Applications and Applied Mathematics: An International Journal (AAM)

Volume 18 | Issue 2 | Article 4

12-2023

(R2066) New Results of Ulam Stabilities of Functional Differential Equations of First Order Including Multiple Retardations

Merve Şengün
Van Yuzuncu Yil University

Cemil Tunç
Van Yuzuncu Yil University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 18, Issue 2 (December 2023), Article 4, 15 pages

New Results of Ulam Stabilities of Functional Differential Equations of First Oder Including Multiple Retardations

[1]Merve Şengün and [2*]Cemil Tunç

Department of Mathematics
Faculty of Sciences
Van Yuzuncu Yil University
65080, Campus, Van, Turkey

[1]mervesengun07@gmail.com; [2]cemtunc@yahoo.com

*Corresponding Author

Received: June 25, 2023; Accepted: August 16, 2023

Abstract

In this study, we pay attention to a functional differential equation (FDE) of first order including *N*-variable delays. We construct new sufficient conditions in relation to the Hyers-Ulam stability (HUS) and the generalized Hyers-Ulam-Rassias stability (GHURS) of the FDE of first order including *N*-variable delays. By using Banach contraction principle (BCP), Picard operator and Gronwall's lemma, we confirm two new theorems with regard to the HUS and the GHURS. The results of this study are new, they extend and improve some earlier results of the HUS and the GHURS.

Keywords: HUS; Generalized HURS; FDE with delay; Fixed point; Banach contraction principle; Picard operator

MSC 2020 No.: 39B82, 45G10, 45J05, 45M10

1. Introduction

Let us recall that S.M. Ulam was a prominent mathematician and physicist. In 1940, S.M. Ulam posed his famous question in relation to approximate homomorphisms, which rise to a long-lasting study of a field which we now call Ulam (or HU) stability and so on. Depending on that question, in numerous researches fields, many researchers, in particular mathematicians have extensively investigated the subjects on Ulam types stabilities, called the HUS, the HURS, the GHURS, the semi-HUS, the semi-HURS, the semi-GHURS and so on (see the books of Brzdęk et al. (2019), Castro and Simões (2019), Jung (2003), Jung (2011), Tripathy (2021), Ulam (1964), and the papers of Akkouchi (2019), Biçer and Tunç (2017), Biçer and Tunç (2018), Castro and Simões (2018), Chauhan et al. (2022), Ciplea et al. (2023), Deep et al. (2020), Graef et al. (2023), Jung (2004), Jung (2006), Khan et al. (2018), Kucche and Shikhare (2019), Li and Shen (2009), Lungu, Popa (2012), Miura et al. (2006), Onitsuka (2019), Otrocol and Ilea (2013), Petru et al. (2011), Popa and Raşa (2011), Tunç and Biçer (2015), Shah et al. (2022) and Tunç and Tunç (2023)). In addition, a large list of references can also be found in these sources, which are not listed here.

In the scientific sources mentioned above, the Ulam type stability problems of several kind of functional equations, FDEs, ODEs, PDEs, etc., have been extensively investigated by a large number of authors. Indeed, the Ulam type stabilities play a crucial role for equations where the exact solutions are not obtainable. Next, increasingly, the Ulam type stability problems emerge in biology, economics, etc., and unlikely places. In general, these kind problems can often be solved and commented using fundamental results of fixed point theory in an elementary and mathematically honest manner; in particular, see the books of Burton (2006), Jung (2003), Jung (2011), Tripathy (2021) and the papers of Graef et al. (2023), Diaz and Margolis (1968), and Tunç and Tunç (2023). In reality, fixed point theory and its main results can be used directly to study the Ulam type stability problems of functional equations, FDEs, ODEs, PDEs, etc., when solutions are being considered on a finite interval case, infinite intervals cases and so on.

We should also state that the qualitative theory of FDEs including multiple constant and variable delays has received and is still receiving the considerable attentions of a lot of researchers (see the sources mentioned above and the recent literature). In this paper, inspiring by the scientific works above, we will consider an FDE including N-variable delays. We will obtain two new theorems, which have new sufficient conditions on the HUS and the GHURS of the considered FDE. The method of the proofs is based on the BCP, Picard operator and Gronwall's lemma. By the results of this paper, we aim to introduce new contributions to the theory of FDEs including several variable delays. Indeed, the outcomes of this paper are new, and they generalize and improve some the earlier relative results regarding the Ulam stabilities. Here, there is also an added benefit to studying stability by fixed point methods. In addition to several types of Ulam stability, we can see several kind of other stabilities in sense of Lyapunov, Lyapunov-Krasowskii, Lyapunov–Razumikhin, etc. (see, for instance, Tunç (2021), Tunç (2023), Tunç and Tunç (2021), Tunç and Tunç (2023), Tunç et al. (2021a), Tunç et al. (2021b), and so on). In this work, we will not discuss these topics.

