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Positive solutions for Hadamard-type fractional differential equations with nonlocal conditions on an infinite interval*

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Abstract. The purpose of this paper is to analyse the local existence and uniqueness of positive solutions for a Hadamard-type fractional differential equation with nonlocal boundary conditions on an infinite interval. The technique used to arrive our results depends on two fixed point theorems of a sum operator in partial ordering Banach spaces. The local existence and uniqueness of positive solution is given, and we can make iterative sequences to approximate the unique positive solution. For the illustration of the main results, we list two concrete examples in the last section.

Keywords: local existence and uniqueness, positive solution, nonlocal boundary conditions, fixed point theorems for a sum operator.

1 Introduction

In present article, we consider the following form of Hadamard-type fractional boundary value problem on an infinite interval:

$$\begin{aligned}&{}^{H}D^{\alpha}_{1+}x(t)+a(t)f\bigl(t,x(t)\bigr)+b(t)g\bigl(t,x(t)\bigr)=0,\qquad t\in(1,+\infty),\\&x(1)=x'(1)=0,\qquad {}^{H}D^{\alpha-1}_{1+}x(+\infty)=\sum_{i=1}^{m}\alpha_i{}^{H}I^{\beta_i}_{1+}x(\eta)+c\sum_{j=1}^{n}\sigma_j x(\xi_j),\end{aligned}\tag{1}$$

where ${}^{H}D^{\alpha}_{1+}$ is the Hadamard-type fractional derivative of order α, $2<\alpha<3$; ${}^{H}I^{\beta_i}_{1+}$ is the Hadamard-type fractional integral of order $\beta_i>0$ ($i=1,2,\dots,m$); $1<\eta<\xi_1<\xi_2<\dots<\xi_n<+\infty$; $c,\alpha_i,\sigma_j\geqslant 0$ ($i=1,2,\dots,m$, $j=1,2,\dots,n$) are given

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constants with

$$\Gamma(\alpha) - \sum_{i=1}^{m} \alpha_i \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)} (\ln \eta)^{\alpha+\beta_i-1} - c \sum_{j=1}^{n} \sigma_j (\ln \xi_j)^{\alpha-1} := \Delta > 0,$$

$a, b \in C(J, \mathbb{R}^+)$, $f, g \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $J = [1, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$. We will investigate the local existence and uniqueness of positive solutions for (1) by using different methods comparing with ones in literature.

Recently, fractional differential equations have aroused an incredible attention among researchers due to their applications for modeling real world problems in areas of mathematical and natural sciences. There has been a rapid growth in the number of fractional differential equations from both theoretical and applied perspectives; see [3–5,9,13,14,16, 17, 19–22] and references therein. Among the class of fractional derivatives, Hadamard-type fractional derivative is an important concept, which was first introduced in 1892 [11]. The integral's kernel in its definition contains a logarithmic function of arbitrary exponent. Hadamard-type fractional differential equations can be used to design and optimized controls for creating more accurate control models, as well as to describe the nonlinear behavior of materials, transport characteristics of media, etc. In the process, many mathematical models that simplify out of practical problems involve solving them on infinite intervals. As it is known well, there have been some papers reported on boundary value problems of Hadamard fractional differential equations; see previous works [1, 6–8, 18, 29]. For example, in [2], by applying fixed point theorems for multivalued mapping, the authors obtained the sufficient conditions for the existence results to a boundary value problem of Hadamard fractional differential inclusions

$$\begin{aligned} &{}^H D^\alpha x(t) \in F\bigl(t, x(t)\bigr), \quad t \in (1, e),\ \alpha \in (1, 2], \\ &x(1) = 0, \qquad x(\mathrm{e}) = {}^H I^\beta x(\eta), \end{aligned}$$

where $F : [1, \mathrm{e}] \times \mathbb{R} \to \varrho(\mathbb{R})$ is a multivalued mapping, $\varrho(\mathbb{R})$ denotes a family of nonempty subsets of $\mathbb{R}$.

In [15], the authors considered a Hadamard fractional nonlocal boundary value problem

$$\begin{aligned} &{}^H\mathcal{D}^\varrho y(\tau) = g\bigl(\tau, y(\tau)\bigr), \quad \tau \in [1, T], \\ &y(1) = 0, \qquad y'(0) = 0, \qquad {}^H\mathcal{D}^\varsigma y(T) = \omega^H \mathcal{J}^\gamma y(\varphi), \quad 1 < \varphi < T, \end{aligned}$$

where $2 < \varrho \leqslant 3$, $1 < \varsigma < 2$, $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ω is a positive real constant. They established the existence and the uniqueness results, and the methods used in their proofs contain the Leray–Schauder nonlinear alternative, Leray–Schauder degree theorem, Krasnoselskii's fixed point theorem, Schaefer's fixed point theorem, Banach fixed point theorem, and nonlinear contractions.

In [21], the authors investigated the following p-Laplacian Hadamard fractional-order three-point boundary value problem:

$$\begin{aligned} &-D^\alpha\bigl(\varphi_p\bigl(D^\beta y(t)\bigr)\bigr) = f\bigl(t, y(t)\bigr), \quad t \in (1, \mathrm{e}), \\ &y(1) = y(\mathrm{e}) = \delta y(1) = \delta y(\mathrm{e}) = 0, \qquad D^\beta y(1) = 0, \qquad D^\beta y(\mathrm{e}) = b D^\beta y(\eta), \end{aligned} \tag{2}$$

where $\alpha \in (1,2]$, $\beta \in (3,4]$ are real numbers, φ_p is p-Laplacian. The sufficient conditions for the existence of positive solutions for (2) are based upon the Avery–Henderson fixed point theorem and the monotone iterative technique.

In a recent paper [28], the authors studied the following fractional differential equation with nonlocal boundary conditions:

$$\begin{aligned} &{}^H D_{1+}^{\alpha} x(t) + a(t) f\bigl(t, x(t)\bigr) = 0, \quad t \in (1, +\infty), \\ &x(1) = x'(1) = 0, \qquad {}^H D_{1+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{m} \alpha_i {}^H I_{1+}^{\beta_i} x(\eta) + b \sum_{j=1}^{n} \sigma_j x(\xi_j), \end{aligned} \tag{3}$$

where ${}^H D_{1+}^{\alpha}$ is the Hadamard-type fractional derivative of order α with $2 < \alpha < 3$; ${}^H I_{1+}^{\beta_i}$ is the Hadamard-type fractional integral of order $\beta_i > 0$ ($i = 1, 2, \dots, m$); $1 < \eta < \xi_1 < \xi_2 < \dots < \xi_n < +\infty$. Some famous methods have been used, which include Schauder's fixed point theorem, Banach's contraction mapping principle, the monotone iterative method, and the Avery–Peterson fixed point theorem, and they got the existence, uniqueness, and multiplicity results for positive solutions to (3).

It is worth noticing that there seems to be a scarcity of literature about the investigation of Hadamard-type fractional differential equation boundary value problems on infinite intervals, and very few have derived the uniqueness results apart from using Banach' theorem. That is because the definition and properties of Hadamard fractional derivatives may be more complicated on infinite intervals. The special case of infinite intervals and the nonlocal boundary value condition need to be taken into account when using the related theorem to solve this type of problem. Therefore, we need to choose a suitable fixed point theorem to solve it.

Different from the above works, we will consider the local unique of positive solutions for (1) by the more recent methods. Motivated by the papers [23, 26, 27], based upon two fixed point theorems of a sum operator, we aim to obtain the sufficient conditions ensuring the local existence and uniqueness of positive solutions for (1). We prepare the following sections of this paper. Section 2 includes some preliminaries, which play a significant role in the study of the given problem. We also summarize some properties of the corresponding Green's function. The main theorems are indicated and proved in Section 3, and we construct two iterative sequences that converge to the local unique positive solutions for (1). In Section 4, two concrete examples are given as applications of our main results.

