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Connectedness and Compactness in Fuzzy Nano Topological Spaces Via Fuzzy Nano Z Open Sets

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Abstract

In this paper, we study the notion of fuzzy nano Z connected spaces, fuzzy nano Z disconnected spaces, fuzzy nano Z compact spaces and fuzzy nano Z separated sets in fuzzy nano topological spaces. We also give some properties and theorems of such concepts with connectedness and compactness in fuzzy nano topological spaces. Finally, a real life application in educational field based on fuzzy score function is examined with graphical representation.

Keywords: Fuzzy nano Z open set; Fuzzy nano Z connected; Fuzzy nano Z disconnected; Fuzzy nano Z separated set; Fuzzy nano Z compact; Fuzzy score function

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1. Introduction

Through his significant theory on fuzzy sets, Zadeh (1965) made the first effective attempt in mathematical modelling to contain non-probabilistic uncertainty, i.e., uncertainty that is not caused by randomness of an event. A fuzzy set is one in which each element of the universe belongs to it, but with a value or degree of belongingness that falls between 0 and 1, and these values are referred to as the membership value of each element in that set. Chang (1968) was the first to propose the concept of fuzzy topology later on.

Pawlak (1982) introduces Rough set theory as a substitute mathematical tool for describing reasoning and deciding how to handle vagueness and uncertainty. This theory uses equivalence relations to approximate sets, and it is used in conjunction with the principal non-statistical techniques to data analysis. Lower and upper approximations are two definite sets that commonly characterize a rough set. The greatest definable set included inside the given collection of objects is the lower approximation, whereas the smallest definable set that contains the provided set is the upper approximation. Rough set concepts are frequently stated in broad terms using topological operations such as interior and closure, which are referred to as approximations.

Lellis Thivagar and Richard (2013) introduced a new topology called Nano topology, which is an extension of rough set theory. He also created Nano topological spaces, which are defined in terms of approximations and the boundary region of a subset of the universe using an equivalence relation. The Nano open sets are the constituents of a Nano topological space, while the Nano closed sets are their complements. The term “nano” refers to anything extremely small. Nano topology, then, is the study of extremely small surfaces. Nano topology is based on the concepts of approximations and indiscernibility relations. In addition, nano delta open sets in nano topological space were investigated by Pankajam and Kavitha (2017).

This paper follows the definition of Lellis Thivagar et al. (2018), generalizations of (fuzzy nano) open sets are a major topic in (fuzzy nano) topology. One of the important generalizations is a Z-open sets which was studied in classical topology by EL-Magharabi and Mubarki (2011). Later on, many studies which investigated a nano topologies have been done such as nano M-open sets by Padma et al. (2019), nano Z-open sets by Arul Selvaraj and Balakrishna (2021), and Z-closed sets in double fuzzy topological spaces by Shiventhiradevi Sathaananthan et al. (2020).

Recently, Thangammal et al. (2022a), Thangammal et al. (2022b), Kalaiyarasan et al. (2023a), and Kalaiyarasan et al. (2023b) introduced some open sets, namely, fuzzy nano Z-open and fuzzy nano M-open sets in fuzzy nano topological spaces, and Vadivel et al. (2022a), Vadivel et al. (2022b) investigated some open sets in neutrosophic nano topological spaces.

Research Gap: No investigation on compactness and connectedness such as fuzzy nano Z connected, fuzzy nano Z disconnected and fuzzy nano Z compactness, fuzzy nano Z separated sets in fuzzy nano topological space has been reported in the fuzzy literature.

This paper introduces the concept of fuzzy nano Z connected, fuzzy nano Z disconnected and fuzzy nano Z compact in fuzzy nano topological spaces. Also, we introduced fuzzy nano Z separated sets. We also give some properties and theorems of such concepts in fuzzy nano topological spaces.

2. Preliminaries

This part explains the concepts and findings that we need to know in order to comprehend the manuscript. The definition of fuzzy set (briefly, $\mathcal{F}s$), fuzzy subsets (briefly, $\mathcal{F}subs$), equality, union, intersection and complement are defined by Zadeh (1965). The fuzzy nano lower approximation ($\underline{\mathcal{FN}}(F)$), fuzzy nano upper approximation ($\overline{\mathcal{FN}}(F)$) and fuzzy nano boundary of F ($B_{\mathcal{FN}}(F)$), fuzzy nano topological space (briefly, $\mathcal{FN}ts$), fuzzy nano open (briefly, $\mathcal{FN}o$) and fuzzy nano closed (briefly, $\mathcal{FN}c$) sets are defined by Lellis Thivagar et al. (2018). The other definitions defined in this paper is studied by the authors Thangammal et al. (2022a), Thangammal et al. (2022b), Kalaiyarasan et al. (2023a), Vadivel et al. (2022a), and Vadivel et al. (2022b).

3. Fuzzy nano Z connected spaces

Definition 3.1.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ is fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) disconnected (briefly, $\mathcal{FN}DCon$ (respectively, $\mathcal{FN}\delta DCon$, $\mathcal{FN}\delta\mathcal{S}DCon$, $\mathcal{FNP}DCon$ and $\mathcal{FN}ZDCon$)) if there exists $\mathcal{FN}o$ (respectively, $\mathcal{FN}\delta o$, $\mathcal{FN}\delta\mathcal{S}o$, $\mathcal{FNP}o$ and $\mathcal{FN}Zo$) sets A, B in U, $A \neq 0_N$, $B \neq 0_N$ $\ni A \vee B = 1_N$ and $A \wedge B = 0_N$. That is, $\mu_A \vee \mu_B = 1_N$ and $\mu_A \wedge \mu_B = 0_N$.

If U is not $\mathcal{FN}DCon$ (respectively, $\mathcal{FN}\delta DCon$, $\mathcal{FN}\delta\mathcal{S}DCon$, $\mathcal{FNP}DCon$ and $\mathcal{FN}ZDCon$) then it is said to be Fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) connected (briefly, $\mathcal{FN}Con$ (respectively, $\mathcal{FN}\delta Con$, $\mathcal{FN}\delta\mathcal{S}Con$, $\mathcal{FNP}Con$ and $\mathcal{FN}ZCon$)).

Example 3.1.

Assume $U = \{s_1, s_2, s_3, s_4\}$ and $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$.

Let $S = \left\{\left\langle \frac{s_1}{0.2}\right\rangle, \left\langle \frac{s_2}{0.3}\right\rangle, \left\langle \frac{s_3}{0.4}\right\rangle, \left\langle \frac{s_4}{0.1}\right\rangle\right\}$ be a $\mathcal{F}subs$ of U. Then,

$$\underline{\mathcal{FN}}(S) = \left\{\left\langle \frac{s_1, s_4}{0.1}\right\rangle, \left\langle \frac{s_2}{0.3}\right\rangle, \left\langle \frac{s_3}{0.4}\right\rangle\right\},$$
$$\overline{\mathcal{FN}}(S) = \left\{\left\langle \frac{s_1, s_4}{0.2}\right\rangle, \left\langle \frac{s_2}{0.3}\right\rangle, \left\langle \frac{s_3}{0.4}\right\rangle\right\},$$
$$B_{\mathcal{FN}}(S) = \left\{\left\langle \frac{s_1, s_4}{0.2}\right\rangle, \left\langle \frac{s_2}{0.3}\right\rangle, \left\langle \frac{s_3}{0.4}\right\rangle\right\}.$$

Thus, $\tau_{\mathcal{F}}(S) = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, \underline{\mathcal{FN}}(S), \overline{\mathcal{FN}}(S) = B_{\mathcal{FN}}(S)\}$.

(i) Let $A = \left\{\left\langle \frac{s_1, s_4}{0.2}\right\rangle, \left\langle \frac{s_2}{0.3}\right\rangle, \left\langle \frac{s_3}{0.7}\right\rangle\right\}$ and $B = \left\{\left\langle \frac{s_1, s_4}{0.7}\right\rangle, \left\langle \frac{s_2}{0.6}\right\rangle, \left\langle \frac{s_3}{0.8}\right\rangle\right\}$ are $\mathcal{FN}Zo$ (respectively, $\mathcal{FNP}o$) sets. Then, U is $\mathcal{FN}ZCon$ (respectively, $\mathcal{FNP}Con$).

