_i)^{\mathrm{op}}} := (\ (s_i)^{\mathrm{op}} \xrightarrow{0} \circ \xrightarrow{1} \circ\ )$ for $i = 1, 2$.

It now suffices to note that for $i = 1, 2$:

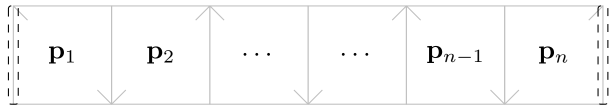
$$
\begin{aligned}
\bowtie \mathbf{p}_i &= (\ (s_i)^{\mathrm{op}} \xrightarrow{\max(0,\, c_{t_i} - ((a_{s_i} - 1) + 1))} \circ \xrightarrow{1 - (1-1)(0-1)} \circ\ ) \\
&= (\ (s_i)^{\mathrm{op}} \xrightarrow{\max(0,\, c_{t_i} - a_{s_i})} \circ \xrightarrow{1} \circ\ ) \\
&= (\ (s_i)^{\mathrm{op}} \xrightarrow{0} \circ \xrightarrow{1} \circ\ ) \qquad \text{as } c_{t_i} = a_{s_i} \text{ when } t_i = s_i - a_{s_i} \text{ and } b_{s_i} = 0 \\
&= \mathbf{e}_{(s_i)^{\mathrm{op}}}
\end{aligned}
$$

Thus we have shown that if $M \in \text{mod-}A$ belongs to this case, then $\Omega(\mathrm{Tr}(M))$ is represented by a strip of the required form. □

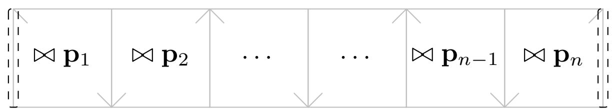
4.1.4. Remark. We note at this point, that this characterisation of $\Omega\,\mathrm{Tr}$ for strips is considerably nicer to work with than the characterisation of Tr for strips given in Propositions 2.4.30–2.4.32, as we can work on a "syllable-by-syllable" basis. We also avoid having to split into cases based on the interior width of the strip.

To describe $\Omega(\mathrm{Tr}(M))$ for band modules $M \in \text{mod-}A$ is slightly more complicated, as the resulting A^{op}-module could be a band module or a string module.

4.1.5. Proposition. Let M be a band module of A represented by a belt, w, of the form:



where all syllables $\mathbf{p}$ of w have $d_{t(\mathbf{p})} = 0$. Then $\Omega(\mathrm{Tr}(M))$ is a band module of A^{op} represented by a band of the form:

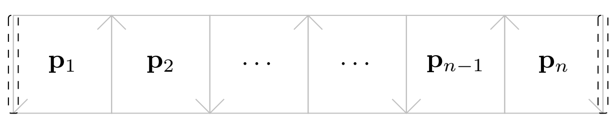


Proof. We first note that if $\mathbf{p} = (\ s \xrightarrow{\ l\ } \circ \xrightarrow{\ 0\ } t\ )^o$ is an interior syllable, then:

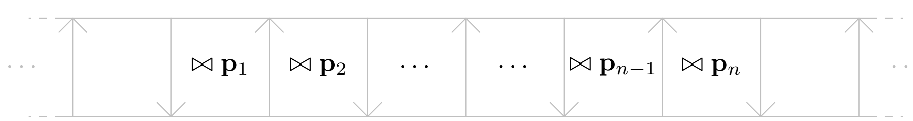
$$\begin{aligned}\nabla_{A^{\mathrm{op}}}(\mathbf{p}^{\mathrm{op}}) &= \nabla_{A^{\mathrm{op}}}((\ t^{\mathrm{op}} \xrightarrow{\ l\ } \circ \xrightarrow{\ 0\ } s^{\mathrm{op}}\ )^{-o}) \\ &= (\ s^{\mathrm{op}} \xrightarrow{\ c_t - l\ } \circ \xrightarrow{\ d_t\ } \circ\ )^o = \bowtie \mathbf{p}\end{aligned}$$

Thus the result follows from the characterisation of the functors Tr and $\Omega_{A^{\mathrm{op}}}$ on band modules represented by belts, given in Propositions 3.1.14 and 3.2.6. (Note that we use the notation c_t instead of $a_{t^{\mathrm{op}}}$ (resp. d_t instead of $b_{t^{\mathrm{op}}}$) to avoid confusion with a_t (resp. b_t).) $\square$

4.1.6. Proposition. Let M be a band module of A represented by a belt of the form:



where $d_{t(\mathbf{p}_1)} = 1$. Then $\Omega(\mathrm{Tr}(M))$ is a string module of A^{op} represented by a flattened family of strips of the form:



Proof. As noted in the proof of Proposition 4.1.5, if $\mathbf{p}$ is an interior syllable, then $\nabla_{A^{\mathrm{op}}}(\mathbf{p}^{\mathrm{op}}) = \bowtie \mathbf{p}$.

Thus the result follows from the characterisation of the functors Tr and $\Omega_{A^{\mathrm{op}}}$ on band modules represented by belts, given in Propositions 3.1.14 and 3.2.8. (Again, note that we use the notation c_t instead of $a_{t^{\mathrm{op}}}$ (resp. d_t instead of $b_{t^{\mathrm{op}}}$) to avoid confusion with a_t (resp. b_t).) $\square$

These characterisations of $\Omega\,\mathrm{Tr}(M)$ give the following:

4.1.7. Proposition. A non-projective string module M has $M \simeq \Omega\Sigma(M)$ if and only if $\Omega\,\mathrm{Tr}(M)$ is not projective and all syllables $\mathbf{p}$ of a strip representing M have $\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) = \mathbf{p}$.

Similarly, a band module M has $M \simeq \Omega\Sigma(M)$ if and only if $\Omega\,\mathrm{Tr}(M)$ is not projective and all syllables $\mathbf{p}$ of a belt representing M have $\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) = \mathbf{p}$.

Proof. This follows immediately from the fact that $\Omega\Sigma = (\Omega\,\mathrm{Tr})(\Omega\,\mathrm{Tr})$. $\square$

We now aim to understand the syllables $\mathbf{p} \in \mathrm{Syll}(A)$ where $\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) = \mathbf{p}$.

4.1.8. Proposition. For a syllable $\mathbf{p} = (s \xrightarrow{l} \circ \xrightarrow{\varepsilon} t)$ of A, $\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) = \mathbf{p}$ if and only if one of the following mutually exclusive conditions holds:

(a) $\varepsilon = 0$ and $d_t = 0$,

(b) $\varepsilon = 1$ and $l = 0$,

(c) $\varepsilon = 1$, $l > 0$ and $\max(l+1, c_t) = a_{t+1+\max(l+1,c_t)}$.

Proof. Let $\mathbf{p} = (s \xrightarrow{l} \circ \xrightarrow{\varepsilon} t)$. We say that $(\star)$ is satisfied if $\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) = \mathbf{p}$.

Case 1: $\mathbf{p}$ is interior, i.e. $\varepsilon = 0$

Since $(s \xrightarrow{l} t)$ is a non-zero path, we know that $c_t \geq l$ and thus $\bowtie_A \mathbf{p} = (s^{\mathrm{op}} \xrightarrow{c_t - l} \circ \xrightarrow{d_t} \circ)$.

Suppose that $d_t = 1$. Since $\bowtie_{A^{\mathrm{op}}}$ sends boundary syllables to boundary syllables, this implies that $\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) \neq \mathbf{p}$, so $(\star)$ is not satisfied.

Now suppose that $d_t = 0$. Thus:

$$\begin{aligned}
\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) &= (\ s \xrightarrow{a_{(s^{\mathrm{op}}-(c_t-l))^{\mathrm{op}}} - (c_t - l)} \circ \xrightarrow{b_{(s^{\mathrm{op}}-(c_t-l))^{\mathrm{op}}}} \circ\ ) \\
&= (\ s \xrightarrow{a_{s+(c_t-l)} - (c_t - l)} \circ \xrightarrow{b_{s+(c_t-l)}} \circ\ ) \\
&= (\ s \xrightarrow{a_{t+c_t} - c_t + l} \circ \xrightarrow{b_{t+c_t}} \circ\ ) \\
&= (\ s \xrightarrow{l} \circ \xrightarrow{0} \circ\ )
\end{aligned}$$

where the last equality follows from the fact that when $d_t = 0$, we have $a_{t+c_t} = c_t$ and $b_{t+c_t} = d_t = 0$. Hence $(\star)$ is satisfied.

Thus we have proved both directions of the equivalence in this case.

Case 2: $\mathbf{p}$ is a stationary syllable, i.e. $\varepsilon = 1$ and $l = 0$

Then $\bowtie_A \mathbf{p} = (\ s^{\mathrm{op}} \xrightarrow{\max(0, c_t - 1)} \circ \xrightarrow{1} s^{\mathrm{op}} - \max(1, c_t)\ )$. Thus:

$$\begin{aligned}
\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) &= (\ s \xrightarrow{\max(0, c_{s^{\mathrm{op}} - \max(1,c_t)} - \max(0, c_t - 1) - 1)} \circ \xrightarrow{1} \circ\ ) \\
&= (\ s \xrightarrow{\max(0, a_{s+\max(1,c_t)} - \max(1, c_t))} \circ \xrightarrow{1} \circ\ )
\end{aligned}$$

Hence:

$$\begin{aligned}
(\star) &\iff 0 = \max(0, a_{s+\max(1,c_t)} - \max(1, c_t)) \\
&\iff 0 \geq a_{s+\max(1,c_t)} - \max(1, c_t) \\
&\iff \max(1, c_t) \geq a_{s+\max(1,c_t)} \\
&\iff \max(1, c_t) \geq a_{t+1+\max(1,c_t)} \\
&\iff 1 + \max(1, c_t) > a_{t+(1+\max(1,c_t))}
\end{aligned}$$

Since $1 + \max(1, c_t) > c_t$, the final equivalent condition in this chain follows from Lemma 2.3.13. Hence $(\star)$ is always satisfied in this case, as required.

Case 3: p is a non-stationary boundary syllable, i.e. $\varepsilon = 1$ and $l > 0$

Then:

$$\bowtie_A \mathbf{p} = \left(s^{\mathrm{op}} \xrightarrow{\max(0,c_t-(l+1))} \circ \xrightarrow{1} s^{\mathrm{op}} - \max(0, c_t - (l+1)) - 1 \right)$$
$$= \left(s^{\mathrm{op}} \xrightarrow{\max(0,c_t-(l+1))} \circ \xrightarrow{1} s^{\mathrm{op}} - \max(1, c_t - l) \right)$$

Thus:

$$\bowtie_{A^{\mathrm{op}}} (\bowtie_A (\mathbf{p})) = \left(s \xrightarrow{\max(0, c_{s^{\mathrm{op}}-\max(1,c_t-l)} - \max(0,c_t-(l+1))-1)} \circ \xrightarrow{1} \circ \right)$$
$$= \left(s \xrightarrow{\max(0, c_{s^{\mathrm{op}}-\max(1,c_t-l)} - \max(1,c_t-l))} \circ \xrightarrow{1} \circ \right)$$

So $(\star)$ is satisfied if and only if $l = \max(0, c_{s^{\mathrm{op}}-\max(1,c_t-l)} - \max(1, c_t - l))$. Hence, in this case:

$$\begin{aligned}
(\star) &\iff l = \max(0, c_{s^{\mathrm{op}}-\max(1,c_t-l)} - \max(1, c_t - l)) \\
&\iff l = c_{s^{\mathrm{op}}-\max(1,c_t-l)} - \max(1, c_t - l) \qquad \text{as } l > 0 \\
&\iff l + \max(1, c_t - l) = a_{s+\max(1,c_t-l)} \\
&\iff \max(l + 1, c_t) = a_{s-l+\max(l+1,c_t)} \\
&\iff \max(l + 1, c_t) = a_{t+1+\max(l+1,c_t)}
\end{aligned}$$

Thus the required equivalence in this case has been shown. $\square$

4.1.9. Remark. Note that condition (c) of Proposition 4.1.8 can be split into:

(i) $\varepsilon = 1$, $l > 0$, $l + 1 \leq c_t$ and $c_t = a_{t+1+c_t}$,

(ii) $\varepsilon = 1$, $l > 0$, $l + 1 > c_t$ and $l + 1 = a_{t+l+2}$.

Further note that for a syllable $\mathbf{p} = (s \xrightarrow{l} \circ \xrightarrow{1} t)$, if $l + 1 > c_t$, then $(s \xrightarrow{l+1} t)$ is not a non-zero $\mathcal{O}$-path. But we know that $(s \xrightarrow{l} \circ)$ is a non-zero $\mathcal{O}$-path, which implies that $l = a_s$. Thus $l + 1 > c_t$ implies that $\mathbf{p} \notin \mathrm{supp}(\nabla)$ and is of the form $(s \xrightarrow{a_s} \circ \xrightarrow{1} \circ)$ for some vertex s of $\mathcal{O}$ with $b_s = 1$.

Therefore, in most instances condition (c) in Proposition 4.1.8 reduces to (i) above.

$$\left(\circ \xrightarrow{\alpha} \circ \xrightarrow{0} \circ\right), \quad \left(\circ \xrightarrow{\gamma} \circ \xrightarrow{0} \circ\right), \quad \left(\circ \xrightarrow{\gamma\alpha} \circ \xrightarrow{0} \circ\right), \quad \left(\circ \xrightarrow{\beta\gamma} \circ \xrightarrow{0} \circ\right),$$

$$\left(\circ \xrightarrow{e_{s(\alpha)}} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{e_{s(\beta)}} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{e_{s(\gamma)}} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{e_{s(\delta)}} \circ \xrightarrow{1} \circ\right),$$

$$\left(\circ \xrightarrow{\alpha} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{\beta} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{\gamma} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{\delta} \circ \xrightarrow{1} \circ\right),$$

$$\left(\circ \xrightarrow{\alpha\beta} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{\beta\gamma} \circ \xrightarrow{1} \circ\right), \quad \left(\circ \xrightarrow{\gamma\alpha} \circ \xrightarrow{1} \circ\right).$$

Figure 4.1: *List of syllables satisfying* $\bowtie_{A^{\text{op}}}(\bowtie_A(\mathbf{p})) = \mathbf{p}$ *for our running example algebra.*
As usual, we suppress mention of the orientations, since they don't impact this property.
The first row consists of syllables that meet condition (a), the second row consists of syllables that meet condition (b) (the stationary syllables), and the remainder are the syllables that meet condition (c). Note that none of the syllables for the running example algebra meet condition (ii) discussed in Remark 4.1.9.

4.1.10. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41, and $\mathcal{O}$ its overquiver. The only vertices i of $\mathcal{O}$ with $d_i = 0$ are $s(\alpha) = t(\gamma)$ and $s(\beta) = t(\alpha)$. All vertices i of $\mathcal{O}$ satisfy $a_{i+c_i+1} = c_i$. It follows that the complete list of syllables, $\mathbf{p}$, satisfying $\bowtie_{A^{\text{op}}}(\bowtie_A(\mathbf{p})) = \mathbf{p}$ is given in Figure 4.1.

Our running example then illustrates the following lemma as well.

4.1.11. Lemma. Suppose $\mathcal{C}$ is a connected component of $\mathcal{O}$ where $a_{i+(c_i+1)} = c_i$ for all $i \in \mathcal{C}$. Then c_i is constant for $i \in \mathcal{C}$, and $a_i = c_i$ for all $i \in \mathcal{C}$.

Proof. By Proposition 2.3.15, it is enough to prove that c_i is constant for $i \in \mathcal{C}$.

Suppose to the contrary that c_i is not constant for $i \in \mathcal{C}$. Then there exists $i_0 \in \mathcal{C}$ such that $c_{i_0} > c_{i_0+1}$. Since $c_{i_0} \leq c_{i_0+1} + d_{i_0+1}$, we have $d_{i_0+1} = 1$ and $c_{i_0} = c_{i_0+1} + 1$.

Then

$$c_{i_0} = a_{i_0+(c_{i_0}+1)} = a_{i_0+(c_{i_0+1}+2)} = a_{(i_0+1)+(c_{i_0+1}+1)} = c_{i_0+1} \neq c_{i_0},$$

which is clearly a contradiction. □

Since Proposition 4.1.7 allows us to characterise $M \in \underline{\text{mod}}\text{-}A$ which satisfy $M \simeq \Omega\Sigma(M)$, we can use this result to characterise $M \in \underline{\text{mod}}\text{-}A$ where $M \simeq \Sigma\Omega(M)$.

First, we prove a small lemma that allows us to simplify the statement of our upcoming characterisation significantly.

4.1.12. Lemma. Let i be a vertex of the overquiver, $\mathcal{O}$. Then

$$\max(c_i + d_i, a_{i+c_i+d_i}) = c_{i+c_i+d_i-1-\max(c_i+d_i,a_{i+c_i+d_i})}$$

if and only if either $d_i = 1$ or $c_i = c_{i-1}$.

Proof. First, we handle the case where $d_i = 1$. In this case we need to show that:

$$\max(c_i + 1, a_{i+c_i+1}) = c_{i+c_i-\max(c_i+1,a_{i+c_i+1})}.$$

Since Lemma 2.3.13 gives us that $c_i + 1 < a_{i+c_i+1}$, this condition simplifies to:

$$a_{i+c_i+1} = c_{i+c_i-a_{i+c_i+1}}.$$

Now, define $j \coloneqq i + c_i + 1 \in \mathcal{O}$ and $k \coloneqq c_i - a_{i+c_i+1} \in \mathbb{Z}$. Then Lemmas 2.3.12 and 2.3.13 respectively give:

$$a_{i+c_i+1} = c_i - k \leq c_{i+k} = c_{i+c_i-a_{i+c_i+1}}$$

$$a_{i+c_i+1} + 1 = a_j + 1 > c_{j-(a_j+1)} = c_{i+c_i+1-a_{i+c_i+1}-1} = c_{i+c_i-a_{i+c_i+1}}$$

Which (since all values in these inequalities are integers), immediately implies $a_{i+c_i+1} = c_{i+c_i-a_{i+c_i+1}}$, as required.

Now we handle the case where $d_i \neq 1$ (i.e. $d_i = 0$) and $c_i = c_i - 1$. Since we have assumed that $d_i = 0$, our original condition simplifies to:

$$\max(c_i, a_{i+c_i}) = c_{i+c_i-1-\max(c_i,a_{i+c_i})}.$$

Now, Lemma 2.3.12 gives us that $c_i \geq a_{i+c_i}$, so this condition simplifies to:

$$c_i = c_{i+c_i-1-c_i},$$

which is clearly equivalent to $c_i = c_{i-1}$, as required. $\square$

We are now in a position to prove our characterisation.

4.1.13. Proposition. We say that a vertex i of $\mathcal{O}$ satisfies condition ♣ if either $d_i = 1$ or $c_i = c_{i-1}$. A non-projective string module M has $M \simeq \Sigma\Omega(M)$ if and only if it is represented by a strip w where each peak $\boxed{\mathbf{p}_1 \uparrow \mathbf{p}_2}$ of w meets one of the following mutually exclusive conditions (possibly after the peak is reflected):

(i) $\mathbf{p}_1, \mathbf{p}_2$ are both interior, and $b_{s(\mathbf{p}_1)} = b_{s(\mathbf{p}_2)} = 0$;

(ii) $\mathbf{p}_1 \in \operatorname{supp}(\nabla_A)$ is boundary, $\mathbf{p}_2$ is interior, $b_{s(\mathbf{p}_1)} = b_{s(\mathbf{p}_2)} = 0$, and $t(\mathbf{p}_1)^\dagger$ satisfies condition ♣;

(iii) $\mathbf{p}_1 \notin \operatorname{supp}(\nabla_A)$ is boundary, $\mathbf{p}_2$ is interior, and $a_{s(\mathbf{p}_2)} = c_{s(\mathbf{p}_2)-1-a_{s(\mathbf{p}_2)}}$;

(iv) $\mathbf{p}_1, \mathbf{p}_2 \in \operatorname{supp}(\nabla_A)$ are both boundary, $b_{s(\mathbf{p}_1)} = b_{s(\mathbf{p}_2)} = 0$ and $t(\mathbf{p}_1)^\dagger, t(\mathbf{p}_2)^\dagger$ both satisfy condition ♣;

(v) $\mathbf{p}_1 \in \operatorname{supp}(\nabla_A), \mathbf{p}_2 \notin \operatorname{supp}(\nabla_A)$ are both boundary, $a_{s(\mathbf{p}_1)} = c_{s(\mathbf{p}_1)-1-a_{s(\mathbf{p}_1)}}$ and $t(\mathbf{p}_1)^\dagger$ satisfies condition ♣;

(vi) $\mathbf{p}_1, \mathbf{p}_2 \notin \operatorname{supp}(\nabla_A)$ are both boundary, and $a_{s(\mathbf{p}_j)} = c_{s(\mathbf{p}_j)-1-a_{s(\mathbf{p}_j)}}$ for $j = 1, 2$.

Proof. By Lemma 2.2.70, it is sufficient to characterise string modules M where $\operatorname{Tr}(M)$ satisfies the conditions of Proposition 4.1.7.

Since the conditions of Proposition 4.1.7 are based entirely on the syllables present in a strip, it is sufficient to characterise when the syllables of a strip representing $\operatorname{Tr}(M)$ meet these conditions. Since the construction for $\operatorname{Tr}(M)$ given in Propositions 2.4.30–2.4.32 can be considered as a "peak-by-peak" operation on M, it is sufficient to characterise which peaks of a strip, w, representing M result in syllables of a strip w' representing $\operatorname{Tr}(M)$ meeting the necessary conditions. Since this operation is compatible with reflection, we can consider these peaks up to reflection, reducing the number of cases we have to consider.

When considering these peaks, we split into cases based on the number of interior and boundary syllables, and whether any of the boundary syllables belong to $\operatorname{supp}(\nabla_A)$ or not. We will denote the syllables in this peak by $\mathbf{p}_1$ and $\mathbf{p}_2$, and their compressions by p_1 and p_2 respectively. We then denote the sources and targets of the compressions respectively by s_i and t_i for $i \in \{1, 2\}$.

Case (i): $\mathbf{p}_1, \mathbf{p}_2$ are both interior.

In this case, we apply the construction of Proposition 2.4.30. By this construction, this peak of w results in a valley $\boxed{\mathbf{q}_1 \mathbf{q}_2}$ of w', where $\mathbf{q}_1 = \mathbf{p}_1^{\mathrm{op}}$ and $\mathbf{q}_2 = \mathbf{p}_2^{\mathrm{op}}$. It is clear that these fit the conditions of Proposition 4.1.8 if and only if $d_{t(\mathbf{q}_1)} = d_{t(\mathbf{q}_2)} = 0$. This is equivalent to $b_{s(\mathbf{p}_1)} = b_{s(\mathbf{p}_2)} = 0$.

Thus we have shown that the stated conditions are sufficient and necessary in this case.

Case (ii): $\mathbf{p}_1 \in \mathrm{supp}(\nabla_A)$ is boundary and $\mathbf{p}_2$ is interior.

In this case, we apply the construction of Proposition 2.4.30. By this construction, this peak of w results in an arrangement of syllables $\boxed{\mathbf{q}_0 \mathbf{q}_1 \mathbf{q}_2}$ in w', where $\mathbf{q}_0 = (\ (t_1^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1} \circ \xrightarrow{1} \circ\ )$, $\mathbf{q}_1 = (\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)} \circ \xrightarrow{0} s_1^{\mathrm{op}}\ )$ and $\mathbf{q}_2 = \mathbf{p}_2^{\mathrm{op}}$.

The interior syllables $\mathbf{q}_1$ and $\mathbf{q}_2$ meet the conditions of Proposition 4.1.8 if and only if $d_{t(\mathbf{q}_1)} = d_{t(\mathbf{q}_2)} = 0$. This is equivalent to $b_{s(\mathbf{p}_1)} = b_{s(\mathbf{p}_2)} = 0$.

If $c_{(t_1)^\dagger} = 0$, then $\mathbf{q}_0$ is stationary, and thus always meets the conditions of Proposition 4.1.8.

On the other hand, if $c_{(t_1)^\dagger} > 0$, then $\mathbf{q}_0$ is not stationary, and meets the conditions of Proposition 4.1.8 if and only if

$$\max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, c_{(t_1^{\mathrm{op}})^\dagger - c_{(t_1)^\dagger} - d_{(t_1)^\dagger}}) = a_{(t_1^{\mathrm{op}})^\dagger - c_{(t_1)^\dagger} - d_{(t_1)^\dagger} + 1 + \max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, c_{(t_1^{\mathrm{op}})^\dagger - c_{(t_1)^\dagger} - d_{(t_1)^\dagger}})}$$

Converting from A^{op} encoding to A encoding gives the following equivalent condition:

$$\max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, a_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger}}) = c_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1 - \max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, a_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger}})}$$

By Lemma 4.1.12, this is equivalent to condition ♣ for $(t_1)^\dagger$.

Thus we have shown that the stated conditions are sufficient and necessary in this case.

Case (iii): $\mathbf{p}_1 \notin \mathrm{supp}(\nabla_A)$ is boundary and $\mathbf{p}_2$ is interior.

In this case, we apply the construction of Proposition 2.4.30. By this construction, this peak of w results in the single syllable $\boxed{\mathbf{q}_2}$ in w', where $\mathbf{q}_2 = (\ t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2) - 1} \circ \xrightarrow{1} s_2^{\mathrm{op}}\ )$.

The boundary syllable $\mathbf{q}_2$ meets the conditions of Proposition 4.1.8 if and only if

$$\max(\mathrm{len}(p_2), c_{s_2^{\mathrm{op}}}) = a_{s_2^{\mathrm{op}} + 1 + \max(\mathrm{len}(p_2), c_{s_2^{\mathrm{op}}})}.$$

Switching from the encoding for A^{op} to the encoding for A gives the equivalent condition

$$\max(\mathrm{len}(p_2), a_{s_2}) = c_{s_2-1-\max(\mathrm{len}(p_2),a_{s_2})}.$$

Since $\mathbf{p}_2$ is interior, we know that $\mathbf{p}_2 \in \mathrm{supp}(\nabla_A)$. It follows that $\mathrm{len}(p_2) < a_{s_2} + b_{s_2}$, and thus that $\mathrm{len}(p_2) \leq a_{s_2}$. Therefore we can simplify the equivalent condition to

$$a_{s_2} = c_{s_2-1-a_{s_2}}.$$

Thus we have shown that the stated conditions are sufficient and necessary in this case.

Case (iv): $\mathbf{p}_1, \mathbf{p}_2 \in \mathrm{supp}(\nabla_A)$ are both boundary.

In this case, we apply the construction of Proposition 2.4.31. By this construction, this peak of w results in a pair of peaks $\boxed{\mathbf{q}_0 \,|\, \mathbf{q}_1 \,|\, \mathbf{q}_2 \,|\, \mathbf{q}_3}$ in w', where

$$\mathbf{q}_0 = \left((t_1^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1} \circ \xrightarrow{1} \circ \right)$$

$$\mathbf{q}_1 = \left(t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)} \circ \xrightarrow{0} s_1^{\mathrm{op}} \right)$$

$$\mathbf{q}_2 = \left(t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2)} \circ \xrightarrow{0} s_2^{\mathrm{op}} \right)$$

$$\mathbf{q}_3 = \left((t_2^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_2)^\dagger} + d_{(t_2)^\dagger} - 1} \circ \xrightarrow{1} \circ \right).$$

By the same logic as case (ii), $\mathbf{q}_0$ satisfies the conditions of Proposition 4.1.8 if and only if

$$\max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, a_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger}}) = c_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1 - \max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, a_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger}})},$$

which, by Lemma 4.1.12, is equivalent to condition ♣ for $(t_1)^\dagger$.

Similarly, $\mathbf{q}_3$ satisfies the conditions of Proposition 4.1.8 if and only if $(t_2)^\dagger$ satisfies condition ♣.

Now, for the interior syllables $\mathbf{q}_1$ and $\mathbf{q}_2$, they satisfy the conditions of Proposition 4.1.8 if and only if $d_{t(\mathbf{q}_1)} = d_{t(\mathbf{q}_2)} = 0$. This is equivalent to $b_{s(\mathbf{p}_1)} = b_{s(\mathbf{p}_2)} = 0$.

Thus we have shown that the stated conditions are sufficient and necessary in this case.

Case (v): $\mathbf{p}_1 \in \operatorname{supp}(\nabla_A), \mathbf{p}_2 \notin \operatorname{supp}(\nabla_A)$ are both boundary.

In this case, we apply the construction of Proposition 2.4.31. By this construction, this peak of w results in a peak $\boxed{\mathbf{q}_0 \,|\, \mathbf{q}_1}$ in w', where $\mathbf{q}_0 = (\ (t_1^{\mathrm{op}})^\dagger \xrightarrow{c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1} \circ \xrightarrow{1} \circ\ )$ and $\mathbf{q}_1 = (\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)-1} \circ \xrightarrow{1} s_1^{\mathrm{op}}\ )$.

By the same logic as case (ii), $\mathbf{q}_0$ satisfies the conditions of Proposition 4.1.8 if and only if

$$\max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, a_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger}}) = c_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger} - 1 - \max(c_{(t_1)^\dagger} + d_{(t_1)^\dagger}, a_{(t_1)^\dagger + c_{(t_1)^\dagger} + d_{(t_1)^\dagger}})},$$

which, by Lemma 4.1.12, is equivalent to condition ♣ for $(t_1)^\dagger$.

The boundary syllable $\mathbf{q}_1$ meets the conditions of Proposition 4.1.8 if and only if

$$\max(\mathrm{len}(p_1), c_{s_1^{\mathrm{op}}}) = a_{s_1^{\mathrm{op}} + 1 + \max(\mathrm{len}(p_1), c_{s_1^{\mathrm{op}}})}.$$

Switching from the encoding for A^{op} to the encoding for A gives the equivalent condition

$$\max(\mathrm{len}(p_1), a_{s_1}) = c_{s_1 - 1 - \max(\mathrm{len}(p_1), a_{s_1})}.$$

Since $\mathbf{p}_1 \in \operatorname{supp}(\nabla_A)$, it follows that $\mathrm{len}(p_1) < a_{s_1} + b_{s_1}$, and thus that $\mathrm{len}(p_1) \leq a_{s_1}$. Therefore we can simplify the equivalent condition to

$$a_{s_1} = c_{s_1 - 1 - a_{s_1}}.$$

Thus we have shown that the stated conditions are sufficient and necessary in this case.

Case (vi): $\mathbf{p}_1, \mathbf{p}_2 \notin \operatorname{supp}(\nabla_A)$ are both boundary.

In this case, we apply the construction of Proposition 2.4.32. By this construction, this peak of w results in a peak $\boxed{\mathbf{q}_1 \,|\, \mathbf{q}_2}$ in w', where $\mathbf{q}_1 = (\ t_1^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_1)-1} \circ \xrightarrow{1} s_1^{\mathrm{op}}\ )$ and $\mathbf{q}_2 = (\ t_2^{\mathrm{op}} \xrightarrow{\mathrm{len}(p_2)-1} \circ \xrightarrow{1} s_1^{\mathrm{op}}\ )$.

As discussed in Proposition 2.4.32, since M is non-projective, it follows that $b_{s_i} = d_{t_i} = 0$ for $i = 1, 2$, and that $a_{s_i} = c_{t_i} > 1$. Since $\mathbf{p}_i \notin \operatorname{supp}(\nabla_A)$, $\mathrm{len}(p_i) = a_{s_i} > 1$. Hence $\mathbf{q}_i$ is not stationary for $i = 1, 2$.

Therefore, for $i = 1, 2$, $\mathbf{q}_i$ meets the conditions of Proposition 4.1.8 if and only if:

$$\max(\mathrm{len}(p_i), c_{s_i^{\mathrm{op}}}) = a_{s_i^{\mathrm{op}}+1+\max(\mathrm{len}(p_i), c_{s_i^{\mathrm{op}}})}.$$

Switching from the encoding for A^{op} to the encoding for A gives the equivalent condition

$$\max(\mathrm{len}(p_i), a_{s_i}) = c_{s_i-1-\max(\mathrm{len}(p_i), a_{s_i})}.$$

Since (as discussed above) $\mathrm{len}(p_i) = a_{s_i}$, we can simplify this condition to:

$$a_{s_i} = c_{s_i-1-a_{s_i}}.$$

Thus we have shown that the stated conditions are sufficient and necessary in this case. □

4.1.14. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41, and $\mathcal{O}$ its overquiver. The only vertices i of $\mathcal{O}$ with $b_i = 0$ are $s(\alpha) = t(\gamma)$ and $s(\beta) = t(\alpha)$. Since $a_i = c_i$ for all vertices i of $\mathcal{O}$, and they are constant on connected components, all vertices i of $\mathcal{O}$ satisfy both ♣ and $a_i = c_{i-1-a_i}$. It follows that the complete list of peaks of A satisfying the conditions of Proposition 4.1.13 is given in Figure 4.2.

4.1.15. Proposition. A band module M has $M \simeq \Sigma\Omega(M)$ if and only if it is represented by a belt w where all syllables $\mathbf{p}$ of w have $b_{s(\mathbf{p})} = 0$.

Proof. By Lemma 2.2.70, it is sufficient to characterise band modules M where $\mathrm{Tr}(M)$ satisfies the conditions of Proposition 4.1.7. Now, by Proposition 3.1.14 and Proposition 4.1.8, the result follows (noting that we use the notation $b_{s(\mathbf{p})}$ rather than $d_{s(\mathbf{p})^{\mathrm{op}}}$ to avoid confusion). □

Combining Proposition 4.1.15 with our construction for syzygies of belts immediately gives:

4.1.16. Corollary. A band module M has $M \simeq \Sigma\Omega(M)$ if and only if $\Omega(M)$ is also a band module.

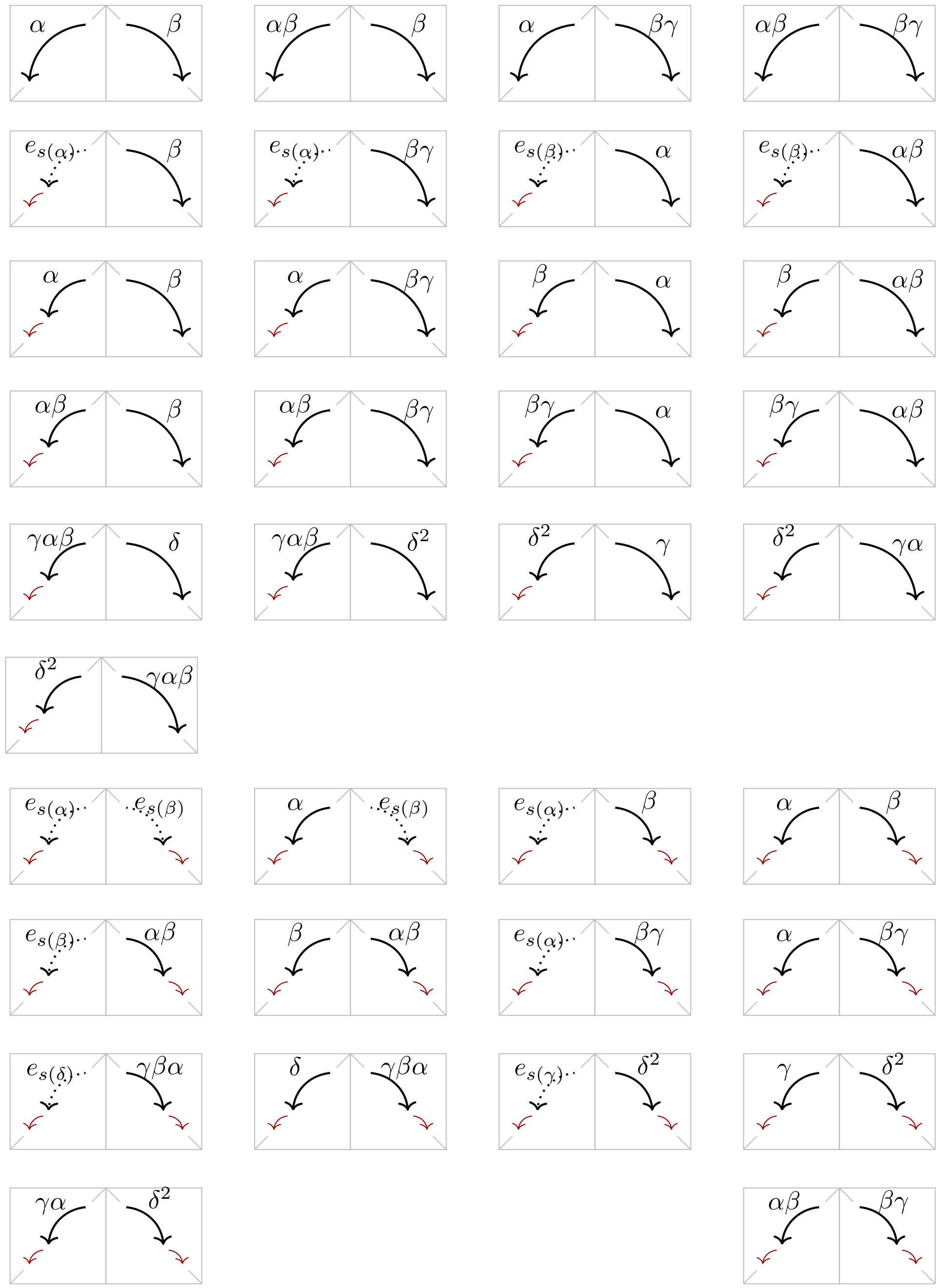


Figure 4.2: *List of peaks satisfying the conditions of Proposition 4.1.13 for our running example algebra.* We only list the peaks up to reflection. The first row consists of peaks that meet condition (i). The second and third rows consist of peaks that meet condition (ii). The fourth, fifth and sixth rows consist of peaks that meet condition (iii). The seventh row consists of peaks that meet condition (iv). The eight and ninths rows, along with the left of the tenth row consists of peaks that meet condition (v). The right of the tenth row is the only peak meeting condition (vi) (noting that we have excluded strips representing projective string modules).

