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Convergence of Lagrange-Hermite Interpolation Using Non-uniform Nodes on the Unit Circle

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Abstract

In this research article, we brought into consideration the set of non-uniformly distributed nodes on the unit circle to investigate a Lagrange-Hermite interpolation problem. These nodes are obtained by projecting vertically the zeros of Jacobi polynomial onto the unit circle along with the boundary points of the unit circle on the real line. Explicitly representing the interpolatory polynomial as well as establishment of convergence theorem are the key highlights of this manuscript. The result proved are of interest to approximation theory.

Keywords: Lagrange interpolation; Hermite interpolation; Unit circle; Jacobi polynomial; Rate of convergence; Non-uniform nodes

MSC 2010 No.: 41A10, 97N50, 41A05, 30E10

1. Introduction

Approximation of continous functions can be done using different methods by constructing algebraic or trigonometric polynomials. Polynomial interpolation has remained a ceaseless topic of research since decades due to its various numerical applications such as forming the basis for algorithms in numerical derivation and numerical quadrature. Lagrange and Hermite interpolation are among the profuse approaches to peruse polynomial interpolation. A lot of literature (see Bahadur and Varun (2018), Berriochoa et al. (2021), Bahadur and Varun (2022) and Bahadur and Bano (2022)) got accumulated discussing the Lagrange and Hermite techniques of interpolation and providing important results on the convergence of interpolatory polynomials. This research article focuses on the intermediate problem between the Lagrange interpolation and the Hermite interpolation.

Lagrange-Hermite interpolation: It is the process of finding a polynomial, which coincides with the continuous function at certain pre-assigned points, called the nodes of interpolation, and its derivative coinciding with not all of the nodes of the nodal system.

Trefethen (2011) broke the myths about the polynomial interpolation and quoted that polynomial interpolants always converge if the function is smooth a little bit in Chebyshev points. Thus, the function is required to be imposed by some restrictions such as altering conditions on its modulus of continuity in order so that sequence of Lagrange-Hermite interpolation polynomials attains finer properties of the convergence.

Kiš (1960) was the initiator of interpolation processes on the unit circle. He considered the Lacunary interpolation on the n^{th} roots of unity. Daruis and González-Vera (2000) took the roots of complex numbers with modulus one as the nodal points and, using a suitable modulus of continuity, obtained a result about convergence of the interpolants for continuous functions.

Berriochoa et al. (2016) studied the Lagrange-Hermite interpolation on the unit circle (LHIUC) by prescribing the Lagrange values at the $2n$ roots of a complex number with modulus one and prescribing values for the first derivative only on half of the nodes. They obtained two different types of expressions for the interpolatory polynomials and provided sufficient conditions in order to obtain convergence in case of continous functions.

Apart from the uniform nodal system (in the sense that nodes are equally spaced on the unit circle), LHIUC have also been studied on some non-uniformly distributed nodes on the unit circle. Bahadur and Varun (2017) studied LHIUC by projecting vertically the zeros of Legendre polynomial of degree n together with the end points of the unit circle on the real line.

In the present paper, we consider a LHIUC problem on the nodal system constituted of projections of the zeros of Jacobi polynomial of degree n vertically onto the unit circle together with the end points of the unit circle on the real line . The novelty of this research article is that it aims to extend the work of Bahadur and Varun (2017) by considering Jacobi polynomial instead of Legendre polynomial in choosing the nodal points.

The paper has been organized in following manner. Preliminaries are given in Section 2. Section 3 introduces the interpolation problem and its regularity. Section 4 covers explicit representation of the interpolatory polynomial. Section 5 is devoted to finding estimates whereas the convergence theorem, and its proof has been assigned Section 6. Numerical experiment and the conclusions of the research article are provided in the Section 7 and Section 8, respectively.

2. Preliminaries

This section includes the following well known results, which we shall use. Throughout this research paper, we denote the Jacobi polynomial of degree n by $P_n^{(\alpha,\beta)}(x)$.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is,

$$(1-x^2)P_n^{(\alpha,\beta)''}(x) + [\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x) + n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = 0, \quad (1)$$

where $x = \frac{1+z^2}{2z}$.

