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The iterative properties of solutions for a singular k-Hessian system[*]

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Abstract. In this paper, we focus on the uniqueness and iterative properties of solutions for a singular k-Hessian system involving coupled nonlinear terms with different properties. Unlike the existing work, instead of directly dealing with the system, we use a coupled technique to transfer the Hessian system to an integral equation, and then by introducing an iterative technique, the iterative properties of solution are derived including the uniqueness of solution, iterative sequence, the error estimation and the convergence rate as well as entire asymptotic behaviour.

Keywords: Hessian equation, uniqueness, iterative properties, singularity.

1 Introduction

In this paper, we shall establish some new results on the uniqueness and iterative properties of radial solutions for the following singular coupled k-Hessian system:

$$\begin{aligned}
&(-1)^k S_k^{1/k}\bigl(\mu(D^2u)\bigr) = f_1\bigl(|x|, v\bigr) \quad \text{in } \Omega \subset \mathbb{R}^N \ (k < N < 2k),\\
&(-1)^k S_k^{1/k}\bigl(\mu(D^2v)\bigr) = f_2\bigl(|x|, u\bigr) \quad \text{in } \Omega \subset \mathbb{R}^N \ (k < N < 2k),\\
&u = v = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{1}$$

where Ω is a unit ball, and the nonlinear terms in the system have the opposite monotonicity, that is, $f_1 \in C((0,1)\times[0,+\infty),[0,+\infty))$ is increasing in the second variable, and

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$f_2 \in C((0,1) \times (0,+\infty), [0,+\infty))$ is decreasing in the second variable, so f_1, f_2 may be singular at $|x| = 0$, $|x| = 1$, and f_2 may have singularity at space variable $u = 0$.

In system (1), the operator $S_k(\mu(D^2u))$ is called Hessian operator, which is defined by the sum of the kth principal minors of the Hessian matrix D^2u, i.e.,

$$S_k\bigl(\mu\bigl(D^2u\bigr)\bigr) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant N} \mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}, \quad k = 1,2,\dots,N.$$

In particular,

$$S_1\bigl(\mu\bigl(D^2u\bigr)\bigr) = \sum_{i=1}^{N} \mu_i = \Delta u, \qquad S_N\bigl(\mu\bigl(D^2u\bigr)\bigr) = \prod_{i=1}^{N} \mu_i = \det\bigl(D^2u\bigr)$$

are the Laplace operator [23,25,32,55] and the Monge–Ampére operator [9,10,16,19,22, 26,28,29], respectively. However, for $1 < k < N$, $S_k(\mu(D^2u))$ is a second-order fully nonlinear differential operator. It is well known that the Hessian operator can describe the local curvature of a function of multiple variables, so has been usually applied to study some geometry problems such as the Weingarten curvature and reflector shape design [40] and Riemannian geometry [31,41] as well as quasilinear parabolic problems [8].

It is necessary to review some existing work related to system (1) for the convenience of readers. In the case $k = 1$, Lair [24] considered the existence of entire large solutions for the following system of semilinear equations:

$$\begin{aligned} &\Delta u = p\bigl(|x|\bigr)v^{\alpha}, \quad x \in R^N, \\ &\Delta v = q\bigl(|x|\bigr)u^{\beta}, \quad x \in R^N, \\ &\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = \infty, \end{aligned} \tag{2}$$

where p, q are nonnegative continuous functions, $0 < \alpha \leqslant 1$ and $0 < \beta \leqslant 1$. This implies that the nonlinear terms of system (2) possess the same monotonicity, and exhibit only a sublinear characteristic. By using an iterative technique and combining with some estimations, Lair proved a sufficient and necessary condition for system (2) to have a nonnegative entire large radial solution (u, v), namely, the functions p and q satisfy the following slow decay conditions:

$$\int_0^\infty tp(t)\Biggl(t^{2-n}\int_0^t s^{n-3}Q(s)\,\mathrm{d}s\Biggr)^{\alpha}\mathrm{d}t = \infty,$$
$$\int_0^\infty tq(t)\Biggl(t^{2-n}\int_0^t s^{n-3}P(s)\,\mathrm{d}s\Biggr)^{\beta}\mathrm{d}t = \infty,$$

where $P(r) = \int_0^r \tau p(\tau)\,\mathrm{d}\tau$ and $Q(r) = \int_0^r \tau q(\tau)\,\mathrm{d}\tau$. In our recent work [58], we generalized the work in [24] to the following more general modified quasilinear Schrödinger elliptic system with a nonconvex diffusion term:

$$\begin{aligned}&\Delta u+\Delta\big(|u|^{2\gamma}\big)|u|^{2\gamma-2}u=p\big(|x|\big)F(v)\chi_\gamma(u),\\&\Delta v+\Delta\big(|v|^{2\delta}\big)|v|^{2\delta-2}v=q\big(|x|\big)G(u)\chi_\delta(v),\\&\lim_{|x|\to\infty}u(x)=\lim_{|x|\to\infty}v(x)=\infty\quad\text{(i.e., } u,v \text{ are large)},\end{aligned}$$

where $x\in\mathbb{R}^N$ ($N\geqslant 3$), $\gamma,\delta>1/2$, $\chi_i(s)=\sqrt{1+2i|s|^{2(2i-1)}}$, $i>1/2$, and the nonnegative functions p and q are continuous on $\mathbb{R}^N$. F, G are also required to be increasing. When $k=N$, Loewner and Nirenberg [27] considered the existence of solutions for the Monge–Ampère equation

$$\det\big(D^2u\big)=u^{-4}\quad\text{in } \Omega,\qquad u=0\quad\text{on } \partial\Omega.$$