This in itself gives a new characterisation of band modules M with $M \simeq \Omega\Sigma(M)$.

4.1.17. Corollary. A band module M has $M \simeq \Omega\Sigma(M)$ if and only if $\Sigma(M)$ is also a band module.

Proof. Suppose first that $M \simeq \Omega\Sigma(M)$. Then $\mathrm{Tr}(\mathrm{Tr}(M)) \simeq \Omega\Sigma(\mathrm{Tr}(\mathrm{Tr}(M)))$, and so $\mathrm{Tr}(M) \simeq \Sigma\Omega(\mathrm{Tr}(M))$, by Lemma 2.2.70. Since M is a band module, so is $\mathrm{Tr}(M)$. Thus we can apply Corollary 4.1.16, showing that $\Omega(\mathrm{Tr}(M))$ is a band module. Hence $\Sigma(M) = \mathrm{Tr}\,\Omega\,\mathrm{Tr}(M)$ is a band module, as required.

For the second implication, suppose that $\Sigma(M) = \mathrm{Tr}\,\Omega\,\mathrm{Tr}(M)$ is a band module. Thus $\Omega\,\mathrm{Tr}(M)$ is a band module too. Thus we can apply Corollary 4.1.16, showing that $\mathrm{Tr}(M) \simeq \Sigma\Omega(\mathrm{Tr}(M))$. Hence, by Lemma 2.2.70, we know that $\mathrm{Tr}(\mathrm{Tr}(M)) \simeq \Omega\Sigma(\mathrm{Tr}(\mathrm{Tr}(M)))$. Thus $M \simeq \Omega\Sigma(M)$, as required. ☐

4.2 Delooping level of band and string modules

We first investigate the delooping level of band modules, as Corollary 3.2.16 makes it easier to understand when they are syzygies, since there is at most one way of expressing a band module as a syzygy.

4.2.1. Proposition. Suppose that M is a band module with $\mathrm{dell}(M) = n > 0$. Then $\Omega^n(M)$ is a string module.

Proof. Suppose to the contrary that $\Omega^n(M)$ is a band module. Then $\Omega^k(M)$ is a band module for all $0 \leq k \leq n$. Thus

$$\begin{aligned}
\Sigma^n\Omega^n(M) &= \Sigma^{n-1}(\Sigma\Omega(\Omega^{n-1}(M))) \simeq \Sigma^{n-1}(\Omega^{n-1}(M)) \\
&= \Sigma^{n-2}(\Sigma\Omega(\Omega^{n-2}(M))) \simeq \Sigma^{n-2}(\Omega^{n-2}(M)) \\
&= \cdots \\
&= \Sigma\Omega(M) \simeq M
\end{aligned}$$

by repeated application of Corollary 4.1.16.

We now handle two cases separately, based on whether $\Sigma(M)$ is a band module or a string module.

Case 1: $\Sigma(M)$ is a band module

Then, by Corollary 4.1.17, $M \simeq \Omega\Sigma(M)$. Therefore $\operatorname{dell}(M) = 0$, which is a contradiction.

Case 2: $\Sigma(M)$ is a string module

Thus $\Omega^{n+1}\Sigma(M) \simeq \Omega^{n+1}\Sigma^{n+1}\Omega^n(M)$ is a string module (where this stable isomorphism follows from the above chain of equalities and stable isomorphisms). Since $\Omega^n(M)$ is a stable retract of $\Omega^{n+1}\Sigma^{n+1}\Omega^n(M)$, it is also a string module. This contradicts our initial assumption. $\square$

4.2.2. Proposition. Let Φ_A be the pin graph of A. Let l be the number of arrows in the largest acyclic component of Φ_A. Let M be a band module, represented by a belt w. Suppose that $\Omega^{2l+2}(M)$ also a band module. Then there exists $k \in \mathbb{Z}_+$ where $\Omega^k(M)$ is represented by w, and we have $\operatorname{dell}(M) = 0$.

Proof. Let w be a belt representing M. Since $\Omega^{2l+2}(M)$ is a band module, so is $\Omega^k(M)$ for all $0 \leq k \leq 2l+2$. Thus $\nabla^k(\mathbf{p})$ is an interior syllable for all $0 \leq k \leq 2l+2$, when $\mathbf{p}$ is any syllable of w.

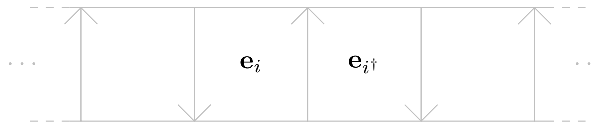
Since $\nabla^k(\mathbf{p})$ is an interior syllable for all $0 \leq k \leq 2l+2$, the Q-vertices corresponding to the source and target of $\mathbf{p}$ both lie on paths of length $l+1$ on the pin graph Φ_A. Thus these vertices must lie on cyclic components of Φ_A, and by Proposition 2.3.47, we know that the length of those cycles are the same, l', and that $\nabla^{2l'}(\mathbf{p}) = \mathbf{p}$. Note that this length, l', is the same for all syllables of w, and that for each syllable $\mathbf{p}$ of w, $\nabla^k(\mathbf{p})$ is an interior syllable for all $k \in \mathbb{N}$.

It follows that $\Omega^k(M)$ is a band module for all $k \in \mathbb{N}$, and that $\Omega^{2l'}(M)$ is represented by w. By Corollary 3.2.14, since w represents both M and $\Omega^{2l'}(M)$, this means that $M \in \Omega(\text{mod-}A)$, and thus $\operatorname{dell}(M) = 0$. $\square$

4.2.3. Proving results that involve the delooping level of string modules is more difficult than for band modules. This is due to the fact that, in general, a string module M may be expressed as a summand of a syzygy in non-trivially different ways. To be more precise, for an indecomposable string module M, there may be non-isomorphic indecomposable modules Y_1, Y_2 such that M is a summand of both $\Omega(Y_1)$ and $\Omega(Y_2)$ (see Remark 3.2.17 for an example of this). This is not the case for band modules, by Corollary 3.2.16.

4.2.4. Lemma. Let A be an SB algebra, and S a simple A-module which is not projective. Then $\operatorname{dell}(S) \neq 0$ if and only if $\Omega\operatorname{Tr}(S) \simeq 0 \in \underline{\text{mod}}\text{-}A^{\text{op}}$.

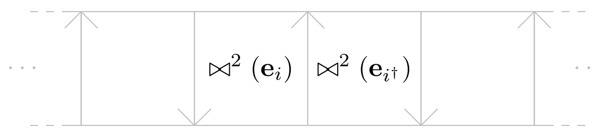
Proof. Let w be a strip representing S of the form:



where $\mathbf{e}_i, \mathbf{e}_{i^\dagger}$ are the stationary syllables corresponding to the two $\mathcal{O}$-vertices associated to the simple S.

Suppose that $\Omega\,\mathrm{Tr}(S) \simeq 0$. Then $\Omega\Sigma(S) = (\Omega\,\mathrm{Tr})^2(S) \simeq 0$. Since S is not projective, it is not a stable retract of 0, and thus $\mathrm{dell}(S) \neq 0$, as required.

Now instead suppose that $\Omega\,\mathrm{Tr}(S) \not\simeq 0$. Then by Proposition 4.1.3, we know that $(\Omega\,\mathrm{Tr})^2(S)$ is a string module represented by the strip:



By Proposition 4.1.8(b), we know that all stationary syllables satisfy $\bowtie^2 \mathbf{e}_j = \mathbf{e}_j$, so $S \simeq (\Omega\,\mathrm{Tr})^2(S) \simeq \Omega\Sigma(S)$. Hence $\mathrm{dell}(S) = 0$, as required. $\square$

4.2.5. Remark. We had originally hoped that characterising the behaviour of $\Omega\,\mathrm{Tr}$ on string and band modules would immediately lead to a much better understanding of the delooping levels of these modules when combined with the construction for computing syzygies. Unfortunately, this has not turned out to be the case, but it turns out that this characterisation is useful when you restrict to the radical-cube-zero case (as we will discuss in Chapter 5) and is also useful in better understanding Gorenstein-projective modules for SB algebras (as we will discuss in Chapter 6).

If better understanding can be achieved of when pin-boundary syllables can appear in a syzygy fabric (and thus when perturbation can occur), then we would expect this representation of Ω and $\Omega\,\mathrm{Tr}$ in terms of descent and twisting would be useful in working towards a general bound on the delooping level of SB algebras.

Chapter 5

Radical-cube-zero special biserial algebras

For this chapter, we focus our attention on radical-cube-zero special biserial algebras. As you would expect, these have a much simpler combinatorial structure than general special biserial algebras. For example, when considering the source encoding of such an algebra, $b_i = 0 \implies a_i = 2$ for $i \in \mathcal{O}$. This is a dramatic simplification of the combinatorial structure compared to more general special biserial algebras.

If we instead restricted to radical-square-zero algebras, then these algebras would necessarily be string algebras (as they could have no pin modules), and would have much less interesting properties. For example, the interior width of any summand of a syzygy would always be zero. Thus radical-cube-zero algebras are the simplest case where we get interesting structures of syzygies.

The fact that these algebras have a simpler combinatorial structure than a general SB algebra makes them easier to study, allowing the building of intuition when working on problems for the more general case. For example, Huisgen-Zimmermann conjectures that the delooping levels of SB algebras are always finite [HZ22, Section 4]. The following is a joint result of Goodearl and Huisgen-Zimmermann that shows that this is true for radical-cube-zero algebras:

Theorem. [HZ22, Thm 4] *If A is an SB algebra with* $\mathrm{rad}^3(A) = 0$*, then* $\mathrm{dell}(A) < \infty$.

(*Note that only the statement of this theorem has been published by Goodearl and Huisgen-Zimmermann; therefore we can't compare the methods used in their proof to those we use in this chapter.*)

The first section of this chapter shows that in fact, such SB algebras are syzygy-finite or self-injective, which immediately implies the above theorem. This is achieved by bounding the interior width of syzygies of sufficiently high degree (over non-self-injective algebras).

The second section of this chapter builds on this by constructing explicit bounds for the delooping level of a radical-cube-zero SB algebra, based on the number of indecomposable projective modules and the number of pin modules (both up to isomorphism). We also conjecture a strict upper bound for these delooping levels.

Throughout, we will often refer to the condition that the regular module A_A is not a direct sum of pin modules. Note that if an algebra fails to meet this condition then we know that A_A is injective, and thus that A is a self-injective algebra; however, the reverse implication does not hold, as some SB algebras have uniserial projective injective modules, which are immediately not pin.

5.1 Syzygy-finiteness

We first prove a lemma that will be particularly useful when applied to radical-cube-zero special biserial algebras.

5.1.1. Lemma. Suppose that A is a connected SB algebra where the regular module A_A is not a direct sum of pin modules. Let k be the number of indecomposable projective A-modules that are pin. Let Q be the underlying sub-2-regular quiver of A, and v_0 be any vertex of Q. For $i \in Q_0$, let $\mathrm{npl}(i)$ denote the length of the shortest oriented path from i to a vertex whose corresponding indecomposable projective is not pin (the "non-pin length"). Then $\mathrm{npl}(v_0) \leq k$ for all vertices v_0 of Q.

In other words, for any vertex v_0 of Q, there is a path starting at v_0 of length at most k, which ends at a vertex whose corresponding indecomposable projective is not pin.

Proof. For all $i \geq 0$, let

$$V_i := \{v \in Q_0 \ : \ \text{there is an oriented path from } v_0 \text{ to } v \text{ of length at most } i\}.$$

Then $V_0 = \{v_0\}$ and $V_i \subseteq V_{i+1} \subseteq Q_0$ for all $i \geq 0$. The claim is now equivalent to the statement that:

$$V_k \cap \{v \in Q_0 \ : \ \text{the projective corresponding to } v \text{ is not pin}\} \neq \varnothing.$$

We now handle two cases separately; $|V_k| > k$ or $|V_k| \leq k$. If $|V_k| > k$ then V_k must contain a vertex whose corresponding projective is not pin (as there are only k indecomposable projectives that are pin).

So we now suppose that $|V_k| \leq k$. Since $|V_0| = 1$, this means that the chain of inequalities:

$$|V_0| \leq |V_1| \leq \cdots \leq |V_k|,$$

must have at least one equality in it. Suppose that $|V_i| = |V_{i+1}|$ for some $0 \leq i < k$. Then since $V_i \subseteq V_{i+1}$, we know that $V_i = V_{i+1}$. Thus there are no arrows from vertices in V_i to vertices outside V_i.

Consider the full subquiver Q' of Q with vertices V_i. Note that Q' can't be the whole of Q, as it has at most k vertices and Q has greater than k vertices. Since Q is sub-2-regular, so is Q'; but Q' can't be 2-regular as then it would be a connected component of Q, which gives a contradiction. Therefore Q' has a vertex v' with out-degree ≤ 1 in Q'. Thus $v' \in V_i$ has out-degree ≤ 1 in Q, as there are no arrows from v' to vertices outside V_i. Hence the projective corresponding to v' is not pin, as required. □

5.1.2. Triangular arrangement of patches. A *triangular arrangement of patches* with height h is the layout of patches (as defined in Subsection 2.4.1) shown in Figure 5.1, where all displayed syllables are interior and there are no restrictions on syllables outside the triangle.

5.1.3. Lemma. Suppose that A is a connected SB algebra with $\mathrm{rad}^3(A) = 0$ such that the regular module A_A is not a direct sum of pin modules. Let k be the number of indecomposable projective A-modules that are pin. Then there does not exist a triangular arrangement of interior patches with height $k + 2$.

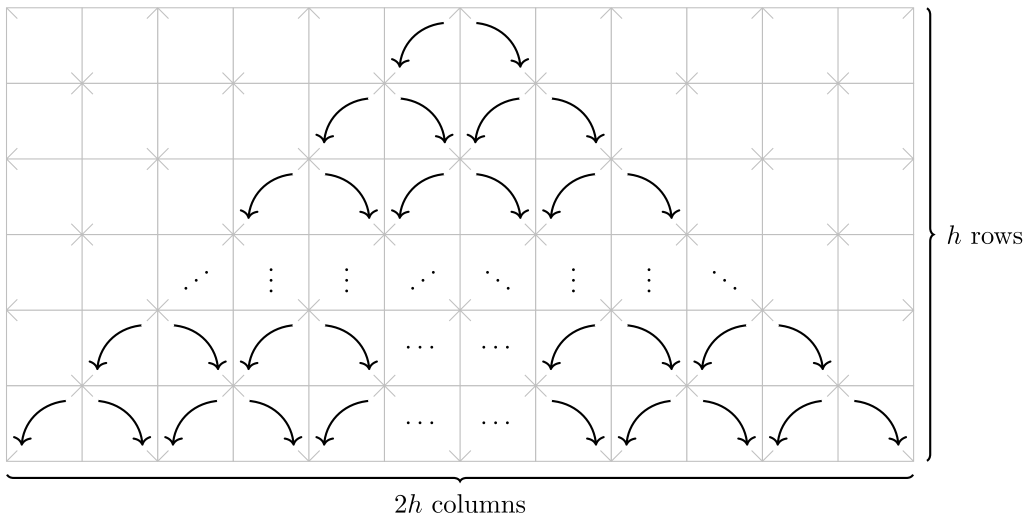


Figure 5.1: *Triangular arrangement of patches.*

Proof. Suppose that there does exist a triangular arrangement of interior patches with height $k+2$. Since $\mathrm{rad}(A)^3 = 0$, if a patch consists only of interior syllables of A, then all of the underlying paths of these syllables must have length exactly one. This means that for any given path of length at most $k+2$ starting at the top vertex, we can construct it as a composition of the underlying paths in a zigzag in our triangular arrangement. Therefore, by applying Lemma 5.1.1, there must be a vertex whose corresponding projective is not pin which appears as the target of a syllable, $\mathbf{s}$, somewhere in the top k rows of our arrangement. Thus there appears a boundary syllable two rows directly beneath $\mathbf{s}$. This proves the claim. ☐

Note that this argument doesn't require the 2 outermost syllables in the bottom row (or the outermost syllables in the second bottom row) to be interior to reach a contradiction. However it would simply complicate subsequent arguments for no benefit to remove them from consideration.

5.1.4. Lemma. Suppose that A is a connected SB algebra with $\mathrm{rad}^3(A) = 0$ such that the regular module A_A is not a direct sum of pin modules. Let k be the number of indecomposable projective A-modules that are pin. Then

$$\Omega^{k+1} M \in \mathrm{add}\{\mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{intwid}(w) \leq 2k+4\},$$

for all $M \in \text{mod-}A$ which are string modules.

Proof. Fix $M \in$ mod-A an indecomposable string module. Suppose that $\Omega^{k+1}M \notin \mathrm{add}\{N \in \text{mod-}A : \mathrm{intwid}\, N \leq 2k+4\}$. Thus $\Omega^{k+1}M$ has a summand M' with $\mathrm{intwid}(M') > 2k+4$. Since $\mathrm{intwid}(M')$ is always even we have $\mathrm{intwid}(M') \geq 2k+6$. Following the notation of [All21, Paragraph 5.2.27], let $w_0, \ldots, w_{k+1}$ be strips such that w_0 represents M, w_{k+1} represents M', and that w_{i+1} represents one of the syzygy strips of w_i for all $i = 0, \ldots, k$. Then, by the discussion in [All21, Paragraph 5.2.27], since $\mathrm{intwid}\, w_{k+1} \geq 2k+6 > 2k+4$ we have

$$\mathrm{int}\, w_{k-r} \supseteq [\min(\mathrm{int}\, w_{k+1}) + r, \max(\mathrm{int}\, w_{k+1}) - r] \qquad \forall r = 0, \ldots, k+1.$$

Therefore, since the interior of a strip is a union of valleys, the collection of syllables corresponding to:

$$[\min(\mathrm{int}\, w_{k+1}) + r + 1, \min(\mathrm{int}\, w_{k+1}) + 2k + 5 - r] \text{ in strip } w_{k-r},$$

(where $r = 0, \ldots, k+1$) forms a triangular arrangement of interior patches with height $k+2$. This is a contradiction to Lemma 5.1.3. $\square$

Note that this proof does not rely on the finiteness of the initial strip w_0, and thus, in particular, it holds for bi-infinite string modules.

This then gives a similar result for band modules.

5.1.5. Corollary. Suppose that A is a connected SB algebra with $\mathrm{rad}^3(A) = 0$ such that the regular module A_A is not a direct sum of pin modules. Let k be the number of indecomposable projective A-modules that are pin. Then

$$\Omega^{k+1}M \in \mathrm{add}\{\mathrm{Str}(w) \in \text{mod-}A : \mathrm{intwid}(w) \leq 2k+4\},$$

for all $M \in$ mod-A which are band modules.

Proof. Fix a band module $M \in$ mod-A and let $\widehat{M} \in$ Mod-A be the corresponding bi-infinite string module. Since $\Omega^{k+1}(\widehat{M}) \in \mathrm{Add}\{\mathrm{Str}(w) \in \text{mod-}A\}$, it follows from Corollary 3.2.11 that $\mathrm{Add}(\Omega^{k+1}(M)) = \mathrm{Add}(\Omega^{k+1}(\widehat{M}))$. Thus

$$\begin{aligned}\Omega^{k+1}(M) \in \mathrm{Add}(\Omega^{k+1}(\widehat{M})) \cap \text{mod-}A &\subseteq \mathrm{Add}\{\mathrm{Str}(w) \in \text{mod-}A : \mathrm{intwid}(w) \leq 2k+4\} \cap \text{mod-}A \\ &= \mathrm{add}\{\mathrm{Str}(w) \in \text{mod-}A : \mathrm{intwid}(w) \leq 2k+4\}\end{aligned}$$

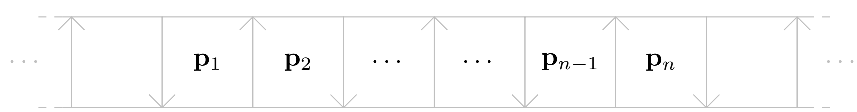
as required. $\square$

Theorem 5.1.6. *Suppose that A is a connected SB algebra with $\operatorname{rad}^3(A) = 0$. Then one of the following mutually exclusive conditions must hold:*

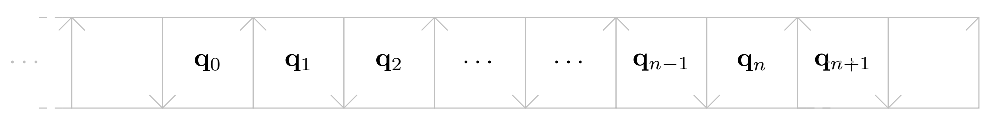
- *A is syzygy-finite,*
- *the regular module A_A is a direct sum of pin modules.*

Proof. Lemma 5.1.4 and Corollary 5.1.5 imply that A must satisfy at least one of the conditions. All that remains is to show that A can't be syzygy-finite if its regular module A_A is a direct sum of pin modules.

Suppose that the regular module A_A is a direct sum of pin modules. Further suppose that M is an indecomposable string module represented by a strip of the form:



where $\mathbf{p}_i$ is interior for $i \in \{2, \ldots, n-1\}$ and $\mathbf{p}_1, \mathbf{p}_n$ are boundary syllables which belong to $\operatorname{supp}(\nabla)$. Then $\Omega(M)$ is an indecomposable string module represented by a strip of the form:



where $\mathbf{q}_i$ is interior for $i \in \{2, \ldots, n-1\}$ and $\mathbf{q}_1, \mathbf{q}_n$ are boundary syllables which belong to $\operatorname{supp}(\nabla)$.

By iterating this logic, we see that $\Omega^n(M)$ is indecomposable for all $n \in \mathbb{N}$ and that:

$$\operatorname{intwid}(M) < \operatorname{intwid}(\Omega(M)) < \ldots < \operatorname{intwid}(\Omega^n(M)) < \ldots.$$

This means that any such M is not syzygy-finite.

Since any simple A-module is of the required form, it is clear that A is not syzygy-finite. □

Unfortunately, this result does not hold if we loosen the restriction from $\operatorname{rad}^3(A) = 0$ to $\operatorname{rad}^4(A) = 0$, as shown in the following example.

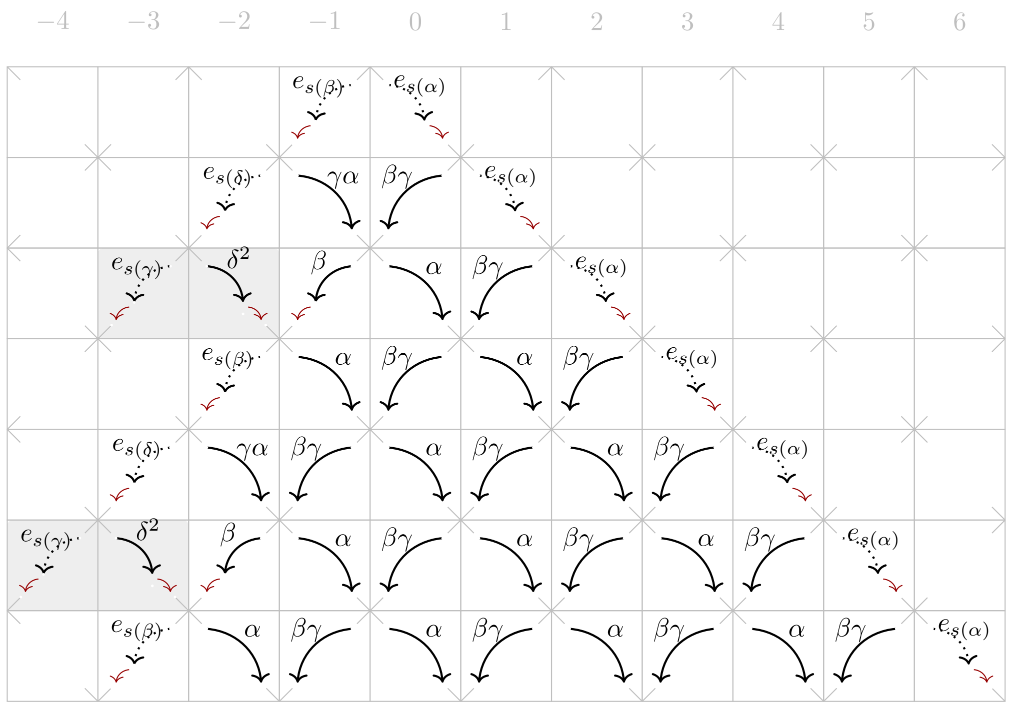


Figure 5.2: *The syzygy fabric associated to a simple module, S_1, of our running example algebra.* The white cells in the k-th row depict a strip representing a summand of $\Omega^k(S_1)$. Other summands of $\Omega^k(S_1)$ are formed in the shaded cells, and we ignore the behaviour of the further syzygies of these summands. Also note that, as usual, the strips in question have infinitely many blank syllables on either side; we don't use the dotted cells on the edges for the purpose of conserving space.

5.1.7. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. It is immediately obvious that the regular module A_A is not a direct sum of pin modules. Thus it suffices to show that A is not syzygy-finite.

If we consider the syzygy fabric associated to S_1, the simple module corresponding to the vertex 1, then we obtain the diagram in Figure 5.2. It is clear that by repeating this procedure sufficiently many times we will obtain a strip of arbitrarily large interior width. This is particularly clear if you consider the sequence of strips shown in the rows indexed by multiples of three. Hence S_1 has infinitely many different summands of syzygies. Therefore A is not syzygy-finite, as required.

Theorem 5.1.6 immediately gives the following by applying Lemmas 2.2.65 and 2.2.71.

5.1.8. Corollary. Suppose that A is an SB algebra with $\mathrm{rad}^3(A) = 0$. Then $\mathrm{dell}(A) < \infty$.

Note that despite not meeting neither of the mutually exclusive conditions from Theorem 5.1.6, our running example algebra still has finite delooping level.

5.1.9. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Then both simple modules, S_1, S_2, can be expressed as syzygies of string modules, as can be easily seen by considering Figure 2.1. Thus $\mathrm{dell}_A(S_1) = \mathrm{dell}_A(S_2) = 0$, and hence $\mathrm{dell}(A) = 0$.

5.2 Bounds for delooping level

While the previous section tells us that the delooping level of a radical-cube-zero special biserial algebra is always finite, it does not give us an explicit bound.

We can apply Corollary 2.2.72 alongside Lemma 5.1.4 to obtain the following bound:

5.2.1. Proposition. Suppose that A is an SB algebra with $\mathrm{rad}^3(A) = 0$. Let n be the number of indecomposable projective A-modules (up to isomorphism) and k be the number of them that are pin. Then $\mathrm{dell}(A) \leq 3n \cdot (2^{2k+4} - 1)$.

Proof. By Lemma 2.2.65, it is sufficient to prove the claim for connected SB algebras, A, with $\mathrm{rad}^3(A) = 0$ where A is not self-injective (and thus the regular module A_A is not a direct sum of pin modules).

To apply Corollary 2.2.72, we require a bound on $|\mathcal{M}_0|$ where

$$\mathcal{M}_0 := \{\mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{intwid}(w) \leq 2k+4\}.$$

Since $\mathrm{rad}^3(A) = 0$, it is clear that $\mathrm{rad}^3(\mathrm{Str}(w)) = 0$ for any strip w as well. Thus we can apply Corollary 2.4.21 to obtain the bound:

$$|\mathcal{M}_0| \leq n \cdot (2+1) \cdot \frac{2^{2k+4}-1}{2-1} = 3n \cdot (2^{2k+4}-1).$$

Now suppose that S is a simple A-module. Then it follows from Lemma 5.1.4 that for $l \geq k+1$ any indecomposable summand Y of $\Omega^l(S)$ belongs $\mathcal{M}_0$. We also know that for $0 \leq l \leq k$ any indecomposable summand Y of $\Omega^l(S)$ satisfies:

$$\mathrm{intwid}(Y) \leq \mathrm{intwid}(S) + 2l = 2l \leq 2k,$$

and thus $Y \in \mathcal{M}_0$ in this case too.

To summarise, $\Omega^l(S) \in \mathrm{add}(\mathcal{M}_0)$ for all $l \geq 0$. Since S was an arbitrary simple A-module, Corollary 2.2.72 then gives:

$$\mathrm{dell}(A) \leq 3n \cdot (2^{2k+4}-1). \qquad \square$$

5.2.2. However, this is not a great bound. For example, Figure 5.3 shows a comparison between the bounds given by Proposition 5.2.1 and the actual maximum values for these algebras, when there are at most 4 simple modules for the algebra. The actual maximum values were calculated using the `SBAlgsFromNumVerticesAndRadLength` function from SBStrips, which allows iteration through all (connected) SB algebras with a given number of vertices and radical length. This computation heavily relies on the idea of a minimally connected overquiver, as introduced in Definition 2.3.24, to reduce the number of duplicate cases that we check. Unfortunately, even with the optimisations from considering only minimally connected overquivers, we do not have access to sufficient computing power and time to calculate the true maxima for cases with 5 or more simple modules; though it is worth noting that none of the hundreds of examples we calculated in the case with 5 simple modules have delooping level greater than 8.

The remainder of this section will focus on improving this bound. While a reader looking to apply

$n\backslash k$	0	1	2	3	4
1	0	0	-	-	-
2	2	2	0	-	-
3	4	4	4	0	-
4	6	6	6	6	0

(a) *True supremum of delooping level, calculated using SBStrips.*

$n\backslash k$	0	1	2	3	4
1	45	189	-	-	-
2	90	378	1530	-	-
3	145	567	2295	9207	-
4	180	756	3060	12276	49140

(b) *Bounds on delooping levels based on Proposition 5.2.1.*

Figure 5.3: *Comparison of initial bound on delooping level to true upper bounds.* Here n denotes the number of (isomorphism classes of) indecomposable projective A-modules, and k denotes the number of those projectives which are pin.

a particular bound could skip to the end for our best bound, we include several other bounds as an illustration of the methods involved when working with syllables and strips.

The first thing to note is that we don't need to consider all of $\mathcal{M}_0$.

5.2.3. Proposition. Suppose that A is an SB algebra with $\mathrm{rad}^3(A) = 0$. Let n be the number of indecomposable projective A-modules (up to isomorphism) and k be the number of them that are pin. Then $\mathrm{dell}(A) \leq 4n \cdot (k + 3)$.

Proof. As before, it is sufficient to prove the claim for connected SB algebras, A, with $\mathrm{rad}^3(A) = 0$ where A is not self-injective (and thus the regular module A_A is not a direct sum of pin modules).

Let

$$\mathcal{M}_1 := \{\mathrm{Str}(w) \in \text{mod-}A \, : \, \mathrm{rad}^2(\mathrm{Str}(w)) = 0, \mathrm{intwid}(w) \leq 2k + 4\}.$$

Since all modules of the form $\Omega(Y) \in \text{mod-}A$ have $\mathrm{rad}^2(\Omega(Y)) = 0$, it follows that $\mathcal{M}_1 \subseteq \mathcal{M}_0 \cap \mathrm{add}(\Omega(\text{mod-}A))$. Thus by the same logic as Proposition 5.2.1, it follows that $\Omega^l(S) \in \mathrm{add}(\mathcal{M}_1)$ for all simple A-modules, S, and $l > 0$. Since $\mathrm{rad}^2(S) = 0$ for all simple $S \in \text{mod-}A$, it follows that $\Omega^l(S) \in \mathrm{add}(\mathcal{M}_1)$ for all simple A-modules, S, and $l \geq 0$.

To bound $|\mathcal{M}_1|$, we apply the $r = 1$ version of Corollary 2.4.21. This gives:

$$|\mathcal{M}_1| \leq n \cdot (1 + 1)^2 \cdot (k + 3) = 4n \cdot (k + 3).$$

Applying Corollary 2.2.72 then gives the required bound. ☐

Figure 5.4 shows a comparison between this improved bound and the true values. While this bound is a significant improvement compared to Proposition 5.2.1, it is still orders of magnitude larger than the true values (at least in small cases).

$n\backslash k$	0	1	2	3	4
1	0	0	-	-	-
2	2	2	0	-	-
3	4	4	4	0	-
4	6	6	6	6	0

(a) *True supremum of delooping level, calculated using SBStrips.*

$n\backslash k$	0	1	2	3	4
1	12	16	-	-	-
2	24	32	40	-	-
3	36	48	60	72	-
4	48	64	80	96	112

(b) *Bounds on delooping levels based on Proposition 5.2.3.*

Figure 5.4: *Comparison of second bound on delooping level to true upper bounds.* Here n denotes the number of (isomorphism classes of) indecomposable projective A-modules, and k denotes the number of those projectives which are pin.

To come up with an improved bound, we can again restrict our collection of modules under consideration by looking at the triangular arrangements of patches induced by syzygies of sufficiently high degree and large width.

5.2.4. Proposition. Suppose that A is an SB algebra with $\mathrm{rad}^3(A) = 0$. Let n be the number of indecomposable projective A-modules (up to isomorphism) and k be the number of them that are pin. Then $\mathrm{dell}(A) \leq 12n + 2k(k+1)$.

Proof. As before, it is sufficient to prove the claim for connected SB algebras, A, with $\mathrm{rad}^3(A) = 0$ where A is not self-injective (and thus the regular module A_A is not a direct sum of pin modules). Let

$$\begin{aligned}\mathcal{M}_2 :=& \{\mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{rad}^2(\mathrm{Str}(w)) = 0, \mathrm{intwid}(w) \leq 4\} \\ &\cup \bigcup_{l=1}^{k} \Big(\big\{ \mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{rad}^2(\mathrm{Str}(w)) = 0, \mathrm{intwid}(w) = 2l+4 \big\} \cap \mathrm{add}(\Omega^l(\text{mod-}A)) \Big).\end{aligned}$$

It is easy to check that $\mathcal{M}_2 \subseteq \mathcal{M}_1$. Also, due to the conditions on intwid in each part of this union, none of the parts of the union intersect. Thus:

$$\begin{aligned}|\mathcal{M}_2| =& |\{\mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{rad}^2(\mathrm{Str}(w)) = 0, \mathrm{intwid}(w) \leq 4\}| \\ &+ \sum_{l=1}^{k} \Big| \big\{ \mathrm{Str}(w) \in \text{mod-}A \,:\, \mathrm{rad}^2(\mathrm{Str}(w)) = 0, \mathrm{intwid}(w) = 2l+4 \big\} \cap \mathrm{add}(\Omega^l(\text{mod-}A)) \Big|.\end{aligned}$$

Corollary 2.4.21 gives a bound on the size of the first term in this sum, but so far we don't have particular bounds for any of the other parts. To aid in calculating such bounds, for $l \in \{1, \dots, k\}$, let us define:

$$\mathcal{M}_2^l := \left\{ \operatorname{Str}(w) \in \text{mod-}A \, : \, \operatorname{rad}^2(\operatorname{Str}(w)) = 0, \operatorname{intwid}(w) = 2l + 4 \right\} \cap \operatorname{add}(\Omega^l(\text{mod-}A)).$$

Suppose that $M \in \mathcal{M}_2^l$ for some $l \in \{1, \dots, k\}$. Let w denote a strip representing M. Then by the same logic as Lemma 5.1.4, the central $2l + 2$ interior syllables of w are the bottom row of a triangular arrangement of patches of height $l+1$. This means that the vertex, v, of Q corresponding to the top of the uppermost patch in this arrangement has $\operatorname{npl}(v) \geq l > 0$.

The first thing to note is that for vertices v' whose corresponding indecomposable projective is not pin, we know that $\operatorname{npl}(v') = 0$. The second thing to note is that for vertices v with $\operatorname{npl}(v) > 0$, there is an arrow $v \to v'$ of Q where $\operatorname{npl}(v') = \operatorname{npl}(v) - 1$ (to see this, consider the first arrow of the shortest path from v to a "non-pin" vertex).

