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Further Result on the Quasilinearization Method For an Initial Value Problem on Time Scales

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Abstract

We have investigated the generalized quasilinearization method under some convenient conditions for nonlinear initial value problem (IVP) of dynamic equation on time scale and constructed by monotone sequences of function by using comparison theorem. The solutions of linear IVPs of dynamic equation on time scale converge uniformly and monotonically to the unique solution of the original problem and the convergence is quadratic.

Keywords: Quasilinearization; Quadratic convergence; Comparison theorem; Time scale; Lower and upper solutions

MSC 2010 No.: 34A12, 34A45, 34C11

1. Introduction

The quasilinearization method technique coupled with the method of lower solution (LS) and upper solution (US) offer a clear analytic representation for the solution of nonlinear dynamic equation which provides (LS) and (US) predictions for the solution of a problem. The calculus theory on time scales (see Bohner and Peterson (2001), Bohner and Peterson (2003) and references mentioned here) was put forward by Stefan Hilger in 1990 to connect continuous and discrete analyses. There are many applications and it has a vast potential. The study of dynamical equations on time scales is a very interesting research area that many researchers are interested in the qualitative study of dynamic equation on time scales. During last years, many studies, books and articles have been discussed and many interesting results were obtained. Some of the papers and books that can be found are given by references to this paper.

Eloe (2002) has developed the method of quasilinearization to a family of two point boundary value problems for dynamic equations on compact measure chains.

Krivec and Mandelzweig (2001) analyzed the convergence, monotonicity and its fast by using the quasilinearization method to quantum mechanics. It was shown that the first few iterations provide very accurate results. The number of iterations necessary to reach a given precision only moderately increases for its larger values. Krivec and Mandelzweig (2003) applied to the quasilinearization method of solving nonlinear differential equations to the quantum mechanics by casting the Schrödinger equation in the nonlinear Riccati form. The authors observed that the quasilinearization method gave excellent results when applied to computation of ground and excited bound state energies and wave functions for a variety of the potentials in quantum mechanics.

Akın et al. (2005) applied the quasilinearization method to the unique solution of the BVP on time scales by monotone convergent sequences of (LS) and (US). They obtained the rate of convergence.

Atici and Topal (2005) studied the convergence of monotone sequences for nonlinear dynamic equations on time scales. They constructed two sequences which converge to the unique solution of BVP.

Yang and Vatsala (2005) have developed generalized quasilinearization method for reaction diffusion systems when the functions are the sum of convex and concave functions and obtained that the corresponding linear systems converge monotonically, uniformly and quadratically to the unique solution of the nonlinear problem.

Jankowski (2007) also used this method to integro differential equations of Volterra type. It was also shown that two monotone sequences converge quadratically to a unique solution of the problem.

Yakar and Yakar (2010) applied the quasilinearization technique in Caputo's fractional differential equation. Under some conditions the authors obtained lower and upper sequences of the solutions of linear differential equations and showed that these sequences converge to the unique solution of the nonlinear differential equation semiquadratically and uniformly. Yakar (2014) applied the method to causal differential equations and obtained upper and lower sequences with initial time difference of the linear causal differential equations. He proved that these sequences converge to the unique solution of the equation superlinearly and uniformly.

Lakshmikantham and Vatsala (2013) applied the method to an initial value problem and obtained lower and upper sequences on ordinary differantial equantions.

Koleva and Vulkov (2013) offered a fast quasilinearization numerical scheme, coupled with Rothe's method, for nonlinear parabolic equations. Under convenient conditions, they obtained the uniform and monotone convergence that provide quadratic on each time level. Namely, numerical results for nonlinear problems of optimal investment were presented and discussed.

Yakar (2015) investigated the qualitative behavior of a perturbed dynamic systems on time scales that differs in initial time with respect to the unperturbed dynamic systems on time scales. He compared the classical notion of stability to the notion of initial time difference stability on time scales. Yakar, Arslan and Çiçek (2015) studied the monotone iterative technique by choosing upper and lower solutions with initial time difference that start at different initial times for the initial value problem. Under convenient conditions, they obtained monotone sequences which converge uniformly and monotonically to minimal and maximal solutions of initial value problem. Yakar and Arslan (2019) constructed new definitions for a causal terminal value problem involving Riemann-Liouville fractional derivatives and considered the Riemann-Liouville fractional causal terminal value problem with the terminal value. They obtained the unique solution by combining techniques from generalized quasilinearization. They obtained monotone sequences which converge uniformly to the unique solution of initial value problem. They also showed the convergence is quadratic.

Wang and Agarwal (2021) have widely applied the time scale calculus to study dynamic systems in both theoretical and practical aspects. Wang and Tian (2015) investigated nonlinear boundary problems for difference equations with causal operators. They obtained a criteria on the existence of extremal solutions by using the method of upper and lower solutions coupled with the monotone iterative technique. Wang and Tian (2014) discussed nonlinear boundary value problems for causal differential equations with two monotone functions.