In the next section, we will present the main FDE to be studied in this paper, which include N-variable delays. Here, we also give the essential information regarding the HUS and the GHURS of the FDE, Picard operator, Gronwall inequality and so on. In the coming section, Section 3, we start to prove the uniqueness, the HUS and the GHURS of the FDE by two theorems. Finally, Section 4 provides a descriptive conclusion of this paper.

2. Preliminaries

The reason to do this study has been inspired by the article of Otrocol and Ilea (2013). We consider the below FDE of first order including N-multiple variable delays:

$$\mathrm{z}'(t) = \sum_{i=1}^{N} F_i(t, \mathrm{z}(t), \mathrm{z}(g_i(t))), \ t \in I = [a,b], \tag{1}$$

$$\mathrm{z}(t) = \phi(t), \quad t \in [a-h, a], \tag{2}$$

where $F_i \in C([a,b] \times \mathbb{R}^2, \mathbb{R}),\ g_i \in C([a,b],[a-h,b]),\ g_i(t) \le t,\ \forall t \in I = [a,b],\ a,\ b,\ h \in \mathbb{R},$

$h > 0,$ and the inequalities in the following lines

$$\left| w'(t) - \sum_{i=1}^{N} F_i(t, w(t), w(g_i(t))) \right| \le \varepsilon,\ t \in I, \tag{3}$$

$$\left| w'(t) - \sum_{i=1}^{N} F_i(t, w(t), w(g_i(t))) \right| \le \psi(t),\ t \in I, \tag{4}$$

where $\varepsilon \in \mathbb{R},\ \varepsilon > 0,$ and the function ψ is defined at the following.

In this work, we discuss the HUS, GHURS and the existence of the solutions of FDE (1). Now, we give some fundamental information which are needed in advance.

The FDE (1) is said to be HU stable if there exists a positive constant C such that for all $\varepsilon > 0$ and for every solution $w \in C^1([a-h,b],\mathbb{R})$ of the inequality (3) there exists a solution $z \in C^1([a-h,b],\mathbb{R})$ of the FDE (1) with $|w(t) - z(t)| \le C\varepsilon$ for $t \in [a-h,b]$.

The FDE (1) is said to be GHUR stable with respect to the positive, nondecreasing continuous function $\psi : [-r,b] \to \mathbb{R}^+,\ \mathbb{R}^+ = [0,\infty),$ if there exists a positive constant C_ψ such that for all $\varepsilon > 0$ and for every solution $w \in C^1([a-h,b],\mathbb{R})$ of the inequality (4) there is a solution $z \in C^1([a-h,b],\mathbb{R})$ of the FDE (1) with $|w(t) - z(t)| \le C\varepsilon_\psi,\ t \in [a-h,b]$.

We observe that a function $w \in C^1(I,\mathbb{R})$ is a solution of the inequality (3) if and only if there exists a function $q_w \in C(I,\mathbb{R})$ depends on y such that

$$|q_w(t)| \le \varepsilon, \forall t \in I, \tag{5}$$

$$w'(t) = \sum_{i=1}^{N} F_i(t, w(t), w(g_i(t))) + q_w(t),\ \forall t \in I. \tag{6}$$

We should state that similar arguments also hold for the inequality (4).

Next, if $w \in C^1(I,\mathbb{R})$ satisfies (3), then w is a solution of the inequality in the following line:

$$\left| w(t) - w(a) - \int_a^t \sum_{i=1}^N F_i(t, w(t), w(g_i(t)))ds \right| \le (t-a)\varepsilon, \quad \forall t \in I. \tag{7}$$

If $w \in C^1(\mathrm{I}, \mathbb{R})$ satisfies (4), then w is a solution of the inequality in the following line:

$$\left| w(t) - w(a) - \int_a^t \sum_{i=1}^N F_i(s, w(s), w(g_i(s)))ds \right| \le \int_a^t \psi(s)ds, \ \forall t \in I.$$

We suppose that (X, d) is a metric space. An operator A from X to X is called a Picard operator if there exist $x^* \in X$ such that the following conditions hold:

$(As1)$ $F_A = \{x_A^*\}$ with $F_A = \{x \in X : A(x) = x\}$. This is a fixed point set of A;

$(As2)$ the sequence $\left(A^n(x_0)\right)$ satisfies $\left(A^n(x_0)\right) \to x^*, \quad n \in \mathbb{N}, \quad \forall x_0 \in X.$

Before giving the Ulam stabilities results of this work, we present two well-known lemmas from the database of present literature, which are used in the proofs of the main results.

Lemma 2.1.

Let $f, p \in C([a,b], \mathbb{R}^+)$ such that the function f is increasing. If $x \in C([a,b], \mathbb{R}^+)$ satisfies the inequality

$$x(t) \le f(t) + \int_a^b x(s)p(s)ds, \ a \le t \le b,$$

then

$$x(t) \le f(t)\exp\left(\int_a^b p(s)ds\right), \ a \le t \le b,$$

(Rus (2009)).

