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Characterization of the boundedness of fractional maximal operator and its commutators in Orlicz and generalized Orlicz–Morrey spaces on spaces of homogeneous type

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Abstract
In this paper, we establish the necessary and sufficient conditions for the boundedness of fractional maximal operator M_α and the fractional maximal commutators $M_{b,\alpha}$ in Orlicz $L^\Phi(X)$ and generalized Orlicz–Morrey spaces $\mathcal{M}^{\Phi,\varphi}(X)$ on spaces of homogeneous type $X = (X, d, \mu)$ in the sense of Coifman-Weiss.

Keywords Orlicz space · Generalized Orlicz–Morrey space · Fractional maximal operator · Commutator · Spaces of homogeneous type

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1 Introduction

In order to extend the traditional Euclidean space to build a general underlying structure for the real harmonic analysis, the notion of spaces of homogeneous type was introduced by Coifman and Weiss [1].

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Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$\begin{aligned} d(x, y) \geq 0 \text{ and } d(x, y) = 0 &\text{ if and only if } x = y, \\ d(x, y) &= d(y, x), \\ d(x, y) &\leq K_1(d(x, z) + d(z, y)), \end{aligned} \tag{1.1}$$

the balls $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x, 2r)) \leq K_2\, \mu(B(x, r)) < \infty, \tag{1.2}$$

where $K_i \geq 1$ $(i = 1, 2)$ are constants independent of $x, y, z \in X$ and $r > 0$. As usual, the dilation of a ball $B = B(x, r)$ will be denoted by $\lambda B = B(x, \lambda r)$ for every $\lambda > 0$.

In the sequel, we always assume that $\mu(X) = \infty$, the space of compactly supported continuous function is dense in $L^1(X, \mu)$ and that X is Q-homogeneous ($Q > 0$), i.e.

$$K_3^{-1} r^Q \leq \mu(B(x, r)) \leq K_3 r^Q, \tag{1.3}$$

where $K_3 \geq 1$ is a constant independent of x and r. The n-dimensional Euclidean space $\mathbb{R}^n$ is n-homogeneous.

In [2], the generalized Orlicz–Morrey space was introduced to unify Orlicz and generalized Morrey spaces. Other definitions of generalized Orlicz–Morrey spaces can be found in [3,4]. Spanne and Adams type boundedness of fractional maximal operator M_α and its commutators $M_{b,\alpha}$ in generalized Orlicz–Morrey spaces on the n-dimensional Euclidean space $\mathbb{R}^n$ was investigated in [5–7]. Moreover, the boundedness of M_α and $M_{b,\alpha}$ in Orlicz spaces on $\mathbb{R}^n$ was characterized in [8]. The purpose of this paper is to extend these Euclidean results to the spaces of homogeneous type setting.

The structure of the remaining part of the present paper is as follows: Sect. 2 provides the definitions and some preliminaries on Young functions and Orlicz spaces. We shall give necessary and sufficient conditions for the boundedness of M_α and $M_{b,\alpha}$ in Orlicz spaces $L^\Phi(X)$ in Sect. 3. In Sect. 4, we investigate the structure of generalized Orlicz–Morrey spaces defined on spaces of homogeneous type $\mathcal{M}^{\Phi,\varphi}(X)$. We give characterizations for the Spanne and Adams type boundedness of M_α and $M_{b,\alpha}$ in $\mathcal{M}^{\Phi,\varphi}(X)$ in Sects. 5 and 6, respectively.

At the end of this section, we make some conventions. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

We recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0}\Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty}\Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty.$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

Let $X = (X, d, \mu)$ be a space of homogeneous type. For a measurable set $\Omega \subset X$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = \mu(\{x \in \Omega : |f(x)| > t\})$. In the case $\Omega = X$, we shortly denote it by $m(f,\ t)$.

The Orlicz spaces and weak Orlicz spaces on spaces of homogeneous type are defined as follows.

Definition 2.2 For a Young function Φ,

$$L^{\Phi}(X) = \left\{ f \in L^1_{\rm loc}(X) : \int_X \Phi(\varepsilon|f(x)|)d\mu(x) < \infty \text{ for some } \varepsilon > 0 \right\},$$

$$\|f\|_{L^\Phi} \equiv \|f\|_{L^\Phi(X)} = \inf\left\{ \lambda > 0 : \int_X \Phi\Big(\frac{|f(x)|}{\lambda}\Big)d\mu(x) \le 1 \right\},$$

$$WL^{\Phi}(X) = \left\{ f \in L^1_{\rm loc}(X) : \sup_{r>0} \Phi(r) m\Big(r, \varepsilon f\Big) < \infty \text{ for some } \varepsilon > 0 \right\},$$

$$\|f\|_{WL^\Phi} \equiv \|f\|_{WL^\Phi(X)} = \inf\left\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \le 1 \right\}.$$

We note that $\|f\|_{WL^\Phi} \le \|f\|_{L^\Phi}$,

$$\sup_{t>0} \Phi(t) m(\Omega,\ f,\ t) = \sup_{t>0} t\, m(\Omega,\ f,\ \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega,\ \Phi(|f|),\ t)$$

and

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\Big)dx \le 1, \qquad \sup_{t>0} \Phi(t) m\Big(\Omega,\ \frac{f}{\|f\|_{WL^\Phi(\Omega)}},\ t\Big) \le 1, \tag{2.1}$$

where $\|f\|_{L^\Phi(\Omega)} = \|f\chi_\Omega\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(\Omega)} = \|f\chi_\Omega\|_{WL^\Phi}$.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < \infty. \tag{2.2}$$

We also note that for $\Phi \in \mathcal{Y}$ we have

$$\|\chi_E\|_{WL^\Phi} = \|\chi_E\|_{L^\Phi} = \frac{1}{\Phi^{-1}\left(\mu(E)^{-1}\right)}, \tag{2.3}$$

where E is a μ-measurable set in X with $\mu(E) < \infty$ and χ_E is the characteristic function of E. Indeed,

$$\begin{aligned}
\|\chi_E\|_{L^\Phi} &= \inf\left\{\lambda > 0 : \int_E \Phi\left(\frac{1}{\lambda}\right) d\mu(y) \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{\lambda} \le \Phi^{-1}\left(\mu(E)^{-1}\right)\right\} \\
&= \inf\left\{\lambda > 0 : \lambda \ge \frac{1}{\Phi^{-1}\left(\mu(E)^{-1}\right)}\right\} \\
&= \frac{1}{\Phi^{-1}\left(\mu(E)^{-1}\right)}, \\
\|\chi_E\|_{WL^\Phi} &= \inf\left\{\lambda > 0 : \sup_{t>0} \Phi\left(\frac{t}{\lambda}\right) \mu(\{x \in X : |\chi_E(x)| > t\}) \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \sup_{0<t<1} \Phi\left(\frac{t}{\lambda}\right) \mu(\{x \in X : |\chi_E(x)| > t\}) \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \Phi\left(\frac{1}{\lambda}\right) \le \mu(E)^{-1}\right\} \\
&= \inf\left\{\lambda > 0 : \lambda \ge \frac{1}{\Phi^{-1}\left(\mu(E)^{-1}\right)}\right\} \\
&= \frac{1}{\Phi^{-1}\left(\mu(E)^{-1}\right)}.
\end{aligned}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$.

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty), \\ \infty, & r = \infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = \infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.4}$$

Note that by the convexity of Φ and concavity of Φ^{-1} we have the following properties

$$\begin{cases} \Phi(\alpha t) \le \alpha\Phi(t), & \text{if } \ 0 \le \alpha \le 1 \\ \Phi(\alpha t) \ge \alpha\Phi(t), & \text{if } \quad \alpha > 1 \end{cases} \text{ and } \begin{cases} \Phi^{-1}(\alpha t) \ge \alpha\Phi^{-1}(t), & \text{if } \ 0 \le \alpha \le 1 \\ \Phi^{-1}(\alpha t) \le \alpha\Phi^{-1}(t), & \text{if } \quad \alpha > 1. \end{cases} \tag{2.5}$$

Remark 2.3 Thanks to (1.3) and (2.5) we have

$$\Phi^{-1}(\mu(B(x,r))^{-1}) \approx \Phi^{-1}(r^{-Q}).$$

The following analogue of the Hölder inequality is known,

$$\int_X |f(x)g(x)|d\mu(x) \le 2\|f\|_{L^\Phi}\|g\|_{L_{\widetilde{\Phi}}}. \tag{2.6}$$

When we prove our main estimates, we use the following lemma, which follows from (2.6), (2.3) and (2.4).

Lemma 2.4 *For a Young function Φ and $B = B(x,r)$, the following inequality is valid*

$$\|f\|_{L^1(B)} \le 2\mu(B)\Phi^{-1}\left(\mu(B)^{-1}\right)\|f\|_{L^\Phi(B)}.$$

3 Fractional maximal operator and its commutators in Orlicz spaces

For a Q-homogeneous space (X, d, μ), let $M_\alpha f$ be the fractional maximal function, i.e.

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))^{1-\alpha/Q}} \int_{B(x,r)} |f(y)|d\mu(y), \qquad 0 \le \alpha < Q.$$

In our definition, we consider balls that are centered at x, but we obtain a noncentered maximal function by taking the supremum over all balls containing x. For doubling measures, these maximal functions are comparable, and it does not matter which one we choose. When $\alpha = 0$ this reduces to the Hardy-Littlewood maximal operator and we write M instead of M_0.

In order to prove our main theorem, we also need the following lemma.

