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Constrained quantization for the Cantor distribution with a family of constraints

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CONSTRAINED QUANTIZATION FOR THE CANTOR DISTRIBUTION WITH A FAMILY OF CONSTRAINTS

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Abstract. In this paper, for a given family of constraints and the classical Cantor distribution we determine the optimal sets of n-points, nth constrained quantization errors for all positive integers n. We also calculate the constrained quantization dimension and the constrained quantization coefficient, and see that the constrained quantization dimension $D(P)$ exists as a finite positive number, but the $D(P)$-dimensional constrained quantization coefficient does not exist.

1. Introduction

Constrained quantization for a Borel probability measure refers to the idea of estimating a given probability by a discrete probability with a finite number of supporting points lying on a specific set. The specific set is known as the constraint of the constrained quantization. A quantization without a constraint is known as an unconstrained quantization, which traditionally in the literature is known as quantization. Constrained quantization has recently been introduced by Pandey and Roychowdhury (see [PR1, PR2]). Recently, they have also introduced the conditional quantization in both constrained and unconstrained quantization (see [PR4]). For some follow up papers in the direction of constrained quantization, one can see [PR3, BCDRV]). With the introduction of constrained quantization, quantization now has two classifications: constrained quantization and unconstrained quantization. For unconstrained quantization and its applications one can see [DFG, DR, GG, GL, GL1, GL2, GL3, GN, KNZ, P, P1, R1, R2, R3, Z1, Z2]. Constrained quantization has much more interdisciplinary applications in the areas such as information theory, machine learning and data compression, signal processing and national security.

Let P be a Borel probability measure on $\mathbb{R}^k$ equipped with a metric d induced by a norm $\|\cdot\|$ on $\mathbb{R}^k$, and $r \in (0, \infty)$. Let $\{S_j : j \in \mathbb{N}\}$ be a family of closed subsets of $\mathbb{R}^k$ such that S_1 is nonempty. The distortion error for P, of order r, with respect to a set $\alpha \subset \mathbb{R}^k$, denoted by $V_r(P; \alpha)$, is defined as

$$V_r(P; \alpha) = \int \min_{a \in \alpha} d(x, a)^r dP(x).$$

Then, for $n \in \mathbb{N}$, the *nth constrained quantization error* for P, of order r, with respect to the family of constraints $\{S_j : j \in \mathbb{N}\}$ is defined as

$$V_{n,r} := V_{n,r}(P) = \inf \Big\{ V_r(P; \alpha) : \alpha \subseteq \bigcup_{j=1}^{n} S_j, \ 1 \leq \text{card}(\alpha) \leq n \Big\}, \tag{1}$$

where card(A) represents the cardinality of the set A. The sets S_j are the constraints in the constrained quantization error. For the probability measure P, we make the standard assumption that $\int d(x, 0)^r dP(x) < \infty$. This ensures that there is a set $\alpha \subseteq \bigcup_{j=1}^{n} S_j$ for which the infimum in (1) exists (see [PR1]). A set $\alpha \subseteq \bigcup_{j=1}^{n} S_j$ for which the infimum in (1) exists and does not contain more than n elements is called an *optimal set of n-points* for P. The elements of an optimal set are called *optimal*

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elements. The two numbers $\underline{D}_r(P)$ and $\overline{D}_r(P)$, defined by

$$\begin{cases} \underline{D}_r(P) := \liminf\limits_{n\to\infty} \frac{r\log n}{-\log(V_{n,r}(P)-V_{\infty,r}(P))}, \text{ and} \\ \overline{D}_r(P) := \limsup\limits_{n\to\infty} \frac{r\log n}{-\log(V_{n,r}(P)-V_{\infty,r}(P))}, \end{cases} \tag{2}$$

where

$$V_{\infty,r}(P) := \lim_{n\to\infty} V_{n,r}(P),$$

are called the *lower* and the *upper constrained quantization dimensions* of the probability measure P of order r, respectively. If $\underline{D}_r(P) = \overline{D}_r(P)$, the common value is called the *constrained quantization dimension* of P of order r and is denoted by $D_r(P)$. The constrained quantization dimension measures the speed at which the specified measure of the constrained quantization error converges as n tends to infinity. For any $\kappa > 0$, the two numbers

$$\liminf_n n^{\frac{r}{\kappa}}(V_{n,r}(P) - V_{\infty,r}(P)) \text{ and } \limsup_n n^{\frac{r}{\kappa}}(V_{n,r}(P) - V_{\infty,r}(P))$$

are, respectively, called the *κ-dimensional lower* and *upper constrained quantization coefficients* for P. If the *κ-dimensional lower* and *upper constrained quantization coefficients* for P exist and are equal, then we call it the *κ-dimensional constrained quantization coefficient* for P.

This paper deals with $r = 2$ and $k = 2$, and the metric on $\mathbb{R}^2$ as the Euclidean metric induced by the Euclidean norm $\|\cdot\|$. Thus, instead of writing $V_r(P;\alpha)$ and $V_{n,r} := V_{n,r}(P)$ we will write them as $V(P;\alpha)$ and $V_n := V_n(P)$. Let us take the family $\{S_j : j \in \mathbb{N}\}$ of constraints, that occurs in (1) as follows:

$$S_j = \{(x,y) : -\frac{1}{j} \leq x \leq 1 \text{ and } y = x + \frac{1}{j}\} \tag{3}$$

for all $j \in \mathbb{N}$. Let $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ be two contractive similarity mappings such that $T_1(x) = \frac{1}{3}x$ and $T_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then, there exists a unique Borel probability measure P on $\mathbb{R}$ such that $P = \frac{1}{2}P\circ T_1^{-1} + \frac{1}{2}P\circ T_2^{-1}$, where $P\circ T_i^{-1}$ denotes the image measure of P with respect to S_i for $i = 1, 2$ (see [H]). If $k \in \mathbb{N}$, and $\sigma := \sigma_1\sigma_2\cdots\sigma_k \in \{1,2\}^k$, then we call σ a word of length k over the alphabet $I := \{1,2\}$, and denote it by $|\sigma| := k$. By I^*, we denote the set of all words including the empty word $\emptyset$. Notice that the empty word has length zero. For any word $\sigma := \sigma_1\sigma_2\cdots\sigma_k \in I^*$, we write

$$T_\sigma := T_{\sigma_1}\circ\cdots\circ T_{\sigma_k} \text{ and } J_\sigma := T_\sigma([0,1]).$$