2 Preliminaries and previous results

For the convenience, we recall some definitions, lemmas, and fixed point theorems that will be used in our discussions.

Definition 1. (See [12].) For a function $f : [1, +\infty) \to \mathbb{R}$, the Hadamard-type fractional derivative of order α is

$${}^H D_{1+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{\mathrm{d}s}{s}, \quad n-1 < \alpha < n,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α, and $\log(\cdot) = \log_e(\cdot)$.

Definition 2. (See [12].) For a function $f : [1, +\infty) \to \mathbb{R}$, the Hadamard-type fractional integral of order α is

$$ {}^{H}I_{1+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{\mathrm{d}s}{s}, \quad \alpha > 0, $$

provided the integral exists.

Lemma 1. (*See* [28].) *Let $y(t) : J \to \mathbb{R}^+$ with $\int_1^{+\infty} y(s)\,\mathrm{d}s/s < +\infty$, then the following Hadamard-type fractional differential equation*

$$ \begin{aligned} &{}^{H}D_{1+}^{\alpha} x(t) + y(t) = 0, \quad 2 < \alpha < 3,\ t \in (1, +\infty), \\ &x(1) = x'(1) = 0, \qquad {}^{H}D_{1+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{m} \alpha_i {}^{H}I_{1+}^{\beta_i} x(\eta) + c \sum_{j=1}^{n} \sigma_j x(\xi_j) \end{aligned} $$

has a solution

$$ x(t) = \int_1^{+\infty} G(t,s) y(s) \frac{\mathrm{d}s}{s}, \quad t \in J, $$

where

$$ G(t,s) = G_1(t,s) + G_2(t,s), \tag{4} $$

$$ G_1(t,s) = g_0(t,s,\alpha) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\ln t)^{\alpha-1} - (\ln \frac{t}{s})^{\alpha-1}, & 1 \leqslant s \leqslant t < +\infty, \\ (\ln t)^{\alpha-1}, & 1 \leqslant t \leqslant s < +\infty, \end{cases} \tag{5} $$

$$ G_2(t,s) = \frac{(\ln t)^{\alpha-1}}{\Delta} \sum_{i=1}^{m} \alpha_i g_0(\eta,\ s,\ \alpha + \beta_i) + \frac{(\ln t)^{\alpha-1}}{\Delta} \sum_{j=1}^{n} c\sigma_j g_0(\xi_j, s, \alpha). \tag{6} $$

Lemma 2. (*See* [28].) *The Green's functions $G(t,s)$, $G_1(t,s)$ defined by* (4) *and* (5) *satisfy the following conditions*:

(i) *$G(t,s)$, $G_1(t,s)$ are nonnegative and continuous for $(t,s) \in J \times J$*;
(ii) *$G_1(t,s)$ is increasing with respect to t*;
(iii) *for all $(t,s) \in J \times J$,*

$$ \frac{G(t,s)}{1 + (\ln t)^{\alpha-1}} \leqslant \frac{1}{\Delta}, \qquad \frac{G_1(t,s)}{1 + (\ln t)^{\alpha-1}} \leqslant \frac{1}{\Gamma(\alpha)}. $$

In the following, we will specifically give two fixed point theorems for a sum operator and some details, which are necessary for our study.

Let $(E, \|\cdot\|)$ be a real Banach space, and let θ be the zero element of E. E is partially ordered by a cone $P \subset E$, i.e., $x \leqslant y$ if and only if $y - x \in P$. A cone P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$. In this case, N is called the normality constant of P.

For $x, y \in E$, the notation $x \sim y$ denotes that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Further, $\sim$ is an equivalence relation. For $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), define $P_h = \{x \in E\colon x \sim h\}$. Clearly, $P_h \subset P$.

Definition 3. (See [10].) An operator $A : E \to E$ is increasing (decreasing) if $x \leqslant y$ implies $Ax \leqslant Ay (Ax \geqslant Ay)$.

Definition 4. (See [10].) Let $0 < \gamma < 1$. An operator $A : P \to P$ is said to be γ-concave if $A(tx) \geqslant t^\gamma Ax$ for $t \in (0, 1)$, $x \in P$. An operator $A : P \to P$ is called to be subhomogeneous if $A(tx) \geqslant tAx$ for $t > 0$, $x \in P$.

In [24, 25], the authors investigated a sum operator equation

$$Ax + Bx = x, \tag{7}$$

where A and B are monotone operators. They gave the existence and uniqueness of positive solutions for (7) and obtained some useful theorems.

Lemma 3. (*See* [25].) *Let E be a real Banach space. P is a normal cone in E, $A, B : P \to P$ are increasing operators, A is γ-concave, and B is subhomogeneous. Suppose that*

(i) *there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$ for all $x \in P$.*

Then the operator equation (7) *has a unique solution $x^* \in P_h$. Further, making the sequence*

$$x_n = Ax_{n-1} + Bx_{n-1}, \quad n = 1, 2 \ldots,$$

for any initial value $x_0 \in P_h$, one has $x_n \to x^*$ as $n \to \infty$.

Lemma 4. (*See* [24].) *Let E be a real Banach space. P is a normal cone in E, $A : P \to P$ is an increasing operator, and $B : P \to P$ is a decreasing operator. In addition,*

(i) *for $x \in P$ and $t \in (0, 1)$, there exist $\varphi_i(t) \in (t, 1)$, $i = 1, 2$, such that*

$$A(tx) \geqslant \varphi_1(t)Ax,\ B(tx) \leqslant \frac{1}{\varphi_2(t)}Bx; \tag{8}$$

(ii) *there is $h_0 \in P_h$ such that $Ah_0 + Bh_0 \in P_h$.*

Then the operator equation (7) *has a unique solution $x^* \in P_h$. Further, for any initial values x_0, $y_0 \in P_h$, making the sequences*

$$x_n = Ax_{n-1} + By_{n-1}, \quad y_n = Ay_{n-1} + Bx_{n-1}, \quad n = 1, 2 \ldots,$$

one has $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.

Remark 1. If B is a null operator, the conclusions in Lemmas 3 and 4 are still true.

3 Main results

In this section, we will apply Lemmas 3 and 4 to obtain the local existence and uniqueness of positive solution for (1).

Let $E = \{x \in C(J, \mathbb{R})\colon \sup_{t\in J} |x(t)|/(1 + (\ln t)^{\alpha-1}) < +\infty\}$ equipped with the norm $\|x\|_E = \sup_{t\in J} |x(t)|/(1 + (\ln t)^{\alpha-1})$, then $(E, \|\cdot\|_E)$ is a Banach space. Define a cone $P = \{x \in E\colon x(t) \geqslant 0 \ on\ J\}$. This space is equipped with a partial order

$$x \leqslant y \quad \Longleftrightarrow \quad x(t) \leqslant y(t), \quad t \in J.$$

If $0 \leqslant x(t) \leqslant y(t)$, then

$$\sup_{t\in J} \frac{|x(t)|}{1 + (\ln t)^{\alpha-1}} \leqslant \sup_{t\in J} \frac{|y(t)|}{1 + (\ln t)^{\alpha-1}} \quad \Longrightarrow \quad \|x\| \leqslant \|y\|,$$

therefore, P is a normal cone in E.

