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Investigation of solitons in magneto-optic waveguides with Kudryashov's law nonlinear refractive index for coupled system of generalized nonlinear Schrödinger's equations using modified extended mapping method

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Abstract. In this work, we investigate the optical solitons and other waves through magneto-optic waveguides with Kudryashov's law of nonlinear refractive index in the presence of chromatic dispersion and Hamiltonian-type perturbation factors using the modified extended mapping approach. Many classifications of solutions are established like bright solitons, dark solitons, singular

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solitons, singular periodic wave solutions, exponential wave solutions, rational wave, solutions, Weierstrass elliptic doubly periodic solutions, and Jacobi elliptic function solutions. Some of the extracted solutions are described graphically to provide their physical understanding of the acquired solutions.

Keywords: optical solitons, magneto-optic waveguides, Kudryashov's law, modified extended mapping technique.

1 Introduction

A soliton is a localized-in-time light pulse in nonlinear optics that propagates through a nonlinear dispersive medium without converting into a different form. Nonlinear partial differential equations (NLPDEs) are commonly used to visualize fundamental physical phenomena such as fluid mechanics, electromagnetism, optics, magnetohydrodynamics, quantum mechanics, superconductivity, thermodynamics, chemical reactions, finance, neuroscience, and elasticity, among others. The mathematics and theoretical background of optical solitons were discussed and explained clearly by Biswas et al. in [11].

When nonlinearity perfectly balances dispersion in optical fibres, an important nonlinear phenomenon known as the soliton arises. A crucial component of the communications industry sector is optical soliton dynamics. The propagation of the optical solitons is usually governed by the nonlinear Schrödinger equation, which is one of the most important models in modern nonlinear science. Optical solitons have promising potential to become information carriers in telecommunication due to their capability of propagating long distance through an optical fiber without attenuation and changing their shapes amplitudes and velocities. Magneto-optic wave guides can help to reduce solitons clutter and to solve the internet bandwidth issue. In addition, magneto-optic waveguides should always be considered instead of standard waveguides. When considering the amount of data that is transmitted across transcontinental and transoceanic distances, such waveguides are always beneficial. The solitons can become less cluttered in the existence of a magnetic field. Only when the solitons are very close together does this prevent information from leaking between them. Addressing the soliton scientific dynamics in nonlinear optics is therefore of utmost importance (see [1, 4–6, 13, 16, 20, 26, 32]).

A variety of models have been examined in the context of magneto-optic solitons. Apart from the Kerr law, the different variants of the non-Kerr law of nonlinearity examined in this context include parabolic law, power law, quadratic-cubic law, dual-power law, and others. Besides, Yildirim et al. investigated many cases of optical solitons with many different approaches (see [27–31]). González-Gaxiola et al. [14, 15] discussed the optical solitons by the Adomian decomposition scheme. In addition, Al Qarni et al. [21, 22] explored the soliton solutions but using the improved Adomian decomposition scheme. Many other researchers discussed a variety of models in order to obtain soliton solutions with many different approaches (see [2, 3, 7–10, 12, 17, 19, 23, 24, 33–35]).

In the present work, we consider a coupled system of generalized nonlinear Schrödinger's equation (NLSE) in magneto-optic waveguides with Kudryashov's law of nonlinear refractive index in the presence of chromatic dispersion and Hamiltonian-type perturbation factors as [18]

$$
\begin{aligned}
& i\Phi_t + \mathcal{A}_1\Phi_{xx} \\
& \quad + \left(\frac{\mathcal{B}_1}{|\Phi|^n} + \frac{\mathcal{C}_1}{|\Phi|^{2n}} + \mathcal{D}_1|\Phi|^n + \mathcal{E}_1|\Phi|^{2n} + \frac{\mathcal{F}_1}{|\Psi|^n} + \frac{\mathcal{G}_1}{|\Psi|^{2n}} + \mathcal{H}_1|\Psi|^n + \mathcal{K}_1|\Psi|^{2n}\right)\Phi \\
& \quad - \mathcal{Q}_1\Psi - i\big(\beta_1\Phi_x + \lambda_1\big(|\Phi|^{2n}\Phi\big)_x + \gamma_1\big(|\Phi|^{2n}\big)_x\Phi + \vartheta_1|\Phi|^{2n}\Phi_x\big) = 0
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
& i\Psi_t + \mathcal{A}_2\Psi_{xx} \\
& \quad + \left(\frac{\mathcal{B}_2}{|\Psi|^n} + \frac{\mathcal{C}_2}{|\Psi|^{2n}} + \mathcal{D}_2|\Psi|^n + \mathcal{E}_2|\Psi|^{2n} + \frac{\mathcal{F}_2}{|\Phi|^n} + \frac{\mathcal{G}_2}{|\Phi|^{2n}} + \mathcal{H}_2|\Phi|^n + \mathcal{K}_2|\Phi|^{2n}\right)\Psi \\
& \quad - \mathcal{Q}_2\Phi - i\big(\beta_2\Psi_x + \lambda_2\big(|\Psi|^{2n}\Psi\big)_x + \gamma_2\big(|\Psi|^{2n}\big)_x\Psi + \vartheta_2|\Psi|^{2n}\Psi_x\big) = 0,
\end{aligned} \tag{2}
$$

where $\Phi(x,t)$ and $\Psi(x,t)$ refer to the profiles of the soliton wave, and $\mathcal{A}_i$ for $i = 1,2$ stem from the coefficients of chromatic dispersion. $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{D}_i$, and $\mathcal{E}_i$ are the self-phase modulation coefficients, while $\mathcal{F}_i$, $\mathcal{G}_i$, $\mathcal{H}_i$, and $\mathcal{K}_i$ arise from the cross-phase modulation. $\mathcal{Q}_i$ represents the magneto-optic parameters, while β_i give the intermodal dispersions, and λ_i emerge from the self-steepening coefficients, while γ_i and ϑ_i come from the nonlinear dispersions, and n denotes the refractive index, which comes from the full nonlinearity.

In this research, the suggested model is handled using the modified extended mapping method, which gives us new and novel solutions not obtained before. This method provided various and new sort of solutions such as bright solitons, dark solitons, singular solitons, exponential wave solutions, rational wave solutions, Weierstrass elliptic doubly periodic solutions, and Jacobi elliptic function solutions. The extracted solutions confirmed the efficacy and strength of the current technique. Furthermore, 3D, contour, and 2D simulations are depicted to demonstrate the nature of the obtained solutions.

2 The proposed technique summary

This part is considered as an introduction for the modified extended mapping scheme [25].