Lemma 2.2.

Suppose that $(X, d, \le)$ is an ordered metric space and $A : X \to X$ is an increasing Picard operator with $F_A = x_A^*$. Then, for $x \in X$, $x \le A(x)$ requires that $x \le x_A^*$, while $x \ge A(x)$ requires that $x \ge x_A^*$.

3. Ulam stabilities

Before stating the new HUS result, let the conditions in the following lines hold:

(C1) $F_i \in C([a,b]\times\mathbb{R}^2,\mathbb{R}),\ g_i \in C([a,b],[a-h,b]),\ g_i(t)\le t,\ \ i=1,2,...,N,$

$h>0,\ \forall t\in I=[a,b];$

(C2) there exist constants $L_{F_1}>0,\ L_{F_2}>0,...,L_{F_N}>0$ with

$$\left|F_1(t,u_1,u_2)-F_1(t,v_1,v_2)\right|\le L_{F_1}\left\{\left|u_1-v_1\right|+\left|u_2-v_2\right|\right\},$$

$$\left|F_2(t,u_1,u_3)-F_2(t,v_1,v_3)\right|\le L_{F_2}\left\{\left|u_1-v_1\right|+\left|u_3-v_3\right|\right\},$$

$$\dots$$

$$\left|F_N(t,u_1,u_{N+1})-F_N(t,v_1,v_{N+1})\right|\le L_{F_N}\left\{\left|u_1-v_1\right|+\left|u_{N+1}-v_{N+1}\right|\right\},$$

$$\forall t\in[a,b],\ \forall u_1,u_{i+1},v_1,v_{i+1}\in\mathbb{R},\quad i=1,2,...,N.$$

Theorem 3.1.

Let $(C1)$ and $(C2)$ hold. If $2(b-a)\sum_{i=1}^{N}\left(L_{F_i}\right)<1$, then the following ideas hold true:

1^0) the IVP (1), (2) has a unique solution in $C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$;

2^0) the FDE (1) is the HU stable.

***Proof*:**

1^0) Firstly, we can observe that under condition (C1), the IVP (1), (2) can be converted to

$$z(t)=\begin{cases}\phi(t), & a-h\le t\le a,\\ \phi(t)+\int_a^t\sum_{i=1}^{N}F_i(s,z(s),z(g_i(s)))ds, & a\le t\le b.\end{cases}$$

Letting $X=C([a-h,b],\mathbb{R})$ as the Banach space with Chebyshev norm, we define the operator $B_F:X\to X$ by

$$B_F(z)(t)=\begin{cases}\phi(t), & a-h\le t\le a,\\ \phi(t)+\int_a^t\sum_{i=1}^{N}F_i(s,z(s),z(g_i(s)))ds, & a\le t\le b.\end{cases}$$

Our idea is that B_F is a Picard operator. Then, by virtue of the conditions of Theorem 3.1, we will show that the operator B_F is a contraction on X, which is Banach space with respect to the Chebyshev norm $\|.\|_C$. Next, it is clear that B_F has a fixed point because of the contraction principle. Then, we derive

$$\left|B_F(z)(t) - B_F(w)(t)\right| = 0, \quad z, w \in C([a-h, b], \mathbb{R}).$$

Next, for any $t \in I$, we follow that

$$\begin{aligned}
\left|B_F(z)(t) - B_F(w)(t)\right| &= \left|\int_a^t \sum_{i=1}^N F_i(s, z(s), z(g_i(s)))ds - \int_a^t \sum_{i=1}^N F_i(s, w(s), w(g_i(s)))ds\right| \\
&\leq \int_a^t \left|F_1(s, z(s), z(g_1(s))) - F_1(s, w(s), w(g_1(s)))\right| ds \\
&\quad + \int_a^t \left|F_2(s, z(s), z(g_2(s))) - F_2(s, w(s), w(g_2(s)))\right| ds \\
&\quad + \ldots + \int_a^t \left|F_N(s, z(s), z(g_N(s))) - F_N(s, w(s), w(g_N(s)))\right| ds \\
&\leq L_{F_1}(b-a)(\max_{a \leq s \leq b}\left|z(s) - w(s)\right| + \max_{a \leq s \leq b}\left|z(g_1(s)) - w(g_1(s))\right|) \\
&\quad + (b-a)L_{F_2}(\max_{a \leq s \leq b}\left|z(s) - w(s)\right| + \max_{a \leq s \leq b}\left|z(g_2(s)) - w(g_2(s))\right|) \\
&\quad + \ldots + (b-a)L_{F_N}(\max_{a \leq s \leq b}\left|z(s) - w(s)\right| + \max_{a \leq s \leq b}\left|z(g_N(s)) - w(g_N(s))\right|) \\
&= (b-a)(L_{F_1} + L_{F_2} + \ldots + L_{F_N}) \max_{a \leq s \leq b}\left|z(s) - w(s)\right| \\
&\quad + L_{F_1}(b-a) \max_{a \leq s \leq b}\left|z(g_1(s)) - w(g_1(s))\right| \\
&\quad + L_{F_2}(b-a) \max_{a \leq s \leq b}\left|z(g_2(s)) - w(g_2(s))\right| \\
&\quad + \ldots + L_{F_N}(b-a) \max_{a \leq s \leq b}\left|z(g_N(s)) - w(g_N(s))\right|.
\end{aligned}$$