To prove the main results, we need the following assumptions:

- (H1) $a, b : J \to \mathbb{R}^+$ are continuous, and $0 < \int_1^{+\infty} a(s)\,\mathrm{d}s/s \int_1^{+\infty} b(s)\,\mathrm{d}s/s < +\infty$;
- (H2) $f, g : J \times \mathbb{R}^+ \to \mathbb{R}^+$ are increasing with respect to the second argument, $f(t, 0), g(t, 0) \not\equiv 0$, $t \in J$;
- (H3) when x is bounded, $f(t, (1+(\ln t)^{\alpha-1})x)$ and $g(t, (1+(\ln t)^{\alpha-1})x)$ are bounded with respect to t for $t \in J$;
- (H4) $g(t, \tau x) \geqslant \tau g(t, x)$ for $\tau \in (0, 1)$, $t \in J$, $x \in \mathbb{R}^+$, and there exists a constant $\gamma \in (0, 1)$ such that $f(t, \tau x) \geqslant \tau^\gamma f(t, x)$ for all $t \in J$, $\tau \in (0, 1)$, $x \in \mathbb{R}^+$;
- (H5) there exists a constant $\delta > 0$ such that $a(t)f(t, x) \geqslant \delta b(t)g(t, x)$, $t \in J$, $x \in \mathbb{R}^+$;
- (H6) $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is increasing in x, $g : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing in x, $f(t, 0), g(t, 0) \not\equiv 0$, $t \in J$;
- (H7) for $\tau \in (0, 1)$, there exist $\varphi_i(\tau) \in (\tau, 1)$ $(i = 1, 2)$ such that for $t \in J$, $x \in \mathbb{R}^+$, $f(t, \tau x) \geqslant f(t, x)\varphi_1(\tau)$, $g(t, \tau x) \leqslant g(t, x)/\varphi_2(\tau)$.

Let $h(t) = (\ln t)^{\alpha-1}$, $t \in J$. As $\sup_{t\in J} |h(t)|/(1 + (\ln t)^{\alpha-1}) = 1 < +\infty$, we have $h \in P$. In the following, we will consider a set $P_h = \{x \in E\colon x \sim h\}$. From Lemma 1 we know that problem (1) has an integral formulation given by

$$x(t) = \int_1^{+\infty} G(t, s)a(s)f\bigl(s, x(s)\bigr)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t, s)b(s)g\bigl(s, x(s)\bigr)\,\frac{\mathrm{d}s}{s},$$

where $G(t, s)$ is given as in (4). Define two operators $A : P \to E$ and $B : P \to E$ by

$$Ax(t) = \int_1^{+\infty} G(t, s)a(s)f\bigl(s, x(s)\bigr)\,\frac{\mathrm{d}s}{s}, \qquad Bx(t) = \int_1^{+\infty} G(t, s)b(s)g\bigl(s, x(s)\bigr)\,\frac{\mathrm{d}s}{s}.$$

Then we can see that x is the solution of problem (1) if and only if $x = Ax + Bx$.

Lemma 5. *Assume that* (H1)–(H3) *hold, then* $A : P \to P$, $B : P \to P$.

Proof. If $x \in P$, then $x(t)/(1 + (\ln t)^{\alpha-1}) < +\infty$ for all $t \in J$. From (H3) there exists $M_x > 0$ such that $f(s, (1 + (\ln s)^{\alpha-1}) \cdot x(s)/(1 + (\ln s)^{\alpha-1})) \leqslant M_x$. Moreover, from (H1), (H2), and Lemma 2 we have

$$\begin{aligned}\frac{Ax(t)}{1+(\ln t)^{\alpha-1}} &= \int\limits_1^{+\infty} \frac{G(t,s)}{1+(\ln t)^{\alpha-1}} a(s) f\bigl(s, x(s)\bigr)\,\frac{\mathrm{d}s}{s} \\ &\leqslant \int\limits_1^{+\infty} \frac{1}{\Delta} a(s) f\biggl(s,\ \bigl(1+(\ln s)^{\alpha-1}\bigr)\cdot\frac{x(s)}{1+(\ln s)^{\alpha-1}}\biggr)\,\frac{\mathrm{d}s}{s} \\ &\leqslant \frac{M_x}{\Delta}\int\limits_1^{+\infty} a(s)\,\frac{\mathrm{d}s}{s} < +\infty,\end{aligned}$$

and by Lemma 2, we know that $Ax \in E$ and $Ax(t) \geqslant 0$ on J, so, $A : P \to P$. Similarly, $B : P \to P$. The proof is complete. □

Lemma 6. *Assume that* f, g *satisfy* (H1), (H2), *and* (H4). *Then* $A : P \to P$ *is an increasing* γ*-concave operator, and* $B : P \to P$ *is an increasing subhomogeneous operator.*

Proof. Firstly, we prove that A and B are two increasing operators. For $x, y \in P$ with $x \geqslant y$, we have $x(t) \geqslant y(t)$, $t \in J$, and by (H1), (H2), and Lemma 2,

$$Ax(t) = \int\limits_1^{+\infty} G(t,s)a(s)f\bigl(s,x(s)\bigr)\,\frac{\mathrm{d}s}{s} \geqslant \int\limits_1^{+\infty} G(t,s)a(s)f\bigl(s,y(s)\bigr)\,\frac{\mathrm{d}s}{s} = Ay(t).$$

So, $Ax \geqslant Ay$. Also, we can get $Bx \geqslant By$.

Secondly, we show that A is a γ-concave operator. For any $\tau \in (0,1)$ and $x \in P$, from (H1), (H2), (H4), and Lemma 2 we obtain

$$\begin{aligned}A(\tau x)(t) &= \int\limits_1^{+\infty} G(t,s)a(s)f\bigl(s,\tau x(s)\bigr)\,\frac{\mathrm{d}s}{s} \geqslant \tau^\gamma \int\limits_1^{+\infty} G(t,s)a(s)f\bigl(s,x(s)\bigr)\,\frac{\mathrm{d}s}{s} \\ &= \tau^\gamma Ax(t).\end{aligned}$$

Hence, $A(\tau x) \geqslant \tau^\gamma Ax$ for $\tau \in (0,1)$, $x \in P$.

Finally, we prove that B is a subhomogeneous operator. For any $\tau \in (0,1)$ and $x \in P$, by (H1), (H2), (H4), and Lemma 2, we obtain

$$\begin{aligned}B(\tau x)(t) &= \int\limits_1^{+\infty} G(t,s)b(s)g\bigl(s,\tau x(s)\bigr)\,\frac{\mathrm{d}s}{s} \geqslant \tau \int\limits_1^{+\infty} G(t,s)b(s)g\bigl(s,x(s)\bigr)\,\frac{\mathrm{d}s}{s} \\ &= \tau Bx(t).\end{aligned}$$

That is, $B(\tau x) \geqslant \tau Bx$ for $\tau \in (0,1)$, $x \in P$. The proof is complete. □

Lemma 7. *Assume that* (H1)–(H3) *are satisfied. Then* $Ah \in P_h$ *and* $Bh \in P_h$.

Proof. As $h \in P$, then $h(t)/(1 + (\ln t)^{\alpha-1}) < +\infty$ for all $t \in J$, from (H3) there exists $M_h > 0$ such that $f(s, (1 + (\ln t)^{\alpha-1})h(s)/(1 + (\ln t)^{\alpha-1})) \leqslant M_h$. Let

$$l_1 = \frac{\sum_{j=1}^{n} c\sigma_j}{\Delta} \cdot \int_1^{\xi_m} G_1(\xi_j, s)a(s)f(s, 0)\,\frac{\mathrm{d}s}{s},$$

$$l_2 = \frac{M_h}{\Gamma(\alpha)} \int_1^{+\infty} a(s)\,\frac{\mathrm{d}s}{s} + \frac{M_h}{\Delta}\sum_{i=1}^{m} \alpha_i \int_1^{+\infty} g_0(\eta,\ s,\ \alpha + \beta_i)a(s)\,\frac{\mathrm{d}s}{s}$$
$$+ \frac{M_h}{\Delta}\sum_{j=1}^{n} c\sigma_j \int_1^{+\infty} G_1(\xi_j, s)a(s)\,\frac{\mathrm{d}s}{s}.$$