Then, the set $C := \bigcap_{k\in\mathbb{N}}\bigcup_{\sigma\in\{1,2\}^k} J_\sigma$ is known as the *Cantor set* generated by the two mappings T_1 and T_2, and equals the support of the probability measure P, where P can be written as

$$P = \sum_{\sigma\in\{1,2\}^k} \frac{1}{2^k} P\circ T_\sigma^{-1}.$$

For this probability measure P, Graf and Luschgy determined the optimal sets of n-means and the nth quantization errors for all $n \in \mathbb{N}$ (see [GL2]). They also showed that the unconstrained quantization dimension of the measure P exists and equals $\frac{\log 2}{\log 3}$, which is the Hausdorff dimension of the Cantor set C, and the unconstrained quantization coefficient does not exist. In fact, in [GL2], they showed that the lower and the upper quantization coefficients exist as finite positive numbers.

Notice that a Borel probability measure P on $\mathbb{R}$ can also be considered as a Borel probability measure on $\mathbb{R}^2$ as $P(\mathbb{R}^2\setminus\mathbb{R}) = 0$. In this paper, with respect to the family of constraints $\{S_j : j \in \mathbb{N}\}$ for the Cantor distribution P we determine the optimal sets of n-points and the nth constrained quantization errors for all positive integers n. We further show that the constrained quantization dimension of the Cantor distribution P exists and equals two. Moreover, the value of the constrained quantization coefficient comes as infinity. From the work in this paper, we see that the constrained quantization dimension and the constrained quantization coefficient for the classical Cantor distribution depend on the family of constraints.

2. Preliminaries

In this section, we give some basic notations and definitions which we have used throughout this paper. As defined in the previous section, let $I := \{1, 2\}$ be an alphabet. For any two words $\sigma := \sigma_1\sigma_2 \cdots \sigma_k$ and $\tau := \tau_1\tau_2 \cdots \tau_\ell$ in I^*, by $\sigma\tau := \sigma_1 \cdots \sigma_k\tau_1 \cdots \tau_\ell$, we mean the word obtained from the concatenation of the two words σ and τ. For $\sigma, \tau \in I^*$, σ is called *an extension of* τ if $\sigma = \tau x$ for some word $x \in I^*$. The mappings $T_i : \mathbb{R} \to \mathbb{R},\ 1 \leq i \leq 2$, such that $T_1(x) = \frac{1}{3}x$ and $T_2 x = \frac{1}{3}x + \frac{2}{3}$ are the generating maps of the Cantor set C, which is the support of the probability measure P on $\mathbb{R}$ given by $P = \frac{1}{2}P \circ T_1^{-1} + \frac{1}{2}P \circ T_2^{-1}$. For $\sigma := \sigma_1\sigma_2 \cdots \sigma_k \in I^k$, write $J_\sigma = T_\sigma[0, 1]$, where $T_\sigma := T_{\sigma_1} \circ T_{\sigma_2} \circ \cdots \circ T_{\sigma_k}$ is a composition mapping. Notice that $J := J_\emptyset = T_\emptyset[0, 1] = [0, 1]$. Then, for any $k \in \mathbb{N}$, as mentioned before, we have

$$C = \bigcap_{k\in\mathbb{N}} \bigcup_{\sigma\in I^k} J_\sigma \text{ and } P = \sum_{\sigma\in I^k} \frac{1}{2^k} P \circ T_\sigma^{-1}.$$

The elements of the set $\{J_\sigma : \sigma \in I^k\}$ are the 2^k intervals in the kth level in the construction of the Cantor set C, and are known as the *basic intervals at the* k*th level.* The intervals $J_{\sigma1}$, $J_{\sigma2}$, into which J_σ is split up at the $(k+1)$th level are called the *children of* J_σ.

With respect to a finite set $\alpha \subset \mathbb{R}^2$, by the *Voronoi region* of an element $a \in \alpha$, it is meant the set of all elements in $\mathbb{R}^2$ which are nearest to a among all the elements in α, and is denoted by $M(a|\alpha)$. For any two elements (a, b) and (c, d) in $\mathbb{R}^2$, we write

$$\rho((a, b), (c, d)) := (a - c)^2 + (b - d)^2,$$

which gives the squared Euclidean distance between the two elements (a, b) and (c, d). Let p and q be two elements that belong to an optimal set of n-points for some positive integer n, and let e be an element on the boundary of the Voronoi regions of the elements p and q. Since the boundary of the Voronoi regions of any two elements is the perpendicular bisector of the line segment joining the elements, we have

$$\rho(p, e) - \rho(q, e) = 0.$$

We call such an equation a *canonical equation.* Notice that any element $x \in \mathbb{R}$ can be identified as an element $(x, 0) \in \mathbb{R}^2$. Thus,

$$\rho : \mathbb{R} \times \mathbb{R}^2 \to [0, \infty) \text{ such that } \rho(x, (a, b)) = (x - a)^2 + b^2, \tag{4}$$

where $x \in \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$, defines a nonnegative real-valued function on $\mathbb{R} \times \mathbb{R}^2$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ such that $\pi(a, b) = a$ for any $(a, b) \in \mathbb{R}^2$ denote the project mapping. For a random variable X with distribution P, let $E(X)$ represent the expected value, and $V := V(X)$ represent the variance of X.

