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Anitta Susan Aniyan
St. Thomas College, Palai

Dais George
Catholicate College, Pathanamthitta

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A New Class of Pareto Distribution: Estimation and its Applications

[1*]Anitta Susan Aniyan and [2]Dais George

[1]Department of Statistics
St. Thomas College, Palai
Kerala, India
anittasusananiyan996@gmail.com

[2]Department of Statistics
Catholicate College, Pathanamthitta
Kerala, India
daissaji@rediffmail.com

*Corresponding Author

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Abstract

The classical Pareto distribution is a positively skewed and right heavy-tailed lifetime distribution having many applications in various fields of science and social science. In this work, via the logarithmic transformed method, a new three-parameter lifetime distribution, an extension of classical Pareto distribution is generated. The different structural properties of the new distribution are studied. The model parameters are estimated by the method of maximum likelihood and Bayesian procedure. When all the three parameters of the distribution are unknown, the Bayes estimators cannot be obtained in a closed form, and hence, the Lindley's approximation under squared error loss function is used to compute the Bayes estimators. A Monte Carlo simulation study is also conducted to compare the performance of these estimators using mean square error. The application of the new distribution for modeling earthquake insurance and reliability data are illustrated using two real data sets.

Keywords: Bayes estimators; Classical Pareto distribution; Heavy-tailed distribution; Lindley's approximation; Logarithmic transformed method; Maximum likelihood method of estimation; Monte Carlo simulation

MSC 2010 No.: 62F15, 60E15, 60E05

1. Introduction

The classical Pareto (Pareto-I) distribution plays a crucial role in various fields such as actuarial science, medical and biological sciences, economics, survival analysis, telecommunication, life testing, hydrology and climatology (see Burroughs and Tebbens (2001), Newman (2005), Verma and Betti (2006), Arnold (2008) and Chhetri et al. (2017)).

Moreover, being positively skewed, heavy right tailed and of decreasing hazard rate, this distribution can also be additionally used for modeling data sets of higher insurance claims and failure times generated from various products and mechanisms. For example, infant mortality and machine life cycles which possess more flexible behavior (see Chahkandi and Ganjali (2009)). Due to its significance and greater versatility, various extensions and generalizations of the Pareto-I distribution have been developed recently by many researchers with added flexibility, viz., Beta-Pareto distribution (Akinsete et al. (2008)), Kumaraswamy Pareto distribution (Bourguignon et al. (2013)), Kumaraswamy transmuted Pareto distribution (Chhetri et al. (2017)), transmuted new Weibull-Pareto distribution (Tahir et al. (2018)), alpha-power Pareto distribution (Ihtisham et al. (2019)), cubic transmuted Pareto distribution (Rahman et al. (2020)) and Pareto-Weibull distribution (Rana et al. (2020)).

In 2012, Pappas et al. introduced Logarithmic Transformed (LT) family of distributions and based on their idea a good deal of work had been done later, such as (P-A-L) extended modified Weibull distribution by Al-Zahrani et al. (2016), alpha logarithmic transformed generalized exponential distribution by Dey et al. (2017), alpha logarithmic transformed Fréchet distribution by Dey et al. (2019), modified logarithmic transformed inverse Lomax distribution by Almarashi (2021) and logarithmic transformed Lomax distribution by Alotaibi et al. (2021).

In this paper, we introduce a new three-parameter extension of Pareto-I distribution with decreasing failure rate namely, logarithmic transformed Pareto-I (LTPa-I) distribution and discuss its various statistical properties. The objective of this paper is twofold: to derive the various statistical properties of the LTPa-I distribution and the parameter estimation of the model under classical and Bayesian paradigms.

The article is composed of seven sections. The next section proposes the logarithmic transformed Pareto-I (LTPa-I) distribution. In Section 3, some of its important mathematical and statistical properties are derived. The model parameters are estimated by the method of maximum likelihood and the Bayesian method of estimation in Section 4. A simulation study is conducted to compare the performance of the proposed estimators in Section 5. In the penultimate section, two real-life data sets are analyzed and the article is concluded in Section 7.

2. LTPa-I Distribution

Let X be a random variable from the Pareto-I distribution with PDF and CDF are given, respectively, as

$$g(x) = \frac{\tau\lambda^{\tau}}{x^{\tau+1}}, \;\; \lambda, \tau > 0, \; x \geq \lambda,$$

and

$$G(x) = 1 - \left(\frac{\lambda}{x}\right)^{\tau}, \tag{1}$$

where λ and τ are the scale and shape parameters, respectively.

Here, we introduce the LTPa-I distribution following the idea of Pappas et al. (2012). The CDF and PDF of the LT family of distributions are, respectively, defined as

$$F(x) = 1 - \frac{\log\left[\rho - (\rho-1)G(x)\right]}{\log(\rho)}, \;\; \rho > 0, \; \rho \neq 1, \tag{2}$$

and

$$f(x) = \frac{(\rho-1)g(x)}{\log(\rho)[\rho - (\rho-1)G(x)]}, \;\; \rho > 0, \rho \neq 1, \tag{3}$$

where $G(x)$ and $g(x)$ denote the CDF and PDF of a baseline distribution.

By inserting the CDF of the Pareto-I distribution, $G(x)$ as the baseline CDF in Equations (2) and (3), then the CDF and PDF of the LTPa-I distribution are, respectively, obtained as

$$F(x;\alpha,\tau,\lambda) = 1 - \frac{\log[\rho - (\rho-1)(1 - \left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}, \;\;\; \rho, \lambda, \tau > 0, \;\; x \geq \lambda, \; \rho \neq 1, \tag{4}$$

and

$$f(x;\rho,\tau,\lambda) = \frac{(\rho-1)\tau\lambda^{\tau})}{x^{\tau+1}\log(\rho)[\rho - (\rho-1)(1 - \left(\frac{\lambda}{x}\right)^{\tau})]}, \;\;\; \rho, \lambda, \tau > 0, \;\; x \geq \lambda, \; \rho \neq 1, \tag{5}$$

where ρ is an extra shape parameter.

When $\rho \to 1$, LTPa-I reduces to Pareto-I distribution. Therefore, LTPa-I is considered as a generalization of the Pareto-I distribution (see Appendix A).

The corresponding survival, hazard rate, reverse hazard rate and cumulative hazard rate functions can be represented as

$$S(x;\rho,\tau,\lambda) = \frac{\log[\rho - (\rho-1)(1 - \left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}, \tag{6}$$

$$h(x;\rho,\tau\lambda) = \frac{(\rho-1)\tau\lambda^{\tau}}{x^{\tau+1}\log[\rho - (\rho-1)(1 - \left(\frac{\lambda}{x}\right)^{\tau})]\left[\rho - (\rho-1)(1 - \left(\frac{\lambda}{x}\right)^{\tau})\right]}, \tag{7}$$

$$\phi(x;\rho,\tau,\lambda) = \frac{(\rho-1)\tau\lambda^{\tau}}{x^{\tau+1}\left[\rho - (\rho-1)(1 - \left(\frac{\lambda}{x}\right)^{\tau})\right]\log\left[\rho\left[\rho - (\rho-1)(1 - \left(\frac{\lambda}{x}\right)^{\tau})\right]^{-1}\right]} \tag{8}$$

and

$$H(x;\rho,\tau,\lambda) = -\ln\left[\frac{\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}\right]. \tag{9}$$

The Mills and the Odds ratios of the LTPa-I distribution are, respectively, given by

$$m(x;\rho,\tau,\lambda) = \frac{x^{\tau+1}\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]\left[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})\right]}{(\rho-1)\tau\lambda^{\tau}}, \text{ and}$$

$$O(x;\rho,\tau,\lambda) = \frac{x^{\tau+1}\left[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})\right]\log\left[\rho\left[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})\right]^{-1}\right]}{(\rho-1)\tau\lambda^{\tau}}.$$

The behavior of the LTPa-I model is illustrated graphically by plotting the shape of PDF and hazard rate function (hrf). Figure 1 and Figure 2 depict the PDF and hrf shapes of LTPa-I distribution respectively. The PDF of LTPa-I distribution is reversed–J shape, right-skewed and decreasing while hrf of LTPa-I distribution has decreasing shape. The decreasing failure rate (dfr) indicates an improvement in the reliability of a certain system with time. The dfr phenomenon is characterized by the terms “infant mortality” or “work hardening.” Thus, LTPa-I may be a good candidate to analyze the data which possess dfr.