Lemma 3.1 *If $B_0 := B(x_0, r_0)$, then $\mu(B_0)^{\frac{\alpha}{Q}} \lesssim M_\alpha \chi_{B_0}(x)$ for every $x \in B_0$.*

Proof For $x \in B_0$, we get

$$M_\alpha \chi_{B_0}(x) \gtrsim \sup_{B \ni x} \mu(B)^{-1+\frac{\alpha}{Q}} \mu(B \cap B_0) \geq \mu(B_0)^{-1+\frac{\alpha}{Q}} \mu(B_0 \cap B_0) = \mu(B_0)^{\frac{\alpha}{Q}}.$$

□

We recall the boundedness property of M on Orlicz spaces since we use it later.

Theorem 3.2 [9] *Let Φ be a Young function.*

(i) The operator M is bounded from $L^\Phi(X)$ to $WL^\Phi(X)$, and the inequality

$$\|Mf\|_{WL^\Phi} \leq C_0 \|f\|_{L^\Phi} \tag{3.1}$$

holds with constant C_0 independent of f.

(ii) The operator M is bounded on $L^\Phi(X)$, and the inequality

$$\|Mf\|_{L^\Phi} \leq C_0 \|f\|_{L^\Phi} \tag{3.2}$$

holds with constant C_0 independent of f if and only if $\Phi \in \nabla_2$.

The following theorem in $\mathbb{R}^n$ case for a more general case of generalized fractional maximal operator was proved in [10]. Moreover, the results of [8] was given for more general Young functions in [10]. The following theorem partially extends the results of [10] to the spaces of homogeneous case. The proof method is essentially the same as in [10].

Theorem 3.3 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and Φ, Ψ be Young functions.*

1. Assume that there exists a positive constant C such that, for all $r > 0$,

$$r^\alpha \leq C \frac{\Psi^{-1}\big(r^{-Q}\big)}{\Phi^{-1}\big(r^{-Q}\big)}. \tag{3.3}$$

Then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^\Phi(X)$ with $f \neq 0$,

$$M_\alpha f(x) \leq C_1 \|f\|_{L^\Phi} (\Psi^{-1} \circ \Phi)\Big(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\Big) \quad (x \in X). \tag{3.4}$$

Consequently, M_α is bounded from $L^\Phi(X)$ to $WL^\Psi(X)$. Moreover, if $\Phi \in \nabla_2$, then M_α is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.

2. Conversely, if M_α is bounded from $L^\Phi(X)$ to $WL^\Psi(X)$, then (3.3) holds.

Proof Let $f \in L^\Phi(X)$. We may assume that $Mf(x) > 0$ for all $x \in X$. For any $x \in X$ and any ball $B = B(z, r) \ni x$, if

$$\Phi\Big(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\Big) \geq r^{-Q},$$

then, by Lemma 2.4 and (3.3), we have

$$\frac{r^\alpha}{\mu(B)}\int_B |f(y)|d\mu(y) \lesssim r^\alpha\, \Phi^{-1}(r^{-Q})\,\|f\|_{L^\Phi} \lesssim \Psi^{-1}(r^{-Q})\,\|f\|_{L^\Phi} \lesssim \Psi^{-1}\left(\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big)\right)\,\|f\|_{L^\Phi}.$$

Conversely, if

$$\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big) \le r^{-Q},$$

then, choosing $t_0 \ge r$ such that

$$\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big) = t_0^{-Q},$$

and using (3.3) and (2.2)

$$r^\alpha \le t_0^\alpha \lesssim \frac{\Psi^{-1}\left(\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big)\right)}{\Phi^{-1}\left(\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big)\right)} \lesssim \frac{\Psi^{-1}\left(\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big)\right)}{\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}},$$

which implies

$$\frac{r^\alpha}{\mu(B)}\int_B |f(y)|d\mu(y) \lesssim \|f\|_{L^\Phi}\,\frac{\Psi^{-1}\left(\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big)\right)}{Mf(x)}\,\frac{1}{\mu(B)}\int_B |f(y)|d\mu(y) \lesssim \Psi^{-1}\left(\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big)\right)\,\|f\|_{L^\Phi},$$

which shows (3.4).

- Let C_0 be as in (3.1). Then by (2.1), (3.1) and (3.4), we have

$$\sup_{r>0}\Psi(r)\,m\Big(\frac{M_\alpha f(x)}{C_1\|f\|_{L^\Phi}}, r\Big) = \sup_{r>0} r\, m\Big(\Psi\Big(\frac{M_\alpha f(x)}{C_1\|f\|_{L^\Phi}}\Big), r\Big) \le \sup_{r>0} r\, m\Big(\Phi\Big(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\Big), r\Big) \le \sup_{r>0}\Phi(r)\,m\Big(\frac{Mf(x)}{\|Mf\|_{\mathrm{W}L^\Phi}}, r\Big) \le 1,$$

  i.e.

$$\|M_\alpha f\|_{\mathrm{W}L^\Psi} \lesssim \|f\|_{L^\Phi}. \tag{3.5}$$

- Assume in addition that $\Phi \in \nabla_2$. Let C_0 be as in (3.2). By (2.1), (3.2) and (3.4), we have

$$\int_X \Psi\left(\frac{M_\alpha f(x)}{C_1\|f\|_{L^\Phi}}\right)d\mu(x) \le \int_X \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right)d\mu(x) \le \int_X \Phi\left(\frac{Mf(x)}{\|Mf\|_{L^\Phi}}\right)d\mu(x) \le 1,$$

i.e.

$$\|M_\alpha f\|_{L^\Psi} \lesssim \|f\|_{L^\Phi}. \tag{3.6}$$

For the necessity, we can concentrate on the boundedness of M_α from $L^\Phi(X)$ to $WL^\Psi(X)$, since the boundedness of M_α from $L^\Phi(X)$ to $L^\Psi(X)$ is stronger than the boundedness of M_α from $L^\Phi(X)$ to $WL^\Psi(X)$. With this in mind, assume that M_α is bounded from $L^\Phi(X)$ to $WL^\Psi(X)$. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.1, we have $r_0^\alpha \lesssim M_\alpha \chi_{B_0}(x)$. Therefore, by (2.3), we have

$$r_0^\alpha \lesssim \Psi^{-1}(\mu(B_0)^{-1})\|M_\alpha\chi_{B_0}\|_{WL^\Psi(B_0)} \lesssim \Psi^{-1}(\mu(B_0)^{-1})\|M_\alpha\chi_{B_0}\|_{WL^\Psi} \lesssim \Psi^{-1}(\mu(B_0)^{-1})\|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-Q})}{\Phi^{-1}(r_0^{-Q})}.$$

Since this is true for every $r_0 > 0$, we are done. □

We can summarize Theorem 3.3 as following:

Corollary 3.4 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and Φ, Ψ be Young functions. Then the condition (3.3) is necessary and sufficient for the boundedness of M_α from $L^\Phi(X)$ to $WL^\Psi(X)$. Moreover, if $\Phi \in \nabla_2$, the condition (3.3) is necessary and sufficient for the boundedness of M_α from $L^\Phi(X)$ to $L^\Psi(X)$.*

To compare, we formulate the following theorem proved in [11,12] and remark below, where

$$I_\alpha f(x) = \int_X \frac{f(y)}{d(x,y)^{Q-\alpha}}d\mu(y), \qquad 0 < \alpha < Q.$$

Theorem 3.5 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and $\Phi, \Psi \in \mathcal{Y}$. If*

$$\int_r^\infty t^{\alpha-1}\Phi^{-1}\left(t^{-Q}\right)dt \lesssim r^\alpha\Phi^{-1}\left(r^{-Q}\right) \qquad \textit{for } 0 < r < \infty, \tag{3.7}$$

holds, then the condition (3.3) is necessary and sufficient for the boundedness of I_α from $L^\Phi(X)$ to $WL^\Psi(X)$. Moreover, if $\Phi \in \nabla_2$, the condition (3.3) is necessary and sufficient for the boundedness of I_α from $L^\Phi(X)$ to $L^\Psi(X)$.

Remark 3.6 Although fractional maximal function M_α is pointwise dominated by the Riesz potential I_α, and consequently, the results for the former could be derived from the results for the latter, we consider them separately, because we are able to study the fractional maximal operator under weaker assumptions than it derived from the results for the potential operator. More precisely, we don't need to regularity condition (3.7) for the boundedness of fractional maximal operator.

We recall that the space $BMO(X) = \{b \in L^1_{\rm loc}(X) \ : \ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in X,\, r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r)}| d\mu(y) < \infty,$$

where $b_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} b(y) d\mu(y)$. We will need the following properties of BMO-functions:

Lemma 3.7 [13, Lemma 7.1] *Let $b \in BMO(X)$.*

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \le C\|b\|_* \ln \frac{6t}{r} \quad for \quad 0 < 2r < t, \tag{3.8}$$

where C does not depend on b, x, r and t.

Lemma 3.8 [11, Lemma 4.7] *Let $b \in BMO(X)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_* \approx \sup_{x\in X,\, r>0} \Phi^{-1}\big(r^{-Q}\big) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L^\Phi(B(x,r))}.$$

Given a measurable function b the operator $M_{b,\alpha}$ is defined by

$$M_{b,\alpha}(f)(x) = \sup_{t>0} \mu(B(x,t))^{-1+\frac{\alpha}{Q}} \int_{B(x,t)} |b(x) - b(y)||f(y)| d\mu(y).$$

If $\alpha = 0$, then $M_{b,0} \equiv M_b$ is called maximal commutator.

The known boundedness statements for the commutator operator M_b on Orlicz spaces run as follows, see [14, Theorem 1.9 and Corollary 2.3]. Note that in [14] a more general case of multilinear commutators was studied.

Theorem 3.9 *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(X)$. Then M_b is bounded on $L^\Phi(X)$ and the inequality*

$$\|M_b f\|_{L^\Phi} \le C_0 \|b\|_* \|f\|_{L^\Phi} \tag{3.9}$$

holds with constant C_0 independent of f.