The following lemmas are well-known (see [GL2]).

Lemma 2.1. *Let* $f : \mathbb{R} \to \mathbb{R}^+$ *be Borel measurable and* $k \in \mathbb{N}$*. Then*

$$\int f dP = \sum_{\sigma\in\{1,2\}^k} p_\sigma \int f \circ S_\sigma dP.$$

Lemma 2.2. *Let* X *be a random variable with probability distribution* P*. Then,* $E(X) = \frac{1}{2}$ *and* $V := V(X) = E\|X - \frac{1}{2}\|^2 = E(X - \frac{1}{2})^2 = \frac{1}{8}$*. Moreover, for any* $x_0 \in \mathbb{R}$*, we have*

$$\int (x - x_0)^2 dP(x) = V(X) + (x - \frac{1}{2})^2.$$

Remark 2.3. For words $\beta, \gamma, \cdots, \delta$ in I^*, by $a(\beta, \gamma, \cdots, \delta)$ we mean the conditional expectation of the random vector X given $J_\beta \cup J_\gamma \cup \cdots \cup J_\delta$, i.e.,

$$a(\beta, \gamma, \cdots, \delta) = E(X : X \in J_\beta \cup J_\gamma \cup \cdots \cup J_\delta) = \frac{1}{P(J_\beta \cup \cdots \cup J_\delta)} \int_{J_\beta\cup\cdots\cup J_\delta} x\, dP.$$

Recall Lemma 2.1. For each $\sigma \in I^*$, since T_σ is a similarity mapping, we have

$$a(\sigma) = E(X : X \in J_\sigma) = \frac{1}{P(J_\sigma)} \int_{J_\sigma} x\, dP = \int_{J_\sigma} x d(P \circ T_\sigma^{-1}) = \int T_\sigma(x)\, dP$$

$$= E(T_\sigma(X)) = T_\sigma(E(X)) = T_\sigma(\frac{1}{2}).$$

In this paper, we investigate the constrained quantization for the family of constraints given by

$$S_j = \{(x, y) : -\frac{1}{j} \leq x \leq 1 \text{ and } y = x + \frac{1}{j}\} \text{ for all } j \in \mathbb{N}, \tag{5}$$

i.e., the constraints S_j are the line segments joining the points $(-\frac{1}{j}, 0)$ and $(1, 1+\frac{1}{j})$ which are parallel to the line $y = x$. The perpendicular on a constraint S_j passing through a point $(x, x + \frac{1}{j}) \in S_j$ intersects the real line at the point $2x + \frac{1}{j}$ if $-\frac{1}{j} \leq x \leq 1$; and it intersects J if $0 \leq 2x + \frac{1}{j} \leq 1$, i.e., if

$$-\frac{1}{2j} \leq x \leq \frac{1}{2} - \frac{1}{2j}. \tag{6}$$

Thus, for all $j \in \mathbb{N}$, there exists a one-one correspondence between the elements $(x, x + \frac{1}{j})$ on S_j and the elements $2x + \frac{1}{j}$ on the real line if $-\frac{1}{j} \leq x \leq 1$. Thus, for all $j \in \mathbb{N}$, there exist bijective mappings U_j such that

$$U_j(x, x + \frac{1}{j}) = 2x + \frac{1}{j} \text{ and } U_j^{-1}(x) = \Big(\frac{1}{2}(x - \frac{1}{j}), \frac{1}{2}(x - \frac{1}{j}) + \frac{1}{j}\Big), \tag{7}$$

where $-\frac{1}{j} \leq x \leq 1$.

The following lemma plays an important role in the paper.

Lemma 2.4. *Let $\alpha_n \subseteq \bigcup_{j=1}^{n} S_j$ be an optimal set of n-points for P such that*

$$\alpha_n := \{(a_j, b_j) : 1 \leq j \leq n\},$$

where $a_1 < a_2 < a_3 < \cdots < a_n$ and π be the projection mapping. Then, $(a_j, b_j) = U_n^{-1}(E(X : X \in \pi(M((a_j, b_j)|\alpha_n))))$, where $M((a_j, b_j)|\alpha_n)$ are the Voronoi regions of the elements (a_j, b_j) with respect to the set α_n for $1 \leq j \leq n$.

Proof. Let $\alpha_n := \{(a_j, b_j) : 1 \leq j \leq n\}$, as given in the statement of the lemma, be an optimal set of n-points. Take any $(a_q, b_q) \in \alpha_n$. Since $\alpha_n \subseteq \bigcup_{j=1}^{n} S_j$, we can assume that $(a_q, b_q) \in S_t$ for some $1 \leq t \leq n$. Since the Voronoi region of (a_q, b_q), i.e., $M((a_q, b_q)|\alpha_n)$ has positive probability, $M((a_q, b_q)|\alpha_n)$ contains some basic intervals from J that generates the Cantor set C. Let $J_{\sigma^{(j)}}$, where $1 \leq j \leq k$ for some positive integer k, be all the basic intervals that are contained in $M((a_q, b_q)|\alpha_n)$. Now, the distortion error contributed by (a_q, b_q) in its Voronoi region $M((a_q, b_q)|\alpha_n)$ is given by