Recently, by using the regularity theory and sub-supersolution method, Lazer and McKenna [26] established a uniqueness result for the following Monge–Ampère equation:

$$\det\big(D^2u\big)=b(x)u^{-\gamma}\quad\text{in } \Omega,\qquad u=0\quad\text{on } \partial\Omega,\tag{3}$$

where $\gamma>1$ and $b\in C^\infty(\overline{\Omega})$ is positive. It was proven that there exist positive constants c_1, c_2 such that the unique solution $u\in C^2(\Omega)\cap C(\overline{\Omega})$ satisfies the following asymptotic property:

$$c_1d^{(N+1)/(N+\gamma)}(x)\leqslant u(x)\leqslant c_2d^{(N+1)/(N+\gamma)}(x)\quad\text{in } \Omega,$$

where $d(x)=\operatorname{dist}(x,\partial\Omega)$. In [30], Mohammed focused on the existence and the global estimates of solutions for the following Monge–Ampère equation:

$$\det\big(D^2u\big)=b(x)f(-u)\quad\text{in } \Omega,\qquad u=0\quad\text{on } \partial\Omega,$$

where $f\in C^\infty((0,\infty),(0,\infty))$ is decreasing, and $b\in C^\infty(\Omega)$ is positive. In our recent work [61], by adopting the sub-supersolution method, we established an eigenvalue interval for the existence of radial solutions for the following singular augmented Hessian equation:

$$\begin{aligned}&S_k^{1/k}\big(\mu\big(D^2u\big)+\sigma(x)I\big)=-f\big(|x|,u\big)\quad\text{in } B_1\subset\mathbb{R}^N\ (k\leqslant N<2k),\\&u=0\quad\text{on } \partial B_1,\end{aligned}$$

where B_1 is a unit ball, $f:[0,1]\times(0,+\infty)\to(0,+\infty)$ is continuous and nonincreasing in $u>0$. For some other related work, we refer the reader to the references [1,2,15,18, 21,33,43,44,54,57,60,62].

However, most of the works in the literature require that the nonlinear terms of the system possess the same character such as [24,58]. We also notice that $f(u)=u^\alpha$ in (2) and $f(u)=u^{-\gamma}$ in (3) belong to different type of problems. The former is increasing and nonsingular, however, the latter is decreasing and can be singular at $u=0$. Thus if the nonlinear terms of the system possess the above different properties, that is, in the coupling case of equations (2) and (3), a question arises, namely, does the solution of the system exist? If so, is the solution unique? To answer this question, in this paper, we will develop a double iterative technique to construct a new iterative process for deriving the uniqueness of solutions, an iterative sequence of solution, error estimation, convergence rate and entire asymptotic properties.

The rest of this paper is organized as follows. Some preliminaries and lemmas are given in Section 2. The main results are stated in Section 3. An example is given to illustrate our main results in Section 4.

2 Preliminary results on radial solutions

It is well known that for fully nonlinear differential equations, the best strategy to deal with them is using the theory and method of nonlinear analysis such as operator theories [7,13,14,17,39], spaces theories [3–5,11,34,35,46,48,49], smoothness theories [6,12,36–38,42,47], variational theory [45,50–52,59], fixed point theorem [20,53,63,64], sub-super solution method [61,62], semigroup approach [50], monotone iterative technique [56] etc. Thus according to this strategy, in this paper, we shall firstly employ operator theories and spaces theories to transform the k-Hessian system (1) to a convenient form and then construct a double iterative process to establish our main results.

Denote the unit open ball $\Omega := \{x \in \mathbb{R}^N\colon |x| < R\}$, let $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, we need the following lemmas.

Lemma 1. (*See* [21].) *Suppose that $\phi(r) \in C^2[0, R)$ is radially symmetric and $\phi'(0) = 0$. Then $u(|x|) = \phi(r) \in C^2(\Omega)$ satisfies*

$$\mu\big(D^2u\big) = \begin{cases} (\phi''(r), \frac{\phi'(r)}{r}, \dots, \frac{\phi'(r)}{r}), & r \in (0, R), \\ (\phi''(0), \phi''(0), \dots, \phi''(0)), & r = 0, \end{cases}$$

and

$$S_k\big(\mu\big(D^2u\big)\big) = \begin{cases} C_{N-1}^{k-1}\phi''(r)(\frac{\phi'(r)}{r})^{k-1} + C_{N-1}^{k}(\frac{\phi'(r)}{r})^k, & r \in (0, R), \\ C_N^k(\phi''(0))^k, & r = 0, \end{cases}$$

where $r = |x| < R$.

Make a radial symmetry transformation $(\phi(r), \psi(r)) = (u(|x|), v(|x|)$ for system (1), then the following lemma is a direct corollary of Lemma 1.

Lemma 2. *The k-Hessian system of equations* (1) *is equivalent to the following system of second-order ordinary differential equations*:

$$\begin{aligned} &C_{N-1}^{k-1}\big(-\phi''(r)\big)\left(-\frac{\phi'(r)}{r}\right)^{k-1} + C_{N-1}^{k}\left(-\frac{\phi'(r)}{r}\right)^{k} = f_1^k\big(r, \psi(r)\big), \quad r \in (0, 1), \\ &C_{N-1}^{k-1}\big(-\psi''(r)\big)\left(-\frac{\psi'(r)}{r}\right)^{k-1} + C_{N-1}^{k}\left(-\frac{\psi'(r)}{r}\right)^{k} = f_2^k\big(r, \phi(r)\big), \quad r \in (0, 1), \\ &\phi'(0) = \psi'(0) = 0, \qquad \phi(1) = \psi(1) = 0, \end{aligned} \tag{4}$$

that is, $(\phi(r), \psi(r))$ *is a solution of* (4) *if and only if* $(u(|x|), v(|x|)$ *is a classical solution of the* k*-Hessian system of equations* (1).

Now rewrite (4) by the following equivalent form:

$$\begin{aligned} &\left[\frac{r^{N-k}}{k}\bigl(-\phi'(r)\bigr)^k\right]' = \frac{r^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(r, \psi(r)\bigr), \quad r \in (0,1), \\ &\left[\frac{r^{N-k}}{k}\bigl(-\psi'(r)\bigr)^k\right]' = \frac{r^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(r, \phi(r)\bigr), \quad r \in (0,1), \\ &\phi'(0) = \psi'(0) = 0, \qquad \phi(1) = \psi(1) = 0. \end{aligned} \tag{5}$$

By integrating (5), one gets

$$\begin{aligned} \phi(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, \psi(s)\bigr)\,\mathrm{d}s\right)^{1/k} \mathrm{d}t, \quad r \in [0,1], \\ \psi(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(s, \phi(s)\bigr)\,\mathrm{d}s\right)^{1/k} \mathrm{d}t, \quad r \in [0,1]. \end{aligned} \tag{6}$$

Let

$$(S\phi)(s) = \psi(s) = \int_s^1 \left(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau, \phi(\tau)\bigr)\,\mathrm{d}\tau\right)^{1/k} d\xi, \quad s \in [0,1], \tag{7}$$

then system (6) can be converted into the following coupled nonlinear integral equation:

$$\phi(r) = \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, (S\phi)(s)\bigr)\,\mathrm{d}s\right)^{1/k} \mathrm{d}t, \quad r \in [0,1], \tag{8}$$

which implies that if $\phi(r)$ is a solution of the integral equation (8), then $(\phi(r), (S\phi)(r))$ is a radial classical solution of equation (4). Consequently, $u(|x|), v(|x|) = (\phi(r), \psi(r)) = (\phi(r), (S\phi)(r))$ is a radial classical solution of equation (1). So, in the following, we shall mainly focus on the integral equation (8).