The following lemma is the analogue of the Hedberg's trick for $[b, I_\alpha]$, see [15, p. 506].

Lemma 3.10 [11, Lemma 5.5] *If (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and $f, b \in L^1_{\rm loc}(X)$, then for all $x \in X$ and $r > 0$ we get*

$$\int_{B(x,r)} \frac{|f(y)|}{d(x,y)^{Q-\alpha}} |b(x) - b(y)| d\mu(y) \lesssim r^\alpha M_b f(x).$$

For proving our main results, we need the following estimate.

Lemma 3.11 *If $b \in L^1_{\rm loc}(X)$ and $B_0 := B(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{B_0}| \lesssim M_{b,\alpha}\chi_{B_0}(x) \text{ for every } x \in B_0.$$

Proof It is well-known that

$$\mathrm{M}_{b,\alpha} f(x) \lesssim M_{b,\alpha} f(x), \tag{3.10}$$

where $\mathrm{M}_{b,\alpha}(f)(x) = \sup\limits_{B \ni x} \mu(B)^{-1+\frac{\alpha}{Q}} \int_B |b(x) - b(y)||f(y)|d\mu(y)$.

Now let $x \in B_0$. By using (3.10), we get

$$\begin{aligned} M_{b,\alpha}\chi_{B_0}(x) &\gtrsim \mathrm{M}_{b,\alpha} f(x) = \sup_{B \ni x} \mu(B)^{-1+\frac{\alpha}{Q}} \int_B |b(x) - b(y)|\chi_{B_0} d\mu(y) \\ &= \sup_{B \ni x} |B|^{-1+\frac{\alpha}{Q}} \int_{B \cap B_0} |b(x) - b(y)| d\mu(y) \\ &\gtrsim \mu(B_0)^{-1+\frac{\alpha}{Q}} \int_{B_0 \cap B_0} |b(x) - b(y)| d\mu(y) \\ &\gtrsim \left|\mu(B_0)^{-1+\frac{\alpha}{Q}} \int_{B_0} (b(x) - b(y)) d\mu(y)\right| = r_0^\alpha |b(x) - b_{B_0}|. \end{aligned}$$

□

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\alpha}$ from $L^\Phi(X)$ to $L^\Psi(X)$.

Theorem 3.12 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $b \in BMO(X)$ and Φ, Ψ be Young functions.*

1. *If $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha \Phi^{-1}\big(r^{-Q}\big) + \sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-Q}\big) t^\alpha \le C \Psi^{-1}\big(r^{-Q}\big) \tag{3.11}$$

   for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(X)$ to $L^\Psi(X)$.
2. *If $\Psi \in \Delta_2$, then the condition (3.3) is necessary for the boundedness of $M_{b,\alpha}$ from $L^\Phi(X)$ to $L^\Psi(X)$.*
3. *Let $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$. If the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-Q}\big) t^\alpha \le C r^\alpha \Phi^{-1}\big(r^{-Q}\big) \tag{3.12}$$

   holds for all $r > 0$, where $C > 0$ does not depend on r, then the condition (3.3) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(X)$ to $L^\Psi(X)$.

Proof (1) For arbitrary $x_0 \in X$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2kB}$ and $f_2 = f\chi_{\complement_{(2kB)}}$, where k is the constant from the triangle inequality (1.1).

Let x be an arbitrary point in B. If $B(x,t) \cap \{^{\complement}(2kB)\} \neq \emptyset$, then $t > r$. Indeed, if $y \in B(x,t) \cap \{^{\complement}(2kB)\}$, then $t > d(x,y) \geq \frac{1}{k} d(x_0, y) - d(x_0, x) > 2r - r = r$.

On the other hand, $B(x,t) \cap \{^{\complement}(2kB)\} \subset B(x_0, 2kt)$. Indeed, if $y \in B(x,t) \cap \{^{\complement}(2kB)\}$, then we get $d(x_0, y) \leq kd(x,y) + kd(x_0,x) < kt + kr < 2kt$.

Hence by (1.3)

$$
\begin{aligned}
M_{b,\alpha}(f_2)(x) &= \sup_{t>0} \frac{1}{\mu(B(x,t))^{1-\frac{\alpha}{Q}}} \int_{B(x,t)\cap ^{\complement}(2kB)} |b(y) - b(x)||f(y)| d\mu(y) \\
&\leq \sup_{t>r} \frac{1}{\mu(B(x,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,2kt)} |b(y) - b(x)||f(y)| d\mu(y) \\
&\lesssim \sup_{t>2r} \frac{1}{\mu(B(x_0,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,t)} |b(y) - b(x)||f(y)| d\mu(y).
\end{aligned}
$$

Therefore, for all $x \in B$ we have

$$
\begin{aligned}
M_{b,\alpha}(f_2)(x) &\lesssim \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b(y) - b(x)||f(y)| d\mu(y) \\
&\lesssim \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)| d\mu(y) \\
&\quad + \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b_{B(x_0,t)} - b_B||f(y)| d\mu(y) \\
&\quad + \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b_B - b(x)||f(y)| d\mu(y) \\
&= J_1 + J_2 + J_3.
\end{aligned}
$$

Applying Hölder's inequality, by (2.4), (3.8) and Lemmas 2.4 and 3.8 we get

$$
\begin{aligned}
J_1 + J_2 &\lesssim \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)| d\mu(y) \\
&\quad + \sup_{t>2r} t^{\alpha-Q} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| d\mu(y) \\
&\lesssim \sup_{t>2r} t^{\alpha-Q} \left\| b(\cdot) - b_{B(x_0,t)} \right\|_{L^{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L^\Phi(B(x_0,t))} \\
&\quad + \sup_{t>2r} t^{\alpha-Q} |b_{B(x_0,r)} - b_{B(x_0,t)}| t^Q \Phi^{-1}\big(\mu(B(x_0,t))^{-1}\big) \|f\|_{L^\Phi(B(x_0,t))} \\
&\lesssim \|b\|_* \sup_{t>2r} \Phi^{-1}(\mu(B(x_0,t))^{-1}) t^\alpha \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^\Phi(B(x_0,t))} \\
&\lesssim \|b\|_* \|f\|_{L^\Phi} \sup_{t>2r} \left(1 + \ln \frac{t}{r}\right) t^\alpha \Phi^{-1}(t^{-Q}).
\end{aligned}
$$

A geometric observation shows $2kB \subset B(x,\delta)$ for all $x \in B$, where $\delta = (2k+1)kr$. Using Lemma 3.10, we get

$$J_0(x) := M_{b,\alpha}(f_1)(x) \lesssim \int_{2kB} \frac{|b(y)-b(x)|}{d(x,y)^{Q-\alpha}}|f(y)|d\mu(y)$$
$$\lesssim \int_{B(x,\delta)} \frac{|b(y)-b(x)|}{d(x,y)^{Q-\alpha}}|f(y)|d\mu(y) \lesssim r^\alpha M_b f(x). \quad (3.13)$$

For all $x \in B$ we get

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_* r^\alpha M_b f(x) + \|b\|_* \|f\|_{L^\Phi} \sup_{t>2r} \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \Phi^{-1}(t^{-Q}).$$

Thus, by (3.11) we obtain

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_* \left(M_b f(x) \frac{\Psi^{-1}(r^{-Q})}{\Phi^{-1}(r^{-Q})} + \Psi^{-1}(r^{-Q}) \|f\|_{L^\Phi} \right).$$

Choose $r > 0$ so that $\Phi^{-1}(r^{-Q}) = \frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}$. Then

$$\frac{\Psi^{-1}(r^{-Q})}{\Phi^{-1}(r^{-Q})} = \frac{(\Psi^{-1} \circ \Phi)\left(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}\right)}{\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}}.$$

Therefore, we get

$$J_0(x) + J_1 + J_2 \le C_1 \|b\|_* \|f\|_{L^\Phi} (\Psi^{-1} \circ \Phi)\Big(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}\Big).$$

Let C_0 be as in (3.9). Consequently by Theorem 3.9 and (2.1) we have

$$\int_B \Psi\left(\frac{J_0(x) + J_1 + J_2}{C_1 \|b\|_* \|f\|_{L^\Phi}}\right) d\mu(x) \le \int_B \Phi\left(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}\right) d\mu(x)$$
$$\le \int_X \Phi\left(\frac{M_b f(x)}{\|M_b f\|_{L^\Phi}}\right) d\mu(x) \le 1,$$

i.e.

$$\|J_0(\cdot) + J_1 + J_2\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \quad (3.14)$$

By Lemma 3.8, Lemma 2.4 and condition (3.11), we also get

$$\|J_3\|_{L^\Psi(B)} = \left\| \sup_{t>2r} \frac{1}{\mu(B(x_0,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,t)} |b(\cdot) - b_B| |f(y)| d\mu(y) \right\|_{L^\Psi(B)}$$
$$\approx \|b(\cdot) - b_B\|_{L^\Psi(B)} \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |f(y)| d\mu(y)$$

$$\begin{aligned}&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(\mu(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}(\mu(B(x_0,t))^{-1})t^\alpha \|f\|_{L^\Phi(B(x_0,t))}\\&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(\mu(B)^{-1}\big)} \|f\|_{L^\Phi} \sup_{t>2r} t^\alpha \Phi^{-1}(\mu(B(x_0,t))^{-1})\\&\lesssim \|b\|_* \|f\|_{L^\Phi}.\end{aligned}$$

Consequently, we have

$$\|J_3\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{3.15}$$

Combining (3.14) and (3.15), we get

$$\|M_{b,\alpha} f\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{3.16}$$

By taking supremum over B in (3.16), we get

$$\|M_{b,\alpha} f\|_{L^\Psi} \lesssim \|b\|_* \|f\|_{L^\Phi},$$

since the constants in (3.16) don't depend on x_0 and r.

