# Covering and Separation for Permutations and Graphs 

Belinda Wickes

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School of Mathematical Sciences<br>Queen Mary, University of London<br>United Kingdom

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## Statement of Originality

I, Belinda Wickes, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:
All the material in Part I was conducted in collaboration with my supervisor Dr. Robert Johnson. The work in Part II is solely my own.

Much of the content of Chapter 1, and all the work in Chapters 2, 3, and 4 have been published previously:

- J. R. Johnson, and B. Wickes. Shattering $\boldsymbol{k}$-sets with Permutations. Order (2023).

The material in Chapter 8 has been accepted for publication, to appear as:

- B. Wickes. Separating Path Systems for the Complete Graph. Discrete Mathematics.


## Abstract

This is a thesis of two parts, focusing on covering and separation topics of extremal combinatorics and graph theory, two major themes in this area. They entail the existence and properties of collections of combinatorial objects which together either represent all objects (covering) or can be used to distinguish all objects from each other (separation). We will consider a range of problems which come under these areas.

The first part will focus on shattering $k$-sets with permutations. A family of permutations is said to shatter a given $k$-set if the permutations cover all possible orderings of the $k$ elements.

In particular, we investigate the size of permutation families which cover $t$ orders for every possible $k$-set as well as study the problem of determining the largest number of $k$-sets that can be shattered by a family with given size. We provide a construction for a small permutation family which shatters every $k$-set. We also consider constructions of large families which do not shatter any triple.

The second part will be concerned with the problem of separating path systems. A separating path system for a graph is a family of paths where, for any two edges, there is a path containing one edge but not the other. The aim is to find the size of the smallest such family.

We will study the size of the smallest separating path system for a range of graphs, including complete graphs, complete bipartite graphs, and lattice-type graphs. A key technique we introduce is the use of generator paths - constructed to utilise the symmetric nature of $K_{n}$. We continue this symmetric approach for bipartite graphs and study the limitations of the method. We consider lattice-type graphs as an example of the most efficient possible separating systems for any graph.

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## Introduction

Covering and separation are both key themes in the area of extremal combinatorics. In general, extremal combinatorics is the study of a collection of objects that satisfy some given property, and in particular is the study of the size of such a collection. The typical framing of a question in this area would be - how large or small can a family of objects be, given that together they satisfy a certain condition? The classical example of this is Mantel's Theorem, which answers the question: How many edges can a graph on $n$ vertices have, given that it contains no triangles?

The objects in question usually have an intrinsic structure, which may not be directly related to the condition we are trying to satisfy, and are comprised of a number of components (e.g. edges in a graph or elements in a set system). They are also usually finite, although we tend to be interested in the cases where they are arbitrarily large. Standard examples of combinatorial objects include graphs, permutations, set systems, codes, and posets. The conditions we consider are typically a requirement for certain relationships between objects to be achieved (e.g. ensuring no set is a subset of another in the collection), but can also be thought of as properties of sub-structures within the objects (e.g. ensuring a graph contains no copy of a triangle).

In this thesis we will consider two types of combinatorial object - permutations and graphs - and two conditions - covering and separating. Both are classical examples of a combinatorial problem and are closely related to each other - where covering is used to represent every object, and separating to distinguish all objects. Each will be self contained in its own part of the thesis, with a thorough introduction of the problem along with all formal definitions and background. Each part will explore a variety of different settings within the confines of the problem, and conclude with open directions for further study.

We will first look at a covering problem on permutations. A covering condition ensures that all copies of a prescribed sub-structure appear in our family of objects. The particular
problem we will be considering is families of permutations of $n$ elements which shatter every $k$-set. Shattering is a covering condition which stipulates that a given set of $k$ elements has all its possible orderings covered by the family.

This problem originated in set theory, where shattering a $k$-set $S$ means that all possible subsets of $S$ can be found by taking the intersection of $S$ with some set from the family. The most well known result in this area is the Sauer-Shelah Lemma on VC-dimension, which determines the maximum number of sets in a family given the maximum size of a shattered set. It also has a direct relation to families which are $k$-independent, another famous problem of set theory. In general, set-theoretic problems often have interesting analogues regarding permutations, which are typically less well understood, and shattering is no exception. Inevitably, the techniques and ideas used to understand the set-theoretic versions do not carry over, and a different approach is required to solve the analogous permutation problems.

We will introduce the notion of partial shattering, an alteration of the problem where only $t$ orderings need to be covered as opposed to all $k$ !. We compare this to other variations of the problem, and determine the minimum size of a family which partially shatters every 3 -set for all values of $t$. We also extend these arguments to the case where $k>3$. We will also consider the question of determining the largest proportion of $k$-sets which can be shattered by a family of fixed size, providing construction techniques and exploring the uses of certain known families.

The order of magnitude for the smallest shattering families is known when $k>3$, but only through probabilistic methods, and there is a lack of any constructions which approach this bound. To this end we provide an example construction which is significantly smaller than any trivial known shattering family, although still does not match the known smallest size. Finally, we look at possible extremal examples of families which are large but do not shatter any 3 -set, constructing many large non-shattering families matching the conjectured bound.

In the second part of the thesis, we consider a separation problem on graphs. A separation condition ensures the objects in our family can be used to differentiate between all their components. Separating systems also have their origins in set theory, with the principal notion being that two elements of some ground set are separated by a subset of that ground set if it contains one element but not the other. It is essentially trivial that separating all pairs of elements from an $n$ element ground set requires $\log _{2}(n)$ subsets, and moreover requires no more than $\log _{2}(n)$ subsets. When we consider our sets as elements of a graph
$G$ however, we gain properties from the inherent structure of $G$. This inherited structure makes the problem more complex but also more interesting.

We will study separating path systems of a graph $G$ - a collection of paths in $G$ such that, for any two edges of $G$, the collection contains a path covering one of the edges but not the other. This means we are able to distinguish every edge of $G$ by the paths which contain it. It is easy to see that every graph has a separating path system, the edge set of $G$ is one for instance since a single edge is also a path. Naturally, we are interested in the smallest possible separating path system of $G$, that is one which uses the fewest paths.

We will investigate the question of finding the smallest separating path system for a variety of natural classes of graph: complete graphs, bipartite graphs, and lattice-type graphs. In each case, providing upper bounds through constructions, and lower bounds by counting arguments. We make use of the symmetry of $G$ in the case of complete graphs and complete bipartite graphs, by introducing the technique of using a generator path. In the case of the complete graph, we are able to completely reduce the problem to that of finding one generator path.

## Part I

## Shattering

## Chapter 1

## Introduction

### 1.1 Background and previous work

Let $S_{n}$ be the set of all permutations of $\{1, \ldots, n\}$ thought of as ordered $n$-tuples. Our aim is to study properties of families of permutations from $S_{n}$ inspired by concepts of shattering from extremal set theory. We begin with the notion of shattering for sets.

Let $\mathcal{F}$ be a family of subsets of $[n]=\{1,2, \ldots, n\}$ and let $A \subseteq[n]$, we say that $A$ is shattered by $\mathcal{F}$ if for each $B \subseteq A$ there exists a set $S \in \mathcal{F}$ such that $A \cap S=B$. The notion of shattered sets has uses throughout combinatorics and computer science, with the focus being on the size of families that shatter certain sets. In particular we have the of VC-dimension of a family $\mathcal{F}$, which is the maximum size of a set $A$ which is shattered by $\mathcal{F}$. For examples of work in this area see [1], [14], [25], [26], and [29]. The most standout result here is the Sauer-Shelah Lemma (found independently in [25], [26], and [29]) which states that $|\mathcal{F}|=O\left(n^{k}\right)$ if $\mathcal{F}$ has VC-dimension $k$.

A family $\mathcal{R}$ of subsets of $[n]$ is $k$-independent if any $R_{1}, R_{2}, \ldots, R_{k} \in \mathcal{R}$ have the property that all $2^{k}$ intersections $\cap_{i=1}^{k} J_{i}$ are non-empty, where $J_{i}$ takes on either $R_{i}$ or its complement $R_{i}^{c}$. A large $k$-independent family $\mathcal{R}$ gives rise to a small family $\mathcal{F}$, where $\mathcal{F} \subseteq 2^{[r]}$ is a family that shatters all the $k$-subsets of $[r]$ and $|\mathcal{R}|=r$. We can think of $\mathcal{R}$ as the dual to the family $\mathcal{F}$. To see this, first note that for each $x \in[n]$ we can define a set $F(x) \subseteq[r]$ by setting $i \in F(x)$ if and only if $x \in R_{i}$. Since $\mathcal{R}$ is $k$-independent, a family consisting of all such $F(x)$ sets must shatter all $k$-subsets of $[r]$. Set $\mathcal{F}=\{F(x): x \in[n]\}$ then clearly $|\mathcal{F}|=n$, so the bigger $r$ is, the greater the number of $k$-subsets shattered by $n$ sets. Kleitman and Spencer [14] posed the question 'How large can a family of $k$-independent sets be?' which is equivalent to the question 'How small can a family that shatters every
$k$-subset of $[n]$ be?'.

Theorem 1.1 (Kleitman and Spencer [14]). For fixed $k$ and $n$ sufficiently large there are absolute constants $d_{1}$ and $d_{2}$ such that

$$
2^{k} d_{2} \log n \leq g_{k}(n) \leq k 2^{k} d_{1} \log n
$$

where $g_{k}(n)$ is the size of the smallest family from $2^{[n]}$ that shatters every $k$-subset of $[n]$.
Our starting point is an analogue of this family size problem using permutations in place of sets. This has been studied under a variety of names ([12], [21], [27], [28], [31]): completely scrambling permutation families, mixing permutations, and sequence covering arrays. We first formalise the problem, establish notation, and summarise the previous work.

Consider the set $S_{n}$ of all permutations of [ $n$ ], any permutation $P \in S_{n}$ corresponds to a particular linear order of the elements of $[n]$, so $P$ can be written as

$$
P=\left(p_{1}, p_{2}, \ldots, p_{n}\right) \text { where }\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}=[n]
$$

Note that this is not cycle notation, rather it can be thought of as the second line of two-row notation. We remark that all permutations and fixed orderings will be written in this format from now on, all sets and $k$-tuples will be written using standard set notation. For all $a, b \in[n]$, we write

$$
a<_{P} b
$$

to mean that $a$ precedes $b$ in the permutation $P \in S_{n}$.
Suppose $R$ is a permutation of $[k]$. For a $k$-tuple $\left\{a_{1}, \ldots, a_{k}\right\} \subseteq[n]$ with $a_{1}<\cdots<a_{k}$ and a $P \in S_{n}$, we say $\left\{a_{1}, \ldots, a_{k}\right\}$ follows the pattern $R \in S_{k}$ in $P$ if $a_{i}<_{P} a_{j}$ for all $i, j \in[k]$ with $i<_{R} j$. That is, $\left\{a_{1}, \ldots, a_{k}\right\}$ follows the pattern $R$ in $P$ if the restriction of $P$ to $\left\{a_{1}, \ldots, a_{k}\right\}$ is order isomorphic to $R$. For any $k$-tuple $X$ from [ $n$ ], we use $P_{X}$ to denote the permutation pattern from $S_{k}$ followed by $X$ in $P \in S_{n}$. We can express our shattering condition through permutation patterns.

Definition 1.2. We say that a family $\mathcal{S} \subseteq S_{n}$ shatters the $k$-tuple $X \subseteq[n]$ if

$$
\left\{P_{X}: P \in \mathcal{S}\right\}=S_{k}
$$

In other words, $\mathcal{S}$ shatters $X$ if every possible ordering of the elements of $X$ appears in the permutations of $\mathcal{S}$.

Example 1.3. The following family of permutations from $S_{5}$ shatters every triple from [5]:

$$
\begin{array}{llll}
(1,2,3,4,5) & (2,4,1,5,3) & (5,3,4,1,2) & (3,1,2,5,4) \\
(1,4,3,5,2) & (4,2,3,5,1) & (5,3,2,1,4) & (5,1,2,4,3) .
\end{array}
$$

We are interested in families that shatter many different sets at the same time, in particular when all sets of the same size are shattered by one family. Let $f_{k}(n)$ be the smallest integer such that there exists a family $\mathcal{S}$ of permutations from $S_{n}$ that shatters every $k$-tuple from $[n]$ and has $|\mathcal{S}|=f_{k}(n)$. Throughout, if a family is said to shatter every $k$-tuple in [ $n$ ], then it is implied that said family is a subset of $S_{n}$.

Clearly we have that $f_{k}(n) \geq\left|S_{k}\right|$, otherwise we certainly cannot shatter any $k$-tuple. It is also plain that $f_{k}(k)=k$ ! since the family $S_{k}$ is suitable.

The order of magnitude of the value $f_{k}(n)$ is known asymptotically. The upper bound given by Spencer [27] can be seen with a simple probabilistic argument, the lower bound is more involved and was shown by Radhakrishnan [21]

$$
\left(\frac{(k-1)!}{\log e}+o(1)\right) \log n \leq f_{k}(n) \leq \frac{k}{\log k!-\log (k!-1)} \log n .
$$

To simplify the notation throughout, $\log (n)=\log _{2}(n)$ unless otherwise stated.
In the case where $k=3$ the best upper bound is actually given by a construction rather than using the probabilistic method. The construction is given by Tarui in [28] and the lower bound from Füredi 12

$$
\frac{2}{\log e} \log n \leq f_{3}(n) \leq 2 \log n+(1+o(1)) \log \log n
$$

For $k>3$ no explicit construction that matches the order of magnitude of the probabilistic upper bound is known.

Our main focus will be partial and fractional variations of this problem, which we introduce in the next section.

The study of shattering with permutations is appealing in its own right, providing a range of interesting combinatorial structures and problems. It also follows in the footsteps of many natural set theoretic questions which have an analogue to combinatorial problem in terms of permutations requiring different techniques to the used. However, shattering as defined in 1.2 also has a particular practical application for manufacturing and computer science in the form of event sequence testing (see [15]).

Event sequence testing aims to provide a means of checking for failures or malfunctions of a sequential process before commencing full operation. Many procedures are event based and ordered, meaning that during a process several steps or events take place one by one, and the order which they occur may have an (unwanted) effect on the outcome or cause the software to fail or become dangerous. For instance a certain fault may only occur after particular devices have been connected.

Often the events are due to user interaction with the system, plugging in or connecting a device or choosing options on an interface. Therefore, even if a certain order of operations is prescribed, human error cannot be eliminated from the procedure. To mitigate the occurrence of such failures the system should be thoroughly tested to check any malfunctions due to the changes in the sequence of events. Clearly for a process which involves $n$ events, $n!$ tests must be conducted to check every eventuality. This is heavily time consuming, especially in the case where a user must input every event manually. If we expect that the errors in sequence ordering are likely to be small compared to the total number of events, as would be the case with user error, then we need only test the orders of a smaller number of events. That is, for a process with a total of $n$ possible events, where we expect there to be at most $k$ ordering errors, we only need to check the orders of every set of $k$ events from the $n$ possible choices. Clearly, this is equivalent to finding small families of permutations from $S_{n}$ which shatter every $k$-tuple from $[n]$.

### 1.2 Partial and fractional shattering

As well as the problem of determining the smallest size of a permutation family that shatters every $k$-tuple, it is natural to ask about small families that cover some subset of orders for every $k$-tuple. Equally it is natural to consider small families that shatter some partial collection of $k$-tuples. Our main aim is to introduce and investigate these variations of the original family size problem. There are two different problems we will consider.

- Given $t \leq k$ !, find the smallest family of permutations from $S_{n}$ which ensures each $k$-tuple appears in at least $t$ orders.
- Given $\alpha \in[0,1]$, find the smallest family that shatters at least $\alpha\binom{n}{k}$ of all $k$-tuples.

Note that there is another immediate definition of partial shattering. The problem is, given some fixed set of patterns $T \subseteq S_{k}$, find the smallest family of permutations from $S_{n}$ such that every $k$-tuple follows all of the patterns in $T$. It turns out that for any $\mathcal{S} \subseteq S_{n}$
in which all $k$-tuples follow a specific non-monotone pattern, we have $|\mathcal{S}|=O(\log n)$ (see Lemma 2.3). This matches the lower bound for total shattering, making this variation uninteresting.

For our first problem we formally define partial shattering as follows, using the same notation as Definition 1.2 .

Definition 1.4. We say that a family $\mathcal{S} \subseteq S_{n} \boldsymbol{t}$-shatters the $k$-tuple $X \subseteq[n]$ if we have $\left|\left\{P_{X}: P \in \mathcal{S}\right\}\right| \geq t$. We define $f_{k}(n, t)$ to be the smallest integer such that there exists a family $\mathcal{S}$, with $|\mathcal{S}|=f_{k}(n, t)$, that $t$-shatters every $k$-tuple in $[n]$.

Example 1.5. The following family of permutations from $S_{5} 3$-shatters every triple from [5]:

$$
(1,2,3,4,5) \quad(2,5,4,1,3) \quad(5,3,2,1,4) .
$$

Indeed, for each triple we have the following orderings contained in the above family:

| $\{1,2,3\}$ | $\{1,2,4\}$ | $\{1,2,5\}$ | $\{1,3,4\}$ | $\{1,3,5\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $(1,2,3)$ | $(1,2,4)$ | $(1,2,5)$ | $(1,3,4)$ | $(1,3,5)$ |
| $(2,1,3)$ | $(2,4,1)$ | $(2,5,1)$ | $(4,1,3)$ | $(5,1,3)$ |
| $(3,2,1)$ | $(2,1,4)$ | $(5,2,1)$ | $(3,1,4)$ | $(5,3,1)$ |
| $\{1,4,5\}$ | $\{2,3,4\}$ | $\{2,3,5\}$ | $\{2,4,5\}$ | $\{3,4,5\}$ |
| $(1,4,5)$ | $(2,3,4)$ | $(2,3,5)$ | $(2,4,5)$ | $(3,4,5)$ |
| $(5,4,1)$ | $(2,4,3)$ | $(2,5,3)$ | $(2,5,4)$ | $(5,4,3)$ |
| $(5,1,4)$ | $(3,2,4)$ | $(5,3,2)$ | $(5,2,4)$ | $(5,3,4)$. |

Clearly $f_{k}(n, k!)$ is the size of the smallest shattering family on $k$-tuples, $f_{k}(n, k!)=f_{k}(n)$. We also have the trivial cases $f_{k}(n, 1)=1$ and $f_{k}(n, 2)=2$ which can be seen by taking only monotone permutations.

Spencer [27] and Füredi [12] also discuss variations which look to cover a subset of patterns of each $k$-tuple where each element appears in a specific place. Neither of these variations can be expressed in terms of partial $t$-shattering or vice versa. However, in [12] Füredi additionally defines an extremely general framework called $\mathcal{S}$-mixing, we note that partial $t$-shattering is one of the most natural special instances of this $\mathcal{S}$-mixing.

For our second problem it is more natural to ask the question in reverse than it is to fix the fraction. We ask for the maximum number of shattered $k$-tuples from a family of
fixed size.

Definition 1.6. Let $F_{k}(n, m)$ be the largest $\alpha \in[0,1]$ such that there exists a collection $\mathcal{R}$ of exactly $m$ permutations from $S_{n}$, with the property that $\alpha\binom{n}{k} k$-tuples are totally shattered by $\mathcal{R}$. We call this fractional shattering of $n$ with $m$ permutations.

Example 1.7. Below is an example of a family of 6 permutations which shatters 8 out of 10 possible triples from [5]. The only triples not shattered by the below permutations are $\{1,2,5\}$ and $\{3,4,5\}$.

$$
\begin{array}{lll}
(5,1,2,3,4) & (2,4,1,3,5) & (3,5,4,1,2) \\
(1,4,5,3,2) & (4,2,5,3,1) & (3,2,1,5,4)
\end{array}
$$

This shows that $F_{3}(5,6) \geq \frac{4}{5}$. In fact we have that $F_{3}(5,6)=\frac{4}{5}$, which we will see in Section 3.3.

It is plain that $F_{k}(n, m)=0$ whenever $m<k!$ and that $F_{k}(k, k!)=1$. In fact, by a result of Levenshtein [18] we have that $F_{k}(k+1, k!)=1$.

This instance of shattering is genuinely different from partial shattering, and showcases different behaviour as a result. The aim of partial shattering is to deal with rates of growth, while fractional shattering enables us to shatter a fixed fraction of all $k$-tuples using only a constant number of permutations.

### 1.3 Structure of Part I

The remainder of Part I will be structured as follows. We conclude this chapter with some known results which will be frequently used throughout the remaining chapters of this part.

We focus on partial shattering in Chapter 2, beginning by showing that fixing a nonmonotone pattern which all $k$-tuples follow results in a large $(\Theta(\log n))$ family of permutations. We then consider the case where $k=3$ and classify the size of $f_{3}(n, t)$ asymptotically for all values of $t \in[6]$. We are able to show that there are three distinct regimes depending on $t$ :

$$
\Theta(\log n), \quad \Theta(\log \log n), \quad \text { or constant. }
$$

We follow up with the extension to larger values of $k$. The same separation into three distinct size categories follows over to the $k>3$ cases, although in this setting there is the possibility that another size class exists for $t \in[k+1,2(k-1)!]$.

A natural question is then, whether or not $f_{k}(n, t)$ always one of the three sizes we see here or if there is another class. This highlights the interesting question of how many size classes there are for such shattering problems in general. We note that for all the permutation shattering variants discussed above and in [12, 21, 28, 27], the smallest known families realising each has size $\Theta(\log n), \Theta(\log \log n)$, or constant.

In Chapter 3 we move on to the fractional version of the problem. Again we focus on the case where $k=3$, the first interesting case is $F_{3}(n, 6)$ where we show that

$$
\frac{17}{42} \leq F_{3}(n, 6) \leq \frac{47}{60}
$$

In fact our method gives a slightly stronger upper bound which is hard to quantify but shows that the upper bound is in fact strict when $n>10$. We also show that in general the value of $F_{k}(n, m)$ is decreasing in $n$. This means that the limit for $F_{k}(n, m)$ exists for fixed $k, m$ and as $n$ tends to infinity. We see that this limit lies strictly between 0 and 1 for all $k$ and $m$.

We also look at a method of iterating small families in such a way as to give a family of permutations on much larger $n$, that preserves most of the shattering conditions from the initial family. We give some bounds using this method and some small starting families we call perfect families.

A perfect family for $k$ on $n$, denoted $\mathcal{Q}_{k}(n)$, is a family of exactly $k$ ! permutations from $S_{n}$ which shatter every $k$-tuple in $[n]$ (a realisation of the family giving $F_{k}(n, k!)=1$ ). An example of a family $\mathcal{Q}_{3}(4)$ is given by the following:

$$
\begin{array}{lll}
Q_{1}=(1,2,3,4) & Q_{2}=(2,4,1,3) & Q_{3}=(3,4,1,2) \\
Q_{4}=(1,4,3,2) & Q_{5}=(4,2,3,1) & Q_{6}=(3,2,1,4) .
\end{array}
$$

We note that Levenshtein's result in [18] can be expressed as showing a perfect family for $k$ on $n$ exists whenever $n=k+1$.

