Applications and Applied Mathematics: An International Journal (AAM)

Volume 18 | Issue 2 | Article 1

12-2022

(R1999) Analysis of a Markovian Retrial Queue with Reneging and Working Vacation under N-control Pattern

P. Manoharan
Annamalai University

S. Pazhani Bala Murugan
Annamalai University

A. Sobanappriya
Annamalai University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 18, Issue 2 (December 2023), Article 1, 17 pages

Analysis of a Markovian Retrial Queue With Reneging and Working Vacation under N-Control Pattern

[1*]**P. Manoharan,** [2]**S. Pazhani Bala Murugan and** [3]**A. Sobanappriya**

Department of Mathematics
Annamalai University
Annamalainagar - 608002
Tamilnadu, India
[1]manomaths.hari@gmail.com; [2]spbm1966@gmail.com; [3]sobanakeerthi465@gmail.com;

*Corresponding Author

Received: July 26, 2022; Accepted: October 30, 2023

Abstract

A Markovian retrial queue with reneging and working vacation under N-control pattern is investigated in this article. To describe the system, we employ a QBD analogy. The model's stability condition is deduced. The stationary probability distribution is generated by utilizing the matrix-analytic technique. The conditional stochastic decomposition of the line length in the orbit is calculated. The performance measures and special cases are designed. The model's firmness is demonstrated numerically.

Keywords: Markovian retrial queue; Working vacation; N-control pattern; Conditional stochastic decomposition; Reneging

MSC 2010 No.: 60K25, 90B22

1. Introduction

Wallace (1969) investigated the Quasi Birth-Death process (QBD) in Queueing Theory using a Markov chain with a tridiagonal generator. Numerical techniques can be used to analyse congestion situations when it is impossible to achieve a explicit solution for queueing problems. The Matrix Geometric technique is ideal for this type of solutions. Neuts (1981) and Latouche and Ramaswami (1999) proposed the matrix geometric solution to the QBD process. Control policies are important for managing queue levels at different epochs. Yadin and Naor (1963) first propose the N - policy.

The queueing system with attendant vacation is noteworthy and can be referred in Tian and Zhang (2006). Servi and Finn (2002) created a modern vacation policy termed as Working Vacation (WV), where the attendant delivers a lesser rate of service than during the engaged period. Wu and Takagi (2006) worked on $M/G/1/MWV$. Kalyanaraman and Pazhani Bala Murugan (2008) have worked on the retrial queue with vacation. Pazhani Bala Murugan and Santhi (2013b) have worked on WV.

Liu (2007) analysed the stochastic decompositions in the $M/M/1/WV$ queue. The $M/M/1/WV$ queue and WV interruptions was analysed by Li and Tian (2007). Tian (2008) considered $M/M/1/SWV$ queue. Analysis for the $M/M/1/MWV$ queue and N-policy was studied by Zhang and Xu (2008). Ye and Liu (2015) discussed the analysis of the $M/M/1$ Queue with two vacation policies.

Recently retrial queues have been studied widely and it was different from normal queues. Due to limited waiting space in the retrial queue the customers are forced to stay in the orbit. Whenever the approaching customers finds that the attendant is engaged they join the orbit and requests service from the orbit. An $M/M/1$ retrial queue with general retrial times was studied by Choi et al. (1993). The retrial queue and WV was simultaneously considered by Do (2010). Li Tao et al. (2012) discussed the $M/M/1$ retrial queue with WV interruption collision under N - Policy.

In retrial queues the customers in the orbit re-attempts for their service and they are unsure of when they will receive service. As a result there is a lack of patience on the customers and they departs from the orbit before getting service and will never come again. This is referred as reneging. On a multiserver Markovian queueing system with balking and reneging was discussed by Haghighi et al. (1986). Azhagappan et al. (2018) considered Transient Solution of an M/M/1 retrial queue with reneging from orbit. Manoharan and Jeeva (2018) examined Impatient customers in an M/M/1/SV and setup times. Manoharan and Ashok (2020) studied Impatient customers in an M/M/1/WV/MV. We consider a Markovian retrial queue with reneging and WV under N-control pattern.

The following are the categories for this article. We present the model and find the infinitesimal generator in Section 2. The stability condition and rate matrix (R) is computed in Section 3. In Section 4, we use a matrix-analytic technique to derive the stationary probability distribution. The line length's conditional stochastic decomposition is computed in Section 5. In Section 6 we calculate performance measures. The special cases are presented in Section 7, and Section 8 has a firmness of the model. The conclusion is given in Section 9.

2. QBD process model

We examine a Markovian retrial queue with reneging and WV under N-control pattern. With the parameter λ, the customer's inter-arrival times are exponentially distributed. The retrial requests from the orbit follows a Poisson process with rate α. The retrying customers from the orbit are unable to wait patiently and start up an unconstrained impatience timer, say T, which is exponentially distributed with the parameter ξ. If the customers gets an opportunity to be served before the timer expires, they receive their service. Otherwise they will depart from the orbit and will never come back. This impatient behaviour of the customer is termed as Reneging. The attendant will take a WV when the system gets clear, which is exponentially distributed with parameter θ. At the time of the regular busy period the service is exponentially distributed with parameters μ. When comparing to the service offered throughout engaged period, the service provided at the time of the WV is at a slower rate. WV service is exponentially distributed with parameter η ($\eta < \mu$). When a WV ends, if the attendant identifies not less than N customers in the orbit, the attendant will terminates his WV and return to the engaged period. Otherwise, the attendant will start another WV. Inter-arrival times, inter-retrial periods, service periods, WV periods and impatience periods are all presumed to be independent of one another. Let the number of customers in the orbit at time t is indicated by $Q(t)$ and $H(t)$ represent attendant's condition at time t. The single attendant might exist in four different states at time t.

