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RBF-PS Method for Eventual Periodicity Of Generalized Kawahara Equation

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Abstract

In engineering and mathematical physics, nonlinear evolutionary equations play an important role. The Kawahara equation is one of the famous nonlinear evolution equations that appeared in the theories of shallow water waves possessing surface tension, capillary-gravity waves and also magneto-acoustic waves in a plasma. Another interesting aspect which has been observed in laboratory experiments when nonlinear evolutionary PDEs are forced periodically from one end of undisturbed stretch of the medium of propagation, the signal eventually becomes temporally periodic at each spatial point. This observation has been confirmed mathematically in the context of Kortewg-de-Vries (KdV) and Benjamin-Bona-Mahony (BBM) equations. In this paper we intend to show the same results hold for the generalized fifth-order Kawahara equation on bounded domain in combination with periodic boundary conditions numerically utilizing meshfree technique known as radial basis function pseudo spectral (RBF-PS) approach.

Keywords: RBFs Meshless Methods; RBF-PS Method; Generalized Kawahara equation; Eventual periodicity

MSC (2010) No.: 33F05, 34K10, 34K13, 34K28, 35Q51, 35Q53

1. Introduction

In engineering and mathematical sciences such as solid state physics, plasma physics, chemical physics, fluid dynamics, chemical kinematics, and geochemistry, nonlinear evolutionary equations play an important role (for more details, look into Ablowitz (1991), Jeffrey (1989), Hunter (1988), Benney (1966), Benjamin (1972), Nagashima (1979), Bona (1981), Ganji (2007), Haghighi (2013), Turkyilmazoglu (2010), Turkyilmazoglu (2018), Turkyilmazoglu (2022)). As an example, the Kawahara equation, one of the famous nonlinear evolution equations, appeared in theories of shallow water waves possessing surface tension (Kawahara (1972), Bridges (2002)). Various physical phenomena, such as plasma magneto-acoustic waves, water waves caused by capillary gravity are described and represented by Kawahara as well as a modified version of the Kawahara equation, respectively (Kawahara (1972), Hunter (1988)). KdV-Kawahara equation is a particular form of Benney-Lin equation that accustomed to clarify the one-dimensional development in diverse media of small but finite amplitude long waves fluid dynamics problems (see Benney (1966), Ak (2018), Biswas (2008), Gazi (2019), Unal (202), Dacscioglu (2021)). Although the most general solution of the Kawahara equation is not available, the analytical solution for a special case in the form of solitary waves is given in Yamamoto (1981). Different analytic and numerical methods including the Tanh-function method (Yusufouglua (2008)), Adomain decomposition method (Kaya (2003)), Sine-cosine method (Yusufouglub (2008)), Variational iteration method, Homotopy perturbation method (Nagashima (1979), Jin (2009)), Crank-Nicolson differential quadrature algorithms (Korkmaz (2009)), Predictor corrector method (Djidjeli (1995)), Dual-petrov Galerkin method (Yuan (2008)) and RBF collocation method (Haq (2011)) have been proposed for solving the Kawahara type equations. It is worth mentioning that the standard mathematical models of integer-order derivatives including nonlinear models do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory signal, image processing and groundwater problems. In the past several decades, the investigation of travelling-wave solutions for nonlinear evolutionary equations has played an important role in the study of nonlinear physical phenomena. An excellent literature of this can be found in fractional differentiation and integration operators used for extensions of the diffusion and wave equations. The Homotopy decomposition method (HDM) was recently applied to solve fractional modified Kawahara equation, fractional complex transform approximate is used for time fractional Kawahara and modified Kawahara equations, method based on the Jacobi elliptic functions for the fractional modified Kawahara equation has been found in Ak (2018), Biazar (2021), Gazi (2019), Haghighi (2012).

Another interesting qualitative characteristic of some evolutionary equations disclosed experimentally on solutions to initial-boundary-value problem (IBVPs) and is related to their large-time behaviour known as eventual time periodicity. A piston-type or paddle-type wave maker fitted at one end of a channel in laboratory experiments show this attractive event. When the wavemaker periodically oscillates with a period $\mathrm{T}_0 > 0$, it is observed that the amplitude of the wave becomes periodic of the same period at each point along the channel after some time. This interesting phenomena of eventual periodicity investigated by Bona and Wu (Bona (1981), Bona (1989)). Various studies have previously addressed this important and interesting eventual periodic phenomena such as Burger-type equations, generalized equations for KdV, BBM, and its dissipating counterparts respectively which include Burger-type term (for more details look at the references, Bona (2009), Shen (2007),Usman (2007), Usman(2009), Al (2018), Uddin (2020), Uddin (2021), Jan (2021) Uddin (2022a), Uddin (2022b), Hussain (2021), Jan (2022a), Jan (2022b)).

Meshfree methods are becoming more popular, emerging and interesting numerical techniques due to its ability to solve those physical and engineering problems with no meshing or minimum of meshing for which the traditionally used mesh-based methods are not suited like Finite volumes, Finite differences, Finite elements, Moving least square, Element free galerkin, Point interpolation method, Reproducing kernel particle method and Boundary element free method. RBFs methods appears to be really consists and most prominent meshless methods among the family of meshless methods while looking at the interpolation of multi dimensional scattered data and have received recently a tremendous and considerable attention in scientific community because of its capacity to achieve spectral accuracy, efficiency and high flexibility in solving complex PDEs, integral equations and fractional equations in comparison to other advanced approaches (see, for example, Belytschko (1996), Buhmann (2003), Fasshauer (2007)). The most commonly used kernel in meshless techniques is the multi-quadric (MQ) kernel suggested by Hardy (1990) using radial basis function to solve PDEs.

In the present work, we investigate the solution as well as the behavior of eventual periodicity of solution to following model for generalized fifth order Kawahara equation alongwith specified initial and boundary condition on bounded domain by using a radial basis function numerical scheme known as RBF-PS meshless method.