$$\begin{aligned}
&\int_{\pi(M((a_q,b_q)|\alpha_n))} \rho(x, (a_q, b_q))\, dP \\
&= \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \int_{J_{\sigma^{(j)}}} \rho(x, (a_q, b_q))\, d(P \circ T_{\sigma^{(j)}}^{-1}) \\
&= \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \frac{1}{9^{\ell(\sigma^{(j)})}} V + \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \rho(T_{\sigma^{(j)}}(\frac{1}{2}), (a_q, a_q + \frac{1}{t})) \\
&= \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \frac{1}{9^{\ell(\sigma^{(j)})}} V + \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \Big((T_{\sigma^{(j)}}(\frac{1}{2}) - a_q)^2 + (a_q + \frac{1}{t})^2\Big) \\
&= \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \frac{1}{9^{\ell(\sigma^{(j)})}} V + \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \Big(2a_q^2 - 2a_q(T_{\sigma^{(j)}}(\frac{1}{2}) - \frac{1}{t}) + (T_{\sigma^{(j)}}(\frac{1}{2}))^2 + \frac{1}{t^2}\Big) \\
&= \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \frac{1}{9^{\ell(\sigma^{(j)})}} V + \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \frac{1}{2}\Big(\Big(2a_q - (T_{\sigma^{(j)}}(\frac{1}{2}) - \frac{1}{t})\Big)^2 + \Big(T_{\sigma^{(j)}}(\frac{1}{2}) + \frac{1}{t}\Big)^2\Big).
\end{aligned}$$

Notice that the above expression is minimum if both the expressions

$$\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \frac{1}{2}\Big(2a_q - (T_{\sigma^{(j)}}(\frac{1}{2}) - \frac{1}{t})\Big)^2 \text{ and } \sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \frac{1}{2}\Big(T_{\sigma^{(j)}}(\frac{1}{2}) + \frac{1}{t}\Big)^2$$

are minimum. Since $1 \leq t \leq n$, both the expressions are minimum if $t = n$. Once $t = n$, the first expression can further be minimized if

$$\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} \Big((2a_q + \frac{1}{n}) - T_{\sigma^{(j)}}(\frac{1}{2})\Big) = 0$$

yielding

$$2a_q + \frac{1}{n} = \frac{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} T_{\sigma^{(j)}}(\frac{1}{2})}{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}}}.$$

Thus, we have

$$a_q = \frac{1}{2}\Big(\frac{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} T_{\sigma^{(j)}}(\frac{1}{2})}{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}}} - \frac{1}{n}\Big) \text{ and } b_q = \frac{1}{2}\Big(\frac{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} T_{\sigma^{(j)}}(\frac{1}{2})}{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}}} - \frac{1}{n}\Big) + \frac{1}{n}$$

implying

$$(a_q, b_q) = U_n^{-1}\Big(\frac{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}} T_{\sigma^{(j)}}(\frac{1}{2})}{\sum_{j=1}^{k} \frac{1}{2^{\ell(\sigma^{(j)})}}}\Big) = U_n^{-1}(\pi(E(\mathbf{X} : \mathbf{X} \in M((a_q, b_q)|\alpha_n)))).$$

Since $(a_q, b_q) \in \alpha_n$ is chosen arbitrarily, the proof of the lemma is complete. □

Remark 2.5. By (6) and (7), and Lemma 2.4, we can conclude that all the elements in an optimal set of n-points must lie on S_n between the two elements $U_n^{-1}(0)$ and $U_n^{-1}(1)$, i.e., between the two elements $(-\frac{1}{2n}, \frac{1}{2n})$ and $(\frac{n-1}{2n}, \frac{n+1}{2n})$. If this fact is not true, then the constrained quantization error can be strictly reduced by moving the elements in the optimal set between the elements $(-\frac{1}{2n}, \frac{1}{2n})$ and $(\frac{n-1}{2n}, \frac{n+1}{2n})$ on S_n, in other words, the x-coordinates of all the elements in an optimal set of n-points must lie between the two numbers $-\frac{1}{2n}$ and $\frac{n-1}{2n}$.

Lemma 2.6. *We have* $1 + 5 + 13 + 17 + 37 + 41 + 49 + 53 + \cdots$ *up to* 2^k*-terms* $= 6^k$*, and* $1^2 + 5^2 + 13^2 + 17^2 + 37^2 + 41^2 + 49^2 + 53^2 + \cdots$ *up to* 2^k*-terms* $= 2^{k-1}(9^k 3 - 1)$.

Proof. For $k \in \mathbb{N} \cup \{0\}$ let us define sets $\mathfrak{C}_0 = 1$ and $\mathfrak{C}_k = \mathfrak{C}_{k-1} \cup (\mathfrak{C}_{k-1} + (3^{k-1}4))$. Then, notice that $\mathfrak{C}_k = \{1, 5, 13, 17, \ldots \text{up to } 2^k\text{-terms}\}$. Let us next define the moment function for the required sum as follows

$$\mathcal{M}_m(k) = \sum_{x \in \mathfrak{C}_k} x^m.$$

Then, for $m = 0$, we get

$$\mathcal{M}_0(k) = \sum_{x \in \mathfrak{C}_k} x^0 = 2^k.$$

For $m = 1$, we have

$$\begin{aligned} \mathcal{M}_1(k) = \sum_{x \in \mathfrak{C}_k} x &= \sum_{x \in \mathfrak{C}_{k-1}} (x + (x + 3^{k-1}4)) \\ &= \sum_{x \in \mathfrak{C}_{k-1}} x + \sum_{x \in \mathfrak{C}_{k-1}} x + \sum_{x \in \mathfrak{C}_{k-1}} 3^{k-1}4 \\ &= 2\mathcal{M}_1(k-1) + 2^{k-1}3^{k-1}4 \\ &= 2^2\mathcal{M}_1(k-2) + 2^{k-1}3^{k-2}4 + 2^{k-1}3^{k-1}4 \\ &\ \vdots \end{aligned}$$

$$
\begin{aligned}
&=2^k\mathcal{M}_1(0)+2^{k-1}4+2^{k-1}3\cdot 4+2^{k-1}3^24+\cdots+2^{k-1}3^{k-2}4+2^{k-1}3^{k-1}4\\
&=2^k\cdot 1+2^{k-1}4(1+3+3^2+\cdots+3^{k-1})\\
&=2^k+2^{k+1}\left(\frac{3^k-1}{2}\right)=6^k.
\end{aligned}
$$