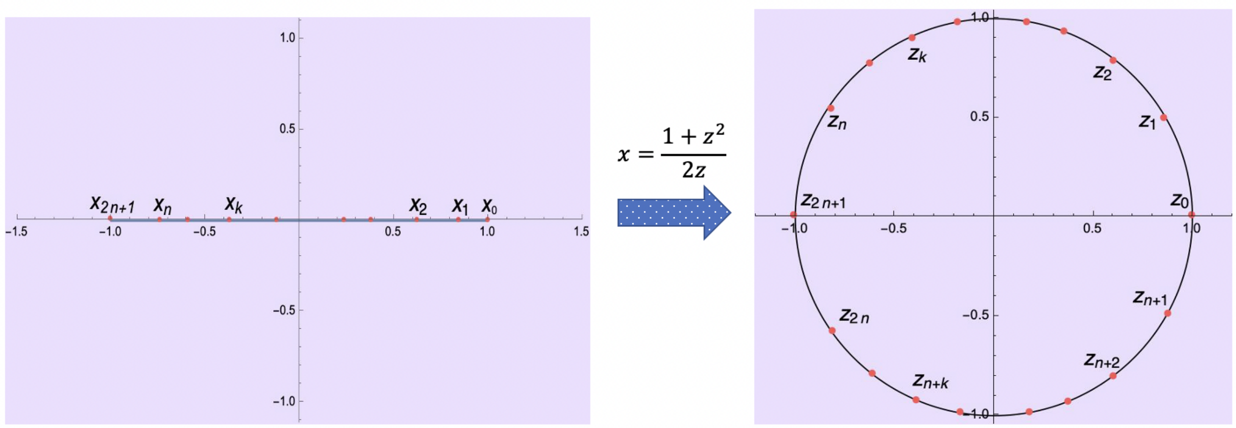

Figure 1. Szegő transformation

Let $\mathbb{Z}_{2n}$ be the set of nodes which are obtained by projecting vertically the zeros of $P_n^{(\alpha,\beta)}(x)$ on the unit circle,

$$\mathbb{Z}_{2n} = \{z_k = x_k + iy_k = \cos\theta_k + i\sin\theta_k\,;\ z_{n+k} = \bar{z}_k\,;\ k = 1, 2, ..., n;\ x_k,\ y_k \in\ R\}, \quad (2)$$

where, R is the set of real numbers. Also, the polynomial defined on $\mathbb{Z}_{2n}$ is given by

$$\mathbb{W}(z) = \prod_{k=1}^{2n}(z - z_k) = K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n, \quad (3)$$

where

$$K_n = 2^{2n} n! \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}.$$

The polynomial defined on $\mathbb{Z}_{2n} \cup \{-1, 1\}$ is given by

$$\mathbb{R}(z) = (z^2-1)\mathbb{W}(z). \quad (4)$$

The fundamental polynomial of Lagrange interpolation on the zeros of $\mathbb{W}(z)$ and $\mathbb{R}(z)$ is given by (5) and (6), respectively,

$$\mathscr{L}_k(z) = \frac{\mathbb{W}(z)}{(z-z_k)\mathbb{W}'(z_k)}, \qquad k = 1(1)2n, \quad (5)$$

$$\mathbb{L}_k(z) = \frac{\mathbb{R}(z)}{(z-z_k)\mathbb{R}'(z_k)}, \qquad k = 0(1)2n+1. \tag{6}$$

Here, $z_0 = 1$ and $z_{2n+1} = -1$.

If $|z| = 1$, then

$$|z^2 - 1| = 2\sqrt{1-x^2}. \tag{7}$$

We also use well known results (Szegő (1975)).

For $-1 \leq x \leq 1$ and $\alpha \geq \beta$,

$$(1-x^2)^{1/2} \mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha-1}), \tag{8}$$

$$\mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha}). \tag{9}$$

Considering the set of nodes $\mathbb{Z}_{2n}$ such that for each k, $x_k \in (-1,1)$, we have

$$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{10}$$

$$\mid P_n^{(\alpha,\beta)'}(x_k) \mid \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \tag{11}$$

$$|z_k - 1| = \sqrt{2(1-x_k)}. \tag{12}$$

3. The Problem and the Regularity

We are interested in determining the interpolatory polynomial $\mathcal{H}_n(z)$ of degree $\leq 2n+3$ satisfying the following conditions:

$$\begin{cases} \mathcal{H}_n(f, z_k) = f(z_k), & k = 0(1)2n+1, \\ \mathcal{H}_n'(f, \pm 1) = \beta_{\pm 1}, \end{cases} \tag{13}$$

where $f(z_k)$ and $\beta_{\pm 1}$ are arbitrary complex constants.