2. Model of Generalized Kawahara type IBVPs equations

Consider the following generalized fifth order Kawahara equation along with specified initial and boundary condition on bounded domain:

$$\begin{cases} w_t + \alpha w_x + (\beta + \delta w) w w_x + \gamma w_{xxx} - \mu w_{xxxxx} = 0,\ x \in [a, b],\ t \in (0, T], \\ w(a, t) = h_a(t),\ t \in (0, T], \\ w(b, t) = w_x(a, t) = w_x(b, t) = w_{xx}(b, t) = 0,\ t \in (0, T], \\ w(x, 0) = w_0(x),\ x \in [a, b], \end{cases} \tag{1}$$

where α, β, δ, γ and μ are known with boundary data $h_a(t)$ supposed to be periodic of period $\mathrm{T}_0 > 0$ such that $h_a(t) = h_a(t + \mathrm{T}_0)$ has asymptotic cycle of periodic behavior at any fixed point in space, supposing amplitude of the boundary forcing term $h_a(t)$ is minimal. So the wave-maker transfers energy from the left boundary ($x = a$, place that mounts the wave-maker) into a finite channel while the channel at the right end ($x = b$ is free and open).

In the model equation (1):

If $\alpha = \delta = 0$, then it is called Kawahara equation.
If $\alpha = \beta = 0$, then it is called Modified Kawahara equation.
If $\delta = 0$, then it is called KdV-Kawahara equation.

3. RBF Pseudo-spectral technique description

Fasshauer combined RBF collocation approach to PS scheme known as RBF-PS scheme, and utilized it to approximate 2D Helmholtz and Laplace models, and Allen-Cahn model with piecewise boundary conditions (see Fasshauer (2005) and Fasshauer (2007)). This approach was utilized and implemented by several authors to evaluate and solve various model PDEs (for example, see the references Ferreira (2006), Ferreira (2007), Roque (2011), Uddin (2016), Uddin (2013), Nikan (2019)). Here in this study we also use this approach for the solution and eventual periodicity of model equation (1).

Assume $\psi_j, j = 1, 2, ..., N$, is a set of arbitrary smooth functions that are linearly independent and serve as a foundation for the purposes of our investigation, and $\Xi = \{x_1, x_2, ..., x_N\}$, be a series of different points in domain $\Omega \subset \mathbf{R}^d,\ d \geq 1$. Now, RBF approximation to unknown solution w of model equation (1) takes the form as follows:

$$w^h(x,t) = \sum_{j=1}^{N} \lambda_j(t)\psi_j(x), x \in \Xi, \tag{2}$$

where $h = h_{x,\Xi} sup_{x\in\Xi} min_{1\leq j\leq N} \|x - x_j\|_2$. The following table, Table 1, lists some of the most utilized radial basis functions (RBFs).

Table 1. Some of the most often used RBFs

Name of RBF	$\psi(r), (r \geq 0), r = \|x - x_j\|_2$
Linear RBF-(LI)	r
Gaussian RBF-(GA)	$e^{-(\varepsilon r)^2}$
Thin Plate Spline RBF-(TPS)	$r^{2\beta} log(r)$
Multiquadric RBF-(MQ)	$\sqrt{1 + (\varepsilon r)^2}$
Inverse Multiquadric RBF-(IMQ)	$\frac{1}{\sqrt{1+(\varepsilon r)^2}}$
Inverse Quadratic RBF-(IQ)	$\frac{1}{1+(\varepsilon r)^2}$

Where the parameter ε is well-known for being the RBF's shape parameter, it is found experimentally to any RBF and is used to modify the shape of functions. Now, collocating Equation (1) on the grid points $x_i \in \Xi$, we obtain,

$$w^h(x_i,t) = \sum_{j=1}^{N} \lambda_j(t)\psi(x_i,x_j), 1 \leq i \leq N. \tag{3}$$

In matrix arrangement, the system outlined above is denoted as

$$\mathbf{w} = \mathbf{E}\lambda, \tag{4}$$

where entries of interpolation matrix $\mathbf{E}$ are found from $\psi(x_i,x_j), 1 \leq i,j \leq N$. The vector of expansion coefficients has unique representation as $\lambda = [\lambda_1, \lambda_2, ..., \lambda_N]^T$. Now differentiation of w that is w_x using Equation (4) is obtained by differentiating and re-evaluating the RBF function at each position $x_i \in \Xi, 1 \leq i \leq N$. Hence, we arrived at the matrix-vector representation

$$\mathbf{w_x} = \mathbf{E_x}\lambda, \tag{5}$$

where entries of matrix $\mathbf{E_x}$ are $\frac{d}{dx}\psi(x,x_j)_{x=x_i}, 1 \leq i,j \leq N$. Upon solution of Equations (4) and (5) in terms of unknown values λ. The differentiation matrix is obtained in the form as under

$$\mathbf{w_x} = \mathbf{E_x}\mathbf{E^{-1}}\mathbf{u} = \mathbf{H_x}\mathbf{w}, \tag{6}$$

where $\mathbf{H_x} = \mathbf{E_x}\mathbf{E^{-1}}$ is known as the differentiation matrix. It is also worth mentioning that this matrix is dependent on invertibility of the matrix $\mathbf{E}$. Also, keep in mind that the matrix $\mathbf{E}$ is always invertible for separate collocation points. Thus, we are able to write in a similar manner

$$\mathbf{w_{xx}} = \mathbf{E_{xx}}\mathbf{E^{-1}}\mathbf{w} = \mathbf{H_{xx}}\mathbf{w}, \tag{7}$$

where $\mathbf{H_{xx}} = \mathbf{E_{xx}}\mathbf{E^{-1}}$ containing form entries $\frac{d^2}{dx^2}\psi(x,x_j)_{x=x_i}, 1 \leq i,j \leq N$. In the same manner, higher-order differentiation matrices can be constructed. The numerical approach for solving equation (1) using the above differentiation matrices is shown below,

$$w' + \alpha\mathbf{H}_x w + (\beta+\delta)w\mathbf{H}_x w + \gamma\mathbf{H}_{xxx}w - \mu\mathbf{H}_{xxxxx}w = 0. \tag{8}$$

This equation can be written in the following form as

$$\mathbf{w}' = -\alpha\mathbf{H}_x w - (\beta+\delta)w\mathbf{H}_x w - \gamma\mathbf{H}_{xxx}w + \mu\mathbf{H}_{xxxxx}w. \tag{9}$$

Equation (9) is represented by

$$\mathbf{w}^{'} = \mathbf{F}(\mathbf{w}). \tag{10}$$

Now ODE solvers like ode45, ode113, ode23 can be utilized to solve the discretize ODE systems Equation (10) in time. w_0 is the initial solution. To address the stiffness of ODE system, each effective ODE solver will choose an appropriate period of time Δt to fix the stiffness of the ODE system.