$$H(t)=\begin{cases} 0 \text{ - attendant is on WV and is unoccupied,} \\ 1 \text{ - attendant is on WV and is engaged,} \\ 2 \text{ - attendant is on engaged period and is unoccupied,} \\ 3 \text{ - attendant is on engaged period and is engaged.} \end{cases}$$

Evidently $\{(Q(t), H(t)); t\geq 0\}$ is a Markov process with state space

$$\Omega=\{(m, h) : m\geq 0, h=0, 1, 2, 3\}.$$

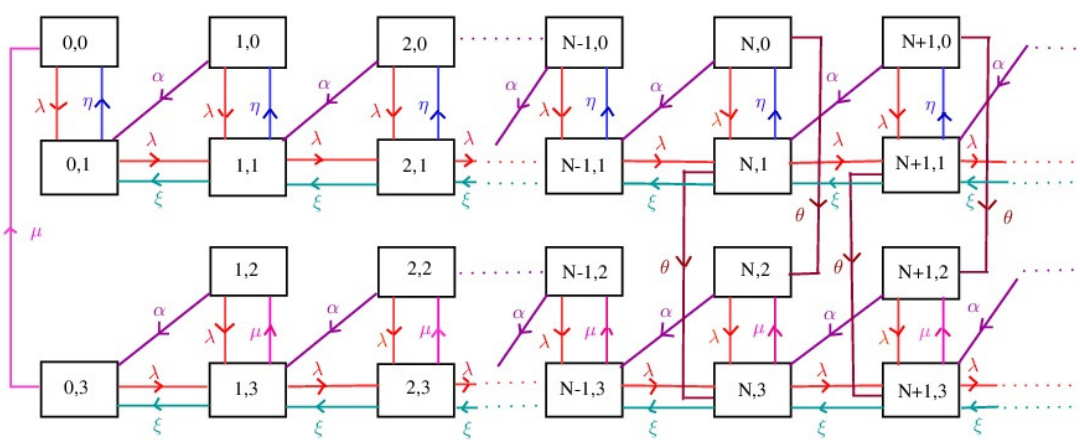

Figure 1. Transition Diagram of the States

The states infinitesimal generator can be described by using *lexicographical sequence* as follows:

$$\widetilde{Q}=\begin{bmatrix} D_0 & F & & & & & & & \\ E & D_1 & F & & & & & & \\ & E & D_1 & F & & & & & \\ & & E & D_1 & F & & & & \\ & & & \vdots & \vdots & \vdots & & & \\ & & & E & D_1 & F & & & \\ & & & & E & D & F & & \\ & & & & & E & D & F & \\ & & & & & & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$D_0=\begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \eta & -\eta-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & -\mu-\lambda \end{bmatrix}, \qquad F=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix},$$

$$D_1=\begin{bmatrix} -\alpha-\lambda & \lambda & 0 & 0 \\ \eta & -\eta-\lambda-\xi & 0 & 0 \\ 0 & 0 & -\alpha-\lambda & \lambda \\ 0 & 0 & \mu & -\mu-\lambda-\xi \end{bmatrix}, \qquad E=\begin{bmatrix} 0 & \alpha & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \xi \end{bmatrix},$$

$$D=\begin{bmatrix} -\alpha-\lambda-\theta & \lambda & \theta & 0 \\ \eta & -\lambda-\eta-\theta-\xi & 0 & \theta \\ 0 & 0 & -\alpha-\lambda & \lambda \\ 0 & 0 & \mu & -\mu-\lambda-\xi \end{bmatrix}.$$

Due to the block structure of matrix $\widetilde{Q}$, $\{(Q(t), H(t));\ t\geq 0\}$ is called a *QBD* process.
Pr{that the attendant is engaged and does not offer a service to a customer while there is no customer in the orbit}= 0.

3. The Model's Stability Condition and Rate Matrix (R)

Theorem 3.1.

The *QBD* process $\{(Q(t), H(t)); t \geq 0\}$ is positive recurrent if and only if $\alpha(\mu-\lambda+\xi) > \lambda(\lambda-\xi)$.

Proof:

Consider

$$S_m{=}E{+}D{+}F{=}\begin{bmatrix} -\alpha{-}\lambda{-}\theta & \alpha{+}\lambda & \theta & 0 \\ \eta & -\theta{-}\eta & 0 & \theta \\ 0 & 0 & -\alpha{-}\lambda & \alpha{+}\lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix}.$$

“Theorem 7.3.1" in Latouche and Ramaswami (1999) offers requirement for positive recurrence of the QBD process, because matrix S_m is reducible. After permutation of rows and columns and hence the *QBD* is positive recurrent if and only if $\pi\begin{bmatrix} 0 & \alpha \\ 0 & \xi \end{bmatrix} e > \pi \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} e.$

Here all the elements of the column vector $e{=}1$ and π is a unique solution of the system $\pi\begin{bmatrix} -\alpha{-}\lambda & \alpha{+}\lambda \\ \mu & -\mu \end{bmatrix} = 0, \quad \pi e{=}1$. The *QBD* process is positive recurrent if and only if $\alpha(\mu-\lambda+\xi) > \lambda(\lambda-\xi)$ after some algebraic manipulations. ■

Theorem 3.2.