Theorem 3.1.

$\mathcal{H}_n(z)$ is regular on $\mathbb{Z}_{2n} \bigcup \{-1, 1\}$.

Proof:

It is sufficient if we show the unique solution of (13) is $\mathcal{H}_n(z) \equiv 0$.

Let us consider

$$\mathcal{H}_n(z) = \mathbb{R}(z)\, q(z), \tag{14}$$

where $q(z)$ is a linear polynomial.

Since, $\mathbb{R}(z_k) = 0$ for $k = 0(1)2n+1$ from (4), this implies $\mathcal{H}_n(z_k) = 0$ for $k = 0(1)2n+1$.

On differentiating (14) with respect to z, we have

$$\mathcal{H}'_n(z) = \mathbb{R}(z)\, q'(z) + \mathbb{R}'(z)\, q(z). \tag{15}$$

Since $\mathcal{H}'_n(\pm 1) = 0$, we get $q(\pm 1) = 0$.

Therefore, we have

$$q(z) = az + b, \tag{16}$$

where a and b are the arbitrary constants independent of n and z. Now, substituting $z = 1$ and $z = -1$ in (16), we get $a = b = 0$. Hence, the theorem follows. ∎

4. Explicit Representation of Interpolatory Polynomials

We shall write

$$\mathcal{H}_n(z) = \sum_{k=0}^{2n+1} f(z_k)\mathcal{C}_k(z) + \sum_{k=0,2n+1} \beta_{\pm 1}\mathcal{D}_k(z), \tag{17}$$

where $\mathcal{C}_k(z)$ and $\mathcal{D}_k(z)$ are fundamental polynomials, each of degree atmost $2n+3$ satisfying the conditions (18) and (19), respectively,

$$\begin{cases} \mathcal{C}_k(z_j) = \delta_{kj} & ; j,k = 0(1)2n+1, \\ \mathcal{C}'_k(z_j) = 0 & ; k = 0(1)2n+1,\ j = 0, 2n+1, \end{cases} \tag{18}$$

$$\begin{cases} \mathcal{D}_k(z_j) = 0 & ; j = 0(1)2n+1,\ k = 0, 2n+1, \\ \mathcal{D}'_k(z_j) = \delta_{kj} & ; j,k = 0, 2n+1. \end{cases} \tag{19}$$

Theorem 4.1.

The fundamental polynomial $\mathcal{D}_k(z)$ is given by

$$\mathcal{D}_k(z) = \frac{\mathbb{R}(z)(z + z_k)}{4K_n}, \tag{20}$$

where $k = 0, 2n+1$.

Proof:

Let us consider

$$\mathcal{D}_k(z) = (z^2 - 1)\mathbb{W}(z)p(z), \tag{21}$$

where $p(z)$ is a linear polynomial. From the first condition in (19), one can verify that $\mathcal{D}_k(z_j) = 0$, for $j = 0(1)2n+1$.

Differentiating Equation (21) with respect to z, we get

$$\mathcal{D}'_k(z) = 2z\mathbb{W}(z)p(z) + (z^2-1)\mathbb{W}'(z)p(z) + (z^2-1)\mathbb{W}(z)p'(z). \tag{22}$$

For $j = 0, 2n+1$, we have

$$\mathcal{D}'_k(z_j) = 2z_j\mathbb{W}(z_j)p(z_j). \tag{23}$$

Using the second set of condition of (19), (6) and (7), we get $p(1) = \dfrac{1}{2K_n}$ and $p(-1) = \dfrac{-1}{2K_n}$.

Therefore, we have

$$p(z) = \frac{z+z_k}{4K_n}. \tag{24}$$

Hence, the theorem follows. ■

Theorem 4.2.

The fundamental polynomial $\mathcal{C}_k(z)$ is defined differently for various values of k.