4. Numerical Results

In this section we use RBF-PS meshless scheme as described above for numerical solution of the generalized fifth order Kawahara equation (1). The accuracy, efficiency and the success of this scheme is tested in terms of L_∞ and L_2 error norms defined as

$$L_\infty = \|w^{ex} - w^{ap}\|_\infty = \max|w_i^{ex} - w_i^{ap}|, \tag{11}$$
$$L_2 = \|w^{ex} - w^{ap}\|_2 = \sqrt{h\sum_{i=0}^{N}|w_i^{ex} - w_i^{ap}|^2},$$

where $h = (b-a)/N$, and two invariants of motion I_1 and I_2 which are defined by

$$I_j = \frac{1}{j}\int_{-\infty}^{\infty} w^j dx \simeq \frac{1}{j}h\sum_{i=1}^{N} w_i^j,\ j = 1,2. \tag{12}$$

Example 4.1.

In this problem, we consider the solution of equation (1) in the form below by Yamamato and Takizawa (Yamamoto (1981)),

$$w(x,t) = \frac{105}{169}\,\text{sech}^4[k(x - \frac{36}{169}t - x_0)]. \tag{13}$$

Initial condition and boundary conditions are chosen from the exact travelling wave solution given in Equation (13) at time $t = 0$ and $w(-20,t) = 0$, $w(30,t) = 0$, respectively. The designed programs are run up to time $t = 25$ with the following parameters $x_0 = 2, \Delta x = 1, N = 51, \Delta t = 0.01, x \in [-20, 30], t \in [0, 25], k = \frac{0.5}{\sqrt{(13)}}$. We compute the solution for the choice of $\alpha = 0$, $\delta = 1$ and $\beta = 1$, $\gamma = 1$, $\mu = 1$. The time integration was carried out by the use of Runge-Kutta of order four (RK-4) scheme. Travelling wave solution at different time level is shown in Figure 1 and the discrete root mean square error norm L_2, maximum error norm L_∞, two lowest conserved quantities I_1, I_2 demonstrated at various times in Table 2, while the absolute error between the approximate solution and exact solution for various space number of grid points are listed in Table 3.

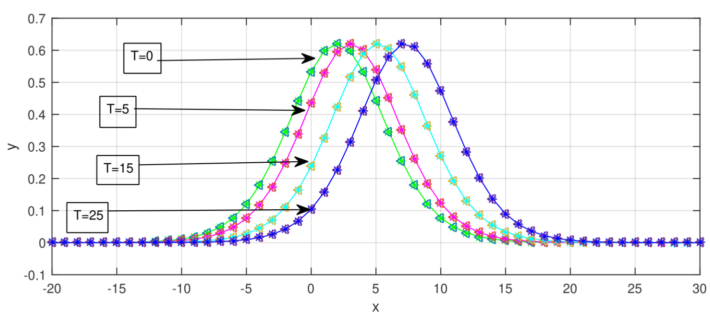

Figure 1. The formed oscillatory waves at different time level corresponding Example 4.1 are seen here in this figure

Table 2. Error norms and conserved quantities corresponding to Example 4.1

Method	t	$L_2 \times 10^3$	$L_\infty \times 10^3$	I_1	I_2
RBF-PS Method	0	0.0000	0.0000	5.85650	1.27250
	5	9.4305×10^{-2}	4.6924×10^{-2}	5.85636	1.27250
	15	1.5391×10^{-1}	5.9294×10^{-2}	5.85631	1.27250
	25	1.6776×10^{-1}	4.7386×10^{-2}	5.85644	1.27250
CDQ (Korkmaz (2009))	0	0.000	0.000	5.97357	1.27250
	5	0.151	0.043	5.97372	1.27250
	15	0.156	0.049	5.97364	1.27250
	25	0.159	0.076	5.97350	1.27250
PDQ (Korkmaz (2009))	0	0.000	0.000	5.97357	1.27250
	5	1.986	0.921	5.97060	1.27250
	15	2.543	1.045	5.97014	1.27250
	25	2.851	0.863	5.97353	1.27250

Table 3. Approximate and Exact solution with Absolute Error corresponding to Example 4.1

x	Approximate Solution	Exact Solution	Absolute Error
-20	0.0000	2.7421×10^{-5}	2.7421×10^{-5}
-19	4.2361×10^{-5}	4.7649×10^{-5}	5.2881×10^{-6}
-18	8.4251×10^{-5}	8.2745×10^{-5}	1.5058×10^{-6}
-17	1.4201×10^{-4}	1.4356×10^{-4}	1.5508×10^{-6}
-16	2.4351×10^{-4}	2.4879×10^{-4}	5.2789×10^{-6}
-15	4.2729×10^{-4}	4.3047×10^{-4}	3.1740×10^{-6}
0	4.3742×10^{-1}	4.3741×10^{-1}	8.6904×10^{-6}
10	5.2848×10^{-1}	5.2847×10^{-1}	6.2847×10^{-6}
15	1.1486×10^{-2}	1.1481×10^{-2}	4.8174×10^{-2}
20	8.0249×10^{-4}	7.9794×10^{-4}	4.5492×10^{-6}
25	4.0306×10^{-5}	5.1201×10^{-5}	1.0894×10^{-5}
30	0.0000	3.2190×10^{-6}	3.2190×10^{-6}

Example 4.2.