(ii) Let $A = \left\{\left\langle \frac{s_1, s_4}{0.2} \right\rangle, \left\langle \frac{s_2}{0.3} \right\rangle, \left\langle \frac{s_3}{0.5} \right\rangle\right\}$ and $B = \left\{\left\langle \frac{s_1, s_4}{0.8} \right\rangle, \left\langle \frac{s_2}{0.5} \right\rangle, \left\langle \frac{s_3}{0.5} \right\rangle\right\}$ are $\mathcal{FN}\delta\mathcal{S}o$ sets. Then, U is $\mathcal{FN}\delta\mathcal{S}Con$.

(iii) Let $A = \left\{\left\langle \frac{s_1, s_4}{0.2} \right\rangle, \left\langle \frac{s_2}{0.3} \right\rangle, \left\langle \frac{s_3}{0.4} \right\rangle\right\}$ and $B = \left\{\left\langle \frac{s_1, s_4}{0.1} \right\rangle, \left\langle \frac{s_2}{0.3} \right\rangle, \left\langle \frac{s_3}{0.4} \right\rangle\right\}$ are $\mathcal{FN}o$ sets. Then, U is $\mathcal{FN}Con$.

(iv) Let $A = \left\{\left\langle \frac{s_1, s_4}{0.2} \right\rangle, \left\langle \frac{s_2}{0.3} \right\rangle, \left\langle \frac{s_3}{0.4} \right\rangle\right\}$ and $B = \left\{\left\langle \frac{s_1, s_4}{1.0} \right\rangle, \left\langle \frac{s_2}{1.0} \right\rangle, \left\langle \frac{s_3}{1.0} \right\rangle\right\}$ are $\mathcal{FN}\delta o$ sets. Then, U is $\mathcal{FN}\delta Con$.

Example 3.2.

In Example 3.1, let $A = \left\{\left\langle \frac{s_1, s_4}{0.0} \right\rangle, \left\langle \frac{s_2}{0.0} \right\rangle, \left\langle \frac{s_3}{1.0} \right\rangle\right\}$ and $B = \left\{\left\langle \frac{s_1, s_4}{1.0} \right\rangle, \left\langle \frac{s_2}{1.0} \right\rangle, \left\langle \frac{s_3}{0.0} \right\rangle\right\}$ be $\mathcal{FN}Zo$ sets. Then, U is $\mathcal{FN}ZDCon$.

Definition 3.2.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ with respect to F where F is a fuzzy subset of U. Let S be a fuzzy subset of U.

(i) If there exists $\mathcal{FN}o$ (respectively, $\mathcal{FN}\delta o$, $\mathcal{FN}\delta\mathcal{S}o$, $\mathcal{FN}\mathcal{P}o$ and $\mathcal{FN}Zo$) sets L and M in U satisfying the following properties, then S is called fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) C_i-disconnected (briefly, $\mathcal{FN}C_iDCon$ (respectively, $\mathcal{FN}\delta C_iDCon$, $\mathcal{FN}\delta\mathcal{S}C_iDCon$, $\mathcal{FN}\mathcal{P}C_iDCon$ and $\mathcal{FN}ZC_iDCon$)) $(i = 1, 2, 3, 4)$:

$C_1:\ S \leq L \vee M, L \wedge M \leq S^c, S \wedge L \neq 0_N, S \wedge M \neq 0_N.$

$C_2:\ S \leq L \vee M, S \wedge L \wedge M = 0_N, S \wedge L \neq 0_N, S \wedge M \neq 0_N.$

$C_3:\ S \leq L \vee M, L \wedge M \leq S^c, L \nleq S^c, M \nleq S^c.$

$C_4:\ S \leq L \vee M, S \wedge L \wedge M = 0_N, L \nleq S^c, M \nleq S^c.$

(ii) S is said to be fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) C_i-connected (briefly, $\mathcal{FN}C_iCon$ (respectively, $\mathcal{FN}\delta C_iCon$, $\mathcal{FN}\delta\mathcal{S}C_iCon$, $\mathcal{FN}\mathcal{P}C_iCon$ and $\mathcal{FN}ZC_iCon$)), $(i = 1, 2, 3, 4)$ if N is not $\mathcal{FN}C_iDCon$ (respectively, $\mathcal{FN}\delta C_iDCon$, $\mathcal{FN}\delta\mathcal{S}C_iDCon$, $\mathcal{FN}\mathcal{P}C_iDCon$ and $\mathcal{FN}ZC_iDCon$), $(i = 1, 2, 3, 4)$. Obviously, we can derive the following inferences from a variety of sources of information $\mathcal{FN}C_iCon$ (respectively, $\mathcal{FN}\delta C_iCon$, $\mathcal{FN}\delta\mathcal{S}C_iCon$, $\mathcal{FN}\mathcal{P}C_iCon$ and $\mathcal{FN}ZC_iCon$), $(i = 1, 2, 3, 4)$.

(i) $\mathcal{FN}C_1Con$ (respectively, $\mathcal{FN}\delta C_1Con$, $\mathcal{FN}\delta\mathcal{S}C_1Con$, $\mathcal{FN}\mathcal{P}C_1Con$ and $\mathcal{FN}ZC_1Con$) $\Rightarrow$ $\mathcal{FN}C_2Con$ (respectively, $\mathcal{FN}\delta C_2Con$, $\mathcal{FN}\delta\mathcal{S}C_2Con$, $\mathcal{FN}\mathcal{P}C_2Con$ and $\mathcal{FN}ZC_2Con$).

(ii) $\mathcal{FN}C_1Con$ (respectively, $\mathcal{FN}\delta C_1Con$, $\mathcal{FN}\delta\mathcal{S}C_1Con$, $\mathcal{FN}\mathcal{P}C_1Con$ and $\mathcal{FN}ZC_1Con$) $\Rightarrow$ $\mathcal{FN}C_3Con$ (respectively, $\mathcal{FN}\delta C_3Con$, $\mathcal{FN}\delta\mathcal{S}C_3Con$, $\mathcal{FN}\mathcal{P}C_3Con$ and $\mathcal{FN}ZC_3Con$).

(iii) $\mathcal{FN}C_3Con$ (respectively, $\mathcal{FN}\delta C_3Con$, $\mathcal{FN}\delta\mathcal{S}C_3Con$, $\mathcal{FN}\mathcal{P}C_3Con$ and $\mathcal{FN}ZC_3Con$) $\Rightarrow$ $\mathcal{FN}C_4Con$ (respectively, $\mathcal{FN}\delta C_4Con$, $\mathcal{FN}\delta\mathcal{S}C_4Con$, $\mathcal{FN}\mathcal{P}C_4Con$ and $\mathcal{FN}ZC_4Con$).

(iv) $\mathcal{FN}C_1Con$ (respectively, $\mathcal{FN}\delta C_1Con$, $\mathcal{FN}\delta\mathcal{S}C_1Con$, $\mathcal{FNP}C_1Con$ and $\mathcal{FN}ZC_1Con$) $\Rightarrow$ $\mathcal{FN}C_4Con$ (respectively, $\mathcal{FN}\delta C_4Con$, $\mathcal{FN}\delta\mathcal{S}C_4Con$, $\mathcal{FNP}C_4Con$ and $\mathcal{FN}ZC_4Con$).

Example 3.3.

In Example 3.1, let $S = \left\{\left\langle\frac{s_1,s_4}{0.8}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$. If $L = \left\{\left\langle\frac{s_1,s_4}{0.8}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\}$ and $M = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.4}\right\rangle, \left\langle\frac{s_3}{0.9}\right\rangle\right\}$ are $\mathcal{FN}Zo$ sets, then S is

(i) $\mathcal{FN}ZC_2Con$ but not $\mathcal{FN}ZC_1Con$.

(ii) $\mathcal{FN}ZC_3Con$ but not $\mathcal{FN}ZC_1Con$.

(iii) $\mathcal{FN}ZC_4Con$ but not $\mathcal{FN}ZC_1Con$.

Example 3.4.

In Example 3.1, let $S = \left\{\left\langle\frac{s_1,s_4}{0.8}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$. If $L = \left\{\left\langle\frac{s_1,s_4}{0.8}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\}$ and $M = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.9}\right\rangle\right\}$ are $\mathcal{FN}Zo$ sets, then S is $\mathcal{FN}ZC_4Con$ but not $\mathcal{FN}ZC_3Con$.