From (H1), (H2), and Lemmas 1, 2

$$\begin{aligned}
Ah(t) &= \int_1^{+\infty} G(t, s)a(s)f\big(s, (\ln s)^{\alpha-1}\big)\,\frac{\mathrm{d}s}{s} \\
&\geqslant \int_1^{+\infty} G(t, s)a(s)f(s, 0)\,\frac{\mathrm{d}s}{s} \geqslant \int_1^{+\infty} G_2(t, s)a(s)f(s, 0)\,\frac{\mathrm{d}s}{s} \\
&= \frac{(\ln t)^{\alpha-1}}{\Delta} \int_1^{+\infty} \left[\sum_{i=1}^{m} \alpha_i g_0(\eta,\ s,\ \alpha + \beta_i) + \sum_{j=1}^{n} c\sigma_j G_1(\xi_j, s)\right] a(s)f(s, 0)\,\frac{\mathrm{d}s}{s} \\
&\geqslant \frac{(\ln t)^{\alpha-1}}{\Delta} \int_1^{+\infty} \sum_{j=1}^{n} c\sigma_j G_1(\xi_j, s)a(s)f(s, 0)\,\frac{\mathrm{d}s}{s} \\
&= \frac{(\ln t)^{\alpha-1}}{\Delta} \sum_{j=1}^{n} c\sigma_j \int_1^{+\infty} G_1(\xi_j, s)a(s)f(s, 0)\,\frac{\mathrm{d}s}{s} \\
&\geqslant \left\{\frac{\sum_{j=1}^{n} c\sigma_j}{\Delta} \int_1^{\xi_m} G_1(\xi_j, s)a(s)f(s, 0)\,\frac{\mathrm{d}s}{s}\right\} \cdot (\ln t)^{\alpha-1} = l_1 \cdot (\ln t)^{\alpha-1} = l_1 \cdot h(t).
\end{aligned}$$

Also, from (H3)

$$\begin{aligned}
Ah(t) &= \int_1^{+\infty} G(t, s)a(s)f\big(s, h(s)\big)\,\frac{\mathrm{d}s}{s} \\
&= \int_1^{+\infty} G(t, s)a(s)f\left(s,\ \big(1 + (\ln t)^{\alpha-1}\big)\frac{h(s)}{1 + (\ln t)^{\alpha-1}}\right)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_1^{+\infty} G(t,s)a(s)M_h \frac{\mathrm{d}s}{s} \\
&\leqslant \int_1^{+\infty} \frac{M_h \cdot (\ln t)^{\alpha-1}}{\Gamma(\alpha)} a(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} \frac{M_h \cdot (\ln t)^{\alpha-1}}{\Delta} \sum_{i=1}^m \alpha_i g_0(\eta,\, s,\, \alpha+\beta_i) a(s) \frac{\mathrm{d}s}{s} \\
&\quad + \int_1^{+\infty} \frac{M_h \cdot (\ln t)^{\alpha-1}}{\Delta} \sum_{j=1}^n c\sigma_j G_1(\xi_j, s) a(s) \frac{\mathrm{d}s}{s} \\
&= \left\{ \frac{M_h}{\Gamma(\alpha)} \int_1^{+\infty} a(s) \frac{\mathrm{d}s}{s} + \frac{M_h}{\Delta} \sum_{i=1}^m \alpha_i \int_1^{+\infty} g_0(\eta, s, \alpha+\beta_i) a(s) \frac{\mathrm{d}s}{s} \right. \\
&\quad \left. + \frac{M_h}{\Delta} \sum_{j=1}^n c\sigma_j \int_1^{+\infty} G_1(\xi_j, s) a(s) \frac{\mathrm{d}s}{s} \right\} \cdot (\ln t)^{\alpha-1} = l_2 \cdot (\ln t)^{\alpha-1} = l_2 \cdot h(t).
\end{aligned}$$

Note that $f(s,0) \not\equiv 0$, $G_1(\xi_j, s) \geqslant 0$, $\int_1^{+\infty} a(s)\,\mathrm{d}s/s > 0$. So, we know $G_1(\xi_j, s)a(s) \times f(s,0) \not\equiv 0$ for $s \in J$, therefore

$$\int_1^{\xi_m} G_1(\xi_j, s) a(s) f(s,0) \frac{\mathrm{d}s}{s} > 0.$$

From (H2), (H3) we have that $f(s,0) \leqslant f(s,h(s)) \leqslant M_h$ for $t \in J$, then combining with $\int_1^{+\infty} a(s)\,\mathrm{d}s/s > 0$,

$$\frac{\sum_{j=1}^n c\sigma_j}{\Delta} \int_1^{\xi_m} G_1(\xi_j, s) a(s) f(s,0) \frac{\mathrm{d}s}{s} \leqslant \frac{M_h}{\Delta} \sum_{j=1}^n c\sigma_j \int_1^{+\infty} G_1(\xi_j, s) a(s) \frac{\mathrm{d}s}{s},$$

$$\frac{M_h}{\Gamma(\alpha)} \int_1^{+\infty} a(s) \frac{\mathrm{d}s}{s} + \frac{M_h}{\Delta} \sum_{i=1}^m \alpha_i \int_1^{+\infty} g_0(\eta,\, s,\, \alpha+\beta_i) a(s) \frac{\mathrm{d}s}{s} > 0.$$

We can conclude $0 < l_1 \leqslant l_2$ and thus $l_1 h(t) \leqslant Ah(t) \leqslant l_2 h(t)$, $t \in J$. So, we have $Ah \in P_h$. Also,

$$\begin{aligned}
Bh(t) &= \int_1^{+\infty} G(t,s) b(s) g\bigl(s, h(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\geqslant \left\{ \frac{\sum_{j=1}^n c\sigma_j}{\Delta} \int_1^{\xi_n} G_1(\xi_j, s) b(s) g(s,0) \frac{\mathrm{d}s}{s} \right\} \cdot h(t),
\end{aligned}$$

$$
\begin{aligned}
Bh(t) &= \int_1^{+\infty} G(t,s)b(s)g\big(s,h(s)\big)\,\frac{\mathrm{d}s}{s} \\
&\leqslant \left\{ \frac{M_h}{\Gamma(\alpha)} \int_1^{+\infty} b(s)\,\frac{\mathrm{d}s}{s} + \frac{M_h}{\Delta} \sum_{i=1}^{m} \alpha_i \int_1^{+\infty} g_0(\eta,\, s,\, \alpha+\beta_i) b(s)\,\frac{\mathrm{d}s}{s} \right. \\
&\quad \left. + \frac{M_h}{\Delta} \sum_{j=1}^{n} c\sigma_j \int_1^{+\infty} G_1(\xi_j, s) b(s)\,\frac{\mathrm{d}s}{s} \right\} \cdot h(t).
\end{aligned}
$$

On the basis of $g(s,0) \not\equiv 0$, $\int_1^{+\infty} b(s)\,\mathrm{d}s/s > 0$, $g(s,0) \leqslant g(s,h(s)) \leqslant M_h$ for $t \in J$, we can easily prove $Bh \in P_h$. The proof is complete. □

Combining Lemmas 5–7, we are in a position to establish the local existence and uniqueness of positive solution for (1).

Theorem 1. *Suppose assumptions* (H1)–(H5) *are satisfied. Then problem* (1) *has a unique positive solution x^* in P_h. For any initial value $x_0 \in P_h$, defining a sequence by*

$$
x_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\big(s,x_n(s)\big)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\big(s,x_n(s)\big)\,\frac{\mathrm{d}s}{s},
$$

$n = 0,1,2\dots$, *we have $x_n(t) \to x^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as in* (4).