Consider Equation (1) with parameters $\alpha = \beta = 0$ and $\delta = \gamma = \mu = 1$, alongwith analytical solitary wave solution (Wazwaz (2007)) given by

$$u(x,t) = D\,\text{sech}^2[k(x - Bt)], \tag{14}$$

where $D = \frac{-3}{\sqrt{10}}$, $B = \frac{4}{25}$ and $k = \frac{1}{2}\sqrt{\frac{1}{5}}$, the initial and boundary conditions are extracted from the exact solution equation (14). Calculation are carried out by taking $[a,b] = [-30,30]$, with $N = 61$, $\Delta t = 0.01$, $MQ_C = 3.5$. Results at $t = 0, 5, 15, 25$ can be seen in Table 4 and Figure 2, respectively.

Table 4. Error norms and Conserved quantities corresponding to Example 4.2

Method	t	L_2	L_∞	I_1	I_2
RBF-PS Method	0	0.0000	0.0000	-8.48526	2.68328
	5	1.9712×10^{-4}	1.0329×10^{-4}	-8.48553	2.68328
	15	2.8386×10^{-4}	8.3509×10^{-5}	-8.48544	2.68328
	25	3.7397×10^{-4}	8.3513×10^{-5}	-8.48547	2.68328
RBF-MOL (bibi (2011))	0	0.0000	0.0000	-8.48525	2.68328
	5	6.1995×10^{-5}	1.7896×10^{-4}	-8.48524	2.68317
	15	1.0717×10^{-4}	2.7337×10^{-4}	-8.48487	2.68296
	25	1.2130×10^{-4}	3.4855×10^{-4}	-8.48464	2.68275
RBF-FD (Jan (2021))	0	0.0000	0.0000	-8.34616	2.63929
	5	4.9580×10^{-4}	9.4940×10^{-4}	-8.34743	2.63929
	15	1.0017×10^{-3}	2.6235×10^{-3}	-8.34357	2.63930
	25	2.8298×10^{-3}	5.9075×10^{-3}	-8.33515	2.63931

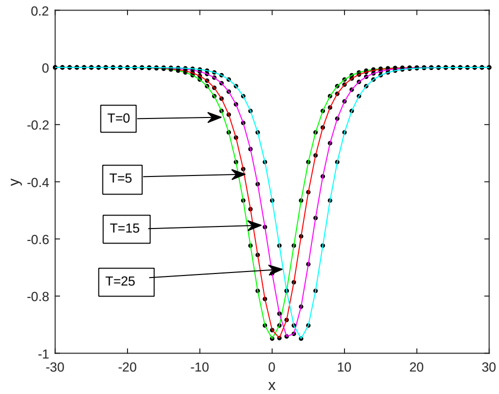

Figure 2. Solitary wave solution corresponding to Example 4.2 contrasted with exact solution equation (14) (solid lines represent exact solution, whereas "." represents numerical solution)

5. Eventual periodicity

Now, we will show the outcomes of our investigation for eventual periodicity of generalized fifth order Kawahara model equation (1) in graphical form along with appropriate boundary data $h_a(t)$. The initial data w_0 is not necessarily necessary in eventual periodicity, so we take it zero. For each problem the amplitudes $w(x,t)$ produced in six graphs at particular points in domain. N indicates total domain points. The X and Y axes are representative in these graphs of time t and amplitude w respectively. The last graph shows the amplitude remains zero in every problem.

5.1. Eventual periodicity of Kawahara equation

We compute the solutions of model equation (1) for Kawahara equation with parameters $\alpha = 0$, $\beta = 1$, $\delta = 0$, $\gamma = 0.027$ and $\mu = 10^{-3}$. The amplitudes $w(x,t)$ for this model is shown in six plots in Figure 3 at given specific points, viz., $x = -19.5$, -17.5, -7.5, 5.0, 17.5 and 30.0 in the domain $[-20, 30]$ and in a time domain $[0, 5]$. The plots in Figure 3 clearly confirm the subsequent periodic behavior of the solution in the specified domain at these particular positions.

5.2. Eventual periodicity of Modified-Kawahara equation

We compute the solutions of model equation (1), for Modified-Kawahara equation using parameters $\alpha = 0$, $\beta = 0$, $\delta = 1$, $\gamma = 0.08$ and $\mu = 10^{-3}$. The amplitudes $u(x,t)$ for this model is shown in six plots in Figure 4 at given specific points, viz., $x = -29.4$, -27.0, -15.0, 0.0, 15.0 and 30.0 in the domain $[-30, 30]$ and in a time domain $[0, 5]$. The plots in Figure 4 clearly confirm that again we discovered the design of eventual periodicity in solution $w(x,t)$ at all selected positions in the specified domain at these particular positions.