Yüzbaşı and Izadi (2022) developed two numerical methods based on the Bessel polynomials to solve the fractional-order HIV-1 infection model of CD4 T-cells considering the impact of antiviral drug treatment. In the first method which is called as Bessel matrix method, they transformed the HIV problem into a system of nonlinear equations by using the Bessel polynomial. The second method, which is called the Bessel-QLM method, transformed the HIV problem to a sequence of linear equations by using the technique of quasilinearization, and they solved this by the direct Bessel matrix method.

2. Main Theorem

Let us consider the (LS) and (US) of nonlinear dynamic equation of IVP:

$$u^{\Delta} = w(\theta, u) = f_3(\theta, u) + f_2(\theta, u) + f_1(\theta, u), \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $f_3(\theta, u), f_2(\theta, u), f_1(\theta, u) \in C_{rd}\left[T^k \times R, R\right]$.

We obtained the monotone sequences on time scale which converge uniformly to the unique solution of the non-linear initial value problem (1). For this aim, the auxiliary linear IVPs were constructed. Each element of the monotone function sequences are the solution of these linear differential equations. By using the technique of upper and lower solutions, under convenient conditions we showed this convergence is quadratic. In addition to this, we will prefer Δt instead of dt in the proof, since we are studying on time scale and not ordinary differential equation.

Theorem 2.1.

Suppose that the following hypotheses are satisfied:

(C_1) Let $\Phi_0, \Psi_0 \in C_{rd}\left[T^k, R\right]$ be (LS) and (US) of (1), respectively, such that $\Phi_0(\theta) \leq \Psi_0(\theta)$ on T^k.
(C_2) $f_{1_u}(\theta, u), f_{2_u}(\theta, u), f_{3_u}(\theta, u), f_{1_{uu}}(\theta, u), f_{2_{uu}}(\theta, u), f_{3_{uu}}(\theta, u)$ exist and continuous functions on $T^k \times R$, where $f_3(\theta, u), f_2(\theta, u), f_1(\theta, u) \in C^2_{rd}\left[T^k \times R, R\right]$, and holds $f_{1_{uu}}(\theta, u) \geq 0$, $f_{2_{uu}}(\theta, u) + \phi_{uu}(\theta, u) \geq 0$, whenever ϕ_{uu} exits and $\phi_{uu}(\theta, u) > 0$, and $f_{3_{uu}}(\theta, u) \leq 0$, on Ω.

Then, there exist the monotone sequences $\{\Phi_n(\theta)\}$ and $\{\Psi_n(\theta)\}$ which converge uniformly to the unique solution of (1) and this convergence is quadratic.

Proof:

Because of (C_2) it can be written that for $\Phi_0(\theta) \leq u_2 \leq u_1 \leq \Psi_0(\theta)$,

$$\begin{aligned} f_1(\theta, u_1) - f_1(\theta, u_2) &\leq L_1(u_1 - u_2), L_1 > 0, \\ f_2(\theta, u_1) - f_2(\theta, u_2) &\leq L_2(u_1 - u_2), L_2 > 0, \\ f_3(\theta, u_1) - f_3(\theta, u_2) &\leq L_3(u_1 - u_2), L_3 > 0. \end{aligned} \tag{3}$$

Let us note that $f_{1_{uu}}(\theta, u) \geq 0, f_{2_{uu}}(\theta, u) + \phi_{uu}(\theta, u) \geq 0$, and $f_{3_{uu}}(\theta, u) \leq 0$, provide the following inequalities. For $u \geq v,\ u, v \in \Omega$,

$$f_1(\theta, u) \geq f_1(\theta, v) + f_{1u}(\theta, v)(u - v), \tag{4}$$

$$f_2(\theta, u) \geq f_2(\theta, v) + [f_{2u}(\theta, v) + \phi_u(\theta, v)](u - v) - [\phi(\theta, u) - \phi(\theta, v)], \tag{5}$$