Hence, we see that

$$\left\|B_F(z)(t) - B_F(w)(t)\right\|_C \leq 2(b-a)\sum_{i=1}^N \left(L_{F_i}\right)\left\|z - w\right\|_C, \; z, w \in C([a-h, b], \mathbb{R}).$$

Since

$$2(b-a)\sum_{i=1}^{N}\left(L_{F_i}\right)<1,$$

then the operator B_F is a contraction on the set X. Hence, the operator B_F includes a fixed point, which is a solution of the IVP (1), (2).

2^0) Let $w\in C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$ be a solution of the inequality (3).

Let $z\in C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$ represent the unique solution of the problem in the following lines:

$$z'(t)=\sum_{i=1}^{N}F_i(t,z(t),z(g_i(t))),\quad a\le t\le b,$$

$$z(t)=w(t),\quad a-h\le t\le a.$$

From (C1), by the integration we can write

$$z(t)=\begin{cases} w(t), & a-h\le t\le a,\\ w(a)+\int_a^t\sum_{i=1}^{N}F_i(s,z(s),z(g_i(s)))ds, & a\le t\le b.\end{cases}$$

According to the (7), we see that

$$\left|w(t)-w(a)-\int_a^t\sum_{i=1}^{N}F_i(s,w(s),w(g_i(s)))ds\right|\le(t-a)\varepsilon,\ a\le t\le b.$$

Clearly, we have

$$|w(t)-z(t)|=0,\ a-h\le t\le a.$$

For $a\le t\le b$, we can write

$$\begin{aligned}|w(t)-z(t)|&\le\left|w(t)-w(a)-\int_a^t\sum_{i=1}^{N}F_i(s,w(s),w(g_i(s)))ds\right|\\&\quad+\left|\int_a^t\sum_{i=1}^{N}\left[F_i(s,w(s),w(g_i(s)))-F_i(s,z(s),z(g_i(s)))\right]\right|ds\\&\le(t-a)\varepsilon+\int_a^t\left|F_1(s,w(s),w(g_1(s)))-F_1(s,z(s),z(g_1(s)))\right|ds\end{aligned}$$

$$+\int_a^t \left|F_2(s,w(s),w(g_2(s))) - F_2(s,z(s),z(g_2(s)))\right| ds$$

$$+\ldots+\int_a^t \left|F_N(s,w(s),w(g_N(s))) - F_N(s,z(s),z(g_N(s)))\right| ds$$

$$\le (t-a)\varepsilon + L_{F_1}\int_a^t |w(s)-z(s)|\,ds + L_{F_1}\int_a^t |w(g_1(s))-z(g_1(s))|\,ds$$

$$+L_{F_2}\int_a^t |w(s)-z(s)|\,ds + L_{F_2}\int_a^t |w(g_2(s))-z(g_2(s))|\,ds$$

$$+\ldots+L_{F_N}\int_a^t |w(s)-z(s)|\,ds + L_{F_N}\int_a^t |w(g_N(s))-z(g_N(s))|\,ds$$

$$= (t-a)\varepsilon + \sum_{i=1}^N L_{F_i}\int_a^t |w(s)-z(s)|\,ds$$

$$+\sum_{i=1}^N L_{F_i}\int_a^t |w(g_i(s))-z(g_i(s))|\,ds. \tag{8}$$

By the virtue of the inequality (8), for $u \in C([a-h,b],\mathbb{R}^+)$, we pay attention to the operator A from $C([a-h,b],\mathbb{R}^+)$ to $C([a-h,b],\mathbb{R}^+)$, which is defined in the following lines by

$$A(u)(t) = \begin{cases} 0, \; a-h \le t \le a, \\ (t-a)\varepsilon + \sum_{i=1}^N L_{F_i}\int_a^t u(s)ds + \sum_{i=1}^N L_{F_i}\int_a^t u(g_i(s))ds, \; a \le t \le b. \end{cases}$$