For $m=2$, we have

$$
\begin{aligned}
\mathcal{M}_2(k)=\sum_{x\in\mathfrak{C}_k}x^2&=\sum_{x\in\mathfrak{C}_{k-1}}(x^2+(x+3^{k-1}4)^2)\\
&=\sum_{x\in\mathfrak{C}_{k-1}}x^2+\sum_{x\in\mathfrak{C}_{k-1}}(x^2+2x\cdot 3^{k-1}4+9^{k-1}16)\\
&=2\sum_{x\in\mathfrak{C}_{k-1}}x^2+3^{k-1}8\sum_{x\in\mathfrak{C}_{k-1}}x+\sum_{x\in\mathfrak{C}_{k-1}}9^{k-1}16\\
&=2\mathcal{M}_2(k-1)+3^{k-1}8\mathcal{M}_1(k-1)+2^{k-1}9^{k-1}16\\
&=2\mathcal{M}_2(k-1)+3^{k-1}6^{k-1}8+18^{k-1}16\\
&=2\mathcal{M}_2(k-1)+18^{k-1}24\\
&=2^2\mathcal{M}_2(k-2)+2\cdot 18^{k-2}24+18^{k-1}24\\
&=2^3\mathcal{M}_2(k-3)+2^218^{k-3}24+18^{k-1}24\\
&\ \vdots\\
&=2^k\mathcal{M}_2(0)+2^{k-1}24+2^{k-2}18\cdot 24+\cdots+2^218^{k-3}24+18^{k-1}24\\
&=2^k\cdot 1+2^{k-1}24(1+9+9^2+\cdots+9^{k-1})\\
&=2^k+2^{k-1}24\left(\frac{9^k-1}{8}\right)=2^{k-1}(9^k3-1).
\end{aligned}
$$

Therefore, $1+5+13+17+37+41+49+53+\cdots$ up to 2^k-terms $=6^k$, and $1^2+5^2+13^2+17^2+37^2+41^2+49^2+53^2+\cdots$ up to 2^k-terms $=2^{k-1}(3\cdot 9^k-1)$. Thus, the proof of the lemma is complete. $\square$

Definition 2.7. For $n\in\mathbb{N}$ with $n\geq 2$, let $\ell(n)$ be the unique natural number with $2^{\ell(n)}\leq n<2^{\ell(n)+1}$. Let U_n be the mappings given by (7). For $I\subset\{1,2\}^{\ell(n)}$ with $\text{card}(I)=n-2^{\ell(n)}$ let $\alpha_n(I)\subseteq S_n$ be the set such that

$$\alpha_n(I)=\{U_n^{-1}(a(\sigma)):\sigma\in\{1,2\}^{\ell(n)}\setminus I\}\cup\{U_n^{-1}(a(\sigma 1)):\sigma\in I\}\cup\{U_n^{-1}(a(\sigma 2)):\sigma\in I\}.$$

Proposition 2.8. *Let $\alpha_n(I)$ be the set given in Definition 2.7. Then, the number of such sets is ${}^{2^{\ell(n)}}C_{n-2^{\ell(n)}}$, and the corresponding distortion error is given by*

$$V(P;\alpha_n(I))=\int\min_{a\in\alpha_n(I)}\rho(x,a)\,dP=\frac{1}{18^{\ell(n)}}V\Big(2^{\ell(n)+1}-n+\frac{1}{9}(n-2^{\ell(n)})\Big)+A,$$

where V is the variance as given by Lemma 2.2, and

$$A=\sum_{\sigma\in\{1,2\}^{\ell(n)}\setminus I}\frac{1}{2^{\ell(n)}}\rho(a(\sigma),U_n^{-1}(a(\sigma)))+\sum_{\sigma\in I}\frac{1}{2^{\ell(n)+1}}\Big(\rho(a(\sigma 1),U_n^{-1}(a(\sigma 1)))+\rho(a(\sigma 2),U_n^{-1}(a(\sigma 2)))\Big).$$

Proof. If $2^{\ell(n)}\leq n<2^{\ell(n)+1}$, then the subset I can be chosen in ${}^{2^{\ell(n)}}C_{n-2^{\ell(n)}}$ different ways, and so, the number of such sets is given by ${}^{2^{\ell(n)}}C_{n-2^{\ell(n)}}$, and the corresponding distortion error is obtained as

$$
\begin{aligned}
V(P;\alpha_n(I))&=\int\min_{a\in\alpha_n(I)}\rho(x,a)\,dP\\
&=\sum_{\sigma\in\{1,2\}^{\ell(n)}\setminus I}\int_{J_\sigma}\rho(x,U_n^{-1}(a(\sigma)))\,dP
\end{aligned}
$$