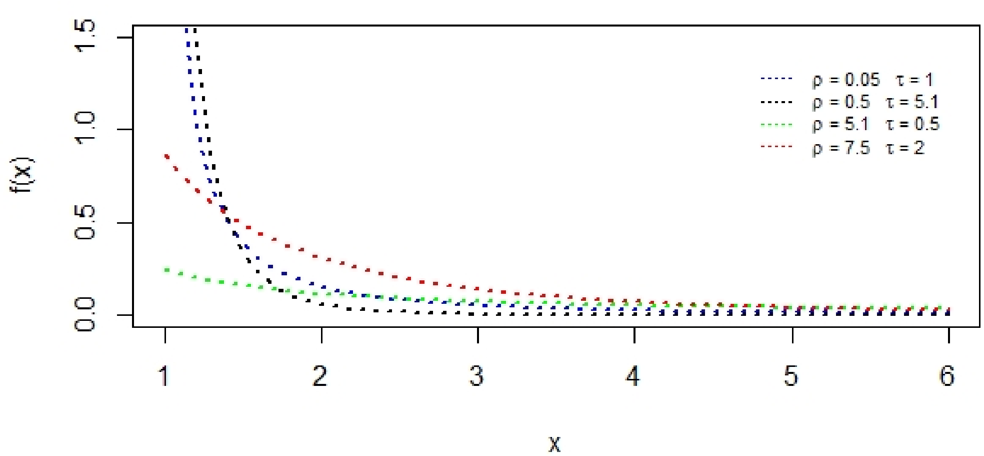

Figure 1. pdf plot of LTPa-I distribution with $\lambda = 1$ for various values of ρ and τ

Remark 2.1.

Dey et al. (2017) showed that the PDF of LT family has the following mixture representation,

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} w_{k,j}c_{j+1}(x), \tag{10}$$

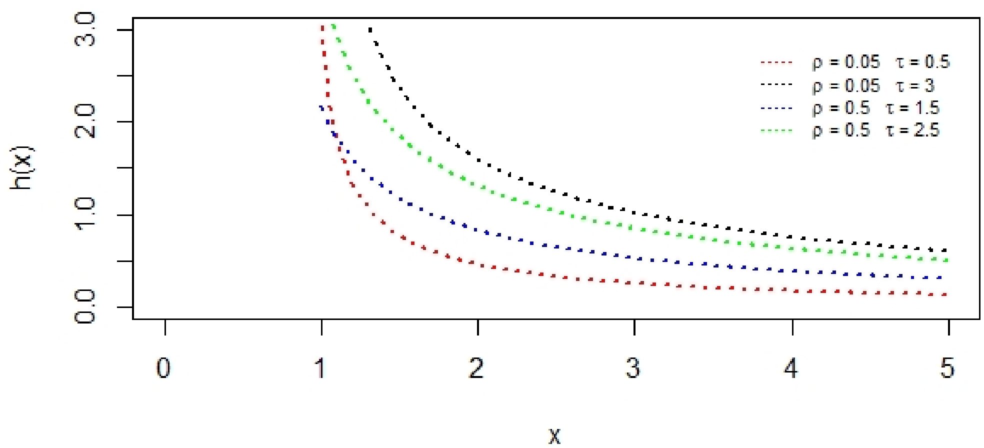

Figure 2. hrf plot of LTPa-I distribution with $\lambda = 1$ for various values of ρ and τ

where $c_{(\theta+1)} = (\theta+1)g(x)G^{(\theta+1)-1}(x)$ is the exp-G PDF with positive power parameter θ and

$$w_{k,j} = \binom{k}{j} \frac{(\rho-1)^{j+1}}{(j+1)(\rho+1)^{k+1}\log(\rho)}.$$

Inserting the PDF and CDF of Pareto-I model in the above equation, we can express Equation (5) as

$$\begin{aligned} f(x;\rho,\tau,\lambda) &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} w_{k,j}(j+1)\frac{\tau\lambda^{\tau}}{x^{\tau+1}}\left[1-\left(\frac{\lambda}{x}\right)^{\tau}\right]^{j} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m}\, g_P(x;\lambda,\tau(m+1)), \end{aligned} \tag{11}$$

where

$$w_{k,j,m} = \binom{j}{m}\frac{w_{k,j}(-1)^m(j+1)}{m+1},$$

and $g_P(x;\lambda,\tau(m+1))$ is the PDF of Pareto-I distribution with parameters $\tau(m+1)$ and λ. Hence, the LTPa-I density function can be expressed as a linear combination of Pareto-I distribution and it is useful for finding various structural properties of the LTPa-I distribution.

3. Properties

This section investigates quantiles, ordinary and incomplete moments, moment generating function and order statistics of LTPa-I distribution.

3.1. Quantiles

The q^{th} quantile x_q, for $0 < q < 1$, of the LTPa-I distribution is given by

$$x_q = Q(q) = \frac{\lambda}{\left[1 - \left(\frac{\rho-\rho^{1-q}}{\rho-1}\right)\right]^{\frac{1}{\tau}}} = \frac{\lambda}{\left(\frac{\rho^{1-q}-1}{\rho-1}\right)^{\frac{1}{\tau}}}, \tag{12}$$

and hence, the median is

$$M = \frac{\lambda}{\left(\frac{\rho^{0.5}-1}{\rho-1}\right)^{\frac{1}{\tau}}}.$$

In particular, the first and third quantiles, $Q(1)$ and $Q(3)$ can be obtained by substituting $q = 0.25$ and $q = 0.75$ in Equation (12) respectively.

Skewness and kurtosis of the LTPa-I distribution can be calculated using the following relations defined respectively by Bowley (1920) and Moors (1988),

$$S = \frac{Q(3) - 2Q(2) + Q(1)}{Q(3) - Q(1)}, \quad \text{and}$$

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

3.2. Moments

The s^{th} ordinary moments of the LTPa-I distribution is defined by,

$$\mu_s^{'} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m} \int_{\lambda}^{\infty} x^s g_P(x;\lambda,\tau(m+1))dx = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m} \frac{\lambda^s \tau(m+1)}{\tau(m+1)-s}, \quad s < \tau. \tag{13}$$

By using the Leibniz test, we can see that $\mu_s^{'}$ is convergent when $s < \tau$ (see Appendix C).