Proof. From Lemmas 5–7, we just need to prove that condition (ii) of Lemma 3 is also satisfied. For $x \in P$, by (H1), (H2), (H5), and Lemma 2,

$$
Ax(t) = \int_1^{+\infty} G(t,s)a(s)f\big(s,x(s)\big)\,\frac{\mathrm{d}s}{s} \geqslant \delta \int_1^{+\infty} G(t,s)b(s)g\big(s,x(s)\big)\,\frac{\mathrm{d}s}{s} = \delta Bx(t).
$$

So, we obtain $Ax \geqslant \delta Bx$, $x \in P$. Therefore, by using Lemma 3, operator equation $Ax + Bx = x$ has a unique solution $x^* \in P_h$. Thus, we get that

$$
x^*(t) = \int_1^{+\infty} G(t,s)\Big[a(s)f\big(s,x^*(s)\big)\,\frac{\mathrm{d}s}{s} + b(s)g\big(s,x^*(s)\big)\Big]\,\frac{\mathrm{d}s}{s},
$$

and it is the unique positive solution of problem (1) in P_h. Moreover, for any initial value $x_0 \in P_h$, the sequence $x_n = Ax_{n-1} + Bx_{n-1}$, $n = 1,2,\dots$, satisfies $x_n(t) \to x^*(t)$ as $n \to \infty$. That is,

$$
x_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\big(s,x_n(s)\big)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\big(s,x_n(s)\big)\,\frac{\mathrm{d}s}{s},
$$

$n = 0,1,2\dots$. For any initial value $x_0 \in P_h$, we have $x_n(t) \to x^*(t)$ as $n \to \infty$. The proof is complete. □

Corollary 1. *Let α, c, α_i, σ_j $(i = 1, 2 \dots m,\ j = 1, 2 \dots n)$ be given in* (1)*, and let*

- (H1′) $0 < \int_1^{+\infty} a(s)\,\mathrm{d}s/s < +\infty$;
- (H2′) $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ *is increasing in x, $f(t, 0) \not\equiv 0$, $t \in J$;*
- (H3′) *when x is bounded, $f(t, 1 + (\ln t)^{\alpha-1}x)$ is bounded with respect to t for $t \in J$;*
- (H4′) *there exists a constant $\gamma \in (0, 1)$ such that $f(t, \tau x) \geqslant \tau^\gamma f(t, x)$ for all $t \in J$, $\tau \in (0, 1)$, $x \in \mathbb{R}^+$.*

Then the following problem

$$\begin{aligned} &{}^H D_{1+}^{\alpha} x(t) + a(t) f\bigl(t, x(t)\bigr) = 0, \quad t \in (1, +\infty), \\ &x(1) = x'(1) = 0, \qquad {}^H D_{1+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{m} \alpha_i {}^H I_{1+}^{\beta_i} x(\eta) + c \sum_{j=1}^{n} \sigma_j x(\xi_j) \end{aligned} \tag{9}$$

has a unique positive solution x^* in P_h, where $h(t) = (\ln t)^{\alpha-1}$, $t \in J$. Further, defining a sequence by

$$x_{n+1}(t) = \int_1^{+\infty} G(t, s) a(s) f\bigl(s, x_n(s)\bigr) \frac{\mathrm{d}s}{s}, \quad n = 0, 1, 2 \dots,$$

for any initial value $x_0 \in P_h$, we have $x_n(t) \to x^*(t)$ as $n \to \infty$, where $G(t, s)$ is given as in (4)*.*

Proof. From Remark 1 and Theorem 1 the conclusions hold. □

Lemma 8. *Assume that f, g satisfy* (H1)*,* (H3)*, and* (H6)*, then $Ah + Bh \in P_h$.*

Proof. Let

$$N_h = \max\Bigl\{\max_{t \in J}\bigl\{f\bigl(t, h(t)\bigr)\bigr\}, \max_{t \in J}\bigl\{g\bigl(t, h(t)\bigr)\bigr\}\Bigr\},$$

$$\begin{aligned} l_3 &= \frac{\sum_{j=1}^{n} c\sigma_j}{\Delta} \int_1^{\xi_m} G_1(\xi_j, s) a(s) f(s, 0) \frac{\mathrm{d}s}{s} \\ &\quad + \frac{\sum_{j=1}^{n} c\sigma_j}{\Delta} \int_1^{\xi_m} G_1(\xi_j, s) b(s) g\bigl(s, h(s)\bigr) \frac{\mathrm{d}s}{s}, \\ l_4 &= \frac{2N_h}{\Gamma(\alpha)} \int_1^{+\infty} \rho(s) \frac{\mathrm{d}s}{s} + \frac{2N_h}{\Delta} \sum_{i=1}^{m} \alpha_i \int_1^{+\infty} g_0(\eta,\ s,\ \alpha + \beta_i) \rho(s) \frac{\mathrm{d}s}{s} \\ &\quad + \frac{2N_h}{\Delta} \sum_{j=1}^{n} c\sigma_j \int_1^{+\infty} G_1(\xi_j, s) \rho(s) \frac{\mathrm{d}s}{s}, \end{aligned}$$

where $\rho(s) = \max_{s \in J}\{a(s), b(s)\}$.

From (H1), (H6), and Lemmas 1, 2

$$
\begin{aligned}
&Ah(t) + Bh(t) \\
&\quad = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, (\ln s)^{\alpha-1}\bigr)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\bigl(s, (\ln s)^{\alpha-1}\bigr)\,\frac{\mathrm{d}s}{s} \\
&\quad \geqslant \int_1^{+\infty} G(t,s)a(s)f(s,0)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\bigl(s, (\ln s)^{\alpha-1}\bigr)\,\frac{\mathrm{d}s}{s} \\
&\quad \geqslant \int_1^{+\infty} \frac{(\ln t)^{\alpha-1}}{\Delta} \sum_{j=1}^{n} c\sigma_j G_1(\xi_j, s)a(s)f(s,0)\,\frac{\mathrm{d}s}{s} \\
&\qquad + \int_1^{+\infty} \frac{(\ln t)^{\alpha-1}}{\Delta} \sum_{j=1}^{n} c\sigma_j G_1(\xi_j, s)b(s)g\bigl(s, (\ln s)^{\alpha-1}\bigr)\,\frac{\mathrm{d}s}{s} \\
&\quad \geqslant \Biggl\{ \frac{\sum_{j=1}^{n} c\sigma_j}{\Delta} \int_1^{\xi_m} G_1(\xi_j, s)a(s)f(s,0)\,\frac{\mathrm{d}s}{s} \\
&\qquad + \frac{\sum_{j=1}^{n} c\sigma_j}{\Delta} \int_1^{\xi_m} G_1(\xi_j, s)b(s)g\bigl(s, (\ln s)^{\alpha-1}\bigr)\,\frac{\mathrm{d}s}{s} \Biggr\} \cdot (\ln t)^{\alpha-1} \\
&\quad = l_3 \cdot (\ln t)^{\alpha-1} = l_3 \cdot h(t).
\end{aligned}
$$

Also, from (H3)