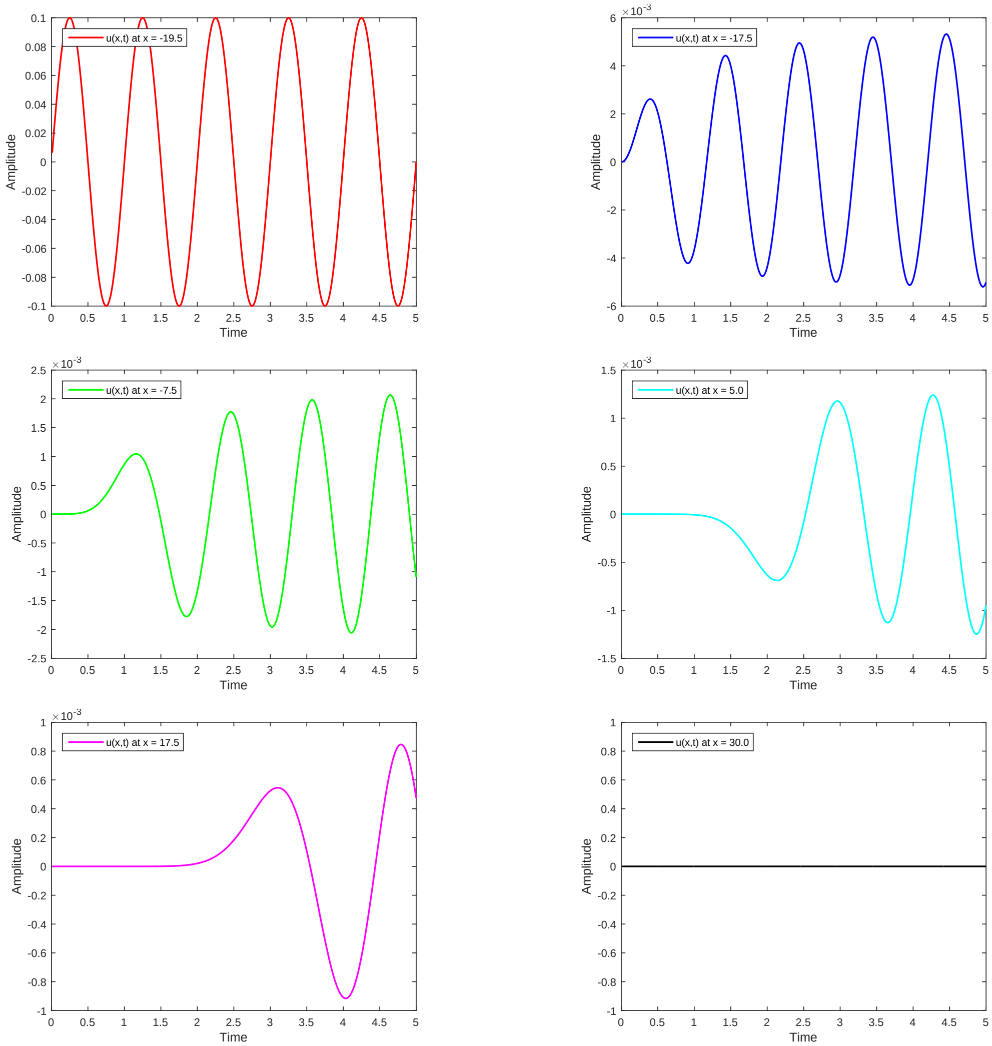

Figure 3. Eventual periodicity of Kawahara equation using $x = -19.5, -17.5, -7.5, 5.0, 17.5$ and $30.0 \in [-20, 30]$, $N = 100$, $\Delta t = 0.01$, $T = 5$, using parameters $\alpha = 0$, $\beta = 1$, $\delta = 0$, $\gamma = 0.027$ and $\mu = 10^{-3}$ and $h_a(t) = 0.1\sin(2\pi t)$

5.3. Eventual periodicity of KdV-Kawahara equation

Finally we compute the solutions of model equation (1), for KdV-Kawahara equation with parameters $\alpha = 0.4$, $\beta = 1.5$, $\delta = 0$, $\gamma = 4$ and $\mu = 10^{-3}$. The amplitudes $u(x,t)$ for this model is shown in six plots in Figure 5 at given particular points, viz., $x = 2, 10, 50, 100, 150$ and 200 in the domain $[0, 200]$ and in a time domain $[0, 5]$. We noticed that the pattern of eventual periodicity at these particular positions in domain is still remained as shown in the plots of Figure 5.

6. Conclusion

In the present work, RBF-PS meshless method is investigated for the approximation of solution and for numerically approximating the eventual periodicity of Kawahara type PDEs model equations.