We also note that these perfect families are of interest in design theory with a more generalised notion of what it means to be perfect. In [31] Yuster defines a perfect shattering family as one where each ordering of every $k$-tuple appears among the permutations exactly $\lambda$ times. Then our notion of a perfect family is the most natural instance of this where $\lambda=1$.

We then have a construction for the original total shattering problem, that is an upper bound for $f_{k}(n)$. The best bound, of order $\log n$, is given by a probabilistic argument and no explicit construction of this size is known. Our construction gives a family with
a power of $\log n$ permutations, this is above the known upper bound for $f_{k}(n)$ but is constructive.

Finally we explore a question in the opposite direction - constructing the largest family of permutations from $\mathcal{S}_{n}$ which does not shatter any triple. This is an analogue of the VC-dimension of sets. In 2000 Raz [23] showed that, for some undetermined universal constant $C$, the upper bound on a set that does not shatter any triple is $C^{n}$. It is suggested in [23] that a possible bound is the Catalan number

$$
C_{n}=\frac{1}{n+1}\binom{2 n}{n} .
$$

This implies that a possible extremal example of such a non-shattering family is the family that contains every permutation of $[n]$ with no decreasing triple. In other words, the family that contains all permutations of $n$ except those that contain a copy of the pattern $(3,2,1)$. For background on this family see [5]. We introduce a construction method for producing non-shattering families of this candidate extremal size, but which have avoid different patterns and have different properties to the known example.

We finish with a summary and some concluding remarks, as well as a few open problems about the topics covered.

### 1.4 Useful results

The following are well known but useful results that will be used throughout.

Theorem 1.8 (Erdős-Szekeres Theorem [9]). Let $r, s \in \mathbb{N}$, then any sequence of real numbers with length at least $n=r s+1$ contains an increasing subsequence of length at least $r+1$ or a decreasing subsequence with length at least $s+1$.

Lemma 1.9. Let $(A, B)$ be a partition of $[n]$, so $A \cup B=[n]$ and $A \cap B=\emptyset$. Any family $\mathcal{U}$ of such partitions with the property that, for every $x, y \in[n]$ there exists $(A, B) \in \mathcal{U}$ where exactly one of $x$ and $y$ is in $A$ and the other is in $B$, also satisfies $|\mathcal{U}| \geq\lceil\log n\rceil$. Furthermore there exists such a family $\mathcal{U}$ where the bound holds with equality.

Lemma 1.10 (Chung, Graham, and Winkler [8]). Let $(A, B)$ be a partition of [ $n$ ], so $A \cup B=[n]$ and $A \cap B=\emptyset$. Any family $\mathcal{U}$ of these partitions with the property that, for every $x, y \in[n]$ there exists $(A, B) \in \mathcal{U}$ where $x \in A$ and $y \in B$, must satisfy
$|\mathcal{U}| \geq\left\lceil\log n+\left(\frac{1}{2}+o(1)\right) \log \log n\right\rceil$. Moreover, there exists such a family with $|\mathcal{U}|=$ $\left\lceil\log n+\left(\frac{1}{2}+o(1)\right) \log \log n\right\rceil$.

### 1.4.1 Levenshtein's perfect family construction

Here we will outline the construction of a perfect family $\mathcal{Q}_{n}(n+1)$ given by Levenshtein in [18]. The inclusion of this is partly for completeness, and partly due to the fact that the paper [18] is not readily available in an English translation.

The construction works by extending each element of $S_{n}$ to a unique element of $S_{n+1}$, therefore the resulting family of permutations has the correct size $\left|S_{n}\right|=n$ !. In fact, the construction partitions $S_{n+1}$ into $n+1$ different perfect families. First we clarify some notation for this section.

We denote continuous sections of any vector $u=\left(u_{1}, \ldots, u_{m}\right)$ between coordinates $i \leq j$ by $u_{i}, \ldots, u_{j}$. Vectors composed of (possibly multiple) subsections of $u$ may then be written as $\left(u_{i}, \ldots, u_{j}\right)$ or $\left(u_{i}, \ldots, u_{j}, u_{k}, \ldots, u_{\ell}\right)$ accordingly, where $i \leq j \leq k \leq \ell$. We denote the number of 1 s in a binary vector $\left(z_{1}, \ldots, z_{j}\right) \in[0,1]^{j}$ by

$$
\left\|\left(z_{i}, \ldots, z_{j}\right)\right\|_{1}=\sum_{k=i}^{j} z_{k},
$$

and the number of 0 s by

$$
\left\|\left(z_{i}, \ldots, z_{j}\right)\right\|_{0}=j-(i-1)-\left\|\left(z_{i}, \ldots, z_{j}\right)\right\|_{1} .
$$

We will also use the value $W(z)$ for a binary vector $z=\left(z_{1}, \ldots, z_{n-1}\right) \in[0,1]^{n-1}$ where

$$
W(z)=\sum_{k=1}^{n-1} k z_{k} .
$$

This construction partitions $S_{n+1}$ into $n+1$ perfect families by choice of the value $a \in$ $[n+1]$. So, for each choice of $a \in[n+1]$, the construction of a perfect family $\mathcal{F}_{a}(n)$ is as follows.

For each binary vector $z=\left(z_{1}, \ldots, z_{n-1}\right)$ we define

$$
r(z)=a-W(z) \quad \bmod n+1
$$

We also set the values $\sigma(z)$ and $j(z)$ by

$$
\sigma(z)= \begin{cases}0 & \text { if } 0 \leq r(z) \leq\|z\|_{1} \\ 1 & \text { if }\|z\|_{1}<r(z) \leq n\end{cases}
$$

and

$$
j(z)= \begin{cases}\text { the largest } j \text { such that }\left\|\left(z_{j}, \ldots, z_{n-1}\right)\right\|_{1}=r(z) & \text { when } \sigma(z)=0 \\ \text { the largest } j \text { such that }\left\|\left(z_{j}, \ldots, z_{n-1}\right)\right\|_{0}=n-r(z) & \text { when } \sigma(z)=1\end{cases}
$$

Let $\left\lfloor S_{n+1}\right\rfloor$ denote the set of $n$ length vectors of distinct elements of $[n-1]$. For any $v=\left(v_{1}, \ldots, v_{n}\right) \in\left\lfloor S_{n+1}\right\rfloor$ let

$$
R(v)=z=\left(z_{1}, \ldots, z_{n-1}\right) \in\{0,1\}^{n-1}
$$

be a binary vector which encodes the increases and decreases of $v$ via

$$
z_{i}= \begin{cases}0 & \text { if } v_{i}<v_{i+1} \\ 1 & \text { if } v_{i}>v_{i+1}\end{cases}
$$

Our perfect family will be given by extending each vector $v \in S_{n}$ to a vector $u \in S_{n+1}$ by adding the element $n+1$ in such a way that if $R(v)=z$ and

$$
x=\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)
$$

then $R(u)=x$.
Moreover, this is equivalent to

$$
\mathcal{F}_{a}(n)=\left\{u \in S_{n+1}: W(R(u)) \equiv a \quad \bmod n+1\right\} .
$$

Example 1.11. Let $n=4$ and choose $a=5$. Take the vector $v=(2,3,1,4) \in S_{4}$, we will extend this vector to a vector of $S_{5}$ by adding the element 5 to $v$ following the construction above.

First note that $R(v)=(0,1,0)=z$ and hence $\|z\|_{1}=1, W(z)=2$ and $r(z)=5-2$ $\bmod 5=3$. Then we have that $\sigma(z)=1$ and $j(z)=3$, giving us the extended vector $x=(0,1,1,0)$.

We must now choose a place to add the element 5 into $v=(2,3,1,4)$ such that the resulting increase/decrease vector is $x$. Adding 5 into the position $(2, \star, 3,1,4)$ does this, giving $u=(2,5,3,1,4)$, and hence $R(u)=(0,1,1,0)=x$.

Note that no other placement of 5 allows the increase/decrease vector to equal $x$. Indeed, $R((5,2,3,1,4))=(1,0,1,0), R((2,3,5,1,4))=(0,0,1,0), R((2,3,1,5,4))=(0,1,0,1)$, and $R((2,3,1,4,5))=(0,1,0,0)$. Moreover, each of these extensions relate to a different
choice of $a \in[5]$. We chose $a=5$ and as a result we have that $W(x)=2+3 \equiv 0 \bmod 5$. Checking the $W$ value for the other placements show that each extension corresponds to a different $a$ value. Observe, $W((1,0,1,0)) \equiv 4, W((0,0,1,0)) \equiv 3, W((0,1,0,1)) \equiv 1$, and $W((0,1,0,0)) \equiv 2$.

The family of vectors from $S_{5}$ generated by applying this method of extension to all vectors of $S_{4}$ with $a=5$ gives us a family containing $u=(2,5,3,1,4)$ which shatters every 4 -tuple and has size $\left|S_{4}\right|=24$. This family is found below.

| $(1,2,3,4,5)$ | $(2,1,3,5,4)$ | $(3,1,2,5,4)$ | $(4,1,2,5,3)$ |
| :--- | :--- | :--- | :--- |
| $(5,1,2,4,3)$ | $(2,1,4,5,3)$ | $(3,1,4,5,2)$ | $(4,1,3,5,2)$ |
| $(1,5,3,2,4)$ | $(2,5,3,1,4)$ | $(3,5,2,1,4)$ | $(4,5,2,1,3)$ |
| $(5,1,3,4,2)$ | $(5,2,3,4,1)$ | $(3,2,4,5,1)$ | $(4,2,3,5,1)$ |
| $(1,5,4,2,3)$ | $(2,5,4,1,3)$ | $(3,5,4,1,2)$ | $(4,5,3,1,2)$ |
| $(1,4,3,2,5)$ | $(2,4,3,1,5)$ | $(3,4,2,1,5)$ | $(5,4,3,2,1)$. |

In order to prove that the construction provided behaves the way we intend, we must show three things:

- That the extension is well defined.
- That the family of extensions $u$ is indeed equivalent to $\left\{u \in S_{n+1}: W(R(u)) \equiv a\right.$ $\bmod n+1\}$.
- That the deletion of any single element from a vector of $\mathcal{F}_{a}(n)$ gives a unique permutation of the remaining elements (i.e. a permutation that cannot be found by deleting an element of another vector of $\mathcal{F}_{a}(n)$ ). This is enough to ensure that each $n$-tuple is shattered.

We will first confirm that $\sigma(z)$ and $j(z)$ are well defined. For $\sigma(z)$, note that $\|z\|_{1}$ is the number of 1 s in the binary vector $z$ of length $n-1$, therefore $\|z\|_{1}$ is a non-negative integer with value $0 \leq\|z\|_{1} \leq n-1$. Now observe that $r(z)$ is an integer modulo $n+1$, and therefore is a non-negative integer with value $0 \leq r(z) \leq n$. Clearly $\sigma(z)$ is well defined, the range covers all possible values of $r(z)$, and the strict inequality is unambiguous since $\|z\|_{1}<n$.

To see that $j(z)$ is well defined, note that if $\sigma(z)=0$ then by definition $r(z) \leq\|z\|_{1}$, and $j(z)$ is defined whenever there exists an integer $j$ where $\left\|\left(z_{j}, \ldots, z_{n-1}\right)\right\|_{1}=r(z)$. Clearly, such a $j$ must exist since $\left\|\left(z_{1}, \ldots, z_{n-1}\right)\right\| \geq r(z)$ and $\left\|\left(z_{n-1}\right)\right\| \leq 1$. When $\sigma(z)=1$, then by definition we have $\|z\|_{1}<r(z) \leq n$. Rearranging gives $n-\|z\|_{1}>n-r(z) \geq 0$ which
is equivalent to $\|z\|_{0} \geq n-r(z) \geq 0$, and in this case $j(z)$ is defined whenever there exists an integer $j$ where $\left\|\left(z_{j}, \ldots, z_{n-1}\right)\right\|_{0}=n-r(z)$. The remainder of this case is analogous to the first case and shows that $j(z)$ is well defined for all values of $\sigma(z)$.

We will now see that the set of all vectors $u \in S_{n+1}$ which are extended from a $v \in S_{n}$ such that $R(u)=\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)$ where $R(v)=z$ is equivalent to the set $\left\{u \in S_{n+1}: W(R(u)) \equiv a \bmod n+1\right\}$.

Let $u$ and $v$ be as above and observe that $W(R(u)) \equiv a \bmod n+1$,

$$
\begin{aligned}
W(x) & =\sum_{k=1}^{j(z)-1} k z_{k}+j(z) \sigma(z)+\sum_{k=j(z)}^{n-1}(k+1) z_{k} \\
& =W(z)+j(z) \sigma(z)+\left\|\left(z_{j(z)}, \ldots, z_{n-1}\right)\right\|_{1} \\
& =W(z)+r(z) \equiv a \bmod n+1 .
\end{aligned}
$$

Now let $u^{\prime} \in S_{n+1}$ and suppose $W\left(R\left(u^{\prime}\right)\right) \equiv a \bmod n$. It is plain that $u^{\prime}$ is an extension of some $v \in S_{n}$, where $v$ is given by deleting the element $n+1$. Therefore we have that $u^{\prime}=\left(v_{1}, \ldots, v_{k}, n+1, v_{k+1}, \ldots, v_{n}\right)$ for some $k \in[n]$ and hence $R\left(u^{\prime}\right)=$ $\left(z_{1}, \ldots, z_{k-1}, 0,1, z_{k+1}, \ldots, z_{n-1}\right)$, where $R(v)=z$. Since we must have $z_{k} \in\{0,1\}$ we see that $R\left(u^{\prime}\right)$ is an extension of $z$ by one element. So we write $R\left(u^{\prime}\right)=y=\left(y_{1}, \ldots, y_{n}\right)$ and moreover, we have

$$
z=\left(y_{1}, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n}\right)
$$

for some $i \in[n]$. We will show that $y=x$, where $x=\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)$. Moreover, this argument holds whenever $y$ is an $n$ length binary vector and $z$ is given by deleting a single element of $y$ (not using properties inherited from $v$ or $u$ ). This means that $\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)$ is the unique extension of any binary vector $\left(z_{1}, \ldots, z_{n-1}\right)$ such that $W\left(\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)\right) \equiv a \bmod n+1$.

Observe that

$$
W(y)=W(z)+i y_{i}+\left\|\left(y_{i+1}, \ldots, y_{n}\right)\right\|_{1} \equiv a \quad \bmod n+1
$$

If $y_{i}=0$ the above gives $\left\|\left(y_{i+1}, \ldots, y_{n}\right)\right\|_{1}=r(z)$. Clearly we have that $\left\|\left(y_{i+1}\right), \ldots, y_{n}\right\|_{1} \leq$ $\|z\|_{1}$ since in this case we have $\|y\|_{1}=\|z\|_{1}$. Meaning that, $r(z) \leq\|z\|_{1}$ and hence $\sigma(z)=0$. By the definition of $j(z)$ we have that $j(z) \geq i+1$ and that $z_{i+1}, \ldots, z_{j(z)-1}=0$, therefore $y=x$. If $y_{i}=1$ we have that $W(z)+n-\left\|\left(y_{i+1}, \ldots, y_{n}\right)\right\|_{0} \equiv a \bmod n$ which gives $n-\left\|\left(y_{i+1}, \ldots, y_{n}\right)\right\|_{0}=r(z)$. Clearly, $\left\|\left(y_{i+1}, \ldots, y_{n}\right)\right\|_{0} \leq\|z\|_{0}=n-1-\|z\|_{1}$, meaning $\|z\|_{1}+1 \leq r(z)$. This gives $\sigma(z)=1$ and by an analogous process as in the first case this forces $y=x$.

Let $v \in\left\lfloor S_{n+1}\right\rfloor$, we will show that there is exactly one vector in $\mathcal{F}_{a}$ which is an extension of $v$. This is equivalent to showing that $\mathcal{F}_{a}$ shatters all every $n$-tuple. Let $R(v)=$ $z$ and let $\lceil v, \sigma(z), j(z)\rceil$ be the set of all extensions $u \in S_{n+1}$ of $v$ such that $R(u)=$ $\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)$. First we will see that $|\lceil v, \sigma(z), j(z)\rceil|=1$, and then that $\lceil v, \sigma(z), j(z)\rceil$ is exactly the set of all extensions of $v$ which are in $\mathcal{F}_{a}$ which proves the result.

Consider the segment of $v$ given by $v_{i}, \ldots, v_{j(z)}$ where $i \leq j(z)$ is such that $z_{i-1} \neq \sigma(z)$ and $z_{i}, \ldots, z_{j(z)-1}=\sigma(z)$. Note that by definition we also have $z_{j(v)} \neq \sigma(v)$. Since $v \in\left\lfloor S_{n+1}\right\rfloor$ the entries of $v$ are $n$ distinct elements of $[n+1]$, and hence there is some $b \in[n+1]$ with $v_{i} \neq b$ for all $i \in[n]$. To extend $v$ to a vector from $S_{n+1}$ we must add the element $b$ to some position of $v$. We need this extension $u$ to be such that $R(u)=\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)$, and therefore $b$ must be added to the segment $v_{i}, \ldots, v_{j(z)}$. Indeed, if $b$ is added to the segment $v_{1}, \ldots, v_{i-2}$ then we have that the $i$ th entry of $R(u)$ is $z_{i-1}$, but we require the $i$ th entry to be $\sigma(z)$. Similarly, if $b$ is added to the segment $v_{j(z)+2}, \ldots, v_{n}$ then the $j(z)$ th entry is $z_{j(z)}$, but we require this entry to be $\sigma(z)$.

Then the value of $|\lceil v, \sigma(z), j(z)\rceil|$ is exactly the number of ways to add $b$ to $\left(v_{i}, \ldots, v_{j(z)}\right)$ such that the resulting vector $w$ has $R(w)=(\sigma(z), \ldots, \sigma(z))$. In other words $w$ is either an increasing or decreasing sequence (depending on the value of $\sigma(z)$ ), it is plain that there is exactly one position to add $b$ which results in such a sequence. Therefore $|\lceil v, \sigma(z), j(z)\rceil|=$ 1.

It remains to show that the set of all vectors of $\mathcal{F}_{a}$ which are extensions of $v$, is the set $\lceil v, \sigma(z), j(z)\rceil$. First, let $u \in\lceil v, \sigma(z), j(z)\rceil$, then we have that

$$
R(u)=\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)
$$

Observe that

$$
\begin{aligned}
W(R(u)) & =W(z)+j(z) \sigma(z)+\left\|\left(z_{j(v)}, \ldots, z_{n-1}\right)\right\|_{1} \\
& =W(z)+r(z) \equiv a \quad \bmod n+1
\end{aligned}
$$

by definition of $\sigma(z)$ and $j(z)$. Since $\mathcal{F}_{a}(n)=\left\{u \in S_{n+1}: W(R(u)) \equiv a \bmod n+1\right\}$ this means that $u \in \mathcal{F}_{a}$.

Now suppose that $u$ is an extension of $v$ and that $u \in \mathcal{F}_{a}$. Since $u$ extends $v$ we may write $u=\left(v_{1}, \ldots, v_{h}, b, v_{h+1}, \ldots, v_{n}\right)$ where $b \neq v_{i}$ for any $i \in[n]$ and $b \in[n+1]$. Let $R\left(\left(v_{h}, b\right)\right)=\alpha$ and $R\left(\left(b, v_{h+1}\right)\right)=\beta$, then

$$
R(u)=\left(z_{1}, \ldots, z_{h-1}, \alpha, \beta, z_{h+1}, \ldots, z_{n-1}\right)
$$

It is plain that either $z_{h}=\alpha$ or $z_{h}=\beta$, and hence

$$
R(u)= \begin{cases}\left(z_{1}, \ldots, z_{h}, \beta, z_{h+1}, \ldots, z_{n-1}\right) & \text { if } z_{h}=\alpha \\ \left(z_{1}, \ldots, z_{h-1}, \alpha, z_{h}, \ldots, z_{n-1}\right) & \text { if } z_{h}=\beta .\end{cases}
$$

So we can rewrite this as $R(u)=\left(z_{1}, \ldots, z_{t-1}, \delta, z_{t+1}, \ldots, z_{n-1}\right)$. Clearly this means that $R(u)$ is an extension of $z$ by one element, and since $u \in \mathcal{F}_{a}$ we have that $W(R(u)) \equiv a$ $\bmod n+1$. Recall that $\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)$ is the unique extension of $z$ with $W(x) \equiv a \bmod n+1$. Therefore

$$
R(u)=\left(z_{1}, \ldots, z_{j(z)-1}, \sigma(z), z_{j(z)}, \ldots, z_{n-1}\right)
$$

and $u \in\lceil v, \sigma(z), j(z)\rceil$.

## Chapter 2

## Partially shattering every $\boldsymbol{k}$-tuple

Recall that we have the following definition, where $P_{X}$ is the permutation pattern which $X$ follows in $P$.

Definition 1.4. We say that a family $\mathcal{S} \subseteq S_{n} \boldsymbol{t}$-shatters the $k$-tuple $X \subseteq[n]$ if we have $\left|\left\{P_{X}: P \in \mathcal{S}\right\}\right| \geq t$. We define $f_{k}(n, t)$ to be the smallest integer such that there exists a family $\mathcal{S}$, with $|\mathcal{S}|=f_{k}(n, t)$, that $t$-shatters every $k$-tuple in $[n]$.

Our aim in this section is to prove the following results, showing that the behaviour of $f_{3}(n, t)$ and $f_{k}(n, t)$ falls into distinct regimes.

Theorem 2.1. We have the following bounds on $f_{3}(n, t)$

$$
f_{3}(n, t)= \begin{cases}t & \text { for } t=1,2 \\ \Theta(\log \log n) & \text { for } t=3,4 \\ \Theta(\log n) & \text { for } t=5,6\end{cases}
$$

Theorem 2.2. We have the following bounds on $f_{k}(n, t)$

$$
f_{k}(n, t)= \begin{cases}t & \text { when } t=1,2 \\ \Theta(\log \log n) & \text { when } t \in[3, k] \\ \Theta(\log n) & \text { when } t \in[2(k-1)!+1, k!] .\end{cases}
$$

The upper bound when $t=k$ ! comes from Spencer's total shattering bound for $f_{k}(n)$ [27]. The value of $f_{k}(t)$ whenever $t=1,2$ is trivial. Indeed note that any one single permutation requires each $k$-tuple to follow some pattern (not necessarily the same pattern). So we
get that $f_{k}(n, 1)=1$ simply by choosing any $P \in S_{n}$. We call $\bar{P}$ the reverse permutation of $P$ if $a<_{\bar{P}} b$ whenever $b<_{P} a$. All $k$-tuples follow a different pattern in $\bar{P}$ than they do in $P$, therefore we must have that $f_{k}(n, 2)=2$.