$$
\begin{aligned}
&Ah(t) + Bh(t) \\
&\quad = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, (\ln s)^{\alpha-1}\bigr)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\bigl(s, (\ln s)^{\alpha-1}\bigr)\,\frac{\mathrm{d}s}{s} \\
&\quad = \int_1^{+\infty} G(t,s)a(s)f\biggl(s,\ \bigl(1 + (\ln s)^{\alpha-1}\bigr)\frac{h(s)}{1 + (\ln s)^{\alpha-1}}\biggr)\,\frac{\mathrm{d}s}{s} \\
&\qquad + \int_1^{+\infty} G(t,s)b(s)g\biggl(s,\ \bigl(1 + (\ln s)^{\alpha-1}\bigr)\frac{h(s)}{1 + (\ln s)^{\alpha-1}}\biggr)\,\frac{\mathrm{d}s}{s} \\
&\quad \leqslant \int_1^{+\infty} G(t,s)a(s)N_h\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)N_h\,\frac{\mathrm{d}s}{s} \\
&\quad \leqslant \int_1^{+\infty} \frac{N_h \cdot (\ln t)^{\alpha-1}}{\Gamma(\alpha)} a(s)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} \frac{N_h \cdot (\ln t)^{\alpha-1}}{\Delta} \sum_{i=1}^{m} \alpha_i g_0(\eta,\ s,\ \alpha + \beta_i)a(s)\,\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_1^{+\infty}\frac{N_h\cdot(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{n}c\sigma_jG_1(\xi_j,s)a(s)\,\frac{\mathrm{d}s}{s}\\
&+\int_1^{+\infty}\frac{N_h\cdot(\ln t)^{\alpha-1}}{\Gamma(\alpha)}b(s)\,\frac{\mathrm{d}s}{s}+\int_1^{+\infty}\frac{N_h\cdot(\ln t)^{\alpha-1}}{\Delta}\sum_{i=1}^{m}\alpha_ig_0(\eta,\,s,\,\alpha+\beta_i)b(s)\,\frac{\mathrm{d}s}{s}\\
&+\int_1^{+\infty}\frac{N_h\cdot(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{n}c\sigma_jG_1(\xi_j,s)b(s)\,\frac{\mathrm{d}s}{s}\\
\leqslant&\int_1^{+\infty}\frac{2N_h\cdot(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\rho(s)\,\frac{\mathrm{d}s}{s}+\int_1^{+\infty}\frac{2N_h\cdot(\ln t)^{\alpha-1}}{\Delta}\sum_{i=1}^{m}\alpha_ig_0(\eta,\,s,\,\alpha+\beta_i)\rho(s)\,\frac{\mathrm{d}s}{s}\\
&+\int_1^{+\infty}\frac{2N_h\cdot(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{n}c\sigma_jG_1(\xi_j,s)\rho(s)\,\frac{\mathrm{d}s}{s}\\
=&\left\{\frac{2N_h}{\Gamma(\alpha)}\int_1^{+\infty}\rho(s)\,\frac{\mathrm{d}s}{s}+\frac{2N_h}{\Delta}\sum_{i=1}^{m}\alpha_i\int_1^{+\infty}g_0(\eta,\,s,\,\alpha+\beta_i)\rho(s)\,\frac{\mathrm{d}s}{s}\right.\\
&\left.+\frac{2N_h}{\Delta}\sum_{j=1}^{n}c\sigma_j\int_1^{+\infty}G_1(\xi_j,s)\rho(s)\,\frac{\mathrm{d}s}{s}\right\}\cdot(\ln t)^{\alpha-1}\\
=&\,l_4\cdot(\ln t)^{\alpha-1}=l_4\cdot h(t).
\end{aligned}
$$

Note that $f(s,0)\not\equiv 0$, $G_1(\xi_j,s)\geqslant 0$, $\int_1^{+\infty}a(s)\,\mathrm{d}s/s>0$. According to the properties of the functions G_1, b, g, we can conclude $l_3>0$. From (H3), (H6) we have $f(s,0)\leqslant f(s,h(s))\leqslant N_h$, $g(s,h(s))\leqslant N_h$ for $t\in J$. Combining

$$
0<\int_1^{\xi_m}a(s)\,\frac{\mathrm{d}s}{s}\leqslant\int_1^{+\infty}\rho(s)\,\frac{\mathrm{d}s}{s},\qquad 0<\int_1^{\xi_m}b(s)\,\frac{\mathrm{d}s}{s}\leqslant\int_1^{+\infty}\rho(s)\,\frac{\mathrm{d}s}{s},
$$

we have

$$
\begin{aligned}
&\frac{\sum_{j=1}^{n}c\sigma_j}{\Delta}\cdot\int_1^{\xi_m}G_1(\xi_j,s)a(s)f(s,0)\,\frac{\mathrm{d}s}{s}+\frac{\sum_{j=1}^{n}c\sigma_j}{\Delta}\cdot\int_1^{\xi_m}G_1(\xi_j,s)b(s)g\bigl(s,h(s)\bigr)\,\frac{\mathrm{d}s}{s}\\
&\quad\leqslant\frac{2N_h}{\Delta}\sum_{j=1}^{n}c\sigma_j\int_1^{+\infty}G_1(\xi_j,s)\rho(s)\,\frac{\mathrm{d}s}{s},\\
&\qquad\frac{2N_h}{\Gamma(\alpha)}\int_1^{+\infty}\rho(s)\,\frac{\mathrm{d}s}{s}+\frac{2N_h}{\Delta}\sum_{i=1}^{m}\alpha_i\int_1^{+\infty}g_0(\eta,s,\alpha+\beta_i)\rho(s)\,\frac{\mathrm{d}s}{s}>0.
\end{aligned}
$$

Therefore, $0 < l_3 \leqslant l_4$ and thus $l_3 h(t) \leqslant Ah(t) + Bh(t) \leqslant l_4 h(t)$, $t \in J$. Therefore, $Ah + Bh \in P_h$. The proof is complete. □

Theorem 2. *Suppose that conditions* (H1)*,* (H3)*, and* (H6)–(H7) *are fulfilled. Then problem* (1) *has a unique positive solution x^* in P_h. For given initial values $x_0, y_0 \in P_h$, constructing the following sequences*

$$x_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, x_n(s)\bigr)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\bigl(s, y_n(s)\bigr)\,\frac{\mathrm{d}s}{s},$$
$$y_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, y_n(s)\bigr)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\bigl(s, x_n(s)\bigr)\,\frac{\mathrm{d}s}{s},$$

$n = 0, 1, 2\ldots$, we have $x_n(t) \to x^*(t)$, $y_n(t) \to x^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as in (4)*.*

Proof. From (H1), (H6), and Lemma 2 we know that $A : P \to P$ is increasing and $B : P \to P$ is decreasing. Further, by (H7), we have

$$\begin{aligned} A(\tau x)(t) &= \int_1^{+\infty} G(t,s)a(s)f\bigl(s, \tau x(s)\bigr)\,\frac{\mathrm{d}s}{s} \geqslant \varphi_1(\tau) \int_1^{+\infty} G(t,s)a(s)f\bigl(s, x(s)\bigr)\,\frac{\mathrm{d}s}{s} \\ &= \varphi_1(\tau) Ax(t), \\ B(\tau x)(t) &= \int_1^{+\infty} G(t,s)b(s)g\bigl(s, \tau x(s)\bigr)\,\frac{\mathrm{d}s}{s} \leqslant \frac{1}{\varphi_2(\tau)} \int_1^{+\infty} G(t,s)b(s)g\bigl(s, x(s)\bigr)\,\frac{\mathrm{d}s}{s} \\ &= \frac{1}{\varphi_2(\tau)} Bx(t). \end{aligned}$$

We can infer that A and B satisfy (8). By Lemma 8, we know that condition (ii) of Lemma 4 holds. Consequently, by Lemma 4, operator equation $Ax+Bx = x$ has a unique solution x^* in P_h. For given initial values $x_0, y_0 \in P_h$, putting the sequences

$$x_n = Ax_{n-1} + By_{n-1}, \quad y_n = Ay_{n-1} + Bx_{n-1} \quad n = 1, 2, \ldots,$$

we have $x_n(t) \to x^*(t), y_n(t) \to x^*(t)$ as $n \to \infty$. Evidently, x^* is the unique positive solution for problem (1) in P_h. For given initial values $x_0, y_0 \in P_h$, making the following sequences

$$x_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, x_n(s)\bigr)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\bigl(s, y_n(s)\bigr)\,\frac{\mathrm{d}s}{s},$$
$$y_{n+1}(t) = \int_1^{+\infty} G(t,s)a(s)f\bigl(s, y_n(s)\bigr)\,\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t,s)b(s)g\bigl(s, x_n(s)\bigr)\,\frac{\mathrm{d}s}{s},$$

$n = 0, 1, 2 \ldots$, we have $x_n(t) \to x^*(t)$, $y_n(t) \to x^*(t)$ as $n \to \infty$. The proof is complete. □

Corollary 2. *Let* α, c, α_i, σ_j $(i = 1, 2 \ldots, m,\ j = 1, 2 \ldots, n)$ *be given in* (1). *Assume that* f *satisfies* (H1′)–(H3′) *and*

(H5′) *for* $\tau \in (0, 1)$, *there exist* $\varphi(\tau) \in (\tau, 1)$ *such that* $f(t, \tau x) \geqslant \varphi(\tau) f(t, x)$ *for* $t \in J$, $x \in \mathbb{R}^+$.