It follows from these two facts that $|\{v \in Q_0 \, : \, \operatorname{npl}(v) > 0\}| \leq k$ and that $|\{v \in Q_0 \, : \, \operatorname{npl}(v) > l'\}| \leq \max(0, |\{v \in Q_0 \, : \, \operatorname{npl}(v) > l' - 1\}| - 1)$ for all $l' \in \{1, \dots, k\}$. Combining these facts means that $|\{v \in Q_0 \, : \, \operatorname{npl}(v) > l'\}| \leq k - l'$.

Since (for algebras A with $\operatorname{rad}^3(A) = 0$) the bottom row of a triangular arrangement of patches is determined by the vertex at the top, there are at most $k - l$ options for the central $2l + 2$ interior syllables of w (up to reflection). Since all the interior syllables of w have underlying paths of length 1, it follows that there are at most $k - l$ choices for the ordered list of interior syllables of w (again, up to reflection). Following the same logic as Proposition 2.4.20 (and Proposition 2.4.20), this means that there are at most $4 \cdot (k - l)$ modules in $\mathcal{M}_2^l$.

Hence:

$$\begin{aligned}
|\mathcal{M}_2| =& |\{\operatorname{Str}(w) \in \text{mod-}A \, : \, \operatorname{rad}^2(\operatorname{Str}(w)) = 0, \operatorname{intwid}(w) \leq 4\}| \\
& + \sum_{l=1}^{k} \left| \left\{ \operatorname{Str}(w) \in \text{mod-}A \, : \, \operatorname{rad}^2(\operatorname{Str}(w)) = 0, \operatorname{intwid}(w) = 2l + 4 \right\} \cap \operatorname{add}(\Omega^l(\text{mod-}A)) \right| \\
\leq& (n \cdot 2^2 \cdot 3) + \sum_{l=1}^{k} 4 \cdot (k - l) = 12n + 4 \cdot \frac{1}{2} k(k+1) \\
=& 12n + 2k(k+1).
\end{aligned}$$

It follows immediately from the discussion in Proposition 5.2.3 that $\Omega^l(S) \in \mathrm{add}(\mathcal{M}_2)$ for all simple $S \in \text{mod-}A$ and $l \geq 0$. Thus Corollary 2.2.72 gives $\mathrm{dell}(A) \leq 12n + 2k(k+1)$, as required. □

5.2.5. Looking at Figure 5.5, there is an obvious elephant in the room. Our bounds so far have managed to show that $\mathrm{dell}(A) = O(n^2)$ (after applying the bound $k \leq n$). However, looking at the true supremums (in any of Figures 5.3–5.5), there is an obvious candidate for a linear bound in n for the delooping level. At least in these small cases the bound $\mathrm{dell}(A) \leq 2n - 2$ works and is sharp. As discussed in Paragraph 5.2.2, this bound has also worked for several hundred examples we have calculated for the $n = 5$ case. Unfortunately, applying Corollary 2.2.72 does not seem to be sufficient to prove a linear bound for the delooping level (in n, the number of simple modules).

However, for any $n \in \mathbb{Z}_+$ we can construct an SB algebra, A, which has n simple modules, satisfies $\mathrm{rad}^3(A) = 0$ and has $\mathrm{dell}(A) = 2n - 2$. This means that if $2n - 2$ is an upper bound for the delooping level of radical-cube-zero algebras (where n is the number of simple modules), then this upper bound is sharp. Since any SB algebra with a single simple module has zero delooping level, it is sufficient to show this for $n \geq 2$.

5.2.6. Example. Let $n \geq 2$ and let Q be the sub-2-regular quiver on n vertices:

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & \cdots \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_{n-1}"] & n \arrow[l, bend left, "\beta_{n-1}"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

$n\backslash k$	0	1	2	3	4
1	0	0	-	-	-
2	2	2	0	-	-
3	4	4	4	0	-
4	6	6	6	6	0

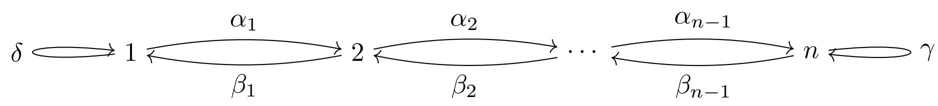
(a) *True supremum of delooping level, calculated using* **SBStrips**.

$n\backslash k$	0	1	2	3	4
1	12	16	-	-	-
2	24	28	36	-	-
3	36	40	48	60	-
4	48	52	60	72	88

(b) *Bounds on delooping levels based on Proposition 5.2.4.*

Figure 5.5: *Comparison of third bound on delooping level to true upper bounds.* Here n denotes the number of (isomorphism classes of) indecomposable projective A-modules, and k denotes the number of those projectives which are pin.

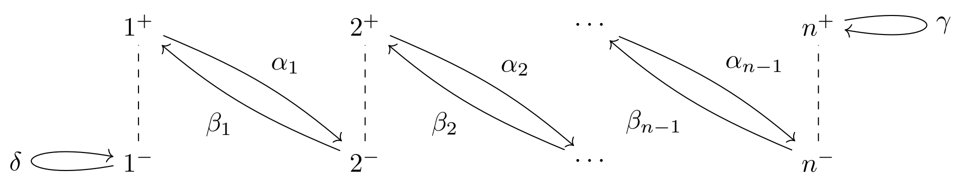
and let $\widetilde{Q}$ be its unique 2-regular augmentation:



Let

$$A \coloneqq \Bbbk Q/\langle \alpha_1\beta_1, \beta_1\alpha_1 - \alpha_2\beta_2, \ldots, \beta_{n-2}\alpha_{n-2} - \alpha_{n-1}\beta_{n-1}, \beta_{n-1}\alpha_{n-1} - \gamma^2 \rangle.$$

Then the overquiver of A is as follows:



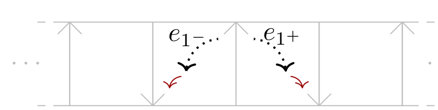
where, as usual, the dashed lines denote the vertex identification, †.

The source and target encoding of the permissible data for this algebra are given by:

$$a_i = c_i = \begin{cases} 0 & i = 1^- \\ 1 & i = 1^+ \\ 2 & \text{otherwise} \end{cases} \qquad b_i = d_i = \begin{cases} 1 & i = 1^- \text{ or } i = 1^+ \\ 0 & \text{otherwise} \end{cases}$$

Clearly all of the simple modules, S, corresponding to Q-vertices other than 1 appear as the socles of pin modules, and thus these simple modules have $\operatorname{dell}(S) = 0$. We claim that the remaining simple module, S_1, (corresponding to the Q-vertex 1) has $\operatorname{dell}(S_1) = 2n - 2$.

To verify this, we first express S_1 as a strip, and apply the syzygy algorithm to calculate its first $2n - 1$ syzygies. The following strip represents S_1:



and the syzygy fabric corresponding to this strip is depicted in Figure 5.6.

Note from the syzygy fabric that row $2n - 2$ is a reflection of row $2n - 1$. Thus the string modules

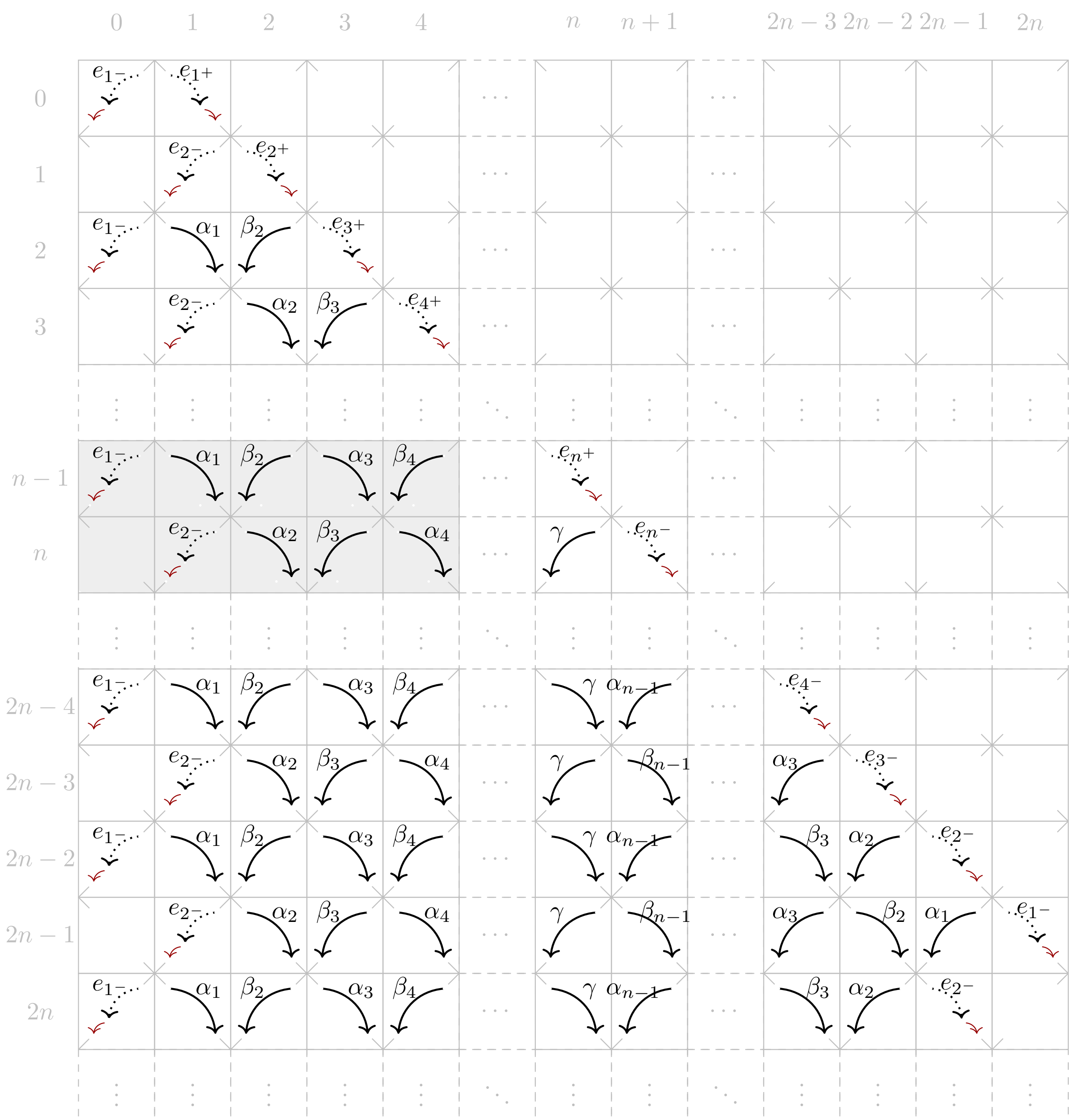


Figure 5.6: *The syzygy fabric associated to a simple module, S_1, in Example 5.2.6.* The k-th row is a strip representing $\Omega^k(S_1)$. Note that the diagram as depicted is only valid when n is odd, but to obtain the even version, just switch the rows in the shaded cells only. Also note that, as usual, the strips in question have infinitely many blank syllables on either side; we don't use the dotted cells on the edges for the purpose of conserving space.

corresponding to these strips ($\Omega^{2n-2}(S_1)$ and $\Omega^{2n-1}(S_1)$ respectively) are isomorphic. This means that $\operatorname{dell}(S_1) \leq 2n-2$. (This may also be noted more simply by the fact that rows $2n-2$ and $2n$ are identical, and thus $\Omega^{2n-2}(S_1) \cong \Omega^{2n}(S_1)$.)

Now, to show that $\operatorname{dell}(S_1) = 2n-2$, it is sufficient to show that the unit map

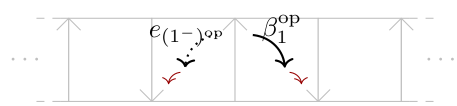
$$\Omega^{2n-3}(S_1) \longrightarrow \Omega^{2n-2}\Sigma^{2n-2}\Omega^{2n-3}(S_1)$$

does not split. To simplify this, we note that all peaks in rows 0 to $2n-4$ meet the conditions of Proposition 4.1.13. This means that, for $k \in \{0, 2n-4\}$, we have $\Omega^k(S_1) \simeq \Sigma\Omega^{k+1}(S_1)$. Hence $\Sigma^{2n-3}\Omega^{2n-3}(S_1) \simeq S_1$. Thus it suffices to show that $\Omega^{2n-3}(S_1)$ is not a stable retract of $\Omega^{2n-2}\Sigma(S_1) = \Omega^{2n-2}\operatorname{Tr}\Omega\operatorname{Tr}(S_1)$.

Applying the twisting function, $\bowtie_A$, to the two syllables in our strip representing S_1 yields:

$$\bowtie_A \mathbf{e}_{1^-} = \left((1^-)^{\mathrm{op}} \xrightarrow{0} \circ \xrightarrow{1} \circ \right) \qquad \bowtie_A \mathbf{e}_{1^+} = \left((1^+)^{\mathrm{op}} \xrightarrow{1} \circ \xrightarrow{1} \circ \right).$$

Thus, by Proposition 4.1.3, we know that $\Omega\operatorname{Tr}(S_1)$ is a string module of A^{op} represented by the strip:



and hence is projective. (Note that this matches our expectations from Lemma 4.2.4.)

Thus $\Omega\operatorname{Tr}(S_1) \simeq 0 \in \underline{\mathrm{mod}}\text{-}A^{\mathrm{op}}$, and so $\Omega^{2n-2}\Sigma(S_1) = \Omega^{2n-2}\operatorname{Tr}\Omega\operatorname{Tr}(S_1) \simeq 0 \in \underline{\mathrm{mod}}\text{-}A$. Since $\Omega^{2n-3}(S_1) \not\simeq 0 \in \underline{\mathrm{mod}}\text{-}A$, this clearly means that the unit map in question does not split, as required.

5.2.7. With the data for small algebras (see Figures 5.3–5.5) and the above example, we feel comfortable stating the following conjecture.

5.2.8. Conjecture. Suppose that A is an SB algebra with $\operatorname{rad}^3(A) = 0$. Let n be the number of (isomorphism classes of) indecomposable projective A-modules. Then $\operatorname{dell}(A) \leq 2n-2$.

Chapter 6

Gorenstein-projective modules of special biserial algebras

In this final chapter, we investigate the structure of finitely generated Gorenstein-projective modules of special biserial algebras (as defined in Definition 2.2.24). To do this we build upon results from the previous chapters, in particular Chapters 3 and 4.

There are several key properties of Gorenstein-projective modules that this chapter relies on, principally the following. As discussed in Subsection 2.2.3, if $M \in \text{mod-}A$ is Gorenstein-projective, then:

- we also know that $\Omega(M), \Sigma(M) \in \text{mod-}A$ are Gorenstein-projective;
- we have stable isomorphisms $\Omega\Sigma(M) \simeq M \simeq \Sigma\Omega(M)$;
- for all $n \in \mathbb{N}$, we have $\text{Ext}^1_A(\Omega^n(M), A) = 0 = \text{Ext}^1_A(\Sigma^n(M), A)$;

The first section of the chapter focuses on characterising modules M with $\text{Ext}^1_A(M, A) = 0$. As highlighted by Proposition 2.2.28, this is a necessary condition for M to be Gorenstein-projective. We identify such modules by considering the structure of the strips and belts that represent them, eventually showing that this condition can be checked almost entirely on a peak-by-peak basis.

The second section of the chapter aims to identify which band modules belong to

$$\Omega^\infty(\text{mod-}A) = \{M \mid \forall n \in \mathbb{N}\, \exists N \in \text{mod-}A \text{ s.t. } M \text{ is a summand of } \Omega^n(N)\}$$

as a first step to fully characterising those that are Gorenstein-projective. It follows immediately from considering the complete resolution of a Gorenstein-projective module that Gproj-$A \subseteq \Omega^\infty(\text{mod-}A)$; if $P^\bullet$ is a complete resolution for $M \in$ Gproj-A, then for any $n \in \mathbb{N}$, we have $M \cong Z^0(P^\bullet) \cong \Omega^n(Z^n(P^\bullet))$. As usual, we characterise the band modules in $\Omega^\infty(\text{mod-}A)$ by considering the belts that represent them and identifying local properties of those belts which are necessary and sufficient for the properties we are interested in. In particular, this heavily relies on the syzygy procedure for belts from Propositions 3.2.6 and 3.2.8. In this case, we can entirely characterise these belts in terms of the syllables present within them.

The third section of the chapter then builds on the results in earlier sections to understand when a special biserial algebra satisfies various Gorenstein homological conditions. For example, in Proposition 6.3.4 we give a necessary and sufficient condition for an SB algebra to be CM-finite (i.e. to have only finitely many indecomposable Gorenstein-projectives). This makes repeated use of the characterisation of Gorenstein-projective band modules from the second section.

6.1 Modules with $\mathrm{Ext}^1_A(M, A) = 0$

Throughout this section, we will use the following characterisation of modules $M \in$ mod-A with $\mathrm{Ext}^1_A(M, A) = 0$, as it is easier to perform combinatorial arguments with.

> **6.1.1. Lemma.** Let A be an SB algebra and $M \in$ mod-A. Then $\mathrm{Ext}^1_A(M, A) = 0$ if and only if for all indecomposable projective string modules, P, the map $\mathrm{Hom}_A(\mathrm{inc}, P) : \mathrm{Hom}_A(\mathbb{P}(M), P) \to \mathrm{Hom}_A(\Omega(M), P)$ is surjective.

Proof. We first note that $\mathrm{Ext}^1_A(M, -)$ is an additive functor that is zero on injective modules. We also note that A_A can be written as a direct sum of indecomposable projective, injective modules and indecomposable projective string modules. Since $\mathrm{Ext}^1_A(-, Q) = 0$ for any injective module Q, it follows that $\mathrm{Ext}^1_A(M, A) = 0$ if and only if $\mathrm{Ext}^1_A(M, P) = 0$ for all indecomposable projective string modules, P.

Fix an indecomposable projective string module, P. We now recall the long exact sequence associated to the right-exact functor $\mathrm{Hom}_A(-, P)$ and the short exact sequence $0 \to \Omega(M) \xrightarrow{\text{inc}} \mathbb{P}(M) \to M \to 0$:

$$0 \to \mathrm{Hom}_A(M, P) \to \mathrm{Hom}_A(\mathbb{P}(M), P) \to \mathrm{Hom}_A(\Omega(M), P) \to \mathrm{Ext}^1_A(M, P) \to \mathrm{Ext}^1_A(\mathbb{P}(M), P) \to \cdots$$

By exactness, it follows that $\mathrm{Ext}^1_A(M, P) = 0$ if and only if the map $\mathrm{Hom}_A(\text{inc}, P) : \mathrm{Hom}_A(\mathbb{P}(M), P) \to \mathrm{Hom}_A(\Omega(M), P)$ is surjective, as required. $\square$

In most previous cases, when we wanted to characterise some property of modules represented by a strip or a belt, we could define some partition of the syllables/peaks/valleys into "good" and "bad" cases, and then say our property is satisfied if all the syllables/peaks/valleys are "good" (or equivalently, there are no "bad" syllables/peaks/valleys present in the strip/belt). However, when trying to characterise when a module, M, represented by a strip or belt has $\mathrm{Ext}^1_A(M, A) = 0$, we need to consider the interaction between nearby peaks to distinguish cases.

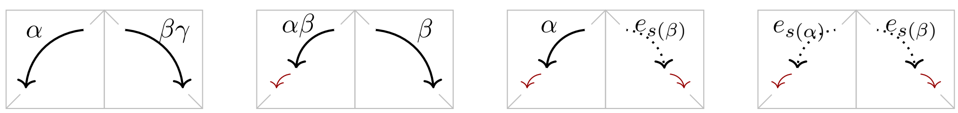
To achieve this, we define a colouring of the peaks; green, yellow and red. The green peaks play the role of the "good" peaks, while the red peaks play the role of the "bad" peaks. The remaining yellow peaks then play an ambiguous role, where they will normally be "good", but when near other yellow peaks, they will be "bad". Fortunately, the yellow peaks will always contain exactly one interior syllable and one boundary syllable, so being adjacent to another yellow peak is equivalent to the whole strip being formed of yellow peaks.

How we characterise these colours will depend on the whether or not the projective associated to the vertex at the top of the peak is pin or not. To simplify future explanations, we thus define pin and non-pin peaks as follows.

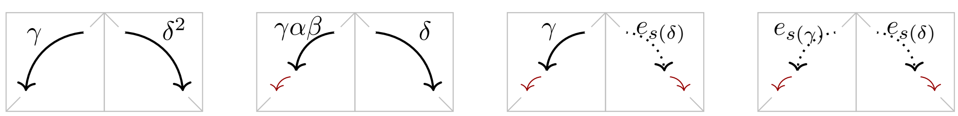
6.1.2. Definition. We call a non-blank peak *pin* (resp. *non-pin*) if $b_{s(\mathbf{p})} = 0$ (resp. $b_{s(\mathbf{p})} = 1$) for all non-blank syllables $\mathbf{p}$ in the peak.

It is clear that all non-blank peaks are either pin or non-pin based on the peak-compatibility conditions.

6.1.3. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. The following are all examples of pin peaks for A:



The following are all examples of non-pin peaks for A:



6.1.1 Characterisation using colouring of peaks

We now define our partition of the collection of all peaks into three groups. As explained above, for simplicity of phrasing we will assign each of these groupings a colour; green, yellow or red.

To do so, we will often refer to the following three conditions for a syllable, $\mathbf{p}$:

Condition $(*)$**:** for all $i \in \mathcal{O}$, $b_i = 1 \wedge (a_i > 0 \vee a_i = a_{i^\dagger} = 0)$ implies $t(\mathbf{p})^\dagger \neq i - a_i$.

Condition $(*')$**:** for all $i \in \mathcal{O}$, $b_i = 1$ implies $t(\mathbf{p})^\dagger \neq i - a_i$.

Condition $(**)$**:** $a_{s(\mathbf{p})} - 1 \neq a_{s(\mathbf{p})-1}$.

Note that we always have an implication $(*') \implies (*)$ and if there are no vertices $i \in \mathcal{O}$ with $b_i = 0$ and $a_i = 0 < a_{i^\dagger}$ then we also have the reverse implication $(*) \implies (*')$. Also note the similarities between condition $(*)$ and Lemma 2.3.16, which characterises standard basis elements of the socle of a projective.

Applying the standard source encoding inequality (Lemma 2.3.14) to the negation of condition $(**)$ gives:

$$a_{s(\mathbf{p})} - 1 = a_{s(\mathbf{p})-1} \geq a_{s(\mathbf{p})} - b_{s(\mathbf{p})-1},$$

so we also have $b_{s(\mathbf{p})-1} = 1$ when $(**)$ does not hold.

6.1.4. Colouring of pin peaks. For pin peaks we assign colours based on the following rules:

Case 0: the peak contains two interior syllables, $\mathbf{p}$ and $\mathbf{p}'$

The peak is assigned the colour green.

Case 1: the peak contains one boundary syllable, $\mathbf{p}$ and one interior syllable, $\mathbf{p}'$

Subcase a: the boundary syllable, $\mathbf{p}$, does not belong to $\mathrm{supp}(\nabla)$

Subcase i: $\mathbf{p}'$ satisfies condition $(**)$

The peak is assigned the colour green.

Subcase ii: $\mathbf{p}'$ does not satisfy condition $(**)$ but has $\mathrm{len}(\mathbf{p}') = 1$

The peak is assigned the colour yellow.

Subcase iii: $\mathbf{p}'$ does not satisfy condition $(**)$ and has $\mathrm{len}(\mathbf{p}') > 1$

The peak is assigned the colour red.

Subcase b: the boundary syllable, $\mathbf{p}$, belongs to $\mathrm{supp}(\nabla)$

Subcase i: $\mathbf{p}$ satisfies condition $(*)$

The peak is assigned the colour green.

Subcase ii: $\mathbf{p}$ does not satisfy condition $(*)$

The peak is assigned the colour red.

Case 2: the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$

Subcase a: neither syllable belongs to $\mathrm{supp}(\nabla)$

Subcase i: $\mathbf{p}$ and $\mathbf{p}'$ both satisfy condition $(**)$

The peak is assigned the colour green.

Subcase ii: at least one of $\mathbf{p}$ and $\mathbf{p}'$ does not satisfy condition $(**)$

The peak is assigned the colour red.

Subcase b: we have $\mathbf{p} \notin \mathrm{supp}(\nabla)$ and $\mathbf{p}' \in \mathrm{supp}(\nabla)$

Subcase i: $\mathbf{p}'$ satisfies conditions $(*)$ and $(**)$

The peak is assigned the colour green.

Subcase ii: $\mathbf{p}'$ has $\mathrm{len}(\mathbf{p}') = 1$, satisfies condition $(*')$, but not $(**)$

The peak is assigned the colour green.

Subcase iii: $\mathbf{p}'$ does not satisfy condition $(*)$

The peak is assigned the colour red.

Subcase iv: $\mathbf{p}'$ has $\mathrm{len}(\mathbf{p}') > 1$, satisfies condition $(*)$, but not $(**)$

The peak is assigned the colour red.

Subcase v: $\mathbf{p}'$ has $\mathrm{len}(\mathbf{p}') = 1$, satisfies condition $(*)$, but not $(*')$ or $(**)$

The peak is assigned the colour red.

Subcase c: both of the syllables belong to $\mathrm{supp}(\nabla)$

Subcase i: $\mathbf{p}$ and $\mathbf{p}'$ both satisfy condition $(*)$

The peak is assigned the colour green.

Subcase ii: at least one of $\mathbf{p}$ and $\mathbf{p}'$ does not satisfy condition $(*)$

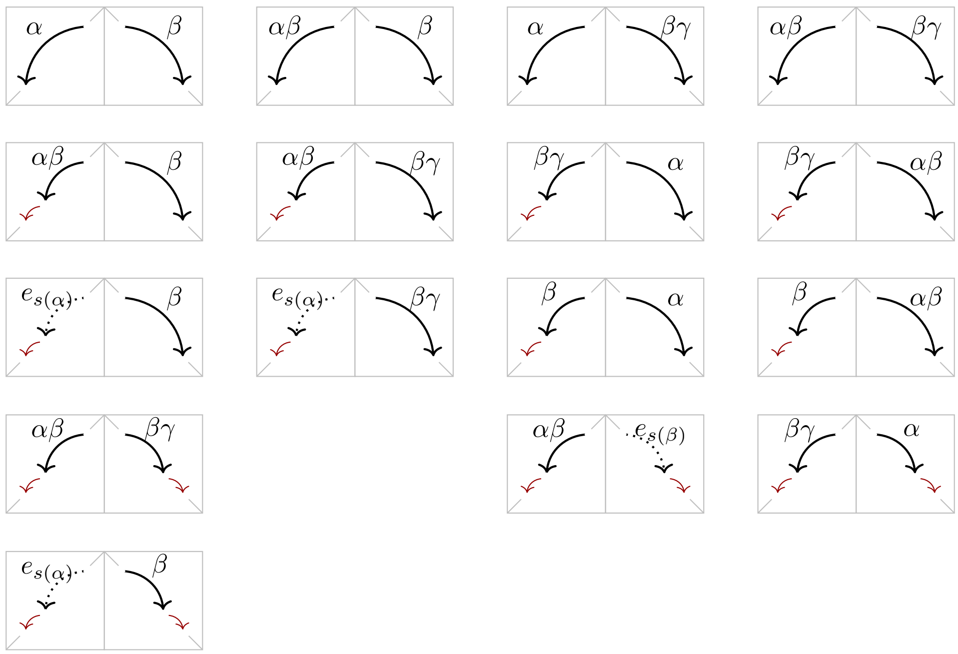
The peak is assigned the colour red.

Since this partitioning based on colours is pretty complicated, we now show how this partitioning works for our running example algebra.

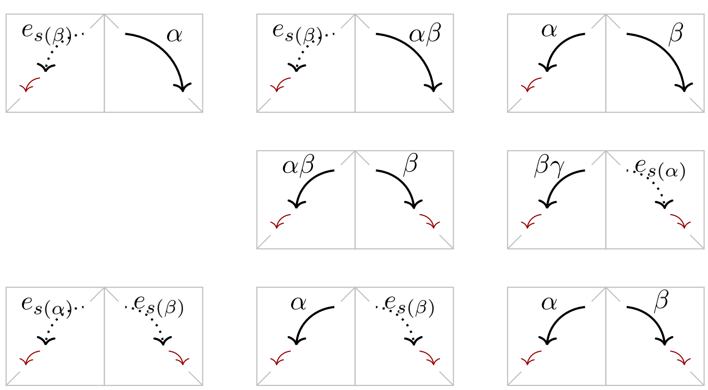
6.1.5. Example. For our running example algebra, there are no vertices $i \in \mathcal{O}$ with $b_i = 0 \wedge (a_i = 0 < a_{i^\dagger})$, and thus a syllable satisfies $(*)$ if and only if it satisfies $(*')$. There are two vertices $i \in \mathcal{O}$ with $b_i = 1$, $s(\gamma)$ and $s(\delta)$. If we consider the corresponding $i - a_i$ we obtain $s(\gamma) - a_{s(\gamma)} = s(\gamma)$ and $s(\delta) - a_{s(\delta)} = s(\delta)$. Therefore $\mathbf{p}$ satisfies condition $(*)$ if and only if $t(\mathbf{p})^\dagger \in \{s(\alpha), s(\beta)\} = \{t(\gamma), t(\alpha)\}$.

Furthermore, since a_j is constant for different vertices j of the same connected component $\mathcal{C}$ of $\mathcal{O}$, all vertices j of $\mathcal{O}$ satisfy $a_j - 1 \neq a_{j-1}$. Therefore any syllable $\mathbf{p}$ of A will satisfy condition $(**)$.

It follows that the colourings of pin peaks for the running example algebra A are given in Figure 6.1.



(a) *List of green pin peaks for our running example algebra.* The first row consists of peaks in case 0. The second row consists of peaks in case 1(ai). The third row consists of peaks in case 1(bi). The left of the fourth row is the unique peak in case 2(ai). The right of the fourth row consists of the peaks in case 2(bi). The fifth row has the unique peak in case 2(ci). Note that there are no peaks in case 2(bii), as all syllables for A satisfy condition $(**)$.



(b) *List of red pin peaks for our running example algebra.* The first row consists of peaks in case 1(bii). The second row consists of peaks in case 2(biii). The third row consists of peaks in case 2(cii). Note that there are no peaks in cases 1(aiii), 2(aii), 2b(iv) or 2b(v), as all syllables for A satisfy condition $(**)$.

Figure 6.1: *Colouring of pin peaks for our running example algebra.* In each case, we only list peaks up to reflection. Since all syllables satisfy condition $(**)$, there are no yellow pin peaks for our example.

6.1.6. Colouring of non-pin peaks. For non-pin peaks we assign colours based on the following rules:

Case 0: the peak contains two interior syllables, $\mathbf{p}$ and $\mathbf{p}'$

The peak is assigned the colour red.

Case 1: the peak contains one boundary syllable, $\mathbf{p}$ and one interior syllable, $\mathbf{p}'$

Subcase a: the boundary syllable, $\mathbf{p}$, does not belong to $\mathrm{supp}(\nabla)$

Subcase i: $\mathbf{p}'$ satisfies condition $(**)$

The peak is assigned the colour green.

Subcase ii: $\mathbf{p}'$ does not satisfy condition $(**)$ but has $\mathrm{len}(\mathbf{p}') = 1$

The peak is assigned the colour yellow.

Subcase iii: $\mathbf{p}'$ does not satisfy condition $(**)$ and has $\mathrm{len}(\mathbf{p}') > 1$

The peak is assigned the colour red.

Subcase b: the boundary syllable, $\mathbf{p}$, belongs to $\mathrm{supp}(\nabla)$

The peak is assigned the colour red.

Case 2: the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$

Subcase a: neither syllable belongs to $\mathrm{supp}(\nabla)$

The peak is assigned the colour green.

Subcase b: we have $\mathbf{p} \notin \mathrm{supp}(\nabla)$ and $\mathbf{p}' \in \mathrm{supp}(\nabla)$

Subcase i: $\mathbf{p}'$ satisfies conditions $(*)$ and $(**)$

The peak is assigned the colour green.

Subcase ii: $\mathbf{p}'$ has $\mathrm{len}(\mathbf{p}') = 1$, satisfies condition $(*')$, but not $(**)$

The peak is assigned the colour green.

Subcase iii: $\mathbf{p}'$ does not satisfy condition $(*)$

The peak is assigned the colour red.

Subcase iv: $\mathbf{p}'$ has $\mathrm{len}(\mathbf{p}') > 1$, satisfies condition $(*)$, but not $(**)$

The peak is assigned the colour red.

Subcase v: $\mathbf{p}'$ has $\mathrm{len}(\mathbf{p}') = 1$, satisfies condition $(*)$, but not $(*')$ or $(**)$

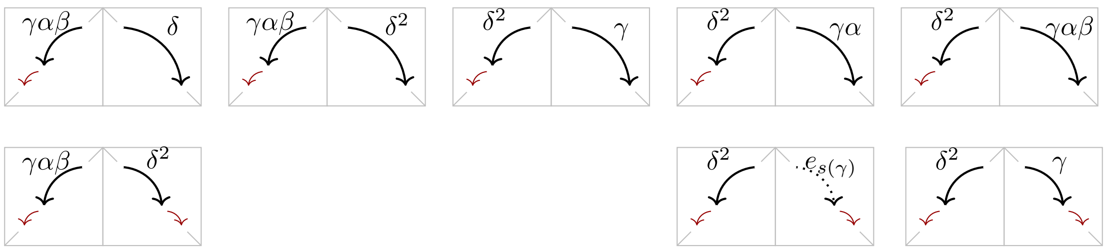
The peak is assigned the colour red.

Subcase c: both of the syllables belong to $\mathrm{supp}(\nabla)$

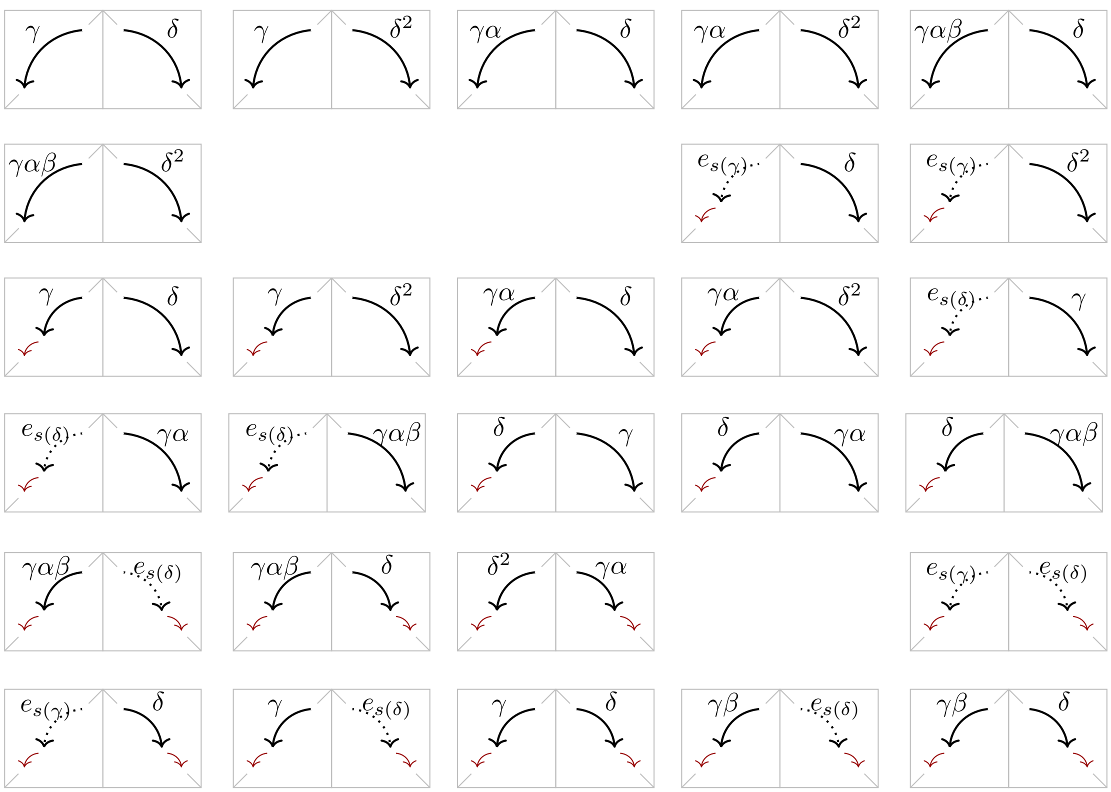
The peak is assigned the colour red.