(2) We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.11, we have $r_0^\alpha |b(x) - b_{B_0}| \lesssim M_{b,\alpha}\chi_{B_0}(x)$. Therefore, by Lemma 3.8 and (2.3)

$$\begin{aligned}r_0^\alpha &\lesssim \frac{\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L^\Psi(B_0)}} \lesssim \Psi^{-1}(\mu(B_0)^{-1})\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}\\&\lesssim \Psi^{-1}(\mu(B_0)^{-1})\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi} \lesssim \Psi^{-1}(\mu(B_0)^{-1})\|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-Q})}{\Phi^{-1}(r_0^{-Q})}.\end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

The following theorem shows that $b \in BMO(X)$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^\Phi(X)$ to $L^\Psi(X)$.

Theorem 3.13 *Let (X, d, μ) be $Q-$homogeneous, $0 \le \alpha < Q$, $b \in L^1_{loc}(X)$, Φ, Ψ be Young functions. Assume that there exists a positive constant C such that, for all $r > 0$,*

$$r^\alpha \ge C \frac{\Psi^{-1}\big(r^{-Q}\big)}{\Phi^{-1}\big(r^{-Q}\big)}. \tag{3.17}$$

Then the condition $b \in BMO(X)$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^\Phi(X)$ to $L^\Psi(X)$.

Proof Suppose that $M_{b,\alpha}$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$. Choose any ball $B = B(x, r)$ in X, by Lemma 2.4, (2.3) and (3.17) we have

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |b(y)-b_B|d\mu(y) &= \frac{1}{\mu(B)}\int_B \Big|\frac{1}{\mu(B)}\int_B (b(y)-b(z))d\mu(z)\Big|d\mu(y)\\ &\le \frac{1}{\mu(B)^2}\int_B\int_B |b(y)-b(z)|d\mu(y)d\mu(z)\\ &= \frac{1}{\mu(B)^{1+\frac{\alpha}{Q}}}\int_B \frac{1}{\mu(B)^{1-\frac{\alpha}{Q}}}\\ &\quad \int_B |b(y)-b(z)|\chi_B(z)d\mu(z)d\mu(y)\\ &\le \frac{1}{\mu(B)^{1+\frac{\alpha}{Q}}}\int_B M_{b,\alpha}\big(\chi_B\big)(y)d\mu(y)\\ &\lesssim \frac{1}{\mu(B)^{\frac{\alpha}{Q}}}\Psi^{-1}(\mu(B)^{-1})\|M_{b,\alpha}\chi_B\|_{L^\Psi}\\ &\lesssim \frac{1}{\mu(B)^{\frac{\alpha}{Q}}}\Psi^{-1}(\mu(B)^{-1})\|\chi_B\|_{L^\Phi}\lesssim r^{-\alpha}\frac{\Psi^{-1}(r^{-Q})}{\Phi^{-1}(r^{-Q})}\lesssim 1.\end{aligned}$$

Thus $b \in BMO(X)$. □

By Theorems 3.12 and 3.13 we have the following characterization of $BMO(X)$.

Theorem 3.14 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $b \in L^1_{\mathrm{loc}}(X)$, $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\frac{\alpha}{Q}}$. If the condition (3.12) holds, then the condition $b \in BMO(X)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(X)$ to $L^\Psi(X)$.*

For comparison, we formulate the following theorem, which was proved in [11], and make a remark, where the commutator generated by $b \in L^1_{\mathrm{loc}}(X)$ and the operator I_α is defined by

$$[b, I_\alpha]f(x) = \int_X \frac{b(x)-b(y)}{d(x,y)^{Q-\alpha}}f(y)d\mu(y), \qquad 0<\alpha<Q$$

and the operator $|b, I_\alpha|$ is defined by

$$|b, I_\alpha|f(x) = \int_X \frac{|b(x)-b(y)|}{d(x,y)^{Q-\alpha}}f(y)d\mu(y), \qquad 0<\alpha<Q.$$

Theorem 3.15 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $b \in BMO(X)$ and $\Phi, \Psi \in \mathcal{Y}$.*

1. *If $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha\Phi^{-1}\big(r^{-Q}\big) + \int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-Q}\big)t^\alpha\frac{dt}{t} \le C\Psi^{-1}\big(r^{-Q}\big) \qquad (3.18)$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(X)$ to $L^\Psi(X)$.

2. *If* $\Psi \in \Delta_2$*, then the condition* (3.3) *is necessary for the boundedness of* $|b, I_\alpha|$ *from* $L^\Phi(X)$ *to* $L^\Psi(X)$*.*
3. *Let* $\Phi \in \Delta_2 \cap \nabla_2$ *and* $\Psi \in \Delta_2$*. If the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-Q}\big) t^\alpha \frac{dt}{t} \le C r^\alpha \Phi^{-1}\big(r^{-Q}\big) \tag{3.19}$$

holds for all $r > 0$*, where* $C > 0$ *does not depend on* r*, then the condition* (3.3) *is necessary and sufficient for the boundedness of* $|b, I_\alpha|$ *from* $L^\Phi(X)$ *to* $L^\Psi(X)$*.*

Remark 3.16 Although $M_{b,\alpha}$ is pointwise dominated by $|b, I_\alpha|$, and consequently, the results for the former could be derived from the results for the latter, we consider them separately, because we are able to study the boundedness of $M_{b,\alpha}$ under weaker assumptions than it derived from the results for the operator $|b, I_\alpha|$. More precisely, integral condition (3.19) imply the supremal condition (3.12). Indeed, by (2.4) we have

$$\Phi^{-1}\big(s^{-Q}\big) \approx \Phi^{-1}\big(s^{-Q}\big) s^Q \int_s^\infty \frac{dt}{t^{Q+1}} \lesssim \int_s^\infty \Phi^{-1}\big(t^{-Q}\big) \frac{dt}{t}.$$

It follows from this inequality

$$\begin{aligned} r^\alpha \Phi^{-1}\big(r^{-Q}\big) &\gtrsim \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \, \Phi^{-1}\big(t^{-Q}\big) \frac{dt}{t} \\ &\gtrsim \int_s^\infty \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \, \Phi^{-1}\big(t^{-Q}\big) \frac{dt}{t} \\ &\gtrsim s^\alpha \Big(1 + \ln \frac{s}{r}\Big) \int_s^\infty \Phi^{-1}\big(t^{-Q}\big) \frac{dt}{t} \\ &\gtrsim \Big(1 + \ln \frac{s}{r}\Big) \Phi^{-1}\big(s^{-Q}\big) s^\alpha, \end{aligned}$$

where we took $s \in (r, \infty)$ arbitrarily, so that

$$\sup_{s>r} \Big(1 + \ln \frac{s}{r}\Big) \Phi^{-1}\big(s^{-Q}\big) s^\alpha \lesssim r^\alpha \Phi^{-1}\big(r^{-Q}\big).$$

The commutators generated by a suitable function b and the operator M_α is formally defined by

$$[b, M_\alpha] f = M_\alpha(bf) - b M_\alpha(f).$$

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid (see, for example, [16]):

Let b be any non-negative locally integrable function. Then for all $x \in X$

$$\begin{aligned} \big|[b, M_\alpha] f(x)\big| &= \big|b(x) M_\alpha f(x) - M_\alpha(bf)(x)\big| \\ &= \big|M_\alpha(b(x) f)(x) - M_\alpha(bf)(x)\big| \\ &\le M_\alpha(|b(x) - b| f)(x) \le M_{b,\alpha}(f)(x) \end{aligned}$$

holds for all $f \in L^1_{\mathrm{loc}}(X)$.

If b is any locally integrable function on X, then

$$|[b, M_\alpha]f(x)| \le M_{b,\alpha}(f)(x) + 2b^-(x)M_\alpha f(x), \qquad x \in X \tag{3.20}$$

holds for all $f \in L^1_{\mathrm{loc}}(X)$, where

$$b^-(x) = \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$.

By (3.20) and Theorems 3.3 and 3.12 we get the following corollary.

Corollary 3.17 *Let $0 < \alpha < Q$, $b \in BMO(X)$, $b^- \in L^\infty(X)$ and Φ, Ψ be Young functions with $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$. Let also the condition (3.11) is satisfied. Then the operator $[b, M_\alpha]$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.*

4 Generalized Orlicz–Morrey spaces

The generalized Orlicz–Morrey spaces and the weak generalized Orlicz–Morrey spaces on spaces of homogeneous type are defined as follows.

Definition 4.1 Let $X = (X, d, \mu)$ be a space of homogeneous type, $\varphi(r)$ be a positive measurable function on $(0, \infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(X)$ the generalized Orlicz–Morrey space, the space of all functions $f \in L^\Phi_{\mathrm{loc}}(X)$ with finite quasinorm

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} \equiv \|f\|_{\mathcal{M}^{\Phi,\varphi}(X)} = \sup_{x\in X, r>0} \varphi(r)^{-1}\Phi^{-1}(\mu(B(x,r))^{-1})\|f\|_{L^\Phi(B(x,r))},$$

where $L^\Phi_{\mathrm{loc}}(X)$ is defined as the set of all functions f such that $f\chi_B \in L^\Phi(X)$ for all balls $B \subset X$.

Also by $W\mathcal{M}^{\Phi,\varphi}(X)$ we denote the weak generalized Orlicz–Morrey space of all functions $f \in WL^\Phi_{\mathrm{loc}}(X)$ for which

$$\|f\|_{W\mathcal{M}^{\Phi,\varphi}} \equiv \|f\|_{W\mathcal{M}^{\Phi,\varphi}(X)} = \sup_{x\in X, r>0} \varphi(r)^{-1}\Phi^{-1}(\mu(B(x,r))^{-1})\|f\|_{WL^\Phi(B(x,r))} < \infty,$$

where $WL^\Phi_{\mathrm{loc}}(X)$ is defined as the set of all functions f such that $f\chi_B \in WL^\Phi(X)$ for all balls $B \subset X$.