Then there is a unique positive solution x^* *in* P_h *for* (9), *where* $h(t) = (\ln t)^{\alpha-1}$, $t \in J$, *and for any initial value* $x_0 \in P_h$, *constructing the sequence*

$$x_{n+1}(t) = \int_1^{+\infty} G(t, s) a(s) f\bigl(s, x_n(s)\bigr) \frac{\mathrm{d}s}{s}, \quad n = 0, 1, 2 \ldots,$$

we have $x_n(t) \to x^*(t)$ *as* $n \to \infty$, *where* $G(t, s)$ *is given as in* (4).

Proof. From Remark 1 and Theorem 2 the conclusions hold. □

Remark 2. In literature, the unique of solutions for fractional equations was obtained always by Banach contractive theorem, and the solution is global. Here we consider fractional problems by using different methods-two interesting fixed point theorems of a sum operator in partial ordering Banach spaces, and further we can get the local existence and uniqueness of positive solutions, which can be seen seldom. For example, letting $b(t) g(t, x(t)) = 0$, (1) is reduced to be problem considered in [21]. Unlike the results obtained in this article, we not only get the uniqueness of positive solutions, but also establish the local uniqueness, which has even more applications in practical problems.

4 Examples

In this section, we present two examples to illustrate our main results.

Example 1. Consider the following Hadamard-type fractional differential equations on an infinite interval:

$$\begin{aligned} &{}^H D_{1+}^{5/2} x(t) + t\mathrm{e}^{-t} \frac{2 + [x(t)]^{1/2}}{1 + (\ln t)^{3/2}} + t\mathrm{e}^{-2t} \frac{1 + [x(t)]^{1/3}}{1 + (\ln t)^{3/2}} = 0, \quad t \in (1, +\infty), \\ &x(1) = x'(1) = 0, \qquad {}^H D_{1+}^{3/2} x(+\infty) = \sum_{i=1}^{2} \alpha_i {}^H I_{1+}^{\beta_i} x(\mathrm{e}^{1/2}) + \frac{\Gamma(\frac{5}{2})}{6} \sum_{j=1}^{3} \sigma_j x(\xi_j). \end{aligned} \tag{10}$$

Notice that (10) is a particular case of (1) with

$$\begin{aligned} &\alpha = \frac{5}{2}, \quad m = 2, \quad n = 3, \quad \alpha_1 = 1, \quad \alpha_2 = 6, \quad \eta = \mathrm{e}^{1/2}, \quad \beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{3}{2}, \\ &c = \frac{\Gamma(\frac{5}{2})}{6}, \quad \sigma_1 = \frac{1}{4}, \quad \sigma_2 = \frac{3}{8\sqrt{2}}, \quad \sigma_3 = \frac{2}{3\sqrt{3}}, \quad \xi_1 = \mathrm{e}, \quad \xi_2 = \mathrm{e}^2, \quad \xi_3 = \mathrm{e}^3. \end{aligned}$$

Take

$$a(t) = t\mathrm{e}^{-t}, \quad b(t) = t\mathrm{e}^{-2t}, \quad f(t,x) = \frac{2 + x^{1/2}}{1 + (\ln t)^{3/2}}, \quad g(t,x) = \frac{1 + x^{1/3}}{1 + (\ln t)^{3/2}}.$$

By calculating, we have

$$\Delta = \Gamma(\alpha) - \sum_{i=1}^{m} \alpha_i \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)} (\ln \eta)^{\alpha+\beta_i-1} - c \sum_{j=1}^{n} \sigma_j (\ln \xi_j)^{\alpha-1} = \frac{3\sqrt{\pi}}{16} > 0.$$

Clearly, $a(t)$, $b(t)$ are continuous with t, and

$$\int_1^{+\infty} a(s)\,\frac{\mathrm{d}s}{s} = \int_1^{+\infty} s\mathrm{e}^{-s}\,\frac{\mathrm{d}s}{s} = \mathrm{e}^{-1} < +\infty,$$
$$\int_1^{+\infty} b(s)\,\frac{\mathrm{d}s}{s} = \int_1^{+\infty} s\mathrm{e}^{-2s}\,\frac{\mathrm{d}s}{s} = \frac{1}{2}\mathrm{e}^{-2} < +\infty.$$

Obviously, $f, g : J \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and increasing with respect to the second argument, $f(t,0) > 0$, $g(t,0) > 0$. So, conditions (H1), (H2) are satisfied.

When $0 \leqslant x \leqslant M$,

$$f\bigl(t, \bigl(1 + (\ln t)^{\alpha-1}\bigr)x\bigr) = \frac{2}{1 + (\ln t)^{3/2}} + \bigl(1 + (\ln t)^{3/2}\bigr)^{1/2-1} x^{1/2} \leqslant 2 + \sqrt{M},$$
$$g\bigl(t, \bigl(1 + (\ln t)^{\alpha-1}\bigr)x\bigr) = \frac{1}{1 + (\ln t)^{3/2}} + \bigl(1 + (\ln t)^{3/2}\bigr)^{1/3-1} x^{1/3} \leqslant 1 + \sqrt[3]{M}$$

for $t \in J$. Hence, condition (H3) is satisfied.

In addition, take $\gamma = 1/2$. For $t \in J$, $\tau \in (0,1)$, $x \in \mathbb{R}^+$, we have

$$f(t,\tau x) = \frac{2 + \tau^{1/2} x^{1/2}}{1 + (\ln t)^{3/2}} \geqslant \tau^{1/2} \frac{2 + x^{1/2}}{1 + (\ln t)^{3/2}} = \tau^{1/2} f(t,x),$$
$$g(t,\tau x) = \frac{1 + \tau^{1/3} x^{1/3}}{1 + (\ln t)^{3/2}} \geqslant \tau \frac{1 + x^{1/3}}{1 + (\ln t)^{3/2}} = \tau g(t,x).$$

So, condition (H4) is satisfied.

Moreover, if we take $\delta \in (0,1]$, for $t \in J$, $x \in \mathbb{R}^+$, we obtain

$$a(t)f(t,x) = \frac{t\mathrm{e}^{-t}(2 + x^{1/2})}{1 + (\ln t)^{3/2}} \geqslant \frac{t\mathrm{e}^{-2t}(1 + x^{1/3})}{1 + (\ln t)^{3/2}} \geqslant \delta b(t) g(t,x).$$

So, all the conditions of Theorem 1 are satisfied. Therefore, problem (10) has a unique positive solution x^* in P_h, where $h(t) = (\ln t)^{3/2}$, $t \in J$. Taking any initial value $x_0 \in P_h$ and making the sequence

$$x_{n+1}(t) = \int_1^{+\infty} G(t,s) \biggl[s\mathrm{e}^{-s} \frac{2 + [x_n(s)]^{1/2}}{1 + (\ln s)^{3/2}} + s\mathrm{e}^{-2s} \frac{1 + [x_n(s)]^{1/3}}{1 + (\ln s)^{3/2}} \biggr] \frac{\mathrm{d}s}{s}, \quad n = 0, 1, 2 \dots,$$

we have $x_n(t) \to x^*(t)$ as $n \to \infty$, where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{11}$$

$$G_1(t,s) = g_0\left(t, s, \frac{5}{2}\right) = \frac{4}{3\sqrt{\pi}} \begin{cases} (\ln t)^{3/2} - (\ln \frac{t}{s})^{3/2}, & 1 \leqslant s \leqslant t < +\infty, \\ (\ln t)^{3/2}, & 1 \leqslant t \leqslant s < +\infty, \end{cases} \tag{12}$$

$$\begin{aligned} G_2(t,s) = {} & \frac{(\ln t)^{3/2}}{\frac{3\sqrt{\pi}}{16}} \left\{ g_0(\mathrm{e}^{1/2}, s, 3) + 6 g_0(\mathrm{e}^{1/2}, s, 4) \right\} + (\ln t)^{3/2} \left\{ \frac{1}{6} g_0\left(\mathrm{e}, s, \frac{5}{2}\right) \right. \\ & \left. + \frac{1}{4\sqrt{2}} g_0\left(\mathrm{e}^2, s, \frac{5}{2}\right) + \frac{4}{9\sqrt{3}} g_0\left(\mathrm{e}^3, s, \frac{5}{2}\right) \right\}. \end{aligned} \tag{13}$$