6.1.7. Remark. Note that cases 1(a) and 2(b) are handled identically for both pin and non-pin peaks.

Again this partitioning based on colours is pretty complicated, so we now show how this partitioning works for our running example algebra.



(a) *List of green non-pin peaks for our running example algebra.* The first row consist of peaks in case 1a(i). The left of the second row has the unique peak in case 2(a). The right of the second row consists of peaks in case 2b(i). Note that there are no peaks in case 2b(ii), as all syllables for A satisfy condition $(**)$.



(b) *List of red non-pin peaks for our running example algebra.* The first row, along with the left half of the second row consists of peaks in case 0. The right half of the second row, along with the third and fourth rows consist of peaks in case 1(b). The left half of the fifth row consists of peaks in case 2b(iii). The right half of the fifth row, along with the sixth row consists of peaks in case 2(c). Note that there are no peaks in cases 1a(iii), 2b(iv) or 2b(v), as all syllables for A satisfy condition $(**)$.

Figure 6.2: *Colouring of non-pin peaks for our running example algebra.* In each case, we only list peaks up to reflection. Since all syllables satisfy condition $(**)$, there are no yellow non-pin peaks for our example.

6.1.8. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. As explained in Example 6.1.5, if $\mathbf{p} \in \mathrm{Syll}(A)$, then $\mathbf{p}$ always satisfies condition $(**)$, and will satisfy condition $(*)$ if and only if $t(\mathbf{p})^\dagger \in \{s(\alpha), s(\beta)\} = \{t(\gamma), t(\alpha)\}$.

It follows that the colourings of pin peaks for the running example algebra A are given in Figure 6.2.

Since our running example algebra has no yellow peaks, we will use a different example algebra to illustrate their properties when they become needed later in our argument (Example 6.1.16).

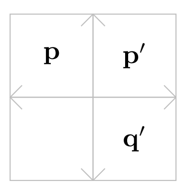
We now begin our examination of how the presence of different coloured peaks in a strip (or belt) representing a module M affects $\mathrm{Ext}^1_A(M, A)$.

> **6.1.9. Lemma.** If an indecomposable string module M is represented by a strip with a red pin peak, then $\mathrm{Ext}^1_A(M, A) \neq 0$.

Proof. We assume that M is an indecomposable string module which is represented by a strip with a red pin peak. We split into the 6 different cases for the form that this red pin peak takes:

Case 1a(iii): the peak contains one boundary syllable, $\mathbf{p}$, which does not belong to $\mathrm{supp}(\nabla)$ and one interior syllable $\mathbf{p}'$ where $\mathbf{p}'$ does not satisfy condition $(**)$ and has $\mathrm{len}(\mathbf{p}') > 1$

Since $\mathbf{p}'$ is interior, it is necessarily valley adjacent to another interior syllable $\mathbf{u}$. The patch, X, associated to this peak takes the following form:



where $\mathbf{q}' = (t(\mathbf{p}') \xrightarrow{a_{s(\mathbf{p}')} - \mathrm{len}(\mathbf{p}')} \circ \xrightarrow{1} \circ)$. We will denote the projective associated to X by P_X.

Let us denote the compression of each of $\mathbf{p}, \mathbf{p}', \mathbf{u}$ by p, p', u respectively.

We now use the bipartite basis of $\mathbb{P}(M)$ constructed in [All21, Prop 4.2.24(b)], and recall that the lower part gives a basis of $\Omega(M)$. We also recall that $p' - u$ is one of the lower basis elements, that it does not belong to $\mathrm{rad}(\Omega(M))$, and that each $\alpha \in Q_1$ annihilates at least one component of $p' - u$ and there exists unique $\alpha_1 \in Q_1$ (resp. $\alpha_2 \in Q_1$) such that it does not annihilate p' (resp. u).

We now define p'' to be the strict suffix of p' with $\mathrm{len}(p'') = \mathrm{len}(p') - 1$. Since $\mathrm{len}(p') =$

$\mathrm{len}(\mathbf{p}') > 1$, we know that $\mathrm{len}(p'') > 0$. Let P be the indecomposable projective corresponding to the vertex $s(p'') = s(p') - 1 \in \mathcal{O}$. Then p'' represents a basis vector of P and it belongs to $\mathrm{rad}^{\mathrm{len}(p'')}(P)$ but not $\mathrm{rad}^{\mathrm{len}(p'')+1}(P) = \mathrm{rad}^{\mathrm{len}(p')}(P)$. Also note that since p'' is a strict suffix of p', and α_1 does not annihilate p', we know that α_1 also doesn't annihilate p''.

We now define a homomorphism $f : \Omega(M) \to P$ by setting $f(p' - u) = p''$, setting $f(p'x) = p''x$ for all $\mathcal{O}$-paths x with $s(x) = t(p') = t(p'')$, and setting $f(y) = 0$ for all of our other basis vectors, y, of $\Omega(M)$. It follows from the fact that $a_{s(\mathbf{p}')} - 1 = a_{s(\mathbf{p}')-1}$ (since condition $(**)$ fails for $\mathbf{p}'$) and the properties of the basis outlined above, that this gives a well-defined A-homomorphism.

We claim that f does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. Suppose to the contrary that there exists $g : \mathbb{P}(M) \to P$ such that $f = g \circ \mathrm{inc}$. Since

$$g(p' - u)\alpha_1 = g((p' - u)\alpha_1) = g(p'\alpha_1) = g(p')\alpha_1,$$
$$g(p' - u)\alpha_2 = g((p' - u)\alpha_2) = g(u\alpha_2) = f(u\alpha_2) = 0 = g(0) = g(p'\alpha_2) = g(p')\alpha_2$$

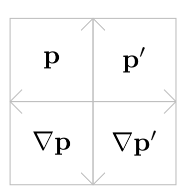
we know that $g(p' - u) = g(p') \in P_X$. Thus $g(u) = g(p') - g(p' - u) = 0 \in P_X$, so we know that g factors through the projection $\phi : \mathbb{P}(M) \to P_X$, i.e. there exists $h : P_X \to P$ such that $g = h \circ \phi$. Thus $f = h \circ \phi \circ \mathrm{inc}$.

So $h(p') = g(p') = g(p' - u) = f(p' - u) = p'' \notin \mathrm{rad}^{\mathrm{len}(p')}(P)$, however $p' \in \mathrm{rad}^{\mathrm{len}(p')}(P_X)$, so $h : P_X \to P$ should send p' to an element of $\mathrm{rad}^{\mathrm{len}(p')}(P)$. This is a contradiction, as required.

(Note that this argument does not rely on the fact that $\mathbf{p}'$ is interior, as anywhere that $p' - u$ appears, it can be replace with p' to give a valid argument when $\mathbf{p}'$ is boundary. Also note that we did not use the fact that the peak is pin anywhere either.)

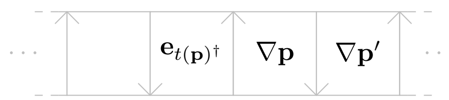
Case 1b(ii): the peak contains one boundary syllable, $\mathbf{p}$, which belongs to $\mathrm{supp}(\nabla)$, and does not satisfy condition $(*)$, and one interior syllable $\mathbf{p}'$

The patch, X, associated to this peak takes the following form:



We will denote the projective associated to X by P_X.

Let us denote the compression of $\mathbf{p}$ by p. Since $\mathbf{p}$ is a boundary syllable, we know (a summand of) $\Omega(M)$ is represented by a strip with the following local picture:



Since $\mathbf{p}$ does not satisfy condition $(*)$, we know that $t(\mathbf{p})^\dagger = i - a_i$ for some $i \in \mathcal{O}$ with $b_i = 1$ and either $a_i > 0$ or $a_i = a_{i^\dagger} = 0$. Fixing such an i, let P be the indecomposable projective string module corresponding to i, and r be the basis element of P corresponding to the path $(i \xrightarrow{a_i} \circ)$. Since $a_i > 0$ or $a_i = a_{i^\dagger} = 0$, we know that $r \in \operatorname{soc}(P)$.

We now use the bipartite basis of $\mathbb{P}(M)$ constructed in [All21, Prop 4.2.24(b)], and recall that the lower part gives a basis of $\Omega(M)$. We also recall that p is one of the lower basis elements, and that it does not belong to $\operatorname{rad}(\Omega(M))$.

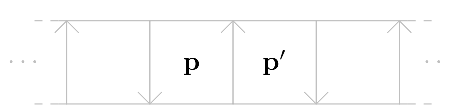
We now define a homomorphism $f : \Omega(M) \to P$ by setting $f(p) = r \in P$ and setting $f(y) = 0$ for all of our other basis vectors, y, of $\Omega(M)$. It follows from the fact that $r \in \operatorname{soc}(P)$ and that $p \notin \operatorname{rad}(\Omega(M))$, that this gives a well-defined A-homomorphism.

We claim that f does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. Suppose to the contrary that there exists $g : \mathbb{P}(M) \to P$ such that $f = g \circ \mathrm{inc}$. Then $r = f(p) = g(p) = g(e_{s(p)}p) = g(e_{s(p)})p$. However $t(r) = t(\mathbf{p})^\dagger \neq t(\mathbf{p}) = t(p)$, which gives a contradiction, as required.

(Note that this argument does not rely on the fact that $\mathbf{p}'$ is interior, or on the fact that the peak is pin.)

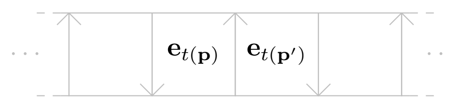
Case 2a(ii): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, which don't belong to $\operatorname{supp}(\nabla)$, at least one of which does not satisfy condition $(**)$

As this peak has two boundary syllables and M is indecomposable, we know that M is represented by a strip of the form:



Let us denote the compression of each of $\mathbf{p}, \mathbf{p}'$ by p, p' respectively.

Since both syllables are pin-boundary, this means that M is a socle-quotient of a pin module, which we will denote by P_X. It also means that $\Omega(M)$ is a simple module represented by a strip of the form:



Without loss of generality, we assume that $\mathbf{p}$ does not satisfy condition $(**)$, in other words $a_{s(\mathbf{p})} - 1 = a_{s(\mathbf{p})-1}$. As previously discussed, this necessarily means that $b_{s(\mathbf{p})-1} = 1$. Thus, let P be the indecomposable projective string module corresponding to $s(\mathbf{p}) - 1$.

Note that using the standard basis for P_X, the paths p and p' both represent the same basis element of $\mathrm{soc}(P_X) = \Omega(M)$.

Let p'' be the strict suffix of p with $\mathrm{len}(p'') = \mathrm{len}(p) - 1$. Since $\mathrm{len}(p) = \mathrm{len}(\mathbf{p}) > 1$ (as $\mathbf{p}$ is pin-boundary), we know that $\mathrm{len}(p'') > 0$. Also, as $s(p'') = s(p) - 1$, we know that p'' represents a basis vector of P, and as $a_{s(\mathbf{p})} - 1 = a_{s(\mathbf{p})-1}$, we know that $p'' \in \mathrm{soc}(P)$. We also define $\alpha \in Q_1$ to be the unique arrow such that $p = \alpha p''$.

We now define a homomorphism $f : \Omega(M) \to P$ by setting $f(p) = p'' \in P$. It follows from the fact that $p'' \in \mathrm{soc}(P)$ that this gives a well-defined A-homomorphism.

We claim that f does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. Suppose to the contrary that there exists $g : \mathbb{P}(M) \to P$ such that $f = g \circ \mathrm{inc}$. Then $p'' = f(p) = g(p) = g(e_{s(p)}p) = g(e_{s(p)})p$. However, there is no element $g(e_{s(p)}) \in P$ that can satisfy this, as $p'' \notin \mathrm{rad}^{\mathrm{len}(p)}(P)$ but $p \in \mathrm{rad}^{\mathrm{len}(p)}(A)$. This gives a contradiction, as required.

Case 2b(iii): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \mathrm{supp}(\nabla)$, $\mathbf{p}' \in \mathrm{supp}(\nabla)$, and $\mathbf{p}'$ does not satisfy condition $(*)$

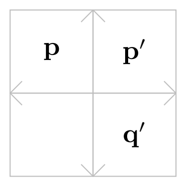
Since the above argument for case 1b(ii) only relies on the fact that there is a boundary syllable which belongs to $\mathrm{supp}(\nabla)$, and does not satisfy condition $(*)$, the same logic applies here. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M, A) \neq 0$.

Case 2b(iv): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \mathrm{supp}(\nabla)$, $\mathbf{p}' \in \mathrm{supp}(\nabla)$, $\mathrm{len}(\mathbf{p}') > 1$, and $\mathbf{p}'$ satisfies condition $(*)$, but not $(**)$

The core of the above argument for case 1a(iii) does not rely on the fact that $\mathbf{p}'$ is interior, as anywhere that $p' - u$ is present in that argument, it can be replaced with p' to make an argument that works when $\mathbf{p}'$ is boundary. Therefore the same logic applies here. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M, A) \neq 0$.

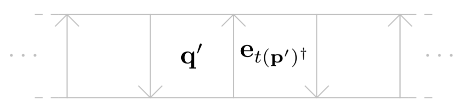
Case 2b(v): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \mathrm{supp}(\nabla)$, $\mathbf{p}' \in \mathrm{supp}(\nabla)$, $\mathrm{len}(\mathbf{p}') = 1$, and $\mathbf{p}'$ satisfies condition $(*)$, but not $(*')$ or $(**)$

The patch, X, associated to this peak takes the following form:



where $\mathbf{q}' = (\; t(\mathbf{p}') \xrightarrow{a_{s(\mathbf{p}')} - \mathrm{len}(\mathbf{p}')} \circ \xrightarrow{1} \circ \;)$. We will denote the projective associated to X by P_X.

Thus $\Omega(M)$ is represented by a strip of the form:



Let the compressions of $\mathbf{p}, \mathbf{p}', \mathbf{q}', \mathbf{e}_{t(\mathbf{p}')^\dagger}$ be denoted by p, p', q', α respectively. Note in particular that that $\mathrm{len}(\alpha) = 1$.

Since $\mathbf{p}'$ does not satisfy condition $(**)$, we know that $a_{s(\mathbf{p}')} - 1 = a_{s(\mathbf{p}')-1} = a_{t(\mathbf{p}')}$. This means that $\mathbf{q}' = (\; t(\mathbf{p}') \xrightarrow{a_{t(\mathbf{p}'}} \circ \xrightarrow{1} \circ \;)$.

Since $\mathbf{p}'$ satisfies condition $(*)$ but not $(*')$, we know that $a_{t(\mathbf{p}')^\dagger} = 0$ and $a_{t(\mathbf{p}')} > 0$. Thus $\Omega(M)$ is a projective string module (in this case it is also a uniserial module).

In particular, this means that there is a homomorphism $f : \Omega(M) \to A_A$ where $\mathrm{im}(f) \not\subseteq \mathrm{rad}(A_A)$. Since the inclusion $\iota : \Omega(M) \to \mathbb{P}(M)$ of the syzygy into the projective cover has $\mathrm{im}(\iota) \subseteq \mathrm{rad}(\mathbb{P}(M))$ and for all homomorphisms $g : \mathbb{P}(M) \to A_A$, we have $g(\mathrm{rad}(\mathbb{P}(M))) \subseteq \mathrm{rad}(A_A)$, it is clear that f does not factor through ι. Thus it follows that $\mathrm{Ext}^1_A(M, A) \neq 0$, as required.

(Note that this argument does not rely on the fact that the peak is pin.)

Case 2c(ii): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, both of which belong to $\mathrm{supp}(\nabla)$, and at least one of which does not satisfy condition $(*)$

Since the above argument for case 1b(ii) only relies on the fact that there is a boundary syllable which belongs to $\mathrm{supp}(\nabla)$, and does not satisfy condition $(*)$, the same logic applies here. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M, A) \neq 0$. $\square$

We now give an example covering some of the key cases reviewed here. Since our running example algebra doesn't have any peaks in case 1a(iii), we use a different algebra.

6.1.10. Example. Let

$$A' \coloneqq \Bbbk\left(\begin{tikzcd} 1 \arrow[loop left, "\eta"] \arrow[r, bend left, "\kappa"] & 2 \arrow[l, bend left, "\mu"] \arrow[loop right, "\nu"] \end{tikzcd}\right)/\langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle.$$

The collection of all A'-paths gives a basis of the regular A'-module as depicted below.

$$\begin{tikzcd}
& e_1 \arrow[dl, "\eta"'] \arrow[dr, "\kappa"] & \\
\eta \arrow[d, "\kappa"'] & & \kappa \arrow[d, "\mu"'] \\
\eta\kappa \arrow[d, "\mu"'] & & \kappa\mu \arrow[d, "\eta"'] \\
\eta\kappa\mu \arrow[d, "\eta"'] & & \kappa\mu\eta \arrow[d, "\kappa"'] \\
\eta\kappa\mu\eta \arrow[d, "\kappa"'] & & \kappa\mu\eta\kappa \\
\eta\kappa\mu\eta\kappa & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& e_2 \arrow[dl, "\mu"'] \arrow[ddr, "\nu"] & \\
\mu \arrow[d, "\eta"'] & & \\
\mu\eta \arrow[d, "\kappa"'] & & \nu \arrow[d, "\nu"] \\
\mu\eta\kappa \arrow[d, "\mu"'] & & \nu\nu \arrow[d, "\nu"] \\
\mu\eta\kappa\mu \arrow[d, "\eta"'] & & \nu\nu\nu \arrow[ddl, "\nu"] \\
\mu\eta\kappa\mu\eta \arrow[dr, "\kappa"'] & & \\
& \mu\eta\kappa\mu\eta\kappa &
\end{tikzcd}$$

The overquiver of A' is isomorphic to the overquiver of the standard running example, A, with a different labelling.

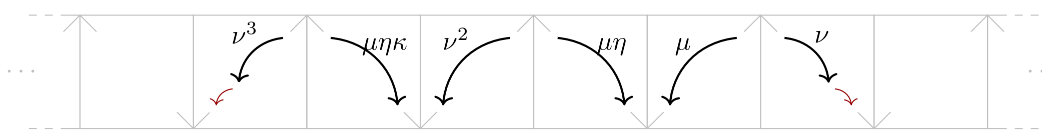
$$\begin{tikzcd}
s(\kappa) \arrow[r, "\kappa"] \arrow[d, dash, dashed] & s(\mu) \arrow[dl, "\mu"] \arrow[d, dash, dashed] \\
s(\eta) \arrow[u, bend left, "\eta"] & s(\nu) \arrow[loop right, "\nu"]
\end{tikzcd}$$

As before, the dashed lines denote the vertex identification, †.

The encoding of the permissible data is as follows:

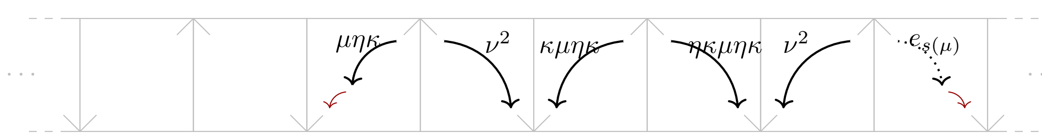
i	a_i	b_i	c_i	d_i
$s(\eta)$	5	1	4	1
$s(\kappa)$	4	1	5	1
$s(\mu)$	6	0	6	0
$s(\nu)$	4	0	4	0

Now consider the strip, w, of A', representing an A'-module, M:



Then the left peak of w falls into pin case 1a(iii), the middle peak of w falls into pin case 0, and the right peak of w falls into pin case 1b(ii).

Applying our syzygy algorithm to this strip leaves a strip, w', representing $\Omega(M) \in$ mod-A, of the following form:



The argument for the peak in pin case 1a(iii) focuses on the obvious homomorphism $f : \Omega(M) \to e_1 A'$ with image $\mathrm{im}(f) = \langle \kappa \rangle_{A'}$ which is zero on all basis elements of $\Omega(M)$ not corresponding to the leftmost syllable of w'. As explained in Lemma 6.1.9, this morphism does not factor through the inclusion inc : $\Omega(M) \to \mathbb{P}(M)$.

Since $a_{s(\mu)-2} = a_{s(\mu)-1} - 1 = a_{s(\mu)} - 2$ and the length of the syllables above ($\circ \xrightarrow{\mu\eta\kappa} \circ \xrightarrow{1} \circ$) has length greater than 2, there is actually another obvious morphism, $f' : \Omega(M) \to e_1 A'$, which does not factor through $\Omega(M) \to \mathbb{P}(M)$. The image of f' is $\mathrm{im}(f) = \langle \eta\kappa \rangle_{A'}$, and f' is also zero on all basis elements of $\Omega(M)$ not corresponding to the leftmost syllable of w'.

Clearly f' and f are linearly independent.

Now, the argument for the peak in pin case 1b(ii) focuses on the obvious homomorphism $f : \Omega(M) \to e_1 A'$ with image $\langle \kappa\mu\eta\kappa \rangle_{A'}$ which is zero on all basis elements of $\Omega(M)$ not corresponding to the rightmost syllable of w'. As explained in Lemma 6.1.9, this morphism does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$.

Since $s(\mu) = s(\eta) - as(\eta)$ as well as $s(\mu) = s(\kappa) - as(\kappa)$, there is actually another obvious morphism, $f' : \Omega(M) \to e_1 A'$, which does not factor through $\Omega(M) \to \mathbb{P}(M)$. The image of f' is $\mathrm{im}(f) = \langle \eta\kappa\mu\eta\kappa \rangle_{A'}$, and f' is also zero on all basis elements of $\Omega(M)$ not corresponding to the rightmost syllable of w'.

It is clear that all four of the morphisms discussed here are linearly independent.

We now return to the setting of general SB algebras.

6.1.11. Lemma. Let A be an SB algebra. If an indecomposable string module M is represented by a strip with a red non-pin peak, then $\mathrm{Ext}_A^1(M, A) \neq 0$.

Proof. We assume that M is an indecomposable string module which is represented by a strip with a red non-pin peak. We split into the 6 different cases for the form that this red non-pin peak takes:

Case 0: the peak contains two interior syllables, $\mathbf{p}$ and $\mathbf{p}'$

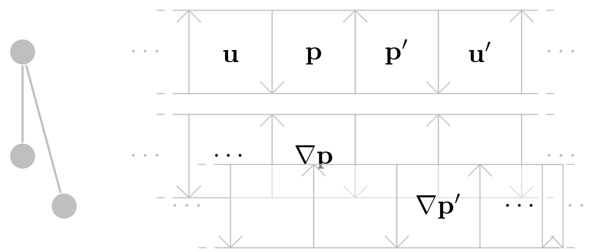
Since $\mathbf{p}, \mathbf{p}'$ are interior, they are necessarily valley adjacent to other interior syllables $\mathbf{u}$ and $\mathbf{u}'$ respectively. The patch, X, associated to this peak takes the following form:

$$\begin{array}{|c|c|}\hline \mathbf{p} & \mathbf{p}' \\ \hline \nabla\mathbf{p} & \nabla\mathbf{p}' \\ \hline \end{array}$$

We will denote the projective associated to X by P_X. Since the peak we are considering is non-pin, P_X is an indecomposable projective string module.

Let us denote the compression of each of $\mathbf{p}, \mathbf{p}', \mathbf{u}, \mathbf{u}'$ by p, p', u, u' respectively.

Since the peak we are considering is non-pin, both $\nabla\mathbf{p}$ and $\nabla\mathbf{p}'$ are boundary syllables. Thus $\Omega(M)$ has at least two direct summands, which fit into the following local diagram of branching:



We now use the bipartite basis of $\mathbb{P}(M)$ constructed in [All21, Prop 4.2.24(b)], and recall that the lower part gives a basis of $\Omega(M)$. We also recall that $p - u$ is one of the lower basis elements, that it does not belong to $\mathrm{rad}(\Omega(M))$, and that each $\alpha \in Q_1$ annihilates at least one component of $p - u$ and there exists unique $\alpha_1 \in Q_1$ (resp. $\alpha_2 \in Q_1$) such that it does not annihilate p (resp. u).

We now define a homomorphism $f : \Omega(M) \to P_X$ by setting $f(p - u) = p \in P_X$, setting $f(px) = px$ for all $\mathcal{O}$-paths x with $s(x) = t(p) \in \mathcal{O}$, and setting $f(y) = 0$ for all of our other basis vectors, y, of $\Omega(M)$. It is immediate that this gives a well-defined A-homomorphism. (For clarity, we point out that f is only non-zero on the background strip in our above diagram, and is zero on all other summands of $\Omega(M)$.)

We claim that f does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. Suppose to the contrary that there exists $g : \mathbb{P}(M) \to P_X$ such that $f = g \circ \mathrm{inc}$. Since

$$g(p - u)\alpha_1 = g((p - u)\alpha_1) = g(p\alpha_1) = g(p)\alpha_1,$$

$$g(p - u)\alpha_2 = g((p - u)\alpha_2) = g(u\alpha_2) = f(u\alpha_2) = 0 = g(0) = g(p\alpha_2) = g(p)\alpha_2$$

we know that $g(p - u) = g(p) \in P_X$. Thus $g(u) = g(p) - g(p - u) = 0 \in P_X$, so we know that g factors through the projection $\phi : \mathbb{P}(M) \to P_X$, i.e. there exists $h : P_X \to P_X$ such that $g = h \circ \phi$. Thus $f = h \circ \phi \circ \mathrm{inc}$.

So $h(e_{s(p)})p = h(e_{s(p)}p) = h(p) = g(p) = f(p) = p \in P_X$. Therefore $h(e_{s(p)}) = e_{s(p)} + r \in P_X$ for some $r \in \mathrm{rad}(P_X)$. Thus

$$\begin{aligned}
0 &= f(p'-u') = g(p'-u') = h(\phi(p'-u')) \\
&= h(p') = h(e_{s(p')}p') = h(e_{s(p')})p' \\
&= h(e_{s(p)})p' \qquad \text{as } e_{s(p')} \text{ and } e_{s(p)} \text{ correspond to the same element of } P_X \\
&= (e_{s(p)} + r)p' = p' + rp'
\end{aligned}$$

So $p' = -rp' \in \mathrm{rad}^{\mathrm{len}(p')+1}(P_X)$, which gives a contradiction, as required.

(Note that the core ideas of this argument don't rely on the fact that both $\mathbf{p}$ and $\mathbf{p}'$ are interior, only that they both belong to $\mathrm{supp}(\nabla)$. For example, if $\mathbf{p}$ was a boundary syllable in $\mathrm{supp}(\nabla)$, then a valid argument could be reached by replacing all instances of $p-u$ above with just p.)

Case 1a(iii): the peak contains one boundary syllable, $\mathbf{p}$, which does not belong to $\mathrm{supp}(\nabla)$ and one interior syllable $\mathbf{p}'$ where $\mathbf{p}'$ does not satisfy condition $(**)$ and has $\mathrm{len}(\mathbf{p}') > 1$

Since the argument in Lemma 6.1.9 for case 1a(iii) does not rely on the fact that the peak is pin at all, the same logic applies here. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M,A) \neq 0$.

Case 1(b): the peak contains one boundary syllable, $\mathbf{p}$, which belongs to $\mathrm{supp}(\nabla)$, and one interior syllable, $\mathbf{p}'$

The core ideas of the argument for case 0 above only rely on the fact that the peak is non-pin, and that its patch has no blank syllables. The same logic works in this case if you replace all instances of $p-u$ with just p. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M,A) \neq 0$.

Case 2b(iii): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \mathrm{supp}(\nabla)$, $\mathbf{p}' \in \mathrm{supp}(\nabla)$, and $\mathbf{p}'$ does not satisfy condition $(*)$

Since the argument in Lemma 6.1.9 for case 1b(ii) does not rely on the fact that the peak is pin at all, as with the pin version of case 2b(iii), the same logic applies. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M,A) \neq 0$.

Case 2b(iv): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \operatorname{supp}(\nabla)$, $\mathbf{p}' \in \operatorname{supp}(\nabla)$, $\operatorname{len}(\mathbf{p}') > 1$, and $\mathbf{p}'$ satisfies condition $(*)$, but not $(**)$

> Since the argument in Lemma 6.1.9 for case 1a(iii) does not rely on the fact that the peak is pin at all, as with the pin version of case 2b(iv), the same logic applies. Thus any M that fits in this case must have $\operatorname{Ext}^1_A(M, A) \neq 0$.

Case 2b(v): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \operatorname{supp}(\nabla)$, $\mathbf{p}' \in \operatorname{supp}(\nabla)$, $\operatorname{len}(\mathbf{p}') = 1$, and $\mathbf{p}'$ satisfies condition $(*)$, but not $(*')$ or $(**)$

> Since the argument in Lemma 6.1.9 for case 2b(v) does not rely on the fact that the peak is pin at all, the same logic applies. Thus any M that fits in this case must have $\operatorname{Ext}^1_A(M, A) \neq 0$.

Case 2(c): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where both belong to $\operatorname{supp}(\nabla)$

> The core ideas of the argument for case 0 above only relies on the fact that the peak is non-pin, and that its patch has no blank syllables. The same logic works in this case if you replace all instances of $p - u$ with just p, and all instances of $p' - u'$ with just p'. Thus any M that fits in this case must have $\operatorname{Ext}^1_A(M, A) \neq 0$. $\square$

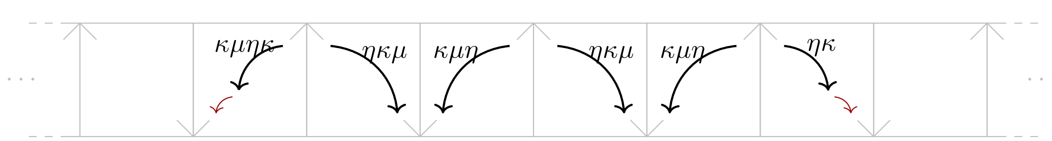
As with the pin case above, our running example algebra doesn't have any peaks in some of the important cases here, so we will use a different algebra for our example.

6.1.12. Example. Let

$$A' := \Bbbk \left(\begin{tikzcd} 1 \arrow[loop left, "\eta"] \arrow[r, bend left, "\kappa"] & 2 \arrow[l, bend left, "\mu"] \arrow[loop right, "\nu"] \end{tikzcd} \right) / \langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle$$

(note that this is the same algebra used in Example 6.1.10).

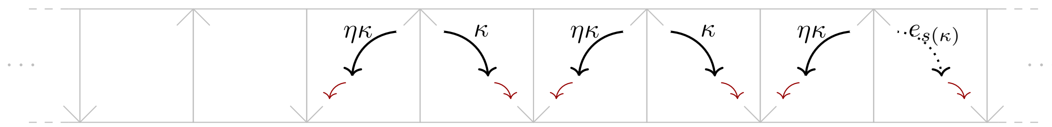
Now consider the strip, w, of A':



Then the left peak of w falls into non-pin case 1a(iii), the middle peak of w falls into non-pin case 0, and the right peak of w falls into non-pin case 1b.

Applying our syzygy algorithm to this strip leaves a flattened family of strips, w', representing

$\Omega(M) \in$ mod-A, of the following form:



The argument for the peak in non-pin case 1a(iii) focuses on the obvious homomorphism $f : \Omega(M) \to e_1 A'$ with image $\mathrm{im}(f) = \langle \kappa\mu \rangle_{A'}$ which is zero on all basis elements of $\Omega(M)$ not corresponding to the leftmost syllable of w'. As explained in Lemma 6.1.11, this morphism does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$.

Now, the argument for the peak in non-pin case 0 focuses on the obvious homomorphism $f : \Omega(M) \to e_1 A'$ with image $\langle \eta\kappa\mu \rangle_{A'}$ which is zero on all basis elements of $\Omega(M)$ not corresponding to the third syllable from the left of w'. As explained in Lemma 6.1.11, this morphism does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$.

Finally, the argument for the peak in non-pin case 0 focuses on the obvious homomorphism $f : \Omega(M) \to e_1 A'$ with image $\langle \eta\kappa\mu \rangle_{A'}$ which is zero on all basis elements of $\Omega(M)$ not corresponding to the second syllable from the right of w'. As explained in Lemma 6.1.11, this morphism does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$.

It is clear that all three of the morphisms discussed here are linearly independent.

Note that the logic applied for case 0 in Lemma 6.1.11 also gives the following result:

6.1.13. Corollary. If an indecomposable band module M has a non-zero projective string module as a summand of its projective cover, then $\mathrm{Ext}^1_A(M, A) \neq 0$.

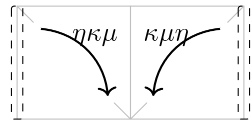
The following gives an example of such a band module:

6.1.14. Example. Let

$$A' := \Bbbk \left(\begin{tikzcd} \eta \curvearrowright 1 \arrow[r, bend left, "\kappa"] & 2 \arrow[l, bend left, "\mu"] \curvearrowleft \nu \end{tikzcd} \right) / \langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle$$

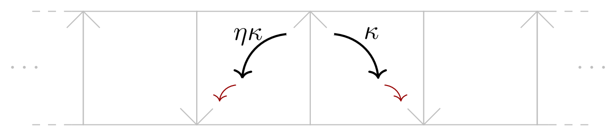
(note that this is the same algebra used in Examples 6.1.10 and 6.1.12).

Now consider the belt, w, of A':



Then the unique peak of w (considered across the wrapping) falls into non-pin case 0.

Applying our syzygy algorithm to this belt leaves a strip, w', representing $\Omega(M) \in \text{mod-}A$, of the following form:

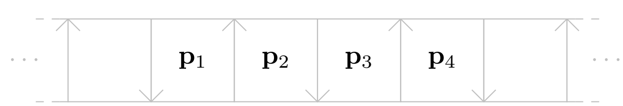


The argument for the peak in non-pin case 0 focuses on the obvious homomorphism $f : \Omega(M) \to e_1 A'$ with image $\text{im}(f) = \langle \eta\kappa\mu \rangle_{A'}$ which is zero on all basis elements of $\Omega(M)$ not corresponding to the leftmost syllable of w'. As explained in Lemma 6.1.11, this morphism does not factor through the inclusion $\text{inc} : \Omega(M) \to \mathbb{P}(M)$.

We now return to the setting of general SB algebras.

6.1.15. Lemma. Let A be an SB algebra. If an indecomposable string module $M \in \text{mod-}A$ is represented by a strip without any green peaks, then $\text{Ext}^1_A(M, A) \neq 0$.

Proof. Suppose that M is a string module represented by a strip without any green peaks, and that $\text{Ext}^1_A(M, A) = 0$. By applying Lemmas 6.1.9 and 6.1.11, this necessarily means that M is represented by a strip consisting entirely of yellow peaks. Since yellow peaks have one boundary syllable and one interior syllable, this means that M is represented by a strip of the form:



where $\mathbf{p}_1, \mathbf{p}_4$ have the form:

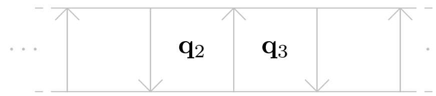
$$(\; s_i \xrightarrow{a_{s_i} + b_{s_i} - 1} \circ \xrightarrow{1} \circ \;),$$

for some choices of $s_i \in \mathcal{O}$, and $\mathbf{p}_2, \mathbf{p}_3$ have the form:

$$(\; s_i \xrightarrow{1} \circ \xrightarrow{0} \circ \;),$$

for some choices of $s_i \in \mathcal{O}$ satisfying $a_{s_i} - 1 = a_{s_i - 1}$.

In particular this means that $\Omega(M)$ is represented by a strip of the form:



where $\mathbf{q}_i = (s_i - 1 \xrightarrow{a_{s_i} - 1} \circ \xrightarrow{1} \circ)$ for $i = 2, 3$. Since $a_{s_i} - 1 = a_{s_i - 1}$ for $i = 2, 3$, $\Omega(M)$ is a non-zero projective string module.

In particular, this means that there is a homomorphism $f : \Omega(M) \to A_A$ where $\mathrm{im}(f) \not\subseteq \mathrm{rad}(A_A)$. Since the inclusion $\iota : \Omega(M) \to \mathbb{P}(M)$ of the syzygy into the projective cover has $\mathrm{im}(\iota) \subseteq \mathrm{rad}(\mathbb{P}(M))$ and for all homomorphisms $g : \mathbb{P}(M) \to A_A$, we have $g(\mathrm{rad}(\mathbb{P}(M))) \subseteq \mathrm{rad}(A_A)$, it is clear that f does not factor through ι. Thus it follows that $\mathrm{Ext}^1_A(M, A) \neq 0$, giving a contradiction as required. $\square$

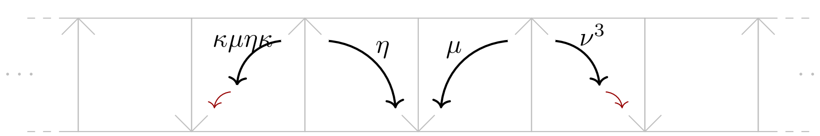
Since the running example algebra has no yellow peaks, we will again use a different algebra as an example for this result.