$$f_3(\theta, u) \geq f_3(\theta, v) + f_{3u}(\theta, u)(u - v). \tag{6}$$

Now, for the construction of the sequences, we take the following auxiliary IVPs

$$\begin{aligned} u^{\Delta} &= g\left(\theta, \Phi_0, \Psi_0; u\right) = f_1\left(\theta, \Phi_0\right) + f_{1u}\left(\theta, \Phi_0\right)\left(u - \Phi_0\right) \\ &\quad + f_2\left(\theta, \Phi_0\right) + \left[f_{2u}\left(\theta, \Phi_0\right) + \phi_u\left(\theta, \Phi_0\right)\right]\left(u - \Phi_0\right) \\ &\quad - \left[\phi\left(\theta, u\right) - \phi\left(\theta, \Phi_0\right)\right] + f_3\left(\theta, \Phi_0\right) + f_{3u}\left(\theta, \Psi_0\right)\left(u - \Phi_0\right), \\ u\left(0\right) &= u_0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} v^{\Delta} &= G\left(\theta, \Phi_0, \Psi_0; v\right) = f_1\left(\theta, \Psi_0\right) + f_{1u}\left(\theta, \Phi_0\right)\left(v - \Psi_0\right) \\ &\quad + f_2\left(\theta, \Psi_0\right) + \left[f_{2u}\left(\theta, \Phi_0\right) + \phi_u\left(\theta, \Phi_0\right)\right]\left(v - \Psi_0\right) \\ &\quad - \left[\phi\left(\theta, v\right) - \phi\left(\theta, \Psi_0\right)\right] + f_3\left(\theta, \Psi_0\right) + f_{3u}\left(\theta, \Psi_0\right)\left(v - \Psi_0\right), \\ v\left(0\right) &= u_0. \end{aligned} \tag{8}$$

Since Φ_0 is (LS) of (1),

$$\Phi_0^{\Delta} \leq f_3\left(\theta, \Phi_0\right) + f_2\left(\theta, \Phi_0\right) + f_1\left(\theta, \Phi_0\right) = g\left(\theta, \Phi_0, \Psi_0; \Phi_0\right),$$

and Ψ_0 is (US) of (1) and by the inequalities (4), (5), and (6), it can be written

$$\begin{aligned} \Psi_0^{\Delta} &\geq f_1\left(\theta, \Phi_0\right) + f_{1u}\left(\theta, \Phi_0\right)\left(\Psi_0 - \Phi_0\right) \\ &\quad + f_2\left(\theta, \Phi_0\right) + \left[f_{2u}\left(\theta, \Phi_0\right) + \phi_u\left(\theta, \Phi_0\right)\right]\left(\Psi_0 - \Phi_0\right) - \left[\phi\left(\theta, \Psi_0\right) - \phi\left(\theta, \Phi_0\right)\right] \\ &\quad + f_3\left(\theta, \Phi_0\right) + f_{3u}\left(\theta, \Psi_0\right)\left(\Psi_0 - \Phi_0\right) = G\left(\theta, \Phi_0, \Psi_0; \Psi_0\right). \end{aligned}$$

Therefore, Ψ_0 and Φ_0 are (LS) and (US) of (7). Then, there exists a unique solution of (7) called Φ_1, such that $\Phi_0\left(\theta\right) \leq \Phi_1\left(\theta\right) \leq \Psi_0\left(\theta\right), \theta \in T^k$. Similarly, we obtain

$$\begin{aligned} \Phi_0^{\Delta} &\leq f_1\left(\theta, \Phi_0\right) + f_2\left(\theta, \Phi_0\right) + f_3\left(\theta, \Phi_0\right) \\ &\leq f_1\left(\theta, \Psi_0\right) + f_{1u}\left(\theta, \Phi_0\right)\left(\Phi_0 - \Psi_0\right) \\ &\quad + f_2\left(\theta, \Psi_0\right) + \left[f_{2u}\left(\theta, \Phi_0\right) + \phi_u\left(\theta, \Phi_0\right)\right]\left(\Phi_0 - \Psi_0\right) - \left[\phi\left(\theta, \Phi_0\right) - \phi\left(\theta, \Psi_0\right)\right] \\ &\quad + f_3\left(\theta, \Psi_0\right) + f_{3u}\left(\theta, \Psi_0\right)\left(\Phi_0 - \Psi_0\right) \\ &= G\left(\theta, \Phi_0, \Psi_0; \Phi_0\right), \end{aligned}$$

and

$$\Psi_0^{\Delta} \geq f_1\left(\theta, \Psi_0\right) + f_2\left(\theta, \Psi_0\right) + f_3\left(\theta, \Psi_0\right) = G\left(\theta, \Phi_0, \Psi_0; \Psi_0\right).$$

Hence, Ψ_0 and Φ_0 are natural (LS) and (US) of (8). Thus, there exists a unique solution of (8) called Ψ_1 such that $\Phi_0\left(\theta\right) \leq \Psi_1\left(\theta\right) \leq \Psi_0\left(\theta\right), \theta \in T^k$. Now, it will be shown that $\Phi_1\left(\theta\right) \leq \Psi_1\left(\theta\right)$. By using (7), we get

$$\begin{aligned} \Phi_1^{\Delta} &= g\left(\theta, \Phi_0, \Psi_0; \Phi_1\right) = f_1\left(\theta, \Phi_0\right) + f_{1u}\left(\theta, \Phi_0\right)\left(\Phi_1 - \Phi_0\right) \\ &\quad + f_2\left(\theta, \Phi_0\right) + \left[f_{2u}\left(\theta, \Phi_0\right) + \phi_u\left(\theta, \Phi_0\right)\right] \\ &\quad + \left[f_{2u}\left(\theta, \Phi_0\right) + \phi_u\left(\theta, \Phi_0\right)\right]\left(\Phi_1 - \Phi_0\right) - \left[\phi\left(\theta, \Phi_1\right) - \phi\left(\theta, \Phi_0\right)\right] \\ &\quad + f_3\left(\theta, \Phi_0\right) + f_{3u}\left(\theta, \Psi_0\right)\left(\Phi_1 - \Phi_0\right) \\ &\leq f_1\left(\theta, \Phi_1\right) + f_2\left(\theta, \Phi_1\right) + f_3\left(\theta, \Phi_1\right) + \left(\Phi_1 - \Phi_0\right)\left[f_{3u}\left(\theta, \Psi_0\right) - f_{3u}\left(\theta, \Phi_1\right)\right]. \end{aligned}$$