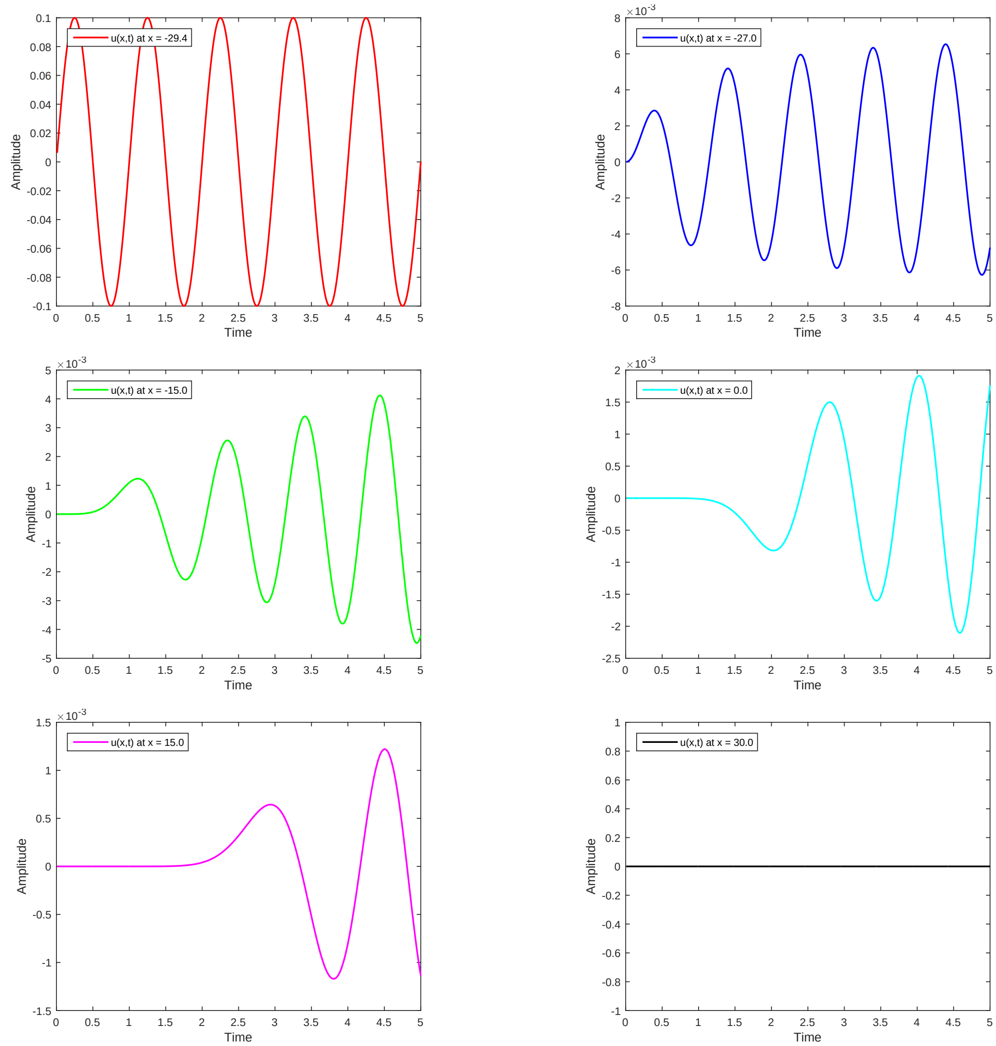

Figure 4. Eventual periodicity of Modified-Kawahara equation for x = -29.4, -27.0, -15.0, 0.0, 15.0 and $30.0 \in [-30, 30]$, $N = 100$, $\Delta t = 0.01$, $T = 5$, using parameters $\alpha = 0$, $\beta = 0$, $\delta = 1$, $\gamma = 0.08$ and $\mu = 10^{-3}$ and $h_a(t) = 0.1\sin(2\pi t)$

To show how good and accurate the present numerical scheme, we computed some error norms and invariants of the model by solving some examples and compared the results of the present method with the available methods in the literature. For execution of temporal variable in the given model equation, Runge-Kutta (RK-4) time steeping approach is utilized. The RBF-PS method has been found to be very accurate and suited to approximate many complicated mechanical problem with ease and accuracy.

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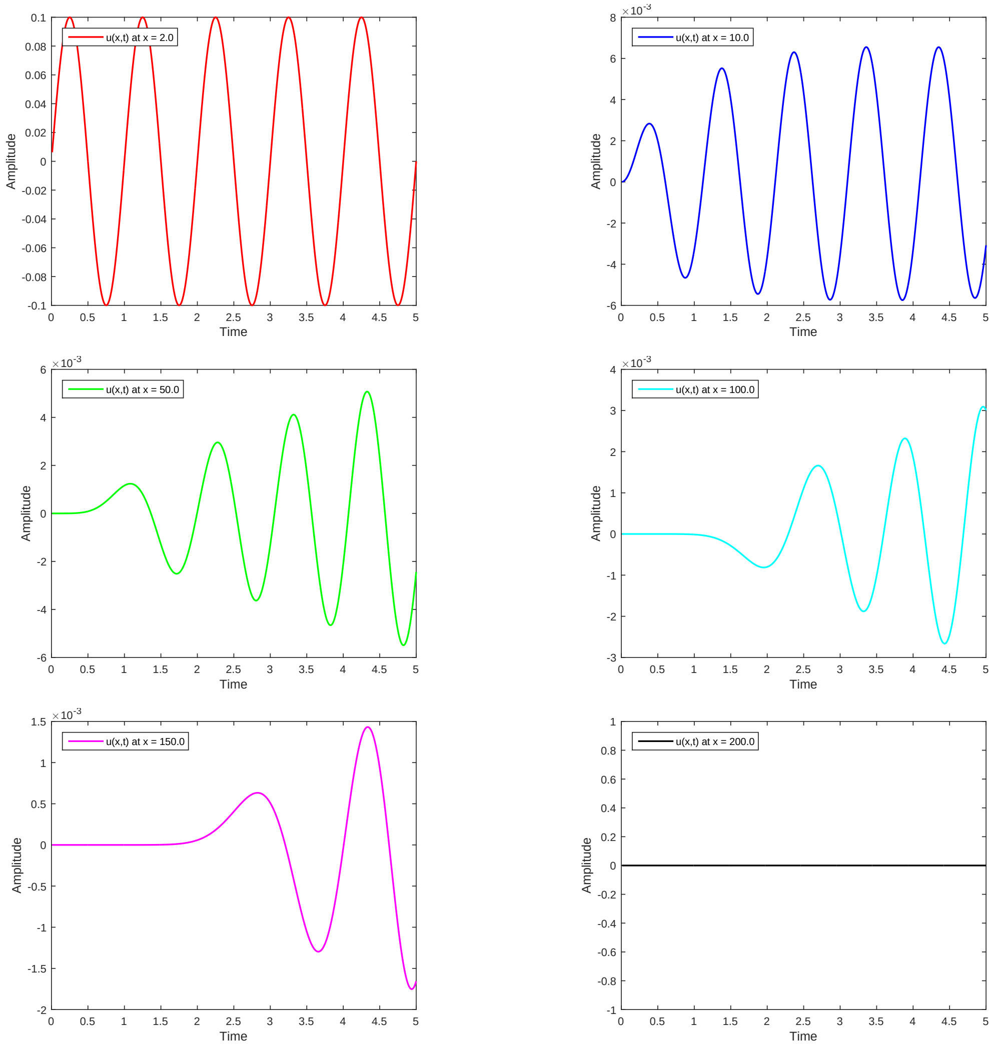

Figure 5. Eventual periodicity of KdV-Kawahara equation for x = 2, 10, 50, 100, 150 and 200 $\in [0, 200]$, $N = 100$, $\Delta t = 0.01$, $T = 5$ with parameters $\alpha = 0.4$, $\beta = 1.5$, $\delta = 0$, $\gamma = 4$ and $\mu = 10^{-3}$ and $h_a(t) = 0.1 \sin(2\pi t)$

Haghighi of Applications and Applied Mathematics (AAM) journal that he has given us a chance to improve our article up to the standard and able to publish.