Definition 3.3.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ is fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) C_5-disconnected (briefly, $\mathcal{FN}C_5DCon$ (respectively, $\mathcal{FN}\delta C_5DCon$, $\mathcal{FN}\delta\mathcal{S}C_5DCon$, $\mathcal{FNP}C_5DCon$ and $\mathcal{FN}ZC_5DCon$)) if there exists fuzzy subset S in U which is both $\mathcal{FN}o$ (respectively, $\mathcal{FN}\delta o$, $\mathcal{FN}\delta\mathcal{S}o$, $\mathcal{FNP}o$ and $\mathcal{FN}Zo$) and $\mathcal{FN}c$ (respectively, $\mathcal{FN}\delta c$, $\mathcal{FN}\delta\mathcal{S}c$, $\mathcal{FNP}c$ and $\mathcal{FN}Zc$) in U, such that $S \neq 0_N$, $S \neq 1_N$. If U is not $\mathcal{FN}C_5DCon$ (respectively, $\mathcal{FN}\delta C_5DCon$, $\mathcal{FN}\delta\mathcal{S}C_5DCon$, $\mathcal{FNP}C_5DCon$ and $\mathcal{FN}ZC_5DCon$) then it is said to be fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) C_5-connected (briefly, $\mathcal{FN}C_5Con$ (respectively, $\mathcal{FN}\delta C_5Con$, $\mathcal{FN}\delta\mathcal{S}C_5Con$, $\mathcal{FNP}C_5Con$ and $\mathcal{FN}ZC_5Con$)).

Example 3.5.

In Example 3.1, let

(i) $S = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.2}\right\rangle, \left\langle\frac{s_3}{0.3}\right\rangle\right\}$ is $\mathcal{FN}ZC_5DCon$ (respectively, $\mathcal{FNP}C_5DCon$).

(ii) $S = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.5}\right\rangle\right\}$ is $\mathcal{FN}\delta\mathcal{S}C_5DCon$.

Theorem 3.1.

$\mathcal{FN}ZC_5DCon$-ness implies $\mathcal{FN}ZCon$-ness.

Proof:

Suppose that there exists non-empty $\mathcal{FN}Zo$ sets A & $B \ni A \vee B = 1_N$ and $A \wedge B = 0_N$. Then, $\mu_A \vee \mu_B = 1_N$ and $\mu_A \wedge \mu_B = 0_N$. In other words, $B^c = A$. Hence, A is $\mathcal{FN}Zclo$ which implies U is $\mathcal{FN}ZC_5DCon$. ■

However, as the following example demonstrates, the opposite may not be true.

Example 3.6.

In Example 3.1, let $A = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.7}\right\rangle\right\}$ and $B = \left\{\left\langle\frac{s_1,s_4}{0.7}\right\rangle, \left\langle\frac{s_2}{0.6}\right\rangle, \left\langle\frac{s_3}{0.8}\right\rangle\right\}$ be $\mathcal{FN}Zo$ sets. Then, U is $\mathcal{FN}ZCon$ but not $\mathcal{FN}ZC_5DCon$.

Theorem 3.2.

Let $f : (U_1, \tau_{\mathcal{F}}(F_1)) \to (U_2, \tau_{\mathcal{F}}(F_2))$ be a $\mathcal{FN}ZIrr$ surjection, U_1 be a $\mathcal{FN}ZCon$. Then, U_2 is $\mathcal{FN}ZCon$.

Proof:

Assume that U_2 is not $\mathcal{FN}ZCon$. Then, there exists nonempty $\mathcal{FN}Zo$ sets U and V in $U_2 \ni U \vee V = 1_N$ and $U \wedge V = 0_N$. Since f is $\mathcal{FN}ZIrr$ mapping, $A = f^{-1}(U) \neq 0_N$, $B = f^{-1}(V) \neq 0_N$, which are $\mathcal{FN}Zo$ sets in U_1 and $f^{-1}(U) \vee f^{-1}(V) = f^{-1}(1_N) = 1_N$, which implies $A \vee B = 1_N$. Also, $f^{-1}(U) \wedge f^{-1}(V) = f^{-1}(0_N) = 0_N$, which implies $A \wedge B = 0_N$. Thus, U_1 is $\mathcal{FN}ZDCon$, which is a contradiction to our hypothesis. Hence, U_2 is $\mathcal{FN}ZCon$. ■

Theorem 3.3.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ is $\mathcal{FN}ZC_5Con$ if and only if there exists no nonempty $\mathcal{FN}Zo$ sets L and M in U such that $L = M^c$.

Proof:

Suppose that L and M are $\mathcal{FN}Zo$ sets in U such that $L \neq 0_N$, $M \neq 0_N$ and $L = M^c$. Since $L = M^c$, M^c is a $\mathcal{FN}Zo$ set and M is $\mathcal{FN}Zc$ set and $L \neq 0_N$ implies $M \neq 1_N$. But this is a contradiction to the fact that U is $\mathcal{FN}ZC_5Con$.

Conversely, let L be a both $\mathcal{FN}Zo$ set and M is $\mathcal{FN}Zc$ set in U such that $L \neq 0_N$, $L \neq 1_N$. Now, take $L^c = M$ is a $\mathcal{FN}Zo$ set and $L \neq 1_N$ which implies $L^C = M \neq 0_N$ which is a contradiction. Hence, U is $\mathcal{FN}ZC_5Con$. ■

Theorem 3.4.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ is $\mathcal{FN}ZCon$ space if and only if there exists no non-zero $\mathcal{FN}Zo$ set L and M in U, $\ni L = M^c$.

Proof:

Necessity: Let L and M be two $\mathcal{FN}Zo$ sets in U such that $L \neq 0_N$, $M \neq 0_N$ and $L = M^c$. Therefore, M^c is a $\mathcal{FN}Zc$ set. Since $L \neq 0_N$, $M \neq 1_N$. This implies M is a proper fuzzy subset, which is both $\mathcal{FN}Zo$ set and $\mathcal{FN}Zc$ set in U. Hence, U is not a $\mathcal{FN}ZCon$ space. But this is a contradiction to our hypothesis. Thus, there exist no non-zero $\mathcal{FN}Zo$ sets L and M in U, such that $L = M^c$.

Sufficiency: Let L be both $\mathcal{FN}Zo$ and $\mathcal{FN}Zc$, U such that $L \neq 0_N$, $L \neq 1_N$. Now let $M = L^c$.

Then, M is a $\mathcal{FN}Zo$ set and $M \neq 1_N$. This implies $L^c = M \neq 0_N$, which is a contradiction to our hypothesis. Therefore, U is $\mathcal{FN}ZCon$ space. ■

Theorem 3.5.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ is $\mathcal{FN}ZCon$ space if and only if there exists no non-zero fuzzy subsets L and M in U, $\ni L = M^c$, $M = (\mathcal{FN}Zcl(L))^c$ and $L = (\mathcal{FN}Zcl(M))^c$.

Proof:

Necessity: Let L and M be two fuzzy subsets in U such that $L \neq 0_N$, $M \neq 0_N$ and $L = M^c$, $M = (\mathcal{FN}Zcl(L))^c$ and $L = (\mathcal{FN}Zcl(M))^c$. Since $(\mathcal{FN}Zcl(L))^c$ and $(\mathcal{FN}Zcl(M))^c$ are $\mathcal{FN}Zo$ sets in U, L and M are $\mathcal{FN}Zo$ set in U. This implies U is not a $\mathcal{FN}ZCon$ space, which is a contradiction. Therefore, there exists no non-zero $\mathcal{FN}Zo$ set L and M in U, $\ni L = M^c$, $M = (\mathcal{FN}Zcl(L))^c$ and $L = (\mathcal{FN}Zcl(M))^c$.