Due to the convexity and concavity properties of the functions f_1, f_2, f_3 for $\Psi_0 \geq \Phi_1$, we get

$$\Phi_1^{\Delta} \leq f_1\left(\theta, \Phi_1\right) + f_2\left(\theta, \Phi_1\right) + f_3\left(\theta, \Phi_1\right).$$

Similarly, by using (8), we obtain

$$\begin{aligned}\Psi_1^{\Delta} &= G(\theta,\Phi_0,\Psi_0;\Psi_1) = f_1(\theta,\Psi_0) + f_{1u}(\theta,\Phi_0)(\Psi_1-\Psi_0)\\ &\quad + f_2(\theta,\Psi_0) + [f_{2u}(\theta,\Phi_0)+\phi_u(\theta,\Phi_0)](\Psi_1-\Psi_0) - [\phi(\theta,\Psi_1)-\phi(\theta,\Psi_0)]\\ &\quad + f_3(\theta,\Psi_1) + f_{3u}(\theta,\Psi_0)(\Psi_1-\Psi_0)\\ &\geq f_1(\theta,\Psi_1) + f_{1u}(\theta,\Psi_1)(\Psi_0-\Psi_1) + f_{1u}(\theta,\Phi_0)(\Psi_1-\Psi_0)\\ &\quad + f_2(\theta,\Psi_1) + [f_{2u}(\theta,\Psi_1)+\phi_u(\theta,\Psi_1)](\Psi_0-\Psi_1) - [\phi(\theta,\Psi_0)-\phi(\theta,\Psi_1)]\\ &\quad + [f_{2u}(\theta,\Phi_0)+\phi_u(\theta,\Phi_0)](\Psi_1-\Psi_0) - [\phi(\theta,\Psi_1)-\phi(\theta,\Psi_0)]\\ &\quad + f_3(\theta,\Psi_1) + f_{3u}(\theta,\Psi_0)(\Psi_0-\Psi_1) + f_{3u}(\theta,\Psi_0)(\Psi_1-\Psi_0).\end{aligned}$$

By the condition (C_2) with $\Psi_1 \geq \Phi_0$, we have

$$\Psi_1^{\Delta} \geq f_1(\theta,\Psi_1) + f_2(\theta,\Psi_1) + f_3(\theta,\Psi_1).$$

The condition (3) and Ψ_1, Φ_1 are (LS) and (US) of (1), respectively, with $\Phi_1(0) \leq \Psi_1(0)$, yield $\Phi_1(\theta) \leq \Psi_1(\theta)$. Therefore, we have established the inequalities such as,

$$\Phi_0(\theta) \leq \Phi_1(\theta) \leq \Psi_1(\theta) \leq \Psi_0(\theta).$$

Let us continue with the next step and consider the following IVPs,

$$\begin{aligned}u^{\Delta} &= g(\theta,\Phi_1,\Psi_1;u) = f_1(\theta,\Phi_1) + f_{1u}(\theta,\Phi_1)(u-\Phi_1)\\ &\quad + f_2(\theta,\Phi_1) + [f_{2u}(\theta,\Phi_1)+\phi_u(\theta,\Phi_1)](u-\Phi_1)\\ &\quad - [\phi(\theta,u)-\phi(\theta,\Phi_1)] + f_3(\theta,\Phi_1) + f_{3u}(\theta,\Psi_1)(u-\Phi_1),\\ u(0) &= u_0,\end{aligned} \tag{9}$$

and

$$\begin{aligned}v^{\Delta} &= G(\theta,\Phi_1,\Psi_1;v) = f_1(\theta,\Psi_1) + f_{1u}(\theta,\Phi_1)(v-\Psi_1)\\ &\quad + f_2(\theta,\Psi_1) + [f_{2u}(\theta,\Phi_1)+\phi_u(\theta,\Phi_1)](v-\Psi_1)\\ &\quad - [\phi(\theta,v)-\phi(\theta,\Psi_1)] + f_3(\theta,\Psi_1) + f_{3u}(\theta,\Psi_1)(v-\Psi_1),\\ v(0) &= u_0.\end{aligned} \tag{10}$$

By (7), we have

$$\Phi_1^{\Delta} \leq f_1(\theta,\Phi_1) + f_2(\theta,\Phi_1) + f_3(\theta,\Phi_1) = g(\theta,\Phi_1,\Psi_1;\Phi_1).$$