For $k = 1(1)2n$,

$$\mathcal{C}_k(z) = \mathbb{L}_k(z) + \frac{(z+z_k)\mathbb{R}(z)}{(z_k^2-1)\mathbb{R}'(z_k)}, \tag{25}$$

and for $k = 0, 2n+1$,

$$\mathcal{C}_k(z) = (z+z_k)\mathbb{L}_k(z)\left[\frac{1}{2z_k} - \left(\frac{1}{4z_k^2} + \frac{\mathbb{L}'_k(z_k)}{2z_k}\right)(z-z_k)\right]. \tag{26}$$

Proof:

Case I: For $k = 1(1)2n$, let us consider

$$\mathcal{C}_k(z) = \mathbb{L}_k(z) + a_k(z+z_k)\mathbb{R}(z), \tag{27}$$

where a_k is a constant independent of n and z. Using (4) and (6), we can verify the first set of conditions in (18).

On differentiating (27) with respect to z, we get

$$\mathcal{C}'_k(z) = \mathbb{L}'_k(z) + a_k(z+z_k)\mathbb{R}'(z) + a_k\mathbb{R}(z).$$

At $z = z_j$, we get

$$\mathcal{C}'_k(z_j) = \mathbb{L}'_k(z_j) + a_k(z_j+z_k)\mathbb{R}'(z_j).$$

To satisfy the second set of conditions in (18), $\mathcal{C}'_k(z_j) = 0$. Thus, we get

$$\mathbb{L}'_k(z_j) + a_k(z_j+z_k)\mathbb{R}'(z_j) = 0, \tag{28}$$

$$a_k = -\frac{\mathbb{L}'_k(z_j)}{(z_j+z_k)\mathbb{R}'(z_j)}. \tag{29}$$

Let us simplify further by substituting the value of $\mathbb{L}'_k(z_j)$.

Using (6), we have

$$\mathbb{L}_k(z)(z - z_k)\mathbb{R}^{'}(z_k) = \mathbb{R}(z). \tag{30}$$

On differentiating with respect to z, we get

$$\mathbb{L}'_k(z)(z - z_k)\mathbb{R}^{'}(z_k) + \mathbb{L}_k(z)\mathbb{R}^{'}(z_k) = \mathbb{R}'(z). \tag{31}$$

Substituting $z = z_j$ in (31), we get

$$\mathbb{L}'_k(z_j)(z_j - z_k)\mathbb{R}^{'}(z_k) = \mathbb{R}'(z_j). \tag{32}$$

Putting this value of $\mathbb{R}'(z_j)$ in (28), we get

$$a_k = \frac{1}{\mathbb{R}'(z_k)(z_k^2 - z_j^2)}. \tag{33}$$

Since $j = 0, 2n + 1$ for the second set of conditions in (19) implies $z_j^2 = (\pm 1)^2 = 1$, we can rewrite (33) as

$$a_k = \frac{1}{\mathbb{R}'(z_k)(z_k^2 - 1)}. \tag{34}$$

Putting value of a_k from (34) in (27), we have

$$\mathcal{C}_k(z) = \mathbb{L}_k(z) + \frac{(z + z_k)\mathbb{R}(z)}{(z_k^2 - 1)\mathbb{R}'(z_k)}. \tag{35}$$

Case II: For $k = 0, 2n + 1$, let us consider

$$\mathcal{C}_k(z) = (z + z_k)\mathbb{L}_k(z)p_k(z), \tag{36}$$

where $p_k(z)$ is a linear polynomial.

On satisfying the first set of conditions in (18), we get

$$\begin{cases} p_0(z_0) & = \frac{1}{2}, \\ p_{2n+1}(z_{2n+1}) & = -\frac{1}{2}. \end{cases} \tag{37}$$

Similarly, satisfying the second set of conditions in (18), we get

$$\begin{cases} p'_0(z_0) & = -\frac{1}{2}\Big(\frac{1}{2} + \mathbb{L}'_0(z_0)\Big), \\ p'_{2n+1}(z_{2n+1}) & = \frac{1}{2}\Big(\frac{1}{2} - \mathbb{L}'_{2n+1}(z_{2n+1})\Big). \end{cases} \tag{38}$$

Combining the results of (37) and (38), we get

$$\begin{cases} p_k(z_k) & = \frac{1}{2z_k}, \\ p'_k(z_k) & = -\frac{p_k(z_k)}{2z_k}\Big(1 + 2z_k\mathbb{L}'_k(z_k)\Big). \end{cases} \tag{39}$$

Since $p_k(z)$ is a linear polynomial, let us consider

$$p_k(z) = c_1 + d_1(z - z_k), \tag{40}$$

where c_1 and d_1 are the constants independent of n and z.