6.1.16. Example. Let

$$A' := \mathbb{k} \left(\eta \circlearrowright 1 \underset{\mu}{\overset{\kappa}{\rightleftarrows}} 2 \circlearrowleft \nu \right) / \langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle$$

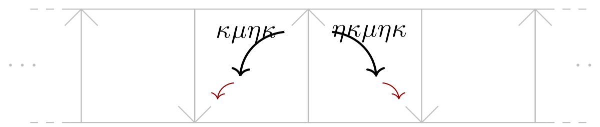
(note that this is the same algebra used in Examples 6.1.10, 6.1.12 and 6.1.14).

Now consider the strip, w, of A':



Then the left peak of w falls into non-pin case 1a(ii), and the right peak of w falls into pin case 1a(ii). They are both examples of “yellow peaks”.

Applying our syzygy algorithm to this strip leaves a strip, w', representing $\Omega(M) \in \text{mod-}A$, of the following form:



As explained in Lemma 6.1.15, this strip represents a projective string module. Thus the identity morphism on this module is a morphism to a projective that does not factor through the projective cover.

We now return to the setting of general SB algebras.

6.1.17. Lemma. Let A be an SB algebra. If an indecomposable string module $M \in \text{mod-}A$ is represented by a strip, w, where w consists of a single peak and that peak is green, then $\mathrm{Ext}^1_A(M, A) = 0$.

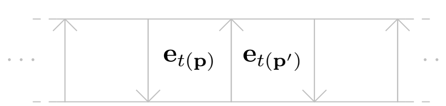
Proof. We assume that M is an indecomposable string module which is represented by a strip consisting of a single peak, which is green. This necessarily means that the peak consists of two boundary syllables.

We first consider the 4 different cases where the peak is pin:

Case 2a(i): the peak consists of two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, neither of which belong to $\mathrm{supp}(\nabla)$ and both of which satisfy condition $(**)$

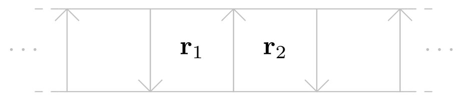
Let us denote the compression of each of $\mathbf{p}, \mathbf{p}'$ by p, p' respectively.

Since both syllables are pin-boundary, this means that M is a socle-quotient of a pin module, and $\Omega(M)$ is a simple module represented by a strip of the form:



Either $\{p\}$ or $\{p'\}$ can be used as a basis of $\Omega(M)$, and can be considered as a subset of the standard basis of $\mathbb{P}(M)$, since both of these paths represent the same element of $\mathbb{P}(M)$.

Suppose for contradiction that $\mathrm{Ext}^1_A(M, A) \neq 0$. This means that there is an indecomposable projective string module P, and a homomorphism $f : \Omega(M) \to P$ that does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. In particular, this means that f is non-zero. We will represent P by the following strip:



where $\mathbf{r}_i = (\ s_i \xrightarrow{a_{s_i}} \circ \xrightarrow{1} \circ \ )$ for $i = 1, 2$ and $s_1^\dagger = s_2$. Let y_i be the underlying $\mathcal{O}$-path of $\mathbf{r}_i$ for $i = 1, 2$. Thus $\{y_1, y_2\}$ is a basis of $\mathrm{soc}(P)$ which is a subset of the standard basis of P.

Since $\Omega(M)$ is a simple module, this means that f is an inclusion into $\mathrm{soc}(P)$. Thus $f(p) = f(p') = \lambda_1 y_1 + \lambda_2 y_2$ for some $\lambda_1, \lambda_2 \in k$, not both zero. Without loss of generality, let us assume that $\lambda_1 \neq 0$. Since f is an A-module homomorphism, this means that $t(y_1) = s_1 - a_{s_1} \in \{t(\mathbf{p}), t(\mathbf{p}')\}$.

Without loss of generality, let us assume that $t(\mathbf{p}) = s_1 - a_{s_1} \in \mathcal{O}$. Since $\mathbf{p}$ is pin-boundary, we have $s(\mathbf{p}) - a_{s(\mathbf{p})} = t(\mathbf{p}) = s_1 - a_{s_1}$. As $b_{s(\mathbf{p})} = 0$ and $b_{s_1} = 1$, it is clear that $s(\mathbf{p}) \neq s_1$ and thus that $a_{s(\mathbf{p})} \neq a_{s_1}$. We now consider the cases where $a_{s(\mathbf{p})} > a_{s_1}$ and $a_{s(\mathbf{p})} < a_{s_1}$ separately.

If $a_{s(\mathbf{p})} > a_{s_1}$, then $l \coloneqq a_{s(\mathbf{p})} - a_{s_1} > 0$, and $s(\mathbf{p}) - l = s_1$. Thus $a_{s(\mathbf{p})} - l = a_{s(\mathbf{p})-l}$, so applying the standard source encoding inequality gives us:

$$a_{s(\mathbf{p})} - l = a_{s(\mathbf{p})-1} - (l-1) = \ldots = a_{s(\mathbf{p})-(l-1)} - 1 = a_{s(\mathbf{p})-l}.$$

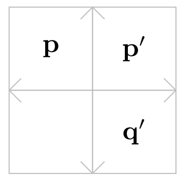
Therefore $a_{s(\mathbf{p})} - 1 = a_{s(\mathbf{p})-1}$, contradicting condition $(**)$ for $\mathbf{p}$.

On the other hand, if $a_{s(\mathbf{p})} < a_{s_1}$, then p is a strict suffix of y_1, as $t(y_1) = t(\mathbf{p}) = t(p)$ and $\mathrm{len}(p) = a_{s(\mathbf{p})} < a_{s_1} = \mathrm{len}(y_1)$. This means that there is an $\mathcal{O}$-path x_1 such that $y_1 = x_1 p$. We now define an A-morphism $g_1 : \mathbb{P}(M) \to P$ by setting $g_1(e_{s(p)}) = g_1(e_{s(p')}) = x_1$, and extending A-linearly (noting that $e_{s(p)}$ and $e_{s(p')}$ represent the same basis element of $\mathbb{P}(M)$). Now we note that $g_1(p) = g_1(e_{s(p)} p) = g_1(e_{s(p)})p = x_1 p = y_1$. Since f doesn't factor through inc and $g_1 \circ \mathrm{inc}$ clearly does, we have that $f - \lambda_1 g_1 \circ \mathrm{inc}$ doesn't factor through inc either. We also note that $(f - \lambda_1 g_1 \circ \mathrm{inc})(p) = \lambda_2 y_2$. Applying the same logic to this new A-homomorphism, results in the obvious contradiction that the zero homomorphism $\Omega(M) \to P$ doesn't factor through inc.

Since we always arrive at a contradiction, we know that $\mathrm{Ext}^1_A(M, A) = 0$ for any M that fits in this case.

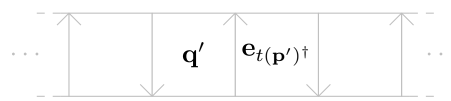
Case 2b(i): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \operatorname{supp}(\nabla)$, $\mathbf{p}' \in \operatorname{supp}(\nabla)$, and $\mathbf{p}'$ satisfies conditions $(*)$ and $(**)$

The patch, X, associated to this peak takes the following form:



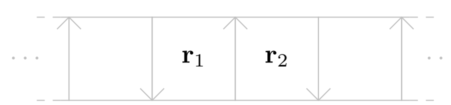
where $\mathbf{q}' = (\ t(\mathbf{p}') \xrightarrow{a_{s(\mathbf{p}')} - \operatorname{len}(\mathbf{p}')} \circ \xrightarrow{1} \circ\ )$. We will denote the projective associated to X by P_X.

Thus $\Omega(M)$ is represented by a strip of the form:



Let the compressions of $\mathbf{p}, \mathbf{p}', \mathbf{q}', \mathbf{e}_{t(\mathbf{p}')^\dagger}$ be denoted by p, p', q', α respectively. Note in particular that $\operatorname{len}(\alpha) = 1$.

Suppose for contradiction that $\operatorname{Ext}^1_A(M, A) \neq 0$. This means that there is an indecomposable projective string module P, and a homomorphism $f : \Omega(M) \to P$ that does not factor through the inclusion $\operatorname{inc} : \Omega(M) \to \mathbb{P}(M)$. In particular, this means that f is non-zero. We will represent P by the following strip:



where $\mathbf{r}_i = (\ s_i \xrightarrow{a_{s_i}} \circ \xrightarrow{1} \circ\ )$ for $i = 1, 2$ and $s_1^\dagger = s_2$. Let r_i be the compression of $\mathbf{r}_i$ for $i = 1, 2$.

We use the standard basis for P, and the standard bi-partitioned basis for $\mathbb{P}(M)$ from the syzygy algorithm. We then use the lower part of the partition as our basis for $\Omega(M)$.

Since f is non-zero, $f(p') = \sum_{j\in J} \lambda_j u_j$ for some index set J where $\lambda_j \in k$ (not all zero) and u_j is an $\mathcal{O}$-path with $s(u_j) \in \{s_1, s_2\}$ (this follows from choice of basis on P). Since $p' = p' e_{t(p')} \in \Omega(M)$ and f is an A-homomorphism, we have

$$\sum_{j\in J} \lambda_j u_j = f(p') = f(p')e_{t(p')} = \sum_{j\in J} \lambda_j u_j e_{t(p')},$$

so we may assume that $t(u_j) \in \{t(p'), t(p')^\dagger\} \subseteq \mathcal{O}_0$ for all $j \in J$. Since $p'q' = 0 \in \Omega(M)$, we also have

$$0 = f(p')q' = \sum_{j\in J} \lambda_j u_j q'.$$

Since, for our choice of basis, multiplying a basis element of P by an A-path either gives zero or another basis element, and the fact that $bq' = b'q' \neq 0$ for two basis elements b, b' implies that $b = b'$, we can assume that $u_j q' = 0$ for all $j \in J$.

To summarise, this means that $f(p') = \sum_{j\in J} \lambda_j u_j$ for some index set J where $\lambda_j \in k$ (not all zero) and u_j is an $\mathcal{O}$-path with $s(u_j) \in \{s_1, s_2\}$, $t(u_j) \in \{t(p'), t(p')^\dagger\}$ and $u_j q' = 0 \in P$ for all $j \in J$.

We claim that all $f : \Omega(M) \to P$ satisfying this must factor through $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. To show this, it is sufficient to show it for f satisfying $f(p') = u$, where $s(u) \in \{s_1, s_2\}$, $t(u) \in \{t(p'), t(p')^\dagger\}$ and $uq' = 0 \in P$, since any of the more general f can be obtained as a linear combination of these.

We first suppose that $t(u) = t(p')^\dagger$. Let $\alpha \in \mathcal{O}_1$ denote the unique arrow of the overquiver with $s(\alpha) = t(p')^\dagger$. By the construction of the overquiver, it is clear that $p'\alpha = 0 \in \Omega(M) \subsetneq \mathbb{P}(M) \subseteq A$. Thus $u\alpha = f(p')\alpha = 0 \in P$, since f is an A-module homomorphism. Hence $u\alpha' = 0 \in P$ for all $\alpha' \in Q_1$. This means that $u \in P$ is thus a basis element of $\mathrm{soc}(P)$. Thus $t(u) = s_i - a_{s_i}$ for one of $i = 1, 2$ with $a_{s_i} > 0$ or $a_{s_i} = a_{s_i^\dagger} = 0$. Since $t(u) = t(p')^\dagger$ and $b_{s_1} = b_{s_2} = 1$, this contradicts the fact that $\mathbf{p}'$ satisfies condition $(*)$.

Therefore we can now assume that $t(u) = t(p')$. We claim that this means that p' is a (not necessarily strict) suffix of u.

Suppose to the contrary, that u is a strict suffix of p'. Let $k := \mathrm{len}(p') - \mathrm{len}(u) > 0$. Then $s(u) = s(p') - k$, and so by applying Lemma 2.3.14 to $s(p') - 1$, it follows that $a_{s(p')-1} \leq a_{s(u)} + (k-1)$. Combining this with the fact that $a_{s(p')} - 1 < a_{s(p')-1}$ (which follows from condition $(**)$ being satisfied by $\mathbf{p'}$), gives $a_{s(p')} - k < a_{s(u)}$. Unpacking our definition of k then gives $a_{s(p')} - \mathrm{len}(p') + \mathrm{len}(u) < a_{s(u)}$. By the definition of $\mathbf{q'}$, this means that $(\mathrm{len}(q') - 1) + \mathrm{len}(u) < a_{s(u)}$ and hence that $\mathrm{len}(uq') \leq a_{s(u)}$. This means that uq' is necessarily a non-zero A-path, and hence a non-zero element of P, contradicting our earlier calculations.

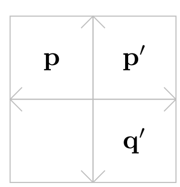
We may now assume that p' is a (not necessarily strict) suffix of u, i.e that $\mathrm{len}(p') \leq \mathrm{len}(u)$ (as we already know that $t(p') = t(u)$. Now let x be the unique $\mathcal{O}$-path satisfying $u = xp'$; note that $s(x) = s(u)$, $t(x) = s(p')$ and that $\mathrm{len}(x) = \mathrm{len}(u) - \mathrm{len}(p') \geq 0$.

We now define an A-morphism $g : \mathbb{P}(M) \to P$ by setting $g(e_{s(p)}) = g(e_{s(p')}) = x$, and extending A-linearly (noting that $e_{s(p)}$ and $e_{s(p')}$ represent the same basis element of $\mathbb{P}(M)$). Now we note that $g(p') = g(e_{s(p')}p') = g(e_{s(p')})p' = xp = u$. Since p' is an A-generator for $\Omega(M)$, it follows that $f = g \circ \mathrm{inc}$, as required.

(Note that this argument does not rely on the fact that the peak is pin.)

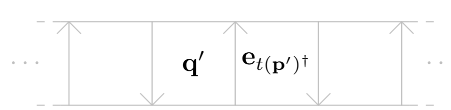
Case 2b(ii): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p'}$, where $\mathbf{p} \notin \mathrm{supp}(\nabla)$, $\mathbf{p'} \in \mathrm{supp}(\nabla)$, $\mathrm{len}(\mathbf{p'}) = 1$ and $\mathbf{p'}$ satisfies condition $(*')$ but not $(**)$

The patch, X, associated to this peak takes the following form:



where $\mathbf{q'} = (t(\mathbf{p'}) \xrightarrow{a_{s(\mathbf{p'})} - \mathrm{len}(\mathbf{p'})} \circ \xrightarrow{1} \circ)$. We will denote the projective associated to X by P_X.

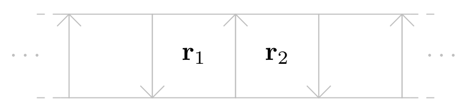
Thus $\Omega(M)$ is represented by a strip of the form:



Let the compressions of $\mathbf{p}, \mathbf{p'}, \mathbf{q'}, \mathbf{e}_{t(\mathbf{p'})^\dagger}$ be denoted by p, p', q', α respectively.

We use the standard basis for P, and the standard bi-partitioned basis for $\mathbb{P}(M)$ from the syzygy algorithm. We then use the lower part of the partition as our basis for $\Omega(M)$.

Suppose for contradiction that $\mathrm{Ext}^1_A(M, A) \neq 0$. This means that there is an indecomposable projective string module P, and a homomorphism $f : \Omega(M) \to P$ that does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. In particular, this means that f is non-zero. We will represent P by the following strip:



where $\mathbf{r}_i = (\; s_i \xrightarrow{a_{s_i}} \circ \xrightarrow{1} \circ \;)$ for $i = 1, 2$ and $s_1^\dagger = s_2$.

As in case 2b(i), we can restrict our focus to $f : \Omega(M) \to P$ satisfying $f(p') = u$, where $s(u) \in \{s_1, s_2\}$, $t(u) \in \{t(p'), t(p')^\dagger\}$ and $uq' = 0 \in P$.

Since $\mathbf{p'}$ satisfies condition $(*)$ like in case 2b(i), we can again assume that $t(u) = t(p')$. We now claim that p' is a (not necessarily strict) suffix of u.

Suppose to the contrary, that u is a strict suffix of p'. Since $\mathrm{len}(p') = \mathrm{len}(\mathbf{p'}) = 1$, this means that $u = e_{t(p')} = e_{s(p')-1}$.

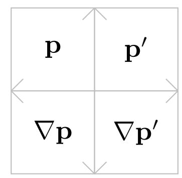
We now note that if $a_{t(p')^\dagger} = 0$ we immediately reach a contradiction for $\mathbf{p'}$ satisfying condition $(*')$. Hence it follows that $a_{t(p')^\dagger} > 0$. By our definition above, $\alpha \in \mathcal{O}_1$ denotes the unique arrow of the overquiver with $s(\alpha) = t(p')$. By the construction of the overquiver, it is clear that $p'\alpha = 0 \in \Omega(M) \subsetneq \mathbb{P}(M) \subseteq A$. Thus $u\alpha = f(p')\alpha = 0 \in P$, since f is an A-module homomorphism. This gives a contradiction as $u = e_{t(p')}$ represents the same basis vector of P as $e_{t(p')^\dagger}$, and $e_{t(p')^\dagger}\alpha = \alpha \neq 0 \in P$ (due to the fact that $a_{t(p')^\dagger} > 0$).

We may now safely assume that p' is a (not necessarily strict) suffix of u, i.e that $\mathrm{len}(p') \leq \mathrm{len}(u)$ (as we already know that $t(p') = t(u)$). Following the same logic as case 2b(i), it is possible to construct a factorisation of f through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$, as required.

(Note that this argument does not rely on the fact that the peak is pin.)

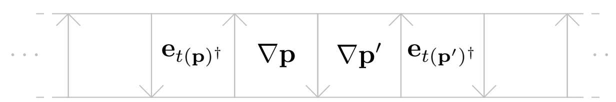
Case 2c(i): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, both of which belong to supp(∇), and both of which satisfy condition ($*$)

The patch, X, associated to this peak takes the following form:



We will denote the projective associated to X by P_X.

Since the peak is pin, both $\mathbf{p}\nabla, \mathbf{p}'\nabla$ are interior syllables. Thus $\Omega(M)$ is represented by a strip of the form:

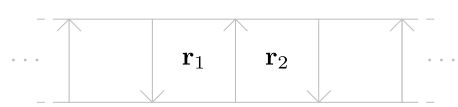


For ease of notation later, we define $\mathbf{q} = \nabla\mathbf{p}$ and $\mathbf{q}' = \nabla\mathbf{p}'$. Let the compressions of $\mathbf{p}, \mathbf{p}', \mathbf{q}, \mathbf{q}'$ be denoted by p, p', q, q' respectively.

We use the standard basis for P, and the standard bi-partitioned basis for $\mathbb{P}(M)$ from the syzygy algorithm. We then use the lower part of the partition as our basis for $\Omega(M)$.

Note that the $\mathcal{O}$-paths pq and $p'q'$ represent the same basis vector of $\Omega(M) \subseteq \mathbb{P}(M)$.

Suppose for contradiction that $\mathrm{Ext}^1_A(M, A) \neq 0$. This means that there is an indecomposable projective string module P, and a homomorphism $f : \Omega(M) \to P$ that does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. In particular, this means that f is non-zero. We will represent P by the following strip:



where $\mathbf{r}_i = (s_i \xrightarrow{a_{s_i}} \circ \xrightarrow{1} \circ)$ for $i = 1, 2$ and $s_1^\dagger = s_2$.

We use the standard basis for P, and the standard bi-partitioned basis for $\mathbb{P}(M)$ from the syzygy algorithm. We then use the lower part of the partition as our basis for $\Omega(M)$.

We first note that $pq = p'q' \in \Omega(M)$, and thus $f(pq) = f(p'q') = 0 \in P$, as $x_1\alpha_1 = x_2\alpha_2 \in P$ and $\alpha_1 \neq \alpha_2$ imply that $x_1\alpha_1 = 0$. Thus $f : \Omega(M) \to P$ can be written as a sum of two morphisms $f_1, f_2 : \Omega(M) \to P$ where f_1 (resp. f_2) is non-zero only on basis vectors corresponding to the syllable $\mathbf{q} = \nabla\mathbf{p}$ (resp. the syllable $\mathbf{q}' = \nabla\mathbf{p}'$). Since the situation is symmetric, we can thus assume without loss of generality that f is non-zero only on basis vectors corresponding to the syllable $\mathbf{q} = \nabla\mathbf{p}$. Thus f is entirely determined by its behaviour on the basis vector p, by A-linearity.

By the same logic as in case 2b(i) for $f(p')$, we may assume that $f(p) = \sum_{j\in J} \lambda_j u_j$ for some index set J where $\lambda_j \in k$ (not all zero) and u_j is an $\mathcal{O}$-path with $s(u_j) \in \{s_1, s_2\}$, $t(u_j) \in \{t(p), t(p)^\dagger\}$ and $u_j q = 0 \in P$ for all $j \in J$.

We claim that all $f : \Omega(M) \to P$ satisfying this must factor through $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. To show this, it is sufficient to show it for f satisfying $f(p) = u$, where $s(u) \in \{s_1, s_2\}$, $t(u) \in \{t(p), t(p)^\dagger\}$ and $uq = 0 \in P$, since any of the more general f can be obtained as a linear combination of these.

Since condition $(*)$ is satisfied by $\mathbf{p}$, by the same argument as case 2b(i), we may further assume that $t(u) = t(p)$. We now claim that p is a (not necessarily strict) suffix of u.

Suppose to the contrary, that u is a strict suffix of p. Let $k := \mathrm{len}(p) - \mathrm{len}(u) > 0$. Then $s(u) = s(p) - k$, so applying Lemma 2.3.14, we obtain $a_{s(p)} - k \leq a_{s(u)}$. Unpacking our definition of k then gives $a_{s(p)} - \mathrm{len}(p) + \mathrm{len}(u) \leq a_{s(u)}$. Then by the definition of $\mathbf{q} = \nabla\mathbf{p}$, it follows that $\mathrm{len}(q) + \mathrm{len}(u) \leq a_{s(u)}$. Hence $uq \neq 0 \in P$, which gives the required contradiction.

We now know that p is a (not necessarily strict) suffix of u. In fact, since $b_{s(p)} = 0$ and $b_{s(u)} = 1$, we know that p is a strict suffix of u. Following the same logic as case 2b(i), it is thus possible to construct a factorisation of f through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$, as required.

We now consider the 3 different cases where the peak is non-pin:

Case 2(a): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, neither of which belongs to $\mathrm{supp}(\nabla)$

Since neither $\mathbf{p}$ or $\mathbf{p}'$ belong to $\mathrm{supp}(\nabla)$, they both take the following form:

$$(\, i \xrightarrow{a_i} \circ \xrightarrow{1} \circ \,),$$

for some compatible choices of $i \in \mathcal{O}$. Now, since this peak is non-pin, this necessarily means that M is a non-zero projective string module. Thus $\mathrm{Ext}^1_A(M, A) = 0$ as required.

Case 2b(i): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \mathrm{supp}(\nabla)$, $\mathbf{p}' \in \mathrm{supp}(\nabla)$, and $\mathbf{p}'$ satisfies conditions $(*)$ and $(**)$

Since the above argument for case 2b(i) with a pin peak does not rely on the fact that the peak is pin, the same logic applies here. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M, A) = 0$.

Case 2b(ii): the peak contains two boundary syllables, $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \notin \mathrm{supp}(\nabla)$, $\mathbf{p}' \in \mathrm{supp}(\nabla)$, $\mathrm{len}(\mathbf{p}') = 1$ and $\mathbf{p}'$ satisfies condition $(*')$ but not $(**)$

Since the above argument for case 2b(ii) with a pin peak does not rely on the fact that the peak is pin, the same logic applies here. Thus any M that fits in this case must have $\mathrm{Ext}^1_A(M, A) = 0$. □

We have already illustrated all of the peaks with two boundary syllables that are green for our running example algebra, A, (see Figures 6.1 and 6.2). Thus we use our other example algebra, A', to give more examples of strips with a single green peak.

6.1.18. Example. Let

$$A' \coloneqq \Bbbk \left(\begin{tikzcd} \eta \circlearrowright 1 \arrow[r, bend left, "\kappa"] & 2 \arrow[l, bend left, "\mu"] \circlearrowleft \nu \end{tikzcd} \right) / \langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle$$

(note that this is the same algebra used in several examples in this section, and initially used in Example 6.1.10).

The $i \in \mathcal{O}$ where $a_i - 1 \neq a_{i-1}$ are $s(\kappa)$ and $s(\nu)$. The $j \in \mathcal{O}$ where $j^\dagger \neq i - a_i$ for any $i \in \mathcal{O}$ with $b_i = 1$ are $s(\eta)$, $s(\kappa)$ and $s(\mu)$. Therefore a syllable of A' satisfies condition $(*)$ if and only if its source is $s(\kappa)$ or $s(\nu)$, and a syllable of A' satisfies condition $(**)$ if and only if its underlying path lies on the larger connected component of A'.

This means that there are no peaks in pin case 2a(i), as $s(\nu)$ is the unique vertex on its component of the overquiver, and $s(\nu)^\dagger = s(\eta) - a_{s(\eta)} = s(\kappa) - a_{s(\kappa)}$. For a similar reason, there are no peaks in pin case 2b(i), as $s(\nu)$ is the unique vertex, i, with $b_i = 0$ and $a_i - 1 \neq a_{i-1}$. Furthermore, there are no peaks in pin case 2c(i), as there is only one vertex $i \in \mathcal{O}$ with $b_i = 0$ and $a_i - 1 \neq a_{i-1}$.

There are also no peaks in non-pin case 2b(ii), as $i - a_i = s(\mu) = s(\nu)^\dagger$ for all $i \in \mathcal{O}$ with $b_i = 1$, and $s(\nu)$ has no vertices $j \in \mathcal{O}$ with $b_j = 1$ on its connected component of the overquiver.

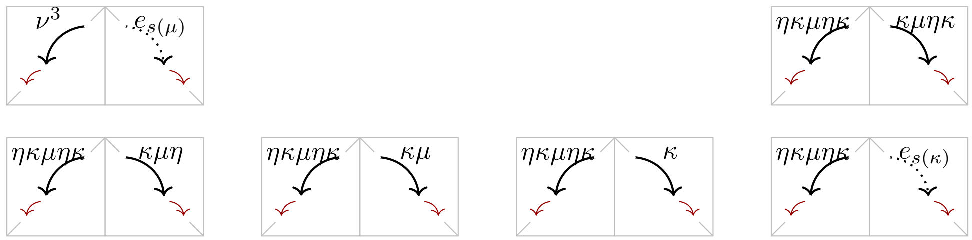


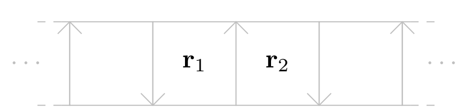
Figure 6.3: *List of all green peaks with two boundary syllables for our secondary example algebra.* The example algebra used is specified in Example 6.1.18. All peaks are given up to reflection. The left peak on the first row belongs to pin case 2b(ii). The right peak on the first row belongs to non-pin case 2(a). The second row consists of peaks belonging to non-pin case 2b(i).

It follows that Figure 6.3 lists all strips of A' with a single green peak. By Lemma 6.1.17 these are exactly the string modules, M, with $\mathrm{intwid}(M) = 0$ and $\mathrm{Ext}_A^1(M, A) = 0$.

We now return to the setting of general SB algebras. Before we continue with our classification of modules $M \in$ mod-A with $\mathrm{Ext}_A^1(M, A) = 0$, we prove a small lemma that will be useful in subsequent arguments.

6.1.19. Lemma. Let A be an SB algebra. Suppose that an indecomposable string module $M \in$ mod-A is represented by a strip, w, where w consists of multiple peaks, none of which are red, and at least one of which is green. Let $f : \Omega(M) \to P$ be a A-homomorphism where P is an indecomposable projective string module. Then f is not surjective.

Proof. We will represent P by the following strip:

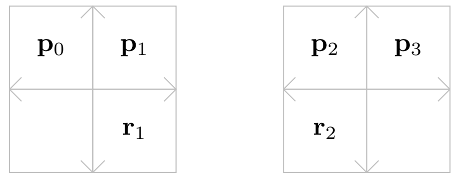


where $\mathbf{r}_i = (s_i \xrightarrow{a_{s_i}} \circ \xrightarrow{1} \circ)$ for $i = 1, 2$ and $s_1^\dagger = s_2$.

Since w contains no red peaks, it is easy to check that applying the syzygy algorithm results in $\Omega(M)$ being indecomposable.

Suppose that f is surjective. Then since P is projective, $f : \Omega(M) \to P$ splits. As $\Omega(M)$ is indecomposable, this means that $\Omega(M) \cong P$, and thus can be represented by the same strip as P.

Since w consists of multiple peaks, it must have exactly two peaks, where their associated patches take the forms:



Since at least one of the peaks of w is green, it is sufficient to check that patches associated to green peaks with one boundary syllable never take this form.

As the patches associated to peaks in pin case 1b(i) have two interior syllables in the bottom row, it remains to show that peaks in case 1a(i) don't have associated patches of this form either (for either pin or non-pin peaks).

Suppose that $(\mathbf{p}_0, \mathbf{p}_1)$ forms a peak in case 1a(i). Let us denote the compression of $\mathbf{p}_1$ and $\mathbf{r}_1$ by p_1 and r_1 respectively. We know from the construction of patches that the $\mathcal{O}$-path $p_1 r_1$ is equal to $(\ s(p_1) \xrightarrow{a_{s(p_1)}+1} \circ\ )$. Thus

$$a_{s(p_1)} + 1 = \mathrm{len}(p_1) + \mathrm{len}(r_1) = \mathrm{len}(p_1) + (a_{s_1} + 1) = \mathrm{len}(p_1) + a_{s(p_1)-\mathrm{len}(p_1)} + 1.$$

So $a_{s(p_1)} - \mathrm{len}(p_1) = a_{s(p_1)-\mathrm{len}(p_1)}$. Repeatedly applying the standard source encoding inequality thus gives:

$$\begin{aligned} a_{s(p_1)} - \mathrm{len}(p_1) &= a_{s(p_1)-\mathrm{len}(p_1)} \\ &\geq a_{s(p_1)-\mathrm{len}(p_1)+1} - 1 \\ &\geq a_{s(p_1)-\mathrm{len}(p_1)+2} - 2 \\ &\geq \dots \\ &\geq a_{s(p_1)-1} - \mathrm{len}(p_1) + 1 \\ &\geq a_{s(p_1)} - \mathrm{len}(p_1) \end{aligned}$$

Hence all of the inequalities here, are actually equalities. It follows that $a_{s(p_1)} - 1 = a_{s(p_1)-1}$, but this means that $\mathbf{p}_1$ does not satisfy condition $(**)$. This contradicts the fact that $(\mathbf{p}_0, \mathbf{p}_1)$ forms a pin peak in case 1a(i). $\square$

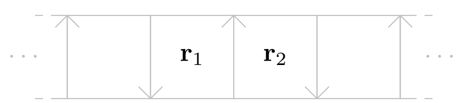
6.1.20. Lemma. Let A be an SB algebra. If an indecomposable string module $M \in \text{mod-}A$ is represented by a strip, w, where w consists of multiple peaks, none of which are red, and at least one of which is green, then $\mathrm{Ext}^1_A(M,A)=0$.

Proof. We assume that M is an indecomposable string module which is represented by a strip consisting of multiple peaks, none of which are red, and at least one of which is green. This necessarily means that all of the peaks have at most one boundary syllable.

We also note that $\Omega(M)$ is indecomposable. This follows immediately from the fact that the patches associated to green and yellow peaks have at most one boundary syllable in their bottom row.

Suppose for contradiction that $\mathrm{Ext}^1_A(M,A) \neq 0$. This means that there is an indecomposable projective string module P, and a homomorphism $f : \Omega(M) \to P$ that does not factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. In particular, this means that f is non-zero.

We will represent P by the following strip:



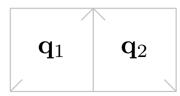
where $\mathbf{r}_i = (\ s_i \xrightarrow{a_{s_i}} \circ \xrightarrow{1} \circ \ )$ for $i=1,2$ and $s_1^\dagger = s_2$.

We use the standard basis for P, and the standard bi-partitioned basis for $\mathbb{P}(M)$ from the syzygy algorithm. We then use the lower part of the partition as our basis for $\Omega(M)$.

By applying Lemma 6.1.19, we see that f is also not surjective. Since f is not surjective, we have an inclusion $\mathrm{im}(f) \subseteq \mathrm{rad}(P)$. As P is a projective string module, $\mathrm{rad}(P)$ is a direct sum of two uniserial modules, one corresponding to each of the $\mathcal{O}$-paths $(\ s_i - 1 \xrightarrow{a_{s_i}-1} \circ \ )$ for $i=1,2$. Thus $f : \Omega(M) \to P$ is the sum of two morphisms, which each factor through one of the inclusions of these uniserial modules. It is thus sufficient to show that any morphism $f : \Omega(M) \to P$ mapping into one of these uniserial submodules must factor through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$. So we now assume that $f : \Omega(M) \to P$ factors through the uniserial module corresponding to $(\ s_1 - 1 \xrightarrow{a_{s_1}-1} \circ \ )$, which we will denote by U. We will denote the morphism in this factoring by $f' : \Omega(M) \to U$ and the inclusion by $\iota : U \to P$. Since f is non-zero, so is f'.

We now note that the only element $u \in U$ where $u = v_1\alpha_1 = v_2\alpha_2$ for two distinct arrows $\alpha_1 \neq \alpha_2$ of Q and elements $v_1, v_2 \in U$, is the zero element. Thus if $b \in \mathrm{soc}(\Omega(M))$ is a standard basis element corresponding to an interior valley, then $f(b) = \iota(f'(b)) = \iota(0) = 0 \in P$. This means that $f' : \Omega(M) \to U$ is the sum of a finite collection of morphisms, each of which is only non-zero on the basis elements corresponding to a single peak of $\Omega(M)$. Since $f = \iota \circ f'$, the same is true of f. So we now assume that f' (and hence also f) is only non-zero on the basis elements corresponding to a single peak of $\Omega(M)$.

Let us denote this peak by



Let $y \in \Omega(M)$ be the standard basis element corresponding to the top of the peak. Since f' is non-zero, $f'(y) \neq 0 \in U$. Thus $f'(y)$ corresponds to a linear combination of $\mathcal{O}$-paths, o_i, indexed by $i \in \mathcal{I}$, where $s(o_i) = s_1 - 1$, and $t(o_i)$ a vertex of $\mathcal{O}$ corresponding to the same Q-vertex as y. We can assume without loss of generality, that this linear combination has only one entry, and that $f'(y)$ is a standard basis element corresponding to an $\mathcal{O}$-path, o, satisfying these conditions. Thus $s(o) = s_1 - 1$ and $t(o) \in \{s(\mathbf{q}_1), s(\mathbf{q}_2)\}$.

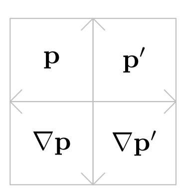
Without loss of generality, we can assume that $t(o) = s(\mathbf{q}_1)$. This means that all of our standard basis elements of $\Omega(M)$ that don't correspond to the syllable $\mathbf{q}_1$, must belong to $\ker(f') = \ker(f)$. Since each syllable of $\Omega(M)$ is determined by a single peak of M, we can handle each type of peak in our classification separately.

To summarise, we need to check that for each of the green or yellow peaks of M with at most one boundary syllable, if f' is non-zero only on the basis elements of $\Omega(M)$ corresponding to a syllable of $\Omega(M)$ that results from their presence (those in the corresponding patch and, in pin case 1b(i), the additional stationary syllable), our morphism f factors through inclusion into the projective cover $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$.

We first handle the pin peaks:

Case 0: the peak contains two interior syllables, $\mathbf{p}$ and $\mathbf{p}'$

The patch, X, associated to this peak takes the following form:



We will denote the projective associated to X by P_X.

Without loss of generality, we can assume that (using the notation of our above argument) $\mathbf{q}_1 = \nabla\mathbf{p}$. Let us denote the compressions of $\mathbf{p}$ and $\mathbf{q_1}$ by p and q_1 respectively (since both of these syllables are interior, these are also the underlying paths of the syllables).

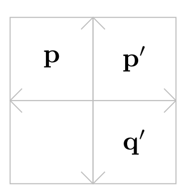
Let $y \in \Omega(M)$ be the standard basis element corresponding to the source of $\mathbf{q}_1$. Let $b \in \mathrm{soc}(\Omega(M))$ be the standard basis element corresponding to the target of $\mathbf{q}_1$. Both of these match our notation in the above general argument. Note that $yq_1 = b$.