Similarly, we can get

$$\begin{aligned}\Psi_1^{\Delta} &\geq f_1(\theta,\Psi_1) + f_2(\theta,\Psi_1) + f_3(\theta,\Psi_1)\\ &\geq f_1(\theta,\Phi_1) + f_{1u}(\theta,\Phi_1)(\Psi_1-\Phi_1)\\ &\quad + f_2(\theta,\Phi_1) + [f_{2u}(\theta,\Phi_1)+\phi_u(\theta,\Phi_1)](\Psi_1-\Phi_1) - [\phi(\theta,\Psi_1)-\phi(\theta,\Phi_1)]\\ &\quad + f_3(\theta,\Phi_1) + f_{3u}(\theta,\Psi_1)(\Psi_1-\Phi_1)\\ &= g(\theta,\Phi_1,\Psi_1;\Psi_1).\end{aligned}$$

Therefore, Ψ_1 and Φ_1 are natural upper and lower solutions of (9). Then, there exists a unique solution of (9) called Φ_2, such that $\Phi_1(\theta) \leq \Phi_2(\theta) \leq \Psi_1(\theta), \theta \in T^k$. Similarly, we can see the following,

$$\begin{aligned}\Phi_1^{\Delta} &\leq f_1(\theta,\Phi_1)+f_2(\theta,\Phi_1)+f_3(\theta,\Phi_1)\\ &\leq f_1(\theta,\Psi_1)+f_{1u}(\theta,\Phi_1)(\Phi_1-\Psi_1)\\ &\quad +f_2(\theta,\Psi_1)+[f_{2u}(\theta,\Phi_1)+\phi_u(\theta,\Phi_1)](\Phi_1-\Psi_1)-[\phi(\theta,\Phi_1)-\phi(\theta,\Psi_1)]\\ &\quad +f_3(\theta,\Psi_1)+f_{3u}(\theta,\Psi_1)(\Phi_1-\Psi_1)\\ &= G(\theta,\Phi_1,\Psi_1;\Phi_1),\end{aligned}$$

and

$$\begin{aligned}\Psi_1^{\Delta} &\geq f_1(\theta,\Psi_1)+f_2(\theta,\Psi_1)+f_3(\theta,\Psi_1)\\ &= G(\theta,\Phi_1,\Psi_1;\Psi_1).\end{aligned}$$

So, Ψ_1 and Φ_1 are natural (LS) and (US) of (10). Then, there exists a unique solution of (10) called Ψ_2, such that $\Phi_1(\theta)\leq\Psi_2(\theta)\leq\Psi_1(\theta), \theta\in T^k$. By similar steps, as in the previous work, we can show $\Phi_2\leq\Psi_2$. As a result, we get

$$\Phi_0(\theta)\leq\Phi_1(\theta)\leq\Phi_2(\theta)\leq\Psi_2(\theta)\leq\Psi_1(\theta)\leq\Psi_0(\theta).$$

If this process is continued in this way, it can be obtain that

$$\Phi_0(\theta)\leq\Phi_1(\theta)\leq\Phi_2(\theta)\leq...\leq\Phi_n(\theta)\leq\Psi_n(\theta)\leq...\leq\Psi_2(\theta)\leq\Psi_1(\theta)\leq\Psi_0(\theta).$$

Here, the elements of the monotone sequences $\{\Phi_n(\theta)\}$ and $\{\Psi_n(\theta)\}$ are the unique solutions of the following linear IVPs,

$$\begin{aligned}\Phi_{n+1}^{\Delta} &= g(\theta,\Phi_n,\Psi_n;\Phi_{n+1}), \Phi_{n+1}(0)=u_0,\\ \Psi_{n+1}^{\Delta} &= G(\theta,\Phi_n,\Psi_n;\Psi_{n+1}), \Psi_{n+1}(0)=u_0.\end{aligned}$$

Since the sequences $\{\Phi_n(\theta)\}$ and $\{\Psi_n(\theta)\}$ are uniformly bounded and equicontinuous, by Arzela-Ascoli Theorem (Green and Valentine (1961)) it is easy to conclude that these sequences converge to the unique solution of (1) uniformly. Now, we shall show that the convergence of the sequences $\{\Phi_n(\theta)\}$ and $\{\Psi_n(\theta)\}$ to the unique solution $u(\theta)$ of (1) is quadratic on T^k. That is, we have to show that

$$\max_J|u(\theta)-\Phi_n(\theta)|\leq k\max_J|u(\theta)-\Phi_n(\theta)|^2, k>0,$$

where $T^k=J$. To show, let us define

$$r_{n+1}(\theta)=u(\theta)-\Phi_{n+1}(\theta)\geq 0,$$

$$s_{n+1}(\theta)=\Psi_{n+1}(\theta)-u(\theta)\geq 0.$$

Note that $r_{n+1}(0)=0$ and $s_{n+1}(0)=0$. Taking delta derivative to both sides and for convenience, $f_2(\theta,u)+\phi(\theta,u)=F(\theta,u)$ is taken and then may be obtained as,