If $\alpha(\mu-\lambda+\xi) > \lambda(\lambda-\xi)$, the matrix quadratic equation $R^2E{+}RD{+}F{=}0$ has the minimal non-negative solution $R{=}\begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1 & r_2 & r_3 & r_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 & r_6 \end{bmatrix}$,

where

$$r_1{=}\frac{\eta r_2}{(\lambda{+}\alpha{+}\theta)}, \qquad r_2{=}\frac{t-\sqrt{t^2-4[\xi(\lambda{+}\alpha{+}\theta){+}\eta\alpha]\lambda(\lambda{+}\alpha{+}\theta)}}{2[\xi(\lambda{+}\alpha{+}\theta){+}\eta\alpha]},$$

and $\quad t{=}[(\lambda{+}\alpha{+}\theta)(\lambda{+}\theta{+}\eta{+}\xi){-}\eta\lambda]$,

$$r_3{=}\frac{r_1\theta{+}r_4\mu}{(\lambda{+}\alpha)}, \qquad r_4{=}\frac{\alpha r_2 r_1\theta{+}r_1\theta\lambda{+}r_2\theta(\lambda{+}\alpha)}{-(\lambda{+}\alpha)[(r_2{+}r_6)\xi{+}\alpha r_5{-}\lambda{-}\mu{-}\xi]{-}r_2\alpha\mu{-}\mu\lambda},$$

$$r_5{=}\frac{\mu\lambda}{\mu\alpha{+}\xi(\lambda{+}\alpha)}, \qquad r_6{=}\frac{\lambda(\lambda{+}\alpha)}{\mu\alpha{+}\xi(\lambda{+}\alpha)}.$$

Proof:

We consider $R{=}\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$ from the matrices E, D, F where R_{11}, R_{12} and R_{22} are all 2×2 matrices. Substituting R into $R^2E{+}RD{+}F{=}0$, we get

$$R_{11}^2\begin{bmatrix} 0 & \alpha \\ 0 & \xi \end{bmatrix} {+}R_{11}\begin{bmatrix} (-\alpha{-}\lambda{-}\theta) & \lambda \\ \eta & (-\lambda{-}\eta{-}\theta{-}\xi) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$(R_{11}R_{12}{+}R_{12}R_{22})\begin{bmatrix} 0 & \alpha \\ 0 & \xi \end{bmatrix} {+}R_{11}\begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} {+}R_{12}\begin{bmatrix} (-\alpha{-}\lambda) & \lambda \\ \mu & (-\mu{-}\lambda{-}\xi) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$R_{22}^2\begin{bmatrix}0 & \alpha\\ 0 & \xi\end{bmatrix}+R_{22}\begin{bmatrix}(-\alpha-\lambda) & \lambda\\ \mu & (-\mu-\lambda-\xi)\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

From the above set of equations with some computations, we get R_{11}, R_{22}, and R_{12} respectively as

$R_{11}=\begin{bmatrix}0 & 0\\ r_1 & r_2\end{bmatrix}$, $R_{22}=\begin{bmatrix}0 & 0\\ r_5 & r_6\end{bmatrix}$ and $R_{12}=\begin{bmatrix}0 & 0\\ r_3 & r_4\end{bmatrix}$. ■

4. The Model's Stationary Probability Distribution

If $\alpha(\mu-\lambda+\xi)>\lambda(\lambda-\xi)$, assign (Q,H) be the stationary probability distribution of the process $\{(Q(t),H(t));t\geq 0\}$. Let us denote,

$$\pi_m=(\pi_{m,0},\pi_{m,1},\pi_{m,2},\pi_{m,3}),\qquad m\geq 0;$$

$$\pi_{m,h}=P\{Q=m,H=h\}=\lim_{t\to\infty}P\{Q(t)=m,H(t)=h\},\qquad (m,h)\in\Omega.$$

It's worth noting that $\pi_{0,2}=0$ from states as we discussed previously.

Theorem 4.1.

If $\alpha(\mu-\lambda+\xi)>\lambda(\lambda-\xi)$, the stationary probability distribution of (Q,H) is indicated by

$$\pi_{m,0}=\pi_{N-1,1}r_1r_2^{m-N},\qquad m\geq N,\tag{1}$$

$$\pi_{m,1}=\pi_{N-1,1}r_2^{m+1-N},\qquad m\geq N,\tag{2}$$

$$\pi_{m,2}=\pi_{N-1,1}\left[r_3r_2^{m-N}+\frac{r_4r_5}{r_6-r_2}\left(r_6^{m-N}-r_2^{m-N}\right)\right]+\pi_{N-1,3}r_5r_6^{m-N},\qquad m\geq N,\tag{3}$$

$$\pi_{m,3}=\pi_{N-1,1}\frac{r_4}{r_6-r_2}(r_6^{m+1-N}-r_2^{m+1-N})+\pi_{N-1,3}r_6^{m+1-N},\qquad m\geq N,\tag{4}$$

$$\pi_{m,0}=\frac{\eta}{\lambda+\alpha}\pi_{0,1}+\frac{\eta}{\lambda+\alpha}(\pi_{1,1}-\pi_{0,1})\frac{1-q_1^m}{1-q_1},\qquad 2\leq m\leq N-2,\tag{5}$$

$$\pi_{m,1}=\pi_{0,1}+(\pi_{1,1}-\pi_{01})\frac{1-q_1^m}{1-q_1},\qquad 2\leq m\leq N-2,\tag{6}$$

$$\pi_{m,2}=\frac{\mu}{\lambda+\alpha}\pi_{0,3}+\frac{\mu}{\lambda+\alpha}(\pi_{1,3}-\pi_{0,3})\frac{1-q_2^m}{1-q_2},\qquad 2\leq m\leq N-2,\tag{7}$$

$$\pi_{m,3}=\pi_{0,3}+(\pi_{1,3}-\pi_{0,3})\frac{1-q_2^m}{1-q_2},\qquad 2\leq m\leq N-2,\tag{8}$$

$$\pi_{N-1,0}=\frac{-\lambda\eta}{[\lambda\eta+(r_1\alpha+r_2\xi-\lambda-\eta-\xi)(\lambda+\alpha)]}\pi_{N-2,1}, \tag{9}$$

$$\pi_{N-1,1}=\frac{\lambda+\alpha}{\eta}\pi_{N-1,0}, \tag{10}$$

$$\pi_{N-1,2}=\frac{\mu}{(\xi\lambda+\mu\alpha+\xi\alpha)}\Big[(r_3\alpha+r_4\xi)\pi_{N-1,1}+\lambda\pi_{N-2,3}\Big], \tag{11}$$