REFERENCES

Ablowitz, Mark J. and Ablowitz, M.A., Clarkson, P.A. and Clarkson, Peter A. (1991). *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge University Press,149.

Ak, T. and Battal Gazi Karakoc, S. (2018). A numerical technique based on collocation method for solving modified Kawahara equation, Journal of Ocean Engineering and Science, Vol. 3, No. 1, pp. 67–75.

Al-Khaled, K., Haynes, N., Schiesser, W. and Usman, M. (2018). Eventual periodicity of the forced oscillations for a Korteweg-de Vries type equation on a bounded domain using a sinc collocation method, Journal of Computational and Applied Mathematics, Vol. 330, pp. 417–428.

Belytschko, T., Krongauz, Y., Organ, D., Fleming, M. and Krysl, P. (1996). Meshless methods: An overview and recent developments, Computer Methods in Applied Mechanics and Engineering, Vol. 139, No. 1-4, pp. 3–47.

Benjamin, T.B., Bona, J.L. and Mahony, J.J. (1972). Model equations for long waves in nonlinear dispersive systems, Philosophical Transactions of the Royal Society of London, Series A, Mathematical and Physical Sciences, Vol. 272, No. 1220, pp. 47–78.

Benney, D.J. (1966). Long waves on liquid films, Journal of Mathematics and Physics, Vol. 45, No. 1-4, pp. 150–155.

Biazar, J. and Asayesh, R. (2021). A new finite difference scheme for high-dimensional heat equation, Applications and Applied Mathematics: An International Journal (AAM), Vol. 16, No. 2, pp. 23.

Bibi, N., Tirmizi, S.I.A. and Haq, S. and others. (2011). Meshless method of lines for numerical solution of Kawahara type equations, Applied Mathematics, Vol. 2, No. 5, pp. 608.

Biswas, A. (2008). Soliton perturbation theory for the modified Kawahara equation, Applications and Applied Mathematics: An International Journal (AAM), Vol. 3, No. 2, pp. 5.

Bona, J.L., Pritchard, W.G. and Scott, L.R. (1981). An evaluation of a model equation for water waves, Philosophical Transactions of the Royal Society of London, Series A, Mathematical and Physical Sciences, Vol. 302, No. 1471, pp. 457–510.

Bona, J.L. and Winther, R. and others. (1989). Korteweg-de Vries equation in a quarter plane, Differential and Integral Equations, Vol. 2, No. 2, pp. 228–250.

Bona, J.L and Wu, J. (2009). Temporal growth and eventual periodicity for dispersive wave equations in a quarter plane, Discrete Contin. Dyn. Syst, Vol. 23, pp. 1141–1168.

Bridges, T.J. and Derks, G. (2002). Linear instability of solitary wave solutions of the Kawahara equation and its generalizations, SIAM Journal on Mathematical Analysis, Vol. 33, No. 6, pp. 1356–1378.

Buhmann, M.D. (2003). *Radial Basis Functions: Theory and Implementations*, Cambridge University Press, Vol. 12.

Daşcıoğlu, A. and Ünal, S.Ç. (2021). *New Exact Solutions for the Space-time Fractional Kawahara Equation*, Applied Mathematical Modelling, Vol. 89.

Djidjeli, K., Price, W.G., Twizell, E.H. and Wang, Y. (1995). Numerical methods for the solution of the third-and fifth-order dispersive Korteweg-de Vries equations, Journal of Computational and Applied Mathematics, Vol. 58, No. 3, pp. 307–336.

Fasshauer, G.E. (2005). *RBF Collocation Methods as Pseudospectral Method*, WIT Transactions on Modelling and Simulation, Vol. 39.

Fasshauer, G.E. (2007). *Meshfree Approximation Methods with MATLAB*, World Scientific.

Ferreira, A.J.M. and Fasshauer, G.E. (2006). Computation of natural frequencies of shear deformable beams and plates by an RBF-pseudospectral method, Computer Methods in Applied Mechanics and Engineering, Vol. 196, No. 1-3, pp. 134–146.

Ferreira, A.J.M. and Fasshauer, G.E. (2007). Analysis of natural frequencies of composite plates by an RBF-pseudospectral method, Composite Structures, Vol. 79, No.2, pp.202–210.

Ganji, D.D., Tari, H. and Jooybari, M,B. (2007). Variational iteration method and homotopy perturbation method for nonlinear evolution equations, Computers & Mathematics with Applications, Vol. 54, No. 7-8, pp. 1018–1027.

Gazi Karakoc, S.B., Kumar Bhowmik, S. and Gao, F. (2019). A numerical study using finite element method for generalized Rosenau-Kawahara-RLW equation, Computational Methods for Differential Equations, Vol. 7, No. 3, pp. 319–333.

Haghighi, A.M., Lian, J. and Mishev, D.P. (2012). *Advanced Mathematics for Engineers with Applications in Stochastic Processes*, Nova Science Publishers, Inc.

Haghighi, A.M. and Mishev, D.P. (2013). *Difference and Differential Equations with Applications in Queueing Theory*, John Wiley & Sons.

Haq, S. and Uddin, M. (2011). RBFs approximation method for Kawahara equation, Engineering Analysis with Boundary Elements, Vol. 35, No. 3, pp. 575–580.

Hardy, R.L. (1990). Theory and applications of the multiquadric-biharmonic method 20 years of discovery 1968–1988, Computers & Mathematics with Applications, Vol. 19, No. 8-9, pp. 163–208.

Hunter, J.K. and Scheurle, J. (1988). Existence of perturbed solitary wave solutions to a model equation for water waves, Physica D: Nonlinear Phenomena, Vol. 32, No. 2, pp. 253–268.

Hussain, A., Uddin, M., Haq, S. and Jan, H.U. (2021). Numerical solution of heat equation in polar cylindrical coordinates by the meshless method of lines, Journal of Mathematics.