We will prove that A is a Picard operator. Hence, we first observe that A is a contraction. For $u,v \in C([a-h,b],\mathbb{R}^+)$ and $a \le t \le b$, we have

$$\left|A(u)(t) - A(v)(t)\right| \le L_{F_1}\int_a^t \left[|u(s)-v(s)| + |u(g_1(s))-v(g_1(s))|\right] ds$$

$$+L_{F_2}\int_a^t \left[|u(s)-v(s)| + |u(g_2(s))-v(g_2(s))|\right] ds$$

$$+\ldots+L_{F_N}\int_a^t \left[|u(s)-v(s)| + |u(g_N(s))-v(g_N(s))|\right] ds$$

$$\leq L_{F_1}(b-a)\left(\max_{a\leq s\leq b}\left|u(s)-v(s)\right|+\max_{a\leq s\leq b}\left|u(g_1(s))-v(g_1(s))\right|\right)$$

$$+L_{F_2}(b-a)\left(\max_{a\leq s\leq b}\left|u(s)-v(s)\right|+\max_{a\leq s\leq b}\left|u(g_2(s))-v(g_2(s))\right|\right)$$

$$+\ldots+L_{F_N}(b-a)\left(\max_{a\leq s\leq b}\left|u(s)-v(s)\right|+\max_{a\leq s\leq b}\left|u(g_N(s))-v(g_N(s))\right|\right)$$

$$\leq 2(b-a)\sum_{i=1}^{N}\left(L_{F_i}\right)\left\|u-v\right\|.$$

As the next step, for all $u,v\in C([a-h,b],\mathbb{R}^+)$, it follows that

$$\left\|A(u)(t)-A(v)(t)\right\|\leq 2(b-a)\sum_{i=1}^{N}\left(L_{F_i}\right)\left\|u-v\right\|.$$

Since $2(b-a)\sum_{i=1}^{N}\left(L_{F_i}\right)\left\|u-v\right\|<1$, A is a contraction on $C([a-h,b],\mathbb{R}^+)$. Employing the BCP, we conclude that A is a Picard operator and $F_A=\left\{u^*\right\}$. Then,

$$u^*(t)=(t-a)\varepsilon+\sum_{i=1}^{N}\left(L_{F_i}\right)\int_a^t\left[u^*(s)+u^*(g_i(s))\right]ds,\quad a\leq t\leq b. \tag{9}$$

The solution u^* is increasing and $(u^*)'\geq 0$. Therefore, we see that

$$u^*(g_1(t))\leq u^*(t),\ldots,u^*(g_N(t))\leq u^*(t). \tag{10}$$

According to (9) and (10), we get

$$u^*\leq (t-a)\varepsilon+2\left(\sum_{i=1}^{N}\left(L_{F_i}\right)\right)\int_a^t u^*(s)ds.$$

From Lemma 2.1, we obtain

$$u^*(t)\leq c\varepsilon,\quad a-h\leq t\leq b,$$

where

$$c=(b-a)\exp\left[2(b-a)\sum_{i=1}^{N}\left(L_{F_i}\right)\right].$$

If $u=\left|w-z\right|$, $u(t)\leq A(u)(t)$, then we obtain

$$u(t)\leq u^*(t),$$

by Lemma 2.2.

This means that A is an increasing Picard operator. Next, we derive that

$$\left|w(t)-z(t)\right| \le c\varepsilon, \quad a-h \le t \le b.$$

Thus, it shows that the FDE (1) is HU stable.

Now, before stating the next GHUR s result, we assume the following conditions meet:

(C3) $F_i \in C([a,\infty)\times\mathbb{R}^2,\mathbb{R}),\ g_i \in C([a,\infty),[a-h,\infty)),\ g_i(t) \le t,$

$i = 1,2,...,N,\ h>0,\ \forall t \in I = [a,\infty);$

(C4) there exist functions $L_{F_1},\ L_{F_2},...,L_{F_N} \in L^1([a,\infty),\mathbb{R}^+)$ such that

$$\left|F_1(t,u_1,u_2)-F_1(t,v_1,v_2)\right| \le L_{F_1}(t)\left\{\left|u_1-v_1\right|+\left|u_2-v_2\right|\right\},$$

$$\left|F_2(t,u_1,u_3)-F_2(t,v_1,v_3)\right| \le L_{F_2}(t)\left\{\left|u_1-v_1\right|+\left|u_3-v_3\right|\right\},$$

$$\cdots$$

$$\left|F_N(t,u_1,u_N)-F_N(t,v_1,v_N)\right| \le L_{F_N}(t)\left\{\left|u_1-v_1\right|+\left|u_{N+1}-v_{N+1}\right|\right\},$$

$$\forall t \in [a,\infty),\ \forall u_1,u_{i+1},v_1,v_{i+1} \in \mathbb{R}, \quad i=1,2,...,N.$$

(C5) There is an increasing function $\psi \in C([a,\infty))$ and there exists λ be positive constant such that $\int_a^t \psi(s)ds \le \lambda\psi(t)$ for all $a \le t < \infty$.

Theorem 3.2.