The mean of LTPa-I distribution is

$$\mu_1^{'} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m} \frac{\lambda \tau(m+1)}{\tau(m+1)-1}, \quad \tau > 1.$$

Also, from Equation (13),

$$\mu_2^{'} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m} \frac{\lambda^2 \tau(m+1)}{\tau(m+1)-2},$$

and hence, the variance is

$$V(X)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}w_{k,j,m}\frac{\lambda^2\tau(m+1)}{\tau(m+1)-2}-\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}w_{k,j,m}\frac{\lambda\tau(m+1)}{\tau(m+1)-1}\right]^2, \quad \tau>2.$$

3.3. Moment Generating Function

The moment generating function of LTPa-I distribution is found out using the mgf of Pareto-I distribution (see Chotikapanich (2008)) and it is given by

$$\begin{aligned}M_X(t)&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}w_{k,j,m}\int_{\lambda}^{\infty}e^{tx}g_P(x;\lambda,\tau(m+1))dx\\&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}w_{k,j,m}(-t\lambda)^{\tau(m+1)}\tau(m+1)\Gamma(-\tau(m+1),-\lambda t), \quad t<0. \qquad (14)\end{aligned}$$

3.4. Incomplete Moments

The p^{th} incomplete moment of the LTPa-I distribution is given by

$$\begin{aligned}\phi_p(t)&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}w_{k,j,m}\int_{\lambda}^{t}x^p g_P(x;\lambda,\tau(m+1))dx\\&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}w_{k,j,m}\frac{\tau(m+1)\lambda^p}{\tau(m+1)-p}\left[1-\left(\frac{\lambda}{t}\right)^{\tau(m+1)-p}\right], \quad p<\tau, \qquad (15)\end{aligned}$$

thereby, the first incomplete moment, $\phi_1(t)$, is

$$\phi_1(t)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}w_{k,j,m}\frac{\tau(m+1)\lambda}{\tau(m+1)-1}\left[1-\left(\frac{\lambda}{t}\right)^{\tau(m+1)-1}\right]. \qquad (16)$$

Some important applications of the first incomplete moment are mean deviations about the mean or median and in calculating some inequality measures such as mean residual life (MRL), mean inactivity time (MIT) and Lorenz and Bonferroni curves (see Gerstenkorn (1975) and Butler and McDonald (1989)). Inequality measures has applications in many fields including life insurance, economics, biomedical sciences and product quality control (see Aaberge (2000) and Giorgi and Crescenzi (2001)).

The mean deviations about the mean, $D_\mu(x)$, and about the median, $D_M(x)$, of the LTPa-I model can be expressed, respectively, as,

$$D_\mu(x)=\int_{\lambda}^{\infty}|x-\mu|f(x)dx=2\mu F(\mu)-2\phi_1(\mu), \quad \text{and}$$

$$D_M(x) = \int_{\lambda}^{\infty} |x - M| f(x) dx = \mu - 2\phi_1(M),$$

where F(.) is the cdf, μ is the mean, M is the median and $\phi_1(.)$ is the first incomplete moment of the LTPa-I distribution. These deviations can be used to measure the amount of scatteredness in a population.

The MRL function $\psi_X(t)$ of a component at time t or the life expectancy at the age $t > 0$ of a LTPa-I random variable is given by,

$$\psi_X(t) = \frac{\left(1 - \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{m=0}^{j} w_{k,j,m} \frac{\tau(m+1)\lambda}{\tau(m+1)-1} \left[1 - \left(\frac{\lambda}{t}\right)^{\tau(m+1)-1}\right]\right) \log(\rho)}{\log[\rho - (\rho - 1)(1 - \left(\frac{\lambda}{t}\right)^{\tau})] - t\log(\rho)}. \quad (17)$$

The mean inactivity time (MIT) of a lifetime random variable follows LTPa-I distribution is given by

$$\psi_X^{'}(t) = t - \frac{\sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{m=0}^{j} w_{k,j,m} \frac{\tau(m+1)\lambda}{\tau(m+1)-1} \left[1 - \left(\frac{\lambda}{t}\right)^{\tau(m+1)-1}\right]}{1 - \frac{\log[\rho - (\rho-1)(1 - \left(\frac{\lambda}{t}\right)^{\tau})]}{\log(\rho)}}. \quad (18)$$

Further, the Lorenz $L(q)$ and Bonferroni $B(q)$ curves are defined as

$$L(q) = \frac{1}{\mu} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{m=0}^{j} w_{k,j,m} \frac{\tau(m+1)\lambda}{\tau(m+1)-1} \left[1 - \left(\frac{\lambda}{x_q}\right)^{\tau(m+1)-1}\right],$$

and

$$B(q) = \frac{1}{q\mu} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{m=0}^{j} w_{k,j,m} \frac{\tau(m+1)\lambda}{\tau(m+1)-1} \left[1 - \left(\frac{\lambda}{x_q}\right)^{\tau(m+1)-1}\right], \quad (19)$$

respectively, where x_q can be computed numerically from Equation (12) for a given probability.

3.5. Order Statistics

Let $X_{(1)} \leq X_{(2)} \leq .... \leq X_{(n)}$ denote the order statistics of a random sample $X_1, X_2, .., X_n$ from LTPa-I (ρ, τ, λ). According to Wilks (1948), the PDF and CDF of the j^{th} order statistics are obtained, respectively, as

$$f_{X_{(j)}}(x;\rho,\tau,\lambda)=\frac{n!}{(j-1)!(n-j)!}\frac{(\rho-1)\tau\lambda^{\tau}}{x^{\tau+1}\log(\rho)[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}$$
$$\times\left[1-\frac{\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}\right]^{j-1}$$
$$\times\left[\frac{\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}\right]^{n-j}$$
$$=\frac{n!}{(j-1)!(n-j)!}\frac{(\rho-1)\tau\lambda^{\tau}}{x^{\tau+1}[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}$$
$$\times\left[1-\frac{\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}\right]^{j-1}\frac{\left[\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]\right]^{n-j}}{(\log\rho)^{n-j+1}},$$

and

$$F_{X_{(j)}}(x;\rho,\tau,\lambda)=\sum_{l=j}^{n}\sum_{u=0}^{l}(-1)^{u}\binom{n}{l}\binom{l}{u}\left[\frac{\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}\right]^{n-l+u}.$$

The PDF of the smallest order statistic, $X_{(1)}$ and the largest order statistic, $X_{(n)}$ are, respectively, given by,

$$f_{X_{(1)}}(x;\rho,\tau,\lambda)=\frac{n(\rho-1)\tau\lambda^{\tau}}{x^{\tau+1}[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}$$
$$\times\frac{\left[\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]\right]^{n-1}}{(\log\rho)^{n}},$$

and

$$f_{X_{(n)}}(x;\rho,\tau,\lambda)=\frac{n(\rho-1)\tau\lambda^{\tau}}{x^{\tau+1}\log(\rho)[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}$$
$$\times\left[1-\frac{\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}\right]^{n-1}.$$

4. Methods of Estimation

In this section, we consider the maximum likelihood and Bayesian estimation methods for estimating the unknown parameters of the LTPa-I distribution.

4.1. Maximum Likelihood Method of Estimation

Let $X_1, X_2, .., X_n$ be a random sample from the LTPa-I distribution with unknown parameter vector $\theta = (\rho, \tau, \lambda)'$, then the likelihood function for θ is given by

$$L(x;\theta) = \frac{(\rho-1)^n \tau^n \lambda^{n\tau}}{\prod_{i=1}^n x_i^{\tau+1} (\log\rho)^n \prod_{i=1}^n [\rho - (\rho-1)(1-\left(\frac{\lambda}{x}\right)^\tau)]}. \tag{20}$$

Then, the corresponding log-likelihood function reduces to

$$\begin{aligned} l(\theta) &= n\log(\rho-1) + n\log\tau + n\tau\log\lambda - n\log(\log\rho) \\ &\quad - (\tau+1)\sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \log\left(1 + (\rho-1)\left(\frac{\lambda}{x_i}\right)^\tau\right). \end{aligned} \tag{21}$$

Since, $x \in (\lambda, \infty)$, the first order statistic $X_{(1)}$ provides the maximum likelihood estimator of λ. Taking the partial derivatives of Equation (21) with respect ρ and τ, we get

$$\frac{\partial l(\theta)}{\partial\rho} = \frac{n}{\rho-1} - \frac{n}{\rho\log\rho} - \frac{1}{(\rho-1)}\sum_{i=1}^n \frac{e_i - 1}{e_i}, \tag{22}$$

and

$$\frac{\partial l(\theta)}{\partial\tau} = \frac{n}{\tau} - \sum_{i=1}^n \log(x_i) - \log\tau \sum_{i=1}^n \frac{e_i - 1}{e_i}, \tag{23}$$

where $e_i = 1 + (\rho-1)\left(\frac{\lambda}{x_i}\right)^\tau$.