If $\Phi(r) = r^p$, $1 \le p < \infty$, then $\mathcal{M}^{\Phi,\varphi}(X)$ coincides with the generalized Morrey space $\mathcal{M}^{p,\varphi}(X)$ equipped with the norm

$$\|f\|_{\mathcal{M}^{p,\varphi}} := \sup_{x\in X, r>0} \varphi(r)^{-1}\mu(B(x,r))^{-\frac{1}{p}}\|f\|_{L^p(B(x,r))}.$$

If $\varphi(r) = \Phi^{-1}(r^{-Q})$, then $\mathcal{M}^{\Phi,\varphi}(X)$ coincides with the Orlicz space $L^{\Phi}(X)$.

A function $\varphi : (0, \infty) \to (0, \infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s. \tag{4.1}$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all $\varphi : (0, \infty) \to (0, \infty)$ functions such that $\frac{\varphi(t)}{\Phi^{-1}(t^{-Q})}$ is almost increasing and $\frac{\varphi(t)}{\Phi^{-1}(t^{-Q})t^Q}$ is almost decreasing. Note that $\varphi \in \mathcal{G}_\Phi$ implies doubling condition of φ.

An observation similar to the one made by Nakai [17, p. 446] it can be assumed that $\varphi \in \mathcal{G}_\Phi$ in the definition of $\mathcal{M}^{\Phi,\varphi}(X)$. See [18, Section 5] for more details.

As the following lemma shows, $\mathcal{G}_\Phi$ is useful:

Lemma 4.2 [11] *Let $B_0 := B(x_0, r_0)$. If $\varphi \in \mathcal{G}_\Phi$ is almost decreasing, then there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{B_0}\|_{W\mathcal{M}^{\Phi,\varphi}} \le \|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \le \frac{C}{\varphi(r_0)},$$

where C is the constant from (4.1).

We need the following boundedness properties of M and M_b to prove our main results.

Theorem 4.3 [11] *Let (X, d, μ) be $Q-$homogeneous.*

1. *Let $\varphi \in \mathcal{G}_\Phi$ be almost decreasing. Then the maximal operator M is bounded from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathrm{W}\mathcal{M}^{\Phi,\varphi}(X)$.*
2. *Let $\Phi \in \nabla_2$ and $\varphi \in \mathcal{G}_\Phi$ be almost decreasing. Then the maximal operator M is bounded on $\mathcal{M}^{\Phi,\varphi}(X)$.*

Theorem 4.4 [11] *Let (X, d, μ) be $Q-$homogeneous, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $b \in BMO(X)$ and the function $\varphi \in \mathcal{G}_\Phi$ be almost decreasing and satisfies the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big)\varphi(t) \le C\,\varphi(r), \tag{4.2}$$

where C does not depend on r. Then the operator M_b is bounded on $\mathcal{M}^{\Phi,\varphi}(X)$.

5 Fractional maximal operator in generalized Orlicz–Morrey spaces

5.1 Spanne-type result

We use the following lemma:

Lemma 5.1 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, Φ, Ψ be Young functions. Assume that the condition* (3.3) *is fulfilled. Then for all $f \in L^{\Phi}_{\mathrm{loc}}(X)$ and $B = B(x, r)$,*

$$\|M_\alpha f\|_{WL^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>r} \Psi^{-1}\big(t^{-Q}\big) \, \|f\|_{L^\Phi(B(x,t))}. \tag{5.1}$$

Moreover if we assume $\Phi \in \nabla_2$, the following inequality is also valid:

$$\|M_\alpha f\|_{L^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>r} \Psi^{-1}\big(t^{-Q}\big) \, \|f\|_{L^\Phi(B(x,t))}. \tag{5.2}$$

Proof Let $\Phi \in \nabla_2$. We put $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2kr)}$ and $f_2 = f\chi_{{}^{\complement}B(x,2kr)}$, where k is the constant from the triangle inequality (1.1).

Estimation of $M_\alpha f_1$: By Theorem 3.3 we have

$$\|M_\alpha f_1\|_{L^\Psi(B)} \le \|M_\alpha f_1\|_{L^\Psi(X)} \lesssim \|f_1\|_{L^\Phi(X)} = \|f\|_{L^\Phi(B(x,2kr))}.$$

By using the monotonicity of the functions $\|f\|_{L^\Phi(B(x,t))}$, $\Psi^{-1}\big(t\big)$ with respect to t and doubling property of Ψ^{-1} we get,

$$\begin{aligned} &\frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-Q}\big)\|f\|_{L^\Phi(B(x,t))} \\ &\quad \ge \frac{\|f\|_{L^\Phi(B(x,2kr))}}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-Q}\big) \gtrsim \|f\|_{L^\Phi(B(x,2kr))}. \end{aligned} \tag{5.3}$$

Consequently we have

$$\|M_\alpha f_1\|_{L^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>r} \Psi^{-1}\big(t^{-Q}\big)\|f\|_{L^\Phi(B(x,t))}. \tag{5.4}$$

Estimation of $M_\alpha f_2$: Let y be an arbitrary point from B. If $B(y,t) \cap {}^{\complement}(B(x,2kr)) \ne \emptyset$, then $t > r$. Indeed, if $z \in B(y,t) \cap {}^{\complement}(B(x,2kr))$, then $t > d(y,z) \ge \frac{1}{k} d(x,z) - d(x,y) > 2r - r = r$.

On the other hand, $B(y,t) \cap {}^{\complement}(B(x,2kr)) \subset B(x,2kt)$. Indeed, if $z \in B(y,t) \cap {}^{\complement}(B(x,2kr))$, then we get $d(x,z) \le kd(y,z) + kd(x,y) < kt + kr < 2kt$.

Therefore,

$$\begin{aligned} M_\alpha f_2(y) &= \sup_{t>0} \frac{t^\alpha}{\mu(B(y,t))} \int_{B(y,t)\cap {}^{\complement}(B(x,2kr))} |f(z)| d\mu(z) \\ &\le \sup_{t>r} \frac{t^\alpha}{\mu(B(y,t))} \int_{B(x,2kt)} |f(z)| d\mu(z) \\ &\lesssim \sup_{t>r} \frac{t^\alpha}{\mu(B(y,2kt))} \int_{B(x,2kt)} |f(z)| d\mu(z) \\ &\approx \sup_{t>2kr} \frac{t^\alpha}{\mu(B(y,t))} \int_{B(x,t)} |f(z)| d\mu(z). \end{aligned}$$

Hence by Lemma 2.4 and (3.3)

$$M_\alpha f_2(y) \lesssim \sup_{t>r} \Psi^{-1}\big(t^{-Q}\big)\|f\|_{L^\Phi(B(x,t))}. \tag{5.5}$$

Thus the function $M_\alpha f_2(y)$, with fixed x and r, is dominated by the expression not depending on y. Then we integrate the obtained estimate for $M_\alpha f_2(y)$ in y over B, we get

$$\|M_\alpha f_2\|_{L^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>r} \Psi^{-1}\big(t^{-Q}\big)\|f\|_{L^\Phi(B(x,t))}. \tag{5.6}$$

Gathering the estimates (5.4) and (5.6) we arrive at (5.2).

Let now Φ be an arbitrary Young function. It is obvious that

$$\|M_\alpha f\|_{WL^\Psi(B)} \le \|M_\alpha f_1\|_{WL^\Psi(B)} + \|M_\alpha f_2\|_{WL^\Psi(B)}.$$

By the boundedness of the operator M_α from $L^\Phi(X)$ to $WL^\Psi(X)$, provided by Theorem 3.3, we have

$$\|M_\alpha f_1\|_{WL^\Psi(B)} \lesssim \|f\|_{L^\Phi(B(x,2kr))}.$$

By using (5.3), (5.5) and (2.3) we arrive at (5.1). □

The following theorem gives a necessary and sufficient condition for Spanne-type boundedness of the operator M_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$: We notice that the requirement is the same as the Orlicz spaces.

Theorem 5.2 *(Spanne-type result) Let (X, d, μ) be $Q-$homogeneous, Φ, Ψ be Young functions, and let $\varphi_1 \in \mathcal{G}_\Phi$ and $\varphi_2 \in \mathcal{G}_\Psi$.*

1. *Assume that the condition (3.3) is satisfied. Then the condition*

$$\sup_{r<t<\infty} \varphi_1(t)\frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \le C\,\varphi_2(r), \tag{5.7}$$

   for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$. Moreover, if $\Phi \in \nabla_2$, then the condition (5.7) is sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.
2. *Let φ_1 be almost decreasing. Then the condition*

$$\varphi_1(r)r^\alpha \le C\varphi_2(r), \tag{5.8}$$

   for all $r > 0$, where $C > 0$ does not depend on r, is necessary for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$ and hence $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.
3. *Let φ_1 be almost decreasing. Assume that conditions (3.3) and*

$$\sup_{r<t<\infty} \varphi_1(t)\frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \le C\,\varphi_1(r)r^\alpha,$$

for all $r > 0$, where $C > 0$ does not depend on r, are satisfied. Then the condition (5.8) is necessary and sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$. Moreover, if $\Phi \in \nabla_2$, then the condition (5.8) is necessary and sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Proof *1.* By (5.1) and (5.7) we have

$$\begin{aligned}\|M_\alpha f\|_{\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}} &\lesssim \sup_{x\in X, r>0} \varphi_2(r)^{-1} \sup_{r<t<\infty} \|f\|_{L^\Phi(B(x,t))}\, \Psi^{-1}\big(t^{-Q}\big) \\ &\lesssim \sup_{x\in X, r>0} \varphi_2(r)^{-1} \sup_{r<t<\infty} \varphi_1(t) \frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \|f\|_{\mathcal{M}^{\Phi,\varphi_1}} \\ &\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}}.\end{aligned}$$

Simply replace $\mathrm{W}L^\Psi(B)$ with $L^\Psi(B)$ and $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$ with $\mathcal{M}^{\Psi,\varphi_2}(X)$ for the strong estimate.
2. We will now prove the necessity. Let $B_0 = B(x_0, t_0)$ and $x \in B_0$. By Lemma 3.1 we have $t_0^\alpha \lesssim M_\alpha \chi_{B_0}(x)$. Therefore, by (2.3) and Lemma 4.2, we have

$$\begin{aligned}t_0^\alpha &\lesssim \Psi^{-1}(\mu(B_0)^{-1})\|M_\alpha \chi_{B_0}\|_{WL^\Psi(B_0)} \lesssim \varphi_2(t_0)\|M_\alpha \chi_{B_0}\|_{W\mathcal{M}^{\Psi,\varphi_2}} \\ &\lesssim \varphi_2(t_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi_1}} \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)}.\end{aligned}$$

Since this is true for every $t_0 > 0$, we are done.
3. The third statement of the theorem follows from the first and second parts of the theorem. □

5.2 Adams-type result

The following theorem is one of our main results.