Since $\mathbf{q}_1$ is an interior syllable, as discussed in the above argument, we know that $f'(b) = 0$. We have also assumed that $f'(y) \in U$ is a basis element represented by an $\mathcal{O}$-path, o, where $s(o) = s_1 - 1$ and $t(o) = s(\mathbf{q}_1) = s(q_1)$. Therefore $0 = f'(b) = f'(yq_1) = f'(y)q_1 = oq_1 \in U$. Let o' be the unique $\mathcal{O}$-path with $\mathrm{len}(o') = \mathrm{len}(o) + 1$ and o as a suffix. Then this means that $o'q_1 = 0 \in P$, and thus that $o'q_1$ is an $\mathcal{O}$-path which is zero when considered as an A-path. Since pq_1 is an $\mathcal{O}$-path which is non-zero when considered as an A-path, this means that pq_1 is a strict suffix of $o'q_1$, and thus that p is a strict suffix of o'. Now let x be the unique $\mathcal{O}$-path satisfying $o' = xp$; note that $s(x) = s(o')$, $t(x) = s(p)$ and that $\mathrm{len}(x) = \mathrm{len}(o') - \mathrm{len}(p) > 0$.

We now define an A-morphism $g : \mathbb{P}(M) \to P$ by setting $g(e_{s(p)}) = g(e_{s(p')}) = x$, setting $g(z) = 0$ for all other generators z of $\mathbb{P}(M)$, and then extending A-linearly (noting that $e_{s(p)}$ and $e_{s(p')}$ represent the same basis element of $\mathbb{P}(M)$). Now we note that $g(p) = g(e_{s(p)}p) = g(e_{s(p)})p = xp = o'$. Since the morphisms agree on a set of generators for $\Omega(M)$, it follows from A-linearity that $f = g \circ \mathrm{inc}$, as required.

Case 1a(i): the peak contains one boundary syllable, $\mathbf{p}$, which does not belong to $\mathrm{supp}(\nabla)$ and one interior syllable $\mathbf{p}'$, where $\mathbf{p}'$ satisfies condition $(**)$

The patch, X, associated to this peak takes the following form:



where $\mathbf{q}' = (t(\mathbf{p}') \xrightarrow{a_{s(\mathbf{p}')} - \mathrm{len}(\mathbf{p}')} \circ \xrightarrow{1} \circ)$.

So (using the notation of our above argument) we have that $\mathbf{q}_1 = \mathbf{q}'$. Let us denote the compressions of $\mathbf{p'}$ and $\mathbf{q_1}$ by p' and q_1 respectively.

Let $y \in \Omega(M)$ be the standard basis element corresponding to the source of $\mathbf{q}_1$. This matches our notation in the above argument. Note that $yq_1 = 0 \in \Omega(M)$.

We have assumed that $f'(y) \in U$ is a basis element represented by an $\mathcal{O}$-path, o, where $s(o) = s_1 - 1$ and $t(o) = s(\mathbf{q_1}) = s(q_1) = t(p')$. Since $yq_1 = 0 \in \Omega(M)$, it follows that $0 = f'(yq_1) = f'(y)q_1 = oq_1$. Let o' be the unique $\mathcal{O}$-path with $\operatorname{len}(o') = \operatorname{len}(o) + 1$ and o as a suffix. Then this means that $o'q_1 = 0 \in P$, and thus that $o'q_1$ is an $\mathcal{O}$-path which is zero when considered as an A-path. Also note that $t(o') = t(o) = t(p')$.

We now claim that p' is a (not necessarily strict) suffix of o'.

Suppose to the contrary, that o' is a strict suffix of p'. Let $k := \operatorname{len}(p') - \operatorname{len}(o') > 0$. Then $s(o') = s(p') - k$, and so by applying Lemma 2.3.14 to $s(p') - 1$, it follows that $a_{s(p')-1} \leq a_{s(o')} + (k-1)$. Combining this with the fact that $a_{s(p')} - 1 < a_{s(p')-1}$ (which follows from condition $(**)$ being satisfied by $\mathbf{p'}$), gives $a_{s(p')} - k < a_{s(o')}$. Unpacking our definition of k then gives $a_{s(p')} - \operatorname{len}(p') + \operatorname{len}(o') < a_{s(o')}$. By the definition of $\mathbf{q'}$, this means that $(\operatorname{len}(q') - 1) + \operatorname{len}(o') < a_{s(o')}$, and hence that $\operatorname{len}(o'q') \leq a_{s(o')}$. This means that $o'q'$ is necessarily an $\mathcal{O}$-path which is non-zero when considered as an A-path, contradicting our earlier calculations.

We may now assume that p' is a (not necessarily strict) suffix of o', i.e. that $\operatorname{len}(p') \leq \operatorname{len}(o')$ (as we already know that $t(p') = t(o')$). Now let x be the unique $\mathcal{O}$-path satisfying $o' = xp'$; note that $s(x) = s(o')$, $t(x) = s(p')$ and that $\operatorname{len}(x) = \operatorname{len}(o') - \operatorname{len}(p') \geq 0$.

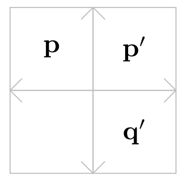
We now define an A-morphism $g : \mathbb{P}(M) \to P$ by setting $g(e_{s(p)}) = g(e_{s(p')}) = x$, setting $g(z) = 0$ for all other generators z of $\mathbb{P}(M)$, and then extending A-linearly (noting that $e_{s(p)}$ and $e_{s(p')}$ represent the same basis element of $\mathbb{P}(M)$). Now we note that $g(p) = g(e_{s(p)}p) = g(e_{s(p)})p = xp = o'$. Since the morphisms agree on a set of generators for $\Omega(M)$, it follows from A-linearity that $f = g \circ \operatorname{inc}$, as required.

(Note that this argument does not rely on the fact that the peak is pin.)

Case 1a(ii): the peak contains one boundary syllable, $\mathbf{p}$, which does not belong to $\mathrm{supp}(\nabla)$ and one interior syllable $\mathbf{p}'$, where $\mathbf{p}'$ does not satisfy condition $(**)$ and has $\mathrm{len}(\mathbf{p}') = 1$

This is the yellow pin peak case.

The patch, X, associated to this peak takes the following form:



where $\mathbf{q}' = (\; t(\mathbf{p}') \xrightarrow{a_{s(\mathbf{p}')} - \mathrm{len}(\mathbf{p}')} \circ \xrightarrow{1} \circ \;)$.

So (using the notation of our above argument) we have that $\mathbf{q_1} = \mathbf{q}'$. Let us denote the compressions of $\mathbf{p}'$ and $\mathbf{q_1}$ by p' and q_1 respectively.

We have assumed that $f'(y) \in U$ is a basis element represented by an $\mathcal{O}$-path, o, where $s(o) = s_1 - 1$ and $t(o) = s(\mathbf{q_1}) = s(q_1) = t(p')$. Since $yq_1 = 0 \in \Omega(M)$, it follows that $0 = f'(yq_1) = f'(y)q_1 = oq_1$. Let o' be the unique $\mathcal{O}$-path with $\mathrm{len}(o') = \mathrm{len}(o) + 1$ and o as a suffix. Then this means that $o'q_1 = 0 \in P$, and thus that $o'q_1$ is an $\mathcal{O}$-path which is zero when considered as an A-path. Also note that $t(o') = t(o) = t(p')$.

We now split into two cases based on whichever of o' and p' is longer.

If $\mathrm{len}(p') \leq \mathrm{len}(o')$, then applying the same logic as in case 1a(i) above shows that f factors through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$.

It remains to handle the case where $\mathrm{len}(p') > \mathrm{len}(o')$. Since $\mathrm{len}(p') = \mathrm{len}(\mathbf{p}')$, which is assumed to be 1, it follows that $\mathrm{len}(o') = 0$. But we know that $\mathrm{len}(o') = \mathrm{len}(o) + 1 \geq 0$, which gives us a contradiction. Hence this case can never occur.

Thus we know that f always factors through the inclusion $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$, as required.

(Note that this argument does not rely on the fact that the peak is pin.)

Case 1b(i): the peak contains one boundary syllable, $\mathbf{p}$, which belongs to $\mathrm{supp}(\nabla)$, and satisfies condition $(*)$, and one interior syllable $\mathbf{p}'$

The patch, X, associated to this peak takes the following form:

$$\begin{array}{|c|c|}\hline \mathbf{p} & \mathbf{p}' \\ \hline \nabla\mathbf{p} & \nabla\mathbf{p}' \\ \hline \end{array}$$

We will denote the projective associated to X by P_X.

Since $\mathbf{p}$ is boundary, the syllables we have to consider are $\nabla\mathbf{p}$, $\nabla\mathbf{p}'$ and $\mathbf{e}_{t(\mathbf{p})^\dagger}$. So (using the notation of our above argument) we have that $\mathbf{q}_1 \in \{\nabla\mathbf{p}, \nabla\mathbf{p}', \mathbf{e}_{t(\mathbf{p})^\dagger}\}$.

Since all syllables in the image of ∇ are the image of an interior syllable under ∇, and Case 0 handles such syllables, we need only consider the case where $\mathbf{q}_1 = \mathbf{e}_{t(\mathbf{p})^\dagger}$.

The following argument follows similar ideas to cases 2b(i), 2b(ii) and pin case 2c(i), but for clarity, we repeat it here.

Let $y \in \Omega(M)$ be the standard basis element corresponding to the source of $\mathbf{q}_1$. This matches our notation in the above argument.

We know that $t(o) = s(\mathbf{q}_1) = t(\mathbf{p})^\dagger$. Let $\alpha \in \mathcal{O}_1$ denote the unique arrow of the overquiver with $s(\alpha) = t(p')$. By the construction of the overquiver, it is clear that $y\alpha = 0 \in \Omega(M)$. Thus $o'\alpha = f(y)\alpha = 0 \in P$, since f is an A-module homomorphism. Hence $o'\alpha' = 0 \in P$ for all arrows $\alpha' \in Q_1$. This means that $o' \in P$ is thus a basis element of $\mathrm{soc}(P)$. Thus $t(o') = s_i - a_{s_i}$ for one of $i = 1, 2$ where $b_{s_1} = b_{s_1} = 1$ and $a_{s_i} > 0$ or $a_{s_i} = a_{s_i^\dagger} = 0$. Since $t(o') = t(\mathbf{p})^\dagger$ this contradicts the fact that $\mathbf{p}'$ satisfies condition $(*)$.

Therefore we know that all f in pin case 1b(i) factor through $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$, as required.

We now handle the non-pin peaks:

Case 1a(i): the peak contains one boundary syllable, $\mathbf{p}$, which does not belong to $\mathrm{supp}(\nabla)$ and one interior syllable $\mathbf{p}'$ where $\mathbf{p}'$ satisfies condition $(**)$

Since the above argument for subcase 1a(i) with a pin peak does not rely on the fact that the peak is pin, the same logic applies here.

Case 1a(ii): the peak contains one boundary syllable, $\mathbf{p}$, which does not belong to $\mathrm{supp}(\nabla)$ and one interior syllable $\mathbf{p}'$ where $\mathbf{p}'$ does not satisfy condition $(**)$ and has $\mathrm{len}(\mathbf{p}') = 1$

This is the yellow non-pin peak case.

Since the above argument for subcase 1a(ii) with a pin peak does not rely on the fact that the peak is pin, the same logic applies here.

Thus we have shown that all such morphisms $f : \Omega(M) \to P$ factor through the inclusion into the projective cover $\mathrm{inc} : \Omega(M) \to \mathbb{P}(M)$.

Hence all modules M of the given form have $\mathrm{Ext}^1_A(M, A) = 0$, as required. □

Note that the logic applied for the pin case 0 also gives the following result:

6.1.21. Corollary. If an indecomposable band module M has no non-zero projective string modules as summands of its projective cover, then $\mathrm{Ext}^1_A(M, A) = 0$.

We now combine our lemmas to obtain the following:

6.1.22. Proposition. Let A be an SB algebra.

An indecomposable string module $M \in \text{mod-}A$ has $\mathrm{Ext}^1_A(M, A) = 0$ if and only if it is represented by a strip with at least one green peak, and without any red peaks.

An indecomposable band module M has $\mathrm{Ext}^1_A(M, A) = 0$ if and only if it is represented by a belt consisting entirely of green peaks.

Proof. The statement for string modules follows immediately from Lemmas 6.1.9, 6.1.11, 6.1.15, 6.1.17 and 6.1.20. The statement for band modules follows immediately from Corollaries 6.1.13 and 6.1.21. □

This characterisation gives us the following property:

6.1.23. Corollary. Suppose that M is a non-projective indecomposable string module represented by a strip w, and $\mathrm{Ext}^1_A(M, A) = 0$. Then $\Omega(M)$ is an indecomposable string module, and we have the following inequalities

$$\mathrm{intwid}(M) - 2 \leq \mathrm{intwid}(\Omega(M)) \leq \mathrm{intwid}(M) + 2.$$

In particular, if $\mathrm{intwid}(M) > 0$, and $x \in \{0, 1, 2\}$ is the number of pin peaks of w in Case 1b(i) then:

$$\mathrm{intwid}(\Omega(M)) = \mathrm{intwid}(M) + 2x - 2$$

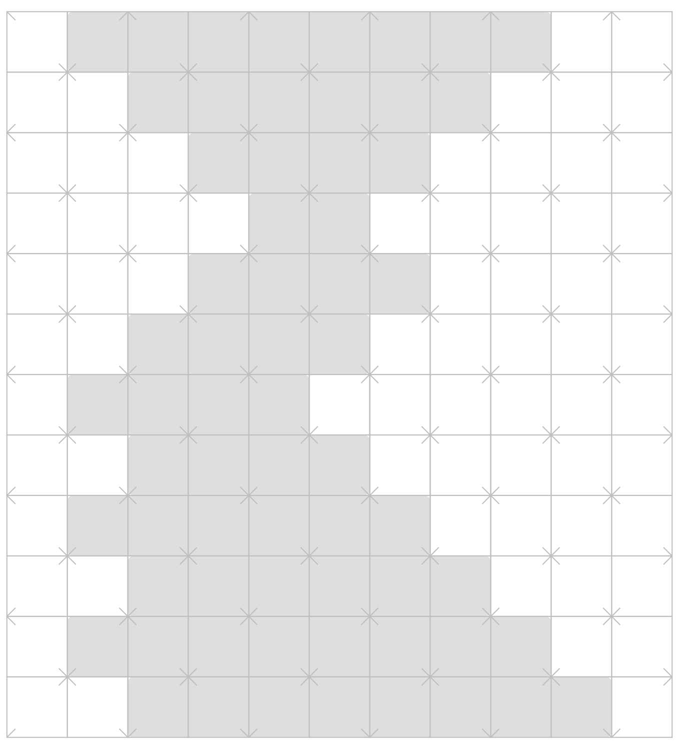
and

$$[\min(\mathrm{int}(M)) + 1, \max(\mathrm{int}(M)) - 1] \subseteq \mathrm{int}(\Omega(M)) \subseteq [\min(\mathrm{int}(M)) - 1, \max(\mathrm{int}(M)) + 1].$$

Whereas, if $\mathrm{intwid}(M) = 0$, and $y \in \{0, 1\}$ is the number of pin peaks of w in subcase 2c(i) then:

$$\mathrm{intwid}(\Omega(M)) = 2y.$$

6.1.24. Remark. One immediate consequence of this result is that if M is a semi-Gorenstein-projective string module, and we consider its syzygy fabric, the interior width of each row can differ by at most 2 from that of the previous row and in particular that the ends of the row can differ in position from those above it by at most 1. Thus if we considered only the non-blank syllables present in the syzygy fabric associated to M, then we would have a sort of "double-edged zigzag" shape; an example of this is illustrated below:



If we could show that, for a given semi-Gorenstein-projective string module there exists a finite upper bound on the interior width of its syzygies, then this would be sufficient to show that this string module was in fact Gorenstein-projective. This follows (though not immediately) from the fact that there are only finitely many string modules with interior width less than this bound, and thus you must eventually obtain a repeat.

If we could show that, for any semi-Gorenstein-projective string module there exists such a bound, then that would immediately show that all SB algebras are weakly Gorenstein. This is discussed more in Lemma 6.3.13.

6.1.2 Consequences of characterisation by colours

Now, if we restrict our focus to Gorenstein-projective modules, $M \in \text{mod-}A$, in addition to knowing that $\mathrm{Ext}^1_A(M, A) = 0$, we also know that $\Sigma\Omega(M) \simeq M \simeq \Omega\Sigma(M)$. Hence we can use results from Chapter 4 characterising when we have such stable isomorphisms to further restrict the types of peaks present in a strip or belt representing such M.

6.1.25. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Then by combining the classifications in Propositions 4.1.7, 4.1.13 and 6.1.22, we know that a string module (resp. band module), $M \in \text{mod-}A$, satisfies all of:

- $\mathrm{Ext}^1_A(M, A) = 0$,
- $M \simeq \Omega\Sigma(M)$, and
- $M \simeq \Sigma\Omega(M)$;

if and only if it is represented by a strip (resp. belt) consisting only of peaks of the types in Figure 6.4 (up to reflection).

If we additionally want $\mathrm{Ext}^2_A(M, A) = \mathrm{Ext}^1_A(\Omega(M), A) = 0$, then we can remove the sixth, eighth and ninth peaks from our list, as applying our syzygy algorithm would result in a red peak, causing $\mathrm{Ext}^1_A(\Omega(M), A) \neq 0$.

6.1.26. As previously mentioned, our running example algebra, A, has no yellow peaks. Thus it is no surprise that there are no yellow peaks in Figure 6.4.

We now look at the corresponding list of peaks for our secondary running example algebra, since this algebra does have yellow peaks.

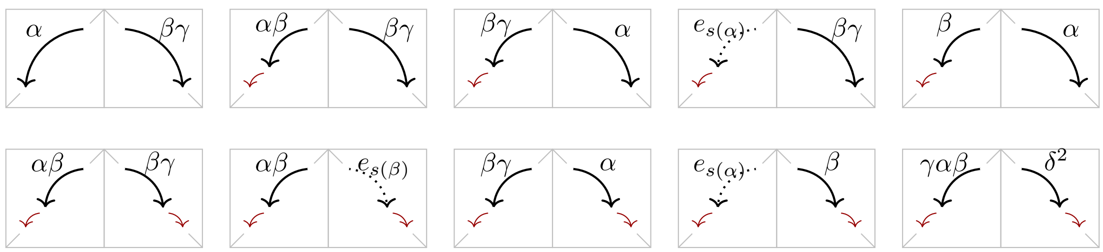


Figure 6.4: *List of green peaks for our running example algebra satisfying conditions of Proposition 4.1.7 and Proposition 4.1.13.* Since no non-pin maximal syllables satisfy the double twist condition, the only non-pin peak in this list corresponds to the projective string module.

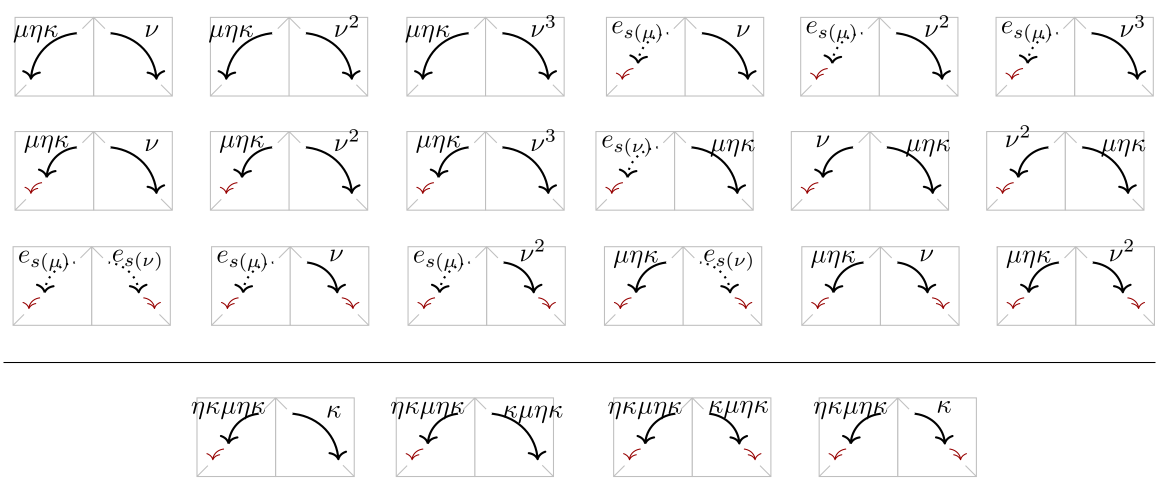


Figure 6.5: *List of green and yellow peaks for our secondary example algebra satisfying conditions of Proposition 4.1.7 and Proposition 4.1.13.* The pin peaks are listed above the line, while the non-pin peaks are listed below. Note that even though this algebra has yellow peaks, none of them appear here.

6.1.27. Example. Let

$$A' \coloneqq \Bbbk\left(\begin{tikzcd} 1 \arrow[loop left, "\eta"] \arrow[r, bend left, "\kappa"] & 2 \arrow[l, bend left, "\mu"] \arrow[loop right, "\nu"] \end{tikzcd} \right) / \langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle.$$

Then by combining the classifications in Propositions 4.1.7, 4.1.13 and 6.1.22, we know that a string module (resp. band module), $M \in \text{mod-}A'$, satisfies all of:

- $\mathrm{Ext}^1_{A'}(M, A') = 0$,
- $M \simeq \Omega\Sigma(M)$, and
- $M \simeq \Sigma\Omega(M)$;

if and only if it is represented by a strip (resp. belt) consisting only of peaks of the types in Figure 6.5 (up to reflection).

If we additionally want $\mathrm{Ext}^2_{A'}(M, A') = \mathrm{Ext}^1_{A'}(\Omega(M), A') = 0$, then we can remove pin peaks 4-9 and 13-18 from our list, along with the final non-pin peak, as applying our syzygy algorithm would result in red peaks, causing $\mathrm{Ext}^1_{A'}(\Omega(M), A') \neq 0$.

Note that there are no yellow peaks in Figure 6.5. We can verify that, no matter which algebra is chosen, this is always the case. This follows immediately from the following result for general SB algebras.

6.1.28. Proposition. Let A be an SB algebra. Then:

- no string module, M, represented by a strip with a yellow peak, satisfies $M \simeq \Sigma\Omega(M)$, and
- any string (resp. band) module, M, represented by a strip (resp. belt) consisting entirely of green peaks (except Case 2b(ii)) satisfies $M \simeq \Sigma\Omega(M)$.

It follows that a string (resp. band) module, M, satisfies both $\mathrm{Ext}^1_A(M, A) = 0$ and $M \simeq \Sigma\Omega(M)$ if and only if it is represented by a strip (resp. belt) consisting entirely of green peaks.

Proof. We first handle the case where M is a projective string module. This means that it is represented by a strip with a single green peak (Case 2(c)), and it is automatic that it satisfies $M \simeq \Sigma\Omega(M)$.

We now focus on the case where M is not projective. Due to the way that the cases are separated in Subsection 6.1.1 and Proposition 4.1.13, it is sufficient to prove that:

(i) a syllable $\mathbf{p}$ satisfies condition $(**)$ from Subsection 6.1.1 if and only if $a_{s(\mathbf{p})} = c_{s(\mathbf{p})-1-a_{s(\mathbf{p})}}$, and

(ii) if a syllable, $\mathbf{p}$, satisfies condition $(*)$ from Subsection 6.1.1, then $t(\mathbf{p})^{\dagger}$ satisfies condition ♣ from Proposition 4.1.13.

We first assume that $\mathbf{p}$ satisfies condition $(**)$, i.e. we assume that $a_{s(\mathbf{p})} - 1 \neq a_{s(\mathbf{p})-1}$. Then by Lemma 2.3.12, this implies $a_{s(\mathbf{p})} < a_{s(\mathbf{p})-1} + 1$. Since all of the values in this inequality are integers, this means that $a_{s(\mathbf{p})} \leq a_{s(\mathbf{p})-1}$. By Lemma 2.3.12, this means that $a_{s(\mathbf{p})} \leq c_{s(\mathbf{p})-1-a_{s(\mathbf{p})}}$. For the other inequality, note that since $a_{s(\mathbf{p})} + 1 > a_{s(\mathbf{p})}$, Lemma 2.3.13 implies that $a_{s(\mathbf{p})} + 1 > c_{s(\mathbf{p})-(a_{s(\mathbf{p})}+1)}$. It follows that $a_{s(\mathbf{p})} = c_{s(\mathbf{p})-1-a_{s(\mathbf{p})}}$, as required.

Now instead assume that $\mathbf{p}$ does not satisfy condition $(**)$, i.e. we assume that $a_{s(\mathbf{p})} - 1 = a_{s(\mathbf{p})-1}$. Thus $a_{s(\mathbf{p})} = a_{s(\mathbf{p})-1} + 1 > a_{s(\mathbf{p})-1}$, so we can apply Lemma 2.3.13 to obtain the inequality $a_{s(\mathbf{p})} > c_{s(\mathbf{p})-1-a_{s(\mathbf{p})}}$. Hence $a_{s(\mathbf{p})} \neq c_{s(\mathbf{p})-1-a_{s(\mathbf{p})}}$, as required.

Therefore, statement (i) holds.

We now prove statement (ii) by contrapositive. Suppose that $j := t(\mathbf{p})^{\dagger}$ does not satisfy condition ♣. Then $d_j = 0$ and $c_j \neq c_{j-1}$. By Lemma 2.3.14, it follows that $c_j > c_{j-1}$. Since $d_j = 0$, we know that $c_j \geq 2$. Therefore Lemma 2.3.13 gives us that $a_{j+c_j-1} < c_j$. Since all of the values in this inequality are integers, this means that $a_{j+c_j-1} \leq c_j - 1$. Now, as $c_j - 1 \leq c_j$, Lemma 2.3.12 implies the reverse inequality, that $a_{j+c_j-1} \geq c_j - 1$. Hence $a_{j+c_j-1} = c_j - 1 \geq 1$, and thus $(j + c_j - 1) - a_{j+c_j-1} = j$.

It remains to show that $b_{j+c_j-1} = 1$. Suppose otherwise that $b_{j+c_j-1} = 0$. Then

$$c_j = c_{j+(c_j-1)-(c_j-1)} = c_{j+c_j-1-(a_{j+c_j-1})} = a_{j+c_j-1} = c_j - 1,$$

which is clearly a contradiction. Hence $b_{j+c_j-1} = 1$, and thus $\mathbf{p}$ does not satisfy condition $(*)$, as required. $\square$

Statements (i) and (ii) in the above proof also give the following:

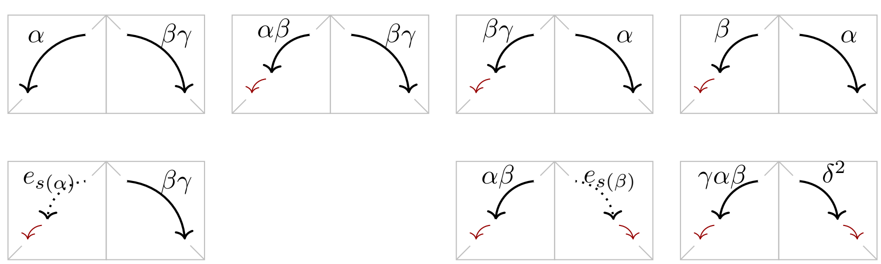
> **6.1.29. Corollary.** Let A be an SB algebra. Then any string (resp. band) module, M, represented by a strip (resp. belt) containing a peak in:
>
> - pin Cases 1a(iii), 2a(ii), 2b(ii), 2b(iv), 2b(v), or
> - non-pin Cases 1a(iii), 2b(ii), 2b(iv), 2b(v), 2(c)
>
> does not satisfy $M \simeq \Sigma\Omega(M)$.

Note that neither of these results say anything about red peaks in pin Cases 1b(ii), 2b(iii), 2c(ii) or in non-pin Case 2b(iii). This is because, in general, peaks in these cases can satisfy the conditions of Proposition 4.1.13 in some algebras, and fail to satisfy them in others.

6.1.30. Using the colouring of peaks and the conditions of Propositions 4.1.7 and 4.1.13, along with the syzygy algorithm, we can characterise the finitely-generated Gorenstein-projective modules by hand for small examples.

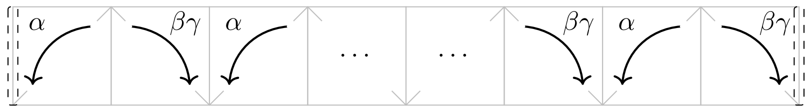
We first do this for our running example algebra, as introduced in Paragraph 2.2.41.

6.1.31. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. As discussed in Example 6.1.25, a non-pin module $M \in \text{Gproj-}A$ must be represented by a strip or band formed entirely of peaks of the following types:



We first handle the peaks with two boundary syllables, as they completely define the strips they belong to. The first of these peaks represents a uniserial module whose syzygy is isomorphic to itself. Since we already know that this module, U, satisfies $\operatorname{Ext}^1_A(U, A) = 0$, it therefore follows that U is Gorenstein-projective. The second of these peaks represents the unique indecomposable projective string module, and is thus automatically Gorenstein-projective.

We now focus our attention on the peaks with at least one interior syllable. The first thing to note is that there is only one peak in our list with two interior syllables. This means that if we have any belts representing Gorenstein-projective band modules, they must be of the form:

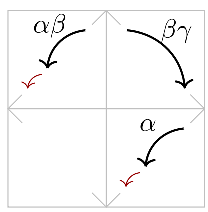


for some non-negative number of peaks.

Fortunately, it is easy to verify that applying the syzygy algorithm to such a belt results in a belt of the same form. Hence all band modules represented by a belt of this form are Gorenstein-projective, and there are no other Gorenstein-projective band modules for this algebra.

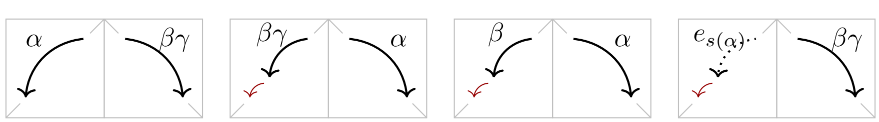
Now, since we have classified the Gorenstein-projective band modules and the Gorenstein-projective string modules whose interior width is zero, we may focus on the Gorenstein-projective string modules whose interior width is greater than zero.

We first note that the patch corresponding to the second peak in our list is:

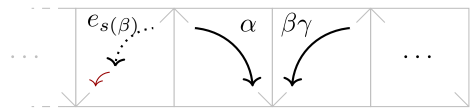


Since the boundary syllable ($\circ \xrightarrow{\alpha} \circ \xrightarrow{1} \circ$) is not present in any of the peaks in our list, it follows that the second peak in our list can't appear in a strip representing a Gorenstein-projective string module.

Therefore a Gorenstein-projective string module of non-zero interior width must be represented by a strip where all of its peaks are of one of the following forms:

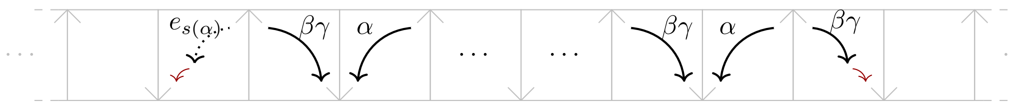


Now suppose that $M \in \text{Gproj-}A$ is a string module with $\operatorname{intwid}(M) > 0$, and that it is represented by a strip containing the third peak in our shortened list. Applying our syzygy algorithm to this peak results in a local diagram of the form:



Now we note that the left peak in this diagram is not in our shortened list, and that $\mathrm{intwid}(\Omega(M)) > 0$. Hence $\Omega(M) \notin \mathrm{Gproj}\text{-}A$, giving us a contradiction.

Therefore a Gorenstein-projective string module of non-zero interior width must be represented by a strip of the form:



for some non-negative number of peaks of the first type in the interior of our strip.

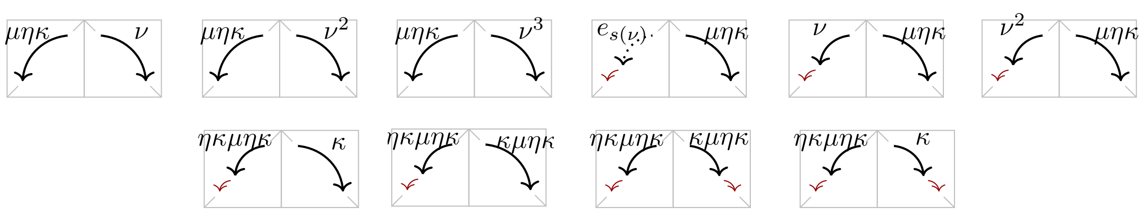
If we apply the syzygy algorithm to such a strip, it becomes clear that for modules M of this form, we have $M \simeq \Omega(M)$. Therefore, it follows that all modules of this form are Gorenstein-projective.

We can also characterise such modules for the secondary example used in Subsection 6.1.1.

6.1.32. Example. Let

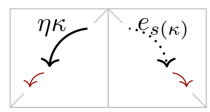
$$A' := \Bbbk \left(\eta \circlearrowright 1 \underset{\mu}{\overset{\kappa}{\rightleftarrows}} 2 \circlearrowleft \nu \right) / \langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle.$$

As discussed in Example 6.1.27, a non-pin module $M \in \mathrm{Gproj}\text{-}A'$ must be represented by a strip or band formed entirely of peaks of the following types:



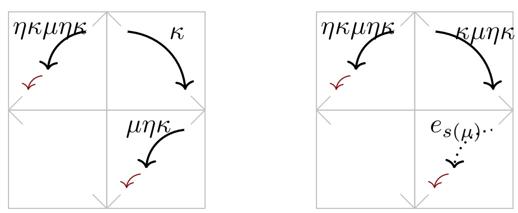
We first handle the peaks with two boundary syllables, as they completely define the strips they belong to. The first of these peaks represents the unique indecomposable projective string module, and is thus automatically Gorenstein-projective.

The second of these peaks represents an indecomposable string module whose syzygy is represented by a strip of the form:



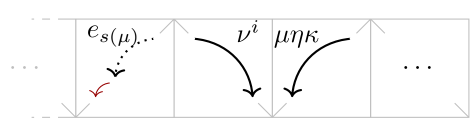
which does not belong to our list. Hence the second of these peaks is not present in any strip representing a Gorenstein-projective string module.

We now consider the two non-pin peaks with one interior syllable. The patches corresponding to these peaks are:



Since the boundary syllables ($\circ \xrightarrow{\mu\eta\kappa} \circ \xrightarrow{1} \circ$) and ($\circ \xrightarrow{e_{s(\mu)}} \circ \xrightarrow{1} \circ$) are not present in any of the peaks in our list, the two non-pin peaks with one interior syllable can't be present in any strip representing a Gorenstein-projective string module.

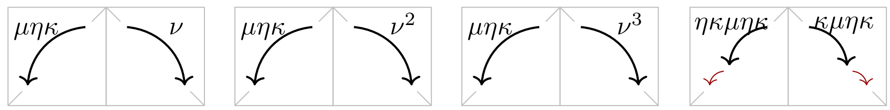
We now consider the pin peaks with one interior syllable. Applying the syzygy algorithm to these peaks results in a local picture of a strip of the form:



for some $i \in \{1, 2, 3\}$.

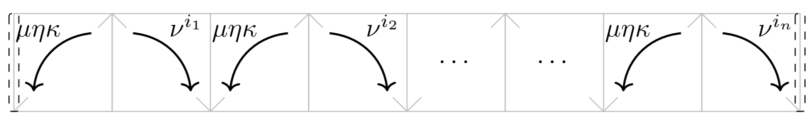
Since the stationary syllable ($\circ \xrightarrow{e_{s(\mu)}} \circ \xrightarrow{1} \circ$) does not appear in any of the peaks in our list, the three pin peaks with one interior syllable can't be present in any strip representing a Gorenstein-projective string module.

This means that any non-pin module $M \in \text{Gproj-}A'$ must be represented by a strip or band formed entirely of peaks of the following types:



Since there are no peaks in this list with one interior and one boundary syllable, there are no Gorenstein-projective string modules with non-zero interior width. Therefore, as there is only one peak in our list with two boundary syllables, the projective string module is the unique indecomposable Gorenstein-projective string module.

It now remains to classify the Gorenstein-projective band modules for this algebra. Since we know that it must be represented by a belt, all of whose peaks are of one of the first three forms in our shortened list, they are represented by belts of the form:



for some $n \in \mathbb{Z}_+$ and $(i_k)_{k=1}^n \subseteq \{1, 2, 3\}$.

Applying our syzygy algorithm to this belt tells us that its syzygy must also be of the same form, with the same value of n, but possible different values for the i_k. Since this process must eventually repeat under repeated application of the syzygy algorithm, it follows that all such band modules have a syzygy which is represented by the same belt as them. Hence all of these band modules are Gorenstein-projective.