WE consider the following NLPDE:

$$\mathcal{Z}(\phi, \phi_t, \phi_x, \phi_{xx}, \phi_{xt}, \phi_{xxt}, \dots) = 0, \tag{3}$$

where $\mathcal{Z}$ stands to a polynomial function of $\phi(x,t)$ and its corresponding partial derivatives for the space and time.

Step 1. The subsequent travelling wave transformation will be used:

$$\phi(x,t) = \mathcal{U}(\xi), \quad \xi = x - \eta t,\ \eta \neq 0, \tag{4}$$

where η is a real constant, and it represents the wave speed that will be evaluated later on. Equation (3) can be transformed into the following nonlinear ordinary differential equation (NLODE) by substituting Eq. (4) into Eq. (3):

$$\mathcal{X}(\mathcal{U},\mathcal{U}',\mathcal{U}'',\mathcal{U}''',\dots)=0. \tag{5}$$

Step 2: The solution of Eq. (5) can be expressed as

$$\begin{aligned}\mathcal{U}(\xi)={}&\sum_{j=0}^{\mathbb{M}}a_j\mathcal{W}^j(\xi)+\sum_{j=-1}^{-\mathbb{M}}b_{-j}\mathcal{W}^j(\xi)+\sum_{j=2}^{\mathbb{M}}c_j\mathcal{W}^{j-2}(\xi)\mathcal{W}'(\xi)\\&+\sum_{j=-1}^{-\mathbb{M}}d_{-j}\mathcal{W}^j(\xi)\mathcal{W}'(\xi),\end{aligned} \tag{6}$$

where a_j, b_{-j}, c_j, d_{-j} are real constants to be estimated, and $\mathcal{W}(\xi)$ satisfies the following auxiliary equation condition:

$$\mathcal{W}'(\xi)=\sqrt{\varrho_0+\varrho_1\mathcal{W}(\xi)+\varrho_2\mathcal{W}^2(\xi)+\varrho_3\mathcal{W}^3(\xi)+\varrho_4\mathcal{W}^4(\xi)+\varrho_6\mathcal{W}^6(\xi)}, \tag{7}$$

where ϱ_i, $(i=0,1,2,3,4,6)$ are constants.

Step 3. By applying the principle of balance through Eq. (5) between the highest-order derivatives and the highest-order nonlinear terms, the positive integer $\mathbb{M}$ can be determined.

Step 4. Inserting the supposed solution in Eq. (6) along with Eq. (7) into Eq. (5), then equalizing the coefficients of $\mathcal{W}'^j(\xi)\mathcal{W}^i(\xi)$ $(j=0,1;\ i=0,\pm1,\pm2,\dots)$ to zero yields a set of nonlinear algebraic equations for a_j, b_{-j}, c_j, d_{-j}, and η that can be solved by Mathematica software packages or Maple. Then we can determine the unknown constants a_j, b_{-j}, c_j, and d_{-j}. After that, we can obtain many exact solutions to Eq. (3).

3 Optical solitons and other wave solutions

The following wave transformations are assumed in order to obtain the optical solutions of (1) and (2):

$$\begin{aligned}\Phi(x,t)&=\mathcal{U}(\xi)\mathrm{e}^{\mathrm{i}Z(x,t)},\\ \Psi(x,t)&=\mathcal{V}(\xi)\mathrm{e}^{\mathrm{i}Z(x,t)},\end{aligned} \tag{8}$$

and

$$\xi=x-\eta t,\quad \eta\neq0,\qquad Z(x,t)=-\kappa x+\omega t+\Delta, \tag{9}$$

where $\mathcal{U}(\xi)$ and $\mathcal{V}(\xi)$ represent the amplitude terms of the solution, and κ, η, ω, and Δ denote the velocity, wave number, frequency, and phase constant in the mentioned sequence.

Substituting Eqs. (8)–(9) into Eqs. (1) and (2), then separating the real and imaginary parts, we get

$$\begin{aligned}
&\mathcal{A}_1\mathcal{U}'' - \left[\omega + \beta_1\kappa + \mathcal{A}_1\kappa^2\right]\mathcal{U} - \kappa(\lambda_1 + \vartheta_1)\mathcal{U}^{2n+1} \\
&\quad - \mathcal{Q}_1\mathcal{V} + \frac{\mathcal{B}_1}{\mathcal{U}^{n-1}} + \frac{\mathcal{C}_1}{\mathcal{U}^{2n-1}} + \mathcal{D}_1\mathcal{U}^{n+1} \\
&\quad + \mathcal{E}_1\mathcal{U}^{2n+1} + \left(\frac{\mathcal{F}_1}{\mathcal{V}^n} + \frac{\mathcal{G}_1}{\mathcal{V}^{2n}} + \mathcal{H}_1\mathcal{V}^n + \mathcal{K}_1\mathcal{V}^{2n}\right)\mathcal{U} = 0, \qquad (10) \\
&\mathcal{A}_2\mathcal{V}'' - \left[\omega + \beta_2\kappa + \mathcal{A}_2\kappa^2\right]\mathcal{V} - \kappa(\lambda_2 + \vartheta_2)\mathcal{V}^{2n+1} \\
&\quad - \mathcal{Q}_2\mathcal{U} + \frac{\mathcal{B}_2}{\mathcal{V}^{n-1}} + \frac{\mathcal{C}_2}{\mathcal{V}^{2n-1}} + \mathcal{D}_2\mathcal{V}^{n+1} \\
&\quad + \mathcal{E}_2\mathcal{V}^{2n+1} + \left(\frac{\mathcal{F}_2}{\mathcal{U}^n} + \frac{\mathcal{G}_2}{\mathcal{U}^{2n}} + \mathcal{H}_2\mathcal{U}^n + \mathcal{K}_2\mathcal{U}^{2n}\right)\mathcal{V} = 0, \qquad (11)
\end{aligned}$$

$$\begin{aligned}
(\eta + 2\mathcal{A}_1\kappa + \beta_1)\mathcal{U}' + \left[(2n+1)\lambda_1 + 2n\gamma_1 + \vartheta_1\right]\mathcal{U}^{2n}\mathcal{U}' &= 0, \\
(\eta + 2\mathcal{A}_2\kappa + \beta_2)\mathcal{V}' + \left[(2n+1)\lambda_2 + 2n\gamma_2 + \vartheta_2\right]\mathcal{V}^{2n}\mathcal{V}' &= 0.
\end{aligned}$$

We can obtain the exact solution under the following constraints:

$$\begin{gathered}
\eta = -2\mathcal{A}_1\kappa - \beta_1, \qquad (12) \\
(2n+1)\lambda_1 + 2n\gamma_1 + \vartheta_1 = 0, \\
\eta = -2\mathcal{A}_2\kappa - \beta_2, \qquad (13) \\
(2n+1)\lambda_2 + 2n\gamma_2 + \vartheta_2 = 0.
\end{gathered}$$