Theorem 5.3 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, Φ be a Young function, $\varphi \in \mathcal{G}_\Phi$ be almost decreasing $\beta \in (0, 1)$, $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.*

1. *If $\Phi \in \nabla_2$, then the condition*

$$t^\alpha \varphi(t) \le C\varphi(t)^\beta, \tag{5.9}$$

   for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$.
2. *The condition (5.9) is necessary for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$.*
3. *Let $\Phi \in \nabla_2$. Then, the condition (5.9) is necessary and sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$.*

Proof For arbitrary ball $B = B(x, r)$ we represent f as

$$f = f_1 + f_2, \quad f_1 = f\chi_B, \quad f_2 = f - f_1, \quad r > 0,$$

so that

$$M_\alpha f(x) \le M_\alpha f_1(x) + M_\alpha f_2(x).$$

Meanwhile by Lemma 2.4,

$$\begin{aligned} M_\alpha f_2(x) &= \sup_{t>0} \frac{t^\alpha}{\mu(B(x,t))} \int_{B(x,t)\cap\{X\setminus B(x,r)\}} |f(z)| d\mu(z) \\ &\le \sup_{r<t<\infty} \frac{t^\alpha}{\mu(B(x,t))} \int_{B(x,t)} |f(z)| d\mu(z) \\ &\lesssim \sup_{r<t<\infty} t^\alpha\, \Phi^{-1}(\mu(B(x,t))^{-1})\, \|f\|_{L^\Phi(B(x,t))}. \end{aligned}$$

Consequently from Hedberg's trick [15] and the last inequality, we have

$$\begin{aligned} M_\alpha f(x) &\lesssim r^\alpha Mf(x) + \sup_{r<t<\infty} \Phi^{-1}(t^{-Q}) t^\alpha \|f\|_{L^\Phi(B(x,t))} \\ &\lesssim r^\alpha Mf(x) + \|f\|_{\mathcal{M}^{\Phi,\varphi}} \sup_{r<t<\infty} t^\alpha \varphi(t). \end{aligned}$$

Thus, using the technique in [19, p. 6492], by (5.9) we obtain

$$\begin{aligned} M_\alpha f(x) &\lesssim \min\{\varphi(r)^{\beta-1} Mf(x), \varphi(r)^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}\} \\ &\lesssim \sup_{s>0} \min\{s^{\beta-1} Mf(x), s^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}\} \\ &= (Mf(x))^\beta\, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}, \end{aligned}$$

where we have used that the supremum is achieved when the minimum parts are balanced. Hence for every $x \in X$ we have

$$M_\alpha f(x) \lesssim (Mf(x))^\beta\, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}. \tag{5.10}$$

Note that from (2.1) we get

$$\int_B \Psi\left(\frac{(Mf(x))^\beta}{\|Mf\|_{L^\Phi(B)}^\beta}\right) d\mu(x) = \int_B \Phi\left(\frac{Mf(x)}{\|Mf\|_{L^\Phi(B)}}\right) d\mu(x) \le 1.$$

Thus $\|(Mf)^\beta\|_{L^\Psi(B)} \le \|Mf\|_{L^\Phi(B)}^\beta$. From this fact and using (5.10) and the boundedness of the maximal operator, we get, for all balls B,

$$\begin{aligned} \eta(r)^{-1}\Psi^{-1}(r^{-Q})\|M_\alpha f\|_{L^\Psi(B)} &\lesssim \eta(r)^{-1}\Psi^{-1}(r^{-Q})\|Mf\|_{L^\Phi(B)}^\beta\, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \\ &= \left(\varphi(r)^{-1}\Phi^{-1}(r^{-Q})\|Mf\|_{L^\Phi(B)}\right)^\beta\, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi}}. \end{aligned}$$

By taking the supremum with respect to B, we get the desired result.

We shall now prove the necessary part. Let $B_0 = B(x_0, t_0)$ and $x \in B_0$. By Lemma 3.1 we have $t_0^\alpha \lesssim M_\alpha \chi_{B_0}(x)$. Therefore, by (2.3) and Lemma 4.2

$$t_0^\alpha \lesssim \Psi^{-1}(t_0^{-Q}) \|M_\alpha \chi_{B_0}\|_{L^\Psi(B_0)} \lesssim \eta(t_0) \|M_\alpha \chi_{B_0}\|_{\mathcal{M}^{\Psi,\eta}}$$
$$\lesssim \eta(t_0) \|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \lesssim \frac{\eta(t_0)}{\varphi(t_0)} \lesssim \varphi(t_0)^{\beta-1}.$$

Since this is true for every $t_0 > 0$, we are done. The third statement of the theorem follows from the first and second parts of the theorem. □

Remark 5.4 As observed in Remark 3.6, we can compare Theorem 5.3 with [11, Theorem 6.1].

The following result is the weak version of Theorem 5.3.

Theorem 5.5 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, Φ be a Young function, $\varphi \in \mathcal{G}_\Phi$ be almost decreasing, $\beta \in (0, 1)$, $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. The condition (5.9) is necessary and sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(X)$ to $W\mathcal{M}^{\Psi,\eta}(X)$.*

Proof By using the inequality (5.10) we have

$$\|M_\alpha f\|_{WL^\Psi(B)} \lesssim \|(Mf)^\beta\|_{WL^\Psi(B)} \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}},$$

where $B = B(x, r)$.

Note that from (2.1) we get

$$\sup_{t>0} \Psi\left(\frac{t^\beta}{\|Mf\|^\beta_{WL^\Phi(B)}}\right) m((Mf)^\beta, t^\beta) = \sup_{t>0} \Phi\left(\frac{t}{\|Mf\|_{WL^\Phi(B)}}\right) m(Mf, t) \le 1.$$

Thus $\|(Mf)^\beta\|_{WL^\Psi(B)} \le \|Mf\|^\beta_{WL^\Phi(B)}$. Consequently by using the weak boundedness of the maximal operator, we get

$$\eta(r)^{-1}\Psi^{-1}(\mu(B)^{-1})\|M_\alpha f\|_{WL^\Psi(B)} \lesssim \eta(r)^{-1}\Psi^{-1}(\mu(B)^{-1})\|Mf\|^\beta_{WL^\Phi(B)} \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}$$
$$= \left(\varphi(r)^{-1}\Phi^{-1}(\mu(B)^{-1})\|Mf\|_{WL^\Phi(B)}\right)^\beta \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}} \lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi}}.$$

By taking the supremum with respect to all B, we get the desired result.

We will now prove the necessity. Let $B_0 = B(x_0, t_0)$ and $x \in B_0$. By Lemma 3.1 we have $t_0^\alpha \lesssim M_\alpha \chi_{B_0}(x)$. Therefore, by (2.3) and Lemma 4.2

$$t_0^\alpha \lesssim \Psi^{-1}(\mu(B_0)^{-1})\|M_\alpha \chi_{B_0}\|_{WL^\Psi(B_0)} \lesssim \eta(t_0)\|M_\alpha \chi_{B_0}\|_{W\mathcal{M}^{\Psi,\eta}}$$
$$\lesssim \eta(t_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \lesssim \frac{\eta(t_0)}{\varphi(t_0)} = \varphi(t_0)^{\beta-1}.$$

Since this is true for every $t_0 > 0$, we are done. □

6 Commutators of fractional maximal operator in generalized Orlicz–Morrey spaces

6.1 Spanne-type result

The following lemma is valid.

Lemma 6.1 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and $b \in BMO(X)$. Let $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$ and the condition (3.11) holds, then the inequality*

$$\|M_{b,\alpha} f\|_{L^{\Psi}(B(x_0,r))} \lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} \Big(1 + \ln \frac{t}{r}\Big) \Psi^{-1}\big(t^{-Q}\big) \|f\|_{L^{\Phi}(B(x_0,t))}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{\Phi}_{\mathrm{loc}}(X)$.