$$\begin{aligned}r_{n+1}^{\Delta} &= u^{\Delta}-\Phi_{n+1}^{\Delta}\\ &= [f_1(\theta,u)-f_1(\theta,\Phi_n)]+[f_3(\theta,u)-f_3(\theta,\Phi_n)]\\ &\quad +[F(\theta,u)-F(\theta,\Phi_n)]-[\phi(\theta,u)-\phi(\theta,\Phi_{n+1})]\\ &\quad +[f_{1u}(\theta,\Phi_n)+f_{3u}(\theta,\Psi_n)+F_u(\theta,\Phi_n)]((u-\Phi_{n+1})-(u-\Phi_n)).\end{aligned}$$

Using the definition Φ_n, Ψ_n together with (C_2) and applying the mean value theorem, we write

$$\begin{aligned}
r_{n+1}^{\Delta} &= f_{1u}(\theta,a)\,r_n - f_{1u}(\theta,\Phi_n)\,r_n + f_{1u}(\theta,\Phi_n)\,r_{n+1} \\
&\quad +F_u(\theta,c)\,r_n - F_u(\theta,\Phi_n)\,r_n + F_u(\theta,\Phi_n)\,r_{n+1} - \phi_u(\theta,d)\,r_{n+1} \\
&\quad +f_{3u}(\theta,e)\,r_n - f_{3u}(\theta,\Psi_n)\,r_n + f_{3u}(\theta,\Psi_n)\,r_{n+1} \\
&= r_n\left[f_{1u}(\theta,a) - f_{1u}(\theta,\Phi_n)\right] + f_{1u}(\theta,\Phi_n)\,r_{n+1} \\
&\quad +r_n\left[F_u(\theta,c) - F_u(\theta,\Phi_n)\right] + r_{n+1}\left[F_u(\theta,\Phi_n) - \phi_u(\theta,d)\right] \\
&\quad +r_n\left[f_{3u}(\theta,e) - f_{3u}(\theta,\Psi_n)\right] + f_{3u}(\theta,\Psi_n)\,r_{n+1} \\
&\le r_n\left[f_{1u}(\theta,u) - f_{1u}(\theta,\Phi_n)\right] + f_{1u}(\theta,\Phi_n)\,r_{n+1} \\
&\quad +r_n\left[F_u(\theta,u) - F_u(\theta,\Phi_n)\right] + r_{n+1}\left[F_u(\theta,\Phi_n) - \phi_u(\theta,\Phi_n)\right] \\
&\quad +r_n\left[f_{3u}(\theta,u) - f_{3u}(\theta,\Psi_n)\right] + f_{3u}(\theta,\Psi_n)\,r_{n+1} \\
&\le r_n f_{1uu}(\theta,b)\,(u-\Phi_n) + f_{1u}(\theta,\Phi_n)\,r_{n+1} \\
&\quad +r_n F_{uu}(\theta,d)\,(u-\Phi_n) + r_{n+1}\left[F_u(\theta,\Phi_n) - \phi_u(\theta,\Phi_n)\right] \\
&\quad +r_n f_{3uu}(\theta,f_1)\,(u-\Psi_n) + f_{3u}(\theta,\Psi_n)\,r_{n+1} \\
&= r_n^2 f_{1uu}(\theta,b) + r_{n+1} f_{1u}(\theta,\Phi_n) \\
&\quad +r_n^2\left[F_{uu}(\theta,d) + \phi_{uu}(\theta,d)\right] + r_{n+1} F_u(\theta,\Phi_n) \\
&\quad -r_n s_n f_{3uu}(\theta,f_1) + r_{n+1} f_{3u}(\theta,\Psi_n),
\end{aligned}$$

where $\Phi_n < a < u,\ \Phi_n < c < u,\ \Phi_{n+1} < d < u,\ \Phi_n < e < u, \Phi_n < b < u,\ \Phi_n < d < u,\ u < f < \Psi_n$. Thus,

$$r_{n+1}^{\Delta} \le r_n^2\,(N+L+B) + r_{n+1}\,(M+K+A) - r_n s_n C,$$

where $|f_{1uu}(\theta,u)| \le N,\ \ |f_{1u}(\theta,u)| \le M,\ \ |f_{2uu}(\theta,u)| \le L,\ \ |f_{2u}(\theta,u)| \le K,\ \ |\phi_{uu}(\theta,u)| \le B,\ \ |f_{3uu}(\theta,u)| \le C,\ \ |f_{3u}(\theta,u)| \le A$. If the Cauchy inequality is applied to the term $p_n q_n C$, then we have

$$r_{n+1}^{\Delta} \le r_n^2\left(N+L+B+\frac{C}{2}\right) + r_{n+1}\,(M+K+A) + s_n^2\frac{C}{2},$$

which is linear in r_{n+1}. Now, by using Gronwall's inequality we get

$$0 \le r_{n+1} \le \int_0^\theta \left[r_n^2\left(N+L+B+\frac{C}{2}\right) + s_n^2\frac{C}{2}\right] e^{(M+K+A)(\theta-s)}\Delta s.$$