$$\pi_{N-1,3}=\frac{\lambda+\alpha}{\mu}\pi_{N-1,2}, \tag{12}$$

where $\quad q_1=\dfrac{\lambda(\lambda+\alpha)}{\alpha\eta+\xi\lambda+\xi\alpha} \quad$ and $\quad q_2=\dfrac{\lambda(\lambda+\alpha)}{\alpha\mu+\xi\lambda+\xi\alpha}$,

$$\pi_{1,1}=-K^{-1}\left[\frac{\lambda(\lambda+\alpha+\eta)}{\lambda+\alpha}+\Delta-K\right]\pi_{0,1}, \tag{13}$$

$$\pi_{1,0}=\frac{\eta}{\lambda+\alpha}\pi_{1,1}, \tag{14}$$

$$\pi_{0,0}=\frac{\lambda+\eta}{\lambda}\pi_{0,1}-\frac{\alpha}{\lambda}\pi_{1,0}-\frac{\xi}{\lambda}\pi_{1,1}, \tag{15}$$

$$\pi_{0,3}=\frac{\lambda}{\mu}\pi_{0,0}-\frac{\eta}{\mu}\pi_{0,1}, \tag{16}$$

$$\pi_{1,2}=\frac{\lambda+\mu}{\alpha}\pi_{0,3}-\frac{\xi}{\alpha}\pi_{1,3}, \tag{17}$$

$$\pi_{1,3}=\frac{(\lambda+\alpha)}{\mu}\pi_{1,2}, \tag{18}$$

where $\quad \Delta=\dfrac{-(\lambda\alpha\eta+\lambda(\lambda+\alpha)\xi)}{[\lambda\eta+(r_1\alpha-r_2\xi-\lambda-\eta-\xi)(\lambda+\alpha)]}-\lambda-\eta-\xi \quad$ and

$$K=\left[\lambda\ \frac{1-q_1^{N-3}}{1-q_1}+\left(\Delta+\frac{\lambda\eta}{\lambda+\alpha}\right)\frac{1-q_1^{N-2}}{1-q_1}\right].$$

The normalization condition can finally be used to determine $\pi_{0,1}$.

Proof:

Using the technique from Neuts (1981), we have

$$\begin{aligned}\pi_m&=(\pi_{m,0},\pi_{m,1},\pi_{m,2},\pi_{m,3})=\pi_{N-1}R^{m+1-N}\\&=(\pi_{N-1,0},\pi_{N-1,1},\pi_{N-1,2},\pi_{N-1,3})R^{m+1-N}, \qquad m\geq N.\end{aligned}$$

For $m \geq N$,

$$R^{m+1-N}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1 r_2^{m-N} & r_2^{m+1-N} & r_3 r_2^{m-N}+\frac{r_4 r_5(r_6^{m-N}-r_2^{m-N})}{r_6-r_2} & \frac{r_4(r_6^{m+1-N}-r_2^{m+1-N})}{r_6-r_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 r_6^{m-N} & r_6^{m+1-N} \end{bmatrix},$$

substituting R^{m+1-N} into the above equation, we get $(1-4)$. However, $\pi_0, \pi_1, ..., \pi_{N-1}$ satisfies the equation ($\pi_0, \pi_1, ..., \pi_{N-1}$)B[R]=0, where

$$B[R]=\begin{bmatrix} D_0 & F & & & \\ E & D_1 & F & & \\ & E & D_1 & F & \\ & & \vdots & \vdots & \vdots \\ & & E & D_1 & F \\ & & & E & RE+D_1 \end{bmatrix}$$

and

$$RE+D_1=\begin{bmatrix} -(\lambda+\alpha) & \lambda & 0 & 0 \\ \eta & r_1\alpha+r_2\xi-\lambda-\eta-\xi & 0 & r_3\alpha+r_4\xi \\ 0 & 0 & -(\lambda+\alpha) & \lambda \\ 0 & 0 & \mu & r_5\alpha+r_6\xi-\lambda-\mu-\xi \end{bmatrix}.$$

The following equations are computed from $B[R]$:

$$-\lambda\pi_{0,0}+\eta\pi_{0,1}+\mu\pi_{0,3}=0, \tag{19}$$

$$\lambda\pi_{0,0}-(\lambda+\eta)\pi_{0,1}+\alpha\pi_{1,0}+\xi\pi_{1,1}=0, \tag{20}$$

$$-(\lambda+\mu)\pi_{0,3}+\alpha\pi_{1,2}+\xi\pi_{1,3}=0, \tag{21}$$

$$-(\lambda+\alpha)\pi_{m,0}+\eta\pi_{m,1}=0, \qquad 1\leq m\leq N-2, \tag{22}$$

$$\lambda\pi_{m-1,1}+\lambda\pi_{m,0}-(\lambda+\eta+\xi)\pi_{m,1}+\alpha\pi_{m+1,0}+\xi\pi_{m+1,1}=0, \qquad 1\leq m\leq N-2, \tag{23}$$

$$-(\lambda+\alpha)\pi_{m,2}+\mu\pi_{m,3}=0, \qquad 1\leq m\leq N-2, \tag{24}$$

$$\lambda\pi_{m-1,3}+\lambda\pi_{m,2}-(\lambda+\mu+\xi)\pi_{m,3}+\alpha\pi_{m+1,2}+\xi\pi_{m+1,3}=0, \qquad 1\leq m\leq N-2, \tag{25}$$

$$-(\lambda+\alpha)\pi_{N-1,0}+\eta\pi_{N-1,1}=0, \tag{26}$$

$$\lambda\pi_{N-2,1}+\lambda\pi_{N-1,0}+(r_1\alpha+r_2\xi-\lambda-\eta-\xi)\pi_{N-1,1}=0, \tag{27}$$

$$-(\lambda+\alpha)\pi_{N-1,2}+\mu\pi_{N-1,3}=0, \tag{28}$$

$$\lambda\pi_{N-2,3}+(r_3\alpha+r_4\xi)\pi_{N-1,1}+\lambda\pi_{N-1,2}+(r_5\alpha+r_6\xi-\lambda-\mu-\xi)\pi_{N-1,3}=0. \tag{29}$$

From (19) to (29), we get (5) to (18) , where $\sum_{h=0}^{3}\sum_{m=0}^{\infty}\pi_{m,h}=1$. Finally, we can get $\pi_{0,1}$. ■

5. The Model's Conditional Stochastic Decomposition

Lemma 5.1.