$$+\sum_{\sigma\in I}\Big(\int_{J_{\sigma 1}}\rho(x, U_n^{-1}(a(\sigma 1)))\,dP+\int_{J_{\sigma 2}}\rho(x, U_n^{-1}(a(\sigma 2)))\,dP\Big)$$
$$=\sum_{\sigma\in\{1,2\}^{\ell(n)}\setminus I}\frac{1}{2^{\ell(n)}}\int\rho(T_\sigma(x), U_n^{-1}(a(\sigma)))\,dP$$
$$+\sum_{\sigma\in I}\frac{1}{2^{\ell(n)+1}}\Big(\int\rho(T_{\sigma 1}(x), U_n^{-1}(a(\sigma 1)))\,dP+\int\rho(T_{\sigma 2}(x), U_n^{-1}(a(\sigma 2))\,dP\Big)$$
$$=\sum_{\sigma\in\{1,2\}^{\ell(n)}\setminus I}\frac{1}{2^{\ell(n)}}\Big(\frac{1}{9^{\ell(n)}}V+\rho(a(\sigma), U_n^{-1}(a(\sigma)))\Big)$$
$$+\sum_{\sigma\in I}\frac{1}{2^{\ell(n)+1}}\Big(\frac{2}{9^{\ell(n)+1}}V+\rho(a(\sigma 1), U_n^{-1}(a(\sigma 1)))+\rho(a(\sigma 2), U_n^{-1}(a(\sigma 2)))\Big)$$
$$=\frac{1}{18^{\ell(n)}}V\Big(2^{\ell(n)+1}-n+\frac{1}{9}(n-2^{\ell(n)})\Big)+A,$$

where

$$A=\sum_{\sigma\in\{1,2\}^{\ell(n)}\setminus I}\frac{1}{2^{\ell(n)}}\rho(a(\sigma), U_n^{-1}(a(\sigma)))+\sum_{\sigma\in I}\frac{1}{2^{\ell(n)+1}}\Big(\rho(a(\sigma 1), U_n^{-1}(a(\sigma 1)))+\rho(a(\sigma 2), U_n^{-1}(a(\sigma 2)))\Big).$$

Thus, the proof of the proposition is complete. □

The following corollary is a consequence of Proposition 2.8.

Corollary 2.9. *Let A be the expression given in Proposition 2.8. Then, if n is of the form $n = 2^{\ell(n)}$ for some positive integer $\ell(n) \in \mathbb{N}$, we have*

$$A=\frac{2^{\ell(n)}+1}{2\cdot 4^{\ell(n)}}+\frac{3\cdot 9^{\ell(n)}-1}{16\cdot 9^{\ell(n)}}.$$

Proof. Let $n \in \mathbb{N}$ be such that n is of the form $n = 2^{\ell(n)}$ for some positive integer $\ell(n) \in \mathbb{N}$. Notice that for $\sigma \in \{1, 2\}^{\ell(n)}$, by (4) we have

$$\rho(a(\sigma), U_n^{-1}(a(\sigma)))=\rho\Big(a(\sigma),\Big(\frac{1}{2}(a(\sigma)-\frac{1}{n}),\frac{1}{2}(a(\sigma)-\frac{1}{n})+\frac{1}{n}\Big)\Big)=\frac{1}{2}(a(\sigma)+\frac{1}{n})^2.$$

Thus, using Lemma 2.6, we have

$$A=\sum_{\sigma\in\{1,2\}^{\ell(n)}}\frac{1}{2^{\ell(n)}}\frac{1}{2}(a(\sigma)+\frac{1}{2^{\ell(n)}})^2=\sum_{\sigma\in\{1,2\}^{\ell(n)}}\frac{1}{2^{\ell(n)}}\frac{1}{2}\Big((a(\sigma))^2+2a(\sigma)\cdot\frac{1}{2^{\ell(n)}}+\frac{1}{4^{\ell(n)}}\Big)$$
$$=\frac{1}{2^{\ell(n)+1}}\cdot\frac{1}{(2\cdot 3^{\ell(n)})^2}\Big(1^2+5^2+13^2+17^2+37^2+41^2+49^2+53^2+\cdots\text{ up to }2^{\ell(n)}\text{-terms}\Big)$$
$$+\frac{1}{4^{\ell(n)}}\cdot\frac{1}{2\cdot 3^{\ell(n)}}\Big(1+5+13+17+37+41+49+53+\cdots\text{ up to }2^{\ell(n)}\text{-terms}\Big)+\frac{1}{2}\cdot\frac{1}{4^{\ell(n)}}$$
$$=\frac{1}{2\cdot 4^{\ell(n)}}+\frac{6^{\ell(n)}}{(2\cdot 3^{\ell(n)})\,4^{\ell(n)}}+\frac{2^{\ell(n)-1}\left(3\cdot 9^{\ell(n)}-1\right)}{2^{\ell(n)+1}\left(2\cdot 3^{\ell(n)}\right)^2}$$
$$=\frac{2^{\ell(n)}+1}{2\cdot 4^{\ell(n)}}+\frac{3\cdot 9^{\ell(n)}-1}{16\cdot 9^{\ell(n)}}.$$

Thus, the proof of the corollary is yielded. □

In the next sections, we give the main results of the paper.