Sufficiency: Let L be both $\mathcal{FN}Zo$ and $\mathcal{FN}Zc$ set in U such that $L \neq 0_N$, $L \neq 1_N$. Now by taking $M = L^c$, we obtain a contradiction to our hypothesis. Hence, U is $\mathcal{FN}ZCon$ space. ■

Definition 3.4.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) strongly connected (briefly, $\mathcal{FN}StCon$ (respectively, $\mathcal{FN}\delta StCon$, $\mathcal{FN}\delta\mathcal{S}StCon$, $\mathcal{FN}\mathcal{P}StCon$ and $\mathcal{FN}ZStCon$)), if there exists no nonempty $\mathcal{FN}c$ (respectively, $\mathcal{FN}\delta c$, $\mathcal{FN}\delta\mathcal{S}c$, $\mathcal{FN}\mathcal{P}c$ and $\mathcal{FN}Zc$) sets A and$B$ in U $\ni \mu_A + \mu_B \geq 1_N$.

In other words, a $\mathcal{FN}ts$ U is $\mathcal{FN}StCon$ (respectively, $\mathcal{FN}\delta StCon$, $\mathcal{FN}\delta\mathcal{S}StCon$, $\mathcal{FN}\mathcal{P}StCon$ and $\mathcal{FN}ZStCon$), if there exists no nonempty $\mathcal{FN}c$ (respectively, $\mathcal{FN}\delta c$, $\mathcal{FN}\delta\mathcal{S}c$, $\mathcal{FN}\mathcal{P}c$ and $\mathcal{FN}Zc$) sets A and B in U such that $A \wedge B = 0_N$.

Example 3.7.

In Example 3.1, let $A = \left\{\left\langle\frac{s_1,s_4}{0.0}\right\rangle, \left\langle\frac{s_2}{0.0}\right\rangle, \left\langle\frac{s_3}{1.0}\right\rangle\right\}$ and $B = \left\{\left\langle\frac{s_1,s_4}{1.0}\right\rangle, \left\langle\frac{s_2}{1.0}\right\rangle, \left\langle\frac{s_3}{0.0}\right\rangle\right\}$ be $\mathcal{FN}Zo$ sets. Then, U is $\mathcal{FN}ZStCon$.

Theorem 3.6.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$ is $\mathcal{FN}ZStCon$, if there exists no nonempty $\mathcal{FN}Zo$ sets A and B in U, $A \neq 1_N \neq B$ such that $\mu_A + \mu_B \geq 1_N$.

Proof:

Let A and B be $\mathcal{FN}Zo$ sets in $U \ni A \neq 1 \neq B$ and $\mu_A + \mu_B \geq 1_N$. If we take $C = A^c$ and $D = B^c$, then C and D become $\mathcal{FN}Zc$ sets in U and $C \neq 0_N \neq D$, $\mu_C + \mu_D \leq 1_N$, a contradiction.

Conversely, use a similar technique as above. ■

Theorem 3.7.

Let $f : (U_1, \tau_{\mathcal{F}}(F_1)) \to (U_2, \tau_{\mathcal{F}}(F_2))$ be a $\mathcal{FN}ZIrr$ surjection, U_1 be a $\mathcal{FN}ZStCon$. Then, U_2 is also $\mathcal{FN}ZStCon$.

Proof:

Assume that U_2 is not $\mathcal{FN}ZStCon$. Then, there exists nonempty $\mathcal{FN}Zc$ sets L and M in U_2 such that $L \neq 0_N$, $M \neq 0_N$ and $L \wedge M = 0_N$. Since f is $\mathcal{FN}ZIrr$ mapping, $A = f^{-1}(L) \neq 0_N$, $B = f^{-1}(M) \neq 0_N$, which are $\mathcal{FN}Zc$ sets in U_1 and $f^{-1}(L) \wedge f^{-1}(M) = f^{-1}(0_N) = 0_N$, which implies $A \wedge B = 0_N$. Thus, U_1 is not a $\mathcal{FN}ZStCon$, which is a contradiction to our hypothesis. Hence, U_2 is $\mathcal{FN}ZStCon$. ■

Remark 3.1.

$\mathcal{FN}ZStCon$ and $\mathcal{FN}ZC_5Con$ are independent.

Example 3.8.

In Example 3.1, let $A = \left\{\left\langle \frac{s_1,s_4}{0.0} \right\rangle, \left\langle \frac{s_2}{0.0} \right\rangle, \left\langle \frac{s_3}{1.0} \right\rangle\right\}$ and $B = \left\{\left\langle \frac{s_1,s_4}{1.0} \right\rangle, \left\langle \frac{s_2}{1.0} \right\rangle, \left\langle \frac{s_3}{0.0} \right\rangle\right\}$ be $\mathcal{FN}Zo$ sets. Then, U is $\mathcal{FN}ZStCon$ but not $\mathcal{FN}ZC_5Con$.

Example 3.9.

In Example 3.1, let $A = \left\{\left\langle \frac{s_1,s_4}{0.8} \right\rangle, \left\langle \frac{s_2}{0.6} \right\rangle, \left\langle \frac{s_3}{0.7} \right\rangle\right\}$ and $B = \left\{\left\langle \frac{s_1,s_4}{0.3} \right\rangle, \left\langle \frac{s_2}{0.6} \right\rangle, \left\langle \frac{s_3}{0.8} \right\rangle\right\}$ be $\mathcal{FN}Zo$ sets. Then, U is $\mathcal{FN}ZC_5Con$ but not $\mathcal{FN}ZStCon$.

Remark 3.2.

Theorems 3.1 to 3.7 and Remark 3.1 are also true for $\mathcal{FN}o$, $\mathcal{FN}\delta o$, $\mathcal{FN}\delta\mathcal{S}o$ and $\mathcal{FNP}o$ sets.

4. Fuzzy nano Z separated sets

This section provides an overview of the concept of fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) separated sets in fuzzy nano topological spaces. Also, we study some of the main results depending on fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) separated sets.

Definition 4.1.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. If A and B are non-zero fuzzy subsets in U, then A and B are said to be

(i) fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) weakly separated (briefly, $\mathcal{FN}WSep$ (respectively, $\mathcal{FN}\delta WSep$, $\mathcal{FN}\delta\mathcal{S}WSep$, $\mathcal{FNP}\ WSep$ and $\mathcal{FN}ZWSep$)) if $\mathcal{FN}cl(A) \leq B^c$ (respectively, $\mathcal{FN}\delta cl(A) \leq B^c$, $\mathcal{FN}\delta\mathcal{S}cl(A) \leq B^c$, $\mathcal{FNP}cl(A) \leq B^c$ and $\mathcal{FN}Zcl(A) \leq B^c$) and $\mathcal{FN}cl(B) \leq A^c$ (respectively, $\mathcal{FN}\delta cl(B) \leq A^c$, $\mathcal{FN}\delta\mathcal{S}cl(B) \leq A^c$, $\mathcal{FNP}cl(B) \leq A^c$ and $\mathcal{FN}Zcl(B) \leq A^c$).

(ii) fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) separated (briefly, $\mathcal{FN}Sep$ (respectively, $\mathcal{FN}\delta Sep$, $\mathcal{FN}\delta\mathcal{S}Sep$, $\mathcal{FNP}Sep$ and $\mathcal{FN}ZSep$)) if $\mathcal{FN}cl(A) \wedge B = A \wedge \mathcal{FN}cl(B) = 0_N$ (respectively, $\mathcal{FN}\delta cl(A) \wedge B = A \wedge \mathcal{FN}\delta cl(B) = 0_N$, $\mathcal{FN}\delta\mathcal{S}cl(A) \wedge B = A \wedge \mathcal{FN}\delta\mathcal{S}cl(B) = 0_N$, $\mathcal{FNP}cl(A) \wedge B = A \wedge \mathcal{FNP}cl(B) = 0_N$ and $\mathcal{FN}Zcl(A) \wedge B = A \wedge \mathcal{FN}Zcl(B) = 0_N$).

Remark 4.1.

Any two disjoint non-empty $\mathcal{FN}Zc$ sets are $\mathcal{FN}ZSep$.

Theorem 4.1.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. If A and B are non-zero fuzzy subsets in U.

(i) If A and B are $\mathcal{FN}ZSep$ and $C \leq A$, $D \leq B$, then C and D are also $\mathcal{FN}ZSep$.

(ii) If A and B are both $\mathcal{FN}Zo$ sets and if $H = A \wedge B^c$ and $G = B \wedge A^c$, then H and G are $\mathcal{FN}ZSep$.