Let $E = C[0,1]$. It is a Banach space with the norm $\|\phi\| = \max_{r\in[0,1]} |\phi(r)|$. Define a cone of E

$$P = \bigl\{\phi \in C[0,1]\colon\ \phi(r) \geqslant 0,\ r \in [0,1]\bigr\}.$$

Obviously, it is a normal cone of E with normality constant 1. Now define a nonlinear operator $T\colon E \to E$ by

$$(T\phi)(r) = \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, (S\phi)(s)\bigr)\,\mathrm{d}s\right)^{1/k} \mathrm{d}t, \quad r \in [0,1],$$

where $(S\phi)(s)$ is defined by (7). Thus a fixed point $\phi(r)$ of the operator T is a solution of the integral equation (8).

3 Main results

In order to proceed the iterative process, the following growth conditions on f_1 and f_2 will be adopted:

(F) $f_1 : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous and increasing in the second variable v, $f_2 : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous and decreasing in the second variable u and satisfies

$$\chi_{f_i} = \inf_{r\in(0,1)} f_i\bigl(r,\, 1 - r^{2-N/k}\bigr) > 0, \quad i = 1,2. \tag{9}$$

(G) For any $\sigma \in (0,1)$, there exist two constants α and $\beta > 0$ with $0 < \alpha\beta < 1$ such that, for any $(t,u) \in (0,1) \times (0,+\infty)$ and for any $(t,v) \in (0,1) \times [0,+\infty)$,

$$f_1(t,\sigma v) \geqslant \sigma^{\beta} f_1(t,v), \qquad f_2(t,\sigma u) \leqslant \sigma^{-\alpha} f_2(t,u). \tag{10}$$

Remark 1. $\sigma \geqslant 1$, (F) and (G) hold, and by simple computation, one has

$$\begin{aligned} &f_1(t,\sigma v) \leqslant \sigma^{\beta} f_1(t,v), \quad (t,v) \in (0,1) \times [0,+\infty), \\ &f_2(t,\sigma u) \geqslant \sigma^{-\alpha} f_2(t,u), \quad (t,u) \in (0,1) \times (0,+\infty). \end{aligned} \tag{11}$$

Remark 2. From condition (F) it is not difficult to see that f_1 and f_2 may be singular at $t = 0$ and $t = 1$, f_2 can be singular at $u = 0$. Let $N/k = 3/2$, a typical example is

$$\begin{aligned} f_1(t,v) &= \bigl(1 + t^{1/2}\bigr)^2 \bigl(1 + v^{1/2}\bigr), \\ f_2(t,u) &= t^{-1/3}\bigl(1 - t^{1/2}\bigr)^{3/2} u^{-3/2}. \end{aligned}$$

In fact, take $\alpha = 1/3$, $\beta = 2$, $\chi_{f_1} = \chi_{f_2} = 1$, then f_1 and f_2 satisfy assumptions (F) and (G).

Remark 3. We also give an example to illustrate the full singularity of f_1 and f_2.
Let $N/k = 3/2$ and

$$\begin{aligned} f_1(t,v) &= t^{-1/2}\bigl(1 - t^{1/2}\bigr)^{-1/4} v^{1/4}, \\ f_2(t,u) &= \bigl(1 - t^{1/2}\bigr)^{3/2}\bigl[t^{-1/2} + (1-t)^{-1/2}\bigr] u^{-3/2}. \end{aligned}$$

Thus f_1 has singularity at both $t = 0$ and $t = 1$, f_2 has singularity at both $t = 0$, $t = 1$ and $u = 0$. By simple calculation, we have

$$\chi_{f_1} = 1, \qquad \chi_{f_2} = 2\sqrt{2}.$$

Thus (F) holds.

Take $\alpha = 1/3$, $\beta = 2$, then $0 < \alpha\beta < 1$. For any $\sigma \in (0,1)$, we have

$$\begin{aligned} f_1(t,\sigma v) &= t^{-1/2}\bigl(1 - t^{1/2}\bigr)^{-1/4} \sigma^{1/4} v^{1/4} \\ &\geqslant t^{-1/2}\bigl(1 - t^{1/2}\bigr)^{-1/4} \sigma^{2} v^{1/4} = \sigma^{\beta} f_1(t,v) \end{aligned}$$

and

$$\begin{aligned} f_2(t,\sigma u) &= \left(1-t^{1/2}\right)^{3/2}\left[t^{-1/2}+(1-t)^{-1/2}\right]\sigma^{-3/2}u^{-3/2} \\ &\geqslant \left(1-t^{1/2}\right)^{3/2}\left[t^{-1/2}+(1-t)^{-1/2}\right]\sigma^{-1/3}u^{-3/2} \\ &= \sigma^{-\alpha} f_2(t,u). \end{aligned}$$

So f_1 and f_2 satisfy assumption (G).