From Eqs. (12) and (13) we can estimate the frequency

$$\kappa = \frac{\beta_2 - \beta_1}{2(\mathcal{A}_1 - \mathcal{A}_2)}, \quad \mathcal{A}_1 \neq \mathcal{A}_2,\ \beta_1 \neq \beta_2.$$

Set

$$\mathcal{V}(\xi) = \varOmega\,\mathcal{U}(\xi), \quad \varOmega \neq 0, 1.$$

Thus, Eqs. (10) and (11) come out as

$$\begin{aligned}
&\mathcal{A}_1\mathcal{U}^{2n-1}\mathcal{U}'' + \left(\mathcal{C}_1 + \mathcal{G}_1\varOmega^{-2n}\right) + \left(\mathcal{B}_1 + \mathcal{F}_1\varOmega^{-n}\right)\mathcal{U}^n \\
&\quad - \left[\omega + \beta_1\kappa + \mathcal{A}_1\kappa^2 + \mathcal{Q}_1\varOmega\right]\mathcal{U}^{2n} + \left(\mathcal{D}_1 + \mathcal{H}_1\varOmega^n\right)\mathcal{U}^{3n} \\
&\quad + \left[\left(\mathcal{E}_1 + \mathcal{K}_1\varOmega^{2n}\right) - \kappa(\lambda_1 + \vartheta_1)\right]\mathcal{U}^{4n} = 0 \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{A}_2\varOmega\mathcal{U}^{2n-1}\mathcal{U}'' + \left(\mathcal{C}_2\varOmega^{1-2n} + \mathcal{G}_2\varOmega\right) + \left(\mathcal{B}_2\varOmega^{1-n} + \mathcal{F}_2\varOmega\right)\mathcal{U}^n \\
&\quad - \left[\omega + \beta_2\kappa + \mathcal{A}_2\kappa^2 + \mathcal{Q}_2\right]\mathcal{U}^{2n} + \left(\mathcal{D}_2\varOmega^{n+1} + \mathcal{H}_2\varOmega\right)\mathcal{U}^{3n} \\
&\quad + \left[\left(\mathcal{E}_2\varOmega^{2n+1} + \mathcal{K}_2\varOmega\right) - \kappa(\lambda_2 + \vartheta_2)\varOmega^{2n+1}\right]\mathcal{U}^{4n} = 0. \qquad (15)
\end{aligned}$$

Then, by comparing the coefficients of Eqs. (14) and (15), they are equivalent under the following constraints:

$$\mathcal{A}_1 = \Omega\mathcal{A}_2, \tag{16}$$

$$\begin{aligned}
\mathcal{C}_1 + \mathcal{G}_1\Omega^{-2n} &= \mathcal{C}_2\Omega^{1-2n} + \mathcal{G}_2\Omega,\\
\mathcal{B}_1 + \mathcal{F}_1\Omega^{-n} &= \mathcal{B}_2\Omega^{1-n} + \mathcal{F}_2\Omega,\\
\mathcal{A}_1\kappa^2 + \beta_1\kappa + \mathcal{Q}_1\Omega + \omega &= \mathcal{Q}_2 + \left[\mathcal{A}_2\kappa^2 + \beta_2\kappa + \omega\right]\Omega, \qquad (17)\\
\mathcal{D}_1 + \mathcal{H}_1\Omega^{n} &= \mathcal{D}_2\Omega^{n+1} + \mathcal{H}_2\Omega,
\end{aligned}$$

and

$$\mathcal{K}_1\Omega^{2n} + \mathcal{E}_1 - \kappa(\vartheta_1 - \lambda_1) = \mathcal{K}_2\Omega + \mathcal{E}_2\Omega^{2n+1} - \kappa(\vartheta_2 - \lambda_2)\Omega^{2n+1}.$$

Through Eqs. (16) and (17), we get the wave number as below:

$$\omega = \frac{\kappa(\beta_1 - \beta_2\Omega) + \Omega\mathcal{Q}_1 - \mathcal{Q}_2}{\Omega - 1}.$$

Suppose that

$$\mathcal{U} = P^{1/n}.$$

Then Eq. (14) can be represented as follows:

$$n\mathcal{A}_1PP'' + (1-n)\mathcal{A}_1(P')^2 + n^2\left(\mathcal{L}_0 + \mathcal{L}_1P + \mathcal{L}_2P^2 + \mathcal{L}_3P^3 + \mathcal{L}_4P^4\right) = 0, \tag{18}$$

where $\mathcal{L}_i$, $(i = 0,1,2,3,4)$ are constants incorporated for notational convenience and given by

$$\begin{aligned}
\mathcal{L}_0 &= \mathcal{C}_1 + \mathcal{G}_1\Omega^{-2n},\\
\mathcal{L}_1 &= \mathcal{B}_1 + \mathcal{F}_1\Omega^{-n},\\
\mathcal{L}_2 &= -\left(\mathcal{A}_1\kappa^2 + \beta_1\kappa + \mathcal{Q}_1\Omega + \omega\right),\\
\mathcal{L}_3 &= \mathcal{D}_1 + \mathcal{H}_1\Omega^{n},\\
\mathcal{L}_4 &= \mathcal{K}_1\Omega^{2n} + \mathcal{E}_1 - \kappa(\kappa\lambda_1 + \vartheta_1).
\end{aligned}$$

According to the proposed scheme in Section 2, the general solution for Eq. (18) can be written as

$$P(\xi) = a_0 + a_1\mathcal{W}(\xi) + b_1\frac{1}{\mathcal{W}(\xi)} + d_1\frac{\mathcal{W}'(\xi)}{\mathcal{W}(\xi)}, \tag{19}$$

where a_i $(i = 0,1)$, b_1, and d_1 are constants, which can be evaluated under the restrictions a_1 or b_1 or $d_1 \neq 0$.