6.1.33. While this method works for algebras with small $\Bbbk$-dimension, for larger algebras it quickly becomes impractical to use these techniques to characterise finitely-generated Gorenstein-projective string modules, due to the larger and larger collections of green peaks. This is due to the fact that we are not aware of any robust way to implement this method in a computer, and thus all calculations must be performed by hand.

However, it is possible to characterise the Gorenstein-projective band modules in general. This will be the focus of the next section.

6.2 Band modules in $\Omega^\infty(\text{mod-}A)$

The aim of this section is to characterise the band modules belonging to Gproj-A for a special biserial algebra, A. In general, we only have an inclusion Gproj-$A \subseteq \Omega^\infty(\text{mod-}A)$. However, we will prove that if we restrict to band modules, we in fact have an equality:

$$\{\mathrm{Bnd}(v^m, \psi) \in \text{Gproj-}A\} = \{\mathrm{Bnd}(v^m, \psi) \in \Omega^\infty(\text{mod-}A)\}.$$

We first characterise the modules belonging to the right side of our claimed equality, and then show that they all belong to the left side as well.

6.2.1. Proposition. Let A be an SB algebra. Let $k \in \mathbb{N}$ and Φ_A be the pin graph of A. A band module, M, represented by a belt, w, belongs to $\Omega^k(\text{mod-}A)$ if and only if both of the following hold:

(i) the Q-vertex corresponding to the bottom of each valley of w is the target of a path of length $\lceil k/2 \rceil$ in Φ_A, and

(ii) the Q-vertex corresponding to the top of each peak of w is the target of a path of length $\lfloor k/2 \rfloor$ in Φ_A.

Proof. We proceed with this proof by induction.

Firstly, we handle the base case, where $k = 0$. In this case,

$$\Omega^k(\text{mod-}A) = \Omega^0(\text{mod-}A) = \text{mod-}A$$
$$\lceil k/2 \rceil = \lceil 0 \rceil = 0 = \lfloor 0 \rfloor = \lfloor k/2 \rfloor.$$

Therefore neither of the conditions we claim to be equivalent place any restriction on the band module in question. Hence the result holds in the base case.

Now for the inductive step we assume that the result holds for $k = l \in \mathbb{N}$.

Suppose that $M \in \Omega^{l+1}(\text{mod-}A)$ is a band module represented by a belt w. Then there exists $M' \in \Omega^l(\text{mod-}A)$ where $\Omega(M') \cong M$. Since the syzygy of a string module is a string module, it follows that M' is a band module, represented by a belt w'. By the inductive hypothesis, this means that:

(i′) the Q-vertex corresponding to the bottom of each valley of w' is the target of a path of length $\lceil l/2 \rceil$ in Φ_A, and

(ii′) the Q-vertex corresponding to the top of each peak of w' is the target of a path of length $\lfloor l/2 \rfloor$ in Φ_A.

Since we know that $\Omega(M') \cong M$ is a band module, we know from our syzygy algorithm that:

- there is an identification of valleys of w' with peaks of w, where the Q-vertices corresponding to their bottom and top respectively are the same,
- there is an identification of peaks of w' with valleys of w, where for each peak of w', the Q-vertex associated to its top is the source of an arrow of Φ_A whose target is the Q-vertex associated to the bottom of the valley of w it is identified with.

Since $\lceil l/2 \rceil = \lfloor (l+1)/2 \rfloor$ and $\lfloor l/2 \rfloor + 1 = \lfloor (l+2)/2 \rfloor = \lceil (l+1)/2 \rceil$, it follows that conditions (i) and (ii) hold for M when $k = l+1$, as required.

Now instead suppose that M is a band module represented by a belt w satisfying conditions (i) and (ii) when $k = l+1$. Since $l \in \mathbb{N}$, we know that $\lceil (l+1)/2 \rceil \in \mathbb{Z}_+$. Hence the Q-vertex associated to the bottom of each valley of w is the target of an arrow in Φ_A. Therefore each syllable of w belongs to $\mathrm{im}(\nabla)$, and thus by Proposition 3.2.13, there exists a band module M' represented by a belt w', such that $\Omega(M') \cong M$. Thus we know from our syzygy algorithm that:

- there is an identification of valleys of w' with peaks of w, where the Q-vertices corresponding to their bottom and top respectively are the same,
- there is an identification of peaks of w' with valleys of w, where for each peak of w', the Q-vertex associated to its top is the source of an arrow of Φ_A whose target is the Q-vertex associated to the bottom of the valley of w it is identified with.

Since the pin graph Φ_A is sub-1-regular, $\lceil l/2 \rceil = \lfloor (l+1)/2 \rfloor$ and $\lfloor l/2 \rfloor + 1 = \lfloor (l+2)/2 \rfloor = \lceil (l+1)/2 \rceil$, it follows that conditions (i) and (ii) hold for M' when $k = l$, so the inductive hypothesis tells us that $M' \in \Omega^{l}(\text{mod-}A)$. Hence $M \cong \Omega(M') \in \Omega^{l+1}(\text{mod-}A)$, as required. $\square$

We can rephrase this result in a way that does not require representing a band module as a belt.

6.2.2. Corollary. Let $k \in \mathbb{N}$ and Φ_A be the pin graph of A. A band module, M, belongs to $\Omega^k(\text{mod-}A)$ if and only if both of the following hold:

(i) the Q-vertex corresponding to each simple summand of $\operatorname{soc}(M)$ is the target of a path of length $\lceil k/2 \rceil$ in Φ_A, and

(ii) the Q-vertex corresponding to each simple summand of $\operatorname{top}(M)$ is the target of a path of length $\lfloor k/2 \rfloor$ in Φ_A.

By "taking the limit" of Proposition 6.2.1, we obtain the following.

6.2.3. Proposition. Let A be an SB algebra. Let Φ_A be the pin graph of A. A band module, M, represented by a belt, w, belongs to $\Omega^\infty(\text{mod-}A)$ if and only if the Q-vertex corresponding to the bottom of each valley and the top of each peak of w lies on a cycle of Φ_A.

Proof. Let M be a band module represented by a belt w.

Suppose that $M \in \Omega^\infty(\text{mod-}A)$. Therefore, for all $k \in \mathbb{N}$, we have $M \in \Omega^k(\text{mod-}A)$. Hence, for all $k \in \mathbb{N}$, Proposition 6.2.1 gives us:

(i) the Q-vertex corresponding to the bottom of each valley of w is the target of a path of length $\lceil k/2 \rceil$ in Φ_A, and

(ii) the Q-vertex corresponding to the top of each peak of w is the target of a path of length $\lfloor k/2 \rfloor$ in Φ_A.

Therefore the Q-vertex corresponding to the bottom of each valley and the top of each peak of w is the target of paths in Φ_A of arbitrarily large finite length. Since Φ_A is finite, this means that the Q-vertex corresponding to the bottom of each valley and the top of each peak of w lies on a cycle of Φ_A, as required.

Now, instead suppose that the Q-vertex corresponding to the bottom of each valley and the top of each peak of w lies on a cycle of Φ_A, as required. Hence, for all $k \in \mathbb{N}$, we know that:

(i) the Q-vertex corresponding to the bottom of each valley of w is the target of a path of length $\lceil k/2 \rceil$ in Φ_A, and

(ii) the Q-vertex corresponding to the top of each peak of w is the target of a path of length $\lfloor k/2 \rfloor$ in Φ_A.

Therefore, for all $k \in \mathbb{N}$, Proposition 6.2.1 implies that $M \in \Omega^k(\text{mod-}A)$. Thus $M \in \Omega^\infty(\text{mod-}A)$, as required. □

Like with Proposition 6.2.1, we can rephrase this result in a way that does not require representing a band module as a belt.

6.2.4. Corollary. Let Φ_A be the pin graph of A. A band module, M, represented by a belt, w, belongs to $\Omega^\infty(\text{mod-}A)$ if and only if the Q-vertex corresponding to each simple summand of $\text{soc}(M) \oplus \text{top}(M)$ lies on a cycle of Φ_A.

This also means that we can rule out the existence of Gorenstein-projective band modules when there are no cycles in the pin graph.

6.2.5. Corollary. Let A be an SB algebra. If Φ_A is acyclic, then there are no band modules in $\Omega^\infty(\text{mod-}A)$.

Hence if Φ_A is acyclic, then there are no band modules in Gproj-A.

We now note that band modules belonging to $\Omega^\infty(\text{mod-}A)$ are almost Ω-periodic.

6.2.6. Lemma. Let A be an SB algebra. Let $M \in \Omega^\infty(\text{mod-}A)$ be a band module represented by a belt w. Then there exists $k \in \mathbb{Z}_+$ such that $\Omega^k(M)$ is also represented by the belt w.

Proof. By Proposition 6.2.3, we know that the Q-vertices associated to the bottom of each valley and the top of each peak of w lies on a cycle of Φ_A. It thus follows from our syzygy algorithm that $\Omega^i(M)$ is a band module for all $i \in \mathbb{N}$. Then the result follows from Proposition 4.2.2. □

We can strengthen this lemma by specifying a particular value of k based on the structure of the pin graph Φ_A and how the band module in question relates to it.

6.2.7. Proposition. Let A be an SB algebra. Let $M \in \Omega^\infty(\text{mod-}A)$ be a band module represented by a belt $w : \mathbb{Z}/n\mathbb{Z} \to \text{Syll}(A)$. Let v be the Q-vertex corresponding to $s(w(0 + n\mathbb{Z})) \in \mathcal{O}_0$. Let l be the length of the cycle of Φ_A that v belongs to. Then $\Omega^{2l}(M)$ is also represented by the belt w.

Proof. By our characterisation of band graphs belonging to $\Omega^\infty(\text{mod-}A)$ (Proposition 6.2.3), we know that the Q-vertices associated to the source and target of each syllable of w lies on a cycle of Φ_A. Thus by Proposition 2.3.47 we know that the Q-vertices associated to the source and target of each syllable lie on cycles of the same length. Recall that the Q-vertices associated to the sources of syllables forming a peak are the same, and that the Q-vertices associated to the target of syllables forming a valley are the same. It follows that the Q-vertices corresponding to the source and target of each syllable lie on a cycle of Φ_A of length l.

Hence by Proposition 2.3.47, it follows that

$$\nabla^{2l}\mathbf{p} = \left(s(\mathbf{p}) \xrightarrow{\mathrm{len}(\mathbf{p})} \circ \xrightarrow{0} t(\mathbf{p}) \right)$$

for each syllable $\mathbf{p}$ of w. Since each syllable of w is interior, it follows that $\nabla^{2l}\mathbf{p} = \mathbf{p}$ for each syllable $\mathbf{p}$ of w. Hence by our syzygy algorithm for belts, we know that $\Omega^{2l}(M)$ is also represented by w. □

If you need a fixed value of k that works for all band modules, then you can use the following result.

6.2.8. Corollary. Let A be an SB algebra. Let $M \in \Omega^\infty(\text{mod-}A)$ be a band module represented by a belt $w : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$. Let

$$k_0 \coloneqq 2 \cdot \mathrm{lcm}\left(\{l \in \mathbb{Z}_+ \ :\ \text{where } l \text{ is the length of a cycle in } \Phi_A\}\right).$$

Then $\Omega^{2l}(M)$ is also represented by the belt w.

We can now combine our characterisation of band modules belonging to $\Omega^\infty(\text{mod-}A)$ with our previous characterisations of band modules M satisfying $M \simeq \Omega\Sigma(M)$, $M \simeq \Sigma\Omega(M)$ and $\mathrm{Ext}^1_A(M, A) = 0$.

6.2.9. Proposition. Let A be an SB algebra. Let $M \in \Omega^\infty(\text{mod-}A)$ be a band module. Then $M \simeq \Omega\Sigma(M)$, $M \simeq \Sigma\Omega(M)$, $\mathrm{Ext}^1_A(M, A) = 0$ and $\Omega(M), \Sigma(M)$ are both also band modules belonging to $\Omega^\infty(\text{mod-}A)$.

Proof. Let w be a belt representing M.

The first condition follows from the fact that the Q-vertex corresponding to the bottom of a valley of w is the target of an arrow in Φ_A (by applying Proposition 4.1.7).

The second and third conditions follow from the fact that the Q-vertex corresponding to the top of a peak of w is the source of an arrow in Φ_A (by applying Proposition 4.1.15 and Proposition 6.1.22 respectively).

The fact that $\Omega(M)$ is a band module belonging to $\Omega^\infty(\text{mod-}A)$ follows from the syzygy algorithm for belts (Proposition 3.2.6) and our characterisation of band modules in $\Omega^\infty(\text{mod-}A)$ (Proposition 6.2.3).

The fact that $\Sigma(M) = \operatorname{Tr}\Omega\operatorname{Tr}(M)$ is a band module belonging to $\Omega^\infty(\text{mod-}A)$ follows from a combination of the syzygy algorithm and the transpose algorithm for belts (Proposition 3.2.6 and Proposition 3.1.14 respectively), along with our characterisation of band modules in $\Omega^\infty(\text{mod-}A)$ (Proposition 6.2.3). $\square$

It then follows that all band modules in $\Omega^\infty(\text{mod-}A)$ are Gorenstein-projective.

6.2.10. Corollary. Let $M \in \Omega^\infty(\text{mod-}A)$ be a band module. Then M is Gorenstein-projective.

Proof. By Proposition 6.2.9, we know that we have two short exact sequences:

$$0 \longrightarrow \Omega(M) \hookrightarrow P^{-1} \twoheadrightarrow M \longrightarrow 0$$

$$0 \longrightarrow M \hookrightarrow P^{0} \twoheadrightarrow \Sigma(M) \longrightarrow 0$$

where $\Omega(M), \Sigma(M)$ are both band modules belonging to $\Omega^\infty(\text{mod-}A)$.

We an repeat this process and splice together the short exact sequences to obtain a long exact sequence of projectives

$$P^\bullet := \qquad \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^{0} \longrightarrow P^{1} \longrightarrow P^{2} \longrightarrow \cdots$$

where $Z^0(P^\bullet) = M$ and for all $i \in \mathbb{Z}$ $Z^i(P^\bullet)$ is a band module belonging to $\Omega^\infty(\text{mod-}A)$. Thus, by Proposition 6.2.9, we have $\operatorname{Ext}^1_A(Z^i(P^\bullet), A) = 0$ for all $i \in \mathbb{Z}$. Therefore, $\operatorname{Ext}^j_A(Z^i(P^\bullet), A) = 0$ for all $i \in \mathbb{Z}, j \in \mathbb{Z}_+$.

This means that $P^\bullet$ is totally acyclic, and hence that M is Gorenstein-projective, as required. $\square$

While at this point we could restate Proposition 6.2.3, replacing $\Omega^\infty(\text{mod-}A)$ with Gproj-A, we will soon have some other equivalent conditions; thus we will hold off for the moment and state them all together.

The way we have gone about characterising the Gorenstein-projective band modules so far can be viewed as approaching the problem "working upwards" in the syzygy fabric; we considered the band modules M' where $\Omega(M') \simeq M$ for a given band module M. While "working downwards" in the syzygy fabric (considering the syzygies of a band module M directly) won't give us a different classification of Gorenstein-projective band modules, the analogous results give us a slightly different viewpoint.

The following is analogous to Proposition 6.2.1, and the proof has a similar structure.

6.2.11. Proposition. Let A be an SB algebra. Let $k \in \mathbb{N}$ and Φ_A be the pin graph of A. A band module, M, represented by a belt, w, has $\operatorname{Ext}^i_A(M, A) = 0$ for all $i \in \mathbb{Z}_+$ with $i \leq k$ if and only if both of the following hold:

(i) the Q-vertex corresponding to the bottom of each valley of w is the source of a path of length $\lfloor k/2 \rfloor$ in Φ_A, and

(ii) the Q-vertex corresponding to the top of each peak of w is the source of a path of length $\lceil k/2 \rceil$ in Φ_A.

Proof. We proceed with this proof by induction.

Firstly, we handle the base case, where $k = 0$. In this case,

$$\lceil k/2 \rceil = \lceil 0 \rceil = 0 = \lfloor 0 \rfloor = \lfloor k/2 \rfloor.$$

Therefore neither of the conditions we claim to be equivalent place any restriction on the band module in question. Hence the result holds in the base case.

Now for the inductive step we assume that the result holds for $k = l \in \mathbb{N}$.

Suppose that $M \in \text{mod-}A$ is a band module represented by a belt w, and that $\operatorname{Ext}^i_A(M, A) = 0$ for all $i \in \mathbb{Z}_+$ with $i \leq l + 1$. This means that $\Omega(M) \in \text{mod-}A$ is a band module represented by a belt w', and that $\operatorname{Ext}^i_A(\Omega(M), A) = 0$ for all $i \in \mathbb{Z}_+$ with $i \leq l$. By the inductive hypothesis, this means that:

(i$'$) the Q-vertex corresponding to the bottom of each valley of w' is the source of a path of length $\lfloor l/2 \rfloor$ in Φ_A, and

(ii$'$) the Q-vertex corresponding to the top of each peak of w' is the source of a path of length $\lceil l/2 \rceil$ in Φ_A.

We know from our syzygy algorithm that:

- there is an identification of valleys of w with peaks of w, where the Q-vertices corresponding to the bottom and top respectively are the same,
- there is an identification of peaks of w with valleys of w', where for each peak of w, the Q-vertex associated to its top is the source of an arrow of Φ_A whose target is the Q-vertex associated to the bottom of the valley of w' it is identified with.

Since $\lceil l/2 \rceil = \lfloor (l+1)/2 \rfloor$ and $\lfloor l/2 \rfloor + 1 = \lfloor (l+2)/2 \rfloor = \lceil (l+1)/2 \rceil$, it follows that conditions (i) and (ii) hold for M when $k = l+1$, as required.

Now instead suppose that M is a band module represented by a belt w satisfying conditions (i) and (ii) when $k = l+1$. Since $l \in \mathbb{N}$, we know that $\lceil (l+1)/2 \rceil \in \mathbb{Z}_+$. Hence the Q-vertex associated to the top of each peak of w is the source of an arrow in Φ_A. Therefore the image of any syllable of w under ∇ is an interior syllable. Thus $\mathrm{Ext}^1_A(M, A) = 0$ and $\Omega(M)$ is a band module represented by a belt w'. We know from our syzygy algorithm that:

- there is an identification of valleys of w with peaks of w', where the Q-vertices corresponding to their bottom and top respectively are the same,
- there is an identification of peaks of w with valleys of w', where for each peak of w, the Q-vertex associated to its top is the source of an arrow of Φ_A whose target is the Q-vertex associated to the bottom of the valley of w' it is identified with.

Since the pin graph Φ_A is sub-1-regular, $\lceil l/2 \rceil = \lfloor (l+1)/2 \rfloor$ and $\lfloor l/2 \rfloor + 1 = \lfloor (l+2)/2 \rfloor = \lceil (l+1)/2 \rceil$, it follows that conditions (i) and (ii) hold for M' when $k = l$, so the inductive hypothesis tells us that $\mathrm{Ext}^i_A(\Omega(M), A) = 0$ for all $i \in \mathbb{Z}_+$ with $i \leq l$. Hence $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \in \mathbb{Z}_+$ with $i \leq l+1$, as required. $\square$

The following is analogous to Proposition 6.2.3, and characterises semi-Gorenstein-projective band modules in the sense of [RZ20] (as we defined earlier in Definition 2.2.35).

6.2.12. Proposition. Let A be an SB algebra. Let Φ_A be the pin graph of A. A band module, M, represented by a belt, w, satisfies $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \in \mathbb{Z}_+$ if and only if the Q-vertex corresponding to the bottom of each valley and the top of each peak of w lies on a cycle of Φ_A.

Proof. Follow the same logic as the proof of Proposition 6.2.3, replacing discussion of the target of paths with the source of paths in Φ_A. □

We can now state the main result of Chapter 6; a collection of equivalent conditions for when a band module is Gorenstein-projective.

Theorem 6.2.13. *Let M be a band module represented by a belt w. Then the following conditions are equivalent:*

(i) *M is Gorenstein-projective,*

(ii) *M is semi-Gorenstein-projective (i.e. $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \in \mathbb{Z}_+$),*

(iii) *M belongs to $\Omega^\infty(\text{mod-}A)$,*

(iv) *the Q-vertex corresponding to the bottom of each valley and the top of each peak of w lies on a cycle of Φ_A,*

(v) *the Q-vertex corresponding to each simple summand of $\mathrm{soc}(M) \oplus \mathrm{top}(M)$ lies on a cycle of Φ_A,*

(vi) *the Q-vertex corresponding to the source and target of each syllable of w lies on a cycle of Φ_A.*

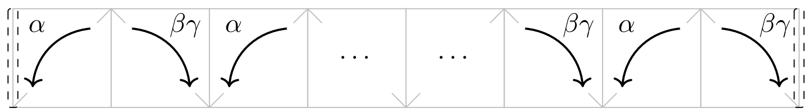
The following two examples show how this result can be used to characterise the Gorenstein-projective band modules for our running and secondary example algebras. It also demonstrates that if you only care about Gorenstein-projective *band* modules, this method is simpler and gives better structural understanding. It avoids the colouring of peaks, and checking whether the presence of a green peak in a belt leads to a red peak in a belt representing a syzygy. Compare with Examples 6.1.31 and 6.1.32.

6.2.14. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Then the pin graph of A, Φ_A, is given by $\left(\circlearrowright 1 \qquad 2 \right)$.

Thus the vertices of the overquiver $\mathcal{O}$ whose corresponding Q-vertices lie on cycles of Φ_A are exactly $s(\alpha) = t(\gamma)$ and $s(\beta) = t(\alpha)$. Therefore the interior syllables $\mathbf{p}$ where the Q-vertices corresponding to $s(\mathbf{p})$ and $t(\mathbf{p})$ lie on cycles of Φ_A are exactly:

$$\left(\circ \xrightarrow{\alpha} \circ \xrightarrow{0} \circ \right) \qquad \left(\circ \xrightarrow{\beta\gamma} \circ \xrightarrow{0} \circ \right).$$

This means that the Gorenstein-projective band modules are exactly those represented by belts of the form:



for some non-negative number of peaks.

6.2.15. Example. Let

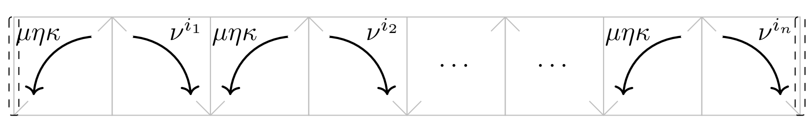
$$A' \coloneqq \Bbbk \left(\eta \circlearrowright 1 \underset{\mu}{\overset{\kappa}{\rightleftarrows}} 2 \circlearrowleft \nu \right) / \langle \eta^2, \kappa\nu, \mu\kappa, \nu\mu, (\mu\eta\kappa)^2 - \nu^4, \kappa\mu\eta\kappa\mu \rangle.$$

Then the pin graph of A', $\Phi_{A'}$, is given by $\left(1 \qquad 2 \circlearrowleft \right)$.

Thus the vertices of the overquiver $\mathcal{O}$ whose corresponding Q-vertices lie on cycles of Φ_A are exactly $s(\mu) = t(\kappa)$ and $s(\nu) = t(\nu)$. Therefore the interior syllables $\mathbf{p}$ where the Q-vertices corresponding to $s(\mathbf{p})$ and $t(\mathbf{p})$ lie on cycles of $\Phi_{A'}$ are exactly:

$$\left(\circ \xrightarrow{\mu\eta\kappa} \circ \xrightarrow{0} \circ \right) \qquad \left(\circ \xrightarrow{\nu} \circ \xrightarrow{0} \circ \right) \qquad \left(\circ \xrightarrow{\nu^2} \circ \xrightarrow{0} \circ \right) \qquad \left(\circ \xrightarrow{\nu^3} \circ \xrightarrow{0} \circ \right).$$

This means that the Gorenstein-projective band modules are exactly those represented by belts of the form:



for some $n \in \mathbb{Z}_+$ and $(i_k)_{k=1}^n \subseteq \{1, 2, 3\}$.

6.2.16. In Corollary 6.2.5, we gave a sufficient condition for there not being any Gorenstein-projective band modules for a given SB algebra (the pin graph being acyclic). However, it is not a necessary condition, as illustrated by the following example:

6.2.17. Example. Let

$$A \coloneqq \Bbbk \left(\begin{tikzcd} 1 \arrow[r, bend left=30, "\alpha_1"] \arrow[r, bend left=10, "\alpha_2"'] & 2 \arrow[l, bend left=10, "\beta_2"'] \arrow[l, bend left=30, "\beta_1"] \end{tikzcd} \right) / \langle \alpha_1\beta_2, \alpha_2\beta_1, \alpha_1\beta_1 - \alpha_2\beta_2, \beta_1\alpha_1, \beta_2\alpha_2 \rangle.$$

Then the pin graph Φ_A associated to A is $\left(\begin{tikzcd} 1 \arrow[loop left] & 2 \end{tikzcd} \right)$ which is clearly not acyclic.

However, there are no interior A-syllables whose source and target are both $\mathcal{O}$-vertices, i, satisfying $b_i = 0$. Thus there are no syllables satisfying the condition required by Theorem 6.2.13(vi), and hence A has no Gorenstein-projective band modules.

The following result gives a sufficient *and* necessary condition for a general SB algebra to have no Gorenstein-projective band modules.

6.2.18. Proposition. Let A be an SB algebra. Let m be the number of vertices on cycles of Φ_A. Then the following conditions are equivalent:

(i) there are no band modules in Gproj-A,

(ii) there are no band modules, $M \in$ Gproj-A, with $\operatorname{intwid}(M) \leq 4m$.

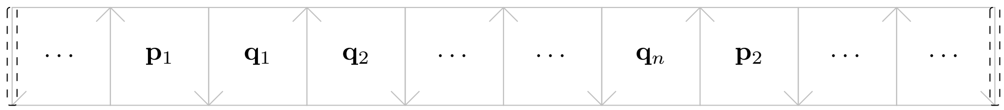
Proof. Clearly (i) $\implies$ (ii), so it remains to show that (ii) $\implies$ (i).

We prove this via the contrapositive. Suppose that (i) does not hold. Then there exists a band module M which belongs to Gproj-A.

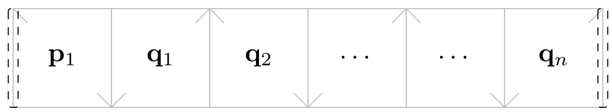
If $\operatorname{intwid}(M) \leq 4m$, then (ii) does not hold, as required.

We now suppose that $\operatorname{intwid}(M) > 4m$. Thus $\operatorname{intwid}(M) \geq 4m + 2$.

Let w be a belt representing M. Since M is a Gorenstein-projective band module, by Theorem 6.2.13, for each syllable $\mathbf{p}$ of w, the Q-vertices associated to $s(\mathbf{p})$ and $t(\mathbf{p})$ lie on cycles of Φ_A. As $\operatorname{intwid}(w) \geq 4m + 2$ there are at least $2m + 1$ positively oriented syllables in w. Since there are $2m$ $\mathcal{O}$-vertices corresponding to Q-vertices which lie on cycles of Φ_A, the pigeonhole principle means that there are two positively oriented syllables in w with the same source, $\mathbf{p}_1$ and $\mathbf{p}_2$.



Since they have the same source, taking the syllables between them, including $\mathbf{p}_1$ but not $\mathbf{p}_2$, gives rise to a smaller belt.



Since we characterised Gorenstein-projective belts in terms of their syllables, this smaller belt is also Gorenstein-projective. If the smaller belt still has intwid greater than $4m$, this process can be repeated until it doesn't.

Thus we know that if there is a Gorenstein-projective band module, there must be a Gorenstein-projective band module whose interior width is at most $4m$. This is exactly the contrapositive of the implication (ii) $\implies$ (i), as required. $\square$

6.2.19. Condition (ii) in Proposition 6.2.18 can then be deterministically checked; either by hand or using a computer algebra system like SBStrips. Checking this by hand would be easiest using condition (vi) of Theorem 6.2.13; calculate the syllables meeting the condition, and then check that there are no possible combinations into belts of interior width at most $4n$.

6.3 Gorenstein homological properties of special biserial algebras

Due to the above classification of Gorenstein-projective band modules, and the association between band modules and belts, we have the following result.

> **6.3.1. Lemma.** Let A be an SB algebra. Suppose that A has a Gorenstein-projective band module M. Then there is an infinite family $(X_i)_{i\in\mathbb{Z}_+}$ of Gorenstein-projective band modules of A, where $\dim_{\Bbbk}(X_i) = i \cdot \dim_{\Bbbk}(M)$. In particular, A is CM-infinite (i.e. there are infinitely many indecomposable Gorenstein-projective A-modules).

Proof. Let $w_1 : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Syll}(A)$ be a belt representing the band module M. By Theorem 6.2.13, the Q-vertices corresponding to the source and target of each syllable of w_1 lie on cycles of Φ_A.

For each $i \in \mathbb{Z}_+$, define

$$\begin{aligned} w_i : \mathbb{Z}/(ni)\mathbb{Z} &\longrightarrow \mathrm{Syll}(A) \\ k + (ni)\mathbb{Z} &\longmapsto w_i(k + (ni)\mathbb{Z}) = w_1(k + n\mathbb{Z}). \end{aligned}$$

It is simple to verify that each of the w_i are well-defined belts.

Note that every syllable of each belt w_i is also present in w_1. Therefore we know that, for each $i \in \mathbb{Z}_+$, the Q-vertices corresponding to the source and target of each syllable of w_i lie on cycles of Φ_A. Hence, by Theorem 6.2.13, any band module represented by any of the w_i is Gorenstein-projective.

By Proposition 3.1.13, we know there exists a band module associated to each belt. Thus, for each $i \in \mathbb{Z}_+$, let X_i be a band module associated to w_i. Then it is clear that each of the X_i is Gorenstein-projective, and that $\dim_{\Bbbk}(X_i) = i \cdot \dim_{\Bbbk}(M)$ for each $i \in \mathbb{Z}_+$, as required. □

Combining the classification of Gorenstein-projective band modules with the classification of string modules M with $\mathrm{Ext}^i_A(M, A) = 0$ for $1 \le i \le k$ gives:

6.3.2. Lemma. Let A be an SB algebra. Let l be the number of arrows in the largest acyclic component of Φ_A, and m be the number of vertices on cycles of Φ_A. Suppose that $M \in$ mod-A is a string module with $\mathrm{intwid}(M) \ge 4m + 4l + 2$ and $\mathrm{Ext}^i_A(M, A) = 0$ for $1 \le i \le 2l$. Then there is a band module $Y \in$ mod-A which is Gorenstein-projective.

Proof. Let w_0 be a strip representing M. For $1 \le i \le 2l$, inductively define w_i by applying the syzygy algorithm for strips to w_{i-1} and "rounding off" as appropriate.

Since $\mathrm{intwid}(w_0) = \mathrm{intwid}(M) \ge 4m+4l+2$, we know from Corollary 6.1.23 that $\mathrm{intwid}(w_{2l}) \ge 4m+2$ and there exists an interval $\mathcal{I} \subsetneq \mathbb{Z}$ with $|\mathcal{I}| \ge 4m + 2$ and $\mathcal{I} \subseteq \mathrm{int}(w_i)$ for all $0 \le i \le 2l$. Thus, for each $k \in \mathcal{I}$ and $0 \le i \le 2l + 2$, the syllable $\nabla^i(w_0(k))$ is interior. Hence, for each $k \in \mathcal{I}$, the Q-vertices corresponding to $s(w_0(k))$ and $t(w_0(k))$ lie on paths of length $l + 1$ on Φ_A. Thus, for each $k \in \mathcal{I}$, the Q-vertices corresponding to $s(w_0(k))$ and $t(w_0(k))$ lie on cycles of Φ_A.

As $|\mathcal{I}| \ge 4m + 2$, we can apply the same logic as Proposition 6.2.18; there exists a belt, w', whose syllables all belong to the set $\{w_0(k) : k \in \mathcal{I}\}$. By Theorem 6.2.13, any band module represented by w' is Gorenstein-projective.

By Proposition 3.1.13, we know there exists a band module associated to each belt. Thus there is a band module $Y \in \text{mod-}A$ which is Gorenstein-projective, as required. □

The next result follows immediately from Lemma 6.3.2.

6.3.3. Corollary. Let A be an SB algebra. Let l be the number of arrows in the largest acyclic component of Φ_A, and m be the number of vertices on cycles of Φ_A. Suppose that $M \in \text{mod-}A$ is a Gorenstein-projective string module with $\text{intwid}(M) \geq 4m + 4l + 2$. Then there is a band module $M \in \text{mod-}A$ which is Gorenstein-projective.

Combining these results gives the following classification of when a special biserial algebra is CM-finite.

6.3.4. Proposition. An SB algebra A is CM-finite if and only if it has no Gorenstein-projective band modules.

Proof. The first implication follows immediately from an application of Lemma 6.3.1.

The second implication follows from the fact that if A has no Gorenstein-projective band modules, then (by Corollary 6.3.3) all Gorenstein-projective string modules must have $\text{intwid}(M) < 4m+4l+2$, where l is the number of arrows in the largest acyclic component of Φ_A, and m is the number of vertices on cycles of Φ_A. There are only finitely many string modules with interior width less than $4m + 4l + 2$. □

Combining this with Corollary 6.2.5 gives:

6.3.5. Corollary. Let A be an SB algebra. Suppose that the pin graph Φ_A of A is acyclic. Then A is CM-finite.

Unfortunately, we are not aware of any characterisation of when an SB algebra is CM-free. As demonstrated by Example 6.1.32, there are SB algebras that have no non-projective Gorenstein-projective string modules, but are not CM-free. The following shows that there are also SB algebras that have no Gorenstein-projective band modules (and hence are CM-finite); but are not CM-free.

6.3.6. Example. Let

$$A \coloneqq \Bbbk \left(\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] \arrow[loop right, "\beta"] \end{tikzcd} \right) / \langle \alpha_1\beta, \beta\alpha_2, (\alpha_1\alpha_2)^2, \beta^2 \rangle.$$

Since this quiver is not 2-regular, we must choose which 2-regular extension to use. We will use

$$\widetilde{Q} \coloneqq \quad \begin{tikzcd} 1 \arrow[loop left, "\gamma"] \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] \arrow[loop right, "\beta"] \end{tikzcd}$$

This means that the overquiver implicit in future discussions (for example with A-syllables), will be:

$$\begin{tikzcd} s(\alpha_1) \arrow[r, bend left, "\alpha_1"] \arrow[d, dash, dashed] & s(\alpha_2) \arrow[l, bend left, "\alpha_2"] \arrow[d, dash, dashed] \\ s(\gamma) \arrow[loop left, "\gamma"] & s(\beta) \arrow[loop right, "\beta"] \end{tikzcd}$$

where, as usual, the dashed lines denote the vertex identification, †.

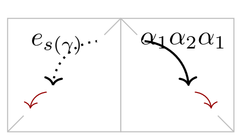
The pin graph of A is $\left(1 \qquad 2 \right)$ which is clearly acyclic, as it is discrete. Thus A has no Gorenstein-projective band modules and is CM-finite.

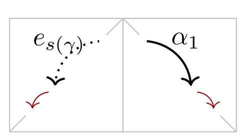
We now classify the Gorenstein-projective string modules for A. If we consider the A-syllables satisfying the conditions of Proposition 4.1.8, these are:

$$\left(\circ \xrightarrow{e_{s(\alpha_1)}} \circ \xrightarrow{1} \circ \right) \quad \left(\circ \xrightarrow{e_{s(\alpha_2)}} \circ \xrightarrow{1} \circ \right) \quad \left(\circ \xrightarrow{e_{s(\beta)}} \circ \xrightarrow{1} \circ \right) \quad \left(\circ \xrightarrow{e_{s(\gamma)}} \circ \xrightarrow{1} \circ \right)$$

$$\left(\circ \xrightarrow{\alpha_1} \circ \xrightarrow{1} \circ \right) \quad \left(\circ \xrightarrow{\alpha_1\alpha_2\alpha_1} \circ \xrightarrow{1} \circ \right) \quad \left(\circ \xrightarrow{\alpha_2\alpha_1} \circ \xrightarrow{1} \circ \right)$$

The only green peaks that can be formed from these syllables (avoiding Case 2b(ii)) are:





Note that the peak corresponding to the second indecomposable projective string module is not in this list, but still constitutes a strip for a valid Gorenstein-projective module (as Proposition 4.1.7 specifically excludes projective string modules from its characterisation).

The first of these peaks corresponds to the first indecomposable projective string module. The second peak corresponds to a uniserial module:

$$U \coloneqq \begin{array}{c} 1 \\ \big\downarrow \alpha_1 \\ 2 \end{array}$$

whose syzygy is isomorphic to itself.

A projective string module is automatically Gorenstein-projective, so we can ignore them for the moment. Since we know that $U \simeq \Omega(U) \simeq \Sigma\Omega(U) \simeq \Omega\Sigma(U)$ and that $\mathrm{Ext}^1_A(U, A) = 0$, it follows that U is a Gorenstein-projective string module. Based on the exclusion of the rest of the peaks, we know that U is the only Gorenstein-projective string module that is not projective.