In this paper, we shall carry out our work in the following subsets of P:

$$\begin{aligned} K = \Big\{&\phi \in P\colon \text{ there exists a number } 0 < l_\phi < 1 \text{ such that} \\ &l_\phi\left(1-r^{2-N/k}\right) \leqslant \phi(r) \leqslant \frac{1}{l_\phi}\left(1-r^{2-N/k}\right),\ r\in[0,1]\Big\}. \end{aligned}$$

Theorem 1. *Assume that* (F)–(G) *hold and* f_i, $i=1,2$, *satisfy the conditions*

$$0 < \int_0^1 s^{N-1} f_i^k\left(s,\ 1-s^{2-N/k}\right)\mathrm{d}s < +\infty, \quad i=1,2. \tag{12}$$

Then we have the following conclusions:

(i) *Uniquness. The singular* k*-Hessian system* (1) *has a unique classical solution* $(\phi^*, S\phi^*)$ *in* K;

(ii) *Iterative sequence. For any initial value* $\tilde\phi_0 \in K$, *construct the iterative sequences*

$$\tilde\phi_m(r) = \int_r^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\left(s, (S\tilde\phi_{m-1})(s)\right)\mathrm{d}s\right)^{1/k}\mathrm{d}t, \quad r\in[0,1],$$

$$(S\tilde\phi_{m-1})(r) = \int_r^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_2^k\left(s, \tilde\phi_{m-1}(s)\right)\mathrm{d}s\right)^{1/k}\mathrm{d}t, \quad r\in[0,1].$$

Then

$$\lim_{m\to+\infty}\tilde\phi_m(r) = \phi^*(r), \qquad \lim_{m\to+\infty}(S\phi_{m-1})(r) = (S\phi^*)(r)$$

uniformly hold for $r\in[0,1]$;

(iii) *Error estimation. For the component* ϕ^* *of the solution of the* k*-Hessian system* (1), *the error estimation between* ϕ^* *and the* m*th iterative value* $\tilde\phi_m$ *can be formulated by*

$$\|\tilde\phi_m - \phi^*\| \leqslant 2\zeta^{-1/2}\left(1-\zeta^{(\alpha\beta)^{2m}}\right),$$

where $0<\zeta<1$ *is a positive constant, and hence, a convergence rate is*

$$\|\tilde\phi_m - \phi^*\| = o\left(1-\zeta^{(\alpha\beta)^{2m}}\right);$$

(iv) *Entire asymptotic behaviour. The unique classical solution* $(\phi^*, S\phi^*)$ *of the* k*-Hessian system* (1) *has entire asymptotic estimation, i.e., there exist two positive constants* $0 < \kappa_1, \kappa_2 < 1$ *such that for any* $r \in [0,1]$,

$$\kappa_1\bigl(1 - r^{2-N/k}\bigr) \leqslant \phi^*(r) \leqslant \kappa_1^{-1}\bigl(1 - r^{2-N/k}\bigr),$$
$$\kappa_2\bigl(1 - r^{2-N/k}\bigr) \leqslant (S\phi^*)(r) \leqslant {\kappa_2}^{-1}\bigl(1 - r^{2-N/k}\bigr).$$

Proof. Firstly, we prove that the operator $T\colon K \to K$ is a completely continuous operator. To do this, for any $\phi \in K$, it follows from the definition of K that there exists a constant $0 < l_\phi < 1$ such that

$$l_\phi\bigl(1 - t^{2-N/k}\bigr) \leqslant \phi(t) \leqslant \frac{1}{l_\phi}\bigl(1 - t^{2-N/k}\bigr), \quad t \in [0,1]. \tag{13}$$

Take

$$l_\phi^* = \min\Biggl\{\frac{1}{3}, \frac{(2-\frac{N}{k})(C_{N-1}^{k-1})^{1/k}}{k^{1/k} l_\phi^{-\alpha}}\Biggl(\int_0^1 \tau^{N-1} f_2^k\bigl(\tau, 1 - \tau^{2-N/k}\bigr)\,\mathrm{d}\tau\Biggr)^{-1/k}, \frac{\chi_{f_2} l_\phi^\alpha k^{1/k}}{2(C_{N-1}^{k-1})^{1/k} N^{1/k}}\Biggr\}.$$

By using (10), (13), (12) and combining with the fact that $f_2(t,\phi)$ is decreasing in ϕ, we have

$$\begin{aligned}
(S\phi)(s) &= \int_s^1 \Biggl(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau, \phi(\tau)\bigr)\,\mathrm{d}\tau\Biggr)^{1/k} \mathrm{d}\xi \\
&\leqslant \int_s^1 \Biggl(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau, l_\phi\bigl(1 - \tau^{2-N/k}\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{1/k} \mathrm{d}\xi \\
&\leqslant l_\phi^{-\alpha} \int_s^1 \Biggl(\frac{k}{\xi^{N-k}} \int_0^1 \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau, 1 - \tau^{2-N/k}\bigr)\,\mathrm{d}\tau\Biggr)^{1/k} \mathrm{d}\xi \\
&\leqslant \frac{k^{1/k} l_\phi^{-\alpha}}{(2-\frac{N}{k})(C_{N-1}^{k-1})^{1/k}} \Biggl(\int_0^1 \tau^{N-1} f_2^k\bigl(\tau, 1 - \tau^{2-N/k}\bigr)\,\mathrm{d}\tau\Biggr)^{1/k} \bigl(1 - s^{2-N/k}\bigr) \\
&\leqslant \frac{1}{l_\phi^*}\bigl(1 - s^{2-N/k}\bigr), \quad s \in [0,1],
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
(S\phi)(s) &= \int_s^1 \Biggl(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau, \phi(\tau)\bigr)\,\mathrm{d}\tau\Biggr)^{1/k} \mathrm{d}\xi \\
&\geqslant \int_s^1 \Biggl(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau, l_\phi^{-1}\bigl(1 - \tau^{2-N/k}\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{1/k} d\xi
\end{aligned}$$

$$
\begin{aligned}
&\geqslant l_\phi^\alpha \int_s^1 \left(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau,\, 1-\tau^{2-N/k}\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}\xi \\
&\geqslant \chi_{f_2} l_\phi^\alpha \int_s^1 \left(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}}\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}\xi \\
&\geqslant \frac{\chi_{f_2} l_\phi^\alpha k^{1/k}}{2(C_{N-1}^{k-1})^{1/k} N^{1/k}} \bigl(1-s^2\bigr) \geqslant l_\phi^*\bigl(1-s^{2-N/k}\bigr) \geqslant 0, \quad s\in[0,1]. \qquad (15)
\end{aligned}
$$

Now according to the monotonicity of f_1 and (11), (14), one has

$$
\begin{aligned}
(T\phi)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, \bigl(S\phi\bigr)(s)\bigr)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\leqslant \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\left(s, \frac{1}{l_\phi^*}\bigl(1-s^{2-N/k}\bigr)\right) \mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\leqslant (l_\phi^*)^{-\beta} \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, 1-s^{2-N/k}\bigr)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\leqslant \frac{k^{1/k}(l_\phi^*)^{-\beta}}{(2-\frac{N}{k})(C_{N-1}^{k-1})^{1/k}} \left(\int_0^1 \tau^{N-1} f_1^k\bigl(\tau,\, 1-\tau^{2-N/k}\bigr)\,\mathrm{d}\tau \right)^{1/k} \bigl(1-r^{2-N/k}\bigr) \\
&< \infty. \qquad (16)
\end{aligned}
$$

Thus the operator T is uniformly bounded.