Proof:

(i) Let A and B be $\mathcal{FN}ZSep$ sets in $\mathcal{FN}ts$ U. Then, $\mathcal{FN}Zcl(A) \wedge B = 0_N = A \wedge \mathcal{FN}Zcl(B)$. Since $C \leq A$ and $D \leq B$, then $\mathcal{FN}Zcl(C) \leq \mathcal{FN}Zcl(A)$ and $\mathcal{FN}Zcl(D) \leq \mathcal{FN}Zcl(B)$. This implies that $\mathcal{FN}Zcl(C) \wedge D \leq \mathcal{FN}Zcl(A) \wedge B = 0_N$ and hence $\mathcal{FN}Zcl(C) \wedge D = 0_N$. Similarly, $\mathcal{FN}Zcl(D) \wedge C \leq \mathcal{FN}Zcl(B) \wedge A = 0_N$ and hence $\mathcal{FN}Zcl(D) \wedge C = 0_N$. Therefore, C and D are $\mathcal{FN}ZSep$.

(ii) Let A and B both $\mathcal{FN}Zo$ subsets in U. Then, A^c and B^c are $\mathcal{FN}Zc$ sets. Since $H \leq B^c$, then $\mathcal{FN}Zcl(H) \leq \mathcal{FN}Zcl(B^c) = B^c$ and so $\mathcal{FN}Zcl(H) \wedge B = 0_N$. Since $G \leq B$, then $\mathcal{FN}Zcl(H) \wedge G \leq \mathcal{FN}Zcl(H) \wedge B = 0_N$. Thus, $\mathcal{FN}Zcl(H) \wedge G = 0_N$. Similarly, $\mathcal{FN}Zcl(G) \wedge H = 0_N$. Hence, H and G are $\mathcal{FN}ZSep$. ■

Theorem 4.2.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. If A and B are non-zero fuzzy subsets in U are $\mathcal{FN}ZSep$ if and only if there exist L and M in $\mathcal{FN}Zo$ set in $U \ni A \leq L$, $B \leq M$ and $A \wedge M = 0_N$ and $B \wedge L = 0_N$.

Proof:

Let A and B be $\mathcal{FN}ZSep$. Then, $A \wedge \mathcal{FN}Zcl(B) = 0_N = \mathcal{FN}Zcl(A) \wedge B$. Take $M = (\mathcal{FN}Zcl(A))^c$ and $L = (\mathcal{FN}Zcl(B))^c$. Then, L and M are $\mathcal{FN}Zo$ sets $\ni A \leq L$, $B \leq M$ and $A \wedge M = 0_N$ and $B \wedge L = 0_N$.

Conversely, let L and M be $\mathcal{FN}Zo$ sets such that $A \leq L$, $B \leq M$ and $A \wedge M = 0_N$, $B \wedge L = 0_N$. Then, $A \leq M^c$ and $B \leq L^c$ and M^c and L^c are $\mathcal{FN}Zc$. This implies $\mathcal{FN}Zcl(A) \leq \mathcal{FN}Zcl(M^c) = M^c \leq B^c$ and $\mathcal{FN}Zcl(B) \leq \mathcal{FN}Zcl(L^c) = L^c \leq A^c$. That is, $\mathcal{FN}Zcl(A) \leq B^c$ and $\mathcal{FN}Zcl(B) \leq A^c$. Therefore, $A \wedge \mathcal{FN}Zcl(B) = 0_N = \mathcal{FN}Zcl(A) \wedge B$. Hence, A and B are $\mathcal{FN}ZSep$. ■

Proposition 4.1.

Each two $\mathcal{FN}ZSep$ sets are always disjoint.

Proof:

Let A and B be $\mathcal{FN}ZSep$. Then, $A \wedge \mathcal{FN}Zcl(B) = 0_N = \mathcal{FN}Zcl(A) \wedge B$. Now, $A \wedge B \leq A \wedge \mathcal{FN}Zcl(B) = 0_N$. Therefore, $A \wedge B = 0_N$ and hence A and B are disjoint. ■

Theorem 4.3.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. Then, U is $\mathcal{FN}ZCon$ if and only if $1_N \neq A \vee B$, where A and B are $\mathcal{FN}ZSep$ sets.

Proof:

Assume that U is a $\mathcal{FN}ZCon$ space. Suppose $1_N = A \vee B$, where A and B are $\mathcal{FN}ZSep$ sets. Then, $\mathcal{FN}Zcl(A) \wedge B = A \wedge \mathcal{FN}Zcl(B) = 0_N$. Since $A \leq \mathcal{FN}Zcl(A)$, we have $A \wedge B \leq \mathcal{FN}Zcl(A) \wedge B = 0_N$. Therefore, $\mathcal{FN}Zcl(A) \leq B^c = A$ and $\mathcal{FN}Zcl(B) \leq A^c = B$. Hence, $A = \mathcal{FN}Zcl(A)$ and $B = \mathcal{FN}Zcl(B)$. Therefore, A and B are $\mathcal{FN}Zc$ sets and hence $A = B^c$ and $B = A^c$ are disjoint $\mathcal{FN}Zo$ sets. Thus, $A \neq 0_N$, $B \neq 0_N \ni A \vee B = 1_N$ and $A \wedge B = 0_N$, A and B are $\mathcal{FN}Zo$ sets. That is, U is not $\mathcal{FN}ZCon$, which is a contradiction to U is a $\mathcal{FN}ZCon$ space. Hence, 1_N is not the union of any two $\mathcal{FN}ZSep$ sets.

Conversely, assume that 1_N is not the union of any two $\mathcal{FN}ZSep$ sets. Suppose U is not $\mathcal{FN}ZCon$. Then, $1_N = A \vee B$, where $A \neq 0_N$, $B \neq 0_N \ni$ and $A \wedge B = 0_N$, A and B are $\mathcal{FN}Zo$ sets in U. Since $A \leq B^c$ and $B \leq A^c$, $\mathcal{FN}Zcl(A) \wedge B \leq B^c \wedge B = 0_N$ and $A \wedge \mathcal{FN}Zcl(B) \leq A \wedge A^c = 0_N$. That is, A and B are $\mathcal{FN}ZSep$ sets. This is a contradiction. Therefore, U is $\mathcal{FN}ZCon$. ■

Definition 4.2.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. Let S be a fuzzy subset of U. Then fuzzy nano

(i) δ (respectively, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) regular open (briefly, $\mathcal{FN}\delta ro$ (respectively, $\mathcal{FN}\delta\mathcal{S}ro$, $\mathcal{FNP}ro$ and $\mathcal{FN}Zro$)) set if $S = \mathcal{FN}\delta int(\mathcal{FN}\delta\ cl(S))$ (respectively, $S = \mathcal{FN}\delta\mathcal{S}int(\mathcal{FN}\delta\mathcal{S}cl(S))$, $S = \mathcal{FNP}int(\mathcal{FNP}cl(S))$ and $S = \mathcal{FN}Zint(\mathcal{FN}Zcl(S))$).

(ii) δ (respectively, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) regular closed (briefly, $\mathcal{FN}\delta rc$ (respectively, $\mathcal{FN}\delta\mathcal{S}rc$, $\mathcal{FNP}rc$ and $\mathcal{FN}Zrc$)) set if $S = \mathcal{FN}\delta cl(\mathcal{FN}\delta\ int(S))$ (respectively, $S = \mathcal{FN}\delta\mathcal{S}cl(\mathcal{FN}\delta\mathcal{S}int(S))$, $S = \mathcal{FNP}cl(\mathcal{FNP}int(S))$ and $S = \mathcal{FN}Zcl(\mathcal{FN}Zint(S))$).

(iii) The complement of $\mathcal{FN}\delta ro$ (respectively, $\mathcal{FN}\delta\mathcal{S}ro$, $\mathcal{FNP}ro$ and $\mathcal{FN}Zro$) set is $\mathcal{FN}\delta rc$ (respectively, $\mathcal{FN}\delta\mathcal{S}rc$, $\mathcal{FNP}rc$ and $\mathcal{FN}Zrc$) set.

Proposition 4.2.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$.

(i) Every $\mathcal{FN}Zro$ set is $\mathcal{FN}Zo$.

(ii) Every $\mathcal{FN}Zrc$ set is $\mathcal{FN}Zc$.