Using (39), we have

$$\begin{cases} p_k(z_k) & = c_1 = \dfrac{1}{2z_k}, \\ p'_k(z_k) & = d_1 = -\Big(\dfrac{1}{4z_k^2} + \dfrac{\mathbb{L}'_k(z_k)}{2z_k}\Big). \end{cases} \tag{41}$$

Substituting the values of constants c_1 and d_1 from (41) in (40), we get

$$p_k(z) = \left[\frac{1}{2z_k} - \Big(\frac{1}{4z_k^2} + \frac{\mathbb{L}'_k(z_k)}{2z_k}\Big)(z - z_k)\right].$$

We get the desired result (26) by putting the value of $p_k(z)$ in (36). Hence, the theorem follows. ∎

5. Estimation of Fundamental Polynomials

Lemma 5.1.

Let $\mathbb{L}_k(z)$ be given by (6), then

$$\max_{|z|=1} \sum_{k=0}^{2n+1} |\ \mathbb{L}_k(z)\ | \leq\ b_1 \sum_{k=0}^{2n+1} \frac{1}{k^{-\alpha+\frac{3}{2}}}, \tag{42}$$

where b_1 is a constant independent of n and z.

Proof:

From (6), we have

$$\begin{aligned} |\mathbb{L}_k(z)| &= \frac{|\mathbb{R}(z)|}{|(z - z_k)|\,|\mathbb{R}'(z_k)|}, \qquad k = 0(1)2n+1, \\ &= \frac{|(z^2-1)\mathbb{W}(z)|}{|(z - z_k)|\,|(z_k^2-1)\mathbb{W}'(z_k) + 2z_k\mathbb{W}(z_k)|}. \end{aligned} \tag{43}$$

For $k = 1(1)2n$ and using (3) above the equation can be written as,

$$\begin{aligned} |\mathbb{L}_k(z)| &= \frac{|(z^2-1)\mathbb{W}(z)|}{|(z - z_k)|\,|(z_k^2-1)\mathbb{W}'(z_k)|}, \\ &= \frac{2|z^2-1|\,|P_n^{(\alpha,\beta)}(x)|\,|z^n|}{|z - z_k|\,|z_k^2-1|^2\,|P_n^{(\alpha,\beta)'}(x_k)|\,|z_k^{n-2}|}. \end{aligned}$$

Since $\max|z| = 1$ and $|z_k| = 1$, we have

$$|\mathbb{L}_k(z)| \leq \frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)|\sqrt{(1-xx_k)}}{|x - x_k|(1-x_k^2)|P_n^{(\alpha,\beta)'}(x_k)|}.$$

Let us consider $| \ x - x_k \ | \geq \dfrac{1}{2}\sqrt{1-x_k^2}$ and using (8), (10) and (11), we have

$$|\mathbb{L}_k(z)| \leq b_1 \frac{1}{k^{-\alpha+\frac{3}{2}}},$$

where b_1 is a constant. Taking summation on both sides, we get

$$\sum_{k=1}^{2n} |\mathbb{L}_k(z)| \leq b_1 \sum_{k=1}^{2n} \frac{1}{k^{-\alpha+\frac{3}{2}}}. \tag{44}$$

For $k = 0$ and $2n + 1$, Equation (43) can be written as,

$$|\mathbb{L}_k(z)| = \frac{|(z^2-1)\mathbb{W}(z)|}{|(z-z_k)|\,|2z_k\mathbb{W}(z_k)|},$$

,

$$|\mathbb{L}_k(z)| = \frac{|(z^2-1)|\,|P_n^{(\alpha,\beta)}(x)|\,|z^n|}{|(z-z_k)|\,|2z_k^{n+1}||P_n^{(\alpha,\beta)}(x_k)|}.$$

Since $\max|z| = 1$ and $|z_k| = 1$, we have

$$|\mathbb{L}_k(z)| \leq \frac{\sqrt{1-x^2}\,|P_n^{(\alpha,\beta)}(x)|\sqrt{(1-xx_k)}}{|x-x_k|\,|P_n^{(\alpha,\beta)}(x_k)|}.$$

Let us consider $| \ x - x_k \ | \geq \frac{1}{2}\sqrt{1-x_k^2}$ and using (8) and (10), we have

$$|\mathbb{L}_k(z)| \leq b_2, \tag{45}$$

■

where b_2 is constant.