Jan, H.U. and Uddin, M. (2021). Approximation and eventual periodicity of Generalized Kawahara equation using RBF-FD method, Punjab University Journal of Mathematics, Vol. 53, No .9.

Jan, H.U., Uddin, M., Shah, I.A. and Khan, S.U. (2022). On the eventual periodicity of fractional order dispersive wave equations using RBFS and transform, EUREKA: Physics and Engineering, No. 3, pp. 133–148.

Jan, H.U., Uddin, M., Norin, A. and others. (2022). On the approximation of eventual periodicity of linearized KdV type equations using RBF-PS method, Applications and Applied Mathematics: An International Journal (AAM), Vol. 17, No. 2, pp. 16.

Jeffrey, A. and Xu, S. (1989). Solitons, nonlinear evolution equations and inverse scattering, International Journal of Non-linear Mechanics, Vol. 24, No. 5, 149, pp. 425–429.

Jin, L. (2009). Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation, Mathematical and Computer Modelling, Vol. 49, No. 3-4, pp. 573–578.

Kawahara, T. (1972). Oscillatory solitary waves in dispersive media, Journal of the Physical Society of Japan, Vol. 33, No. 1, pp. 260–264.

Kaya, D. (2003). An explicit and numerical solutions of some fifth-order KdV equation by decomposition method, Applied Mathematics and Computation, Vol. 144, No. 2-3, pp. 353–363.

Korkmaz, A. and Dağ, İ. (2009). Crank-Nicolson–differential quadrature algorithms for the Kawahara equation, Chaos, Solitons & Fractals, Vol. 42, No. 1, pp. 65–73.

Nagashima, H. (1979). Experiment on solitary waves in the nonlinear transmission line described by the equation (remark: Graphics omitted.), Journal of the Physical Society of Japan, Vol. 47, No. 4, pp. 1387–1388.

Nikan, O., Machado, J.A.T., Golbabai, A. and Nikazad, T. (2019). Numerical investigation of the nonlinear modified anomalous diffusion process, Nonlinear Dynamics, Vol. 97, No. 4, pp. 2757–2775.

Roque, C.M.C., Ferreira, A.J.M., Neves, A.M.A., Soares, C.M.M., Reddy, J.N. and Jorge, R.M.N. (2011). Transient analysis of composite and sandwich plates by radial basis functions, Journal of Sandwich Structures & Materials, Vol. 13, No. 6, pp.681–704.

Shen, J., Wu, J. and Yuan, J.-M. (2007). Eventual periodicity for the KdV equation on a half-line, Physica D: Nonlinear Phenomenat, Vol. 227, No. 2, pp. 105–119.

Turkyilmazoglu, M. (2010). Unsteady mhd flow with variable viscosity, International Journal of Thermal Sciences, Vol. 49, No. 3, pp. 563–570.

Turkyilmazoglu, M. (2018). Unsteady flow over a decelerating rotating sphere, Physics of Fluids, Vol. 30, No. 3, pp. 033601.

Turkyilmazoglu, M. (2022). Maximum wave run-up over beaches of convex/concave bottom profiles, Continental Shelf Research, Vol. 232, pp. 104610.

Uddin, M. and Jan, H.U. (2021). Eventual periodicity of linearized BBM equation using RBFs meshless method, Punjab University Journal of Mathematics, Vol. 53, No. 3, pp. 9–19.

Uddin, M., Ullah Jan, H. and Usman, M. (2020). RBF-FD method for some dispersive wave equations and their eventual periodicity, Computer Modeling in Engineering & Sciences, Vol. 123, No. 2, pp. 797–819.

Uddin, M., Jan, H.U. and Usman, M. (2022a). RBF-PS method for approximation and eventual periodicity of fractional and integer type KdV equations, Partial Differential Equations in Applied Mathematics, Vol. 5, p. 100288.

Uddin, M., Jan, H.U. and Usman, M. (2022b). On the solution of fractional order KdV equation and its periodicity on bounded domain using radial basis functions, Mathematical Problems in Engineering.

Uddin, M. (2013). RBF-PS scheme for solving the equal width equation, Applied Mathematics and Computations, Vol. 222, pp.619–631.

Uddin, M., Jan, H.U., Ali, A. and Shah, I.A. (2016). Soliton kernels for solving PDEs, Journal of Computational and Applied Mathematics, Vol. 13, No. 2, pp. 1640009.

Ünal, S.Ç., Daşcıoğlu, A. and Bayram, D.V. (2020). New exact solutions of space and time fractional modified Kawahara equation, Physica A: Statistical Mechanics and its Applications, Vol. 551, No. 3, pp. 124550.

Usman, M. (2007). *Forced Oscillations of the Korteweg-de Vries Equation and Their Stability*, University of Cincinnati.

Usman, M. and Zhang, B. (2009). Forced oscillations of the Korteweg-de Vries equation on a bounded domain and their stability, Discrete and Continuous Dynamical Systems-Series A (DCDS-A), Vol. 26, No. 4.

Wazwaz, A.-M. (2007). The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, Applied Mathematics and Computation, Vol. 184, No. 2, pp. 1002–1014.

Yamamoto, Y. and Takizawa, É.I. (1981). On a solution on non-linear time-evolution equation of fifth order, Journal of the Physical Society of Japan, Vol. 50, No. 5, pp. 1421–1422.

Yuan, J.-M., Shen, J. and Wu, J. (2008). A dual-Petrov-Galerkin method for the Kawahara-type equations, Journal of Scientific Computing, Vol. 34, No. 1, pp. 48–63.

Yusufoğlu, E. and Bekir, A. (2008). Symbolic computation and new families of exact travelling solutions for the Kawahara and modified Kawahara equations, Computers & Mathematics with Application, Vol. 55, No. 6, pp. 1113–1121.

Yusufoğlu, E., Bekir, A. and Alp, M. (2008). Periodic and solitary wave solutions of Kawahara and modified Kawahara equations by using Sine–Cosine method, Chaos, Solitons & Fractals, Vol. 37, No. 4, pp. 1193–1197.