If $\alpha(\mu-\lambda+\xi)>\lambda(\lambda-\xi)$, let Q_0 be the conditional line length of an $M/M/1$ retrial queue with reneging in the orbit when the attendant is engaged. Then, Q_0 has a PGF $G_{Q_0}(z)=\dfrac{1-r_6}{1-r_6z}$.

Proof:

Consider a Markovian retrial queue with reneging. Two inter-valued random variables are used to explain the system at time t. Let $Q^\bullet(t)$ be the number of customers in the orbit at time t,

$$H^\bullet(t)=\begin{cases}0 \text{ - attendant is unoccupied,}\\ 1 \text{ - attendant is engaged.}\end{cases}$$

Then, $\{(Q^\bullet(t),H^\bullet(t));t\geq 0\}$ is a Markov process with state space $\{(m,h):m\geq 0,h=0,1\}$. The *infinitesimal generator* can be expressed as

$$\widetilde{Q^\bullet}=\begin{bmatrix} D_0 & F & & \\ E & D & F & \\ & E & D & F \\ & & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$D_0=\begin{bmatrix}-\lambda & \lambda\\ \mu & -\mu-\lambda\end{bmatrix},\quad F=\begin{bmatrix}0 & 0\\ 0 & \lambda\end{bmatrix},\quad E=\begin{bmatrix}0 & \alpha\\ 0 & \xi\end{bmatrix},\quad D=\begin{bmatrix}-\alpha-\lambda & \lambda\\ \mu & -\mu-\lambda-\xi\end{bmatrix}.$$

The *QBD* process $\{(Q^\bullet(t),H^\bullet(t));t\geq 0\}$ is positive recurrent if and only if $\alpha(\mu-\lambda+\xi)>\lambda(\lambda-\xi)$.

Express

$$\pi_{m,h}=P\{Q^\bullet=m,H^\bullet=h\}=\lim_{t\to\infty}P\{Q^\bullet(t)=m,H^\bullet(t)=h\}.$$

The stationary probability distribution is

$$\widetilde{\pi}_{m,0}=\widetilde{\pi}_{0,1}r_5r_6^{m-1},\qquad m\geq 1,$$

$$\widetilde{\pi}_{m,1}=\widetilde{\pi}_{0,1}r_6^{m},\qquad m\geq 0,$$

$$\widetilde{\pi}_{0,0}{=}\left(1{+}\frac{1{+}r_5}{1{-}r_6}\frac{\lambda}{\mu}\right)^{-1},$$

$$\widetilde{\pi}_{0,1}{=}\frac{\lambda}{\mu}\widetilde{\pi}_{0,0}.$$

The normalization condition is used to determine the value of $\pi_{0,0}$.

Therefore,

$$G_{Q_0}(z){=}\sum_{m=0}^{\infty}z^m P\{Q_0{=}m\}{=}\frac{\sum_{m=0}^{\infty}\widetilde{\pi}_{0,1}r_6^m z^m}{\sum_{m=0}^{\infty}\widetilde{\pi}_{0,1}r_6^m}{=}\frac{1{-}r_6}{1{-}r_6 z}.$$

Establishing $Q^N = \{$Difference of Q and N such that the state of the attendant is either 1 *or* 3 and $Q \geq N\}$ and Q^N is the line length which depends on the condition that the attendant is engaged and there are not less than N customers in the orbit.

Let $P_b^{\bullet}$ denotes *Pr{the server is occupied given that at least N customers present in the orbit}*.

$$P_b^{\bullet}{=}P\{Q \geq N, H{=}1 \ or \ 3\}{=}\sum_{m=N}^{\infty}\pi_{m,1}{+}\sum_{m=N}^{\infty}\pi_{m,3},$$

$$={\sum_{m=N}^{\infty}}\pi_{N-1,1}r_2^{m+1-N}+\sum_{m=N}^{\infty}\frac{r_4}{r_6{-}r_2}(r_6^{m+1-N}{-}r_2^{m+1-N})\pi_{N-1,1}{+}\sum_{m=N}^{\infty}r_6^{m+1-N}\pi_{N-1,3},$$

$$={\frac{r_4{+}r_2(1{-}r_6)}{(1{-}r_2)(1{-}r_6)}}\pi_{N-1,1}{+}\frac{r_6}{(1{-}r_6)}\pi_{N-1,3}.$$

■

Theorem 5.1.

If $\alpha(\mu - \lambda + \xi) > \lambda(\lambda - \xi)$, then we can disintegrate $Q^N{=}Q_0{+}Q_c$, where Q_0 go along with a geometric distribution with specification $1{-}r_6$. Subsidiary line length Q_c has a distribution

$$P\{Q_c{=}0\}{=}\frac{1}{P_b^{\bullet}}\frac{(r_2{+}r_4)\pi_{N-1,1}{+}r_6\pi_{N-1,3}}{1{-}r_6},$$

$$P\{Q_c{=}m\}{=}\frac{\pi_{N-1,1}}{P_b^{\bullet}}\frac{r_2(r_2{+}r_4{-}r_6)}{1{-}r_6}r_2^{m-1}, \qquad m{\geq}1.$$

Proof:

The PGF of Q^N is given below:

$$G_{Q^N}(z){=}\sum_{m=0}^{\infty}z^m P\{Q^N{=}m\}{=}\frac{1}{p_b^{\bullet}}\left(\sum_{m=0}^{\infty}z^m\pi_{N+m,1}{+}\sum_{m=0}^{\infty}z^m\pi_{N+m,3}\right),$$

$$=\frac{1}{p_b^\bullet}\left[\pi_{N-1,1}\frac{r_2}{1-r_2z}+\pi_{N-1,1}\frac{r_4}{(1-r_2z)(1-r_6z)}+\pi_{N-1,3}\frac{r_6}{1-r_6z}\right],$$

$$=\frac{1}{p_b^\bullet}\frac{1-r_6}{1-r_6z}\left[\pi_{N-1,1}\frac{r_2(1-r_6z)}{(1-r_2z)(1-r_6)}+\pi_{N-1,1}\frac{r_4}{(1-r_2z)(1-r_6)}+\pi_{N-1,3}\frac{r_6}{1-r_6}\right],$$

$$=\frac{1}{p_b^\bullet}\frac{1-r_6}{1-r_6z}\left[\frac{(r_2+r_4)\pi_{N-1,1}+r_6\pi_{N-1,3}}{1-r_6}+\pi_{N-1,1}\frac{r_2(r_2+r_4-r_6)z}{(1-r_2z)(1-r_6)}\right],$$

$$=\frac{1-r_6}{1-r_6z}\left[\frac{1}{p_b^\bullet}\frac{(r_2+r_4)\pi_{N-1,1}+r_6\pi_{N-1,3}}{1-r_6}+\pi_{N-1,1}\frac{1}{p_b^\bullet}\frac{r_2(r_2+r_4-r_6)z}{(1-r_2z)(1-r_6)}\right],$$

$$=G_{Q_0}(z)G_{Q_c}(z).$$

■

6. The Model's Performance Measures

From Theorem 4.1, we have

$$\textit{Pr\{that the attendant is engaged\}} = P_b = \sum_{m=0}^{\infty}\pi_{m,1}+\sum_{m=0}^{\infty}\pi_{m,3},$$

$$=(N-1)\left(\frac{\pi_{1,1}}{1-q_1}-\frac{q_1\pi_{0,1}}{1-q_1}\right)-\frac{\pi_{1,1}-\pi_{0,1}}{(1-q_1)^2}(1-q_1^{N-1})$$
$$+(N-1)\left(\frac{\pi_{1,3}}{1-q_2}-\frac{q_2\pi_{0,3}}{1-q_2}\right)-\frac{\pi_{1,3}-\pi_{0,3}}{(1-q_2)^2}(1-q_2^{N-1})$$
$$+\frac{1-r_6+r_4}{(1-r_2)(1-r_6)}\pi_{N-1,1}+\frac{1}{(1-r_6)}\pi_{N-1,3}.$$

$$\textit{Pr\{that the attendant is unoccupied\}}=P_f=\sum_{m=0}^{\infty}\pi_{m,0}+\sum_{m=1}^{\infty}\pi_{m,2}=1-P_b.$$

Assume that $E[L]$ *denotes the mean number of customers in the orbit*, then

$$E[L]=\sum_{m=1}^{\infty}m(\pi_{m,0}+\pi_{m,1}+\pi_{m,2}+\pi_{m,3}),$$
$$=\sum_{m=1}^{N-1}m(\pi_{m,0}+\pi_{m,2})+\sum_{m=1}^{N-2}m(\pi_{m,1}+\pi_{m,3})+(N-1)\pi_{N-1,3}\frac{1+r_5}{1-r_6}$$
$$+\pi_{N-1,3}\frac{r_5+r_6}{(1-r_6)^2}+(N-1)\pi_{N-1,1}\frac{(1+r_1+r_3)(1-r_6)+r_4(1+r_5)}{(1-r_2)(1-r_6)}$$
$$+\pi_{N-1,1}\frac{(r_1+r_2+r_3)(1-r_6)^2+r_4r_5(2-r_2-r_6)+r_4(1-r_2r_6)}{(1-r_6)^2(1-r_2)^2}.$$

Let $E[L_s]$ *be the mean number of customers in the system*, then

$$E[L_s]=\sum_{m=1}^{\infty}m(\pi_{m,0}+\pi_{m,2})+\sum_{m=0}^{\infty}(m+1)(\pi_{m,1}+\pi_{m,3}).$$

We have the following assumptions and results:

Let W $-$ orbit customer's waiting time,

$E[W_s]$ $-$ expected stopover time of orbit customer in the system.

Then

$$E[W]=\frac{E[L]}{\lambda}, \quad E[W_s]=\frac{E[L_s]}{\lambda} \quad \text{and} \quad \pi_{0,0}=\frac{E[T_{0,0}]}{E[T]+1/\lambda},$$

where

$E[T_{0,0}]$- absolute time in the idle state throughout a *regenerative cycle*, and

T $-$ engaged period.

Also $E[T_{0,0}]=\dfrac{1}{\lambda}$, $E[T]=(\pi_{0,0}^{-1}-1)\lambda^{-1}$.

7. Special Cases

(a) If $\xi=0$ this model is remodeled as "An M/M/1 retrial queue with working vacation under N-policy".
(b) If $\alpha \to \infty$ this model is remodeled as "Analysis for the M/M/1 queue with multiple working vacations and N-policy".
(c) If $\alpha \to \infty$, $\eta=0$ this model is remodeled as "An M/M/1 queue with multiple vacation under N-policy".
(d) If $\alpha \to \infty$, $\eta=0$, $\theta=0$ this model is remodeled as "Standard M/M/1 queue under N-policy".

8. Numerical Results

By fixing the values of $N=2$, $\mu=8.6$, $\theta=1.6$, $\eta=0.3$, $\xi=0.8$ and extending the value of λ from 1.0 to 2.0 incremented with 0.2 and extending the values of α from 2.9 to 4.9 insteps of 1 subject to the stability condition the values of $E(L)$ are calculated and tabulated in Table 1 and the corresponding line graphs are drawn in the Figure 2. From the graph it is inferred that as λ rises, $E(L)$ rises as expected.