If $(C3)-(C5)$ hold, then the following results hold:

3^0) the IVP (1), (2) has a unique solution in $C([a-h],\infty),\mathbb{R}) \cap C^1([a,\infty),\mathbb{R})$;

4^0) The FDE (1) is GHUR stable.

***Proof*:**

The steps in the proof of Theorem 3.1 are followed. Letting

$$w \in C([a-h],\infty),\mathbb{R}) \cap C^1([a,\infty),\mathbb{R}),$$

as a solution of the inequality (4), we will show the FDE (1) has a unique solution in $C([a-h],\infty),\mathbb{R})\cap C^1([a,\infty),\mathbb{R})$ for all $t\in[a,\infty)$.

Denote the unique solution by $z\in C([a-h],\infty),\mathbb{R})\cap C^1([a,\infty),\mathbb{R})$ of the below IVP,

$$z'(t)=\sum_{i=1}^{N}F_i(t,\mathrm{z}(t),\mathrm{z}(g_i(t))),\ t\in I=[a,\infty),$$

$$\mathrm{z}(a)=w(t),\quad a-h\le t\le a.$$

Hence, integrating the FDE above, we have

$$z(t)=\begin{cases}\phi(t), & t\in[a-h,a],\\ \phi(t)+\int_a^t\sum_{i=1}^{N}F_i(s,z(s),z(g_i(s))ds, & t\in[a,\infty).\end{cases}$$

From the integration of the (4) and the condition $\int_a^t\psi(s)ds\le\lambda\psi(t)$ of Theorem 3, we obtain

$$\left|w(t)-w(a)-\int_a^t\sum_{i=1}^{N}F_i(s,w(s),w(g_i(s)))ds\right|\le\int_a^t\psi(s)ds\le\lambda\psi(t),\ t\in[a,\infty).$$

It is clear that for all $t\in[a-h,a]$, we have $|w(t)-z(t)|=0$. Next, for $t\in[a,\infty)$, we derive

$$\begin{aligned}|w(t)-z(t)|\le&\left|w(t)-w(a)-\int_a^t\sum_{i=1}^{N}F_i(s,w(s),w(g_i(s))ds\right|\\&+\left|\int_a^t\sum_{i=1}^{N}\left[F_i(s,w(s),w(g_i(s)))-F_i(s,z(s),z(g_i(s)))\right]\right|ds\\\le&\lambda\phi(t)+\int_a^t\sum_{i=1}^{N}\left|F_i(s,w(s),w(g_i(s)))-F_i(s,z(s),z(g_i(s)))\right|ds\\\le&\lambda\phi(t)+\int_a^t L_{F_1}(s)|w(s)-z(s)|ds+\int_a^t L_{F_1}(s)|w(g_1(s))-z(g_1(s))|ds\\&+\int_a^t L_{F_2}(s)|w(s)-z(s)|ds+\int_a^t L_{F_2}(s)|w(g_2(s))-z(g_2(s))|ds\end{aligned}$$

$$+\ldots+\int_a^t L_{F_N}(s)\left|w(s)-z(s)\right|ds+\int_a^t L_{F_N}(s)\left|w(g_N(s))-z(g_N(s))\right|ds.$$

According to the above information, we conclude that

$$\left|w(t)-z(t)\right|\le\lambda\psi(t)+\sum_{i=1}^{N}\int_a^t L_{F_i}(s)\left|w(s)-z(s)\right|ds+\sum_{i=1}^{N}\int_a^t L_{F_i}(s)\left|w(g_i(s))-z(g_i(s))\right|ds.$$

Using the Gronwall lemma, we obtain

$$\left|w(t)-z(t)\right|\le\lambda\psi(t)\exp\left(2\sum_{i=1}^{N}\left(L_{F_i}(s)\right)\right)=C_\psi\psi(t),\ \ t\in[a,\infty),$$

where $C_\psi=\lambda\exp\left(2\sum_{i=1}^{N}\left(L_{F_i}(s)\right)\right)$. Therefore, the FDE (1) is generalized Hyers-Ulam-Rassias stable.

4. Numerical Applications

Example 4.1.

We consider the below FDE of first order including two multiple variable delays and the initial condition:

$$\mathrm{z}'(t)=F_1\left(t,\mathrm{z}(t),\mathrm{z}\left(\frac{t^2}{2}\right)\right)+F_1\left(t,\mathrm{z}(t),\mathrm{z}\left(\frac{t^2}{4}\right)\right),\ t\in I=[0,1], \tag{11}$$

$$\mathrm{z}(0)=z_0, \tag{12}$$

together with the inequalities in the following lines:

$$\left|w'(t)-F_1\left(t,w(t),w\left(\frac{t^2}{2}\right)\right)-F_2\left(t,w(t),w\left(\frac{t^2}{4}\right)\right)\right|\le\varepsilon,$$