Inserting Eqs. (19) and (7) into Eq. (18), then grouping the coefficients of the same powers and setting them to zero, the result is a set of nonlinear algebraic equations that can be solved with Mathematica to provide the following outcomes:

Case 1. When $\varrho_0 = \varrho_1 = \varrho_3 = \varrho_6 = 0$, we found the following set of solution:

$$\begin{aligned}
&a_0 = a_1 = b_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0,\\
&d_1 = \frac{2\sqrt{\mathcal{A}_1\mathcal{L}_0}}{n\mathcal{L}_2\sqrt{n-1}}, \qquad \varrho_2 = \frac{n^2\mathcal{L}_2}{2\mathcal{A}_1}, \qquad \mathcal{L}_4 = \frac{(1-n^2)\mathcal{L}_2^2}{4\mathcal{L}_0}.
\end{aligned}$$

Through the above solution set, we can express the solutions of Eqs. (1) and (2) as follows:

If $\varrho_2 > 0$, $\varrho_4 < 0$, $\mathcal{L}_2 > 0$, and $\mathcal{A}_1\mathcal{L}_0 > 0$, then the solutions are:

$$\Phi_{1.1}(x,t) = \left(\frac{2\sqrt{\mathcal{A}_1\mathcal{L}_0\varrho_2}}{n\mathcal{L}_2\sqrt{n-1}}\tanh\bigl[(x-\eta t)\sqrt{\varrho_2}\bigr]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)}, \tag{20}$$

$$\Psi_{1.1}(x,t) = \Omega\left(\frac{2\sqrt{\mathcal{A}_1\mathcal{L}_0\varrho_2}}{n\mathcal{L}_2\sqrt{n-1}}\tanh\bigl[(x-\eta t)\sqrt{\varrho_2}\bigr]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the dark soliton solutions.

Case 2. When $\varrho_1 = \varrho_3 = \varrho_6 = 0$, $\varrho_0 = \varrho_2^2/(4\varrho_4)$, we obtained the following sets of solutions:

(2.1) $$a_0 = b_1 = d_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0,$$
$$a_1 = \frac{\sqrt{-(n+1)\mathcal{A}_1\varrho_4}}{n\sqrt{\mathcal{L}_4}}, \qquad \mathcal{L}_0 = -\frac{(n^2-1)\mathcal{L}_2^2}{4\mathcal{L}_4}, \qquad \varrho_2 = -\frac{n^2\mathcal{L}_2}{\mathcal{A}_1},$$

(2.2) $$a_0 = a_1 = d_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0,$$
$$b_1 = \frac{n\mathcal{L}_2\sqrt{-(n+1)}}{2\sqrt{\mathcal{A}_1\mathcal{L}_4\varrho_4}}, \qquad \mathcal{L}_0 = -\frac{(n^2-1)\mathcal{L}_2^2}{4\mathcal{L}_4}, \qquad \varrho_2 = -\frac{n^2\mathcal{L}_2}{\mathcal{A}_1},$$

(2.3) $$a_0 = a_1 = d_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0,$$
$$b_1 = -\frac{n\mathcal{L}_2\sqrt{-(n+1)}}{2\sqrt{\mathcal{A}_1\mathcal{L}_4\varrho_4}}, \qquad \mathcal{L}_0 = -\frac{(n^2-1)\mathcal{L}_2^2}{4\mathcal{L}_4}, \qquad \varrho_2 = -\frac{n^2\mathcal{L}_2}{\mathcal{A}_1},$$

(2.4) $$a_0 = d_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0,$$
$$a_1 = \frac{\sqrt{-(n+1)\mathcal{A}_1\varrho_4}}{n\sqrt{\mathcal{L}_4}}, \qquad b_1 = \frac{n\mathcal{L}_2\sqrt{-(n+1)}}{2\sqrt{\mathcal{A}_1\mathcal{L}_4\varrho_4}},$$
$$\mathcal{L}_0 = -\frac{(n^2-1)\mathcal{L}_2^2}{4\mathcal{L}_4}, \qquad \varrho_2 = -\frac{n^2\mathcal{L}_2}{4\mathcal{A}_1},$$

(2.5) $$a_0 = a_1 = b_1 = \mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}_3 = 0,$$
$$d_1 = \frac{\sqrt{-(n+1)\mathcal{A}_1}}{n\sqrt{\mathcal{L}_4}}, \qquad \varrho_2 = \frac{n^2\mathcal{L}_2}{2\mathcal{A}_1}.$$

Through the solution set (2.1), we can express the solutions of Eqs. (1) and (2) as follows:

If $\varrho_2 < 0$, $\varrho_4 > 0$, and $\mathcal{L}_2\mathcal{L}_4 < 0$, then the solutions are:

$$\Phi_{2.1}(x,t) = \left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{2\mathcal{L}_4}}\tanh\left[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}\,\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

$$\Psi_{2.1}(x,t) = \Omega\left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{2\mathcal{L}_4}}\tanh\left[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}\,\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the dark soliton solutions.

Through the solution set (2.2), we can express the solutions of Eqs. (1) and (2) as follows:

If $\varrho_2 < 0$, $\varrho_4 > 0$, and $\mathcal{L}_2\mathcal{L}_4 < 0$, then the solutions are:

$$\Phi_{2.2}(x,t) = \left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}} \coth\left[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)}, \tag{21}$$
$$\Psi_{2.2}(x,t) = \Omega\left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}} \coth\left[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the singular soliton solutions.

Through the solution set (2.3), we can express the solutions of Eqs. (1) and (2) as follows:

If $\varrho_2 < 0$, $\varrho_4 > 0$, and $\mathcal{L}_2 < 0$, $\mathcal{L}_4 > 0$, then the solutions are:

$$\Phi_{2.3}(x,t) = \left(-\mathcal{L}_2\sqrt{-\frac{n+1}{\mathcal{L}_2\mathcal{L}_4}} \coth\left[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{2.3}(x,t) = \Omega(-\mathcal{L}_2\sqrt{-\frac{n+1}{\mathcal{L}_2\mathcal{L}_4}} \coth\left[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}\right]\Bigg)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the singular soliton solutions.

Through the solution set (2.4), we can express the solutions of Eqs. (1) and (2) as follows:

If $\varrho_2 < 0$, $\varrho_4 > 0$, and $\mathcal{L}_2\mathcal{L}_4 < 0$, then the solutions are:

$$\Phi_{2.4}(x,t) = \left(\frac{\sqrt{-2(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}}\left(\frac{3\cosh^2[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}]+1}{2\sinh[2(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}]}\right)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{2.4}(x,t) = \Omega\left(\frac{\sqrt{-2(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}}\left(\frac{3\cosh^2[(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}]+1}{2\sinh[2(x-\eta t)\sqrt{-\frac{\varrho_2}{2}}]}\right)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the hyperbolic wave solutions.

Through the solution set (2.5), we can express the solutions of Eqs. (1) and (2) as follows: If $\varrho_2 < 0$, $\varrho_4 > 0$, and $\mathcal{L}_2\mathcal{L}_4 > 0$, then the solutions are:

$$\Phi_{2.5}(x,t) = \left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}} \operatorname{csch}\left[(x-\eta t)\sqrt{-2\varrho_2}\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{2.5}(x,t) = \Omega\left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}} \operatorname{csch}\left[(x-\eta t)\sqrt{-2\varrho_2}\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the singular soliton solutions.