### 2.1 Lower bounds

We begin by providing a lemma which shows that whenever we require a family of permutations to follow a fixed non-monotone pattern, that family has the same order of magnitude as a totally shattering family. This is one of the reasons we have chosen to define partially shattering as in Definition 1.4 .

Lemma 2.3. Let $n \geq 3$ and $R \in S_{k}$ be any non-monotone permutation pattern. If $\mathcal{S} \subseteq S_{n}$ is a family of permutations for which every $k$-tuple follows $R$ in at least one $P \in \mathcal{S}$, then we must have $|\mathcal{S}| \geq \log (n-k+2)$. Hence $|\mathcal{S}|=\Omega(\log n)$.

Proof. Let $\mathcal{S}$ and $R$ be as described in the statement of the lemma. Note that whenever $R$ is non-monotone there is some element $x \in[k]$ such that when $R$ is restricted to $\{x, x+1, x+2\}$ the triple is non-monotone.

If $R$ induces $(x+1, x, x+2)$ or $(x+2, x, x+1)$ then set $y=x$. If $R$ induces $(x, x+2, x+1)$ or $(x+1, x+2, x)$ then set $y=n-k+x+2$.

For each $P \in \mathcal{S}$ we generate a partition of the set $W:=[n] \backslash([x-1] \cup[n-k+x+3, n] \cup y)$ into two parts $A_{P}$ and $B_{P}$, where $A_{P}$ contains every $w \in W$ such that $y<_{P} w$ and $B_{P}=W \backslash A_{P}$.

Note that when $y=x$ the set $W$ contains only elements larger than or equal to $y$, and when $y=n-k+x+2$ the set $W$ only contains smaller or equal elements. Hence for any pair $a, b \in W$ we must have one of the orders $(a, y, b)$ or ( $b, y, a$ ) appearing in $\mathcal{S}$ since exactly one of them follows $R$ as part of the $k$-tuple $\{1,2, \ldots, x-1, y, a, b, n-k+x+3, \ldots, n\}$. To see this note that the elements that correspond to $\{x, x+1, x+2\}$ are exactly $\{y, a, b\}$. The family of partitions $\left\{\left(A_{P}, B_{P}\right): P \in \mathcal{S}\right\}$ satisfies the conditions for Lemma 1.9 and we must have that $|\mathcal{S}| \geq \log |W|$, which gives the result.

To get a lower bound for our partial shattering problem we need to work a little harder when $t \geq 3$. It does not complicate the method to consider the general $k$-tuples rather than just triples so we proceed in this generality. Observe that taking $P$ to be the increasing permutation $(1,2, \ldots, n-1, n)$ we get that $\bar{P}$ is the decreasing permutation. Now consider
a third permutation $P^{\prime}$, any $k$-tuple contained in a monotone subpermutation of $P^{\prime}$ will not follow a new pattern. We know by the Erdős-Szekeres Theorem that when $n$ is large we must have some reasonably large monotone subpermutation. We use this idea to get the lower bound in this case.

Theorem 2.4. For any $n \geq 3$ and every $t \geq 3$, we have $f_{k}(n, t) \geq \log \log n-C$ where $C$ is a constant dependant on $k$ and $t$. More precisely we have

$$
f_{k}(n, t) \geq \log \log (n-1)-\log \log (k-1)+t-3
$$

Proof. Let $\mathcal{S}$ be a family of permutations of $[n]$ that $t$-shatters every $k$-tuple. Suppose for a contradiction that $|\mathcal{S}| \leq \log \log (n-1)-\log \log (k-1)+t-4$.

Choose any $t-3$ permutations from $\mathcal{S}$, and set $\mathcal{S}^{\prime}$ to be the $\mathcal{S}$ with the chosen permutations removed. This is a family of permutations from $S_{n}$ that 3 -shatters every $k$-tuple. Note that $\left|\mathcal{S}^{\prime}\right| \leq \log \log (n-1)-\log \log (k-1)-1=m$.

Take any $P_{1} \in \mathcal{S}^{\prime}$, then by the Erdős-Szekeres Theorem $P_{1}$ must contain an increasing subsequence of length $r=\left\lfloor(n-1)^{\frac{1}{2}}\right\rfloor+1$ (or a decreasing subsequence of length $r$ ). Let the elements in this monotone subsequence be written as $X_{1}=\left\{x_{1}, \ldots, x_{r}\right\}$, and let $P_{1}\left(X_{1}\right)$ be the restriction of $P_{1}$ to the elements of $X_{1}$. Then $P_{1}\left(X_{1}\right)$ is simply a permutation of $X_{1}$ where the elements appear in the same order that they appear in $P_{1}$.

Look at another permutation $P_{2} \in \mathcal{S}^{\prime}$ restricted to the elements of $X_{1}, P_{2}\left(X_{1}\right)$. Applying Erdős-Szekeres again, this time to $P_{2}\left(X_{1}\right)$, we see that there must be an monotonic subsequence of length $\left\lfloor(r-1)^{\frac{1}{2}}\right\rfloor+1$. Let $X_{2} \subseteq X_{1}$ be the set of elements in this subsequence. Then consider $P_{3}\left(X_{2}\right)$ and generate $X_{3} \subseteq X_{2}$ in an analogous manner.

Take each permutation from $\mathcal{S}^{\prime}$ into consideration one by one, at step $i$ generate a set of 'bad' elements $X_{i} \subseteq X_{i-1}$ by applying Erdős-Szekeres to $P_{i}\left(X_{i-1}\right)$ and finding a monotone subsequence of length at least $\left\lfloor\left(\left|X_{i-1}\right|-1\right)^{\frac{1}{2}}\right\rfloor+1$.

Consider the set $X_{m}$, it must contain elements that appear in a monotone subsequence of every permutation in $\mathcal{S}^{\prime}$. In other words, the elements of $X_{m}$ appear in a maximum of 2 possible orders. Therefore if $\left|X_{m}\right| \geq k$ then $X_{m}$ contains a $k$-tuple that does not appear in 3 orders across $\mathcal{S}^{\prime}$. Hence, from our initial conditions we must have $\left|X_{m}\right|<k$.

Note that the number of elements we are restricting to in the final stage is at most

$$
(n-1)^{\left(\frac{1}{2}\right)^{m}}+1 .
$$

Then observe

$$
\begin{aligned}
(n-1)^{\frac{1}{2^{m}}}+1 & <k \\
\log \log (n-1)-\log \log (k-1) & <m
\end{aligned}
$$

On the other hand, based on the assumed size of $\mathcal{S}$ we have that $m=\log \log (n-1)-$ $\log \log (k-1)-1$, a contradiction. Therefore we must have

$$
|\mathcal{S}| \geq \log \log (n-1)-\log \log (k-1)+t-3
$$

From this we can see that if $t \geq 3$ then $f_{k}(n, t)$ is always between $\Omega(\log \log n)$ and $O(\log n)$. Recall that $O(\log n)$ is the bound for $f_{k}(n)$ - the total shattering problem.

### 2.2 The case $k=3$

Recall we wish to prove the following.
Theorem 2.1. We have the following bounds on $f_{3}(n, t)$

$$
f_{3}(n, t)= \begin{cases}t & \text { for } t=1,2 \\ \Theta(\log \log n) & \text { for } t=3,4 \\ \Theta(\log n) & \text { for } t=5,6\end{cases}
$$

We have seen the lower bounds for $f_{3}(n, t)$ when $t=1,3$ and the upper bounds when $t=2,6$. It remains to lower bound $f_{3}(n, t)$ when $t=5$ and upper bound $f_{3}(n, t)$ for $t=4$.

We are able to show that there are values of $t \geq 3$ which allow $f_{3}(n, t)$ to match the order of magnitude of Theorem 2.4.

The next result gives an upper bound on $f_{3}(n, 4)$, first we show the bound in Theorem 2.6 using the recursion of Lemma 2.5, then we give the construction that provides the recursion.

Lemma 2.5. For $n \geq 3$ we have that $f_{3}\left(n^{n}, 4\right) \leq f_{3}(n, 4)+\log n+\left(\frac{1}{2}+o(1)\right) \log \log n+2$. Therefore we get the following bound.

Theorem 2.6. For large $n$ we have $\log \log n \leq f_{3}(n, 4) \leq 2 \log \log n$

Proof. The lower bound is directly from Theorem [2.4 with $k=3$ and $t=4$.
For the upper bound, write $n=\left(m^{m}\right)^{m^{m}}$ for some real number $m$, noting that $m$ may not be an integer. Then from Lemma 2.5 we have

$$
\begin{aligned}
f_{3}(n, 4) & \leq f_{3}\left(\left\lceil m^{m}\right\rceil, 4\right)+\log \left\lceil m^{m}\right\rceil+\left(\frac{1}{2}+o(1)\right) \log \log \left\lceil m^{m}\right\rceil+2 \\
& \leq f_{3}\left((m+1)^{m+1}, 4\right)+\log m^{m}+\log \log m^{m}+3 \\
& \leq f_{3}(m+1,4)+\log (m+1)+\log \log (m+1)+3+\log m^{m}+\log \log m^{m}+3 \\
& \leq f_{3}(m+1,4)+\log m+\log \log m+\log m^{m}+\log \log m^{m}+8
\end{aligned}
$$

we can now use Tarui's upper bound for $f_{3}(n)$ in [28]

$$
\begin{aligned}
& \leq 2(\log (m+1)+\log \log (m+1))+\log \log m^{m}+\log \log \left(m^{m}\right)^{m^{m}}+8 \\
& \leq 2(\log m+\log \log m)+\log \log m^{m}+\log \log \left(m^{m}\right)^{m^{m}}+12 \\
& \leq 2 \log \log m^{m}+\log \log m^{m}+\log \log \left(m^{m}\right)^{m^{m}}+12 \\
& \leq \log \log n+3 \log \log m^{m}+12 \\
& \leq 2 \log \log n
\end{aligned}
$$

Now we see the construction that gives the recursion.

Proof of Lemma 2.5. Let $\mathcal{S}$ be a 4 -shattering family for triples in $[n]$ with $|\mathcal{S}|=f_{3}(n, 4)$. We will use this family to construct a new family from $S_{n^{n}}$ that 4 -shatters every triple.

Assign each $x \in\left[n^{n}\right]$ to a unique string $\left(x_{1}, \ldots, x_{n}\right) \in[n]^{n}$, by equating the standard order on $\left[n^{n}\right]$ with the lexicographic order on $[n]^{n}$. That is, $x=1$ is assigned to $(1,1, \ldots, 1,1,1)$, $x=n$ is assigned $(1,1, \ldots, 1,1, n), x=n+1$ is assigned $(1,1, \ldots, 1,2,1)$, and so on. We call $\left(x_{1}, \ldots, x_{n}\right)$ the code (or unique code) for $x$. Note that this is equivalent to the base $n$ notation of elements of $\left[n^{n}\right]$ but shifted by 1 , this shift makes notation simpler later on. Let $d:\left[n^{n}\right]^{2} \rightarrow[n]$ be the function giving the first coordinate that differs between two elements, so $d(x, y)=\min \left\{i: x_{i} \neq y_{i}\right\}$. We will use these unique codes to generate two types of permutations on $\left[n^{n}\right]$.

Type 1. Here we apply permutations from $\mathcal{S}$. We will generate one permutation $P^{n}$ of [ $n^{n}$ ] from each permutation $P \in \mathcal{S}$.

Consider any $P \in \mathcal{S}$, we can apply $P$ to any set of $n$ objects. In particular we can apply $P$ to each coordinate of the unique code of every $x \in\left[n^{n}\right]$, call the resulting string the $P$-permuted unique code of $x$. Having found the $P$-permuted unique code of every $x \in\left[n^{n}\right]$, we get a permutation $P^{n} \in S_{n^{n}}$ by considering the order on $\left[n^{n}\right]$ given by the lexicographic order on the $P$-permuted unique codes of each $x \in\left[n^{n}\right]$.

The result is that for elements $x, y \in\left[n^{n}\right]$ with $d(x, y)=i$, we have that $x$ precedes $y$ in $P^{n}$ if and only if $x_{i}$ precedes $y_{i}$ in $P$. In other words, $x<_{P^{n}} y$ if and only if $x_{i}<_{P} y_{i}$.

We do this for all $P \in \mathcal{S}$ which gives us $|\mathcal{S}|=f_{3}(n, 4)$ permutations of $\left[n^{n}\right]$, call this collection of permutations $\mathcal{S}^{n}$.

To see which triples are now 4 -shattered, consider the triple $\{x, y, z\}$ with the codes $\left(x_{1}, x_{2}, \ldots, x_{n}\right),\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ and $\left(z_{1}, z_{2}, \ldots, z_{n}\right)$ respectively. Suppose all three elements agree in the first $k \in[0, n-1]$ coordinates, so $x_{i}=y_{i}=z_{i}$ for $i \leq k$, and further suppose that none agree in coordinate $k+1$, that is $d(x, y)=d(x, z)=d(y, z)=k+1$. Then note that the order of $x, y, z$ in $P^{n}$ relies only on the order of $x_{k+1}, y_{k+1}, z_{k+1}$ in $P$. Since $\mathcal{S} 4$-shatters triples in [n] there must be permutations $P_{1}, P_{2}, P_{3}, P_{4} \in \mathcal{S}$ that 4 -shatter $\left\{x_{k+1}, y_{k+1}, z_{k+1}\right\}$, hence $P_{1}^{n}, P_{2}^{n}, P_{3}^{n}, P_{4}^{n}$ must 4-shatter $\{x, y, z\}$

The only triples that do not have 4 orders covered by permutations in $\mathcal{S}^{n}$ are those that have two elements that agree in the first $k$ coordinates of their unique code and the final element only agrees in the first $r$ coordinates where $r<k$. This is equivalent to triples $\{x, y, z\}$ with $d(x, y)=k$ and $d(x, z)=d(y, z)=r$, we will call such triples 'bad'. Note that for a 'bad' triple $\{x, y, z\}$ where $x<y<z$ we must have either $d(x, y)=k$ and $d(x, z)=d(y, z)<k$ or $d(y, z)=k$ and $d(x, y)=d(x, z)<k$. It cannot be that $d(x, z)=k$ and $d(x, y)=d(y, z)=r<k$ because $x<y<z$ means that $x_{r}<y_{r}<z_{r}$ but $d(x, z)=k$ implies $x_{r}=z_{r}$.

We can assume without loss of generality that $\mathcal{S}$ contains the monotone increasing order, therefore we can assume $\mathcal{S}^{n}$ contains it. We now construct the other collection of permutations to cover orders on these 'bad' triples.

Type 2. By Lemma 1.10 we are able to find $\left\lceil\log n+\left(\frac{1}{2}+o(1)\right) \log \log n\right\rceil$ partitions of $[n]$ into two sets $I$ and $D$ that satisfy the conditions in 1.10. In this section we will view $[n]$ as the set of coordinates for the unique codes, and therefore $I$ and $D$ as partitions of the coordinates. For each partition we will define one permutation in $S_{n^{n}}$.

Let $(I, D)$ be any of our partitions and start by considering the first coordinate. If
$1 \in I$ then we order the elements increasing by coordinate 1 , namely $x$ will precede $y$ if $x_{1}<y_{1}$. If $1 \in D$ then order elements with decreasing coordinates, meaning $x$ will precede $y$ if $x_{1}>y_{1}$. Next look at the second coordinate, if $2 \in I$ order such that $x$ precedes $y$ if $x_{2}<y_{2}$, otherwise order so $x$ precedes $y$ if $x_{2}>y_{2}$. Continue in this manner until the elements have been ordered with respect to each of their coordinates.

Formally, consider $x, y \in\left[n^{n}\right]$ and let $d(x, y)=i$, we have that $x$ precedes $y$ if

$$
\begin{cases}x_{i}<y_{i} & \text { and } i \in I \\ x_{i}>y_{i} & \text { and } i \in D\end{cases}
$$

We use the above process to construct $\left\lceil\log n+\left(\frac{1}{2}+o(1)\right) \log \log n\right\rceil$ permutations with the property that for any coordinates $i$ and $j$ we can always find two permutations such that one has $i$ increasing and $j$ decreasing and the other has $i$ decreasing and $j$ increasing.

Let $\mathcal{T}$ be the set of these Type 2 permutations along with the permutation of $\left[n^{n}\right]$ that is totally decreasing (if this is not already included by the two types), then $|\mathcal{T}| \leq\lceil\log n+$ $\left.\left(\frac{1}{2}+o(1)\right) \log \log n\right\rceil+1$.

To identify the triples that are partially shattered by $\mathcal{T}$, consider a triple $\{x, y, z\}$ where $x<y<z$ which was not 4 -shattered by $\mathcal{S}^{n}$. As discussed after Type 1 there are two cases, either $d(x, y)=i$ and $d(x, z)=d(y, z)=j$ where $j<i$, or $d(y, z)=i$ and $d(x, y)=d(x, z)=j$ for $j<i$.

Suppose $d(x, y)=i$ and $d(x, z)=d(y, z)=j$ where $j<i$. We know from our assumption that the order $(x, y, z)$ appears in $\mathcal{S}^{n}$, we also know that $(z, y, x)$ appears in $\mathcal{T}$ since we included the decreasing permutation here. Furthermore we know that there is some permutation in $\mathcal{T}$ where $i$ is increasing and $j$ is decreasing. Since $x<z$ and $d(x, z)=j$ we must have $x_{j}<z_{j}$ from the construction of the unique codes, similarly we have $x_{i}<y_{i}$. Then for any permutation with $i$ increasing, $x$ must appear before $y$, and $j$ decreasing means $x$ (and $y$ since $x_{j}=y_{j}$ ) comes after $z$. This means that the order $(z, x, y)$ is covered. Similarly there is a permutation where $i$ is decreasing and $j$ is increasing, giving the order $(y, x, z)$. By the same reasoning, if we are in the second case where $d(y, z)=i$ and $d(x, y)=d(x, z)=j$, then we find the orders $(x, y, z),(z, y, x),(y, z, x)$ and $(x, z, y)$.

We now have our desired partial shattering condition, the family $\mathcal{S}^{n} \cup \mathcal{T}$ covers 4 orders for each triple in $\left[n^{n}\right]$.

Thus

$$
f_{3}\left(n^{n}, 4\right) \leq\left|\mathcal{S}^{n}\right|+|\mathcal{T}|=f_{3}(n, 4)+\left\lceil\log n+\left(\frac{1}{2}+o(1)\right) \log \log n\right\rceil+1
$$

We now have all the ingredients needed to prove Theorem 2.1.

Proof of Theorem 2.1. Clearly $f_{3}(n, 1)=1$ as any $P \in S_{n}$ forces all triples from [ $n$ ] to appear in one order. Recall the reverse permutation pf $P, \bar{P}$, then we must have that all triples follow a different pattern in $\bar{P}$ as they did in $P$, hence $f_{3}(n, 2)=2$.

The lower bound $f_{3}(n, 3) \geq \log \log (n-1)$ comes directly from Theorem 2.4. The upper bound $f_{3}(n, 4) \leq 2 \log \log n$ comes directly from Theorem 2.6. This gives us $f_{3}(n, t)=$ $\Theta(\log \log n)$ when $t=3,4$.

To see $f_{3}(n, 5) \geq \log (n-1)$, let $\mathcal{S}$ be a family of permutations from $S_{n}$ that 5 -shatters every triple. Consider any triple of the form $\{n, x, y\}$, we must have at least one of the orders $(x, n, y)$ and $(y, n, x)$ appearing in some permutation from $\mathcal{S}$ otherwise we have at most 4 orders for $\{n, x, y\}$. For each $P \in \mathcal{S}$ generate a partition of $[n-1]$ by having

$$
\begin{aligned}
& A_{P}:=\{x \in[n-1]: x \text { appears after } n \text { in } P\} \\
& B_{P}:=\{x \in[n-1]: x \text { appears before } n \text { in } P\} .
\end{aligned}
$$

Then using Lemma 1.9 in order to ensure at least one of the orders $(x, n, y)$ and $(y, n, x)$ is seen we must have at least $\log (n-1)$ permutations in $\mathcal{S}$.

Finally we have that $f_{3}(n, 6) \leq(2+o(1)) \log n$ from [28] since $f_{3}(n, 6)=f_{3}(n)$.

### 2.3 The case $k>3$

For triples, the different values of $t$ feed equally into the three size classifications. In the general $k$-tuple case, we actually have that for most values of $t$ we require $O(\log n)$ permutations.

Theorem 2.2. We have the following bounds on $f_{k}(n, t)$

$$
f_{k}(n, t)= \begin{cases}t & \text { when } t=1,2 \\ \Theta(\log \log n) & \text { when } t \in[3, k] \\ \Theta(\log n) & \text { when } t \in[2(k-1)!+1, k!]\end{cases}
$$

The value of $f_{k}(n, t)$ for $t \in[k+1,2(k-1)!]$ is unknown but does lie between $\log \log n$ and $O(\log n)$. An interesting further question here is if the cases always split into exactly these three orders, or is there a different behaviour for some $t \in[k+1,2(k-1)!]$ ?

We again have the trivial cases $t=1,2$. Since Theorem 2.4 was for general $k$ that result is still giving us the lower bound when $t=3$.

It is a direct consequence of a result of Spencer [27] that $f_{k}(n, k)=O(\log \log n)$. In fact Spencer proved the stronger claim that there exists a family $\mathcal{F}$ of permutations from $S_{n}$ with size $O(\log \log n)$ such that for every $k$-tuple $X$, and every $x \in X$, there is some $P \in \mathcal{F}$ with $x<_{P} y$ for all $y \in X \backslash x$. That is, not only does any $k$-tuple appear in at least $k$ orders, but each element in the $k$-tuple appears first in at least one order.

That leaves us with only the following result left to prove Theorem 2.2 .

Theorem 2.7. For fixed $k$ and when $n$ is large, we have that $f_{k}(n, t)=\Theta(\log n)$ whenever $t>\frac{2(k!)}{k}$.

Proof. Let $\mathcal{S}$ be a family that $(2(k-1)!+1)$-shatters every $k$-tuple. Consider $k$-tuples of the form $X:=\{x, y, n-k+3, \ldots, n-1, n\}$ for any $x, y \in[n-k+2]$.

Note that $x$ and $y$ must be split by at least one of $\{n-k+3, \ldots, n\}$ in some $P \in \mathcal{S}$. Indeed, there are only $2(k-1)$ ! ways to order $X$ such that $x$ and $y$ are consecutive, yet we know that $X$ appears in at least $2(k-1)!+1$ orders in $\mathcal{S}$.

Consider the following sets

$$
A_{x}^{i}:=\{P \in \mathcal{S}: x \text { appears after } i\}
$$

where $i \in[n-k+3, n]$. Then for any $x$ and $y$ there exists an $i \in[n-k+3, n]$ such that $A_{x}^{i} \neq A_{y}^{i}$.