By fixing the values of $N=2$, $\mu=7$, $\theta=3.3$, $\alpha=5.2$, $\xi=1.7$ and extending the value of λ from 1.0 to 2.0 incremented with 0.2 and extending the values of η from 0.3 to 1.1 insteps of 0.4 subject

Table 1. $E(L)$ with turnover of λ

λ	α =2.9	α =3.9	α =4.9
1.0	0.2143	0.1780	0.1555
1.2	0.2788	0.2314	0.2019
1.4	0.3509	0.2907	0.2534
1.6	0.4322	0.3571	0.3108
1.8	0.5252	0.4321	0.3752
2.0	0.6331	0.5176	0.4479

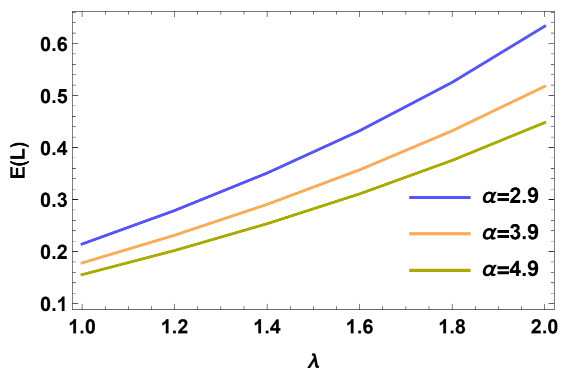

Figure 2. $E(L)$ with turnover of λ

to the stability condition the values of $E(L)$ are calculated and tabulated in Table 2 and the corresponding line graphs are drawn in the Figure 3. From the graph it is inferred that as λ rises, $E(L)$ rises as expected.

Table 2. $E(L)$ with turnover of λ

λ	η =0.3	η =0.7	η =1.1
1.0	0.1476	0.1289	0.1144
1.2	0.1921	0.1720	0.1556
1.4	0.2416	0.2204	0.2025
1.6	0.2969	0.2748	0.2557
1.8	0.3590	0.3361	0.3160
2.0	0.4290	0.4056	0.3845

By fixing the values of N=2, μ =8.5, θ =3.6, η =4.3, ξ =1.9 and extending the value of λ from 1.0 to 2.0 incremented with 0.2 and extending the values of α from 0.1 to 0.3 insteps of 0.1 subject to the stability condition the values of P_b are calculated and tabulated in Table 3 and the corresponding line graphs are drawn in the Figure 4. From the graph it is inferred that as λ rises, P_b rises as expected.

By fixing the values of N=2, μ =5.7, θ =1.8, η =2.1, ξ =0.9 and extending the values of λ from 1.0 to 2.0 incremented with 0.2 and extending the values α from 1 to 2 insteps of 0.5 subject to the

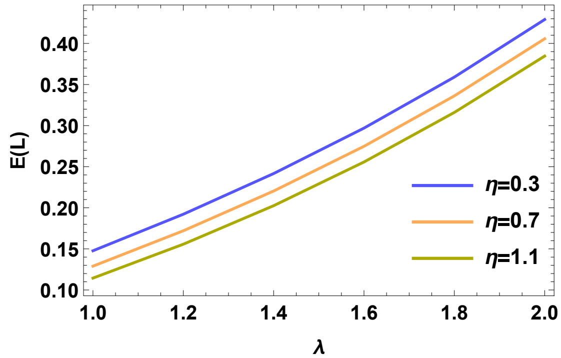

Figure 3. $E(L)$ with turnover of λ

Table 3. P_b with turnover of λ

λ	α =0.1	α =0.2	α =0.3
1.0	0.1481	0.1572	0.1630
1.2	0.1632	0.1741	0.1814
1.4	0.1761	0.1883	0.1971
1.6	0.1873	0.2007	0.2105
1.8	0.1974	0.2115	0.2223
2.0	0.2065	0.2212	0.2327

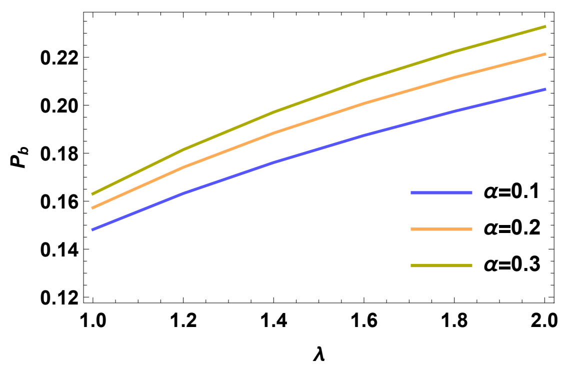

Figure 4. P_b with turnover of λ

stability condition the values of P_f are calculated and tabulated in Table 4 and the corresponding line graphs are drawn in the Figure 5. From the graph it is inferred as λ rises, P_f falls as expected.

9. Conclusion

In this article, a Markovian retrial queue with reneging and WV under N- control pattern is evaluated. We calculate stability condition and rate matrix of the model. We went on the stationary probability distribution by adopting the matrix-analytic methods. We also derive the conditional stochastic decomposition and performance measures. We perform some special cases. We illustrate some numerical examples under the stability condition.