The MLE's of the parameters ρ and τ are obtained by equating Equations (22) and (23) to zero and solve them simultaneously. But it is difficult to obtain their explicit solutions. In this situation, Newton-Raphson algorithm is used to obtain the desired MLE's and are computed by using MaxLik package (Henningsen and Toomet (2011)) in RStudio (RStudio Team (2022)).

4.2. Bayesian Estimation Method

Here, we discuss the Bayesian estimates of the unknown parameters of our model using the improper prior (non-informative prior) as the priors for all the three parameters under squared error loss function (SELF).

The joint prior distribution of ρ, τ and λ is written as

$$\pi(\rho,\tau,\lambda) \propto \frac{1}{\rho\tau\lambda},\ \rho,\tau,\lambda > 0. \tag{24}$$

Then, based on Equations (20) and (24), the joint posterior distribution is given by,

$$\pi(\rho,\tau,\lambda|x) = \frac{1}{K}\left(\frac{1}{\alpha\beta\lambda}\right)\left(\frac{(\alpha-1)^n\beta^n\lambda^{n\beta}}{\log^n(\alpha)}\right) \times \prod_{i=1}^{n}\left(\frac{1}{x_i^{\beta+1}\left(\alpha-(\alpha-1)(1-\left(\frac{\lambda}{x_i}\right)^{\beta})\right)}\right), \tag{25}$$

where

$$K = \int_0^\infty\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta\lambda}\right)\left(\frac{(\alpha-1)^n\beta^n\lambda^{n\beta}}{\log^n(\alpha)}\right) \times \prod_{i=1}^{n}\left(\frac{1}{x_i^{\beta+1}\left(\alpha-(\alpha-1)(1-\left(\frac{\lambda}{x_i}\right)^{\beta})\right)}\right) d\rho d\tau d\lambda.$$

The Bayes estimator of a function $\omega(\rho,\tau,\lambda)$ under SELF is the posterior mean and is given by,

$$\hat{\omega}(\rho,\tau,\lambda) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \omega(\rho,\tau,\lambda)L(\rho,\tau,\lambda)\times\pi(\rho,\tau,\lambda)d\rho d\tau d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty L(\rho,\tau,\lambda)\times\pi(\rho,\tau,\lambda)d\rho d\tau d\lambda}. \tag{26}$$

Here, it is observed that Equation (26) is not in a closed form. Hence, it is difficult to acquire the Bayes estimators for each parameter. Therefore, we employ Lindley's approximation technique to derive the approximate Bayes estimators.

4.2.1. *Lindley's Approximation*

The Bayes estimators in Equation (26) takes a ratio form of two integrals which cannot be expressed in an explicit form. Therefore, approximate Bayes estimators can be found out by a procedure named Lindley's approximation, which was proposed by Lindley (1980).

Now, the posterior expectation of a function $\omega(\rho,\tau,\lambda)$ can be written as

$$I(x) = E(\omega(\rho,\tau,\lambda)|x) = \frac{\int_{\rho,\tau,\lambda}\omega(\rho,\tau,\lambda)e^{l(\rho,\tau,\lambda)+\delta(\rho,\tau,\lambda)}d(\rho,\tau,\lambda)}{\int_{\rho,\tau,\lambda}e^{l(\rho,\tau,\lambda)+\delta(\rho,\tau,\lambda)}d(\rho,\tau,\lambda)},$$

where $\omega(\rho,\tau,\lambda)$ is a function of ρ, τ and λ only, $l(\rho,\tau,\lambda)$ is the log-likelihood and $\delta(\rho,\tau,\lambda) = log(\pi(\rho,\tau,\lambda))$.

According to Lindley (1980), the above equation can be approximately written as:

$$\begin{aligned} I(x) = {} & \omega(\hat{\rho},\hat{\tau},\hat{\lambda}) + (\hat{\omega}_1\hat{v}_1 + \hat{\omega}_2\hat{v}_2 + \hat{\omega}_3\hat{v}_3 + \hat{v}_4 + \hat{v}_5) \\ & + 0.5[\hat{A}(\hat{\omega}_1\hat{\sigma}_{11} + \hat{\omega}_2\hat{\sigma}_{12} + \hat{\omega}_3\hat{\sigma}_{13}) + \hat{B}(\hat{\omega}_1\hat{\sigma}_{21} + \hat{\omega}_2\hat{\sigma}_{22} + \hat{\omega}_3\hat{\sigma}_{23}) \\ & + \hat{C}(\hat{\omega}_1\hat{\sigma}_{31} + \hat{\omega}_2\hat{\sigma}_{32} + \hat{\omega}_3\hat{\sigma}_{33})], \end{aligned} \tag{27}$$

where $\hat{\rho}$, $\hat{\tau}$ and $\hat{\lambda}$ are the MLE's of ρ, τ and λ, respectively,

$$\begin{aligned}
\hat{\upsilon}_i &= \hat{\delta}_1\sigma_{i1} + \hat{\delta}_2\sigma_{i2} + \hat{\delta}_3\sigma_{i3}, \text{ for } i = 1, 2 \text{ and } 3,\\
\hat{\upsilon}_4 &= \hat{\omega}_{12}\hat{\sigma}_{12} + \hat{\omega}_{13}\hat{\sigma}_{13} + \hat{\omega}_{23}\hat{\sigma}_{23},\\
\hat{\upsilon}_5 &= 0.5(\hat{\omega}_{11}\hat{\sigma}_{11} + \hat{\omega}_{22}\hat{\sigma}_{22} + \hat{\omega}_{33}\hat{\sigma}_{33}),\\
\hat{A} &= \hat{\sigma}_{11}\hat{l}_{111} + 2\hat{\sigma}_{12}\hat{l}_{121} + 2\hat{\sigma}_{13}\hat{l}_{131} + 2\hat{\sigma}_{23}\hat{l}_{231} + \hat{\sigma}_{22}\hat{l}_{221} + \hat{\sigma}_{33}\hat{l}_{331},\\
\hat{B} &= \hat{\sigma}_{11}\hat{l}_{112} + 2\hat{\sigma}_{12}\hat{l}_{122} + 2\hat{\sigma}_{13}\hat{l}_{132} + 2\hat{\sigma}_{23}\hat{l}_{232} + \hat{\sigma}_{22}\hat{l}_{222} + \hat{\sigma}_{33}\hat{l}_{332}, \quad \text{and}\\
\hat{C} &= \hat{\sigma}_{11}\hat{l}_{113} + 2\hat{\sigma}_{12}\hat{l}_{123} + 2\hat{\sigma}_{13}\hat{l}_{133} + 2\hat{\sigma}_{23}\hat{l}_{233} + \hat{\sigma}_{22}\hat{l}_{223} + \hat{\sigma}_{33}\hat{l}_{333}.
\end{aligned}$$