On the other hand, for any $0\leqslant r_1<r_2\leqslant 1$ and $\phi\in K$, it follows from (11), (12), (14) that

$$
\begin{aligned}
&\bigl|(T\phi)(r_1)-(T\phi)(r_2)\bigr| \\
&\quad\leqslant \left| \int_{r_1}^{r_2} \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, (S\phi)(s)\bigr)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t \right| \\
&\quad\leqslant \frac{(l_\phi^*)^{-\beta}}{(C_{N-1}^{k-1})^{1/k}} \left| \int_{r_1}^{r_2} \left(\frac{k}{t^{N-k}}\right)^{1/k} \mathrm{d}t \left(\int_0^1 s^{N-1} f_1^k\bigl(s,\, 1-s^{2-N/k}\bigr)\mathrm{d}s \right)^{1/k} \right| \\
&\quad\leqslant \frac{k^{1/k}(l_\phi^*)^{-\beta}}{(2-\frac{N}{k})(C_{N-1}^{k-1})^{1/k}} \left(\int_0^1 \tau^{N-1} f_1^k\bigl(s,\, 1-s^{2-N/k}\bigr)\,\mathrm{d}\tau \right)^{1/k} \\
&\qquad\times \bigl|r_2^{2-N/k}-r_1^{2-N/k}\bigr| \to 0, \quad |r_1-r_2|\to 0,
\end{aligned}
$$

which implies that $T(K)$ is equicontinuous.

Next, we show that $T(K) \subset K$. In fact, for any $\phi \in K$, it follows from (9), (11), (13) that

$$
\begin{aligned}
T\phi)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s, (S\phi)(s)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\geqslant \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s,\, l_\phi^*\big(1 - s^{2-N/k}\big)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\geqslant (l_\phi^*)^\beta \chi_{f_1} \left(\frac{k}{C_{N-1}^{k-1}} \right)^{1/k} \int_r^1 \left(\frac{1}{t^{N-k}} \int_0^t s^{N-1}\,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\geqslant \frac{1}{2} (l_\phi^*)^\beta \chi_{f_1} \left(\frac{k}{N C_{N-1}^{k-1}} \right)^{1/k} \big(1 - r^2\big) \\
&\geqslant \frac{1}{2} (l_\phi^*)^\beta \chi_{f_1} \left(\frac{k}{N C_{N-1}^{k-1}} \right)^{1/k} \big(1 - r^{2-N/k}\big).
\end{aligned} \tag{17}
$$

Take

$$
\begin{aligned}
\tilde{l}_{T_\phi} = \min\Biggl\{ \frac{1}{3},\, &\left\{ \frac{k^{1/k} (l_\phi^*)^{-\beta}}{(2 - \frac{N}{k})(C_{N-1}^{k-1})^{1/k}} \left(\int_0^1 \tau^{N-1} f_1^k\big(s,\, 1 - s^{2-N/k}\big)\,\mathrm{d}\tau \right)^{1/k} \right\}^{-1}, \\
&\frac{1}{2} (l_\phi^*)^\beta \chi_{f_1} \left(\frac{k}{N C_{N-1}^{k-1}} \right)^{1/k} \Biggr\}.
\end{aligned} \tag{18}
$$

Then it follows from (16) and (17) that

$$
\tilde{l}_{T_\phi}\big(1 - r^{2-N/k}\big) \leqslant (T\phi)(r) \leqslant \tilde{l}_{T_\phi}^{-1}\big(1 - r^{2-N/k}\big),
$$

which implies that $T(K) \subset K$. It is clear that T is a continuous operator, thus $T\colon K \to K$ is a completely continuous operator.

Secondly, by finding a special initial value, we shall construct an iterative process to establish the result of the uniqueness of classical solution of the k-Hessian system (1).

Take $\rho(r) = 1 - r^{2-N/k}$ and $l_\rho = 1/2$, then $\rho \in K$. Since $T(K) \subset K$, we have $T\rho \in K$. Thus by (18), we can choose a constant $0 < l_{T_\rho} < 1$ such that

$$
l_{T_\rho} \rho(r) \leqslant (T\rho)(r) \leqslant \frac{1}{l_{T_\rho}} \rho(r). \tag{19}
$$

Notice $0 < \alpha\beta < 1$, we have $\lim_{\gamma \to +\infty} 2^{-\gamma(1-\alpha\beta)} = 0$, which implies that there exists a sufficiently large positive constant γ_0 such that

$$
2^{-\gamma_0(1-\alpha\beta)} \leqslant l_{T_\rho}. \tag{20}
$$

We fix the initial value $\phi_0 = 2^{-\gamma_0}\rho(r)$ and denote

$$(S\phi_0)(s) = \int_s^1 \left(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\big(\tau,\, 2^{-\gamma_0}\rho(\tau)\big)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}\xi, \quad s \in [0,1]. \tag{21}$$

Now let us construct an iterative sequence

$$\begin{aligned}
\phi_1(r) = (T\phi_0)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s, (S\phi_0)(s)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t,\\
\phi_2(r) = (T\phi_1)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s, (S\phi_1)(s)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t,\\
&\dots\\
\phi_m(r) = (T\phi_{m-1})(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s, (S\phi_{m-1})(s)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t.
\end{aligned} \tag{22}$$

We assert that the above iterative sequence $\{\phi_m\}|_{m=0}^\infty$ satisfies the following inequalities:

$$\phi_0 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_{2m} \leqslant \cdots \leqslant \phi_{2m+1} \leqslant \cdots \leqslant \phi_3 \leqslant \phi_1. \tag{23}$$