Case 3. When $\varrho_3 = \varrho_4 = \varrho_6 = 0$, we found the following sets of solutions:

(3.1) $$a_0 = a_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0, \qquad b_1 = \frac{2n\sqrt{\mathcal{L}_0\varrho_0}}{\varrho_2\sqrt{(n-1)\mathcal{A}_1}}, \qquad d_1 = \frac{2n\sqrt{\mathcal{L}_0}}{\varrho_2\sqrt{(n-1)\mathcal{A}_1}},$$
$$\mathcal{L}_4 = -\frac{\mathcal{A}_1^2(n^2-1)\varrho_2^2}{16n^4\mathcal{L}_0}, \qquad \mathcal{L}_2 = \frac{\mathcal{A}_1\varrho_2}{2n^2},$$

(3.2) $$a_0 = a_1 = b_1 = \varrho_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0, \qquad d_1 = \frac{n\sqrt{\mathcal{L}_0}}{\varrho_2\sqrt{(n-1)\mathcal{A}_1}},$$
$$\mathcal{L}_4 = -\frac{\mathcal{A}_1^2(n^2-1)\varrho_2^2}{n^4\mathcal{L}_0}, \qquad \mathcal{L}_2 = \frac{2\mathcal{A}_1\varrho_2}{n^2}.$$

Through the solution set (3.1), we can express the solutions of Eqs. (1) and (2) as follows: If $\varrho_0 > 0$, $\varrho_2 > 0$, $\varrho_1 = 0$, and $\mathcal{A}_1\mathcal{L}_0 > 0$, then the solutions are:

$$\varPhi_{3.1,1}(x,t) = \left(\frac{2n\sqrt{\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{A}_1\varrho_2}}\coth\left[\frac{1}{2}(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$
$$\varPsi_{3.1,1}(x,t) = \varOmega\left(\frac{2n\sqrt{\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{A}_1\varrho_2}}\coth\left[\frac{1}{2}(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$

these solutions represent the singular soliton solutions.

If $\varrho_2 > 0$, $\varrho_0 = \varrho_1^2/(4\varrho_2)$, and $\mathcal{A}_1\mathcal{L}_0 > 0$, then the solutions are:

$$\varPhi_{3.1,2}(x,t) = \left(\frac{2n\sqrt{\mathcal{L}_0}}{\sqrt{(n-1)\varrho_2\mathcal{A}_1}}\,\frac{2\varrho_2\mathrm{e}^{(x-\eta t)\sqrt{\varrho_2}}+\varrho_1}{|2\varrho_2\mathrm{e}^{(x-\eta t)\sqrt{\varrho_2}}-\varrho_1|}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$
$$\varPsi_{3.1,2}(x,t) = \varOmega\left(\frac{2n\sqrt{\mathcal{L}_0}}{\sqrt{(n-1)\varrho_2\mathcal{A}_1}}\,\frac{2\varrho_2\mathrm{e}^{(x-\eta t)\sqrt{\varrho_2}}+\varrho_1}{|2\varrho_2\mathrm{e}^{(x-\eta t)\sqrt{\varrho_2}}-\varrho_1|}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$

these solutions represent the exponential wave solutions.

Through the solution set (3.2), we can express the solutions of Eqs. (1) and (2) as follows:

If $\varrho_0 > 0$, $\varrho_2 > 0$, $\varrho_1 = 0$, and $\mathcal{A}_1\mathcal{L}_0 > 0$, then the solutions are:

$$\varPhi_{3.2,1}(x,t) = \left(\frac{n\sqrt{\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{A}_1\varrho_2}}\coth\left[(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$
$$\varPsi_{3.2,1}(x,t) = \varOmega\left(\frac{n\sqrt{\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{A}_1\varrho_2}}\coth\left[(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$

these solutions represent the singular soliton solutions.

If $\varrho_2 > 0$, $\varrho_0 = \varrho_1^2/(4\varrho_2)$, and $\mathcal{A}_1\mathcal{L}_0 > 0$, then the solutions are:

$$\varPhi_{3.2,2}(x,t) = \left(\frac{2n\sqrt{\mathcal{L}_0\varrho_2}}{\sqrt{(n-1)\mathcal{A}_1}}\left(\frac{\mathrm{e}^{(x-\eta t)\sqrt{\varrho_2}}}{|\varrho_1 - 2\varrho_2 e^{(x-\eta t)\sqrt{\varrho_2}}|}\right)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$
$$\varPsi_{3.2,2}(x,t) = \varOmega\left(\frac{2n\sqrt{\mathcal{L}_0\varrho_2}}{\sqrt{(n-1)\mathcal{A}_1}}\left(\frac{\mathrm{e}^{(x-\eta t)\sqrt{\varrho_2}}}{|\varrho_1 - 2\varrho_2 e^{(x-\eta t)\sqrt{\varrho_2}}|}\right)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$

these solutions represent the exponential wave solutions.

Case 4. When $\varrho_0 = \varrho_1 = \varrho_2 = \varrho_6 = 0$, then we found the following set of solution:

$$a_0 = b_1 = d_1 = \mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}_2 = 0,$$
$$a_1 = \frac{\sqrt{-(n+1)\mathcal{A}_1\varrho_4}}{n\sqrt{\mathcal{L}_4}}, \qquad \mathcal{L}_3 = \frac{(n+2)\varrho_3\sqrt{-\mathcal{A}_1\mathcal{L}_4}}{2n\sqrt{(n+1)\varrho_4}}.$$

Then the solutions are:

$$\varPhi_4(x,t) = \left(\frac{4\varrho_3\sqrt{-(n+1)\mathcal{A}_1\varrho_4}}{n\sqrt{\mathcal{L}_4}(\varrho_3^2(x-\eta t)^2 - 4\varrho_4)}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$
$$\varPsi_4(x,t) = \varOmega\left(\frac{4\varrho_3\sqrt{-(n+1)\mathcal{A}_1\varrho_4}}{n\sqrt{\mathcal{L}_4}(\varrho_3^2(x-\eta t)^2 - 4\varrho_4)}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$

which represent rational wave solutions under the conditions that $\mathcal{A}_1\mathcal{L}_4 > 0$, $\varrho_4 < 0$, and $\varrho_3 > 0$.