Proof:

(i) Let A be a $\mathcal{FN}Zro$ in U. Then, $A = \mathcal{FN}Zint(\mathcal{FN}Zcl(A))$ since $\mathcal{FN}Zint(\mathcal{FN}Zcl(A))$ is $\mathcal{FN}Zo$. Therefore, A is $\mathcal{FN}Zo$.

(ii) Similar proof of (i). ■

Definition 4.3.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. Then, U is fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) super disconnected (briefly, $\mathcal{FN}superDCon$ (respectively, $\mathcal{FN}\delta superDCon$, $\mathcal{FN}\delta\mathcal{S}superDCon$, $\mathcal{FNP}superDCon$ and $\mathcal{FN}ZsuperDCon$)) if there exists a $\mathcal{FN}ro$ (respectively, $\mathcal{FN}\delta ro$, $\mathcal{FN}\delta\mathcal{S}ro$, $\mathcal{FNP}ro$ and $\mathcal{FN}Zro$) set A in U such that $A \neq 0_N$ and $A \neq 1_N$.

A $\mathcal{FN}ts$ U is called fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) super connected (briefly, $\mathcal{FN}superCon$ (respectively, $\mathcal{FN}\delta superCon$, $\mathcal{FN}\delta\mathcal{S}superCon$, $\mathcal{FNP}superCon$ and $\mathcal{FN}ZsuperCon$)) if U is not $\mathcal{FN}superDCon$ (respectively, $\mathcal{FN}\delta superDCon$, $\mathcal{FN}\delta\mathcal{S}$ $superDCon$, $\mathcal{FNP}superDCon$ and $\mathcal{FN}ZsuperDCon$).

Example 4.1.

In Example 3.1, Let

(i) $S = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.2}\right\rangle, \left\langle\frac{s_3}{0.3}\right\rangle\right\}$ is $\mathcal{FN}ZsuperDCon$ (respectively, $\mathcal{FNP}superDCon$).

(ii) $S = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.5}\right\rangle\right\}$ is $\mathcal{FN}\delta\mathcal{S}superDCon$.

Theorem 4.4.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$, the equivalents are as follows:

(i) U is $\mathcal{FN}ZsuperCon$.

(ii) For each $\mathcal{FN}Zo$ set $L \neq 0_N$ in U, we have $\mathcal{FN}Zcl(L) = 1_N$.

(iii) For each $\mathcal{FN}Zc$ set $L \neq 1_N$ in U, we have $\mathcal{FN}Zint(L) = 0_N$.

(iv) There exists no $\mathcal{FN}Zo$ subsets L and M in U, such that $L \neq 0_N$, $M \neq 0_N$ and $L \leq M^c$.

(v) There exists no $\mathcal{FN}Zo$ subsets L and M in U, such that $L \neq 0_N$, $M \neq 0_N$, $M = (\mathcal{FN}Zcl(L))^c$ and $L = (\mathcal{FN}Zcl(M))^c$.

(vi) There exists no $\mathcal{FN}Zc$ subsets L and M in U such that $L \neq 1_N$, $M \neq 1_N$, $M = (\mathcal{FN}Zcl(L))^c$ and $L = (\mathcal{FN}Zcl(M))^c$.

Proof:

(i) $\Rightarrow$ (ii) Assume that there exists a $\mathcal{FN}Zo$ set $A \neq 0_N \ni \mathcal{FN}Zcl(A) \neq 1_N$. Now, take $B = \mathcal{FN}Zint(\mathcal{FN}Zcl(A))$. Then, B is proper $\mathcal{FN}Zro$ set in U which contradicts that U is $\mathcal{FN}ZsuperCon$-ness.

(ii) $\Rightarrow$ (iii) Let $A \neq 1_N$ be a $\mathcal{FN}Zc$ set in U. If $B = A^c$, then B is $\mathcal{FN}Zo$ set in U and $B \neq 0_N$. Hence, $\mathcal{FN}Zcl(A) = 1_N$, $(\mathcal{FN}Zcl(B))^c = 0_N \Rightarrow \mathcal{FN}Zint(B^c) = 0 \Rightarrow \mathcal{FN}Zint(A) = 0_N$.

(iii) $\Rightarrow$ (iv) Let A and B be $\mathcal{FN}Zo$ sets in $U \ni A \neq 0_N \neq B$ and $A \leq B^c$. Since B^c is $\mathcal{FN}Zc$ set in U and $B \neq 0_N$ implies $B^c \neq 1_N$, we obtain $\mathcal{FN}Zint(B^c) = 0_N$. But, from $A \leq B^c$, $0_N \neq A = \mathcal{FN}Zint(A) \leq \mathcal{FN}Zint(B^c) = 0_N$, which is a contradiction.

(iv) $\Rightarrow$ (i) Let $0_N \neq A \neq 1_N$ be $\mathcal{FN}Zro$ set in U. If we take $B = (\mathcal{FN}Zcl(A))^c$, we get $B \neq 0_N$. Otherwise, we have $B \neq 0_N$ implies $(\mathcal{FN}Zcl(A))^c = 0_N$. That implies $\mathcal{FN}Zcl(A) = 1_N$. That shows $A = \mathcal{FN}Zint(\mathcal{FN}Zcl(A)) = \mathcal{FN}Zint(1_N) = 1_N$. But this is to a contradiction to $A \neq 1_N$.

Further, $A \leq B^c$, this is also a contradiction.

(i) $\Rightarrow$ (v) Let A and B be $\mathcal{FN}Zo$ sets in $U \ni A \neq 0_N \neq B$ and $B = (\mathcal{FN}Zcl(A))^c$, $A = (\mathcal{FN}Zint(B))^c$. Now, $\mathcal{FN}Zint(\mathcal{FN}Zcl(A)) = \mathcal{FN}Zint(B^c) = (\mathcal{FN}Zcl(B))^c = A$ and $A \neq 0_N$, $A \neq 1_N$. Suppose not; if $A = 1_N$, then $1_N = (\mathcal{FN}Zcl(B))^c$ implies $0 = \mathcal{FN}Zcl(B) \Rightarrow B = 0$. This is a contradiction.

(v) $\Rightarrow$ (i) Let A be $\mathcal{FN}Zo$ set in $U \ni A = \mathcal{FN}Zint(\mathcal{FN}Zcl(A))$, $0_N \neq A \neq 1_N$. Now $B = (\mathcal{FN}Zcl(A))^c$ and $(\mathcal{FN}Zcl(B))^c = (\mathcal{FN}Zcl(\mathcal{FN}Zcl(A))^c)^c = \mathcal{FN}Zint(\mathcal{FN}Zcl(A)) = A$. This is a contradiction.

(v) $\Rightarrow$ (vi) Let A and B be $\mathcal{FN}Zc$ set in U such that $A \neq 1_N \neq B$. $B = (\mathcal{FN}Zint(A))^c$, $A = (\mathcal{FN}Zint(B))^c$. Taking $C = A^c$ and $D = B^c$, C and D become $\mathcal{FN}Zo$ set in U and $C \neq 0_N \neq D$, $(\mathcal{FN}Zcl(C))^c = (\mathcal{FN}Zcl(A^c))^c = ((\mathcal{FN}Zint(A))^c)^c = \mathcal{FN}Zint(A) = B^c = D$ and similarly $(\mathcal{FN}Zcl(D))^c = C$. But this is a contradiction.

(vi) $\Rightarrow$ (v) Similar as in above. ■

Remark 4.2.

Theorems 4.1 to 4.4, Remark 4.1 and Propositions 4.1 and 4.2 are also true for $\mathcal{FN}o$, $\mathcal{FN}\delta o$, $\mathcal{FN}\delta\mathcal{S}o$ and $\mathcal{FNP}o$ sets.

5. Fuzzy nano Z compact spaces

In this section, we present fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$, $\delta\gamma$ and Z) compact spaces and examine some of their fundamental properties using fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$, $\delta\gamma$ and Z) open sets.

Definition 5.1.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. A collection B of $\mathcal{FN}o$ (respectively, $\mathcal{FN}\delta o$, $\mathcal{FN}\delta\mathcal{S}o$, $\mathcal{FNP}o$ and $\mathcal{FN}Zo$) sets in U is called a fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) open cover (briefly, $\mathcal{FN}OCov$ (respectively, $\mathcal{FN}\delta OCov$, $\mathcal{FN}\delta\mathcal{S}OCov$, $\mathcal{FNP}OCov$ and $\mathcal{FN}ZOCov$)) of a subset B of U if $B \leq \bigvee\{L_\alpha : L_\alpha \in B\}$.