Now we shall prove the above fact. Firstly, note that the operator T is decreasing in ϕ, thus, by using (19)–(22), we have

$$\begin{aligned}
&\phi_0(r) \leqslant \rho(r),\\
&\phi_1(r) = (T\phi_0)(r) \geqslant (T\rho)(r) \geqslant l_{T_\rho}\rho(r)\\
&\qquad \geqslant 2^{-\gamma_0(1-\alpha\beta)}\rho(r) = 2^{\alpha\beta\gamma_0}2^{-\gamma_0}\rho(r) \geqslant \phi_0(r)
\end{aligned}$$

and then

$$\phi_2(r) = (T\phi_1)(r) \leqslant (T\phi_0)(r) = \phi_1(r).$$

Consequently, it follows from (10) and (19)–(20) that

$$\begin{aligned}
\phi_1(r) = (T\phi_0)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s, (S\phi_0)(s)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t\\
&\leqslant \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s,\, 2^{\alpha\gamma_0}(S\rho)(s)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t\\
&\leqslant 2^{\alpha\beta\gamma_0} \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\big(s, (S\rho)(s)\big)\,\mathrm{d}s \right)^{1/k} \mathrm{d}t\\
&= 2^{\alpha\beta\gamma_0}(T\rho)(r) \leqslant 2^{\alpha\beta\gamma_0} l_{T_\rho}^{-1}\rho(r) \leqslant 2^{\gamma_0(1-\alpha\beta)}2^{\alpha\beta\gamma_0}\rho(r)\\
&= 2^{\gamma_0}\rho(r).
\end{aligned} \tag{24}$$

Now by using the monotonicity of T as well as (11), (19) and (24), we have

$$\begin{aligned}\phi_2(r) &= T\phi_1(r) \geqslant T\bigl(2^{\gamma_0}\rho(r)\bigr)\\ &= \int_r^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, \bigl(S2^{\gamma_0}\rho\bigr)(s)\bigr)\,\mathrm{d}s\right)^{1/k}\mathrm{d}t\\ &\geqslant \int_r^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, 2^{-\alpha\gamma_0}(S\rho)(s)\bigr)\,\mathrm{d}s\right)^{1/k}\mathrm{d}t\\ &\geqslant 2^{-\alpha\beta\gamma_0}T\rho(r) \geqslant 2^{-\alpha\beta\gamma_0}l_{T_\rho}\rho(r) \geqslant 2^{-\gamma_0}\rho(r) = \phi_0,\end{aligned}$$

which implies

$$\phi_0 \leqslant \phi_2 \leqslant \phi_1.$$

By induction, inequality (23) holds.

On the other hand, for any $c \in (0,1)$, by (G) and (11), one gets

$$T(c\phi) \leqslant c^{-\alpha\beta}T\phi, \qquad T^2(c\phi) \geqslant c^{(\alpha\beta)^2}T^2\phi. \tag{25}$$

Obviously, the operator T^2 is nondecreasing with respect to ϕ, thus it follows from (24) and (25) that

$$\begin{aligned}\phi_{2m} &= T\phi_{2m-1}(r) = T^{2m}\phi_0 = T^{2m}\bigl(2^{-\gamma_0}\rho(r)\bigr)\\ &= T^{2m}\bigl(2^{-2\gamma_0}2^{\gamma_0}\rho(r)\bigr) \geqslant T^{2m-2}\bigl(T^2\bigl(2^{-2\gamma_0}\phi_1(r)\bigr)\bigr)\\ &\geqslant T^{2m-2}\bigl(\bigl(2^{-2\gamma_0}\bigr)^{(\alpha\beta)^2}T^2\phi_1(r)\bigr) = T^{2m-4}T^2\bigl(\bigl(2^{-2\gamma_0}\bigr)^{(\alpha\beta)^2}T^2\phi_1(r)\bigr)\\ &\geqslant T^{2m-4}\bigl(\bigl(2^{-2\gamma_0}\bigr)^{(\alpha\beta)^4}T^4\phi_1(r)\bigr) \geqslant \cdots \geqslant \bigl(2^{-2\gamma_0}\bigr)^{(\alpha\beta)^{2m}}T^{2m}\phi_1(r)\\ &= (2^{-2\gamma_0})^{(\alpha\beta)^{2m}}T^{2m+1}\phi_0(r) = 2^{-2\gamma_0(\alpha\beta)^{2m}}\phi_{2m+1},\end{aligned}$$

which implies

$$2^{-2\gamma_0(\alpha\beta)^{2m}}\phi_{2m+1} \leqslant \phi_{2m} \leqslant \phi_{2m+1}.$$

Thus, for all $m, p \in \mathbb{N}$, one has

$$\begin{aligned}0 &\leqslant \phi_{2(m+p)}(r) - \phi_{2m}(r) \leqslant \phi_{2m+1}(r) - \phi_{2m}(r)\\ &\leqslant \bigl(1 - 2^{-2\gamma_0(\alpha\beta)^{2m}}\bigr)\phi_{2m+1} \leqslant \bigl(1 - 2^{-2\gamma_0(\alpha\beta)^{2m}}\bigr)\phi_1\\ &\leqslant \bigl(1 - 2^{-2\gamma_0(\alpha\beta)^{2m}}\bigr)2^{\gamma_0}\rho(r)\end{aligned} \tag{26}$$

and

$$\begin{aligned}0 &\leqslant \phi_{2m+1}(r) - \phi_{2(m+p)+1}(r) \leqslant \phi_{2m+1}(r) - \phi_{2m}(r)\\ &\leqslant \bigl(1 - 2^{-2\gamma_0(\alpha\beta)^{2m}}\bigr)2^{\gamma_0}\rho(r).\end{aligned} \tag{27}$$

Since P is a normal cone with normality constant 1, in view of (26), (27), we obtain

$$\|\phi_{m+p} - \phi_m\| \leqslant \bigl(1 - 2^{-2\gamma_0(\alpha\beta)^{2m}}\bigr)2^{\gamma_0} \to 0, \quad m \to +\infty,$$

which implies that $\{\phi_m\}$ is a Cauchy sequence of compact set K. Therefore there exists some $\phi^* \in K$ such that $\{\phi_m\} \to \phi^*$ as $m \to \infty$ satisfying $\phi_{2m} \leqslant \phi^* \leqslant \phi_{2m+1}$. Thus according to the monotonicity of T, we have

$$\phi_{2m+2} = T\phi_{2m+1} \leqslant T\phi^* \leqslant T\phi_{2m} = \phi_{2m+1}. \tag{28}$$

Taking the limit on both sides of (28), one gets $\phi^*(r) = T\phi^*(r)$, i.e., ϕ^* is a solution of the integral equation (8), and then $(\phi^*, S\phi^*)$ is a classical solution of the k-Hessian system (1).