Case 5. When $\varrho_0 = \varrho_1 = \varrho_6 = 0$, then we found the following set of solutions:

$$a_0 = b_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0,\ a_1 = -\frac{\sqrt{\mathcal{A}_1\mathcal{L}_0\varrho_4}}{n\mathcal{L}_2\sqrt{n-1}},$$
$$d_1 = -\frac{\sqrt{\mathcal{A}_1\mathcal{L}_0}}{n\mathcal{L}_2\sqrt{n-1}}, \qquad \varrho_2 = \frac{2n^2\mathcal{L}_2}{\mathcal{A}_1},\ \mathcal{L}_4 = -\frac{(n^2-1)\mathcal{L}_2^2}{4\mathcal{L}_0}.$$

Then the corresponding solutions of Eqs. (1) and (2) will be as follows:

If $\varrho_2 > 0$, $\varrho_3^2 = 4\varrho_2\varrho_4$, and $\mathcal{L}_0\mathcal{L}_2 > 0$, then the solutions are:

$$\varPhi_{5.1}(x,t) = \left(\frac{\sqrt{2\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{L}_2}}\tanh\left[\frac{1}{2}(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$
$$\varPsi_{5.1}(x,t) = \varOmega\left(\frac{\sqrt{2\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{L}_2}}\tanh\left[\frac{1}{2}(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\varDelta)},$$

these solutions represent the dark soliton solutions.

If $\varrho_2 > 0$, $\varrho_3^2 = 4\varrho_2\varrho_4$, and $\mathcal{L}_0\mathcal{L}_2 > 0$, then the solutions are:

$$\Phi_{5.2}(x,t) = \left(\frac{\sqrt{2\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{L}_2}}\coth\left[\frac{1}{2}(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{5.2}(x,t) = \Omega\left(\frac{\sqrt{2\mathcal{L}_0}}{\sqrt{(n-1)\mathcal{L}_2}}\coth\left[\frac{1}{2}(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the singular soliton solutions.

Case 6. When $\varrho_2 = \varrho_4 = \varrho_6 = 0$, then we found the following set of solution:

$$a_0 = a_1 = d_1 = \mathcal{L}_0 = \mathcal{L}_2 = 0, \qquad b_1 = -\frac{(n+2)\mathcal{A}_1\varrho_1}{2n^2\mathcal{L}_3},$$
$$\varrho_3 = -\frac{4n^4\mathcal{L}_1\mathcal{L}_3}{(n^2-4)\mathcal{A}_1^2\varrho_1}, \qquad \mathcal{L}_4 = -\frac{4n^2(n+1)\mathcal{L}_3^2\varrho_0}{(n+2)^2\mathcal{A}_1\varrho_1^2}.$$

Then the solutions are:

$$\Phi_6(x,t) = \left(-\frac{(n+2)\mathcal{A}_1\varrho_1}{2n^2\mathcal{L}_3\wp[\frac{1}{2}(x-\eta t)\sqrt{\varrho_3}, -\frac{4\varrho_1}{\varrho_3}, -\frac{4\varrho_0}{\varrho_3}]}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_6(x,t) = \Omega\left(-\frac{(n+2)\mathcal{A}_1\varrho_1}{2n^2\mathcal{L}_3\wp[\frac{1}{2}(x-\eta t)\sqrt{\varrho_3}, -\frac{4\varrho_1}{\varrho_3}, -\frac{4\varrho_0}{\varrho_3}]}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the Weierstrass elliptic doubly periodic solutions under the conditions that $\varrho_3 > 0$ and $\mathcal{A}_1\mathcal{L}_3\varrho_1 < 0$.

Case 7. When $\varrho_1 = \varrho_3 = 0$, then we found the following sets of solutions:

(7.1) $a_0 = a_1 = b_1 = \mathcal{L}_1 = \mathcal{L}_3 = \varrho_6 = 0$,
$$d_1 = \frac{\sqrt{-(n+1)\mathcal{A}_1}}{n\sqrt{\mathcal{L}_4}}, \qquad \mathcal{L}_2 = \frac{2\mathcal{A}_1\varrho_2}{n^2}, \qquad \mathcal{L}_0 = -\frac{(n^2-1)\mathcal{A}_1^2(\varrho_2^2 - 4\varrho_0\varrho_4)}{n^4\mathcal{L}_4},$$

(7.2) $a_0 = b_1 = d_1 = \mathcal{L}_1 = \mathcal{L}_3 = \varrho_6 = 0$,
$$a_1 = \frac{\sqrt{-(n+1)\mathcal{A}_1\varrho_4}}{n\sqrt{\mathcal{L}_4}}, \qquad \mathcal{L}_0 = -\frac{(n^2-1)\mathcal{A}_1^2\varrho_0\varrho_4}{n^4\mathcal{L}_4}, \qquad \mathcal{L}_2 = -\frac{\mathcal{A}_1\varrho_2}{n^2}.$$

Through the solution set (7.1), we can express the solutions of Eqs. (1) and (2) as follows: If $\varrho_2 > 0$ and $\mathcal{A}_1\mathcal{L}_4 < 0$, so, soliton solutions of the singular type can be found as

$$\Phi_{7.1}(x,t) = \left(\frac{\sqrt{-(n+1)\mathcal{A}_1\varrho_2}}{n\sqrt{\mathcal{L}_4}}\coth\left[(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{7.1}(x,t) = \Omega\left(\frac{\sqrt{-(n+1)\mathcal{A}_1\varrho_2}}{n\sqrt{\mathcal{L}_4}}\coth\left[(x-\eta t)\sqrt{\varrho_2}\right]\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)}.$$

Through the solution set (7.2), we can express the solutions of Eqs. (1) and (2) as follows: If $\varrho_2 > 0$ and $\mathcal{A}_1\mathcal{L}_4 < 0$, so, soliton solutions of the hyperbolic wave type can be found as

$$\Phi_{7.2}(x,t) = \left(\frac{\sqrt{-2(n+1)\mathcal{A}_1}}{n\sqrt{\mathcal{L}_4}}\sqrt{\frac{\varrho_2}{\cosh\bigl[2(x-\eta t)\sqrt{\varrho_2}\bigr]-1}}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

$$\Psi_{7.2}(x,t) = \Omega\left(\frac{\sqrt{-2(n+1)\mathcal{A}_1}}{n\sqrt{\mathcal{L}_4}}\sqrt{\frac{\varrho_2}{\cosh\bigl[2(x-\eta t)\sqrt{\varrho_2}\bigr]-1}}\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)}.$$

Case 8. When $\varrho_1 = \varrho_3 = \varrho_6 = 0$, then we found the following sets of solutions:

(8.1) $a_0 = a_1 = d_1 = \mathcal{L}_1 = \mathcal{L}_3 = 0,$

$$b_1 = \frac{\sqrt{(n+1)\mathcal{L}_2\varrho_0}}{\sqrt{\mathcal{L}_4\varrho_2}}, \qquad \mathcal{A}_1 = -\frac{n^2\mathcal{L}_2}{\varrho_2}, \qquad \mathcal{L}_0 = -\frac{(n^2-1)\mathcal{L}_2^2\varrho_0\varrho_4}{\mathcal{L}_4\varrho_2^2},$$