For each $i \in[n-k+3, n]$ we define a partition of $[n-k+2]$ into at most $m$ parts, where elements $x, y \in[n-k+2]$ are in the same part if and only if $A_{x}^{i}=A_{y}^{i}$. Label the parts arbitrarily with labels $B_{1}^{i}, \ldots, B_{m}^{i}$ noting that some labels may not be used at all. Then we must have that

$$
m^{k-2} \geq n-k+2
$$

Indeed, we are able to write each element $x$ uniquely as a $k-2$ length string from [ $m$ ], $x=\left(x_{1}, x_{2}, \ldots, x_{k-2}\right)$ where $x_{i}=r$ if $x \in B_{r}^{i}$. To see that this does create a unique identification, consider a pair $x, y$ with the same string. We must have that $x$ and $y$ are
in the same $B^{i}$ part for all $i$, then from the definition of $B^{i} s$ that means $A_{y}^{i}=A_{x}^{i}$ for all $i$. We have already established that distinct $x, y$ must have $A_{y}^{i} \neq A_{x}^{i}$ for some $i$ so conclude that $y=x$.

By choosing the smallest possible $m$ we can assume that there is some $i$ such that the partition has exactly $m$ parts, that is

$$
\left|\left\{B_{1}^{i}, \ldots, B_{m}^{i}\right\}\right|=m
$$

Notice that the set

$$
\left\{A_{x}^{i}: x \in[n-k+2]\right\}
$$

must also have size $m$ since each $x \in B_{r}^{i}$ gives rise to the same set $A_{x}^{i}$. Therefore this set has size at least $(n-k+2)^{\frac{1}{k-2}}$ by our above bound on $m$ and hence

$$
(n-k+2)^{\frac{1}{k-2}} \leq 2^{|\mathcal{S}|}
$$

Giving us the result

$$
|\mathcal{S}| \geq \frac{1}{k-2} \log (n-k+2)
$$

## Chapter 3

## Totally shattering a fraction of all $k$-tuples

For this problem we have a fixed number of permutations and wish to know the largest proportion of $k$-tuples that can be shattered. Recall the definition.

Definition 1.6. Let $F_{k}(n, m)$ be the largest $\alpha \in[0,1]$ such that there exists a collection $\mathcal{R}$ of exactly $m$ permutations from $S_{n}$, with the property that $\alpha\binom{n}{k} k$-tuples are totally shattered by $\mathcal{R}$. We call this fractional shattering of $n$ with $m$ permutations.

We first focus on the case where $k=3$, the first interesting case is $F_{3}(n, 6)$ where we will show the following bounds.

Theorem 3.1. For any $n \geq 10$ we have

$$
\frac{17}{42} \leq F_{3}(n, 6) \leq \frac{47}{60}
$$

We also show that in general the value of $F_{k}(n, m)$ is decreasing in $n$.

Theorem 3.2. For fixed $k$ and fixed $m \geq k$ ! we have

$$
F_{k}(n, m) \geq F_{k}(n+1, m)
$$

This means that the limit for $F_{k}(n, m)$ exists for fixed $k, m$ and as $n$ tends to infinity, we see that this limit lies strictly between 0 and 1 for all $k$ and $m$.

### 3.1 Upper bound

First we show that the function $F_{k}(n, m)$ is weakly decreasing in $n$ when $k$ and $m$ are fixed such that $m \geq k!$.

Theorem 3.2. For fixed $k$ and fixed $m \geq k$ ! we have $F_{k}(n, m) \geq F_{k}(n+1, m)$.
Proof. Suppose $F_{k}(n, m)=\alpha$ and consider a family $\mathcal{S}$ of $m$ permutations from $S_{n+1}$. Let $X \subseteq[n+1]$ with $|X|=n$, then by considering the permutations of $\mathcal{S}$ restricted to the elements of $X$, we see there are at most $\alpha\binom{n}{k} k$-tuples shattered. In other words, at least $(1-\alpha)\binom{n}{k} k$-tuples from $X$ remain un-shattered by $\mathcal{S}$. Since this is true for any such $X$ we get that the number of un-shattered $k$-tuples in $[n+1]$ is at least

$$
\frac{(1-\alpha)\binom{n}{k}\binom{n+1}{n}}{\binom{n+1-k}{n-k}}=(1-\alpha)\binom{n+1}{k} .
$$

Therefore the number of $k$-tuples that are shattered by $\mathcal{S}$ is at most $\alpha\binom{n+1}{k}$. Therefore, $F_{k}(n+1, m) \leq \alpha$.

It is an easy observation that, for fixed $k$ and $m$, we can always find a suitably large value of $n$ such that every family of $m$ permutations from $S_{n}$ fails to shatter one $k$-tuple. Indeed, when $n \geq m^{2^{m}}$ we can apply the Erdős-Szekeres Theorem to see that there must be at least one un-shattered $k$-tuple. Therefore the limit of $F_{k}(n, m)$ in $n$ lies strictly between 0 and 1 for all $k$ and $m$, it is an interesting question to determine the value of this limit.

A direct consequence of this weakly decreasing behaviour is that the value of $F_{k}(N, m)$ for given fixed $N$ provides an upper bound on $F_{k}(n, m)$ where $n \geq N$. Consider the case when $k=3$, in particular we fix our family size at $m=6$ since this is the first non-trivial case.

Theorem 3.3. For any $n \geq 10$ we have that $F_{3}(n, 6) \leq \frac{47}{60}$.
Proof. We use the fact that a family of 6 permutations of [5] can shatter at most 8 triples, so $F_{3}(5,6)=\frac{4}{5}$, which we have checked by hand (see Section 3.3). Applying Theorem 3.2 with $F_{3}(5,6)=\frac{4}{5}$ gives us $F_{3}(n, 6) \leq \frac{4}{5}$ for any $n \geq 5$.

Note that the number of triples shattered must be an integer and is given by $F_{3}(n, 6)\binom{n}{3}$. So for any $N$ with $F_{3}(N, 6) \leq \alpha$ where $\alpha\binom{N}{3}$ is not an integer we have at most $\left\lfloor\alpha\binom{N}{3}\right\rfloor$
shattered triples. Using this and the weakly decreasing property we get a slightly lower upper bound on any $n>N$.

In this case, when $n=8$ we have $F_{3}(8,6) \leq \frac{4}{5}$ and the maximum number of triples shattered by 6 permutations is given by $F_{3}(8,6)\binom{8}{3} \leq \frac{4}{5} \times 56=44.8$, then we must shatter at most 44 out of a possible 56 triples, meaning $F_{3}(8,6) \leq \frac{44}{56}=\frac{11}{14}<\frac{4}{5}$.

We note that we can repeat this rounding down argument indefinitely for a smaller upper bound. The next step gives $F_{3}(n, 6) \leq \frac{47}{60}$ whenever $n \geq 10$ by observing that $\frac{11}{14}\binom{10}{3}$ is not an integer. This along with Theorem 3.2 gives the desired result.

We remark the following, let $\alpha_{5}=\frac{4}{5}, \alpha_{8}=\frac{11}{14}$, and $\alpha_{10}=\frac{47}{60}$ be the fractions from the first three steps in this process, and suppose $\alpha_{x}$ is the proportion given in $i$ th step. Then the fraction given in the $(i+1)$ th step will be one of $\alpha_{x+1}, \alpha_{x+2}$, or $\alpha_{x+3}$. In other words, for at least one of $N=x+1, x+2, x+3$ we have $\alpha_{x}\binom{N}{3}$ is not an integer. This argument therefore continues indefinitely. However the actual bound it gives is hard to pin down and the numerical improvement is small and does not provide additional context, so we leave it at this.

All our upper bounds come from analysing small $n$ and the above rounding argument. To improve these significantly seems to require a different and less case based approach.

### 3.2 Lower bound

To find a lower bound for $F_{k}(n, m)$ we show that some initial family on small $n$ can be used in an iterative process, giving a family on $n^{r}$ which preserves much of the shattering from the initial family. Specific lower bounds can then be given by choosing a suitable initial family which shatters a high proportion of $k$-tuples.

Recall from the proof of Lemma 2.5 a method of upscaling permutations on $N$ to permutations on $N^{r}$ (for any integer $r \geq 1$ ) known as Type 1 permutations, we will use this method again in the next result.

Theorem 3.4. Suppose $\mathcal{S}$ is a family which shatters $\alpha\binom{N}{k} k$-tuples from [ $N$ ], then for any integer $r \geq 1$, the set of Type 1 permutations $\mathcal{S}^{r}=\left\{P^{r}: P \in \mathcal{S}\right\}$ shatters at least

$$
\begin{equation*}
\alpha\binom{N}{k} N^{(r-1) k} \frac{1-N^{r(1-k)}}{1-N^{1-k}} \tag{3.1}
\end{equation*}
$$

$k$-tuples from $\left[N^{r}\right]$. Hence, for all $n \geq N$

$$
\begin{equation*}
F_{k}(n,|\mathcal{S}|) \geq \frac{\alpha(N-1)!}{(N-k)!\left(N^{k-1}-1\right)} \tag{3.2}
\end{equation*}
$$

Proof. Assign to each $x \in\left[N^{r}\right]$ a code $\left(x_{1}, x_{2}, \ldots, x_{r}\right) \in[N]^{r}$ by equating the standard order on $\left[N^{r}\right]$ with the lexicographic order on $[N]^{r}$ (just as in the proof of Lemma 2.5).

Let $P \in \mathcal{S}$ and recall that $P^{r}$ is the permutation from $S_{N^{r}}$ where $x<_{P^{r}} y$ if and only if $x_{i}<_{P} y_{i}$ where $i=d(x, y)=\min \left\{i: x_{i} \neq y_{i}\right\}$.

Note that any $k$-tuple whose elements have unique codes which differ for the first time in the same coordinate $i$, is shattered by $\mathcal{S}^{r}=\left\{P^{r}: P \in \mathcal{S}\right\}$ if the $k$-tuple of $i$ th coordinates is shattered by $\mathcal{S}$. Indeed, consider $a_{1}, \ldots, a_{k} \in\left[N^{r}\right]$ and write $\left(a_{1}^{j}, \ldots, a_{r}^{j}\right)$ for the unique code of $a_{j}$ for each $j \in[k]$. If there exists a coordinate $i \in[r]$ such that $a_{\ell}^{1}=a_{\ell}^{2}=\cdots=a_{\ell}^{k}$ whenever $\ell<i$ and where $a_{i}^{1}, a_{i}^{2}, \ldots, a_{i}^{k}$ are all distinct, then the $k$-tuple $\left\{a_{1}, \ldots, a_{k}\right\}$ will have its order in $P^{r}$ defined by the order of the $\left\{a_{i}^{1}, a_{i}^{2}, \ldots, a_{i}^{k}\right\}$ in $P$. Hence such triples where $\left\{a_{i}^{1}, a_{i}^{2}, \ldots, a_{i}^{k}\right\}$ is shattered by $\mathcal{S}$ are shattered by $\mathcal{S}^{r}$.

Therefore, we may count the minimum number of shattered $k$-tuples by counting exactly those $k$-tuples whose codes have the above property. The number of such $k$-tuples is given by

$$
\sum_{i=1}^{r} N^{i-1} \alpha\binom{N}{k}\left(N^{r-i}\right)^{k}=\alpha\binom{N}{k} N^{(r-1) k} \sum_{i=0}^{r-1}\left(N^{1-k}\right)^{i} .
$$

This proves the first statement.
The second statement follows by selecting an integer $r$ such that $n \leq N^{r}$. We will show that $F_{k}\left(N^{r},|\mathcal{S}|\right)$ follows the statement whenever $r \geq 1$, then the decreasing property of Theorem 3.2 gives the statement for $F_{k}(n,|\mathcal{S}|)$.

Observe that equation (3.1) gives us the minimum number of $k$-tuples shattered, to get an expression for the proportion $F_{k}\left(N^{r},|\mathcal{S}|\right)$ we simply divide by the total number of $k$-tuples.

$$
\begin{aligned}
F_{k}\left(N^{r},|\mathcal{S}|\right) & \geq \alpha\binom{N}{k}\binom{N^{r}}{k}^{-1} N^{(r-1) k} \frac{1-N^{r(1-k)}}{1-N^{1-k}} \\
& =\frac{N^{r k}\left(1-N^{r(1-k)}\right)}{N^{r}\left(N^{r}-1\right) \cdots\left(N^{r}-k+1\right)} \times \frac{\alpha(N-1)!}{(N-k)!\left(N^{k-1}-1\right)}
\end{aligned}
$$

To prove the second statement of the theorem, it is enough to show that

$$
\begin{equation*}
\frac{N^{r k}\left(1-N^{r(1-k)}\right)}{N^{r}\left(N^{r}-1\right) \cdots\left(N^{r}-k+1\right)} \geq 1 . \tag{3.3}
\end{equation*}
$$

To do this we can use induction on $k$. Note that when $k=2$ the left hand side of (3.3) is equal to 1 , so the statement holds for $k=2$. Then note the difference in the expression when considering $k+1$
$\frac{N^{r(k+1)}\left(1-N^{r(1-(k+1))}\right)}{N^{r}\left(N^{r}-1\right) \cdots\left(N^{r}-(k+1)+1\right)}=\frac{N^{r k}\left(1-N^{r(1-k)}\right)}{N^{r}\left(N^{r}-1\right) \cdots\left(N^{r}-k+1\right)} \times \frac{N^{r k}-1}{\left(N^{r}-k\right)\left(N^{r(k-1)}-1\right)}$.
Then (3.3) must hold for all $k>2$ as long as

$$
\frac{N^{r k}-1}{\left(N^{r}-k\right)\left(N^{r(k-1)}-1\right)} \geq 1 .
$$

Observe that this holds whenever $k N^{r(k-1)}+N^{r} \geq k+1$, and since $N^{r}>1$ we have satisfied all the conditions.

We know from Levenshtein [18] that a perfect family $\mathcal{Q}_{k}(k+1)$ exists. By setting $\mathcal{S}=$ $\mathcal{Q}_{k}(k+1)$, we can apply Theorem 3.4 with $n=k+1$ and $\alpha=1$ to get the following result.

Corollary 3.5. For any $n$ we have that

$$
F_{k}(n, k!) \geq \frac{k!}{(k+1)^{k-1}-1} .
$$

When $k=3$ it is known that the largest value $n$ for which a perfect family $\mathcal{Q}_{k}(n)$ exists is 4. Indeed, the family $\mathcal{Q}_{3}(4)$ Chapter 1 is perfect, but $F_{3}(5,3!)=\frac{4}{5}$ (see Section 3.3) and $F_{k}(n, m)$ is decreasing (Theorem 3.2), meaning that $\mathcal{Q}_{3}(n)$ does not exists whenever $n \geq 5$. However, in general it is not known which values (if any) of $n>k+1$ admit a perfect family for any given $k$. This is an interesting open question in its own right, but perfect families for large values of $n$ would give better lower bounds for $F_{k}(N, k!)$ for all $N<n$.

The family $\mathcal{S}$ used in Theorem 3.4 need not be perfect to give a lower bound. Note that the bound of Corollary 3.5 when $k=3$ gives $F_{3}(n, 6) \geq \frac{2}{5}$ for all $n$. We can improve this bound by using a non-perfect family on 8 points for $\mathcal{S}$.

Corollary 3.6. For all $n$ we have $F_{3}(n, 6) \geq \frac{17}{42}$.

Proof. Apply Theorem 3.4 with $n=8, k=3$, and $\mathcal{S}$ given by the 6 permutations below:

$$
\begin{array}{ll}
P_{1}=(6,1,2,5,8,3,4,7) & P_{2}=(5,2,1,6,7,4,3,8) \\
P_{3}=(7,3,6,2,8,4,5,1) & P_{4}=(3,7,1,5,4,8,2,6) \\
P_{5}=(8,4,5,1,7,3,6,2) & P_{6}=(4,8,2,6,3,7,1,5) .
\end{array}
$$

Note that $\mathcal{S}$ shatters 34 out of 56 possible triples.

Combining the results of Theorem 3.3 and Corollary 3.6 gives Theorem 3.1, and we see that $\lim _{n \rightarrow \infty} F_{3}(n, 6)$ lies in the interval $\left[\frac{17}{42}, \frac{47}{60}\right)$.

Theorem 3.1. For any $n \geq 10$ we have

$$
\frac{17}{42} \leq F_{3}(n, 6) \leq \frac{47}{60}
$$

### 3.3 Showing that $F_{3}(5,6)=\frac{4}{5}$

Recall that $F_{3}(5,6) \geq \frac{4}{5}$ as seen in Example 1.7. It remains to show that $F_{3}(4,5) \leq$ $\frac{4}{5}$.

To show that $F_{3}(5,6) \leq \frac{4}{5}$ we must show that any 6 permutations from $S_{5}$ shatter at most 8 triples out of a possible 10. Therefore it is sufficient to show that in any 6 permutations there are at least 2 distinct triples that are not shattered, and hence have a repeated pattern.

So we look for a family of 6 permutations from $S_{5}$ with at most 1 triple un-shattered (i.e. with a pattern repeated), if there is no such family then we have proved the statement.

If such a family exists, we may assume that the triple that is not shattered contains the element 5 . From this we see that the family of 6 permutations of $S_{4}$ generated by omitting 5 must shatter every triple from [4]. Since the family $\mathcal{Q}_{k}(4)$ is unique up to isomorphism, to prove the statement we show that adding 5 in any position to permutations from $\mathcal{Q}_{k}(4)$ results in a family that leaves 2 triples un-shattered.

Here is the family $\mathcal{Q}_{k}(4)$ :

$$
\begin{array}{lll}
Q_{1}=(1,2,3,4) & Q_{2}=(2,4,1,3) & Q_{3}=(3,4,1,2) \\
Q_{4}=(1,4,3,2) & Q_{5}=(4,2,3,1) & Q_{6}=(3,2,1,4) .
\end{array}
$$

The permutation generated by adding 5 into $Q_{i}$ in some position will be denoted $Q_{i}^{\prime}$.
We split into 5 cases, one for each location of element 5 in $Q_{1}$.
Case 1: $Q_{1}^{\prime}=(1,2,3,4,5)$.
So we have fixed $Q_{1}^{\prime}$, consider the options for $Q_{2}^{\prime}$.
If $Q_{2}^{\prime}=(2,4,1,3,5)$ the triples $\{1,3,5\}$ and $\{2,4,5\}$ appear in the same order in both $Q_{1}^{\prime}$ and $Q_{2}^{\prime}$. Any family that contains $Q_{1}^{\prime}$ and $Q_{2}^{\prime}=(2,4,1,3,5)$ is not the family we search for, so we look at a different option for $Q_{2}^{\prime}$.

If $Q_{2}^{\prime}=(2,4,1,5,3)$ then the triple $\{2,4,5\}$ appears in the same order in $Q_{1}^{\prime}$ and $Q_{2}^{\prime}$. Consider now $Q_{3}^{\prime}=(3,4, \star, 1, \star, 2, \star)$ where 5 appears in any location denoted by $\star$, the triple $\{3,4,5\}$ appears in the same order in $Q_{1}^{\prime}$ and $Q_{3}^{\prime}$. This forces both triples $\{2,4,5\}$ and $\{3,4,5\}$ to repeat. So we assume instead that $Q_{3}^{\prime}=(\star, 3, \star, 4,1,2)$.

So we have fixed $Q_{1}^{\prime}=(1,2,3,4,5), Q_{2}^{\prime}=(2,4,1,5,3)$, and $Q_{3}^{\prime}=(\star, 3, \star, 4,1,2)$ and we know that $\{2,4,5\}$ is repeated already. Consider $Q_{5}^{\prime}=(4,2,3, \star, 1, \star)$, then $\{2,3,5\}$ appears in the same order in $Q_{1}^{\prime}$ and $Q_{5}^{\prime}$, meaning both $\{2,4,5\}$ and $\{2,3,5\}$ are not shattered. Similarly if $Q_{5}^{\prime}=(4, \star, 2, \star, 3,1)$ the triple $\{3,4,5\}$ is copied in $Q_{2}^{\prime}$ and $Q_{5}^{\prime}$, meaning $\{2,4,5\}$ and $\{3,4,5\}$ are not shattered. Finally, if $Q_{5}^{\prime}=(5,4,2,3,1)$ we have $\{1,4,5\}$ appearing in the same pattern in $Q_{3}^{\prime}$ and $Q_{5}^{\prime}$ giving the pair $\{2,4,5\}$ and $\{1,4,5\}$.

We record the above information as follows.

$$
\begin{gather*}
Q_{2}^{\prime}=(2,4,1,5,3) \quad\{2,4,5\} \\
Q_{3}^{\prime}=(3,4, \star, 1, \star, 2, \star) \\
Q_{3}^{\prime}=(\star, 3, \star, 4,1,2) \\
Q_{5}^{\prime}=(4,2,3, \star, 1, \star) \\
Q_{5}^{\prime}=(4, \star, 2, \star, 3,1) \\
Q_{5}^{\prime}=(5,4,2,3,1)
\end{gather*}
$$

This means that the family we search for cannot contain $Q_{1}^{\prime}$ and $Q_{2}^{\prime}=(2,4,1,5,3)$, so we assume next that $Q_{2}^{\prime}=(2,4,5,1,3)$. It happens that for $Q_{2}^{\prime}=(2,4,5,1,3)$ the case is the same as that of $Q_{2}^{\prime}=(2,4,1,5,3)$.

For the remaining two options for $Q_{2}^{\prime}$ we have the following.

$$
\begin{align*}
& Q_{2}^{\prime}=(2,5,4,1,3) \\
& \begin{array}{cr}
Q_{4}^{\prime}=(1,4,3, \star, 2, \star) & \\
Q_{4}^{\prime}=(1,4,5,3,2) & \{1,4,5\},\{1,3,5\} \\
Q_{5}^{\prime}=(4,2,3, \star, 1, \star) & \{1,4,5\} \\
Q_{5}^{\prime}=(\star, 4, \star, 2, \star, 3,1) & \{2,3,5\} \\
Q_{4}^{\prime}=(1,5,4,3,2) & \{3,4,5\} \\
Q_{3}^{\prime}=(3,4,1, \star, 2, \star) & \{3,4,5\} \\
Q_{3}^{\prime}=(3,4,5,1,2) & \{1,2,5\} \\
Q_{5}^{\prime}=(4,2, \star, 3, \star, 1, \star) & \{2,3,5\} \\
Q_{5}^{\prime}=(\star, 4, \star, 2,3,1) & \{1,4,5\} \\
Q_{3}^{\prime}=(\star, 3, \star, 4,1,2) & \{1,4,5\}
\end{array}
\end{align*}
$$

$$
Q_{4}^{\prime}=(5,1,4,3,2)
$$

$$
\begin{align*}
& Q_{2}^{\prime}=(5,2,4,1,3) \\
& Q_{4}^{\prime}=(1,4,3, \star, 2, \star) \\
& \{1,4,5\},\{1,3,5\} \\
& Q_{4}^{\prime}=(1,4,5,3,2) \\
& Q_{5}^{\prime}=(4,2,3, \star, 1, \star) \\
& Q_{5}^{\prime}=(4,2,5,3,1) \\
& Q_{5}^{\prime}=(\star, 4, \star, 2,3,1) \\
& Q_{4}^{\prime}=(1,5,4,3,2) \\
& Q_{5}^{\prime}=(4,2,3, \star, 1, \star) \\
& Q_{5}^{\prime}=(4,2,5,3,1) \\
& Q_{3}^{\prime}=(3,4,1, \star, 2, \star) \\
& Q_{3}^{\prime}=(\star, 3, \star, 4, \star, 1,2) \\
& \{3,4,5\} \\
& \{2,3,5\} \\
& Q_{4}^{\prime}=(5,1,4,3,2)
\end{align*}
$$

These show that if $Q_{2}^{\prime}=(2,5,4,1,3)$ or $Q_{2}^{\prime}=(5,2,4,1,3)$, there is no position 5 can take in $Q_{4}^{\prime}$ without causing a pair of un-shattered triples. With all of these pieces we see that no matter where 5 is found in $Q_{2}^{\prime}$ it leads to a family with at most 8 triples shattered.