Next, we show that the solution $(\phi^*, S\phi^*)$ of the k-Hessian system (1) is unique in K. Clearly, we only need to prove that $\phi^*(r)$ is unique in K. To do this, suppose that $(\tilde{\phi}, S\tilde{\phi})$ is another solution of the k-Hessian system (1). Let $\varrho_1 = \sup\{\varrho > 0 | \tilde{\phi} \geqslant \varrho\phi^*\}$, obviously, $\varrho_1 \in (0, +\infty)$. Next, we show $\varrho_1 \geqslant 1$. If not, one has $0 < \varrho_1 < 1$, which leads to

$$\tilde{\phi} = T\tilde{\phi} = T^2\tilde{\phi} \geqslant T^2(\varrho_1\phi^*) \geqslant \varrho_1^{(\alpha\beta)^2} T^2\phi^* = \varrho_1^{(\alpha\beta)^2}\phi^*.$$

It follows from definition of ϱ that $\varrho_1 > \varrho_1^{(\alpha\beta)^2}$. On the other hand, since $0 < \alpha\beta < 1$ and $0 < \varrho_1 < 1$, we have $\varrho_1^{(\alpha\beta)^2} > \varrho_1$, that is a contradiction. Therefore $\varrho_1 \geqslant 1$, which yields $\tilde{\phi} \geqslant \phi^*$. Following the same strategy, we also have $\tilde{\phi} \leqslant \phi^*$. Thus $\tilde{\phi} = \phi^*$, i.e., $\phi^*(r)$ is unique in K. Consequently, the solution $(\phi^*, S\phi^*)$ of the k-Hessian system (1) is unique in K.

Finally, we prove the iterative properties of the unique solution $(\phi^*, S\phi^*)$ to the k-Hessian system (1). We choose an initial value $\tilde{\phi}_0 \in K$ and construct an iterative sequence

$$\begin{aligned}\tilde{\phi}_m(r) &= (T\tilde{\phi}_{m-1})(r)\\ &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} f_1^k\bigl(s, \bigl(S\tilde{\phi}_{m-1}\bigr)(s)\bigr)\,\mathrm{d}s\right)^{1/k} \mathrm{d}t, \quad r \in [0,1],\end{aligned}$$

$$\begin{aligned}&(S\tilde{\phi}_{m-1})(s)\\ &\quad= \int_s^1 \left(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{\tau^{N-1}}{C_{N-1}^{k-1}} f_2^k\bigl(\tau, \tilde{\phi}_{m-1}(\tau)\bigr)\,\mathrm{d}\tau\right)^{1/k} \mathrm{d}\xi, \quad s \in [0,1],\end{aligned}$$

where $m = 1, 2, 3, \dots$. It follows from $T(K) \subset K$ that $\tilde{\phi}_1 = T\tilde{\phi}_0 \in K$, which implies that there exist two constants $l_{\tilde{\phi}_0}, l_{\tilde{\phi}_1} \in (0,1)$ such that

$$l_{\tilde{\phi}_0}\rho(r) \leqslant \tilde{\phi}_0(r) \leqslant \frac{1}{l_{\tilde{\phi}_0}}\rho(r), \quad l_{\tilde{\phi}_1}\rho(r) \leqslant \tilde{\phi}_1 = T\tilde{\phi}_0 \leqslant \frac{1}{l_{\tilde{\phi}_1}}\rho(r), \quad r \in [0,1]. \tag{29}$$

Since $\lim_{\gamma\to+\infty} 2^{-\gamma\alpha} = 0$, we can take a sufficiently large constant γ_0 such that

$$2^{-\alpha\gamma_0} \leqslant \min\{l_{\tilde{\phi}_0}, l_{\tilde{\phi}_1}, l_{T_\rho}\}. \tag{30}$$

Thus by (29) and (30), we have

$$\phi_0 = 2^{-\gamma_0}\rho(r) \leqslant 2^{-\alpha\gamma_0}\rho(r) \leqslant l_{\tilde{\phi}_0}\rho(r) \leqslant \tilde{\phi}_0,$$
$$\phi_0 = 2^{-\gamma_0}\rho(r) \leqslant 2^{-\alpha\gamma_0}\rho(r) \leqslant l_{\tilde{\phi}_1}\rho(r) \leqslant \tilde{\phi}_1,$$

which lead to

$$\tilde{\phi}_1 = T\tilde{\phi}_0 \leqslant T\phi_0 = \phi_1, \quad \phi_0 \leqslant \tilde{\phi}_1 \leqslant \phi_1, \quad \phi_2 \leqslant \tilde{\phi}_2 \leqslant \phi_1. \tag{31}$$

Thus by continuous iteration for (31), one derives

$$\begin{aligned} \phi_{2m}(r) &\leqslant \tilde{\phi}_{2m+1}(r) \leqslant \phi_{2m+1}(r), \\ \phi_{2m+2}(r) &\leqslant \tilde{\phi}_{2m+2}(r) \leqslant \phi_{2m+1}(r). \end{aligned} \tag{32}$$

Letting $m \to \infty$ in (32), we have that $\tilde{\phi}_m \to \phi^*$ and $S\tilde{\phi}_m \to S\phi^*$ uniformly hold for $r \in [0,1]$.