Hence A is CM-finite but not CM-free.

6.3.7. The above example algebra is a string algebra; a subclass of SB algebras where the Gorenstein-projective modules have been characterised in [CSZ18]. This characterisation shows that there are string algebras which are CM-free and there are those that are not. This range of behaviour is not restricted to string algebras. The following four examples show that there are SB algebras with non-discrete pin graphs (SB algebras which are not string algebras) and that are CM-finite satisfying all four combinations of:

- CM-free (or not), and
- having an acyclic pin graph (or not).

6.3.8. Example. Let

$$A \coloneqq \Bbbk \left(\begin{tikzcd} 1 \arrow[loop left, "\alpha_1"] \arrow[r, bend left, "\alpha_2"] & 2 \arrow[l, bend left, "\alpha_4"] \arrow[loop right, "\alpha_3"] \end{tikzcd} \right) / \langle \alpha_1^2, \alpha_2\alpha_4, \alpha_3^2, \alpha_4\alpha_2, \alpha_1\alpha_2 - \alpha_2\alpha_3, \alpha_4\alpha_1 \rangle.$$

The pin graph of A is $\left(1 \longrightarrow 2 \right)$ which is clearly acyclic. Thus A has no Gorenstein-projective band modules and is CM-finite.

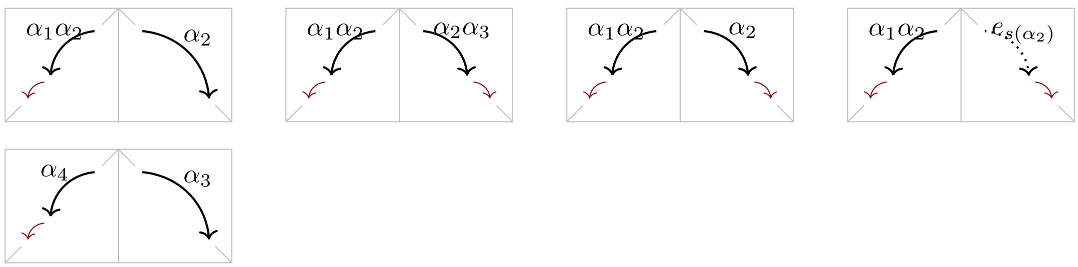
We now classify the Gorenstein-projective string modules for A. If we consider the A-syllables satisfying the conditions of Proposition 4.1.8, these are:

$$\left(\circ \xrightarrow{\alpha_2} \circ \xrightarrow{0} \circ\right) \quad \left(\circ \xrightarrow{\alpha_3} \circ \xrightarrow{0} \circ\right) \quad \left(\circ \xrightarrow{\alpha_1\alpha_2} \circ \xrightarrow{1} \circ\right)$$

$$\left(\circ \xrightarrow{\alpha_2} \circ \xrightarrow{1} \circ\right) \quad \left(\circ \xrightarrow{\alpha_2\alpha_3} \circ \xrightarrow{1} \circ\right) \quad \left(\circ \xrightarrow{\alpha_3} \circ \xrightarrow{1} \circ\right) \quad \left(\circ \xrightarrow{\alpha_4} \circ \xrightarrow{1} \circ\right)$$

$$\left(\circ \xrightarrow{e_{s(\alpha_1)}} \circ \xrightarrow{1} \circ\right) \quad \left(\circ \xrightarrow{e_{s(\alpha_2)}} \circ \xrightarrow{1} \circ\right) \quad \left(\circ \xrightarrow{e_{s(\alpha_3)}} \circ \xrightarrow{1} \circ\right) \quad \left(\circ \xrightarrow{e_{s(\alpha_4)}} \circ \xrightarrow{1} \circ\right)$$

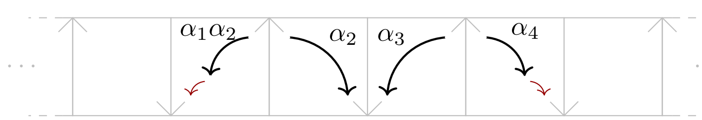
The only green peaks that can be formed from these syllables (avoiding Case 2b(ii)) are:



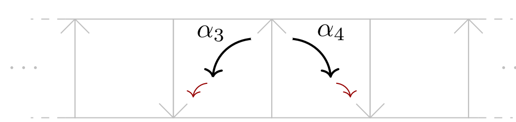
Note that the peak corresponding to the indecomposable projective string module is not in this list, but still constitutes a strip for a valid Gorenstein-projective module (as Proposition 4.1.7 specifically excludes projective string modules from its characterisation).

Observe that applying our syzygy algorithm to any of the peaks in this list with two boundary syllables results in a peak that is not in our list. Thus none of these peaks can be present in a strip representing a Gorenstein-projective string module.

The only strip that can be constructed from the remaining two peaks in our list is:



Applying the syzygy algorithm to this strip gives a strip of the following form:



which clearly contains peaks that are not on our shortened list.

It follows that the only indecomposable Gorenstein-projective string module is the projective one.

Thus A is CM-free (and has a pin graph which *is* acyclic).

6.3.9. Example. Let

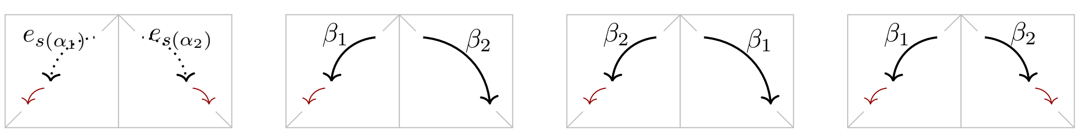
$$A \coloneqq \Bbbk\left(\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend left=10, "\alpha_2"'] & 2 \arrow[l, bend left=10, "\beta_2"'] \arrow[l, bend left, "\beta_1"] \end{tikzcd}\right) / \langle \alpha_1\beta_2, \alpha_2\beta_1, \alpha_1\beta_1 - \alpha_2\beta_2, \beta_1\alpha_1, \beta_2\alpha_2 \rangle.$$

As discussed in Example 6.2.17, the pin graph of A is $(\circlearrowleft 1 \qquad 2)$ (which is clearly not acyclic), and this algebra has no Gorenstein-projective band modules (so is CM-finite).

We now classify the Gorenstein-projective string modules for A. If we consider the A-syllables satisfying the conditions of Proposition 4.1.8, these are:

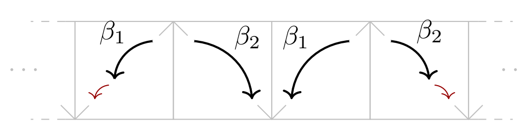
$$\begin{array}{cccc}
(\circ \xrightarrow{\beta_1} \circ \xrightarrow{0} \circ) & (\circ \xrightarrow{\beta_2} \circ \xrightarrow{0} \circ) & (\circ \xrightarrow{\beta_1} \circ \xrightarrow{1} \circ) & (\circ \xrightarrow{\beta_2} \circ \xrightarrow{1} \circ) \\
(\circ \xrightarrow{e_{s(\alpha_1)}} \circ \xrightarrow{1} \circ) & (\circ \xrightarrow{e_{s(\alpha_2)}} \circ \xrightarrow{1} \circ) & (\circ \xrightarrow{e_{s(\beta_1)}} \circ \xrightarrow{1} \circ) & (\circ \xrightarrow{e_{s(\beta_2)}} \circ \xrightarrow{1} \circ)
\end{array}$$

The only green peaks that can be formed from these syllables (avoiding Case 2b(ii)) are:



Applying the syzygy algorithm to the first of these peaks results in peaks that aren't in our list, and thus the first peak can't be present in a strip representing a Gorenstein-projective string module.

The only peak in our list with an interior syllable that is valley compatible with the second peak in our list is the third peak (and vice versa). Thus if either of these peaks were present in a strip representing a Gorenstein-projective string module, the strip must be of the form:



Applying the syzygy algorithm to this strip results in a strip containing a peak of the first form, which we have already shown can't be present in a strip representing a Gorenstein-projective string module. Thus neither the second or third peak can be present in a strip representing a Gorenstein-projective string module.

It follows that the only indecomposable Gorenstein-projective string module is the one represented by our fourth peak; the indecomposable projective string module.

Thus A is CM-free (and has a pin graph which *is not* acyclic).

6.3.10. Example. Let

$$A := \Bbbk \left(\begin{tikzcd} & & 3 \arrow[dl, bend right, "\beta_1"'] \arrow[dd, bend left, "\gamma_1"] \\ 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] \arrow[dr, bend right, "\beta_2"'] & \\ & & 4 \arrow[uu, bend left, "\gamma_2"] \end{tikzcd} \right) / \langle \alpha_1\beta_2, \beta_1\alpha_2, \beta_2\gamma_2, \gamma_2\beta_1, (\alpha_1\alpha_2)^2, \beta_1\beta_2 - \gamma_1\gamma_2\gamma_1 \rangle.$$

Since this quiver is not 2-regular, we must choose which 2-regular extension to use. We will use

$$\widetilde{Q} := \begin{tikzcd} & & 3 \arrow[dl, bend right, "\beta_1"'] \arrow[dd, "\gamma_1"] \\ 1 \arrow[loop left, "\delta"] \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] \arrow[dr, bend right, "\beta_2"'] & \\ & & 4 \arrow[uu, bend left=60, "\gamma_2"] \arrow[uu, bend right=60, "\varepsilon"'] \end{tikzcd}$$

This means that the overquiver implicit in future discussions (for example with A-syllables), will be:

$$\begin{tikzcd} s(\alpha_1) \arrow[r, bend left, "\alpha_1"] \arrow[d, dash, dashed] & s(\alpha_2) \arrow[l, bend left, "\alpha_2"] \arrow[d, dash, dashed] & s(\gamma_1) \arrow[r, bend left, "\gamma_1"] \arrow[d, dash, dashed] & s(\gamma_2) \arrow[l, bend left, "\gamma_2"] \arrow[d, dash, dashed] \\ s(\delta) \arrow[loop left, "\delta"] & s(\beta_2) \arrow[rr, bend right, "\beta_2"'] & s(\beta_1) \arrow[l, "\beta_1"'] & s(\varepsilon) \arrow[l, "\varepsilon"'] \end{tikzcd}$$

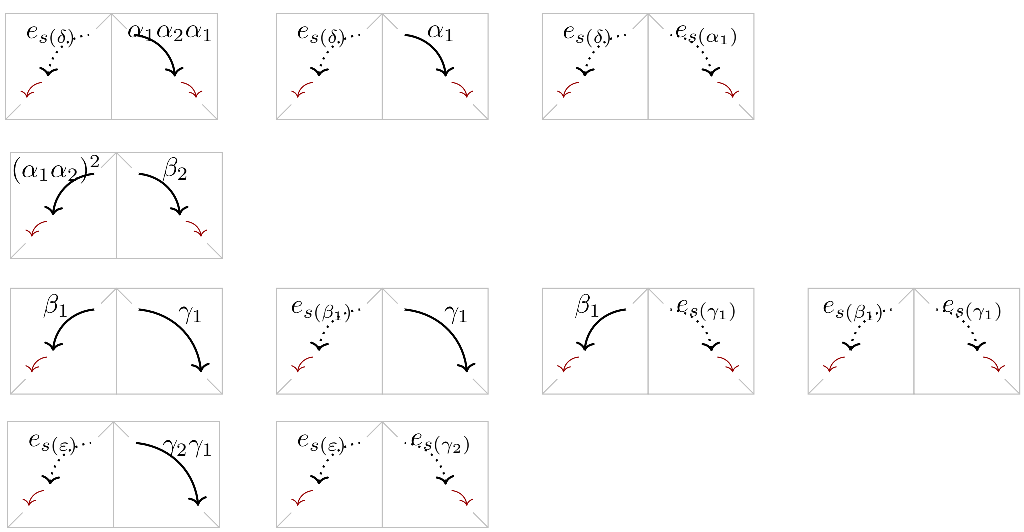
where, as usual, the dashed lines denote the vertex identification, †.

The pin graph of A is $\left(1 \quad 2 \quad 3 \longrightarrow 4 \right)$ which is clearly acyclic. Thus A has no Gorenstein-projective band modules and is CM-finite.

We now classify the Gorenstein-projective string modules for A. If we consider the A-syllables satisfying the conditions of Proposition 4.1.8, these are:

$(\circ \xrightarrow{\beta_2} \circ \xrightarrow{0} \circ)$	$(\circ \xrightarrow{\gamma_1} \circ \xrightarrow{0} \circ)$	$(\circ \xrightarrow{\gamma_2\gamma_1} \circ \xrightarrow{0} \circ)$	
$(\circ \xrightarrow{\alpha_1} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{\alpha_1\alpha_2\alpha_1} \circ \xrightarrow{1} \circ)$		$(\circ \xrightarrow{\alpha_2\alpha_1} \circ \xrightarrow{1} \circ)$
$(\circ \xrightarrow{\beta_1} \circ \xrightarrow{1} \circ)$			$(\circ \xrightarrow{\beta_2} \circ \xrightarrow{1} \circ)$
$(\circ \xrightarrow{\gamma_1} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{\gamma_1\gamma_2} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{\gamma_2} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{\gamma_2\gamma_1} \circ \xrightarrow{1} \circ)$
$(\circ \xrightarrow{e_{s(\alpha_1)}} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{e_{s(\alpha_2)}} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{e_{s(\beta_1)}} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{e_{s(\beta_2)}} \circ \xrightarrow{1} \circ)$
$(\circ \xrightarrow{e_{s(\gamma_1)}} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{e_{s(\gamma_2)}} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{e_{s(\delta)}} \circ \xrightarrow{1} \circ)$	$(\circ \xrightarrow{e_{s(\varepsilon)}} \circ \xrightarrow{1} \circ)$

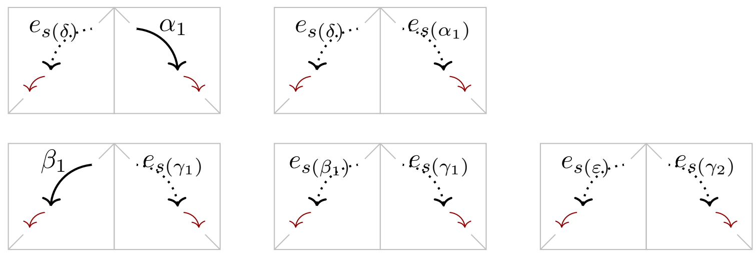
The only green peaks that can be formed from these syllables (avoiding Case 2b(ii)) are:



Note that this list does not contain all peaks corresponding to the indecomposable projective string modules, but they are still valid for Gorenstein-projective strips (as Proposition 4.1.7 specifically excludes projective string modules from its characterisation).

Also note that all interior syllables $\mathbf{p}$ in our list have $t(\mathbf{p}) = t(\gamma_1)$, and thus can not form interior valleys with each other. Hence we can remove all peaks in our list containing an interior syllable.

If we also remove the peaks in our list that correspond to indecomposable projective string modules, the remaining peaks are:



For all of these peaks apart from the first, applying the syzygy algorithm results in peaks that do not belong to this list. For the first peak in the list, applying the syzygy algorithm gives a strip consisting only of the first peak. This corresponds to the fact that the uniserial module:

$$U \coloneqq \begin{tikzcd} 1 \arrow[d, "\alpha_1"] \\ 2 \end{tikzcd}$$

is isomorphic to its syzygy.

Since we know that $U \simeq \Omega(U) \simeq \Sigma\Omega(U) \simeq \Omega\Sigma(U)$ and that $\mathrm{Ext}^1_A(U, A) = 0$, it follows that U is a Gorenstein-projective string module. Based on the exclusion of the rest of the peaks, we know that U is the only Gorenstein-projective string module that is not projective.

Thus A is CM-finite but not CM-free (and has a pin graph which *is* acyclic).

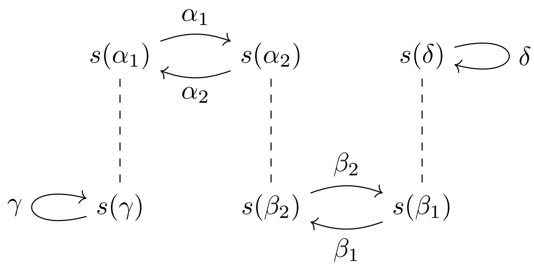
6.3.11. Example. Let

$$A \coloneqq \Bbbk \left(\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] \arrow[r, bend left, "\beta_2"] & 3 \arrow[l, bend left, "\beta_1"] \end{tikzcd} \right) / \langle \alpha_1\beta_2, \beta_1\alpha_2, (\alpha_1\alpha_2)^2, (\alpha_2\alpha_1)^2 - \beta_2\beta_1, \beta_1\beta_2 \rangle.$$

Since this quiver is not 2-regular, we must choose which 2-regular extension to use. We will use

$$\widetilde{Q} \coloneqq \begin{tikzcd} \gamma \; 1 \arrow[loop left] \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_2"] \arrow[r, bend left, "\beta_2"] & 3 \arrow[l, bend left, "\beta_1"] \arrow[loop right] \; \delta \end{tikzcd}$$

This means that the overquiver implicit in future discussions (for example with A-syllables), will be:



where, as usual, the dashed lines denote the vertex identification, †.

The pin graph of A is $\big(\ 1 \qquad 2 \circlearrowleft \ 3\ \big)$ which is clearly not acyclic. Thus we can't rely on Corollary 6.3.5 to prove that A is CM-finite.

The Q-vertex 2 is the only one lying on a cycle of Φ_A. Therefore an interior syllable $\mathbf{p}$ meets the syllable condition of Theorem 6.2.13(vi) if and only if the Q-vertices corresponding to $s(\mathbf{p})$ and $t(\mathbf{p})$ are both 2. The only interior syllable meeting this condition is $(\circ \xrightarrow{\alpha_2\alpha_1} \circ \xrightarrow{0} \circ)$. Since no syllable is peak- or valley-compatible with itself, we can't use this single syllable to form a belt. Hence A has no Gorenstein-projective band modules, and thus A is CM-finite.

We now classify the Gorenstein-projective string modules for A. If we consider the A-syllables satisfying the conditions of Proposition 4.1.8, these are:

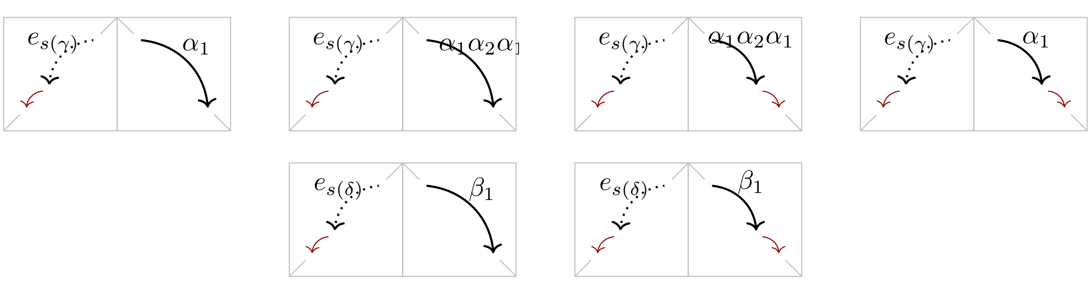
$$(\circ \xrightarrow{\alpha_1} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\alpha_2\alpha_1} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\alpha_1\alpha_2\alpha_1} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\beta_1} \circ \xrightarrow{0} \circ)$$

$$(\circ \xrightarrow{\alpha_1} \circ \xrightarrow{1} \circ) \quad (\circ \xrightarrow{\alpha_2\alpha_1} \circ \xrightarrow{1} \circ) \quad (\circ \xrightarrow{\alpha_1\alpha_2\alpha_1} \circ \xrightarrow{1} \circ) \quad (\circ \xrightarrow{\beta_1} \circ \xrightarrow{1} \circ)$$

$$(\circ \xrightarrow{e_{s(\alpha_1)}} \circ \xrightarrow{1} \circ) \quad (\circ \xrightarrow{e_{s(\alpha_2)}} \circ \xrightarrow{1} \circ) \quad (\circ \xrightarrow{e_{s(\beta_1)}} \circ \xrightarrow{1} \circ) \quad (\circ \xrightarrow{e_{s(\beta_2)}} \circ \xrightarrow{1} \circ)$$

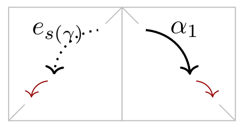
$$(\circ \xrightarrow{e_{s(\gamma)}} \circ \xrightarrow{1} \circ) \quad (\circ \xrightarrow{e_{s(\delta)}} \circ \xrightarrow{1} \circ)$$

The only green peaks that can be formed from these syllables (avoiding Case 2b(ii)) are:



Applying the syzygy algorithm to any of the peaks in this list with an interior syllable results in a boundary syllable which is not present in any of the other peaks in our list. Hence we can remove all peaks in our list containing an interior syllable.

If we also remove the peaks in our list that correspond to indecomposable projective string modules, there is a single remaining peak:



Since this peak has two boundary syllables, if it is present in a strip then it is the only non-blank peak in the strip. If we apply the syzygy algorithm to a strip containing this peak, we obtain another strip containing only this peak.

This corresponds to the fact that the uniserial module:

$$U := \quad \begin{tikzcd} 1 \arrow[d, "\alpha_1"] \\ 2 \end{tikzcd}$$

is isomorphic to its syzygy.

Since we know that $U \simeq \Omega(U) \simeq \Sigma\Omega(U) \simeq \Omega\Sigma(U)$ and that $\mathrm{Ext}^1_A(U, A) = 0$, it follows that U is a Gorenstein-projective string module. Based on the exclusion of the rest of the peaks, we know that U is the only Gorenstein-projective string module that is not projective.

Thus A is CM-finite but not CM-free (and has a pin graph which *is not* acyclic).

6.3.12. Recall from Definition 2.2.39 that a finite-dimensional algebra, A, is called weakly Gorenstein if all semi-Gorenstein-projective A-modules are Gorenstein-projective. In other words, a finite-dimensional algebra, A, is weakly Gorenstein if for any $M \in \text{mod-}A$, we have an implication:

$$\mathrm{Ext}^i_A(M, A) = 0 \text{ for all } i \in \mathbb{Z}_+ \implies M \in \text{Gproj-}A.$$

If we restrict our attention to special biserial algebras, we recall from Theorem 6.2.13 that if $M \in \text{mod-}A$ is a band module, this implication holds (i.e. all semi-Gorenstein-projective band modules are also Gorenstein-projective). Thus (due to Theorem 2.2.58), we have the following result:

6.3.13. Lemma. Let A be an SB algebra. Then A is weakly Gorenstein if and only if for all string modules $M \in \text{mod-}A$ we have an implication:

$$\mathrm{Ext}^i_A(M, A) = 0 \text{ for all } i \in \mathbb{Z}_+ \implies M \in \mathrm{Gproj}\text{-}A.$$

In other words, A is weakly Gorenstein if and only if all semi-Gorenstein-projective string modules are also Gorenstein-projective.

We now use Lemma 6.3.2 again in the context of weakly Gorenstein SB algebras, to classify a case of interest where there are finitely many semi-Gorenstein-projective modules.

6.3.14. Proposition. Let A be an SB algebra. Let l be the number of arrows in the largest acyclic component of Φ_A, and m be the number of vertices on cycles of Φ_A. Suppose that A has no semi-Gorenstein-projective band modules. Then all semi-Gorenstein-projective string modules, $M \in \text{mod-}A$, satisfy $\mathrm{intwid}(M) \leq 4m + 4l$. In particular, there are finitely many semi-Gorenstein-projective A-modules.

Proof. Suppose that $M \in \text{mod-}A$ is a semi-Gorenstein-projective string module with $\mathrm{intwid}(M) > 4m + 4l$. Since $\mathrm{intwid}(M) \in 2\mathbb{N}$, this means that $\mathrm{intwid}(M) \geq 4m + 4l + 2$. Now, by Lemma 6.3.2, we know that there exists a band module $Y \in \text{mod-}A$ which is Gorenstein-projective. This contradicts our assumption that A has no semi-Gorenstein-projective band modules. □

Due to our classification of CM-finiteness in Proposition 6.3.4, we immediately have the following:

6.3.15. Corollary. Let A be an SB algebra. If A is CM-finite, then A is also weakly Gorenstein and any indecomposable non-projective semi-Gorenstein-projective module is Ω-periodic.

Proof. Suppose that A is CM-finite. By Proposition 6.3.4, this means that there are no Gorenstein-projective band modules. Since all semi-Gorenstein-projective band modules are Gorenstein-projective (by Theorem 6.2.13), this means that there are no semi-Gorenstein-projective band modules. By Proposition 6.3.14, this means that there are finitely many indecomposable semi-Gorenstein-projective A-modules.

By [RZ20, Thm 1.3], this means that A is weakly Gorenstein and any indecomposable non-projective semi-Gorenstein-projective module is Ω-periodic, as required. $\square$

This answers [RZ20, Question 9.2] in the affirmative for special biserial algebras.

We now give a slight strengthening of part of [RZ20, Thm 1.2] in the case of special biserial algebras. The proof makes extensive use of the arguments Ringel and Zhang, applying the characterisation of weakly Gorenstein SB algebras (Lemma 6.3.13) where necessary. Note that [RZ20] uses the notation $\mho$ whereas we use Σ.

6.3.16. Proposition. Let A be an SB algebra. The following statements are equivalent:

(1) A is weakly Gorenstein.

(2) Any semi-Gorenstein-projective string module is torsionfree.

(3) Any semi-Gorenstein-projective string module is reflexive.

(4) For any semi-Gorenstein-projective string module M, the map $\phi_M : M \to M^{**}$ is surjective.

(5) For any semi-Gorenstein-projective string module M, the module M^* is semi-Gorenstein-projective.

(6) Any semi-Gorenstein-projective string module M satisfies $\mathrm{Ext}^1_A(M^*, A_A) = 0$.

(7) Any semi-Gorenstein-projective string module M satisfies $\mathrm{Ext}^1_A(\mathrm{Tr}(M), A_A) = 0$.

Proof. The fact that (1) implies (2) to (7) follows immediately from [RZ20, Thm 1.2].

To show that any one of (3) to (7) implies (2), apply the same argument as [RZ20, Thm 1.2], replacing any instance of the word "semi-Gorenstein-projective" with the phrase "a semi-Gorenstein-projective string module".

It remains to show that (2) implies (1). By Lemma 6.3.13, it is sufficient to show that (2) implies that any semi-Gorenstein-projective string module is also Gorenstein-projective. Now apply the same argument as [RZ20, Thm 1.2], replacing any instance of the word "semi-Gorenstein-projective" with the phrase "a semi-Gorenstein-projective string module". $\square$

Of particular interest is the equivalence between the first two conditions, which is based on a result initially published by Huang-Huang [HH12, Prop 4.2]. This is because we have already characterised torsionfree string modules in terms of the syllables present in strips representing them (see Proposition 2.2.69 and Proposition 4.1.7).

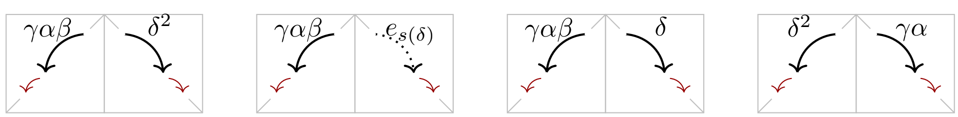
> **6.3.17. Corollary.** Let A be an SB algebra. Then A is weakly Gorenstein if and only if for any indecomposable non-projective semi-Gorenstein-projective string module, M, there is a strip, w, representing M, all of whose syllables satisfy the conditions of Proposition 4.1.7.

We now use this characterisation to check if our running example algebra is weakly Gorenstein.

6.3.18. Example. Let A be our running example algebra, as defined in Paragraph 2.2.41. Using our list of green peaks for A (see Figures 6.1 and 6.2), we will show that all semi-Gorenstein-projective string modules are represented by a strip, all of whose syllables satisfy the conditions of Proposition 4.1.7.

First we show that all semi-Gorenstein-projective string modules with interior width zero are torsionfree. We do this by reviewing the green peaks with 2 boundary syllables, checking if any of them contain syllables which fail to satisfy the conditions of Proposition 4.1.7, and then applying the syzygy algorithm to the ones that do to find out if they are semi-Gorenstein-projective.

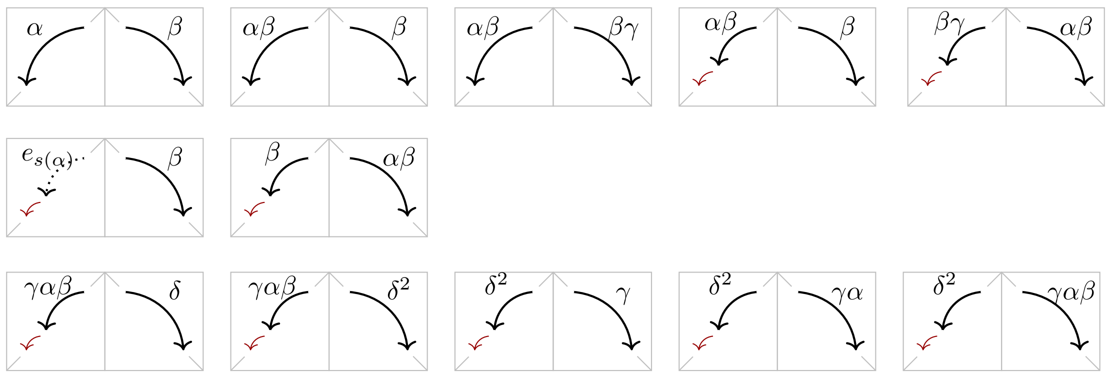
The green peaks with 2 boundary syllables where at least one of them fail to satisfy the syllable conditions of Proposition 4.1.7 are:



We can immediately ignore the first peak, as it represents the unique indecomposable projective string module for this algebra. For the remainder, applying the syzygy algorithm results in a strip with a single red peak. Thus none of these peaks (except the first) correspond to modules with $\mathrm{Ext}^2_A(M, A) = 0$; thus they don't correspond to semi-Gorenstein-projective modules. Hence we know that all semi-Gorenstein-projective string modules with interior width zero are torsionfree.

Checking for non-torsionfree semi-Gorenstein-projective string modules with non-zero interior width directly is harder, as the strips representing them are formed of multiple peaks. We instead characterise all semi-Gorenstein-projective string modules with non-zero interior width, and then show that they are all Gorenstein-projective (and hence torsionfree).

The green peaks with at least one interior syllable where one of the syllables fails to satisfy the conditions of Proposition 4.1.7 are:

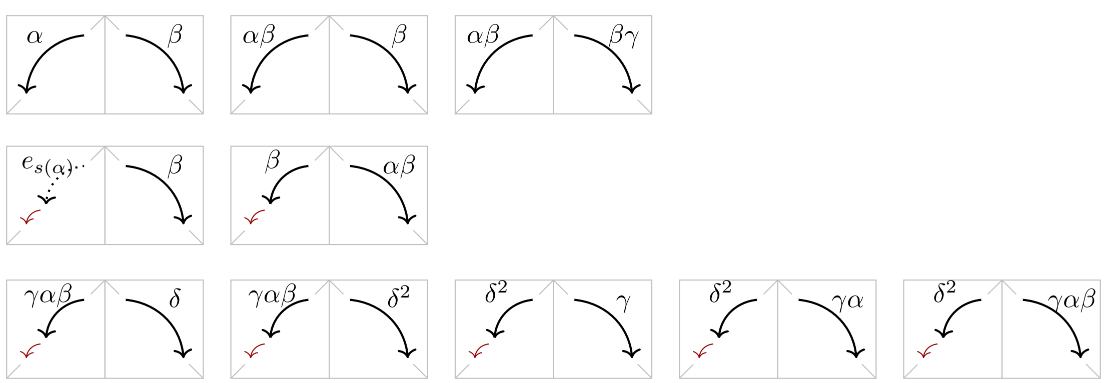


Note that this list contains all instances of the following syllables in a green peak:

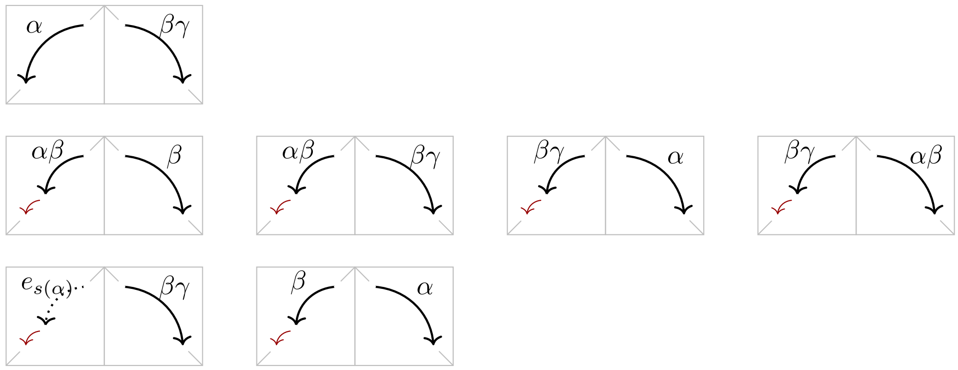
$$(\circ \xrightarrow{\beta} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\alpha\beta} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\delta} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\delta^2} \circ \xrightarrow{0} \circ)$$

$$(\circ \xrightarrow{\gamma} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\gamma\alpha} \circ \xrightarrow{0} \circ) \quad (\circ \xrightarrow{\gamma\alpha\beta} \circ \xrightarrow{0} \circ)$$

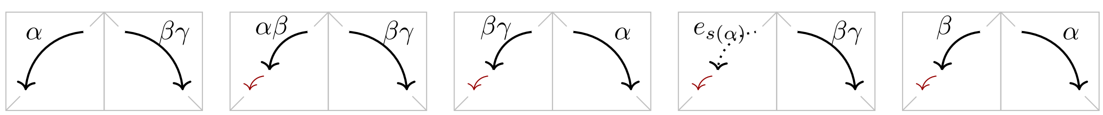
Thus no torsionfree string module M with $\mathrm{Ext}^1_A(M, A) = 0$ can have any of these syllables in a strip representing it. Thus if M is semi-Gorenstein-projective, $\Omega(M)$ can't have any of these syllables in a strip representing it. Applying the syzygy algorithm to any one of the following green peaks results in interior non-pin syllables:



Thus none of these peaks can be present in a strip representing a semi-Gorenstein-projective string module. If we take the full list of green peaks and remove those above, the remaining green peaks with at least one interior syllable are:

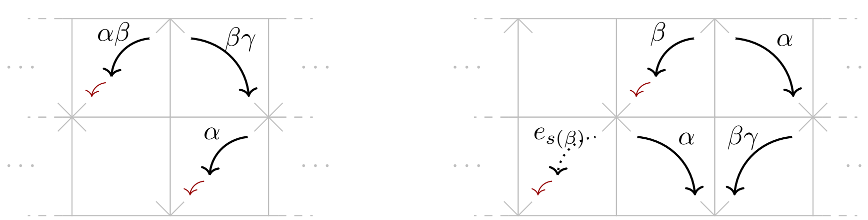


Note that none of the peaks in this list contain an interior syllable $\mathbf{p}$ with $t(\mathbf{p}) = t(\delta) = t(\beta)^\dagger$. Thus none of the peaks in this list containing an interior syllable $\mathbf{p}'$ with $t(\mathbf{p}') = t(\beta)$ can be present in a strip representing a semi-Gorenstein-projective string module. The remaining green peaks with at least one interior peak are:



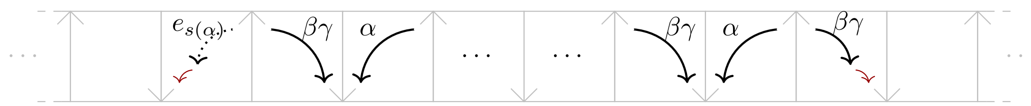
Thus any semi-Gorenstein-projective string module with non-zero interior width must be represented by a strip formed entirely of these peaks.

Applying the syzygy algorithm to the second and fifth peaks in this shortened list gives rise to the following local diagrams:



Note that the boundary syllables ($\circ \xrightarrow{\alpha} \circ \xrightarrow{1} \circ$) and ($\circ \xrightarrow{e_{s(\beta)}} \circ \xrightarrow{1} \circ$) don't appear anywhere in our shortened list of peaks. Therefore, if M is a string module represented by a strip containing either the second or fourth peak, we know that $\Omega(M)$ is not semi-Gorenstein-projective, and thus that M is not semi-Gorenstein-projective.

Hence any semi-Gorenstein-projective string module with non-zero interior width must be represented by a strip of the form:



We showed in Example 6.1.31 that all such string modules are Gorenstein-projective. Hence we have shown that our running example algebra A is weakly Gorenstein.

Index of Terms

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