We continue like this for the remaining cases, showing that there is no possible location for 5 without having a pair of un-shattered triples. Therefore the family we search for does not exist and we must have that $F_{3}(5,6)=\frac{4}{5}$.

Case 2: $Q_{1}^{\prime}=(1,2,3,5,4)$

$$
\begin{array}{rr}
Q_{6}^{\prime}=(3,2,1,4,5) & \\
Q_{4}^{\prime}=(1,4,3, \star, 2, \star) & \{1,4,5\},\{1,3,5\} \\
Q_{4}^{\prime}=(1, \star, 4, \star, 3,2) & \{1,4,5\} \\
Q_{3}^{\prime}=(3, \star, 4, \star, 1, \star, 2, \star) & \{3,4,5\} \\
Q_{3}^{\prime}=(5,3,4,1,2) & \{2,3,5\} \\
Q_{4}^{\prime}=(5,1,4,3,2) & \\
Q_{2}^{\prime}=(2,4,1, \star, 3, \star) & \{2,4,5\},\{1,2,5\}
\end{array}
$$

$$
\begin{align*}
& Q_{2}^{\prime}=(2,4,5,1,3) \\
& Q_{2}^{\prime}=(\star, 2, \star, 4,1,3) \\
& Q_{6}^{\prime}=(3,2, \star, 1, \star, 4) \\
& Q_{6}^{\prime}=(3,5,2,1,4) \\
& Q_{4}^{\prime}=(1,4,3, \star, 2, \star) \\
& Q_{4}^{\prime}=(1,4,5,3,2) \\
& Q_{3}^{\prime}=(3,4,1,2,5) \\
& Q_{3}^{\prime}=(\star, 3, \star, 4, \star, 1, \star, 2) \\
& Q_{4}^{\prime}=(\star, 1, \star, 4,3,2) \\
& \{2,4,5\},\{1,3,5\} \\
& \{3,4,5\},\{1,3,5\} \\
& \{2,4,5\},\{3,4,5\} \\
& Q_{6}^{\prime}=(5,3,2,1,4) \\
& Q_{5}^{\prime}=(4,2,3,1,5) \\
& Q_{2}^{\prime}=(2,4,1, \star, 3, \star) \\
& Q_{2}^{\prime}=(2,4,5,1,3) \\
& Q_{3}^{\prime}=(3,4,1, \star, 2, \star) \\
& Q_{3}^{\prime}=(3,4,5,1,2) \\
& Q_{3}^{\prime}=(3,5,4,1,2) \\
& Q_{3}^{\prime}=(5,3,4,1,2) \\
& Q_{2}^{\prime}=(\star, 2, \star, 4,1,3) \\
& Q_{5}^{\prime}=(4,2,3,5,1) \\
& Q_{2}^{\prime}=(2,4,1,3,5) \\
& Q_{2}^{\prime}=(2,4,1,5,3) \\
& Q_{3}^{\prime}=(3,4, \star, 1, \star, 2, \star) \\
& Q_{3}^{\prime}=(\star, 3, \star, 4,1,2) \\
& Q_{2}^{\prime}=(2, \star, 4, \star, 1,3) \\
& Q_{2}^{\prime}=(5,2,4,1,3) \\
& Q_{5}^{\prime}=(4,2,5,3,1) \\
& Q_{2}^{\prime}=(2, \star, 4, \star, 1, \star, 3, \star) \\
& Q_{2}^{\prime}=(5,2,4,1,3) \\
& \{3,4,5\} \\
& \{2,4,5\} \\
& \{1,3,5\} \\
& \{2,3,5\} \\
& \{2,4,5\}
\end{align*}
$$

$$
Q_{5}^{\prime}=(\star, 4, \star, 2,3,1)
$$

Case 3: $Q_{1}^{\prime}=(1,2,5,3,4)$

$$
\begin{array}{rr}
Q_{4}^{\prime}=(1,4,3,2,5) & \{1,2,5\} \\
Q_{2}^{\prime}=(2,4,1,3,5) & \{1,3,5\} \\
Q_{2}^{\prime}=(2, \star, 4, \star, 1, \star, 3) & \{2,3,5\} \\
Q_{2}^{\prime}=(5,2,4,1,3) & \\
Q_{5}^{\prime}=(4,2, \star, 3, \star, 1, \star) & \{2,4,5\} \\
Q_{5}^{\prime}=(\star, 4, \star, 2,3,1) & \{2,3,5\}
\end{array}
$$

$$
\begin{array}{rrr}
Q_{4}^{\prime}=(1,4,3,5,2) & \\
Q_{2}^{\prime}=(2,4,1,3,5) & \{4,3,5\},\{1,3,5\} \\
Q_{2}^{\prime}=(2,4,1,5,3) & \{2,3,5\},\{1,3,5\} \\
Q_{2}^{\prime}=(2,4,5,1,3) & \{2,3,5\} \\
Q_{6}^{\prime}=(3,2, \star, 1, \star, 4, \star) & \{2,4,5\} \\
Q_{6}^{\prime}=(3,5,2,1,4) & \\
Q_{3}^{\prime}=(3,4,1,2,5) & \{1,2,5\} \\
Q_{3}^{\prime}=(3,4, \star, 1, \star, 2) & \{2,4,5\} \\
Q_{3}^{\prime}=(\star, 3, \star, 4,1,2) & \{3,4,5\} \\
Q_{6}^{\prime}=(5,3,2,1,4) & \{3,4,5\} \\
Q_{2}^{\prime}=(2,5,4,1,3) & \{2,3,5\},\{2,4,5\} \\
Q_{2}^{\prime}=(5,2,4,1,3) & \\
Q_{5}^{\prime}=(4,2,3, \star, 1, \star) & \{3,4,5\} \\
Q_{6}^{\prime}=(3,2,1,4,5) & \{1,4,5\} \\
Q_{6}^{\prime}=(3,2, \star, 1, \star, 4) & \{2,4,5\} \\
Q_{6}^{\prime}=(\star, 3, \star, 2,1,4) & \{1,2,5\} \\
Q_{5}^{\prime}=(4,2,5,3,1) & \{2,3,5\} & \\
Q_{3}^{\prime}=(3,4,1,2,5) & \{1,2,5\} \\
Q_{3}^{\prime}=(3,4, \star, 1, \star, 2) & \{2,4,5\}
\end{array}
$$

$$
\begin{array}{cr}
Q_{3}^{\prime}=(\star, 3, \star, 4,1,2) & \{1,4,5\} \\
Q_{5}^{\prime}=(\star, 4, \star, 2,3,1) & \{1,2,5\},\{2,3,5\} \\
Q_{4}^{\prime}=(1,4,5,3,2) & \{1,3,5\} \\
Q_{3}^{\prime}=(3,4,1,2,5) & \{1,2,5\} \\
Q_{3}^{\prime}=(3,4, \star, 1, \star, 2) & \{2,4,5\} \\
Q_{3}^{\prime}=(3,5,4,1,2) & \{1,4,5\} \\
Q_{6}^{\prime}=(3,2,1,4,5) & \{3,4,5\} \\
Q_{6}^{\prime}=(\star, 3, \star, 2, \star, 1, \star, 4) & \{3,4,5\} \\
Q_{3}^{\prime}=(5,3,4,1,2) & \\
Q_{4}^{\prime}=(1,5,4,3,2) & \{1,3,5\},\{1,4,5\} \\
Q_{4}^{\prime}=(5,1,4,3,2) & \\
Q_{3}^{\prime}=(3,4,1,2,5) & \{1,4,5\} \\
Q_{2}^{\prime}=(2,4,1, \star, 3, \star) & \{1,3,5\} \\
Q_{2}^{\prime}=(\star, 2, \star, 4, \star, 1,3) & \\
Q_{3}^{\prime}=(3,4,1,5,2) & \{1,2,5\} \\
Q_{2}^{\prime}=(2,4,1, \star, 3, \star) & \{1,4,5\} \\
Q_{6}^{\prime}=(3,2, \star, 1, \star, 4, \star) & \{2,4,5\} \\
Q_{6}^{\prime}=(\star, 3, \star, 2,1,4) & \{2,3,5\} \\
Q_{2}^{\prime}=(2,4,5,1,3) & \{1,3,5\},\{2,3,5\} \\
Q_{2}^{\prime}=(\star, 2, \star, 4,1,3) & \{1,3,5\},\{3,4,5\} \\
Q_{3}^{\prime}=(\star, 3, \star, 4, \star, 1,2) & \{1,2,5\} \\
Q_{2}^{\prime}=(2,4,1,3,5) & \{1,3,5\} \\
Q_{6}^{\prime}=(3,2, \star, 1, \star, 4, \star) & \\
Q_{6}^{\prime}=(\star, 3, \star, 2,1,4) & \{2,4,5\} \\
Q_{2}^{\prime}=(5,2,4,1,3) & \{1,4,5\} \\
\hline
\end{array}
$$

Case 4: $Q_{1}^{\prime}=(1,5,2,3,4)$

$$
\begin{array}{cr}
Q_{4}^{\prime}=(1,4,3,2,5) & \\
Q_{6}^{\prime}=(3,2,1, \star, 4, \star) & \{2,3,5\},\{1,4,5\} \\
Q_{6}^{\prime}=(3,2,5,1,4) & \{2,3,5\} \\
Q_{2}^{\prime}=(2,4,1, \star, 3, \star) & \{1,3,5\} \\
Q_{2}^{\prime}=(2, \star, 4, \star, 1,3) & \{1,2,5\} \\
Q_{2}^{\prime}=(5,2,4,1,3) & \{2,4,5\} \\
Q_{6}^{\prime}=(3,5,2,1,4) & \{2,4,5\} \\
Q_{3}^{\prime}=(3,4,1,2,5) & \{1,2,5\} \\
Q_{3}^{\prime}=(3, \star, 4, \star, 1, \star, 2) & \{2,3,5\} \\
Q_{3}^{\prime}=(5,3,4,1,2) & \{3,4,5\} \\
Q_{6}^{\prime}=(5,3,2,1,4) & \{2,4,5\},\{3,4,5\} \\
Q_{4}^{\prime}=(1,4,3,5,2) & \\
Q_{3}^{\prime}=(3,4,1,2,5) & \{1,2,5\} \\
Q_{6}^{\prime}=(3, \star, 2, \star, 1, \star, 4, \star) & \\
Q_{6}^{\prime}=(5,3,2,1,4) & \{2,3,5\} \\
Q_{3}^{\prime}=(3, \star, 4, \star, 1, \star, 2) & \{2,4,5\} \\
Q_{3}^{\prime}=(5,3,4,1,2) & \{2,3,5\} \\
& \\
Q_{4}^{\prime}=(1, \star, 4, \star, 3,2) & \{3,4,5\} \\
Q_{4}^{\prime}=(5,1,4,3,2) & \\
Q_{6}^{\prime}=(3,2,1,4,5) & \{1,2,5\},\{1,3,5\} \\
Q_{3}^{\prime}=(3,4,1, \star, 2, \star) & \{3,4,5\},\{1,3,5\} \\
Q_{3}^{\prime}=(3,4,5,1,2) & \{3,4,5\},\{1,2,5\} \\
Q_{3}^{\prime}=(\star, 3, \star, 4,1,2) & \{2,4,5\},\{1,2,5\} \\
Q_{6}^{\prime}=(3,2, \star, 1, \star, 4) & \{1,4,5\} \\
\hline & \\
\hline
\end{array}
$$

$$
Q_{6}^{\prime}=(\star, 3, \star, 2,1,4) \quad\{1,4,5\},\{2,4,5\}
$$

Case 5: $Q_{1}^{\prime}=(5,1,2,3,4)$

\[

\]

$$
\begin{array}{cr}
Q_{3}^{\prime}=(3,4,1,5,2) & \{1,4,5\} \\
Q_{4}^{\prime}=(1,4,3,2,5) & \{2,3,5\} \\
Q_{6}^{\prime}=(3,2, \star, 1, \star, 4, \star) & \{2,4,5\} \\
Q_{6}^{\prime}=(\star, 3, \star, 2,1,4) & \{1,2,5\} \\
Q_{4}^{\prime}=(\star, 1, \star, 4, \star, 3, \star, 2) & \{1,2,5\} \\
Q_{3}^{\prime}=(3, \star, 4, \star, 1,2) & \{1,4,5\} \\
Q_{5}^{\prime}=(4,2,3,1,5) & \{1,3,5\} \\
Q_{5}^{\prime}=(4,2,3,5,1) & \{2,3,5\} \\
Q_{5}^{\prime}=(\star, 4, \star, 2, \star, 3,1) & \{1,2,5\},\{3,4,5\} \\
Q_{3}^{\prime}=(5,3,4,1,2) & \\
Q_{2}^{\prime}=(2,4,5,1,3) & \{1,3,5\} \\
Q_{3}^{\prime}=(3,4,1, \star, 2, \star) & \{1,4,5\} \\
Q_{5}^{\prime}=(4, \star, 2, \star, 3, \star, 1, \star) & \{2,3,5\} \\
Q_{5}^{\prime}=(5,4,2,3,1) & \{1,2,5\} \\
Q_{3}^{\prime}=(\star, 3, \star, 4, \star, 1,2) & \\
Q_{2}^{\prime}=(2,5,4,1,3) & \{1,3,5\} \\
Q_{3}^{\prime}=(3,4,1, \star, 2, \star) & \\
Q_{6}^{\prime}=(3,2,1,4,5) & \{3,4,5\} \\
Q_{6}^{\prime}=(\star, 3, \star, 2, \star, 1, \star, 4) & \{2,4,5\} \\
Q_{3}^{\prime}=(\star, 3, \star, 4, \star, 1,2) & \{1,2,5\} \\
Q_{2}^{\prime}=(5,2,4,1,3) & \{1,3,5\},\{2,3,5\}
\end{array}
$$

## Chapter 4

## Completely shattering

Recall that Spencer [27] gives a simple probabilistic argument showing that $f_{k}(n)=$ $O(\log n)$, which is the best known upper bound for the total shattering problem. The bound given is

$$
f_{k}(n) \leq \frac{k}{\log k!-\log (k!-1)} \log n .
$$

However, there is only a construction matching this size for $k=3$, provided by Tarui [28]. This gives the bound

$$
f_{3}(n) \leq 2 \log n+(1+o(1)) \log \log n .
$$

We have been unable to extend any of the separation ideas that work in the special $k=3$ case to give a constructive proof that $O(\log n)$ for arbitrary values of $k$. Therefore finding such a construction is still an open problem.

It is of particular interest to have constructions of small shattering families due to the link with event sequence testing. In order to design a thorough test for any given process in a practical setting, we must be able to provide each permutation so that the sequence may be followed exactly.

The aim of this chapter is to give an iterative construction for small shattering families that applies to any value of $k$. The main idea is to identify each element of $[n]$ with a point in the $k$-dimensional lattice, and take permutations by grouping the points in each of the $k$ directions. Unfortunately the bound this gives is a power of $\log n$ rather than the known $O(\log n)$ bound. However, this is a constructive method which gives a relatively small family which shatters every $k$-tuple.

Lemma 4.1. Given a family that shatters every $k$-tuple in $\left[n^{k-1}\right]$ and has size $S$, we can give an explicit construction of a family that shatters every $k$-tuple from $\left[n^{k}\right]$ and has size $k S$.

Proof. Let $\mathcal{S}$ be a shattering family for $\left[n^{k-1}\right]$, we use the permutations in this family to construct permutations of $\left[n^{k}\right]$. For simplicity, let $m=n^{k-1}$.

We consider the elements of $\left[n^{k}\right]$ viewed geometrically as points on an integer lattice of dimension $k$. Label each element $x \in\left[n^{k}\right]$ by a unique string $\left(i_{1}, \ldots, i_{k}\right)$ with all $i_{j} \in[n]$. Without loss of generality, we may assume that all elements of $[\mathrm{m}]$ are labelled with $\left(1, i_{2}, \ldots, i_{k}\right)$ and hence can be thought of instead as the $k-1$ string $\left(i_{2}, \ldots, i_{k}\right)$. This means every $k-1$ string is associated to an element of $[m]$.

Let $r_{j}(x)=\left(i_{1}, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{k}\right)$ be $x$ with the $j$ th coordinate omitted. Note that $r_{j}(x)$ is therefore associated to an element of $[m]$.

For each $P \in \mathcal{S}$ we will create $k$ permutations of $\left[n^{k}\right], P_{1}^{\prime}, \ldots, P_{k}^{\prime}$. To generate the permutation $P_{j}^{\prime}$ order $x$ before $y$ if and only if $r_{j}(x)$ appears before $r_{j}(y)$ in $P$, if $r_{j}(x)=$ $r_{j}(y)$ then order arbitrarily.

This generates $k|\mathcal{S}|$ permutations of $\left[n^{k}\right]$, it is left to show that these are sufficient to shatter all $k$-tuples.

Claim : Given $k$ points in $[n]^{k}, A=\left\{a_{1}, \ldots, a_{k}\right\}$, there exists a direction $j \in[k]$ such that the projection of $A$ in direction $j$ has $k$ points.

Indeed, suppose not for a contradiction. For all $j \in[k]$ there is some pair $a_{\ell}, a_{t} \in A$ such that $r_{j}\left(a_{\ell}\right)=r_{j}\left(a_{q}\right)$. Define a graph $G$ with vertex set $A$, and with one edge for each direction $j \in[k]$ between some pair of vertices $u, v \in A$ with $r_{j}(u)=r_{j}(z)$. By our assumption there is at least one such pair for each $j \in[k]$, if there is a choice then pick arbitrarily. Note that if $r_{j}(u)=r_{j}(z)$ for some $j$ then we cannot have $r_{\ell}(u)=r_{\ell}(v)$ for any $\ell \in[k] \backslash j$ by definition, so we can never pick the same edge more than once. Hence our graph on $k$ vertices has exactly $k$ edges, therefore $G$ must contain a cycle. Let $v_{1}, \ldots, v_{t}$ be a cycle in $G$, then the edge $v_{1} v_{2}$ demonstrates a change in one coordinate, say $j_{1}$. Similarly edge $v_{2} v_{3}$ demonstrates a change in coordinate $j_{2}$. Observe that $j_{1}$ and $j_{2}$ are distinct since there is only one edge for each direction. Continuing, we find $t$ distinct directions $j_{1}, \ldots, j_{t}$ where $j_{t}$ is the direction of the edge $v_{t} v_{1}$. This is equivalent to starting with $v_{1}$, changing $t$ different coordinates and ending up back at $v_{1}$. Clearly this cannot happen and therefore we have a contradiction. This proves the claim.

We have shown that for any $k$-tuple $A=\left\{a_{1}, \ldots, a_{k}\right\}$, there is a coordinate $j$ such that $r_{j}\left(a_{\ell}\right) \neq r_{j}\left(a_{q}\right)$ for all $\ell, q \in[k]$. Hence $A$ is shattered by the collection of permutations of the form $P_{j}^{\prime}$ where $P \in \mathcal{S}$.

Therefore our collection of $P^{\prime} \mathrm{s}$ does indeed shatter all the $k$-tuples in $\left[n^{k}\right]$, and we used $k|\mathcal{S}|$ permutations in total.

Repeatedly applying the construction in 4.1 gives an upper bound $f_{k}(n) \leq(\log n)^{c_{k}}$ where $c_{k} \approx \frac{\log k}{\log k-\log (k-1)}$.

## Chapter 5

## Families which shatter no triples

In this chapter we will consider the problem of constructing a family of permutations which is as large as possible, yet does not shatter any triple. That is, we are interested in the size of the largest $\mathcal{F}(n) \subseteq S_{n}$ where every triple $\{x, y, z\} \subseteq[n]$ appears in at most 5 orders in $\mathcal{F}(n)$.

Definition 5.1. We will call a family of permutations non-shattering if there is no triple which appears in all six orders.

The aim of this chapter is to introduce a method of constructing large non-shattering families, as well as making a few observations about the pattern avoiding behaviour of large non-shattering families.

The problem was introduced by Raz in [23], and is a reformulation of the VC-dimension (see [29]) in terms of permutations. Raz was able to show the following about the size of such families.

Theorem 5.2 (Raz [23]). There exists a universal constant $C$ such that, for any nonshattering family $\mathcal{F}(n) \subseteq S_{n}$ we have $|\mathcal{F}(n)| \leq C^{n}$.

The value of $C$ is unknown and [23] does not attempt to quantify the value. However, it is remarked by Raz that a possible bound is given by $|\mathcal{F}(n)| \leq C_{n}=\frac{1}{n+1}\binom{2 n}{n}$, the $n$th Catalan number.

We remark that the analogous problem for $k$-tuples with $k>3$ seems to have very different behaviour, with constructions of non-shattering families reaching super-exponential size. Raz notes that the methods in [23] do not give a similar bound to Theorem 5.2 when
$k>3$. We will only consider the case $k=3$ here.
Let $\mathcal{A}_{321}(n)$ be the family of all permutations of $S_{n}$ which contain no decreasing triple, that is, the family that contains all permutations of $[n]$ except those that contain a copy of the pattern $(3,2,1)$. It is well known that $\left|\mathcal{A}_{321}(n)\right|=C_{n}$ (see [5]), and clearly the family $\mathcal{A}_{321}(n)$ is non-shattering. In fact if $\sigma$ is any permutation of $S_{3}$, and $\mathcal{A}_{\sigma}(n)$ is the family containing all permutations of $S_{n}$ which do not contain the order $\sigma$, then $\left|\mathcal{A}_{\sigma}(n)\right|=C_{n}$ and plainly $\mathcal{A}_{\sigma}(n)$ is non-shattering (again see [5]). At present, there are no known non-shattering families which are larger than $C_{n}$.