and

$$\left|w'(t)-F_1\left(t,w(t),w\left(\frac{t^2}{2}\right)\right)-F_2\left(t,w(t),w\left(\frac{t^2}{4}\right)\right)\right|\le\psi(t),\ t\in[0,1],$$

where $F_1,\ F_2 \in C([0,1]\times\mathbb{R}^2,\mathbb{R}),\ g_1,\ g_2 \in C([0,1],[-h,1]),\ g_1(t)=\dfrac{t^2}{2}\le t,\ g_2(t)=\dfrac{t^2}{4}\le t,$ $t\in[0,1],\ h\in\mathbb{R},\ h>0,$ and ψ is nondecreasing continuous function $\psi:[-r,1]\to\mathbb{R}^+$, $\mathbb{R}^+=[0,\infty)$. Hence, depending on the conditions of Theorem 3.1, the IVP (11), (12) has a unique solution in $C([-h,1],\mathbb{R})\cap C^1([0,1],\mathbb{R})$ and the FDE (11) is the HU stable provided that $2\sum_{i=1}^{2}\left(L_{F_i}\right)<1.$

We should also note that similar studies can be done depending on the conditions of Theorem 3.2 such that the FDE (11) has a unique solution in $C([-h,\infty],\mathbb{R})\cap C^1([0,\infty],\mathbb{R})$ and the FDE (11) is GHUR stable.

5. Conclusions

This work deals with the uniqueness of solutions, the UHS and the GUHRS of a FDE of first order including N-variable delays. In the present work, two new results including the sufficient conditions in relation to these qualitative concepts for the considered FDE are constructed. The main techniques and basic results of the outcomes are the BCP, Picard operator, Gronwall's lemma and so on. The outcomes of this paper have new inputs to qualitative theory of FDEs.

REFERENCES

Akkouchi, M. (2019). On the Hyers-Ulam-Rassias stability of a nonlinear integral equation, Appl. Sci., Vol. 21, pp. 1–10.

Biçer, E. and Tunç, C. (2017). On the Hyers-Ulam stability of certain partial differential equations of second order, Nonlinear Dyn. Syst. Theory, Vol. 17, Issue 2, pp. 150–157.

Biçer, E. and Tunç, C. (2018). New theorems for Hyers-Ulam stability of Lienard equation with variable time lags, Int. J. Math. Comput. Sci., Vol. 13, Issue 2, pp. 231–242.

Brzdęk, J., Popa, D. and Themistocles, M. R. (2019). *Ulam Type Stability*, Selected papers from the CUTS Conferences held in Cluj-Napoca, July 4–9, 2016 and Timişoara, October 8–13, 2018, Springer, Cham.

Burton, T. A. (2006). *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, Inc., Mineola, NY.

Castro, L.P. and Simões, A.M. (2018). Hyers-Ulam-Rassias stability of nonlinear integral equations through the Bielecki metric, Math. Methods Appl. Sci., Vol. 41, Issue 17, pp. 7367–7383.

Castro, L.P. and Simões, A.M. (2019). *Hyers-Ulam and Hyers-Ulam-Rassias stability for a class of integro-differential equations*, Mathematical Methods in Engineering, pp. 81–94, Nonlinear Syst. Complex., 23, Springer, Cham.

Chauhan, H.V.S., Singh, B., Tunç, C. and Tunç, O. (2022). On the existence of solutions of non-linear 2D Volterra integral equations in a Banach space, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, Vol. 116, Issue 3, Paper No. 101, 11 pp.

Ciplea, S.A., Lungu, N., Marian, D. and Rassias, T. M. (2023). Hyers-Ulam stability of a general linear partial differential equation, Aequationes Math., Vol. 97, Issue. 4, pp. 649–657.

Deep, A., Deepmala and Tunç, C. (2020). On the existence of solutions of some non-linear functional integral equations in Banach algebra with applications, Arab Journal of Basic and Applied Sciences, Vol. 27, Issue. 1, pp. 279–286.

Diaz, J. B. and Margolis, B. (1968). A fixed point theorem of the alternative, for contractions on a generalized complete metric space, Bull. Amer. Math. Soc., Vol. 74, pp. 305–309.

Graef, J. R., Tunç, C., Şengün, M. and Tunç, O. (2023). The stability of nonlinear delay integro-differential equations in the sense of Hyers-Ulam, Nonauton. Dyn. Syst., Vol. 10, Issue 1, msds–2022–0169. https://doi.org/10.1515/msds-2022-0169

Jung, S.-M. (2003). *On the Hyers-Ulam-Rassias Stability of a Functional Equation*, Kluwer Academic Publishers, Dordrecht, pp. 67–71.

Jung, S.-M. (2004). Hyers-Ulam stability of linear differential equations of first order, Appl. Math. Lett., Vol. 17, pp. 1135–1140.

Jung, S.-M. (2006). Hyers-Ulam stability of a system of first order linear differential equations with constant coefficient, J. Math. Anal. Appl., Vol. 320, Issue 2, pp. 549–561.