Proof For arbitrary $x_0 \in X$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2kB}$ and $f_2 = f\chi_{\complement_{(2kB)}}$, where k is the constant from the triangle inequality (1.1). Hence

$$\big\|M_{b,\alpha} f\big\|_{L^{\Psi}(B)} \le \big\|M_{b,\alpha} f_1\big\|_{L^{\Psi}(B)} + \big\|M_{b,\alpha} f_2\big\|_{L^{\Psi}(B)}.$$

From the boundedness of $M_{b,\alpha}$ from $L^{\Phi}(X)$ to $L^{\Psi}(X)$ (see, Theorem 3.12) it follows that

$$\begin{aligned}\|M_{b,\alpha} f_1\|_{L^{\Psi}(B)} &\le \|M_{b,\alpha} f_1\|_{L^{\Psi}(X)} \\ &\lesssim \|b\|_* \, \|f_1\|_{L^{\Phi}(X)} = \|b\|_* \, \|f\|_{L^{\Phi}(2kB)}.\end{aligned}$$

As we proceed in Theroem 3.12, for all $x \in B$ we have

$$M_{b,\alpha}(f_2)(x) \lesssim \sup_{t>2r} \frac{1}{\mu(B(x_0,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,t)} |b(y) - b(x)||f(y)| d\mu(y). \tag{6.1}$$

Then

$$\begin{aligned}\|M_{b,\alpha} f_2\|_{L^{\Psi}(B)} &\lesssim \left\| \sup_{t>2r} \frac{1}{\mu(B(x_0,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,t)} |b(y) - b(\cdot)||f(y)| d\mu(y) \right\|_{L^{\Psi}(B)} \\ &\lesssim \left\| \sup_{t>2r} \frac{1}{\mu(B(x_0,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,t)} |b(y) - b_B||f(y)| d\mu(y) \right\|_{L^{\Psi}(B)} \\ &\quad + \left\| \sup_{t>2r} \frac{1}{\mu(B(x_0,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,t)} |b(\cdot) - b_B||f(y)| d\mu(y) \right\|_{L^{\Psi}(B)} \\ &= J_1 + J_2.\end{aligned}$$

Let us estimate J_1.

$$J_1 = \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} \frac{1}{\mu(B(x_0,t))^{1-\frac{\alpha}{Q}}} \int_{B(x_0,t)} |b(y) - b_B||f(y)|d\mu(y)$$
$$\approx \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b(y) - b_B||f(y)|d\mu(y).$$

Applying Hölder's inequality, by Lemma 3.8 and (3.8) we get

$$\begin{aligned} J_1 &\lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)|d\mu(y) \\ &\quad + \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} t^{\alpha-Q} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|d\mu(y) \\ &\lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} t^{\alpha-Q} \left\| b(\cdot) - b_{B(x_0,t)} \right\|_{L_{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L^\Phi(B(x_0,t))} \\ &\quad + \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} t^{\alpha-Q} |b_{B(x_0,r)} - b_{B(x_0,t)}| t^Q \Phi^{-1}\big(t^{-Q}\big) \|f\|_{L^\Phi(B(x_0,t))} \\ &\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-Q}\big)\Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))}. \end{aligned}$$

In order to estimate J_2 note that

$$\begin{aligned} J_2 &\approx \|b(\cdot) - b_B\|_{L^\Psi(B)} \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |f(y)|d\mu(y) \\ &\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-Q}\big) \|f\|_{L^\Phi(B(x_0,t))}. \end{aligned}$$

Summing up J_1 and J_2 we get

$$\|M_{b,\alpha} f_2\|_{L^\Psi(B)} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-Q}\big)\Big(1+\ln \frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))}. \quad (6.2)$$

Finally,

$$\begin{aligned} \|M_{b,\alpha} f\|_{L^\Psi(B)} &\lesssim \|b\|_* \|f\|_{L^\Phi(2kB)} \\ &\quad + \frac{\|b\|_*}{\Psi^{-1}\big(r^{-Q}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-Q}\big)\Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))}, \end{aligned}$$

and the statement of Lemma 6.1 follows by (5.3). □

The following theorem gives a necessary and sufficient condition for Spanne-type boundedness of the operator $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$. We notice that the requirement is the same as for Orlicz spaces.

Theorem 6.2 *(Spanne-type result) Let* (X, d, μ) *be* $Q-$*homogeneous,* $0 < \alpha < Q$ *and* $b \in BMO(X)$*,* Φ, Ψ *be Young functions, and let* $\varphi_1 \in \mathcal{G}_\Phi$ *and* $\varphi_2 \in \mathcal{G}_\Psi$.

1. *Assume that* $\Phi \in \Delta_2 \cap \nabla_2$ *and* $\Psi \in \Delta_2$ *and the condition* (3.11) *is satisfied. Then the condition*

$$\sup_{r<t<\infty} \varphi_1(t)\Big(1+\ln\frac{t}{r}\Big)\frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \le C\,\varphi_2(r), \tag{6.3}$$

for all $r > 0$*, where* $C > 0$ *does not depend on* r*, is sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi_1}(X)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(X)$.

2. *Let* φ_1 *be almost decreasing and* $\Psi \in \Delta_2$*. Then the condition* (5.8) *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi_1}(X)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(X)$.
3. *Let* φ_1 *be almost decreasing,* $\Phi \in \Delta_2 \cap \nabla_2$ *and* $\Psi \in \Delta_2$*. Assume that conditions* (3.11) *and*

$$\sup_{r<t<\infty} \varphi_1(t)\Big(1+\ln\frac{t}{r}\Big)\frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \le C\,\varphi_1(r) r^{\alpha}, \tag{6.4}$$

for all $r > 0$*, where* $C > 0$ *does not depend on* r*, are satisfied. Then the condition* (5.8) *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi_1}(X)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Proof *1.* By (6.1) and (6.3) we have

$$\begin{aligned}\|M_{b,\alpha} f\|_{\mathcal{M}^{\Psi,\varphi_2}} &\lesssim \sup_{x\in X, r>0} \varphi_2(r)^{-1} \sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L^\Phi(B(x,t))}\,\Psi^{-1}\big(t^{-Q}\big)\\ &\lesssim \sup_{x\in X, r>0} \varphi_2(r)^{-1} \sup_{r<t<\infty} \varphi_1(t)\Big(1+\ln\frac{t}{r}\Big)\frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)}\|f\|_{\mathcal{M}^{\Phi,\varphi_1}}\\ &\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}}.\end{aligned}$$

2. We will now prove the necessity. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.11 we have $r_0^\alpha |b(x) - b_{B_0}| \lesssim M_{b,\alpha}\chi_{B_0}(x)$. Therefore, by Lemma 3.8 and Lemma 4.2

$$\begin{aligned} r_0^\alpha &\lesssim \frac{\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}}{\|b(\cdot)-b_{B_0}\|_{L^\Psi(B_0)}} \lesssim \frac{1}{\|b\|_*}\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}\Psi^{-1}(\mu(B_0)^{-1})\\ &\lesssim \frac{1}{\|b\|_*}\varphi_2(r_0)\|M_{b,\alpha}\chi_{B_0}\|_{\mathcal{M}^{\Psi,\varphi_2}} \lesssim \varphi_2(r_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi_1}} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}.\end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.
3. The third statement of the theorem follows from the first and second parts of the theorem.

□

By (3.20) and Theorems 5.2 and 6.2 we get the following corollary.

Corollary 6.3 *Let* $0 < \alpha < Q$*,* $\Phi \in \Delta_2 \cap \nabla_2$*,* $\Psi \in \Delta_2$*,* $\varphi_1 \in \mathcal{G}_\Phi$*,* $\varphi_2 \in \mathcal{G}_\Psi$*,* $b \in BMO(X)$ *and* $b^- \in L^\infty(X)$*. Let also the conditions* (3.11) *and* (6.3) *are satisfied. Then the operator* $[b, M_\alpha]$ *is bounded from* $\mathcal{M}^{\Phi,\varphi_1}(X)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(X)$.

The following theorem shows that $b \in BMO(X)$ is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Theorem 6.4 *Let (X, d, μ) be $Q-$homogeneous, $0 \le \alpha < Q$, $b \in L^1_{\mathrm{loc}}(X)$, Φ, Ψ be Young functions, $\varphi_1 \in \mathcal{G}_\Phi$ is almost decreasing and $\varphi_2 \in \mathcal{G}_\Psi$. Assume that there exists a positive constant C such that, for all $r > 0$,*

$$\varphi_1(r)r^\alpha \ge C\varphi_2(r). \tag{6.5}$$

Then the condition $b \in BMO(X)$ is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Proof Suppose that $M_{b,\alpha}$ is bounded from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$. Choose any ball $B = B(x, r)$ in X, by Lemmas 2.4 and 4.2 and (6.5) we have

$$\begin{aligned}
&\frac{1}{\mu(B)}\int_B |b(y) - b_B| d\mu(y) \\
&\quad \le \frac{1}{\mu(B)^{1+\frac{\alpha}{Q}}}\int_B \frac{1}{\mu(B)^{1-\frac{\alpha}{Q}}}\int_B |b(y) - b(z)|\chi_B(z)d\mu(z)d\mu(y) \\
&\quad \le \frac{1}{\mu(B)^{1+\frac{\alpha}{Q}}}\int_B M_{b,\alpha}\big(\chi_B\big)(y)d\mu(y) \lesssim \frac{1}{\mu(B)^{\frac{\alpha}{Q}}}\Psi^{-1}(\mu(B)^{-1})\|M_{b,\alpha}\chi_B\|_{L^\Psi} \\
&\quad \lesssim \frac{\varphi_2(r)}{\mu(B)^{\frac{\alpha}{Q}}}\|M_{b,\alpha}\chi_B\|_{\mathcal{M}^{\Psi,\varphi_2}} \lesssim \frac{\varphi_2(r)}{r^\alpha}\|\chi_B\|_{\mathcal{M}^{\Phi,\varphi_1}} \lesssim \frac{\varphi_2(r)}{\varphi_1(r)r^\alpha} \lesssim 1.
\end{aligned}$$

Thus $b \in BMO(X)$. □

By Theorems 6.2 and 6.4 we have the following characterization of $BMO(X)$.