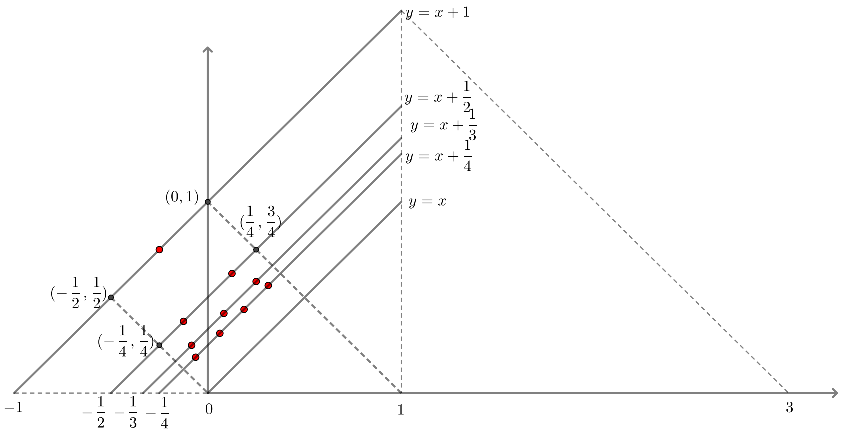

FIGURE 1. Points in the optimal sets of n-points for $1 \leq n \leq 4$.

3. Main Results

In this section, Theorem 3.4, Theorem 3.6, and Theorem 3.7 contain all the main results of the paper.

Proposition 3.1. *An optimal set of one-point is* $\{(-\frac{1}{4}, \frac{3}{4})\}$ *with constrained quantization error* $V_1 = \frac{5}{4}$.

Proof. Let $\alpha := \{(a, b)\}$ be an optimal set of one-point. Since $\alpha \subseteq S_1$, we have $b = a + 1$. Now, the distortion error for P with respect to the set α is give by

$$V(P; \alpha) = \int \rho((x, 0), (a, a + 1))dP = 2a^2 + a + \frac{11}{8},$$

the minimum value of which is $\frac{5}{4}$ and it occurs when $a = -\frac{1}{4}$. Thus, an optimal set of one-point is $\{(-\frac{1}{4}, \frac{3}{4})\}$ with constrained quantization error $V_1 = \frac{5}{4}$, which is the proposition. $\square$

The following proposition is known.

Proposition 3.2. *(see [GL2]) For $n \geq 2$, let $\alpha_n(I)$ be the set given by Definition 2.7, and for each $j \in \mathbb{N}$, let U_j be the bijective mapping as defined by (7). Then, the set*

$$U_n(\alpha_n(I)) = \{a(\sigma) : \sigma \in \{1, 2\}^{\ell(n)} \setminus I\} \cup \{a(\sigma 1) : \sigma \in I\} \cup \{a(\sigma 2) : \sigma \in I\}$$

forms an optimal set of n-means for the Cantor distribution P with the nth unconstrained quantization error

$$V(P; U_n(\alpha_n(I))) = \frac{1}{18^{\ell(n)}} V\Big(2^{\ell(n)+1} - n + \frac{1}{9}(n - 2^{\ell(n)})\Big).$$

Proposition 3.3. *The bijective mappings U_n preserves the Voronoi regions with respect to the probability measure P, i.e., for any discrete $\beta \subset \mathbb{R}$, and $a \in \beta$, we have*

$$P(M(a|\beta)) = P(M(U_n^{-1}(a)|U_n^{-1}(\beta))).$$

Proof. Since $\beta \subset \mathbb{R}$, for any $a \in \beta$, we can write $M(a|\beta) = [c, d]$ for some $c, d \in \mathbb{R}$ with $c < d$. Notice that the bijective mapping U_n preserves the order, i.e., for any $(e, e + \frac{1}{n})$, $(f, f + \frac{1}{n}) \in S_n$ if $e < f$, then $U_n(e, e + \frac{1}{n}) < U_n(f, f + \frac{1}{n})$. Moreover, for any $(e, e + \frac{1}{n}) \in S_n$, $U_n(e, e + \frac{1}{n})$ represents the point on J where the perpendicular on S_n at $(e, e + \frac{1}{n})$ intersects J. Hence, we can say that the boundary of the Voronoi region $M(U_n^{-1}(a)|U_n^{-1}(\beta))$ intersects S_n at the points given by $U_n^{-1}(c)$ and $U_n^{-1}(d)$, i.e., $M(U_n^{-1}(a)|U_n^{-1}(\beta))$ contains the closed interval $[c, d]$ as a subset, i.e.,

$$M(a|\beta) \subset M(U_n^{-1}(a)|U_n^{-1}(\beta)).$$

Since $P(M(U_n^{-1}(a)|U_n^{-1}(\beta)) \setminus M(a|\beta)) = 0$, we have $P(M(a|\beta)) = P(M(U_n^{-1}(a)|U_n^{-1}(\beta)))$. Thus, the proof of the proposition is complete. $\square$

The following theorem gives the optimal sets of n-points for all positive integers $n \geq 2$ for the Cantor distribution P with respect to the family of constraints $\{S_j : j \in \mathbb{N}\}$.

Theorem 3.4. *For $n \geq 2$, let $\alpha_n(I)$ be the set given by Definition 2.7. Then, $\alpha_n(I)$ forms an optimal set of n-points for P with nth constrained quantization error*

$$V_n = V(P; \alpha_n(I)) = V(P; U_n(\alpha_n(I))) + A.$$

Proof. To prove that $\alpha_n(I)$ forms an optimal set of n-points for P, it is enough to prove the fact that $\alpha_n(I)$ forms an optimal set of n-points for P if and only if $U_n(\alpha_n(I))$ forms an optimal set of n-means for P. The fact is clearly true by Proposition 3.2 and Proposition 3.3. Hence, $\alpha_n(I)$ forms an optimal set of n-points for P (see Figure 1). Then, by Proposition 2.8 and Proposition 3.2, we have the nth constrained quantization error as

$$V_n = V(P; \alpha_n(I)) = V(P; U_n(\alpha_n(I))) + A.$$

Thus, the proof of the theorem is complete. □

We need the following proposition, which is a special case of Theorem 3.4, to prove Theorem 3.6 and Theorem 3.7.