Jung, S.-M. (2011). *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optim. Appl., 48, Springer, New York.

Khan, H., Tunc, C., Chen, W. and Khan, A. (2018). Existence theorems and Hyers-Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator, J. Appl. Anal. Comput., Vol. 8, Issue 4, pp. 1211–1226.

Kucche, K.D. and Shikhare, P.U. (2019). Ulam stabilities for nonlinear Volterra delay integro-differential equations, Izv. Nats. Akad. Nauk Armenii Mat. 54 (2019), No. 5, pp. 27–43; reprinted in J. Contemp. Math. Anal., Vol. 54, Issue 5, pp. 276–287.

Li, Y. and Shen, Y. (2009). Hyers-Ulam stability of nonhomogeneous linear differential equations of second order, Int. J. Math. Math. Sci., Art. ID 576852, 7 pp.

Lungu, N. and Popa, D. (2012). Hyers-Ulam stability of a first order partial differential equation, J. Math. Anal. Appl., Vol. 385, Issue 1, pp. 86–91.

Miura, T., Hirasawa, G. and Takahasi, S.-E. (2006). Note on the Hyers-Ulam-Rassias stability of the first order linear differential equation, Nonlinear Funct. Anal. Appl., Vol. 11, Issue 5, pp. 851–858.

Onitsuka, M. (2019). Hyers-Ulam stability of first order linear differential equations of Carathéodory type and its application, Appl. Math. Lett., Vol. 90, pp. 61–68.

Otrocol, D. and Ilea, V. (2013). Ulam stability for a delay differential equation, Cent. Eur. J. Math., Vol. 11, Issue 7, pp. 1296–1303.

Petru, T. P., Petruşel, A. and Yao, J.-C. (2011). Ulam-Hyers stability for operatorial equations and inclusions via nonself operators, Taiwanese J. Math., Vol. 15, Issue 5, pp. 2195–2212.

Popa, D. and Raşa, I. (2011). On the Hyers-Ulam stability of the linear differential equation, J. Math. Anal. Appl., Vol. 381, Issue 2, pp. 530–537.

Rus, I., (2009). Gronwall lemmas: Ten open problems, Sci. Math. Jpn., Vol. 70, pp. 221–228.

Shah, S.O., Tunç, C., Rizwan, R., Zada, A., Khan, Q.U., Ullah, I. and Ullah, I. (2022). Bielecki-Ulam's types stability analysis of Hammerstein and mixed integro-dynamic systems of non-linear form with instantaneous impulses on time scales, Qual. Theory Dyn. Syst. 21, No. 4, Paper No. 107, 21 pp.

Tripathy, A. K. (2021). *Hyers-Ulam Stability of Ordinary Differential Equations*, CRC Press.

Tunç, C. and Biçer, E. (2015). Hyers-Ulam-Rassias stability for a first order functional differential equation, J. Math. Fundam. Sci., Vol. 47, Issue 2, pp.143–153.

Tunç, C. and Tunç, O. (2021). On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation, RACSAM, Vol. 115, Issue 3, Paper No. 115, 17 pp. https://doi.org/10.1007/s13398-021-01058-8

Tunç, C., Tunç, O., Wang, Y. and Yao, J.-C. (2021). Qualitative analyses of differential systems with time-varying delays via Lyapunov–Krasovskiĭ approach, Mathematics, Vol. 2021, Issue 9, Art. ID 1196, 20 pp. https://doi.org/10.3390/math9111196

Tunç, O. (2021). On the behaviors of solutions of systems of non-linear differential equations with multiple constant delays, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. Vol. 115, Issue 4, Paper No. 164, 22 pp. https://doi.org/10.1007/s13398-021-01104-5

Tunç, O. (2023). Stability tests and solution estimates for non-linear differential equations, Int. J. Optim. Control. Theor. Appl. IJOCTA, Vol. 13, Issue 1, pp. 92–103.

Tunç, O., Atan, Ö., Tunç, C and Yao, J.-C. (2021). Qualitative analyses of integro-fractional differential equations with Caputo derivatives and retardations via the Lyapunov–Razumikhin method, Axioms, Vol. 10, Issue 58, 19 pp. https://doi.org/10.3390/axioms10020058

Tunç, O. and Tunç, C. (2023). Solution estimates to Caputo proportional fractional derivative delay integro-differential equations, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., Vol. 117, Issue 1, Paper No. 12, 13 pp. https://doi.org/10.1007/s13398-022-01345-y

Tunç, O. and Tunç, C. (2023). Ulam stabilities of nonlinear iterative integro-differential equations, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., Vol. 117, Issue 1, Paper No. 118, 18 pp. https://doi.org/10.1007/s13398-023-01450-6

Ulam, S.M. (1964). *Problems in Modern Mathematics*, Science Editions, John Wiley & Sons, Inc., New York.