Subscripts 1, 2 and 3 on the R. H. S. of the above equations denote ρ, τ and λ, respectively:

$$\begin{aligned}
\hat{\delta}_i &= \frac{\partial\delta}{\partial\theta_i} \text{ and } \hat{\omega}_i = \frac{\partial\omega}{\partial\theta_i}, \text{ for i} = 1, 2, 3, \theta_1 = \rho, \theta_2 = \tau \text{ and } \theta_3 = \lambda,\\
\hat{\omega}_{ij} &= \frac{\partial^2\omega(\theta_1, \theta_2, \theta_3)}{\partial\theta_i\partial\theta_j}, \text{ for i, j} = 1, 2, 3,\\
\hat{l}_{ij} &= \frac{\partial^2 l(\theta_1, \theta_2, \theta_3)}{\partial\theta_i\partial\theta_j}, \text{ for i, j} = 1, 2, 3,\\
\hat{l}_{ijk} &= \frac{\partial^3 l(\theta_1, \theta_2, \theta_3)}{\partial\theta_i\partial\theta_j\partial\theta_k}, \text{ for i, j} = 1, 2, 3, \quad \text{and}\\
\hat{\sigma_{ij}} &= \frac{-1}{L_{ij}}, \text{ for i, j} = 1, 2, 3.
\end{aligned}$$

Using Equation (4.5), $\delta(\rho, \tau, \lambda)$ can be written as

$$\delta(\rho, \tau, \lambda) = -\log\rho - \log\tau - \log\lambda, \tag{28}$$

and then we obtain

$$\hat{\delta}_1 = \frac{-1}{\hat{\rho}}, \ \hat{\delta}_2 = \frac{-1}{\hat{\tau}} \quad \text{and } \hat{\delta}_3 = \frac{-1}{\hat{\lambda}}.$$

Other quantities are obtained as follows:

$$\begin{aligned}
\hat{l}_{11} &= \frac{-n}{(\hat{\rho}-1)^2} + \frac{n(1+\log\hat{\rho})}{\hat{\rho}^2\log^2(\hat{\rho})} + \sum_{i=1}^{n}\frac{\left(\frac{\hat{\lambda}}{x_i}\right)^{2\hat{\tau}}}{\hat{e}_i^2},\\
\hat{l}_{12} = \hat{l}_{21} &= -\sum_{i=1}^{n}\frac{\log\left(\frac{\hat{\lambda}}{x_i}\right)\left(\frac{\hat{\lambda}}{x_i}\right)^{\hat{\tau}}}{\hat{e}_i^2},\\
\hat{l}_{13} = \hat{l}_{31} &= -\sum_{i=1}^{n}\frac{\hat{\tau}\left(\frac{\hat{\lambda}}{x_i}\right)^{\hat{\tau}}}{\hat{\lambda}\hat{e}_i^2},\\
\hat{l}_{22} &= \frac{-n}{\hat{\tau}^2} - \sum_{i=1}^{n}\frac{\log^2\left(\frac{\hat{\lambda}}{x_i}\right)(\hat{e}_i - 1)}{\hat{e}_i^2},
\end{aligned}$$

$$\hat{l}_{23} = \hat{l}_{32} = \frac{n}{\hat{\lambda}} - \sum_{i=1}^{n} \frac{\hat{\tau}\log\left(\frac{\hat{\lambda}}{x_i}\right)(\hat{e}_i - 1) + \hat{e}_i}{\hat{\lambda}\hat{e}_i^2},$$

$$\hat{l}_{33} = \frac{-n\hat{\tau}}{\hat{\lambda}^2} + \sum_{i=1}^{n} \frac{\hat{\tau}(\hat{e}_i - 1)(\hat{e}_i - \hat{\tau})}{\hat{\lambda}^2\hat{e}_i^2},$$

$$\hat{l}_{111} = \frac{2n}{(\hat{\rho}-1)^3} - \frac{n(2\log^2(\hat{\rho}) + 3\log\hat{\rho} + 2)}{\hat{\rho}^3\log^3(\hat{\rho})} - \sum_{i=1}^{n} \frac{2\left(\frac{\hat{\lambda}}{x_i}\right)^{3\hat{\tau}}}{\hat{e}_i^3},$$

$$\hat{l}_{112} = \hat{l}_{121} = \hat{l}_{211} = \sum_{i=1}^{n} \frac{2\log\left(\frac{\hat{\lambda}}{x_i}\right)\left(\frac{\hat{\lambda}}{x_i}\right)^{2\hat{\tau}}}{\hat{e}_i^3},$$

$$\hat{l}_{113} = \hat{l}_{131} = \hat{l}_{311} = \sum_{i=1}^{n} \frac{2\hat{\tau}\left(\frac{\hat{\lambda}}{x_i}\right)^{2\hat{\tau}}}{\hat{\lambda}\hat{e}_i^3},$$

$$\hat{l}_{123} = \sum_{i=1}^{n} \frac{\left(\frac{\hat{\lambda}}{x_i}\right)^{\hat{\tau}}\left[\hat{\tau}\log\left(\frac{\hat{\lambda}}{x_i}\right)(\hat{e}_i - 2) - \hat{e}_i\right]}{\hat{\lambda}\hat{e}_i^3},$$

$$\hat{l}_{221} = \sum_{i=1}^{n} \frac{\log^2\left(\frac{\hat{\lambda}}{x_i}\right)\left(\frac{\hat{\lambda}}{x_i}\right)^{\hat{\tau}}(\hat{e}_i - 2)}{\hat{e}_i^3},$$

$$\hat{l}_{222} = \sum_{i=1}^{n} \frac{\log^3\left(\frac{\hat{\lambda}}{x_i}\right)(\hat{e}_i - 1)(\hat{e}_i - 2)}{\hat{e}_i^3},$$

$$\hat{l}_{223} = \sum_{i=1}^{n} \frac{(\hat{e}_i - 1)\log\left(\frac{\hat{\lambda}}{x_i}\right)\left[(\hat{e}_i - 2)\hat{\tau}\log\left(\frac{\hat{\lambda}}{x_i}\right) - 2\hat{e}_i\right]}{\hat{\lambda}\hat{e}_i^3},$$

$$\hat{l}_{133} = \sum_{i=1}^{n} \frac{\hat{\tau}\left(\frac{\hat{\lambda}}{x_i}\right)^{\hat{\tau}}\left[(\hat{\tau}+1)(\hat{e}_i - 1) - \hat{\tau} + 1\right]}{\hat{\lambda}^2\hat{e}_i^3},$$

$$\hat{l}_{233} = \sum_{i=1}^{n} \frac{1}{\hat{\lambda}^2\hat{e}_i^3}(\hat{e}_i - 1)[((\hat{\tau}^2 + \hat{\tau})(\hat{e}_i - 1) - \hat{\tau}^2 + \hat{\tau})\log\left(\frac{\hat{\lambda}}{x_i}\right) + (\hat{e}_i - 1)^2 + 2(1-\hat{\tau})(\hat{e}_i - 1) - 2\hat{\tau} + 1], \quad \text{and}$$

$$\hat{l}_{333} = -\sum_{i=1}^{n} \frac{\hat{\tau}(\hat{e}_i - 1)[2(\hat{e}_i - 1)^2 - (\hat{\tau}^2 + 3\hat{\tau} - 4)(\hat{e}_i - 1) + \hat{\tau}^2 - 3\hat{\tau} + 2]}{\hat{\lambda}^3\hat{e}_i^3},$$

where $\hat{e}_i = 1 + (\hat{\rho} - 1)\left(\frac{\hat{\lambda}}{x_i}\right)^{\hat{\tau}}$.

If $\omega(\rho, \tau, \lambda) = \rho$, then $\hat{\omega}_1 = 1$, $\hat{\omega}_{11} = \hat{\omega}_{12} = \hat{\omega}_{13} = 0$ so that v_4 and $v_5 = 0$.

Similarly, we can obtain $\hat{\omega}_2 = \hat{\omega}_3 = 1$ and $\hat{\omega}_{22} = \hat{\omega}_{23} = \hat{\omega}_{33} = 0$.