Every non-shattering family has at least one forbidden pattern of $S_{3}$ for each triple. That is, for every triple of $[n]$ we must have one ordering that does not occur in a family which is non-shattering, this order is described by one pattern from $S_{3}$. This leads to the following easy observation. Let $\mathcal{F} \subseteq S_{n}$ be a non-shattering family of maximum size, and for every triple $X \subseteq[n]$ let $\sigma(X) \in S_{3}$ denote a forbidden pattern of $X$. Then $\mathcal{F}=\left\{P \in S_{n}: P_{X} \neq \sigma(X)\right.$ for all triples $\left.X \subseteq[n]\right\}$.

In other words, when looking for the largest non-shattering family of $[n]$, it is enough to specify a single forbidden pattern of $S_{3}$ for each triple in $[n]$. Note that if we forbid the same pattern $\sigma$ for every triple, then we end up with the family $\mathcal{A}_{\sigma}(n)$.

Let $\mathcal{F} \subseteq S_{n}$ be a non-shattering family (of any size), our main aim in this chapter is to introduce a method of constructing a non-shattering family of $S_{n+1}$ from $\mathcal{F}$. Therefore, after recursively applying the construction we can generate a non-shattering family of [ $N$ ] from any non-shattering family on $[n]$ where $N>n$. Using this we are able to construct a range of different non-shattering families of size $C_{N}$.

Definition 5.3. Let $F=\left(f_{1}, \ldots, f_{n}\right)$ be any permutation from $S_{n}$. We call the longest final segment of $F$ which is an increasing sequence the tail of $F$. More precisely, the segment $f_{i}, \ldots, f_{n}$ is the tail if and only if $f_{i}<\cdots<f_{n}$ and $f_{i-1}>f_{i}$ or $i=1$. We denote the tail length of $F$ by $t(F)$, which is the number of elements in the tail of $F$.

Now we can construct a non-shattering family of permutations $\mathcal{F}^{\prime} \subseteq S_{n+1}$ from a nonshattering family $\mathcal{F} \subseteq S_{n}$ by the following method. For each $F \in \mathcal{F}$, generate $t(F)+1$ permutations of $\mathcal{F}^{\prime}$ by inserting the element $n+1$ into each of the final $t(F)+1$ positions in $F$.

Example 5.4. Let $n=9$ and $F=(5,7,4,1,9,2,3,6,8)$. The tail of $F$ is the segment $2,3,6,8$ and therefore $t(F)=4$. We take 5 permutations of [10] by inserting the element

10 into the final 5 positions of $F$, giving us

$$
\begin{gathered}
(5,7,4,1,9,2,3,6,8,10), \quad(5,7,4,1,9,2,3,6,10,8), \quad(5,7,4,1,9,2,3,10,6,8), \\
(5,7,4,1,9,2,10,3,6,8), \quad(5,7,4,1,9,10,2,3,6,8)
\end{gathered}
$$

Note that the tail each of these permutations has a distinct tail length from [5].
To see that this does indeed give us a non-shattering family, note first that we need only consider triples which contain the element $n+1$, since we have not changed any orders given by the non-shattering family $\mathcal{F}$. Then observe that we have added $n+1$ in such a way as to avoid making any decreasing triple, and hence all triples containing $n+1$ are missing at least the order $(3,2,1)$. Consider any triple $\{x, y, n+1\}$ with $x<y<n+1$, to get the order $(n+1, y, x)$ we need to have inserted $n+1$ before the decreasing pair $(y, x)$ in $F$, but we have added $n+1$ so that everything to the right of it is part of the tail of $F$ and is therefore increasing.

Further note that inserting $n+1$ to any other position guarantees the formation of a decreasing triple. Indeed, if $f_{i}, \ldots, f_{n}$ is the tail of $F$ then $f_{i-1}>f_{i}$, any placement of $n+1$ outside the final $t(F)+1$ places puts $n+1$ to the left of $f_{i-1}$, forcing the triple $\left\{f_{i}, f_{i-1}, n+1\right\}$ to appear decreasing.

We also remark that the tail lengths of the new permutations generated by following the above construction are entirely predictable. This means that all the necessary information is contained in the tail length, and we do not need to know anything else about the structure of the permutation in order to construct the non-shattering family $\mathcal{F}^{\prime}$. We formalise this in the following Lemma.

Lemma 5.5. Let $\mathcal{F} \subseteq S_{n}$ be a non-shattering family, and let $\mathcal{F}^{\prime}$ be constructed from $\mathcal{F}$ by the above process. Then we have that

$$
\left|\mathcal{F}^{\prime}\right|=\sum_{F \in \mathcal{F}} t(F)+1,
$$

and that the number of permutations $F^{\prime} \in \mathcal{F}^{\prime}$ with $t\left(F^{\prime}\right)=t$ is given by the number of $F \in \mathcal{F}$ with $t(F) \geq t-1$.

Proof. Let $F \in \mathcal{F}$ and note that $F$ will generate one permutation with tail length $i$ for each $i \in[t(F)+1]$. This is due to the tails being dictated by the element $n+1$. Since $n+1$ is always larger than any element of $F$, if $F^{\prime} \in \mathcal{F}^{\prime}$ is a permutation generated from $F$, then $F^{\prime}$ has tail length $t(F)+1$ if and only if the last element of $F^{\prime}$ is $n+1$. If $n+1$ is
in any other position then the tail of any $F^{\prime} \in \mathcal{F}^{\prime}$ generated from $F$ must be the segment immediately to the right of $n+1$. Therefore, $t\left(F^{\prime}\right)$ is simply equal to the number of entries to the right of $n+1$. By the construction we take the permutations which have $n+1$ in the last $t(F)+1$ positions, giving the result.

Since both the input and output of the construction is a non-shattering family, we can apply the method repeatedly to get a non-shattering family on $[N]$ for any $N>n$. In particular, since the construction forces all new triples to avoid the decreasing order, setting $\mathcal{F}=\mathcal{A}_{321}(n)$ gives $\mathcal{F}^{\prime}=\mathcal{A}_{321}(n+1)$, and hence repeated application always results in $\mathcal{A}_{321}(N)$. However, Lemma 5.5 implies that the size of $\mathcal{F}^{\prime}$ is determined only by the tails lengths of $\mathcal{F}$, meaning that starting with any $\mathcal{F}$ whose multiset of tail lengths matches the multiset of tail lengths of $\mathcal{A}_{321}(n)$, results in a non-shattering family of size $\left|\mathcal{A}_{321}(N)\right|=C_{N}$. Moreover, unless $\mathcal{F}=\mathcal{A}_{321}(n)$, the resulting family (after any number of iterations of the construction) is not equal to $\mathcal{A}_{\sigma}(N)$ for any pattern $\sigma \in S_{3}$.

Example 5.6. Consider $\mathcal{F}:=\mathcal{A}_{132}(3)=\{(1,2,3),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ and note that the multiset $\left\{t(F): F \in \mathcal{A}_{132}(3)\right\}=\{3,2,1,2,1\}=\left\{t(F): F \in \mathcal{A}_{321}(3)\right\}$. Let $\mathcal{F}^{\prime}$ be the non-shattering family generated by applying the construction to $\mathcal{F}$. We have

$$
\begin{aligned}
& \mathcal{F}^{\prime}=\{(1,2,3,4),(1,2,4,3),(1,4,2,3),(4,1,2,3),(2,1,3,4),(2,1,4,3),(2,4,1,3), \\
&(2,3,1,4),(2,3,4,1),(3,1,2,4),(3,1,4,2),(3,4,1,2),(3,2,1,4),(3,2,4,1)\} .
\end{aligned}
$$

Then the multiset $\left\{t(F): F \in \mathcal{F}^{\prime}\right\}=\{4,1,2,3,3,1,2,2,1,3,1,2,2,1\}=\{t(F): F \in$ $\left.\mathcal{A}_{321}(4)\right\}$, and therefore the construction can be applied again to give a non-shattering family of size $\left|\mathcal{A}_{321}(5)\right|=C_{5}$ with the multiset of tail lengths being equal to $\{t(F): F \in$ $\left.\mathcal{A}_{321}(5)\right\}$. Running multiple iterations of this gives the family defined by forbidding the pattern $\sigma(X)$ for each triple $X$ where

$$
\sigma(X)= \begin{cases}(1,3,2) & \text { if } X=\{1,2,3\} \\ (3,2,1) & \text { otherwise }\end{cases}
$$

In particular, for any $n>3$ we have $C_{n}$ size family from $S_{n}$ which is non-shattering and is distinct from $\mathcal{A}_{\sigma}(n)$.

In general, for any starting family $\mathcal{F}$ we remark that, a permutation $F \in \mathcal{F}$ with $t(F)=t$, after one iteration of the construction $F$ has contributed $t+1$ permutations to the new larger family. Each of these permutations has a distinct tail length from $[t+1]$, therefore after two iterations $F$ will have contributed $\sum_{i=2}^{t+2} i=\frac{1}{2}\left(t^{2}+5 t+4\right)$. This information is summarised in Table 5.1.

| Tail length | 1 iteration | 2 iterations |
| :---: | :---: | :---: |
| $t+2$ | 0 | $1=\sum_{i=1}^{1} 1$ |
| $t+1$ | 1 | $2=\sum_{i=1}^{2} 1$ |
| $t$ | 1 | $3=\sum_{i=1}^{3} 1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 3 | 1 | $t=\sum_{i=1}^{t} 1$ |
| 2 | 1 | $t+1=\sum_{i=1}^{t+1} 1=T_{1}$ |
| 1 | 1 | $t+1=\sum_{i=1}^{t+1} 1=T_{1}$ |
| Total | $\sum_{i_{1}=1}^{t+1} 1=t+1:=T_{1}$ | $\sum_{i_{2}=1}^{t+1} \sum_{i_{1}=1}^{i_{2}} 1+T_{1}:=T_{2}$ |

Table 5.1: Number of permutations after 1 and 2 iterations generated by a single permutation depending on its tail length.

After three iterations we have a contribution of $\frac{1}{6}\left(t^{3}+12 t^{2}+41 t+30\right)$ permutations. To see this, note that the number of contributed permutations with tail length $t+3$ is exactly the number of permutations from the previous iteration which have tail length $t+2$, of which there was one. Continuing in the manner, we see that the number of new permutations with tail length $k \geq 3$ is exactly $\sum_{i=1}^{t+4-k} i$, and every previous permutation contributes one new permutation of tail length 2 and 1 respectively. Similarly, we can calculate that the number of contributions in the fourth iteration is $\frac{1}{24}\left(17 t^{4}+162 t^{3}+547 t^{2}+834 t+432\right)$. This information is summarised in Table 5.2 .

| Tail length | 3 iterations | 4 iterations |
| :---: | :---: | :---: |
| $t+4$ | 0 | $\sum_{i_{3}=1}^{1} \sum_{i_{2}=1}^{i_{3}} \sum_{i_{1}=1}^{i_{2}} 1$ |
| $t+3$ | $\sum_{i_{2}=1}^{1} \sum_{i_{1}=1}^{i_{2}} 1$ | $\sum_{i_{3}=1}^{2} \sum_{i_{2}=1}^{i_{3}} \sum_{i_{1}=1}^{i_{2}} 1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 4 | $\sum_{i_{2}=1}^{t} \sum_{i_{1}=1}^{i_{2}} 1$ | $\sum_{i_{3}=1}^{t+1} \sum_{i_{2}=1}^{i_{3}} \sum_{i_{1}=1}^{i_{2}} 1$ |
| 3 | $\sum_{i_{2}=1}^{t+1} \sum_{i_{1}=1}^{i_{2}} 1$ | $\sum_{i_{3}=1}^{t+1} \sum_{i_{2}=1}^{i_{3}} \sum_{i_{1}=1}^{i_{2}} 1+T_{2}$ |
| 2 | $\sum_{i_{2}=1}^{t+1} \sum_{i_{1}=1}^{i_{2}} 1+\sum_{i_{1}=1}^{t+1} 1=T_{2}$ | $T_{3}$ |
| 1 | $\sum_{i_{2}=1}^{t+1} \sum_{i_{1}=1}^{i_{2}} 1+\sum_{i_{1}=1}^{t+1} 1=T_{2}$ | $T_{3}$ |
| Total | $\sum_{i_{3}=1}^{t+1} \sum_{i_{2}=1}^{i_{3}} \sum_{i_{1}=1}^{i_{2}} 1+2 T_{2}:=T_{3}$ | $\sum_{i_{4}=1}^{t+1} \sum_{i_{3}=1}^{i_{4}} \sum_{i_{2}=1}^{i_{3}} \sum_{i_{1}=1}^{i_{2}} 1+T_{2}+2 T_{3}$ |

Table 5.2: Number of permutations after 3 and 4 iterations generated by a single permutation depending on its tail length.

In general, a formula for the number of contributions from $F$ after $k$ iterations is fairly
unwieldy and not particularly insightful, so we do not attempt to give such a formula here. However, a key observation about these contributions is that the function which gives the number of contributions is convex when $t>0$. It is therefore possible that applying the construction to two non-shattering families $\mathcal{F}$ and $\mathcal{H}$, with $|\mathcal{F}|<|\mathcal{H}|$, achieves after $k$ iterations non-shattering families $\mathcal{F}^{\prime}$ and $\mathcal{H}^{\prime}$ with $\left|\mathcal{F}^{\prime}\right|>\left|\mathcal{H}^{\prime}\right|$, as long as $\mathcal{F}$ contains longer tails.

This motivates finding non-shattering families which have a large number of permutations with long tails.

Definition 5.7. Let $\mathcal{F} \subseteq S_{n}$ be any family of permutations and fix an integer $k>0$. We say the number of $\boldsymbol{k}+$ tails of $\mathcal{F}$ to mean the quantity

$$
T_{k}(\mathcal{F}):=|\{F \in \mathcal{F}: t(F) \geq k\}| .
$$

Let $T_{k}(n)=\max \left\{T_{k}(\mathcal{F}): \mathcal{F} \subseteq S_{n}\right.$ is non-shattering $\}$.
We wish to know how large $T_{k}(\mathcal{F})$ can be for a non-shattering $\mathcal{F}$. First observe that, by this definition and the nature of tails, we have that $T_{1}(\mathcal{F})=|\mathcal{F}|$ for all families $\mathcal{F}$. Therefore when $k=1$ the problem is equivalent to the original problem of how large $\mathcal{F}$ can be given that it is non-shattering. Secondly, note that there is exactly one permutation of $S_{n}$ with tail length $n$, namely the increasing permutation $(1,2, \ldots, n-1, n)$. Therefore, it is trivial that $T_{n}(n)=1$. It is also trivial that $T_{n-1}(n)=n$, since the set $\left\{F \in S_{n}: t(F) \geq n-1\right\}$ is itself non-shattering and has size $n$.

We remark that $T_{k}\left(\mathcal{A}_{321}(n)\right)=T_{k}(n)$ whenever $k=n-1, n$. However, if it is the case that $T_{k}\left(\mathcal{A}_{321}(n)\right)<T_{k}(n)$ for some $k$, then iterating the construction on a non-shattering family $\mathcal{F} \subseteq S_{n}$ with $T_{k}(F)=T_{k}(n)$ may, after a number of steps, lead to a non-shattering family from $S_{N}$ which is larger that $\mathcal{A}_{321}(N)$ and therefore has more than $C_{N}$ permutations.

However, it may be the case that the largest non-shattering family of $S_{n}$ really does have size $C_{n}$. In particular, should it be the case that $T_{k}\left(\mathcal{A}_{321}(n)\right)=T_{k}(n)$ for all values of $k$, then there will be no counter-example found by iterating our construction. In this case, it is valuable to have a wider range of non-shattering families with size $C_{n}$, not just the six families of the form $\mathcal{A}_{\sigma}(n)$.

Recall that starting with some non-shattering $\mathcal{F} \subseteq S_{n}$ whose multiset of tail lengths matches the multiset of tail lengths of $\mathcal{A}_{321}(n)$, and repeatedly applying our construction results in a non-shattering $\mathcal{F}^{\prime} \subseteq S_{N}$ with $\left|\mathcal{F}^{\prime}\right|=C_{N}$ (see Lemma 5.5 and Example 5.6).

In particular, setting $\mathcal{F}$ to $\mathcal{A}_{132}(n)$ or $\mathcal{A}_{231}(n)$ produces non-shattering families distinct from $\mathcal{A}_{\sigma}(N)$ but with the same size.

We are interested in finding such an $\mathcal{F}$ for $n>3$. Due to the nature of the construction, in $\mathcal{F}^{\prime}$ the pattern $(3,2,1)$ is forbidden for all triples with an element larger than $n$. We would like to be able to choose the value of $n$ which has a different pattern forbidden, and to do this we must find a valid family $\mathcal{F}$ for arbitrary $n$. The next result does this, in fact we find a family which not only has the same tail lengths as $\mathcal{A}_{321}(n)$, but the same tails.

Lemma 5.8. Let $\mathcal{F}_{n} \subseteq S_{n}$ be the non-shattering family which forbids the pattern $(2,3,1)$ for every triple containing $n$, and forbids $(3,2,1)$ in all other cases. Then we have $\left|\mathcal{F}_{n}\right|=$ $C_{n}$, and moreover, the multiset of tails of $\mathcal{F}_{n}$ is the same as the multiset of tails of $\mathcal{A}_{321}(n)$.

Proof. Will will use induction on $n$. First note that the claim holds when $n=3$.

| $\mathcal{F}_{3}:$ | $(1,2,3)$ | $\mathcal{A}_{321}(3):$ | $(1,2,3)$ |
| ---: | :--- | ---: | :--- |
|  | $(1,3,2)$ |  | $(1,3,2)$ |
|  | $(2,1,3)$ |  | $(2,1,3)$ |
|  | $(3,1,2)$ |  | $(3,1,2)$ |
|  | $(3,2,1)$ |  | $(2,3,1)$. |

Assume that the statement holds for all values up to $n-1$, we will show that it then also holds for $n$. From our induction assumption we have that $\left|\mathcal{F}_{n-1}\right|=\left|\mathcal{A}_{321}(n-1)\right|$ and both families have the same multiset of tails. Therefore there is a bijection from $\mathcal{A}_{321}(n-1)$ to $\mathcal{F}_{n-1}$ which preserves tails. Let $g$ be such a function, then we have that $g(P)$ and $P$ have the same tail whenever $P \in \mathcal{A}_{321}(n-1)$. To prove the statement holds for $n$ we will define a bijection $f$ between $\mathcal{A}_{321}(n)$ and $\mathcal{F}_{n}$ which also preserves tails.

First, for each $x \in \mathbb{N}$ we define a function $h_{x}$ which acts on segments of permutations of any length, and increases by 1 the value of any element larger than $x$. More precisely, if $a_{1}, \ldots, a_{m}$ is a segment of a permutation from $S_{M}$ (where $M>m$ ) then we define

$$
h_{x}\left(a_{1}, \ldots, a_{m}\right)=b_{1}, \ldots, b_{m} \quad \text { where } \quad b_{i}= \begin{cases}a_{i} & \text { when } a_{i}<x \\ a_{i}+1 & \text { when } a_{i} \geq x\end{cases}
$$

We then also have the inverse, which acts on segments that do not contain the entry $x$, formally let $b_{1}, \ldots, b_{m}$ be a segment of a permutation from $S_{M}$ where $x \neq b_{i}$ for every
$i \in[m]$, then define

$$
h_{x}^{-1}\left(b_{1}, \ldots, b_{m}\right)=a_{1}, \ldots, a_{m} \quad \text { where } \quad a_{i}= \begin{cases}b_{i} & \text { when } b_{i}<x \\ b_{i}-1 & \text { when } b_{i}>x\end{cases}
$$

Now we will define the function $f: \mathcal{A}_{321}(n) \rightarrow \mathcal{F}_{n}$ by the following, let $P \in \mathcal{A}_{321}(n)$ and

$$
f(P)= \begin{cases}P & \text { when } P=\left(a_{1}, \ldots, a_{n-2}, n, n-1\right) \\ \left(h_{n-1}(g(P \backslash n)) \backslash x, n-1, x\right) & \text { when } P=\left(a_{1}, \ldots, a_{n-2}, n, x\right) \quad x \neq n-1 \\ P & \text { when } P=\left(a_{1}, \ldots, a_{n-1}, n\right) \\ \left(h_{x_{t}}\left(g\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right)\right), x_{t}\right) & \text { when } P=\left(a_{1}, \ldots, a_{i}, n, x_{1}, \ldots, x_{t}\right) .\end{cases}
$$

Here we use the notation $Q \backslash y$ where $Q \in S_{m}$ and $y \in[m]$ to mean the permutation of the elements $[m] \backslash\{y\}$ given by omitting $y$ from $Q$.

First note that $f$ is defined for every $P \in \mathcal{A}_{321}(n)$. Indeed, let $t(P)=t$ and note that $P$ must take one of the following forms

$$
\left(a_{1}, \ldots, a_{i}, n, x_{1}, \ldots, x_{t}\right) \quad \text { or } \quad\left(a_{1}, \ldots, a_{i}, x_{1}, \ldots, x_{t-1}, n\right),
$$

where $x_{1}<\cdots<x_{t}$. Since $P \in \mathcal{A}_{321}(n)$ there can be no decreasing triples in $P$, therefore everything to the right of $n$ must be increasing. Then, either $n$ is in the tail (and by definition at the end) or the tail begins immediately after $n$.

Fist we will confirm that $f$ is well defined and does indeed preserve tails, then we will show that it is bijective. It is trivial that tails are preserved when $f(P)=P$.

When $P=\left(a_{1}, \ldots, a_{n-2}, n, x\right)$ we have that the tail of $P$ is just $x$, and that $f(P)=$ $\left(h_{n-1}(g(P \backslash n)) \backslash x, n-1, x\right)$ which clearly also has tail $x$. Note that $P \backslash n \in \mathcal{A}_{321}(n-1)$, then we are indeed able to apply $g$. Therefore $f$ is well defined in this case.

When $P=\left(a_{1}, \ldots, a_{i}, n, x_{1}, \ldots, x_{t}\right)$, we must have that $x_{1}<\cdots<x_{t}$ and hence the tail of $P$ is $x_{1}, \ldots, x_{t}$. By definition we have that $f(P)=\left(h_{x_{t}}\left(g\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right)\right), x_{t}\right)$. It is plain that $P \backslash x_{t}$ does not contain the element $x_{t}$ and therefore $h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)$ is defined and contains all the elements of $[n-1]$. In fact $\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right) \in \mathcal{A}_{321}(n-1)$, otherwise any decreasing triple ( $q, r, s$ ) would appear as $\left(h_{x_{t}}(q, r, s)\right)$ in $P,\left(h_{x_{t}}(q, r, s)\right)$ can only take the form $(q+1, r, s),(q+1, r+1, s)$, or $(q+1, r+1, s+1)$ all of which are also decreasing. Moreover, $\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right)$ has tail $x_{1}, \ldots, x_{t-1}$, then $g\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right)$ is defined and also has tail $x_{1}, \ldots, x_{t-1}$. Finally, $\left(h_{x_{t}}\left(g\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right)\right)\right)$ must have tail $x_{1}, \ldots, x_{t-1}$ too, since $x_{1}<\cdots<x_{t}$. Meaning that $f(P)$ is well defined and has tail $x_{1}, \ldots, x_{t}$.