Example 2. Consider the following Hadamard-type fractional differential equations on an infinite interval:

$$\begin{gathered} {}^H D_{1+}^{5/2} x(t) + t\mathrm{e}^{-t} \cdot \frac{1 + [x(t)]^{\gamma_1}}{1 + (\ln t)^{3/2}} + t\mathrm{e}^{-2t} \cdot \frac{1 + [1 + x(t)]^{-\gamma_2}}{1 + (\ln t)^{3/2}} = 0, \quad t \in (1, +\infty), \\ x(1) = x'(1) = 0, \qquad {}^H D_{1+}^{3/2} x(+\infty) = \sum_{i=1}^{2} \alpha_i {}^H I_{1+}^{\beta_i} x(\mathrm{e}^{1/2}) + \frac{\Gamma(\frac{5}{2})}{6} \sum_{j=1}^{3} \sigma_j x(\xi_j), \end{gathered} \tag{14}$$

where $\gamma_1, \gamma_2 \in (0,1)$, and

$$\begin{gathered} \alpha = \frac{5}{2}, \quad m = 2, \quad n = 3, \quad \alpha_1 = 1, \quad \alpha_2 = 6, \quad \eta = \mathrm{e}^{1/2}, \quad \beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{3}{2}, \\ c = \frac{\Gamma(\frac{5}{2})}{6}, \quad \sigma_1 = \frac{1}{4}, \quad \sigma_2 = \frac{3}{8\sqrt{2}}, \quad \sigma_3 = \frac{2}{3\sqrt{3}}, \quad \xi_1 = \mathrm{e}, \quad \xi_2 = \mathrm{e}^2, \quad \xi_3 = \mathrm{e}^3. \end{gathered}$$

Take

$$a(t) = t\mathrm{e}^{-t}, \quad b(t) = t\mathrm{e}^{-2t}, \quad f(t,x) = \frac{1 + x^{\gamma_1}}{1 + (\ln t)^{3/2}}, \quad g(t,x) \frac{1 + (1+x)^{-\gamma_2}}{1 + (\ln t)^{3/2}}.$$

By calculating, we have

$$\Delta = \Gamma(\alpha) - \sum_{i=1}^{m} \alpha_i \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta_i)} (\ln \eta)^{\alpha + \beta_i - 1} - c \sum_{j=1}^{n} \sigma_j (\ln \xi_j)^{\alpha - 1} = \frac{3\sqrt{\pi}}{16} > 0.$$

Clearly, $a(t)$ and $b(t)$ are continuous with t, and

$$\begin{gathered} \int_1^{+\infty} a(s) \frac{\mathrm{d}s}{s} = \int_1^{+\infty} s\mathrm{e}^{-s} \frac{\mathrm{d}s}{s} = \mathrm{e}^{-1} < +\infty, \\ \int_1^{+\infty} b(s) \frac{\mathrm{d}s}{s} = \int_1^{+\infty} s\mathrm{e}^{-2s} \frac{\mathrm{d}s}{s} = \frac{1}{2}\mathrm{e}^{-2} < +\infty. \end{gathered}$$

Obviously, $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is increasing in x, $g : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing in x, $f(t, 0) > 0$, $g(t, 0) > 0$. So, conditions (H1) and (H6) are satisfied.

When $0 \leqslant x \leqslant M$,

$$f\bigl(t, \bigl(1 + (\ln t)^{\alpha-1}\bigr)x\bigr) = \frac{1}{1 + (\ln t)^{3/2}} + \bigl(1 + (\ln t)^{3/2}\bigr)^{\gamma_1 - 1} x^{\gamma_1} \leqslant 1 + M^{\gamma_1},$$
$$g\bigl(t, \bigl(1 + (\ln t)^{\alpha-1}\bigr)x\bigr) = \frac{1 + (1 + (1 + (\ln t)^{3/2})x)^{-\gamma_2}}{1 + (\ln t)^{3/2}} \leqslant 2$$

for $t \in J$. Hence, condition (H3) is satisfied.

Take $\varphi_1(\tau) = \tau^{\gamma_1}$, $\varphi_2(\tau) = \tau^{\gamma_2}$, then $\varphi_1(\tau), \varphi_2(\tau) \in (\tau, 1)$ for $\tau \in (0, 1)$. Thus,

$$f(t, \tau x) = \frac{1 + \tau^{\gamma_1} x^{\gamma_1}}{1 + (\ln t)^{3/2}} \geqslant \tau^{\gamma_1} \frac{1 + x^{\gamma_1}}{1 + (\ln t)^{3/2}} = \varphi_1(\tau) f(t, x),$$
$$g(t, \tau x) = \frac{1 + (1 + \tau x)^{-\gamma_2}}{1 + (\ln t)^{3/2}} \leqslant \frac{1}{\tau^{\gamma_2}} \frac{1 + (1 + x)^{-\gamma_2}}{1 + (\ln t)^{3/2}} = \frac{1}{\varphi_2(\tau)} g(t, x).$$

So, Theorem 2 implies that problem (14) has a unique positive solution x^* in P_h, where $h(t) = (\ln t)^{3/2}$, $t \in J$. For given initial values $x_0, y_0 \in P_h$, putting the sequences

$$x_{n+1}(t) = \int_1^{+\infty} G(t, s) s \mathrm{e}^{-s} \frac{1 + [x_n(s)]^{\gamma_1}}{1 + (\ln s)^{3/2}} \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t, s) s \mathrm{e}^{-2s} \frac{1 + (1 + y_n(s))^{-\gamma_2}}{1 + (\ln t)^{3/2}} \frac{\mathrm{d}s}{s},$$
$$y_{n+1}(t) = \int_1^{+\infty} G(t, s) s \mathrm{e}^{-s} \frac{1 + [y_n(s)]^{\gamma_1}}{1 + (\ln s)^{3/2}} \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G(t, s) s \mathrm{e}^{-2s} \frac{1 + (1 + x_n(s))^{-\gamma_2}}{1 + (\ln t)^{3/2}} \frac{\mathrm{d}s}{s},$$

$n = 0, 1, 2 \ldots$, we have $x_n(t) \to x^*(t)$, $y_n(t) \to x^*(t)$ as $n \to \infty$, where $G(t, s)$ is given as in (11)–(14).

5 Conclusions

In this article, we obtained the sufficient conditions for the existence and uniqueness of positive solutions for (1) by using two fixed point theorems of a sum operator, and we can construct iterative sequences to approximate the unique solutions. In the last section, we give two illustrative examples that effectively show the applicability of the obtained theoretical results. According to the literature, the unique solution for fractional equations was obtained always by Banach contractive theorem, and the solution is global. Here we get the uniqueness of positive solution and it is local. In addition, through Corollaries 1 and 2, we show that our results are superior to those previously presented in the literature. For future work, we intend to explore other different types of fractional differential equations. In the meanwhile, we try to study other fixed point theorems for better use in solving differential equations.

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