Thus, the approximate Bayesian estimators of $\theta = (\rho, \tau, \lambda)$ under SELF that we obtained are

$$\hat{\rho}_{BS} = \hat{\rho} + \hat{v}_1 + 0.5(\hat{\sigma}_{11}\hat{A} + \hat{\sigma}_{21}\hat{B} + \hat{\sigma}_{31}\hat{C}),$$
$$\hat{\tau}_{BS} = \hat{\beta} + \hat{v}_2 + 0.5(\hat{\sigma}_{12}\hat{A} + \hat{\sigma}_{22}\hat{B} + \hat{\sigma}_{32}\hat{C}), \quad \text{and}$$
$$\hat{\lambda}_{BS} = \hat{\lambda} + \hat{v}_3 + 0.5(\hat{\sigma}_{13}\hat{A} + \hat{\sigma}_{23}\hat{B} + \hat{\sigma}_{33}\hat{C}).$$

5. Simulation Study

A Monte Carlo simulation study is implemented to evaluate the performance of the proposed estimators of the parameters for the LTPa-I distribution. A comparison between MLE's and Bayesian estimators are also made on the basis of their MSEs. All the computations were performed in R software (R Core Team (2022)). 1000 random samples of sizes $n = (20, 50, 100, 200)$ are generated from the LTPa-I model for $\lambda = (1, 2), \tau = (0.5, 2.5)$ and $\rho = (0.5, 2.5)$. The average values of estimates and their corresponding MSEs are reported in Tables 3 through 6 (in Appendix D).

It is evident that for both the method of estimations, the MSEs decreases with increasing sample size which exhibits the consistency property of all the estimates. It is also revealed that for all the parameters except λ, Bayes estimators outperform ML estimators, in terms of their smaller MSEs.

6. Applications

In this section, we analyse two datasets for illustrating the applicability and flexibility of the LTPa-I model and to compare the efficacy of different estimation approaches discussed in Section 4.

The first data set represents 29 observations on time to repair of a piece of construction equipment in chronological order was initially reported by Fan and Fan (2015).

Almarashi (2021) analyzed this data set using a modified logarithmic transformed inverse Lomax distribution and compared with inverse Weibull, alpha power inverse Lomax and alpha power inverse Weibull distributions. According to him, among them the modified logarithmic transformed inverse Lomax distribution is a better model for this data. We also fit our model to this data set and compare the results with modified logarithmic transformed inverse Lomax whose PDF is given by

$$\text{MLTIL} : f(x; \rho, \tau, \lambda) = \frac{(\rho - 1)\tau\lambda x^{-2}\left(1 + \frac{\lambda}{x}\right)^{-(\tau+1)}}{log(\rho)\left[\rho - (\rho - 1)\left(1 + \frac{\lambda}{x}\right)^{-\tau}\right]}.$$

The second data set corresponds to 19 observations on the loss ratios (yearly data in billlion of dollars) for earthquake insurance in California from the period 1971-1993 for the values larger than zero taken from Jaffe and Russell (1996).

Ghitany et al. (2018) modeled this data set with generalized truncated log-gamma distribution and it seems to be a better model. Here, we fit the proposed model LTPa-I to this data set and compare the results with generalized truncated log-gamma model with PDF given by

$$\text{GLTG} : f(x;\rho,\tau,\lambda) = \frac{\tau^{\rho}}{\lambda\Gamma\rho}\left(\frac{x}{\lambda}\right)^{-\tau-1}\left(log\frac{x}{\lambda}\right)^{\rho-1}.$$

The model comparison is done based on the four goodness of fit statistics: AIC (Akaike's Information criteria), BIC (Bayesian Information criteria), Kolmogorov–Smirnov (KS) statistic and its associated p–value via maximum likelihood estimates. The model with the smallest AIC, BIC, KS and the highest p-value is considered to be the best fit model for the given data sets. Table 1 lists the values of MLEs, AIC, BIC, KS and its respective p-values of the fitted distributions for the data sets 1 and 2. Plots of the fitted PDF of the LTPa-I distribution and other competitive distributions to the data sets 1 and 2 are displayed in Figures 3 and 4 respectively. From Table 1, it can be observed that the LTPa-I distribution yields better fit to both the data sets. The plots in Figures 3 and 4 also supports the argument of the suitability of LTPa-I distribution to the data sets in terms of model fitting.

Table 1. Parameter estimates and the goodness of fit statistics of both data sets

Data set	Distribution	Estimates	AIC	BIC	KS	p-value
Data 1 ($n = 29$)	LTPa-I(ρ,τ,λ)	(370.4763,1.1675,0.33)	209.1444	213.2463	0.0923	0.9657
	MLTIL(ρ,τ,λ)	(52.5769,41.521,0.0153)	215.0714	219.1733	0.0962	0.9513
Data 2 ($n = 19$)	LTPa-I(ρ,τ,λ)	(100.501,1.157,0.6)	134.5812	137.4145	0.1149	0.9388
	GTLG(ρ,τ,λ)	(7.401,1.845,0.1)	137.9738	140.8071	0.1488	0.7450

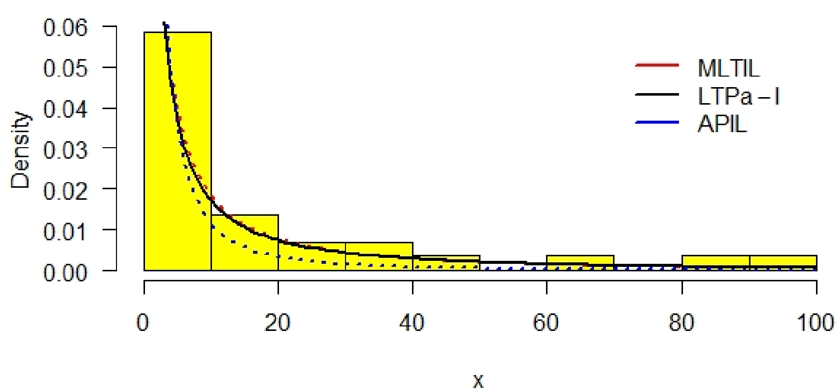

Figure 3. Fitted distributions for the data set 1.

Now, we compare the maximum likelihood and Bayesian estimation method by using the idea of Pradhan and Kundu (2011) on the basis of KS statistic and its p-value. Table 2 reports the ML and Bayes estimates, KS and its p-value for both the data sets. It can be noticed that for the considered data sets, Bayes estimates gives a better fit in terms of smaller KS value and larger p-value. So,

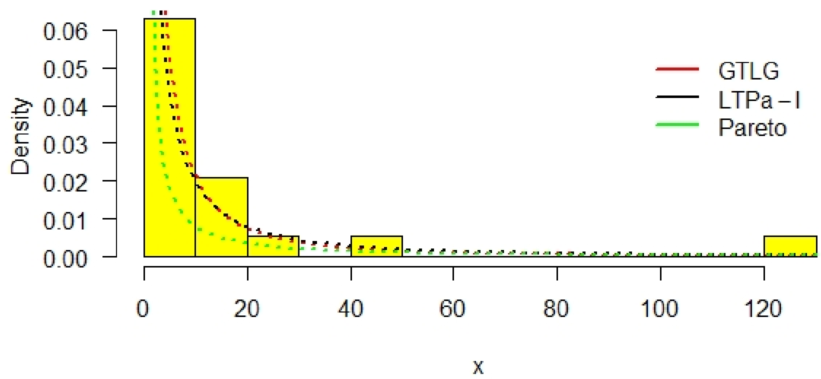

Figure 4. Fitted distributions for the data set 2.

we recommend to use the Bayes estimates to estimate the unknown parameters of the LTPa-I distribution. The plots of the fitted pdfs for the proposed methods of estimation are displayed in Figures 5 and 6 for both the data sets. These figures also validate the results in Table 2.