It remains to check that $f: \mathcal{A}_{321}(n) \rightarrow \mathcal{F}_{n}$ is bijective. First, let $P, Q \in \mathcal{A}_{321}(n)$ and suppose that $f(P)=f(Q)$. We have seen that $f$ preserves tails, so $f(P)=f(Q)$ implies that $P$ and $Q$ have the same tail. Suppose first that this tail is $x_{1}, \ldots, x_{t-1}, n$, then we can write $P=\left(a_{1}, \ldots, a_{i}, x_{1}, \ldots, x_{t-1}, n\right)$ and $Q=\left(b_{1}, \ldots, b_{i}, x_{1}, \ldots, x_{t-1}, n\right)$. Then, by the definition of $f$ we have $f(P)=P$ and $f(Q)=Q$, hence $P=Q$. An analogous argument applies when the tail of $P$ and $Q$ is just $n-1$, so $P=Q$ in this case too.

Now suppose that $P$ and $Q$ both have tail $x$ where $x \neq n-1$, then we can write $P=$ $\left(a_{1}, \ldots, a_{n-2}, n, x\right)$ and $Q=\left(b_{1}, \ldots, b_{n-2}, n, x\right)$. Observe that $P \backslash n=\left(a_{1}, \ldots, a_{n-2}, x\right)$ and $Q \backslash n=\left(b_{1}, \ldots, b_{n-2}, x\right)$, these may not have the same tail but both tails end in $x$. Let $g(P \backslash n)=R$ and $g(Q \backslash n)=S$, since $g$ preserves tails we have that $R$ and $S$ both have $x$ as their final element.

The definition of $f$ along with the fact that $f(P)=f(Q)$ imply that $h_{n-1}(R) \backslash x=$ $h_{n-1}(S) \backslash x$. Since $x<n-1$ and both $R$ and $S$ end in $x$, we must have that $h_{n-1}(R)$ and $h_{n-1}(S)$ end in $x$, therefore if $h_{n-1}(R) \backslash x=h_{n-1}(S) \backslash x$ we must have $h_{n-1}(R)=h_{n-1}(S)$. Since $R, S \in \mathcal{F}_{n-1} \subseteq S_{n-1}$, the contain the same elements and therefore we must have $R=S$. This means $g(P \backslash n)=g(Q \backslash n)$, and as $g$ is bijective $P \backslash n=Q \backslash n$. Hence $a_{i}=b_{i}$ for all $i \in[n-2]$ and $P=Q$.

Suppose now that the tail of $P$ and $Q$ is $x_{1}, \ldots, x_{t}$, then $x_{1}<\cdots<x_{t}$ and we can write $P=\left(a_{1}, \ldots, a_{i}, n, x_{1}, \ldots, x_{t}\right)$ and $Q=\left(b_{1}, \ldots, b_{i}, n, x_{1}, \ldots, x_{t}\right)$. Recall that $\left(h_{x_{t}}^{-1}(P \backslash\right.$ $\left.\left.x_{t}\right)\right) \in \mathcal{A}_{321}(n-1)$ and therefore $R:=g\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right) \in \mathcal{F}_{n-1} \subseteq S_{n-1}$, similarly $S:=$ $g\left(h_{x_{t}}^{-1}\left(Q \backslash x_{t}\right)\right) \in S_{n-1}$. This means $R$ and $S$ contain the same elements.

From the definition of $f$ and the fact that $f(P)=f(Q)$, we have $h_{x_{t}}(R)=h_{x_{t}}(S)$. Since $R$ and $S$ contain the same elements this implies $R=S$, and as $g$ is bijective we have further that $h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)=h_{x_{t}}^{-1}\left(Q \backslash x_{t}\right)$. Then observe that

$$
\begin{aligned}
h_{x_{t}}^{-1}\left(P \backslash x_{t}\right) & =h_{x_{t}}^{-1}\left(a_{1}, \ldots, a_{i}, n, x_{1}, \ldots, x_{t-1}\right) \\
& =h_{x_{t}}^{-1}\left(a_{1}, \ldots, a_{i}, n\right), x_{1}, \ldots, x_{t-1}
\end{aligned}
$$

since $x_{1}<\cdots<x_{t}$, and similarly for $h_{x_{t}}^{-1}\left(Q \backslash x_{t}\right)$. Therefore we must have $h_{x_{t}}^{-1}\left(a_{1}, \ldots, a_{i}, n\right)=$ $h_{x_{t}}^{-1}\left(b_{1}, \ldots, b_{i}, n\right)$, again these are acting on the same elements and hence $a_{1}, \ldots, a_{i}, n=$ $b_{1}, \ldots, b_{i}, n$. This implies that $P=Q$. This means we have $P=Q$ whenever $f(P)=f(Q)$.

Finally, we show that for every $Q \in \mathcal{F}_{n}$ there is some $P \in \mathcal{A}_{321}(n)$ such that $f(P)=Q$. Let $Q \in \mathcal{F}_{n}$ and suppose first that $Q$ has tail length one. Then $Q$ must have one of the following forms

$$
\left(b_{1}, \ldots, b_{n-2}, n, n-1\right) \quad \text { or } \quad\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}, n-1, x\right) .
$$

Indeed, $n$ cannot be the final element as it will always be larger than the element to its left, meaning the tail length is at least two. Then we must have $n-1$ to the right of $n$, otherwise we will have a $(2,3,1)$ pattern with $n, n-1$, and anything to the right of $n$. Finally, everything to the right of $n-1$ must be increasing to avoid a decreasing triple containing $n-1$, since this will make the tail there can be at most one element to the right of $n-1$.

Note first that if $Q=\left(b_{1}, \ldots, b_{n-2}, n, n-1\right)$, then $Q \in \mathcal{A}_{321}(n)$ and $f(Q)=Q$. So assume that $Q=\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}, n-1, x\right)$ and consider $h_{n-1}^{-1}(Q \backslash n-1)=$ $b_{1}, \ldots, b_{i}, n-1, b_{i+1}, \ldots, b_{k}, x$. We must have $\left(h_{n-1}^{-1}(Q \backslash n-1)\right) \in \mathcal{F}_{n-1}$. Indeed, if $(r, n-1, s)$ follows $(2,3,1)$ in $\left(h_{n-1}^{-1}(Q \backslash n-1)\right)$, then $\left(h_{n-1}(r, n-1, s)\right)=(r, n, s)$ appears in $Q$ and clearly still follows $(2,3,1)$. Similarly, if ( $q, r, s$ ) follows $(3,2,1)$ in $\left(h_{n-1}^{-1}(Q \backslash n-1)\right)$ with $q, r, s<n-1$, then $\left(h_{n-1}(q, r, s)\right)=(q, r, s)$ appears in $Q$.

Since $\left(b_{1}, \ldots, b_{i}, n-1, b_{i+1}, \ldots, b_{k}, x\right) \in \mathcal{F}_{n-1}$ and $g: \mathcal{A}_{321}(n-1) \rightarrow \mathcal{F}_{n-1}$ is a bijection, there exists a permutation $R \in \mathcal{A}_{321}(n-1)$ such that $g(R)=\left(b_{1}, \ldots, b_{i}, n-\right.$ $\left.1, b_{i+1}, \ldots, b_{k}, x\right)$, moreover $R$ ends with the element $x$ since $g$ preserves tails. Let $P=$ ( $R \backslash x, n, x$ ) and observe that $P$ is simply $R$ with the additional element $n$ inserted between the final two elements, it is plain that $P \in \mathcal{A}_{321}(n)$. Observe

$$
\begin{aligned}
f(P) & =\left(h_{n-1}(g(P \backslash n)) \backslash x, n-1, x\right) \\
& =\left(h_{n-1}(g(R)) \backslash x, n-1, x\right) \\
& =\left(h_{n-1}\left(b_{1}, \ldots, b_{i}, n-1, b_{i+1}, \ldots, b_{k}, x\right) \backslash x, n-1, x\right) \\
& =\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}, n-1, x\right)=Q .
\end{aligned}
$$

Now suppose that $Q$ has tail length $t$, then it must take one of the following forms

$$
\left(b_{1}, \ldots, b_{k}, x_{1}, \ldots, x_{t-1}, n\right) \quad \text { or } \quad\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}, x_{1}, \ldots, x_{t}\right) \text {, }
$$

where $x_{1}<\cdots<x_{t}$ and $b_{1}, \ldots, b_{i}$ or $b_{i+1}, \ldots, b_{k}$ may be empty. We are not using any properties of $Q$ here, all permutations of $S_{n}$ with tail length $t$ must have the above forms - either the tail contains $n$ or it does not.

First note that if $Q=\left(b_{1}, \ldots, b_{k}, x_{1}, \ldots, x_{t-1}, n\right)$ then $Q \in \mathcal{A}_{321}(n)$ and $f(Q)=Q$. Assume then that $Q=\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}, x_{1}, \ldots, x_{t}\right)$ and consider $h_{x_{t}}^{-1}\left(Q \backslash x_{t}\right)$. Note that

$$
h_{x_{t}}^{-1}\left(Q \backslash x_{t}\right)=h_{x_{t}}^{-1}\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}, x_{1}, \ldots, x_{t-1}\right)
$$

$$
=h_{x_{t}}^{-1}\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}\right), x_{1}, \ldots, x_{t-1}=: S
$$

Since the tail of $Q$ is $x_{1}, \ldots, x_{t}$ we must have that $b_{k}>x_{1}$, in particular we must also have $h_{x_{t}}^{-1}\left(b_{k}\right)>x_{1}$. Indeed, $h_{x_{t}}^{-1}\left(b_{k}\right)=b_{k}$ if $b_{k}<x_{t}$, and if $b_{k}>x_{t}$ then $h_{x_{t}}^{-1}\left(b_{k}\right)=b_{k}-1 \geq$ $x_{t}>x_{1}$. Therefore the tail of $S$ is $x_{1}, \ldots, x_{t-1}$.

Further note that $S \in \mathcal{F}_{n-1}$. Indeed, if $(r, n-1, s)$ follows $(2,3,1)$ in $S$, then $\left(h_{x_{t}}(r, n-\right.$ $1, s))$ appears in $Q$. This can take only the forms $(r, n, s),(r+1, n, s)$, or $(r+1, n, s+1)$ since $n-1>r>s$, each gives a triple containing $n$ following $(2,3,1)$ contradicting $Q \in \mathcal{F}_{n}$. Similarly, if $(q, r, s)$ follows $(3,2,1)$ in $S$ with $q, r, s<n-1$, then $\left(h_{x_{t}}(q, r, s)\right)$ appears in $Q$. Again, this can only take the forms $(q, r, s),(q+1, r, s),(q+1, r+1, s)$, or $(q+1, r+1, s+1)$ as $q>r>s$.

Having $S \in \mathcal{F}_{n-1}$ implies there is an $R \in \mathcal{A}_{321}(n-1)$ such that $g(R)=S$ and $R$ has tail $x_{1}, \ldots, x_{t-1}$. Since $R \in \mathcal{A}_{321}(n-1)$, it must have the form $R=\left(a_{1}, \ldots, a_{k}, n-\right.$ $\left.1, x_{1}, \ldots, x_{t-1}\right)$. Set $P=\left(h_{x_{t}}(R), x_{t}\right)=\left(h_{x_{t}}\left(a_{1}, \ldots, a_{k}\right), n, x_{1}, \ldots, x_{t}\right)$, and note that $P \in \mathcal{A}_{321}(n)$. Indeed, suppose $(q, r, s)$ is found in $P$ with $q>r>s$ and $x_{t} \notin\{q, r, s\}$, then $\left(h_{x_{t}}^{-1}(q, r, s)\right)$ is found in $R$. This means ones of $(q, r, s),(q-1, r, s),(q-1, r-1, s)$, or $(q-1, r-1, s-1)$ is in $R$, but all of these form a decreasing triple contradicting $R \in \mathcal{A}_{321}(n-1)$. Suppose instead that $(q, r, s)$ is found in $P$ with $q>r>s$ and $x_{t} \in\{q, r, s\}$, then since $x_{t}$ is the last element of $P$ we must have $s=x_{t}$. This means, $x_{t-1} \notin\{q, r, s\}$ as $q>r>x_{t}>x_{t-1}$, and therefore $\left(h_{x_{t}}^{-1}(q, r), x_{t-1}\right)$ is found in $R$. The only forms $\left(h_{x_{t}}^{-1}(q, r), x_{t-1}\right)$ can take are $\left(q, r, x_{t-1}\right),\left(q-1, r, x_{t-1}\right),\left(q-1, r-1, x_{t-1}\right)$ all of which are decreasing triples.

Since $P=\left(h_{x_{t}}(R), x_{t}\right)=\left(h_{x_{t}}\left(a_{1}, \ldots, a_{k}\right), n, x_{1}, \ldots, x_{t}\right)$ from the definition of $f$ we have that

$$
\begin{aligned}
f(P) & =\left(h_{x_{t}}\left(g\left(h_{x_{t}}^{-1}\left(P \backslash x_{t}\right)\right)\right), x_{t}\right) \\
& =\left(h_{x_{t}}\left(g\left(h_{x_{t}}^{-1}\left(h_{x_{t}}(R)\right)\right)\right), x_{t}\right) \\
& =\left(h_{x_{t}}(g(R)), x_{t}\right) \\
& =\left(h_{x_{t}}\left(h_{x_{t}}^{-1}\left(Q \backslash x_{t}\right)\right), x_{t}\right) \\
& =\left(Q \backslash x_{t}, x_{t}\right)=\left(b_{1}, \ldots, b_{i}, n, b_{i+1}, \ldots, b_{k}, x_{1}, \ldots, x_{t-1}, x_{t}\right)=Q .
\end{aligned}
$$

The above result, along with Lemma 5.5, gives the following.

Corollary 5.9. For any $m \in[3, n]$, there is a non-shattering family $\mathcal{F} \subseteq S_{n}$ of size $C_{n}$ where each triple $X \subseteq[n]$ avoids the pattern $\sigma(X)$ given by

$$
\sigma(X)= \begin{cases}(2,3,1) & \text { if } \max (X)=m \\ (3,2,1) & \text { otherwise }\end{cases}
$$

This observation opens the question: How many forbidden patterns can a non-shattering family from $S_{n}$ have while still having size $C_{n}$ ?

We are able to find small cases with a large number of forbidden patterns, indeed the following family from $S_{4}$ is non-shattering and has size $C_{4}=14$.

$$
\begin{array}{ccccccc}
(2,1,3,4) & (1,4,2,3) & (2,4,1,3) & (4,3,1,2) & (2,1,4,3) & (1,4,3,2) & (4,1,3,2) \\
(4,1,2,3) & (2,3,1,4) & (4,2,1,3) & (1,3,4,2) & (3,1,4,2) & (2,4,3,1) & (4,2,3,1)
\end{array}
$$

Here each triple avoids a different pattern, meaning four patterns are forbidden. This is the maximum possible for a family on this many elements. Observe that

$$
\begin{array}{ll}
\{1,2,3\} \text { avoids }(3,2,1) & \{1,2,4\} \text { avoids }(1,2,3) \\
\{1,3,4\} \text { avoids }(2,3,1) & \{2,3,4\} \text { avoids }(2,1,3) .
\end{array}
$$

However, the multiset of tail lengths of the above is not the same as the tail lengths of $\mathcal{A}_{321}(4)$, and therefore using our construction on this family will produce a non-shattering family smaller than Catalan size. In fact, the above family contains a high proportion of short tailed permutations and is missing those with the longest tails. It is therefore still an open problem to find a non-shattering family from $S_{n}$ with size $C_{n}$ and more than two forbidden triple patterns when $n$ is arbitrary. It is also open to find any Catalan sized non-shattering family with more than four forbidden patterns regardless of $n$, although it is plain that $n>4$.

## Chapter 6

## Summary and open problems

We have investigated a problem of set theoretical origin expressed in terms of permutations, thinking of a permutation as an ordering of a set of elements. We touched on the differences between the two versions of the problem and how the permutation version allows for some interesting variations on what it means to shatter a $k$-tuple. We introduced two different relaxations of the problem, partial and fractional shattering, as well as outlining the general behaviour of each.

We also saw a construction for a completely shattering family, which is the purest translation of shattering into the world of permutations. The study of this, in particular constructions of such families, is an ongoing area of research in this topic. We were also able to consider an inverted formulation of this problem, rather than a small family which shatters everything we look for a large family which shatters nothing. Again we were able to add new construction technique, which allows many different non-shattering families to be generated with the same size as the conjectured extremal family.

In the course of our investigation into this topic we opened up many interesting open problems and areas for further study which we will now discuss in more detail.

For the partial shattering variant, does the size of $f_{k}(n, t)$ always fall into one of the three sizes as classified in Theorem 2.1 and 2.2? We know that there are values of $t$ that put $f_{k}(n, t)$ in each of these regimes, but is there another size bracket in between $\log \log n$ and $\log n$ ? In particular we ask the following question.

Question 6.1. When $k>3$, what is the value of $f_{k}(n, t)$ for $k+1 \leq t \leq 2(k-1)$ ??
It is not difficult to see that when $t$ is odd we can bound $f_{k}(n, t+1) \leq 2 f_{k}(n, t)$ by taking
the family that realises $f_{k}(n, t)$ along with all its reverse permutations. This means that for odd $k$ we know $f_{k}(n, k+1)$ is $O(\log \log n)$. We also ask the following, slightly weaker question.

Question 6.2. Is it true that $f_{k}(n, t)$ is one of three sizes $\Theta(\log n), \Theta(\log \log n)$, or constant for all values of $t$ ?

We remark that in all the previous work on permutation shattering and in the literature, the size of the smallest family which 'shatters' all $k$-tuples falls into one of these size brackets. Here we use 'shatters' to mean any altered formulation of shattering, including the original version, where some prescribed covering of orders is achieved. Examples of this from this thesis are the partial shattering we are discussing, but also the families where all $k$-tuples follow a fixed order. Recall Lemma 2.3 where we saw that a family which covers the order $R \in S_{k}$ for each $k$-tuple has size $\Theta(\log n)$, unless $R$ is monotone in which case trivially the family size is constant. Examples from outside this thesis include, but are not limited to, the original completely shattering problem and the problem of Spencer [27] in which each element of the $k$-tuple must appear before the other $k-1$ in some permutation.

Moving on to fractional shattering, where $\alpha\binom{n}{k} k$-tuples are completely shattered, we ask the following.

Question 6.3. For fixed $k$ and $m$, what is the limit of $F_{k}(n, m)$ as $n$ increases?
We saw that $F_{k}(n, m)$ is decreasing, and by choosing $m \geq k$ ! we know that the limit as $n$ increases exists and is strictly between 0 and 1 . Progress on this question seems to require a method which does not just rely on using fixed small $n$. It would be interesting to find another method for finding upper bounds on $F_{k}(n, m)$ using an alternative approach. The first interesting case of Question 6.3 is the following.

Question 6.4. What is $\lim _{n \rightarrow \infty} F_{3}(n, 6)$ ?

We saw in Chapter 3 that we can use perfect families to give lower bounds for $F_{k}(n, k!)$. Knowing more about when perfect families occur is not only useful for bounds on Fractional Shattering, but is also interesting and worthwhile in its own right. We therefore highlight the question below.

Question 6.5. For which values of $n \geq k$ does the perfect family $\mathcal{Q}_{k}(n)$ exist? In partic-
ular, what is the largest value of $n$ for which it exists?
We have seen that trivially the families $\mathcal{Q}_{k}(k)=S_{k}$ exist, and also it is plain that they exist whenever $k=2$ since at most two permutations are required for any $n$. We also saw in Section 1.4.1 that Levenshtein [18] showed that the families $\mathcal{Q}_{k}(k+1)$ exist. So the question is answered in the positive whenever $n=k, k+1$, however, Levenshtein conjectured that in general no perfect family will exist when $n>k+1$. A counter example was found by Mathon and Van Trung [19] in that $\mathcal{Q}_{4}(6)$ exists. In fact the authors show that two non-equivalent such families exist, however this is the only known counterexample. By a computer search it was found in [19 that $\mathcal{Q}_{4}(7)$ does not exist. See below the the two families realising $\mathcal{Q}_{4}(6)$, firstly

| $(1,2,3,4,5,6)$ | $(6,1,2,5,4,3)$ | $(5,1,4,6,2,3)$ | $(4,1,5,2,6,3)$ |
| :--- | :--- | :--- | :--- |
| $(1,5,3,6,2,4)$ | $(1,6,3,5,4,2)$ | $(1,4,3,2,6,5)$ | $(5,6,4,1,3,2)$ |
| $(6,5,2,1,3,4)$ | $(2,1,6,4,5,3)$ | $(2,4,6,1,3,5)$ | $(4,2,5,1,3,6)$ |
| $(2,5,6,3,1,4)$ | $(2,3,6,5,4,1)$ | $(5,2,4,3,1,6)$ | $(6,4,2,3,1,5)$ |
| $(3,5,1,2,6,4)$ | $(3,6,1,4,5,2)$ | $(3,4,1,6,2,5)$ | $(4,6,5,3,1,2)$ |
| $(3,2,1,5,4,6)$ | $(6,3,2,4,5,1)$ | $(5,3,4,2,6,1)$ | $(4,3,5,6,2,1)$. |

Secondly,

| $(1,2,3,4,5,6)$ | $(6,1,2,5,4,3)$ | $(1,5,3,2,6,4)$ | $(3,4,1,2,6,5)$ |
| :--- | :--- | :--- | :--- |
| $(1,6,3,5,4,2)$ | $(6,3,2,4,5,1)$ | $(4,3,5,2,6,1)$ | $(5,1,4,2,6,3)$ |
| $(2,1,6,4,5,3)$ | $(1,4,3,6,2,5)$ | $(5,3,4,6,2,1)$ | $(4,3,5,2,6,1)$ |
| $(2,3,6,5,4,1)$ | $(4,1,5,6,2,3)$ | $(3,5,1,6,2,4)$ | $(1,5,3,2,6,4)$ |
| $(3,2,1,5,4,6)$ | $(5,1,4,2,6,3)$ | $(4,1,5,6,2,3)$ | $(3,5,1,6,2,4)$ |
| $(3,6,1,4,5,2)$ | $(3,4,1,2,6,5)$ | $(1,4,3,6,2,5)$ | $(5,3,4,6,2,1)$. |

In general it is possible to get a non-existence result by application of the Erdős-Szekeres Theorem, meaning we know there is no perfect family whenever $n>k^{4}$. However, this is weak and clearly far from the conjectured bound.