The estimate remains the same in the case where $|x - x_k| > \frac{1}{2}\sqrt{1-x_k^2}$. Combining (44) and (45), we get (42).

Lemma 5.2.

Let $\mathcal{D}_k(z)$ be given by (20). Then for $k = 0, 2n + 1$,

$$| \ \mathcal{D}_k(z) \ | \leq \ O(n^{\alpha-1}). \tag{46}$$

Proof:

From (20), we have

$$| \ \mathcal{D}_k(z) \ | = \left|\frac{\mathbb{R}(z)(z+z_k)}{4K_n}\right|.$$

Using (3) and (4), we get

$$| \ \mathcal{D}_k(z) \ | = \frac{|z^2-1||P_n^{(\alpha,\beta)}(x)||z^n||z+z_k|}{4}.$$

Since, $\max|z| = 1$. Using (7), we have

$$| \ \mathcal{D}_k(z) \ | \leq \sqrt{1-x^2}\,|P_n^{(\alpha,\beta)}(x)|.$$

Thus, using (8), we have our lemma. ∎

Lemma 5.3.

Let $\mathcal{C}_k(z)$ be given in Theorem (35), then

$$\sum_{k=0}^{2n+1} \mid \mathcal{C}_k(z) \mid \leq c_2\, n^{\alpha+1},$$

where c_2 is a constant independent of n and z.

Proof:

From (35), the fundamental polynomial $\mathcal{C}_k(z)$ for $k = 1(1)2n$ is,

$$\mathcal{C}_k(z) = \mathbb{L}_k(z) + \frac{(z+z_k)\mathbb{R}(z)}{(z_k^2-1)\mathbb{R}'(z_k)}.$$

Taking modulus on both the sides, we have

$$|\mathcal{C}_k(z)| \leq |\mathbb{L}_k(z)| + \frac{|z+z_k||\mathbb{R}(z)|}{|z_k^2-1||\mathbb{R}'(z_k)|}.$$

Using (3) and (4), we get

$$|\mathcal{C}_k(z)| \leq |\mathbb{L}_k(z)| + \frac{2|z+z_k||z^2-1||P_n^{(\alpha,\beta)}(x)||z^n|}{|z_k^2-1|^3|P_n^{(\alpha,\beta)'}(x_k)||z_k^n|}.$$

Since $\max|z| = 1$ and $|z_k| = 1$, using (7), we have

$$|\mathcal{C}_k(z)| \leq |\mathbb{L}_k(z)| + \frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)|}{(1-x_k^2)^{\frac{3}{2}}|P_n^{(\alpha,\beta)'}(x_k)|}.$$

Using (8), (10) and (11), we get

$$|\mathcal{C}_k(z)| \leq |\mathbb{L}_k(z)| + \frac{1}{k^{-\alpha+\frac{3}{2}}}.$$

Taking summation on both the sides, we obtain

$$\sum_{k=1}^{2n} |\mathcal{C}_k(z)| \leq \sum_{k=1}^{2n} |\mathbb{L}_k(z)| + \sum_{k=1}^{2n} \frac{1}{k^{-\alpha+\frac{3}{2}}}. \tag{47}$$

Now, for $k = 0, 2n+1$, from (26), we have

$$\mathcal{C}_k(z) = (z+z_k)\mathbb{L}_k(z)\left[\frac{1}{2z_k} - \left(\frac{1}{4z_k^2} + \frac{\mathbb{L}_k'(z_k)}{2z_k}\right)(z-z_k)\right].$$

Taking modulus on both the sides, we get

$$|\mathcal{C}_k(z)| \leq |\mathbb{L}_k(z)| + \left|\left(\frac{1}{4z_k^2} + \frac{\mathbb{L}_k'(z_k)}{2z_k}\right)(z^2-z_k^2)\mathbb{L}_k(z)\right|.$$

Taking summation, we have

$$\sum_{k=0,2n+1} |\mathcal{C}_k(z)| \leq \sum_{k=0,2n+1} |\mathbb{L}_k(z)| + \sum_{k=0,2n+1} \left| \left(\frac{1}{4z_k^2} + \frac{\mathbb{L}_k'(z_k)}{2z_k} \right) (z^2 - z_k^2) \mathbb{L}_k(z) \right|.$$

Using (3) and (4), we get

$$\sum_{k=0,2n+1} |\mathcal{C}_k(z)| \leq \sum_{k=0,2n+1} |\mathbb{L}_k(z)| + c_2 n^{\alpha+1}, \tag{48}$$

where c_2 is a constant independent of z and n.