Table 2. ML, Bayes estimates and the goodness of fit statistics for both the data sets

Data set	Method	ρ	τ	λ	KS	p-value
Data 1 ($n = 29$)	MLE	370.4763	1.1675	0.33	0.0923	0.9657
	BS	372.3121	1.1629	0.3297	0.0909	0.9703
Data 2 ($n = 19$)	MLE	100.501	1.157	0.6	0.1149	0.9388
	BS	103.561	1.121	0.532	0.1134	0.9447

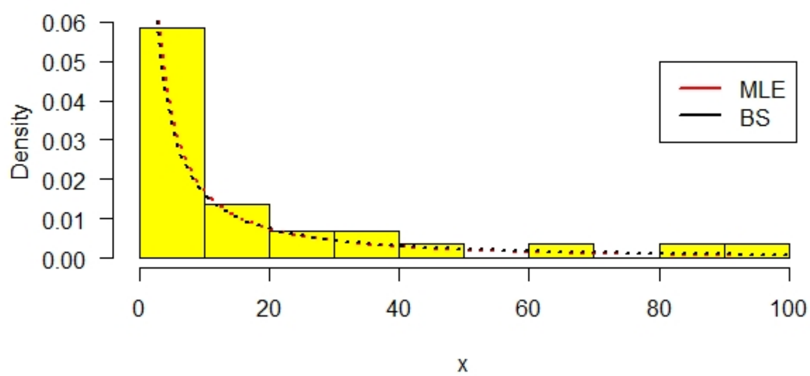

Figure 5. Fitted densities of the LTPa-I distribution using different estimators for data set 1

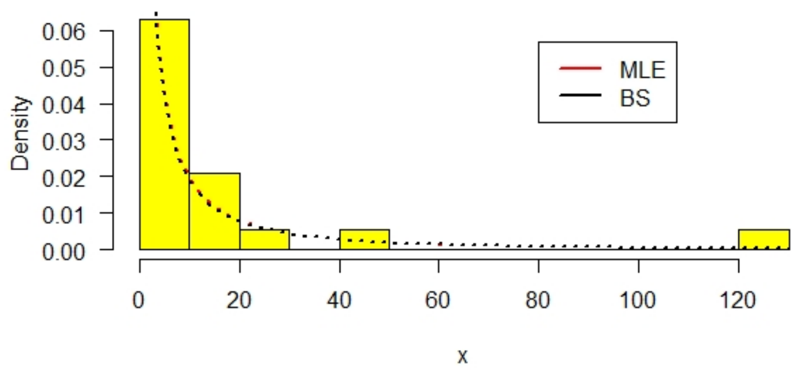

Figure 6. Fitted densities of the LTPa-I distribution using different estimators for data set 2

7. Conclusion

In the present article, a new three parameter lifetime distribution with decreasing failure rate namely, logarithmic transformed Pareto-I (LTPa-I) distribution, is introduced as a generalization of Pareto-I distribution. Some statistical and mathematical properties of the LTPa-I model are derived. The parameters of the distribution are estimated using MLE and Bayesian methods and the performance of the estimators are validated and compared through a Monte Carlo simulation study. The results show that the Bayes estimates through Lindley approximation technique outperforms the MLEs in terms of smaller MSEs. The applications of the new LTPa-I distribution is demonstrated with the help of two real data sets. Based on certain goodness of fit statistics, it is revealed that the LTPa-I provides better fit than the other competitive distributions for these data sets. The real data applications also show that Bayes estimates perform better than MLEs for both the data sets. In this article, the LTPa-I distribution illustrates its flexibility in modeling insurance and failure data sets. So, we can conclude that the LTPa-I distribution seems to be a competitive model even in reliability and survival analysis.

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Appendix A: Proof of LTPa-I as a Generalization of Pareto-I

When $\rho \to 1$, the limits of the numerator and the denominator of Equation (5) are both zero, hence we invoke L'Hopital's rule to obtain

$$\begin{aligned}
\lim_{\rho\to 1} f(x;\rho,\tau,\lambda) &= \lim_{\rho\to 1} \frac{(\rho-1)\tau\lambda^{\tau}}{x^{\tau+1}\log(\rho)[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]} \\
&= \lim_{\rho\to 1} \frac{\tau\lambda^{\tau}}{x^{\tau+1}\left[\log(\rho)\left(\frac{\lambda}{x}\right)^{\tau} + \frac{\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})}{\rho}\right]} \\
&= \frac{\tau\lambda^{\tau}}{x^{\tau+1}},
\end{aligned}$$

which is the pdf of Pareto-I distribution.

Similarly, we apply L'Hopital's rule for Equation (4) and thereby,

$$\begin{aligned}\lim_{\rho\to 1} F(x;\rho,\tau,\lambda) &= \lim_{\rho\to 1} 1 - \frac{\log[\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})]}{\log(\rho)}\\ &= 1-\lim_{\rho\to 1}\frac{\rho(1-(1-\left(\frac{\lambda}{x}\right)^{\tau}))}{\rho-(\rho-1)(1-\left(\frac{\lambda}{x}\right)^{\tau})}\\ &= 1-\left(\frac{\lambda}{x}\right)^{\tau}\end{aligned}$$

which is the cdf of Pareto-I distribution.

Appendix B

- Mixture representation in Equation (11):

$$f(x;\rho,\tau,\lambda)=\sum_{k=0}^{\infty}\sum_{j=0}^{k} w_{k,j}(j+1)\frac{\tau\lambda^{\tau}}{x^{\tau+1}}\left[1-\left(\frac{\lambda}{x}\right)^{\tau}\right]^{j}.$$

Expanding the last term in R.H.S., we have

$$\begin{aligned}f(x;\rho,\tau,\lambda) &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}(-1)^{m} w_{k,j}\binom{j}{m}(j+1)\frac{\tau\lambda^{\tau}}{x^{\tau+1}}\left(\frac{\lambda}{x}\right)^{m\tau}\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m}\, g_P(x;\lambda,\tau(m+1)),\end{aligned}$$

where $w_{k,j,m}=\binom{j}{m}\frac{w_{k,j}(-1)^m(j+1)}{m+1}$.

- Moments:

The s^{th} ordinary moments of the LTPa-I distribution is given by

$$\begin{aligned}\mu_s^{'} &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m}\int_{\lambda}^{\infty} x^s g_P(x;\lambda,\tau(m+1))dx\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m}\int_{\lambda}^{\infty} x^s \frac{\tau(m+1)\lambda^{\tau(m+1)}}{x^{\tau(m+1)+1}}dx\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m}\tau(m+1)\lambda^{\tau(m+1)}\int_{\lambda}^{\infty} x^{s-\tau(m+1)-1}dx\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m}\frac{\lambda^s\tau(m+1)}{\tau(m+1)-s}, \quad s<\tau.\end{aligned}$$

Appendix C: Convergence of the Moments in Equation (13)

$$\mu_s^{'} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j} w_{k,j,m}\frac{\lambda^s\tau(m+1)}{\tau(m+1)-s}, \quad s<\tau$$
$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}(-1)^m\binom{j}{m}kj\frac{(\rho-1)^{j+1}\lambda^s\tau}{(\rho+1)^{k+1}\log(\rho)(\tau(m+1)-s)}. \tag{29}$$

By applying the constant multiplication rule, $\sum ca_n = c\sum a_n$, Equation (7.1) can be rewritten as

$$\mu_s^{'} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{j\frac{k}{j}(\rho-1)^{j+1}\lambda^s\tau}{\log(\rho)(\rho+1)^{k+1}}\sum_{m=0}^{j}\frac{\frac{(-1)^m}{m}}{\tau(m+1)-s}. \tag{30}$$

Again as $k\to\infty$, consider the last term of the R.H.S in Equation (7.2) become $\sum_{m=0}^{\infty}\frac{\frac{(-1)^m}{m}}{\tau(m+1)-s}$ where $\frac{\frac{1}{m}}{\tau(m+1)-s}$ is positive and monotone decreasing.