Proposition 3.5. *Let $n \in \mathbb{N}$ be such that $n = 2^{\ell(n)}$ for some positive integer $\ell(n)$. Then, the set*

$$\alpha_n(I) = \{U_n^{-1}(a(\sigma)) : \sigma \in \{1,2\}^{\ell(n)}\}$$

forms an optimal set of $2^{\ell(n)}$-points with constrained quantization error

$$V_{2^{\ell(n)}}(P) = \frac{1}{16}\left(2^{3-2\ell(n)} + 2^{3-\ell(n)} + 9^{-\ell(n)} + 3\right).$$

Proof. Let $n = 2^{\ell(n)}$ for some positive integer $\ell(n)$. By Theorem 3.4, it follows that the set $\{U_n^{-1}(a(\sigma)) : \sigma \in \{1,2\}^{\ell(n)}\}$ forms an optimal set of n-points. By Proposition 3.2 and Theorem 3.4, and Corollary 2.9, it follows that the nth constrained quantization error is

$$V_{2^{\ell(n)}}(P) = \frac{V}{9^{\ell(n)}} + \frac{2^{\ell(n)}+1}{2\cdot 4^{\ell(n)}} + \frac{3\cdot 9^{\ell(n)}-1}{16\cdot 9^{\ell(n)}},$$

which yields

$$V_{2^{\ell(n)}}(P) = \frac{1}{16}\left(2^{3-2\ell(n)} + 2^{3-\ell(n)} + 9^{-\ell(n)} + 3\right).$$

Thus, the proof of the proposition is complete. □

Theorem 3.6. *The constrained quantization dimension $D(P)$ of the probability measure P exists, and $D(P) = 2$.*

Proof. For $n \in \mathbb{N}$ with $n \geq 2$, let $\ell(n)$ be the unique natural number such that $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. Then, $V_{2^{\ell(n)+1}} \leq V_n \leq V_{2^{\ell(n)}}$. By Proposition 3.5, we see that $V_{2^{\ell(n)+1}} \to \frac{3}{16}$ and $V_{2^{\ell(n)}} \to \frac{3}{16}$ as $n \to \infty$, and so $V_n \to \frac{3}{16}$ as $n \to \infty$, i.e., $V_\infty = \frac{3}{16}$. We can take n large enough so that $(V_{2^{\ell(n)}} - V_\infty) < 1$. Then,

$$0 < -\log(V_{2^{\ell(n)}} - V_\infty) \leq -\log(V_n - V_\infty) \leq -\log(V_{2^{\ell(n)+1}} - V_\infty)$$

yielding

$$\frac{2\ell(n)\log 2}{-\log(V_{2^{\ell(n)+1}} - V_\infty)} \leq \frac{2\log n}{-\log(V_n - V_\infty)} \leq \frac{2(\ell(n)+1)\log 2}{-\log(V_{2^{\ell(n)}} - V_\infty)}.$$

Notice that

$$\lim_{n\to\infty} \frac{2\ell(n)\log 2}{-\log(V_{2^{\ell(n)+1}} - V_\infty)} = \lim_{n\to\infty} \frac{2\ell(n)\log 2}{-\log(\frac{1}{16}\left(2^{2-\ell(n)} + 2^{3-2(\ell(n)+1)} + 9^{-\ell(n)-1} + 3\right) - \frac{3}{16})}$$

implying

$$\lim_{n\to\infty} \frac{2\ell(n)\log 2}{-\log(V_{2^{\ell(n)+1}} - V_\infty)} = 2. \text{ Similarly, } \lim_{n\to\infty} \frac{2(\ell(n)+1)\log 2}{-\log(V_{2^{\ell(n)}} - V_\infty)} = 2.$$

Hence, $\lim_{n\to\infty} \frac{2\log n}{-\log(V_n - V_\infty)} = 2$, i.e., the constrained quantization dimension $D(P)$ of the probability measure P exists and $D(P) = 2$. Thus, the proof of the theorem is complete. □

Theorem 3.7. *The $D(P)$-dimensional constrained quantization coefficient for P is infinity.*

Proof. For $n \in \mathbb{N}$ with $n \geq 2$, let $\ell(n)$ be the unique natural number such that $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. Then, $V_{2(\ell(n)+1)} \leq V_n \leq V_{2\ell(n)}$, and $V_\infty = \lim_{n\to\infty} V_n = \frac{3}{16}$. Since

$$\lim_{n\to\infty} n^2(V_n - V_\infty) \geq \lim_{n\to\infty} (2^{\ell(n)})^2 (V_{2^{\ell(n)+1}} - V_\infty)$$
$$= \lim_{n\to\infty} (2\ell(n))^2 \Big(\frac{1}{16}\left(2^{2-\ell(n)} + 2^{3-2(\ell(n)+1)} + 9^{-\ell(n)-1} + 3\right) - \frac{3}{16}\Big) = \infty, \text{ and}$$
$$\lim_{n\to\infty} n^2(V_n - V_\infty) \leq \lim_{n\to\infty} (2^{\ell(n)+1})^2 (V_{2^{\ell(n)}} - V_\infty)$$
$$= \lim_{n\to\infty} (2^{\ell(n)+1})^2 \Big(\frac{1}{16}\left(2^{3-2\ell(n)} + 2^{3-\ell(n)} + 9^{-\ell(n)} + 3\right) - \frac{3}{16}\Big) = \infty,$$

by squeeze theorem, we have $\lim_{n\to\infty} n^2(V_n - V_\infty) = \infty$, which is the theorem. □

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