Combining (47) and (48), we obtain

$$\sum_{k=0}^{2n+1} |\mathcal{C}_k(z)| \leq \sum_{k=0}^{2n+1} |\mathbb{L}_k(z)| + c_2 n^{\alpha+1} + \sum_{k=1}^{2n} \frac{1}{k^{-\alpha+\frac{3}{2}}}. \tag{49}$$

Using Lemma 5.1, we have our desired Lemma 5.3 . ∎

6. Convergence

Theorem 6.1.

Let $f(z)$ be a function continuous on closed unit disk and analytic on open unit disk. Let the arbitrary numbers $\beta_{\pm 1}$'s be such that

$$| \beta_{\pm 1} | = O(n\, \omega_r(f, n^{-1})). \tag{50}$$

Then, the sequence of interpolatory polynomial $\{\mathcal{H}_n(z)\}$ defined by

$$\mathcal{H}_n(z) = \sum_{k=0}^{2n+1} f(z_k)\mathcal{C}_k(z) + \sum_{k=0,2n+1} \beta_{\pm 1}\mathcal{D}_k(z), \tag{51}$$

satisfies the relation

$$| \mathcal{H}_n(z) - f(z) | = O(\omega_r(f, n^{-1}) n^{\alpha+1}), \tag{52}$$

where $\omega_r(f, n^{-1})$ denotes the r^{th} modulus of continuity of $f(z)$.

Remark 6.1.

Let $f(z)$ be a function continuous on closed unit disk and analytic on open unit disk and $f^{(r)} \epsilon \, Lip\, \nu$, $\nu > 0$. Then, the sequence $\{\mathcal{H}_n(z)\}$ converges uniformly to $f(z)$ on closed unit disk, which follows from (52) as

$$\omega_r(f, n^{-1}) = O(n^{-r-\nu+1}), \qquad \{\nu > \alpha - r + 2\}. \tag{53}$$

To prove Theorem 6.1, we shall need following.

Let $f(z)$ be a function continuous on closed unit disk and analytic on open unit disk. Then, there exists a polynomial $\mathbb{F}_n(z)$ of degree $\leq 2n+3$ satisfying the inequality (Jackson (1911))

$$| f(z) - \mathbb{F}_n(z) | \leq C\, \omega_r(f, n^{-1}), \qquad z = e^{i\theta}(0 \leq \theta < 2\pi), \tag{54}$$

and also an inequality (Kiš (1960))

$$| \mathbb{F}_n^{(m)}(z) | \leq C\, n^m\, \omega_r(f, n^{-1}), \qquad m\, \varepsilon\, \mathbb{Z}^+, \tag{55}$$

where C is a constant independent of n and z.

Proof:

Since $\mathcal{H}_n(z)$ is the uniquely determined polynomial of degree $\leq 2n+3$ and the polynomial $\mathbb{F}_n(z)$ satisfying equation (54) and (55) can be expressed as

$$\mathbb{F}_n(z) = \sum_{k=0}^{2n+1} \mathbb{F}_n(z_k)\mathcal{C}_k(z) + \sum_{k=0,2n+1} \mathbb{F}_n^{'}(z_k)\mathcal{D}_k(z), \tag{56}$$

then, we can write

$$| \mathcal{H}_n(z) - f(z) | \leq | \mathcal{H}_n(z) - \mathbb{F}_n(z) | + | \mathbb{F}_n(z) - f(z) | .$$