(8.2) $a_0 = a_1 = b_1 = \mathcal{L}_1 = \mathcal{L}_3 = \varrho_4 = 0,$

$$d_1 = \frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{2\mathcal{L}_4\varrho_2}}, \qquad \mathcal{A}_1 = \frac{n^2\mathcal{L}_2}{2\varrho_2}, \qquad \mathcal{L}_0 = -\frac{(n^2-1)\mathcal{L}_2^2}{4\mathcal{L}_4}.$$

Through the solution set (8.1), we can express the solutions of Eqs. (1) and (2) as follows: If $\varrho_0 = 1$, $\varrho_2 = -m^2-1$, $\varrho_4 = m^2$, $\mathcal{L}_2\mathcal{L}_4 < 0$, and $0 \leqslant m \leqslant 1$, then the solutions are:

$$\Phi_{8.1,1}(x,t) = \left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{-(m^2+1)\mathcal{L}_4}}\,\mathrm{ns}(x-\eta t)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

$$\Psi_{8.1,1}(x,t) = \Omega\left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{-(m^2+1)\mathcal{L}_4}}\,\mathrm{ns}(x-\eta t)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)}$$

or

$$\Phi_{8.1,2}(x,t) = \left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{-(m^2+1)\mathcal{L}_4}}\,\mathrm{dc}(x-\eta t)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

$$\Psi_{8.1,2}(x,t) = \Omega\left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{-(m^2+1)\mathcal{L}_4}}\,\mathrm{dc}(x-\eta t)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the Jacobi elliptic function solutions.

If $\varrho_0 = m^2-1$, $\varrho_2 = 2-m^2$, $\varrho_4 = -1$, $\mathcal{L}_2\mathcal{L}_4 < 0$, and $0 \leqslant m < 1$, then the solutions are:

$$\Phi_{8.1,3}(x,t) = \left(\frac{\sqrt{(m^2-1)(n+1)\mathcal{L}_2}}{\sqrt{-(m^2-2)\mathcal{L}_4}}\,\mathrm{nd}(x-\eta t)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

$$\Psi_{8.1,3}(x,t) = \Omega\left(\frac{\sqrt{(m^2-1)(n+1)\mathcal{L}_2}}{\sqrt{-(m^2-2)\mathcal{L}_4}}\,\mathrm{nd}(x-\eta t)\right)^{1/n}\mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the Jacobi elliptic function solutions.

If $\varrho_0 = -m^2$, $\varrho_2 = 2m^2 - 1$, $\varrho_4 = 1 - m^2$, $\mathcal{L}_2\mathcal{L}_4 < 0$, and $0 < m \leqslant 1$, then the solutions are:

$$\Phi_{8.1,4}(x,t) = \left(\frac{\sqrt{-m^2(n+1)\mathcal{L}_2}}{\sqrt{(2m^2-1)\mathcal{L}_4}}\,\mathrm{cn}(x-\eta t)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{8.1,4}(x,t) = \Omega\left(\frac{\sqrt{-m^2(n+1)\mathcal{L}_2}}{\sqrt{(2m^2-1)\mathcal{L}_4}}\,\mathrm{cn}(x-\eta t)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the Jacobi elliptic function solutions.

If we set $m = 1$, then

$$\Phi_{8.1,5}(x,t) = \left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}}\,\mathrm{sech}[x-\eta t]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)}, \tag{22}$$
$$\Psi_{8.1,5}(x,t) = \Omega\left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{\mathcal{L}_4}}\,\mathrm{sech}[x-\eta t]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the bright soliton solutions.

If $\varrho_0 = -1$, $\varrho_2 = 2 - m^2$, $\varrho_4 = m^2 - 1$, $\mathcal{L}_2\mathcal{L}_4 < 0$, and $0 \leqslant m \leqslant 1$, then the solutions are:

$$\Phi_{8.1,6}(x,t) = \left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{-(m^2-2)\mathcal{L}_4}}\,\mathrm{dn}(x-\eta t)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{8.1,6}(x,t) = \Omega\left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{-(m^2-2)\mathcal{L}_4}}\,\mathrm{dn}(x-\eta t)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the Jacobi elliptic function solutions.

If $\varrho_0 = 1/4$, $\varrho_2 = (m^2 - 2)/2$, $\varrho_4 = m^4/4$, $\mathcal{L}_2\mathcal{L}_4 < 0$, and $0 \leqslant m \leqslant 1$, then the solutions are:

$$\Phi_{8.1,7}(x,t) = \left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{2(m^2-2)\mathcal{L}_4}}\big(\mathrm{dn}(x-\eta t)+1\big)\,\mathrm{ns}(x-\eta t)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{8.1,7}(x,t) = \Omega\left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{2(m^2-2)\mathcal{L}_4}}\big(\mathrm{dn}(x-\eta t)+1\big)\,\mathrm{ns}(x-\eta t)\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)}$$

or

$$\Phi_{8.1,8}(x,t) = \left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{2(m^2-2)\mathcal{L}_4}}\big(\mathrm{dn}(x-\eta t)+\sqrt{1-m^2}\big)\,\mathrm{nc}(x-\eta t)\right)^{1/n} \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$
$$\Psi_{8.1,8}(x,t) = \Omega\left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{2(m^2-2)\mathcal{L}_4}}\big(\mathrm{dn}(x-\eta t)+\sqrt{1-m^2}\big)\,\mathrm{nc}(x-\eta t)\right)^{1/n} \times \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the Jacobi elliptic function solutions.

If we set $m = 1$, then

$$\Phi_{8.1,9}(x,t) = \left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{-2\mathcal{L}_4}} \coth\left[\frac{1}{2}(x-\eta t)\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

$$\Psi_{8.1,9}(x,t) = \Omega\left(\frac{\sqrt{(n+1)\mathcal{L}_2}}{\sqrt{-2\mathcal{L}_4}} \coth\left[\frac{1}{2}(x-\eta t)\right]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the singular soliton solutions.

Through the solution set (8.2), we can express the solutions of Eqs. (1) and (2) as follows:

If $\varrho_0 = -m^2$, $\varrho_2 = 2m^2 - 1$, $\varrho_4 = 1 - m^2$, and $\mathcal{L}_2\mathcal{L}_4 < 0$, then the solutions are:

$$\Phi_{8.2}(x,t) = \left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{2\mathcal{L}_4}} \tanh[x-\eta t]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

$$\Psi_{8.2}(x,t) = \Omega\left(\frac{\sqrt{-(n+1)\mathcal{L}_2}}{\sqrt{2\mathcal{L}_4}} \tanh[x-\eta t]\right)^{1/n} \mathrm{e}^{\mathrm{i}(-\kappa x+\omega t+\Delta)},$$

these solutions represent the dark soliton solutions.