Therefore, one can see

$$\max_J |u-\Phi_{n+1}| \le \left(N+L+B+\frac{C}{2}\right)\frac{e^{(M+K+A)\theta}}{M+K+A}\max_J |u-\Phi_n|^2 + \frac{C}{2}\frac{e^{(M+K+A)\theta}}{M+K+A}\max_J |\Psi_n - u|^2.$$

Similarly, we get

$$\begin{aligned}
s_{n+1}^{\Delta} &= \Psi_{n+1}^{\Delta} - u^{\Delta} \\
&= \left[f_1(\theta,\Psi_n) - f_1(\theta,u)\right] + f_{1u}(\theta,\Phi_n)\,(\Psi_{n+1}-\Psi_n) \\
&\quad + \left[F(\theta,\Psi_n) - F(\theta,u)\right] + F_u(\theta,\Phi_n)\,(\Psi_{n+1}-\Psi_n) - \left[\phi(\theta,\Psi_{n+1}) - \phi(\theta,u)\right] \\
&\quad + \left[f_3(\theta,\Psi_n) - f_3(\theta,u)\right] + f_{3u}(\theta,\Psi_n)\,(\Psi_{n+1}-\Psi_n).
\end{aligned}$$

By using the definition $\Phi_n,\ \Psi_n$ with the condition (C_2) and by applying the mean value theorem, we obtain

$$\begin{aligned}
s_{n+1}^{\Delta} &= f_{1u}(\theta,a)\,s_n + f_{1u}(\theta,\Phi_n)(s_{n+1}-s_n)\\
&\quad + F_u(\theta,c)\,s_n + F_u(\theta,\Phi_n)(s_{n+1}-s_n) - \phi_u(\theta,d)\,s_{n+1}\\
&\quad + f_{3u}(\theta,f_1)\,s_n + f_{3u}(\theta,\Psi_n)(s_{n+1}-s_n)\\
&= s_n\left[f_{1u}(\theta,a) - f_{1u}(\theta,\Phi_n)\right] + s_{n+1}f_{1u}(\theta,\Phi_n)\\
&\quad + s_n\left[F_u(\theta,c) - F_u(\theta,\Phi_n)\right] + s_{n+1}\left[F_u(\theta,\Phi_n) - \phi_u(\theta,d)\right]\\
&\quad + s_n\left[f_{3u}(\theta,f_1) - f_{3u}(\theta,\Psi_n)\right] + s_{n+1}f_{3u}(\theta,\Psi_n)\\
&\le s_n\left[f_{1u}(\theta,\Psi_n) - f_{1u}(\theta,\Phi_n)\right] + s_{n+1}f_{1u}(\theta,\Phi_n)\\
&\quad + s_n\left[F_u(\theta,\Psi_n) - F_u(\theta,\Phi_n)\right] + s_{n+1}\left[F_u(\theta,\Phi_n) - \phi_u(\theta,\Phi_n)\right]\\
&\quad + s_n\left[f_{3u}(\theta,u) - f_{3u}(\theta,\Psi_n)\right] + s_{n+1}f_{3u}(\theta,\Psi_n)\\
&\le s_n^2 f_{1uu}(\theta,b) + r_n s_n f_{1uu}(\theta,b) + f_{1u}(\theta,\Phi_n)\,s_{n+1}\\
&\quad + s_n^2 F_{uu}(\theta,e) + r_n s_n F_{uu}(\theta,e) + s_{n+1}\left[F_u(\theta,\Phi_n) - \phi_u(\theta,\Phi_n)\right]\\
&\quad - s_n^2 f_{3uu}(\theta,f_3) + f_{3u}(\theta,\Psi_n)\,s_{n+1}\\
&= s_n^2 f_{1uu}(\theta,b) + r_n s_n f_{1uu}(\theta,b) + f_{1u}(\theta,\Phi_n)\,s_{n+1}\\
&\quad + s_n^2\left[F_{uu}(\theta,e) + \phi_{uu}(\theta,e)\right] + r_n s_n\left[F_{uu}(\theta,e) + \phi_{uu}(\theta,e)\right] + s_{n+1}F_u(\theta,\Phi_n)\\
&\quad - s_n^2 f_{3uu}(\theta,f_3) + f_{3u}(\theta,\Psi_n)\,s_{n+1},
\end{aligned}$$

where $u < a < \Psi_n,\ u < c < \Psi_n, u < d < \Psi_{n+1},\ u < f_1 < \Psi_n, \Phi_n < b < \Psi_n,\ \Phi_n < e < \Psi_n,\ u < h < \Psi_n$. Hence,