Using (51) and (56), we have

$$\begin{aligned} | \mathcal{H}_n(z) - f(z) | \leq & \sum_{k=0}^{2n+1} | f(z_k) - \mathbb{F}_n(z_k) || \mathcal{C}_k(z) | \\ & + \sum_{k=0,2n+1} | \beta_{\pm 1} - \mathbb{F}'_n(z_k) || \mathcal{C}_k(z) | + | \mathbb{F}_n(z) - f(z) |, \end{aligned}$$

$$\begin{aligned} | \mathcal{H}_n(z) - f(z) | \leq & \underbrace{\sum_{k=0}^{2n+1} | f(z_k) - \mathbb{F}_n(z_k) || \mathcal{C}_k(z) |}_{A_1} + \underbrace{\sum_{k=0,2n+1} | \beta_{\pm 1} || \mathcal{D}_k(z) |}_{A_2} \\ & + \underbrace{\sum_{k=0,2n+1} | \mathbb{F}'_n(z_k) || \mathcal{D}_k(z) |}_{A_3} + \underbrace{| \mathbb{F}_n(z) - f(z) |}_{A_4}. \end{aligned} \tag{57}$$

Using (54) and Lemma 5.3, we get

$$A_1 = O(\omega_r(f, n^{-1})n^{\alpha+1}). \tag{58}$$

Using (50) and Lemma 5.2, we get

$$A_2 = O(n^\alpha\, \omega_r(f, n^{-1})). \tag{59}$$

Using (55) and Lemma 5.2, we get

$$A_3 = O(n^\alpha\, \omega_r(f, n^{-1})). \tag{60}$$

Using (54), we have

$$A_4 = O(\omega_r(f, n^{-1})). \tag{61}$$

Using (58), (59), (60), (61) in (57), we get

$$| \mathcal{H}_n(z) - f(z) | = O(\omega_r(f, n^{-1})n^{\alpha+1}). \tag{62}$$

Hence, Theorem 6.1 follows. ■

7. Numerical Experiments

To visualize the contribution of this research work, we carried out numerical experiments. We work in the following way for the included example.

- Nodal system $\mathbb{Z}_{2n}$ is used with varying values of n.
- We detail a test function $f(z)$ and the Lagrange interpolating polynomial $\mathcal{H}_n(z)$ in 1000 random points of $\mathbb{T} \cup \mathbb{D}$.
- To showcase the interpolation behavior on the boundary and within the unit circle, maximum error has been estimated in 1000 random points.

Example 7.1.

We consider a function f continuous on $z \in \mathbb{T} \cup \mathbb{D}$ and analytic in $\mathbb{D}$ satisfying the hypothesis of Theorem 6.1 defined by

$$f(z) = 0.5 + \left(\frac{z + \frac{1}{z}}{2}\right) \sin\left(\frac{2}{z + \frac{1}{z}}\right)$$

We choose Jacobi polynomial parameters as $\alpha = \beta = 0$. Calculations performed has been arranged in Table 1. Column 1 of the Table 1 contains the varying values of n. Column 2 contains the estimated maximum error $| \mathcal{H}_n(z) - f(z) |$. Notice that, in accordance with Theorem 6.1, the order of the error must be $O(\omega_r(f, n^{-1})n)$. So, for any modulus of continuity, the rate of convergence is calculated to be $\frac{1}{n^p}$ where $p > 0$.

It can be clearly seen that with the varying values of n, values in column 2 shows resemblance with the values in coloumn 3. This clearly indicates that, numerically obtained results shows closeness with the analytically obtained results.

So, $f(z)$ analytic in $z \in \mathbb{T} \cup \mathbb{D}$ can be very well approximated via the interpolatory polynomial created on the projected nodes of the zeros of Jacobi polynomial.

Table 1		
n	max $\mid \mathcal{H}_n(z) - f(z) \mid$	$\frac{1}{n^p}$ $(p = 1)$
2	0.11620	0.5
4	0.22701	0.25
8	0.11198	0.125
16	0.05729	0.0625
32	0.03103	0.03125
64	0.02000	0.015625
128	0.01061	0.0078125

8. Conclusion

Bahadur and Varun in 2017 considered a Lagrange-Hermite interpolation problem making use of the zeros of Legendre polynomial for the nodal system whereas this research paper poses a problem, which is an extension to the same problem, since it involves the more general Jacobi polynomial zeros for the constuction of the nodal system. By putting the value of α equal to zero in our main convergence result and comparing it with the convergence theorem of the paper published by Bahadur and Varun in 2017, we can conclude that when α equals to zero, results are comparable. Since, we are not restricted to use different values of α, we get a good approximation of a function, which is continuous on the closed unit disk and analytic on the open unit disk.

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