Table 4. P_f with turnover of λ

λ	$\alpha=1$	$\alpha=1.5$	$\alpha=2$
1.0	0.7246	0.7149	0.7096
1.2	0.7003	0.6866	0.6788
1.4	0.6807	0.6630	0.6526
1.6	0.6648	0.6431	0.6301
1.8	0.6515	0.6262	0.6106
2.0	0.6404	0.6116	0.5935

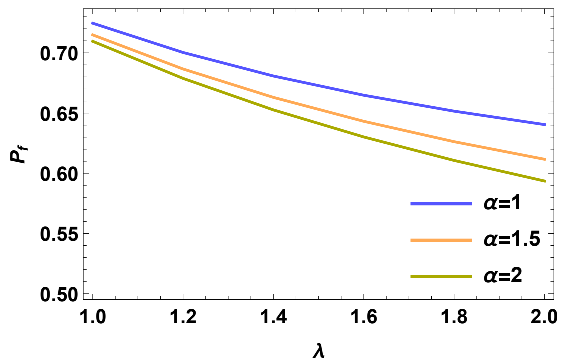

Figure 5. P_f with turnover of λ

Acknowledgment:

The authors would like to thank the Reviewer, former Chief Editor and Founder of the journal Dr. Aliakbar Montazer Haghighi and Editor-in-Chief Dr. James R. Valles, for their valuble comments and suggestions to shape the article in the present format.

REFERENCES

Haghighi, A.M., Medhi, J. and Mohanty, S.G. (1986). On a multiserver Markovian queueing system with balking and reneging, Comput. and Ops. Res., Vol. 13, No. 4, pp. 421-425.

Haghighi, A.M. and Mishev, D.P. (2016). *Delayed and Network Queues*, John Wiley and Sons, Inc., New Jersey, USA.

Artalejo, J. and Corral, A. (2008). *Retrial Queueing Systems*, Springer, Berlin.

Azhagappan, A., Veeramani, E., Monica, W. and Sonabharathi, K. (2018). Transient solution of an M/M/1 retrial queue with reneging from orbit, Appl. Appl. Math., Vol. 13, No. 2, pp. 628-638.

Choi, B., Park, K. and Pearce, C. (1993). An M/M/1 retrial queue with control policy and general retrial times, Queueing Syst., Vol. 14, pp. 275-292.

Do, T. (2010). M/M/1 retrial queue with working vacations, Acta Inform., Vol. 47, pp. 67-75.

Kalyanaraman, R. and Pazhani Bala Murugan, S. (2008). A single server retrial queue with vacation, J. Appl. Math. and Informatics, Vol. 26, No. (3-4), pp. 721-732.

Latouche, G. and Ramaswami, V. (1999). *Introduction to Matrix Analytic Methods in Stochastic Modelling*, ASA-SIAM Series on Applied Probability, USA.

Li, J. and Tian, N. (2007). The M/M/1 queue with working vacations and vacation interruptions, J. Syst. Sci. Syst. Eng., Vol. 16, pp. 121-127.

Liu, W.-Y. (2007). Stochastic decompositions in the M/M/1 queue with working vacations, Operation Research Letters, Vol. 35, pp. 595-600.

Manoharan, P. and Ashok, A. (2020). Impatient customers in an M/M/1 queue with working vacation and multiple vacation, International Journal of Future Generation Communication and Networking, Vol. 13, No. 12, pp. 56-63.

Manoharan, P. and Jeeva, T. (2018). Impatient customers in an M/M/1 queue with single vacation and setup times, International Journal of Information and Computing Science, Vol. 5, No. 7, pp. 204-211.

Manoharan, P. and Majid, S. (2019). Analysis of M/M/1 queue with working vacation and vacation interruption, Appl. Appl. Math., Vol. 14, No. 1, pp. 19-33.

Neuts, M .F. (1981). *Matrix - Geometric Solutions in Stochastic Models*, Johns Hopkins University Press, Baltimore.

Pazhani Bala Murugan, S. and Santhi, K. (2013a). An M/G/1 queue with two stage heterogeneous service and single working vacation, International Mathematical Forum, Vol. 8, No. 27, pp. 1323-1336.

Pazhani Bala Murugan, S. and Santhi, K. (2013b). An M^X/G/1 queue with two stage heterogeneous service and multiple working vacation, International Journal of Mathematics Sciences and Applications, Vol. 3, No. 1, pp. 233-249.

Pazhani Bala Murugan, S. and Santhi, K. (2017). An M/G/1 retrial queue with single working vacation, Appl. Appl. Math., Vol. 12, No. 1, pp. 1-22.

Servi, L. and Finn, S. (2002). M/M/1 queue with working vacations (M/M/1/WV), Perform. Eval., Vol. 50, pp. 41-52.

Tao, L., Liu, Z. and Wang, Z. (2012). M/M/1 retrial queue with collisions and working vacation interruption under N-policy, RAIRO Operations Research, Vol. 46, pp. 355-371.

Tian, N. (2008). The M/M/1 queue with single working vacation, International Journal of Information and Management Sciences, Vol. 19, No. 14, pp. 621-634.

Tian, N. and Zhang, Z. (2006). *Vacation Queueing Models-Theory and Applications*, Springer-Verlag, New York.

Wallace, P. (1969). The solution of birth and death processes arising from multiple access computer systems, Ph.D. dissertation, Systems Engineering Laboratory, University of Michigan. Tech. Dept. 07742-6T.

Wu, D. and Takagi, H. (2006). M/G/1 queue with multiple working vacations, Perform. Eval, Vol. 63, pp. 654-681.

Yadin, M. and Naor, P. (1963). Queueing system with a removable service station, Operational Research Quarterly, Vol. 14, No. 4, pp. 393-405.

Ye, Q. and Liu, L. (2015). The analysis of the M/M/1 queue with two vacation policies, International Journal of Computer Mathematics, Vol. 1, No. 20.

Yue, D., Yue, W. and Zhao, G. (2016). Analysis of an M/M/1 queue with vacations and impatience timers which depend on server's state, Journal of Industrial and Management Optimization, Vol. 12, No. 2, pp. 653-666.

Zhang, M. and Hou, Z. (2010). Performance analysis of M/G/1 queue with working vacations and vacation interruption, J. Comput. Appl. Math, Vol. 234, pp.2977-2985.

Zhang, Z. and Xu, X. (2008). Analysis for the M/M/1 queue with multiple working vacations and N-policy, Infor. Manag. Sci., Vol. 19, pp. 495-506.