Theorem 6.5 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $b \in L^1_{\mathrm{loc}}(X)$, $\Phi \in \Delta_2 \cap \nabla_2$, $\Psi \in \Delta_2$, $\varphi_1 \in \mathcal{G}_\Phi$ is almost decreasing and $\varphi_2 \in \mathcal{G}_\Psi$. Assume that conditions (3.11), (6.4) and $\varphi_1(r)r^\alpha \approx \varphi_2(r)$ hold. Then the condition $b \in BMO(X)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.*

6.2 Adams-type result

The following theorem gives a characterization for the boundedness of the operator $M_{b,\alpha}$ on generalized Orlicz–Morrey spaces.

Theorem 6.6 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $\Phi \in \Delta_2$, $\varphi \in \mathcal{G}_\Phi$ be almost decreasing, $b \in BMO(X)$, $\beta \in (0, 1)$, $\eta(r) \equiv \varphi(r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$.*

1. *If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies (4.2), then the condition*

$$r^\alpha\varphi(r) + \sup_{r<t<\infty}\Big(1 + \ln\frac{t}{r}\Big)\varphi(t)t^\alpha \le C\varphi(r)^\beta \tag{6.6}$$

   for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$.

2. *The condition* (5.9) *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi}(X)$ *to* $\mathcal{M}^{\Psi,\eta}(X)$.
3. *Let* $\Phi \in \nabla_2$. *If* φ *satisfies the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\varphi(t)t^{\alpha} \le C r^{\alpha}\varphi(r) \tag{6.7}$$

for all $r>0$, *where* $C>0$ *does not depend on* r, *then the condition* (5.9) *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi}(X)$ *to* $\mathcal{M}^{\Psi,\eta}(X)$.

Proof For arbitrary $x_0 \in X$, set $B := B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 := f\chi_{2kB}$ and $f_2 := f\chi_{\complement_{(2kB)}}$, where k is the constant from the triangle inequality (1.1).

If we proceed as in Theorem 3.12, for all $x \in B$ we have

$$M_{b,\alpha}(f_2)(x) \lesssim J_1 + J_2 + J_3,$$

where J_1, J_2 and J_3 are same as in Theorem 3.12.

Applying Hölder's inequality, by (2.4), (3.8), Lemma 3.8 and Lemma 2.4 we get

$$\begin{aligned}
J_1 + J_2 &\lesssim \sup_{t>2r} t^{\alpha-Q} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)|d\mu(y) \\
&\quad + \sup_{t>2r} t^{\alpha-Q} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|d\mu(y) \\
&\lesssim \sup_{t>2r} t^{\alpha-Q} \left\| b(\cdot) - b_{B(x_0,t)} \right\|_{L^{\tilde{\Phi}}(B(x_0,t))} \|f\|_{L^{\Phi}(B(x_0,t))} \\
&\quad + \sup_{t>2r} t^{\alpha-Q} |b_{B(x_0,r)} - b_{B(x_0,t)}|\, t^{Q}\, \Phi^{-1}\big(t^{-Q}\big) \|f\|_{L^{\Phi}(B(x_0,t))} \\
&\lesssim \|b\|_* \sup_{t>2r} \Phi^{-1}(t^{-Q})\, t^{\alpha} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L^{\Phi}(B(x_0,t))} \\
&\lesssim \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}} \sup_{t>2r} \Big(1+\ln\frac{t}{r}\Big) t^{\alpha}\varphi(t).
\end{aligned}$$

Taking into account (3.13), for all $x \in B$ we get

$$\begin{aligned}
J_0(x) + J_1 + J_2 &\lesssim \|b\|_* r^{\alpha} M_b f(x) \\
&\quad + \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}} \sup_{t>2r} \Big(1+\ln\frac{t}{r}\Big) t^{\alpha}\, \varphi(t).
\end{aligned}$$

Thus, by (6.6) we obtain

$$\begin{aligned}
J_0(x) + J_1 + J_2 &\lesssim \|b\|_* \min\{\varphi(r)^{\beta-1} M_b f(x), \varphi(r)^{\beta} \|f\|_{\mathcal{M}^{\Phi,\varphi}}\} \\
&\lesssim \|b\|_* \sup_{s>0} \min\{s^{\beta-1} M_b f(x), s^{\beta} \|f\|_{\mathcal{M}^{\Phi,\varphi}}\} \\
&= \|b\|_* (M_b f(x))^{\beta}\, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}.
\end{aligned}$$

Hence for every $x \in B$ we have

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_*(M_b f(x))^\beta \, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}. \tag{6.8}$$

By using the inequality (6.8) we have

$$\|J_0(\cdot) + J_1 + J_2\|_{L^\Psi(B)} \lesssim \|b\|_* \|(M_b f)^\beta\|_{L^\Psi(B)} \, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}.$$

Note that from (2.1) we get

$$\int_B \Psi\left(\frac{(M_b f(x))^\beta}{\|M_b f\|_{L^\Phi(B)}^\beta}\right) d\mu(x) = \int_B \Phi\left(\frac{M_b f(x)}{\|M_b f\|_{L^\Phi(B)}}\right) d\mu(x) \le 1.$$

Thus $\|(M_b f)^\beta\|_{L^\Psi(B)} = \|M_b f\|_{L^\Phi(B)}^\beta$. Therefore, we have

$$\|J_0(\cdot) + J_1 + J_2\|_{L^\Psi(B)} \lesssim \|b\|_* \|M_b f\|_{L^\Phi(B)}^\beta \, \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}.$$

By Lemma 3.8, Lemma 2.4 and condition (6.6), we also get

$$\begin{aligned}
\|J_3\|_{L^\Psi(B)} &= \left\| \sup_{t>2r} \frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}} \int_{B(x_0,t)} |b(\cdot) - b_B||f(y)|d\mu(y) \right\|_{L^\Psi(B)} \\
&\approx \|b(\cdot) - b_B\|_{L^\Psi(B)} \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |f(y)|d\mu(y) \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(\mu(B)^{-1}\big)} \sup_{t>2r} \Phi^{-1}(t^{-Q})\, t^\alpha \, \|f\|_{L^\Phi(B(x_0,t))} \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(\mu(B)^{-1}\big)} \|f\|_{\mathcal{M}^{\Phi,\varphi}} \sup_{t>2r} t^\alpha \varphi(t) \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(\mu(B)^{-1}\big)} \|f\|_{\mathcal{M}^{\Phi,\varphi}} \varphi(r)^\beta.
\end{aligned}$$

Consequently by using Theorem 4.4, we get

$$\begin{aligned}
\|M_{b,\alpha} f\|_{\mathcal{M}^{\Psi,\eta}} &= \sup_{x_0\in X, r>0} \eta(r)^{-1} \Psi^{-1}(\mu(B)^{-1}) \|M_{b,\alpha} f\|_{L^\Psi(B)} \\
&\lesssim \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \left(\sup_{x_0\in X, r>0} \varphi(r)^{-1} \Phi^{-1}(\mu(B)^{-1}) \|M_b f\|_{L^\Phi(B)} \right)^\beta \\
&\quad + \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}} \\
&\lesssim \|b\|_* \|f\|_{\mathcal{M}^{\Phi,\varphi}}.
\end{aligned}$$

We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.11 we have $r_0^\alpha |b(x) - b_{B_0}| \lesssim M_{b,\alpha}\chi_{B_0}(x)$. Therefore, by Lemma 3.8 and Lemma 4.2

$$
\begin{aligned}
r_0^\alpha &\lesssim \frac{\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L^\Psi(B_0)}} \lesssim \frac{1}{\|b\|_*}\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi(B_0)}\Psi^{-1}(\mu(B_0)^{-1}) \\
&\lesssim \frac{1}{\|b\|_*}\eta(r_0)\|M_{b,\alpha}\chi_{B_0}\|_{\mathcal{M}^{\Psi,\eta}} \lesssim \eta(r_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \\
&\lesssim \frac{\eta(r_0)}{\varphi(r_0)} \lesssim \varphi(r_0)^{\beta-1}.
\end{aligned}
$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and second parts of the theorem. □

Remark 6.7 As observed in Remark 3.16, we can compare Theorem 6.6 with [11, Theorem 6.4].

By (3.20) and Theorems 5.3 and 6.6 we get the following corollary.

Corollary 6.8 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $\Phi \in \Delta_2 \cap \nabla_2$, $\varphi \in \mathcal{G}_\Phi$ be almost decreasing, $b \in BMO(X)$, $b^- \in L^\infty(X)$, $\beta \in (0, 1)$, $\eta(r) \equiv \varphi(r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$. Let also conditions (4.2) and (6.6) are satisfied. Then the operator $[b, M_\alpha]$ is bounded from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$.*

Similar to Theorem 6.4 we can show that $b \in BMO(X)$ is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$. The details are omitted.

Theorem 6.9 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $b \in L^1_{\mathrm{loc}}(X)$, Φ, Ψ be Young functions, $\varphi \in \mathcal{G}_\Phi$ is almost decreasing, $\beta \in (0, 1)$, $\eta(r) \equiv \varphi(r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$. Assume that there exists a positive constant C such that, for all $r > 0$,*

$$
r^\alpha\varphi(r) \geq C\varphi(r)^\beta.
$$

Then the condition $b \in BMO(X)$ is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$.

By Theorems 6.6 and 6.9 we have the following characterization of $BMO(X)$.

Theorem 6.10 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$, $b \in L^1_{\mathrm{loc}}(X)$, $\Phi \in \Delta_2 \cap \nabla_2$, $\varphi \in \mathcal{G}_\Phi$ is almost decreasing and $\beta \in (0, 1)$, $\eta(r) \equiv \varphi(r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$. Assume that conditions (6.7) and $r^\alpha\varphi(r) \approx \varphi(r)^\beta$ hold. Then the condition $b \in BMO(X)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(X)$ to $\mathcal{M}^{\Psi,\eta}(X)$.*

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Compliance with ethical standards

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