Also, $\lim\limits_{m\to\infty}\frac{\frac{1}{m}}{\tau(m+1)-s}=0.$

Using the above results and by applying the Leibniz test, we have $\mu_s^{'}$ converges when $s<\tau$.

Appendix D: Simulation Results

Table 3. Average values of estimates and the corresponding MSEs (in parentheses) for $n = 20$

λ	τ	ρ	$\hat{\tau}_M$	$\hat{\rho}_M$	$\hat{\lambda}_M$	$\hat{\tau}_{BS}$	$\hat{\rho}_{BS}$	$\hat{\lambda}_{BS}$
1	0.5	0.5	0.729 (0.221)	0.575 (0.471)	1.153 (0.035)	0.309 (0.049)	0.702 (0.317)	1.207 (0.095)
		2.5	0.664 (0.149)	2.755 (1.756)	1.120 (0.031)	0.409 (0.011)	3.071 (0.438)	1.134 (0.045)
	2.5	0.5	3.419 (1.439)	0.539 (0.839)	1.011 (0.024)	2.985 (0.545)	0.714 (0.632)	1.021 (0.032)
		2.5	3.405 (0.862)	2.458 (1.459)	1.063 (0.008)	2.824 (0.209)	2.564 (0.053)	1.068 (0.008)
2	0.5	0.5	0.701 (0.144)	0.541 (0.477)	2.078 (0.045)	0.292 (0.054)	0.833 (0.235)	2.107 (0.084)
		2.5	0.614 (0.131)	2.453 (1.302)	2.115 (0.087)	0.369 (0.019)	3.154 (0.985)	2.125 (0.158)
	2.5	0.5	2.787 (1.677)	0.948 (0.298)	2.017 (0.052)	0.692 (0.462)	0.354 (0.151)	2.025 (0.059)
		2.5	2.981 (1.249)	3.014 (1.236)	2.036 (0.006)	2.318 (0.123)	3.338 (0.772)	2.044 (0.007)

Table 4. Average values of estimates and the corresponding MSEs (in parentheses) for $n = 50$

λ	τ	ρ	$\hat{\tau}_M$	$\hat{\rho}_M$	$\hat{\lambda}_M$	$\hat{\tau}_{BS}$	$\hat{\rho}_{BS}$	$\hat{\lambda}_{BS}$
1	0.5	0.5	0.546 (0.031)	0.692 (0.497)	1.029 (0.012)	0.475 (0.007)	0.677 (0.144)	1.058 (0.043)
		2.5	0.549 (0.041)	2.398 (0.984)	1.066 (0.029)	0.459 (0.002)	2.721 (0.106)	1.069 (0.034)
	2.5	0.5	2.714 (0.737)	0.571 (0.758)	1.016 (0.006)	1.917 (0.333)	0.644 (0.423)	1.017 (0.009)
		2.5	2.657 (0.230)	2.731 (0.896)	1.012 (0.003)	2.405 (0.014)	2.657 (0.129)	1.015 (0.004)
2	0.5	0.5	0.543 (0.032)	0.623 (0.432)	2.056 (0.008)	0.413 (0.008)	0.783 (0.081)	2.071 (0.009)
		2.5	0.556 (0.051)	2.416 (1.217)	2.078 (0.007)	0.467 (0.001)	2.836 (0.116)	2.080 (0.008)
	2.5	0.5	2.664 (0.617)	0.634 (0.284)	2.011 (0.003)	2.644 (0.041)	0.589 (0.055)	2.017 (0.005)
		2.5	2.814 (0.314)	2.952 (1.025)	2.022 (0.001)	2.561 (0.008)	2.989 (0.351)	2.024 (0.002)

Table 5. Average values of estimates and the corresponding MSEs (in parentheses) for $n = 100$

λ	τ	ρ	$\hat{\tau}_M$	$\hat{\rho}_M$	$\hat{\lambda}_M$	$\hat{\tau}_{BS}$	$\hat{\rho}_{BS}$	$\hat{\lambda}_{BS}$
1	0.5	0.5	0.521 (0.013)	0.571 (0.392)	1.014 (0.0004)	0.459 (0.002)	0.638 (0.019)	1.013 (0.0004)
		2.5	0.532 (0.014)	2.416 (0.866)	1.031 (0.002)	0.489 (0.0001)	2.632 (0.018)	1.028 (0.001)
	2.5	0.5	2.614 (0.329)	0.586 (0.336)	1.003 (0.0002)	2.604 (0.003)	0.598 (0.024)	1.004 (0.0002)
		2.5	2.603 (0.098)	2.610 (0.342)	1.009 (0.0003)	2.483 (0.0008)	2.666 (0.028)	1.011 (0.0005)
2	0.5	0.5	0.512 (0.011)	0.588 (0.182)	2.029 (0.0007)	0.450 (0.003)	0.659 (0.026)	2.027 (0.0005)
		2.5	0.5234 (0.010)	2.493 (1.223)	2.066 (0.004)	0.480 (0.0004)	2.719 (0.048)	2.064 (0.003)
	2.5	0.5	2.571 (0.290)	0.603 (0.247)	2.006 (0.0007)	2.422 (0.001)	0.565 (0.021)	2.011 (0.0001)
		2.5	2.634 (0.147)	2.827 (0.627)	2.011 (0.0002)	2.509 (0.0006)	2.895 (0.157)	2.014 (0.0005)

Table 6. Average values of estimates and the corresponding MSEs (in parentheses) for $n = 200$

λ	τ	ρ	$\hat{\tau}_M$	$\hat{\rho}_M$	$\hat{\lambda}_M$	$\hat{\tau}_{BS}$	$\hat{\rho}_{BS}$	$\hat{\lambda}_{BS}$
1	0.5	0.5	0.507 (0.007)	0.541 (0.076)	1.007 (8.619×10^{-5})	0.477 (0.0005)	0.561 (0.005)	1.005 (8.607×10^{-5})
		2.5	0.504 (0.005)	2.734 (0.459)	1.016 (0.0005)	0.483 (0.0003)	2.560 (0.130)	1.014 (0.0004)
	2.5	0.5	2.544 (0.144)	0.549 (0.088)	1.001 (0.0003)	2.510 (0.0003)	0.543 (0.002)	1.009 (0.0004)
		2.5	2.560 (0.041)	2.604 (0.318)	1.003 (9.778×10^{-5})	2.469 (5.95×10^{-5})	2.633 (0.018)	1.012 (0.0001)
2	0.5	0.5	0.505 (0.005)	0.550 (0.073)	2.014 (0.0004)	0.475 (0.0006)	0.581 (0.006)	2.012 (0.0003)
		2.5	0.507 (0.006)	2.602 (0.398)	2.033 (0.002)	0.486 (1.974×10^{-4})	2.522 (0.049)	2.032 (0.002)
	2.5	0.5	2.522 (0.125)	0.533 (0.067)	2.003 (0.0001)	2.513 (0.008)	0.532 (0.010)	2.013 (0.0001)
		2.5	2.559 (0.049)	2.752 (0.325)	2.006 (9.068×10^{-5})	2.497 (6.869×10^{-5})	2.586 (0.082)	2.014 (9.125×10^{-5})