Definition 5.2.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. Then, U is said to be fuzzy nano (respectively, δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) compact (briefly, $\mathcal{FN}Comp$ (respectively, $\mathcal{FN}\delta Comp$, $\mathcal{FN}\delta\mathcal{S}Comp$, $\mathcal{FNP}Comp$ and $\mathcal{FN}ZComp$)) if every $\mathcal{FN}OCov$ (respectively, $\mathcal{FN}\delta OCov$, $\mathcal{FN}\delta\mathcal{S}OCov$, $\mathcal{FNP}OCov$ and $\mathcal{FN}ZOCov$) of U has a finite subcover.

Definition 5.3.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$. A subset A of U is said to be $\mathcal{FN}Comp$ (respectively, $\mathcal{FN}\delta Comp$, $\mathcal{FN}\delta\mathcal{S}Comp$, $\mathcal{FNP}Comp$, $\mathcal{FN}\delta\gamma Comp$ and $\mathcal{FN}ZComp$) relative to U if every $\mathcal{FN}OCov$ (respectively, $\mathcal{FN}\delta OCov$, $\mathcal{FN}\delta\mathcal{S}OCov$, $\mathcal{FNP}OCov$ and $\mathcal{FN}ZOCov$) of U has a finite subcover.

Theorem 5.1.

Let $(U, \tau_{\mathcal{F}}(F))$ be a $\mathcal{FN}ts$.

(i) Every $\mathcal{FN}Comp$ space is $\mathcal{FN}\delta Comp$.

(ii) Every $\mathcal{FN}\delta\mathcal{S}Comp$ is a $\mathcal{FN}Comp$.

(iii) Every $\mathcal{FNP}Comp$ is a $\mathcal{FN}Comp$.

(iv) Every $\mathcal{FN}ZComp$ is a $\mathcal{FN}\delta\mathcal{S}Comp$.

(v) Every $\mathcal{FN}ZComp$ is a $\mathcal{FNP}Comp$.

Proof:

(i) Let U be $\mathcal{FN}Comp$. Suppose U is not $\mathcal{FN}\delta Comp$. Then, there exists a $\mathcal{FN}\delta OCov\ B$ of U has no finite subcover. Since every $\mathcal{FN}\delta o$ set is $\mathcal{FN}o$ set, then we have $\mathcal{FN}OCov\ B$ of U, which has no finite subcover. This is a contradiction to U is $\mathcal{FN}Comp$. Hence, U is $\mathcal{FN}\delta Comp$.

(ii) Let U be $\mathcal{FN}\delta\mathcal{S}Comp$. Suppose U is not $\mathcal{FN}Comp$. Then, there exists a $\mathcal{FN}OCov\ B$ of U has no finite subcover. Since every $\mathcal{FN}o$ set is $\mathcal{FN}\delta\mathcal{S}o$ set, then we have $\mathcal{FN}\delta\mathcal{S}OCov\ B$ of U, which has no finite subcover. This is a contradiction to U is $\mathcal{FN}\delta\mathcal{S}Comp$. Hence, U is $\mathcal{FN}Comp$.

(iii) Let U be $\mathcal{FNP}Comp$. Suppose U is not $\mathcal{FN}Comp$. Then there exists a $\mathcal{FN}OCov\ B$ of U has no finite subcover. Since every $\mathcal{FN}o$ set is $\mathcal{FNP}o$ set, then we have $\mathcal{FNP}OCov\ B$ of U, which has no finite subcover. This is a contradiction to U is $\mathcal{FNP}Comp$. Hence U is $\mathcal{FN}Comp$.

(iv) Let U be $\mathcal{FN}ZComp$. Suppose U is not $\mathcal{FN}\delta\mathcal{S}Comp$. Then, there exists a $\mathcal{FN}\delta\mathcal{S}OCov\ B$

of U has no finite subcover. Since every $\mathcal{FN}\delta\mathcal{S}o$ set is $\mathcal{FN}Zo$ set, then we have $\mathcal{FN}ZOCov$ B of U, which has no finite subcover. This is a contradiction to U is $\mathcal{FN}ZComp$. Hence, U is $\mathcal{FN}\delta\mathcal{S}Comp$.

(v) Let U be $\mathcal{FN}ZComp$. Suppose U is not $\mathcal{FNP}Comp$. Then, there exists a $\mathcal{FNP}OCov$ B of U has no finite subcover. Since every $\mathcal{FNP}o$ set is $\mathcal{FN}Zo$ set, then we have $\mathcal{FN}ZOCov$ B of U, which has no finite subcover. This is a contradiction to U is $\mathcal{FN}ZComp$. Hence, U is $\mathcal{FNP}Comp$. ■

Theorem 5.2.

A $\mathcal{FN}Zc$ subset of a $\mathcal{FN}ZComp$ space U is $\mathcal{FN}ZComp$ relative to U.

Proof:

Let A be a $\mathcal{FN}Zc$ subset of a $\mathcal{FN}ZComp$ space U. Then, A^c is $\mathcal{FN}Zo$ in U. Let $B = \{A_i : i \in I\}$ be a $\mathcal{FN}ZOCov$ of A. Then, $B \bigvee \{A^c\}$ is a $\mathcal{FN}ZOCov$ of U. Since U is $\mathcal{FN}ZComp$, it has a finite subcover say $\{P_1, P_2, \cdots P_n, A^c\}$. Then, $\{P_1, P_2, \ldots, P_n\}$ is a finite $\mathcal{FN}ZOCov$. Thus, A is $\mathcal{FN}ZComp$ relative to U. ■

Theorem 5.3.

Let $f : (U_1, \tau_{\mathcal{F}}(F_1)) \to (U_2, \tau_{\mathcal{F}}(F_2))$ be a $\mathcal{FN}ZCts$ surjection and U_1 be $\mathcal{FN}ZComp$. Then, U_2 is $\mathcal{FN}Comp$.

Proof:

Let $f : U_1 \to U_2$ be a $\mathcal{FN}ZCts$ surjection and U_1 be $\mathcal{FN}ZComp$. Let $\{M_\alpha\}$ be a $\mathcal{FN}ZOCov$ for U_2. Since f is $\mathcal{FN}ZCts$, $\{f^{-1}(M_\alpha)\}$ is a $\mathcal{FN}ZOCov$ of U_1. Since U_1 is $\mathcal{FN}ZComp$, $\{f^{-1}(M_\alpha)\}$ contains a finite subcover, namely $\{f^{-1}(M_{\alpha_1}), f^{-1}(M_{\alpha_2}), \ldots, f^{-1}(M_{\alpha_n})\}$. Since f is surjection, $\{M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}\}$ is a finite subcover for U_2. Thus, U_2 is $\mathcal{FN}Comp$. ■

Theorem 5.4.

Let $f : (U_1, \tau_{\mathcal{F}}(F_1)) \to (U_2, \tau_{\mathcal{F}}(F_2))$ be a $\mathcal{FN}ZO$ function and U_2 be $\mathcal{FN}ZComp$. Then, U_1 is $\mathcal{FN}Comp$.

Proof:

Let $f : U_1 \to U_2$ be a $\mathcal{FN}ZO$ function and U_2 be $\mathcal{FN}ZComp$. Let $\{M_\alpha\}$ be a $\mathcal{FN}ZOCov$ for U_1. Since f is $\mathcal{FN}ZO$, $\{f(M_\alpha)$ is a $\mathcal{FN}ZOCov$ of U_2. Since U_2 is $\mathcal{FN}ZComp$, $\{f(M_\alpha)\}$ contains a finite sub $\mathcal{FN}ZOCov$, namely $\{f(M_{\alpha_1}), f(M_{\alpha_2}), \ldots, f(M_{\alpha_n})\}$. Then, $\{M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}\}$ is a finite subcover for U_1. Thus U_1 is $\mathcal{FN}Comp$. ■

Theorem 5.5.

The image of a $\mathcal{FN}ZComp$ space under a $\mathcal{FN}ZCts$ map is $\mathcal{FN}Comp$.