4 Illustrations of the solutions graphically

To fully comprehend the physical structures of some extracted solutions to be shown, 2D, contour, and 3D figures of some special solutions are presented.

Figure 1 displays a dark soliton solution of Eq. (20) when $\mathcal{A}_1 = -1.4$, $\mathcal{C}_1 = -1.6$, $\Omega = 1.8$, $\mathcal{G}_1 = 1.8$, $n = 2$, $\Delta = 1.5$, $\varrho_2 = 0.9$, $\beta_1 = -1.9$, $\mathcal{Q}_1 = -1.1$, $\mathcal{Q}_2 = 1.4$, $\beta_2 = 1.5$, $\mathcal{A}_2 = 1.4$, and $-10 < x < 10$.

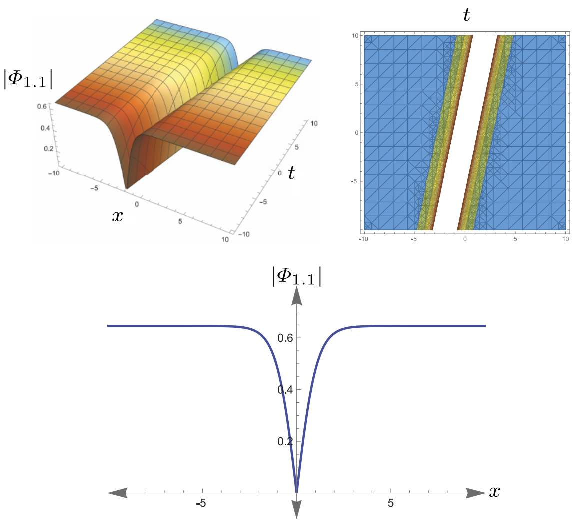

Figure 1. Equation (20): the graphical representation of 3D, contour, and 2D plots of dark soliton solution.

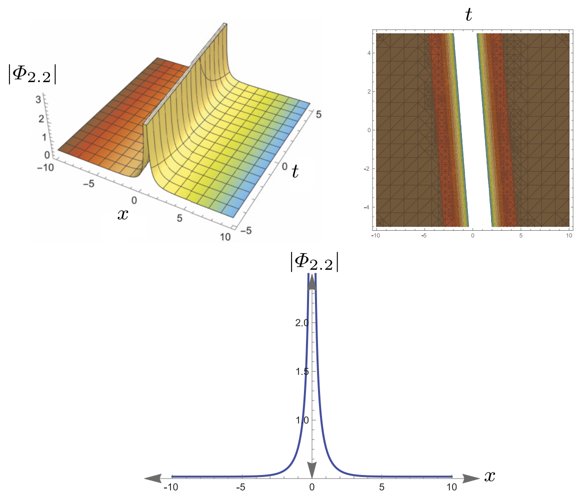

Figure 2. Equation (21): the graphical representation of 3D, contour, and 2D plots of singular soliton solution.

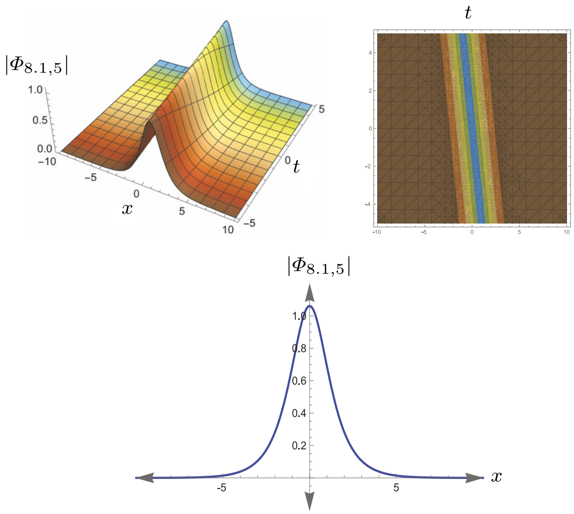

Figure 3. Equation (22): the graphical representation of 3D, contour, and 2D plots of bright soliton solution.

Figure 2 exhibits a singular soliton solution of Eq. (21) with selecting the parameters to be $\mathcal{A}_1 = -1.8$, $\Omega = -1.3$, $n = 3$, $\Delta = 1.5$, $\varrho_2 = -0.9$, $\beta_1 = 1.7$, $\mathcal{Q}_1 = 1.2$, $\vartheta_1 = 1.1$, $\lambda_1 = -1.9$, $\mathcal{K}_1 = -1.7$, $\mathcal{E}_1 = -1.1$, $\mathcal{Q}_2 = 1.5$, $\beta_2 = 1.1$, $\mathcal{A}_2 = -1.1$, and $-10 < x < 10$.

In addition, Eq. (22) is a bright soliton solution that is represented in Fig. 3 when $\mathcal{A}_1 = 1.7$, $\Omega = 1.5$, $n = 2.5$, $\Delta = 1.6$, $\beta_1 = 1.9$, $\vartheta_1 = 1.4$, $\lambda_1 = -1.5$, $\mathcal{K}_1 = 1.5$, $\mathcal{E}_1 = 1.6$, $\mathcal{Q}_1 = 1.8$, $\mathcal{Q}_2 = 1.7$, $\beta_2 = 1.5$, $\mathcal{A}_2 = 1.3$, and $-10 < x < 10$.

5 Conclusion

Magneto-optic elements can move light solitons from the state of attraction to the state of mutual isolation, thereby controlling cluttering of the soliton. Thus, the streamline flow of a soliton pulse across a transcontinental distance is realised, and the smooth propagation of a magneto-optical waveguide across an intercontinental distance is secured. The findings are useful in understanding soliton dynamics in magneto-optic waveguides. In the current work, we obtained the optical solitons and other wave solutions in magneto-optic waveguides with Kudryashov's law of refractive index by using the modified extended mapping method. Several different types of solutions such as bright solitons, dark solitons, singular solitons, singular periodic wave solutions, exponential wave solutions, rational wave solutions, Weierstrass elliptic doubly periodic solutions, and Jacobi elliptic function solutions were extracted. The graphs of different shapes have been sketched for the attained solutions, and some physical properties have been raised. The obtained results show that it is possible to balance the dispersive effects and nonlinearities to produce various solitary wave solutions that propagate while maintaining their speed and shape. By setting the different parameters to the suitable values, these solitary wave solutions can be produced and controlled. The obtained solutions are new, and the governing model is studied at first time by applying the proposed method.

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Conflicts of interest. The authors declare no conflicts of interest.

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