Moreover, it follows from (26), (27) and (32) that

$$\begin{aligned} \|\tilde{\phi}_{2m+1} - \phi^*\| &\leqslant \|\tilde{\phi}_{2m+1} - \phi_{2m}(r)\| + \|\phi_{2m}(r) - \phi^*\| \\ &\leqslant \|\phi_{2m+1} - \phi_{2m}(r)\| + \|\phi_{2m}(r) - \phi^*\| \\ &\leqslant 2\bigl(1 - 2^{-2\gamma_0(\alpha\beta)^{2m}} 2^{\gamma_0} \\ &= 2\zeta^{-1/2}\bigl(1 - \zeta^{(\alpha\beta)^{2m}}\bigr) \end{aligned}$$

and

$$\begin{aligned} \|\tilde{\phi}_{2m+2} - \phi^*\| &\leqslant \|\tilde{\phi}_{2m+2} - \phi_{2m+2}(r)\| + \|\phi_{2m+2}(r) - \phi^*\| \\ &\leqslant \|\phi_{2m+1} - \phi_{2m+2}(r)\| + \|\phi_{2m+2}(r) - \phi^*\| \\ &\leqslant 2\bigl(1 - 2^{-2\gamma_0(\alpha\beta)^{2m}}\bigr) 2^{\gamma_0} \\ &= 2\zeta^{-1/2}\bigl(1 - \zeta^{(\alpha\beta)^{2m}}\bigr), \end{aligned}$$

which imply that

$$\|\tilde{\phi}_m - \phi^*\| \leqslant 2\zeta^{-1/2}\bigl(1 - \zeta^{(\alpha\beta)^{2m}}\bigr),$$

where $0 < \zeta = (1/4)^{\gamma_0} < 1$ is a positive constant, which is determined by ρ and initial value $\tilde{\phi}_0$. In addition, we also have an exact convergence rate for ϕ^* that can be formulated by $\|\tilde{\phi}_m - \phi^*\| = o(1 - \zeta^{(\alpha\beta)^{2m}})$.

In the end, it follows from $\phi^* \in K$ and (14)–(17) that there exist two constants $0 < \kappa_1, \kappa_2 < 1$ such that for any $r \in [0,1]$,

$$\kappa_1\bigl(1 - r^{2-N/k}\bigr) \leqslant \phi^*(r) \leqslant {\kappa_1}^{-1}\bigl(1 - r^{2-N/k}\bigr),$$
$$\kappa_2\bigl(1 - r^{2-N/k}\bigr) \leqslant (S\phi^*)(r) \leqslant {\kappa_2}^{-1}\bigl(1 - r^{2-N/k}\bigr).$$

The proof is completed. □

4 An example

In this section, we give an example to illustrate our main results.

Example. Let $f_1(t,v) = (1+t^{1/2})^2(1+v^{1/2})$, $f_2(t,u) = t^{-1/3}(1-t^{1/2})^{3/2}u^{-3/2}$, and consider the following singular 3-Hessian system:

$$\begin{aligned}
&(-1)^3 S_3^{1/3}\big(\mu\big(D^2u\big)\big) = \big(1+t^{1/2}\big)^2\big(1+v^{1/2}\big) \quad \text{in } \Omega \subset \mathbb{R}^5,\\
&(-1)^3 S_3^{1/3}\big(\mu\big(D^2v\big)\big) = t^{-1/3}\big(1-t^{1/2}\big)^{3/2}u^{-3/2} \quad \text{in } \Omega \subset \mathbb{R}^5,\\
&u = v = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{33}$$

where Ω is an open unit ball. For the singular 3-Hessian system, the following conclusions hold:

(i) *Uniqueness*. The 3-Hessian system (33) has a unique classical radial solution $(\phi^*, S\phi^*)$ in K;

(ii) *Iterative schemes*. For any initial value $\tilde{\phi}_0 \in K$, construct the iterative sequences

$$\tilde{\phi}_m(r) = \int_r^1 \left(\frac{3}{t^2}\int_0^t \frac{s^4}{6}\big(1+s^{1/2}\big)^6\big(1+(S\tilde{\phi}_{m-1})^{1/2}(s)\big)^3\,\mathrm{d}s\right)^{1/3}\mathrm{d}t, \quad r \in [0,1],$$

$$(S\tilde{\phi}_{m-1})(r) = \int_r^1 \left(\frac{3}{t^2}\int_0^t \frac{s^3}{6}\big(1-s^{1/2}\big)^{9/2}\big(\tilde{\phi}_{m-1}(s)\big)^{-9/2}\,\mathrm{d}s\right)^{1/3}\mathrm{d}t, \quad r \in [0,1].$$

Then

$$\lim_{m\to+\infty} \tilde{\phi}_m(r) = \phi^*(r), \qquad \lim_{m\to+\infty} (S\phi_{m-1})(r) = (S\phi^*)(r)$$

uniformly hold for $r \in [0,1]$;

(iii) *Error estimation*. For the component ϕ^* of the solution $(\phi^*, S\phi^*)$ of the k-Hessian system (1), the error estimation between ϕ^* and the mth iterative value $\tilde{\phi}_m$ can be formulated by

$$\|\tilde{\phi}_m - \phi^*\| \leqslant 2\zeta^{-1/2}\big(1-\zeta^{(4/9)^m}\big),$$

where $0 < \zeta < 1$ is a positive constant. Moreover, there is a convergence rate

$$\|\tilde{\phi}_m - \phi^*\| = o\big(1-\zeta^{(4/9)^m}\big);$$

(iv) *Entire asymptotic behaviour*. The unique classical solution $(\phi^*, S\phi^*)$ of the k-Hessian system (33) has entire asymptotic estimation, i.e., there exist two positive constants $0 < \kappa_1, \kappa_2 < 1$ such that for any $r \in [0,1]$,

$$\begin{aligned}
&\kappa_1\big(1-r^{1/3}\big) \leqslant \phi^*(r) \leqslant {\kappa_1}^{-1}\big(1-r^{1/3}\big),\\
&\kappa_2\big(1-r^{1/3}\big) \leqslant (S\phi^*)(r) \leqslant {\kappa_2}^{-1}\big(1-r^{1/3}\big).
\end{aligned}$$

Proof. It follows from Remark 2 that f_1 and f_2 satisfy conditions (F) and (G). So we only need to check condition (12). In fact, since $k = 3$, $N = 5$, we have

$$0 < \int_0^1 s^4 f_1^3\bigl(s,\, 1 - s^{1/3}\bigr)\,\mathrm{d}s = \int_0^1 s^4\bigl(1 + s^{1/2}\bigr)^6\bigl(1 + (1 - s^{1/3})^{1/2}\bigr)^3\,\mathrm{d}s$$
$$= 4.1647 < +\infty$$

and

$$0 < \int_0^1 s^4 f_2^3\bigl(s,\, 1 - s^{1/3}\bigr)\,\mathrm{d}s = \int_0^1 s^3\bigl(1 - s^{1/2}\bigr)^{9/2}\bigl(1 - s^{1/3}\bigr)^{-9/2}\,\mathrm{d}s$$
$$= 0.0072 < +\infty,$$

which imply that (12) holds. Thus it follows from Theorem 1 that the above conclusions hold. □

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