Proof:

Let $f : (U_1, \tau_{\mathcal{F}}(F_1)) \to (U_2, \tau_{\mathcal{F}}(F_2))$ be a $\mathcal{FN}ZCts$ map from a $\mathcal{FN}ZComp$ space U_1 onto U_2. Let $\{A_i : i \in I\}$ be a $\mathcal{FN}ZOCov$ of U_2. Since f is $\mathcal{FN}ZCts$, $\{f^{-1}(A_i) : i \in I\}$ is a $\mathcal{FN}ZOCov$ of U_1. As U_1 is $\mathcal{FN}ZComp$, the $\mathcal{FN}ZOCov$ $\{f^{-1}(A_i) : i \in I\}$ of U_1 has a finite subcover $\{f^{-1}(A_i) : i = 1, 2, 3, \ldots, n\}$. Therefore, $L = \bigcup_{i \in I} f^{-1}(A_i)$. Then, $f(L) = \bigcup_{i \in I} A_i$, that is $M = \bigcup_{i \in I} A_i$. Thus, $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for U_2. Hence, U_2 is $\mathcal{FN}Comp$. ■

Remark 5.1.

Theorems 5.2, 5.3, 5.4 and 5.5 are also true for $\mathcal{FN}o$, $\mathcal{FN}\delta o$, $\mathcal{FN}\delta\mathcal{S}o$ and $\mathcal{FNP}o$ sets.

6. Application Using Fuzzy Score Function

In this section, we present a fuzzy score function based on methodical approach for decision-making problem with fuzzy information.

The following essential steps are proposed the precise way to deal with select the proper attributes and alternative in the decision-making situation.

Step 1: Consider the universe of discourse (set of objects) $t_1, t_2, \cdots, t_m$, the set of alternatives $\tau_1, \tau_2, \cdots, \tau_n$, the set of decision attributes $\nu_1, \nu_2, \cdots, \nu_p$.

Step 2: Construct a fuzzy matrix of alternative verses objects and object verses decision attributes.

Step 3: Construct the fuzzy topologies $\tau_{\mathcal{F}_i}$.

Step 4: Find the fuzzy score values using fuzzy score function (briefly, $\mathcal{FSF}$) by Definition 13 in Kalaiyarasan et al. (2023a).

Step 5: Selection zone

The fuzzy score values are classified into three zones.

	$\mathcal{FSF}$
Highly acceptable zone	$0.50 \leq \mathcal{FSF}(\tau_{\mathcal{F}_j}) \leq 1$
Tolerable acceptable zone	$0.25 \leq \mathcal{FSF}(\tau_{\mathcal{F}_j}) \leq 0.50$
Unacceptable acceptable zone	$0.00 \leq \mathcal{FSF}(\tau_{\mathcal{F}_j}) \leq 0.25$

Table 1. Selection zone

Step 6: Ranking of attributes

Arrange all the values in descending order for fuzzy score values. Select significant attributes (values) in MADM processes, taking into account as highly acceptable zone and acceptable zone on both cases.

Step 7: End

Example 6.1.

In this example, a MADM problem of real life application on student engagement and attitude in mathematics achievement of 4^{th}, 5^{th} and 6^{th} grade students based on fuzzy score function along with graphical representation to show an extremely acceptable zone.

Step 1: Problem field selection:

Consider the following attributes about student's engagement and attitude are Time Spent on Homework in Mathematics (Ma_1), Like Learning in Mathematics (Ma_2), Struggle in Mathematics (Ma_3), Confident in Mathematics (Ma_4) and Value in Mathematics (Ma_5) of 4^{th}, 5^{th} and 6^{th} grade students. The data in Table 2 are the fuzzy numbers of the attributes and alternatives respectively.

Step 2: Construct a fuzzy matrix of alternative verses attributes:

	Ma_1	Ma_2	Ma_3	Ma_4	Ma_5
4^{th} grade	0.17	0.28	0.04	0.39	0.10
5^{th} grade	0.22	0.24	0.13	0.34	0.19
6^{th} grade	0.23	0.20	0.03	0.50	0.09

Table 2. Fuzzy set decision matrix

Step 3: Construct the fuzzy nano topologies $(\tau_{\mathcal{F}_j})$:

(i) $\tau_{\mathcal{F}_1} = \{0, 1, 0.17, 0.22, 0.23\}$.

(ii) $\tau_{\mathcal{F}_2} = \{0, 1, 0.28, 0.24, 0.20\}$.

(iii) $\tau_{\mathcal{F}_3} = \{0, 1, 0.04, 0.13, 0.03\}$.

(iv) $\tau_{\mathcal{F}_4} = \{0, 1, 0.39, 0.34, 0.50\}$.

(v) $\tau_{\mathcal{F}_5} = \{0, 1, 0.10, 0.19, 0.09\}$.

Step 4: Find the fuzzy score values:

Step 5: Selection zone

The arrangement of score values are

$$\mathcal{F}SF(\tau_{\mathcal{F}_4}) \geq \mathcal{F}SF(\tau_{\mathcal{F}_2}) \geq \mathcal{F}SF(\tau_{\mathcal{F}_1}) \geq \mathcal{F}SF(\tau_{\mathcal{F}_5}) \geq \mathcal{F}SF(\tau_{\mathcal{F}_3}).$$

attributes	fuzzy score value
Ma_1	$\mathcal{F}SF(\tau_{\mathcal{F}_1}) = 0.324$
Ma_2	$\mathcal{F}SF(\tau_{\mathcal{F}_2}) = 0.344$
Ma_3	$\mathcal{F}SF(\tau_{\mathcal{F}_3}) = 0.24$
Ma_4	$\mathcal{F}SF(\tau_{\mathcal{F}_4}) = 0.446$
Ma_5	$\mathcal{F}SF(\tau_{\mathcal{F}_5}) = 0.276$

Table 3. Fuzzy score values

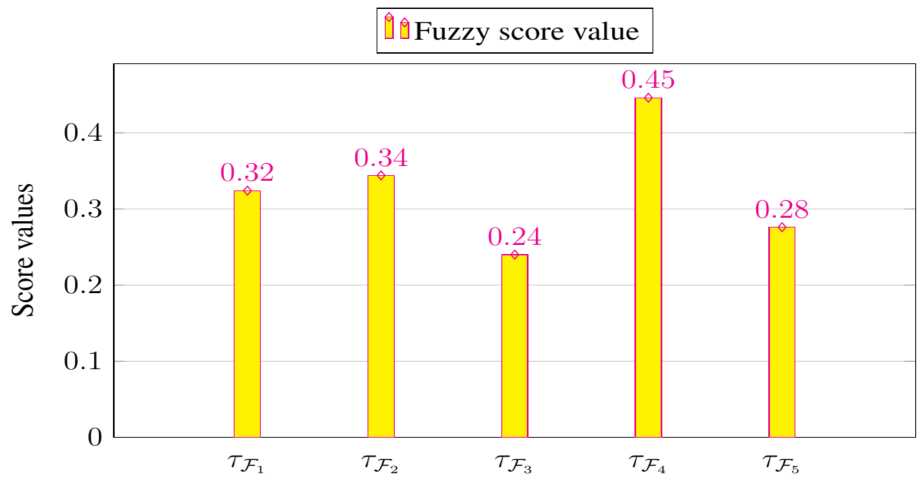

Figure 1. Fuzzy score values

Step 6: Ranking of attributes

From Table 3, Confidence in Mathematics [$\mathcal{F}SF(\tau_{\mathcal{F}_4})$] shows the more impact on students achievement and follows Like Learning [$\mathcal{F}SF(\tau_{\mathcal{F}_2})$] and Times Spent on Mathematics [$\mathcal{F}SF(\tau_{\mathcal{F}_1})$] based on fuzzy score values.

Step 7: End

7. Conclusion

In this paper, fuzzy nano Z connected spaces and fuzzy nano Z disconnected spaces in the fuzzy nano topological space have been introduced. Also, we have introduced the fuzzy nano Z compactness and fuzzy nano Z separated sets in the fuzzy nano topological space and we expect that the findings in this paper will aid researchers in improving and promoting additional research on fuzzy nano topology in order to develop a broad framework for their practical applications.

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