$$\begin{aligned}
s_{n+1}^{\Delta} &\le s_n^2 N + r_n s_n N + M s_{n+1}\\
&\quad + s_n^2\left[L+B\right] + r_n s_n\left[L+B\right] + s_{n+1}K\\
&\quad - s_n^2 C + A s_{n+1},
\end{aligned}$$

where $|f_{1uu}(\theta,u)| \le N,\ \ |f_{1u}(\theta,u)| \le M,\ \ |f_{2uu}(\theta,u)| \le L,\ \ |f_{2u}(\theta,u)| \le K,\ \ |\phi_{uu}(\theta,u)| \le B,\ \ |f_{3uu}(\theta,u)| \le C,\ \ |f_{3u}(\theta,u)| \le A$. If the Cauchy inequality is applied to the term $r_n s_n C$, then we can obtain

$$s_{n+1}^{\Delta} \le \left(\frac{3N}{2} + \frac{3}{2}(L+B) - C\right)s_n^2 + \left(\frac{N}{2} + \frac{1}{2}(L+B)\right)r_n^2 + (M+K+A)\,s_{n+1},$$

which is linear in s_{n+1}. When we apply the Gronwall's inequality, then we have

$$0 \le s_{n+1} \le \int_0^{\theta}\left[\left(\frac{3N}{2} + \frac{3}{2}(L+B) - C\right)s_n^2 + \left(\frac{N}{2} + \frac{1}{2}(L+B)\right)r_n^2\right]e^{(M+K+A)(\theta-s)}\Delta s.$$

Similarly, one can write for s_{n+1}. This yields the desired result

$$\begin{aligned}
\max_J |\Psi_{n+1} - u| &\le \left(\frac{3N}{2} + \frac{3}{2}(L+B) - C\right)\frac{e^{(M+K+A)\theta}}{M+K+A}\max_J |\Psi_n - u|^2\\
&\quad + \left(\frac{N}{2} + \frac{1}{2}(L+B)\right)\frac{e^{(M+K+A)\theta}}{M+K+A}\max_J |u - \Phi_n|^2.
\end{aligned}$$

These complete the proof of the theorem. ■

In the particular case of Equation (1), we now give a theoretical example as the following.

Example 2.1.

For the particular case of the system (1), let us consider the dynamic initial value problem

$$u^{\Delta}=\theta^{2}-\frac{\theta}{2}-\theta^{3},$$
$$u(0)=1, \theta\in[0,1], \tag{11}$$

where $f_1(\theta,u)=\theta^2$, $f_2(\theta,u)=-\frac{\theta}{2}$, $f_3(\theta,u)=-\theta^3$ and $\phi(\theta,u)=\theta$. Let

$$\Phi_0(\theta)=-1, \Psi_0(\theta)=1,$$

for all $\theta\in[0,1]$. It can be obtained that

$$\Phi_0^{\Delta}=0\leq f_1(\theta,\Phi_0)+f_2(\theta,\Phi_0)+f_3(\theta,\Phi_0)=\frac{5}{2},$$
$$\Phi_0(0)=-1\leq u(0)=1,$$

and

$$\Psi_0^{\Delta}=0\geq f_1(\theta,\Psi_0)+f_2(\theta,\Psi_0)+f_3(\theta,\Psi_0)=-\frac{1}{2},$$
$$\Psi_0(0)=1\geq u(0)=1.$$

This implies that Φ_0 and Ψ_0 (LS) and (US) of natural type for (1). Consider the functions

$$\Phi_1(\theta)=\frac{1}{11}\left(17e^{\frac{-11\theta}{2}}-6\right),$$
$$\Phi_1(0)=1,$$

and

$$\Psi_1(\theta)=\frac{1}{3}\left(2+e^{\frac{-3\theta}{2}}\right),$$
$$\Psi_1(0)=1.$$

Hence, we have

$$\Phi_0(\theta)\leq\Phi_1(\theta)\leq\Psi_1(\theta)\leq\Psi_0(\theta), \theta\in[0,1].$$

Similarly, $\Phi_2(\theta)$ and $\Psi_2(\theta)$ can be obtained and it can be continued like this. Then, all the conditions of Theorem 3.1 are satisfied. Thus, we obtain the existence of monotone sequences which converge uniformly to the unique solution of (1).

The models discussed in this example are directly related to real life. Especially, the equations are very common for modeling problems in mechanical engineering, physical engineering, electric and electronics engineering. In fact, the term for causal operators was adopted from the engineering literature, and the theory these operators have is the powerful quality.

3. Conclusion

We have applied the well-known quasilinearization method to a given nonlinear differential equations on time scale. It was seen that similar results were obtained, parallel to the results obtained by applying this method for a nonlinear differential equation given by the classical derivative. Under some conditions we have constructed monotone squences which converge uniformly and monotonically to the unique solution of the original problem. The most important advantage of this method is that each element of the monotone function sequence is the solution of linear differential equations. Also, it has been shown that the convergence is quadratic.

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