In Chapter 4 we saw a construction of a completely shattering family. As discussed in the chapter, this construction offers a small shattering family but does not match the best known size of such a family, which is $O(\log n)$. No known construction offers this value, and so it is an open problem to find one.

Question 6.6. Is there a construction which shows $f_{k}(n)=O(\log n)$ ?
Finally, there are a number of interesting questions pertaining to the VC-dimension and non-shattering families. Firstly, we saw a way of constructing non-shattering families
which have size $C_{n}=\frac{1}{n+1}\binom{2 n}{n}$, but are not equal to $\mathcal{A}_{\sigma}$ for any pattern $\sigma \in S_{3}$. In their paper [23], Raz conjectures that $\mathcal{A}_{\sigma}$ is the largest non-shattering family for any $n$. If this is the case, then using our construction gives lots of examples of maximumly sized non-shattering families which are non-equivalent. Since there are so many of these nonshattering families with size $C_{n}$, it might suggest that this is not in fact the extremal solution.

Question 6.7. Is there a non-shattering family $\mathcal{F} \subseteq S_{n}$ with $|\mathcal{F}|>C_{n}$ ?
As an intermediate step for Question 6.7, we noted the relationship between our construction and the tail lengths of the initial family. In particular, the convex nature of the number of permutations supplied by a given starting permutation after $k$ iterations of the construction. This observation suggest that our construction could, after enough iterations, produce a non-shattering family with size larger than $C_{n}$, as only as some properties about the tails in the original family are satisfied. This motivates the following question.

Question 6.8. How large can $T_{k}(\mathcal{F})$ be for a non-shattering family $\mathcal{F}$ ?
In Chapter 5 we also discussed the number of different patterns of $S_{3}$ that can be avoided by at least one triple of $[n]$ in a non-shattering family of size $C_{n}$. Corollary 5.9 tells us that we can find a non-shattering family of Catalan size which forbids two patterns of $S_{3}$ and where we have a certain degree of choice over which triples avoid which pattern. We also saw an example of a non-shattering family of $S_{4}$ which has four forbidden patterns and has size $C_{4}$, but this could not be extended to family on larger $n$. This gives us the following open questions.

Question 6.9. For any $n$, is there a non-shattering family of $S_{n}$ with size $C_{n}$ that avoids 5 or 6 patterns of $S_{3}$ ?

Question 6.10. How many distinct patterns of $S_{3}$ can be avoided by a non-shattering family of $S_{n}$ which has size $C_{n}$ ?

## Part II

## Separating

## Chapter 7

## Introduction

### 7.1 Background and definitions

The study of separation problems was initiated by Rényi in the 1960s [24]. The problem is to find a minimal family $\mathcal{F}$ of subsets of ground set $[n]=\{1,2, \ldots, n\}$, so that for every ordered pair of distinct $x, y \in[n]$ there is some $F \in \mathcal{F}$ with either $x \in F$ and $y \notin F$, or $y \in F$ and $x \notin F$.

It is trivial that in this case $|\mathcal{F}|=\left\lceil\log _{2}(n)\right\rceil$. However, by applying some structure to our ground set or various restrictions to the members of $\mathcal{F}$, the question opens up to be an interesting problem. One particularly interesting way of doing this is to have the ground set be vertices or edges of a graph, and the separators inherit certain properties from the graph (see [2], [3], [4], [6], [7], [10], [11], [17]).

Here we will focus on separation in the context of a ground set of graph edges, and where all separators are restricted to being paths on these edges. Problems with such a focus were introduced by Balogh, Csaba, Martin, and Pluhár [3] as well as Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [10]. We begin by outlining exactly what we mean to separate a graph by paths.

Definition 7.1. Let $G$ be a graph and e, $e^{\prime} \in E(G)$, and let $P \subseteq E(G)$. We say that $P$ separates $e$ and $e^{\prime}$ if we have $e \in P$ and $e^{\prime} \notin P$, or $e^{\prime} \in P$ and $e \notin P$. Let $\mathcal{S}$ be a family of subsets of $E(G)$ such that for any distinct edges e, $e^{\prime} \in E(G)$ there is some $P \in \mathcal{S}$ which separates $e$ and $e^{\prime}$, then we say that $\mathcal{S}$ is a separating system for $G$. If we also have the condition that every element of $\mathcal{S}$ is a path in $G$, then we call $\mathcal{S}$ a separating path system of $G$.

Example 7.2. Here the base graph $G$ is shown in black, together the three coloured paths form a separating path system of $G$.


The definition naturally leads to questions about the size of the set $\mathcal{S}$. We will use the notation $f(G)$ to mean the size of the smallest path separating system for a graph $G$. It is immediate that $E(G)$ is itself a separating path system where all the paths consist of a single edge. So in particular we get the bound $f(G) \leq|E(G)|$. We can also 'forget' the structure of the paths and consider this separating problem in a purely set theoretical way, giving us the lower bound $f(G) \geq \log _{2}(|E(G)|)$ as discussed earlier for the set case.

There are several key observations from the definition that highlight the structure of small separating path systems. Let $\mathcal{S}$ be a separating path system for a graph $G$. Firstly, there can be no more than one edge that is not found in any $P \in \mathcal{S}$. This means our family of paths must cover all but one of the edges of $G$. Looking at Example 7.2 we see that the edge $(2,4)$ is not covered by the system, yet it is separated from each other edge by a path which covers the other.

Secondly, at most one edge in $G$ can appear in every path of $\mathcal{S}$. If not, say $e, f \in P$ for all $P \in \mathcal{S}$, then there is clearly no $P$ that separates $e$ and $f$, therefore $\mathcal{S}$ cannot be a separating path system for $G$. Again in Example 7.2 the edge $(2,3)$ is the only one to appear in every path.

Finally, looking at some path $P$ in $\mathcal{S}$, if any two edges appear exclusively in $P$ then they cannot be separated by $\mathcal{S}$. Hence, for any path in $\mathcal{S}$ there is at most one unique edge, that is, at most one edge which does not appear in any other path. Note in Example 7.2 the edges $(1,2),(4,5)$, and $(5,2)$ are unique to their paths.

### 7.2 Weak separation and strong separation

In the literature, two different varieties of separating system exist, we have defined above what is sometimes known as 'weak separation'. The case where we have all pairs $x, y$ separated by two separators in an identical way is referred to as 'strong separation'. Equivalently, unordered distinct $x, y \in[n]$ are strongly separated by $\mathcal{F}^{\star}$ if there exists $F, F^{\prime} \in \mathcal{F}^{\star}$ with $x \in F, y \notin F$ and $y \in F^{\prime}, x \notin F^{\prime}$. For paths this gives us the following.

Definition 7.3. Let $G$ be a graph and $e, e^{\prime} \in E(G)$, and let $P, P^{\prime} \subseteq E(G)$. We say that $P$ and $P^{\prime}$ strongly separate $e$ and $e^{\prime}$ if we have $e \in P$ and $e^{\prime} \notin P$, and $e^{\prime} \in P^{\prime}$ and $e \notin P^{\prime}$. Let $\mathcal{S}^{\star}$ be a family of subsets of $E(G)$ such that for any distinct edges e, $e^{\prime} \in E(G)$ there exists $P, P^{\prime} \in \mathcal{S}^{\star}$ which separate e and $e^{\prime}$, then we say that $\mathcal{S}^{\star}$ is a strongly separating system for $G$. If we also have the condition that every element of $\mathcal{S}^{\star}$ is a path in $G$, then we call $\mathcal{S}^{\star}$ a strongly separating path system of $G$.

It is plain that a strong separating system is also a weak separating system, and therefore any constructions or upper bounds for the strong variant provide bounds for the weak formulation of the problem. We denote the smallest size of a strongly separating path system of a graph $G$ by $f^{\prime}(G)$, so we have that $f(G) \leq f^{\prime}(G)$.

To consider the differences between the two problems, we will go back to the basic set setting first. Let $\mathcal{F}=\left\{F_{1}, F_{2}, \ldots, F_{k}\right\}$ be a family of subsets of $[n]$ which (weakly) separates all pairs in $[n]$. We can assign to each element of $[n]$ a $k$-tuple which encodes which sets from $\mathcal{F}$ contain it. Consider $x \in[n]$, let $x$ be represented by the vector $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ where $x_{i}=1$ if $x \in F_{i}$ and otherwise $x_{i}=0$. Then the necessary and sufficient condition for $\mathcal{F}$ to be a (weakly) separating system for $[n]$ is that each $x \in[n]$ is assigned a unique vector. Indeed, if any pair of elements are assigned the same vector then they agree on their inclusion or exclusion of all the sets in $\mathcal{F}$ and are therefore not separated. We conclude that $n \leq 2^{k}$ since there are at most $2^{k}$ distinct binary $k$ length vectors. This gives us the bound

$$
|\mathcal{F}|=k \geq \log n
$$

Now consider the strong version, let $\mathcal{F}^{\star}=\left\{F_{1}, F_{2}, \ldots, F_{k}\right\}$ be a family of subsets of $[n]$ which strongly separates all pairs in $[n]$. It is no longer enough that each element receives a unique vector encoding the inclusions. Instead, for any pair $x, y \in[n]$ we must have
that $\left\{i: x_{i}=1\right\} \nsubseteq\left\{i: y_{i}=1\right\}$ and $\left\{i: y_{i}=1\right\} \nsubseteq\left\{i: x_{i}=1\right\}$. To bound the size of this we can directly apply a result of Chung, Graham and Winkler [8, Lemma 1.10.

More precisely, if $A_{x}=\left\{i: x \in F_{i}\right\}$ then $A_{1}, \ldots, A_{n}$ must be an antichain of subsets from $[k]$. Therefore, $n$ is restricted by the size of the largest antichain of $[k]$. By Sperner's Theorem, the size of the largest antichain is $\binom{k}{\left\lfloor\frac{k}{2}\right\rfloor}$ and therefore the necessary condition for $\mathcal{F}^{\star}$ to be a strong separating system is $\binom{k}{\left\lfloor\frac{k}{2}\right\rfloor} \geq n$.

Using a well known lower bound from Stirling's formula we have that

$$
\left|\mathcal{F}^{\star}\right|=k \geq \log n+\log \log n
$$

In both cases these bounds are attainable and therefore give the smallest possible size of a separating system.

Clearly, the two problems are very similar, and there is no great discrepancy between the sizes of the smallest separating systems for the two versions. It is more natural to consider the weak version in this setting, where one classification covers one object and not the other thereby setting them apart. In this thesis we will keep in line with the set-separating inspiration and focus on the weak version.

However, we should note that for separating path systems the two versions of the problem do have some more significant differences. Most of these differences affect the way certain techniques or constructions work, meaning that many techniques that work well for the weak problem do not work as efficiently in the strong variation. Crucially the impact and importance of covering edges varies between the two. The occurrence of uncovered edges and path-unique edges cannot happen in a strong system. We will touch more on the impact of this when we see the techniques later.

### 7.3 Previous results

There are many ways to give the ground set or separators additional structure. For instance, staying within set theory, restricting the size of the separating sets (see [13], [16], [22], [30]). There are also examples of graph structure being imposed on separating systems that do not take the form of the separating path systems we have defined. In particular we could set up an analogous system with vertices being separated rather than edges, or use some sub-graph which is not a path as the separator (see [2], 4], 7], [11).

The first to formulate the problem in terms of graph edges and separating paths were Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [10] and independently at the same time Balogh, Csaba, Martin, and Pluhár [3]. The authors of 10 focus on the weak variant of separation, while the paper [3] investigates strong separation.

In [10] the authors find bounds on $f(G)$ for a selection of graphs $G$ including trees and certain random graphs. For a tree T, they show the following.

Theorem 7.4 (Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [10]). Let $T$ be a tree on $n \geq 4$ vertices. Then

$$
\left\lceil\frac{n+1}{3}\right\rceil \leq f(T) \leq\left\lfloor\frac{2(n-1)}{3}\right\rfloor
$$

Furthermore, these bounds are best possible.
The extremal tree for the upper bound is the star of order $n$, which is simply a vertex with $n-1$ neighbours and no other edges. The longest possible path in the star has length 2 , and each path of a separating path system contains an edge that must appear in another path in the system. We can cover and separate any three edges of a star from all edges by taking two paths which share exactly one edge. A separating path system can then be made by partitioning the edges of the star into triples and for each triple covering and separating them in two paths.

The extremal tree for the lower bound is the 'hair comb' of order $3 n$. This graph is made up of an $n-1$ length path for the 'spine' of the comb, and from each vertex of the spine there is a 'tooth' which is path of length 2. A separating path system for this tree consists of $n-1$ equivalent paths of length 4 , each containing two edges from a single tooth, one spine edge, and one edge from the next tooth. Plus a further two paths, one consisting of the spine edges and nothing else, the other containing a further edge from the first tooth and another from the last.

The other results of [10] work towards answering a very nice conjecture made by FalgasRavry, Kittipassorn, Korándi, Letzter, and Narayanan in the same paper.

Conjecture 7.5 (Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [10]). There exists and absolute constant $C$ such that, for every graph $G$ on $n$ vertices, $f(G) \leq$ $C n$.

They prove that the conjecture holds for graphs with linear minimum degree as well as random graphs. The strategy for proving these graphs have a separating path system of
linear size involves partitioning the edges of $G$ between two subgraphs $G_{1}$ and $G_{2}$. We then take a path decomposition of $G_{1}$, and find a decomposition into matchings where no two edges from the same path are in the same matching. The matching can then be extended into a path in $G$ using edges from $G_{2}$. The process is then done with $G_{1}$ and $G_{2}$ swapped, giving a separating path system of $G$.

The authors in fact suggest that the value of $C$ could in fact be arbitrarily close to 1 . There is no graph known to have $f(G)>n$, and the best known methods yield a lower bound of $f(G) \geq n-1$

The strong version of the problem was introduced at the same time by Balogh, Csaba, Martin, and Pluhár in [3]. In this paper the authors also work towards identifying the size of the smallest (strongly) separating path system for various classes of graph including trees, complete graphs, hypercubes, and random graphs.

In [3] the value of $f^{\prime}(T)$ for a tree $T$ is given directly for $T$ based on the vertex degrees within the graph, rather than bounds for all trees given by Theorem 7.4 in [10].

Theorem 7.6 (Balogh, Csaba, Martin, and Pluhár [3]). Let $F$ be a forest with $v_{i}$ vertices of degree $i$, and $p$ path-components. Then $f^{\prime}(F)=v_{1}+v_{2}-p$.

This puts the smallest possible size of a strongly separating path system for a tree $T$ on $n$ vertices at $f^{\prime}(T)=\left\lceil\frac{n}{2}\right\rceil+1$. This is achieved when $T$ only has vertices with degree 1 or 3 , unless $n$ is odd in which case there is also a single vertex of degree 2 or 4 . On the other hand, $f^{\prime}(T)$ can be as large as $n-1$. Just as in the weak case (Theorem 7.4) the extremal example for this largest case is again the star on $n$ vertices. The difference here is that for each path in the star, there are at most two edges and both must appear in a further path of the separating system. Therefore simply taking each path to be a single edge is the most efficient.

Interestingly, the authors of [3 also conjecture that the every graph on $n$ vertices admits a (strongly) separating path system with size $O(n)$. Meaning that although the problems exhibit different behaviours (as seen by the difference between Theorems 7.4 and 7.6) overall the values of $f(G)$ and $f^{\prime}(G)$ have the same order of magnitude.

Conjecture 7.7 (Balogh, Csaba, Martin, and Pluhár [3]). There exists a constant $C$ such that, for every positive integer $n$ and for every graph $G$ on $n$ vertices, $f^{\prime}(G) \leq C n$.

The authors of [3] prove that Conjecture 7.7 holds for complete graphs, hypercubes, and
random graphs by providing upper bounds in these cases. They use a range of probabilistic and entropy arguments to achieve these bounds. In particular we highlight the bound on the complete graph, since this is the only previous work towards separating path systems of the complete graph which we will see in Chapter 8 .

Theorem 7.8 (Balogh, Csaba, Martin, and Pluhár [3). For $n \geq 10$ we have $f^{\prime}\left(K_{n}\right) \leq$ $4\left\lceil\frac{n}{2}\right\rceil+2 \leq 2 n+4$.

Both of the papers [10] and [3] state that the best known upper bound for $f(G)$ or $f^{\prime}(G)$ when $G$ is any $n$ vertex graph is $O(n \log n)$. This can be seen easily using set separation of trivial size $(\log (|E(G)|))$ and path decompositions of size $n$. Neither paper make any improvement to this general bound beyond providing improved bounds for a range of graphs $G$. More recently, Letzter further developed the methods of [10] to provide an improved general upper bound. In [17], Letzter shows the following.

Theorem 7.9 (Letzter [17]). Let $G$ be an $n$ vertex graph, then $f^{\prime}(G)=O\left(n \log ^{\star} n\right)$.
Where $\log ^{\star} n$ is the minimum number of times the logarithm must be applied iteratively, in order to get a result less than 1.

Shortly after, a further improvement was given by Bonamy, Botler, Dross, Naia, and Skokan in [6]. This breakthrough construction proves Conjectures 7.5 and 7.7 with a constant of $C=19$.

Theorem 7.10 (Bonamy, Botler, Dross, Naia, and Skokan [6]). Every graph on n vertices has a strongly separating path system of size $19 n$.

This construction makes use of Pósa rotation-extension and induction techniques. A key part of the proof is being able to partition the edges of $G$, using Pósa rotation-extension to get a class of edges which are suited to being covered efficiently by a clever path system, and using induction on the remaining part.

Finally, the most recent work on this problem can be found in [2], where Arrepol, Asenjo, Astete, Cartes, Gajardo, Henríquez, Opazo, Sanhueza-Matamala, and Thraves Caro provide further bounds for trees in the weak setting. They prove a result which directly gives the size of $f(T)$ based on the number of 1 and 2 degree vertices, similar to Theorem 7.6 .

Theorem 7.11 (Arrepol, Asenjo, Astete, Cartes, Gajardo, Henríquez, Opazo, Sanhueza-
-Matamala, and Thraves Caro [2]). Let $T^{a}$ be the binary tree of depth 2, then $f\left(T^{a}\right)=4$. Let $T \neq T^{a}$ be a tree with $v_{i}$ vertices of degree $i$, then

$$
f(T)=\max \left\{\left\lceil\frac{2 v_{1}+v_{2}}{3}\right\rceil,\left\lceil\frac{v_{1}+v_{2}}{2}\right\rceil\right\} .
$$

The authors also investigate trees in the analogous problem of vertex separation, where the aim is that every pair of vertices of the graph is separated by a path which contains one vertex and not the other. They provide a similar result in this case.

### 7.4 Structure of Part II

The remainder of Part II will be structured as follows. Each chapter will be concerned with finding bounds for $f(G)$ for a particular graph $G$. Chapter 8 will focus on the the complete graph on $n$ vertices $G=K_{n}$, Chapter 9 will consider the complete bipartite graph $G=K_{n, n}$, and finally Chapter 10 will focus on graphs with lattice structure - primarily ladders and grids. We finish with some open problems and directions for further study in Chapter 11.

Chapter 8 will begin by formalising the problem and showing the lower bound given in [10]. We then move on to the upper bound in Section 8.2, first focusing on construction methods which utilise the symmetric nature of $K_{n}$ in order to improve the upper bound of [3] (Theorem 7.8).

A key tool for our methods is the notion of generator paths, a special path in $K_{n}$ with nice properties that allow us to create a separating path system by taking only rotated copies of the generator. We formally define this in Section 8.2 and show that they exist for small values of $n$. We show that existence of such a path gives rise to a separating path system of size $n$. In Section 8.3 we show that these paths exist whenever $n$ is prime, and therefore a separating path system of $K_{n}$ with size $n$ exists when $n$ is prime. Moreover, we show that a separating path system of $K_{n}$ with size $n$ exists whenever $n=p+1$ for all odd primes $p$.

Our generalised upper bound given in Section 8.4 comes from constructing an approximate version of this special generator path, and correcting any problems with a small number of additional paths. The bulk of the work is in finding this approximation path, which is the content of Section 8.5.

We finish this chapter with some discussion about the methods and potential extensions we considered. This is the content of Section 8.6.

In Chapter 9 we consider additional uses of generator paths, in particular how they relate to the complete bipartite graph. We first consider the lower bound of $f\left(K_{n, n}\right)$ in before moving on to provide an upper bound which shows a separating path system of $K_{n, n}$ with size smaller than $2 n$ exists. This shows that in the weak setting, complete graphs are 'worse' than complete bipartite graphs in terms of efficient separating families of paths.

Finally, in Chapter 10 we consider the other extreme of the problem. Graphs which admit a separating path system with the smallest possible order of magnitude. We have already seen that in set theoretic terms with no structure on the separators or ground set, a separating system must have size at least $\log n$ where $n$ is the size of the ground set. We look at constructions for separating path systems on certain graphs which have size very close to this $\log |E(G)|$ value.

## Chapter 8

## Complete graphs

In [10] the authors find bounds on $f(G)$ for a selection of graphs $G$ including trees and certain random graphs. They also ask about the case where $G$ is the complete graph on $n$ vertices, that is to determine the exact value of $f\left(K_{n}\right)$. The current best known bounds are

$$
n-1 \leq f\left(K_{n}\right) \leq 2 n+4
$$

The lower bound is a simple counting argument found in [10, and the upper bound here is from Theorem 7.8. The upper bound is given by a probabilistic argument and is actually an upper bound on the strong version of the problem. So there is a strongly separating path system for $K_{n}$ with size at most $2 n+4$. However, there has been no other work towards an upper bound for the complete graph for either variation of the problem. Currently there are no results, or works toward a result, for the complete graph based in the weak setting.

In this chapter we will consider separating path systems on complete graphs. This class of graphs is a natural case to consider for this problem. Firstly, not that much is known about the problem for complete graphs beyond two very simple arguments giving the above bounds. Given that complete graphs are a very well studied and often interesting class of graphs it is very natural to want to explore this case deeper.

Secondly, it is unclear at first glance whether complete graphs should be efficient or not in terms of the size of a separating path system. On one hand, there are many more edges for the number of vertices when compared to other graphs, from a counting standpoint this could indicate that complete graphs might require a large number of paths to separate its edges. On the other hand, every possible path between vertices is available to use, there are no limits at all on where the paths can go or how long/short they must be. This
flexibility might imply that we are able to choose paths very cleverly and optimally in order to form a small separating path system.

### 8.1 Lower bound

As discussed, a firm lower bound is the set theoretic minimum of $\log _{2}\binom{n}{2}$. Currently, the best method for getting lower bounds is to use relatively simple counting arguments. We will only be able to make use of one property from the restriction that our separators must be paths, which is the number of edges that a path may have.

Recall the simple observations about separating path systems made in the introduction. In particular the fact that there is at most one edge of the base graph that may be left out of all paths, and the fact that each path has up to one unique edge. We will use these to form an argument giving the lower bound, the authors of [10] use an equivalent method to give this bound. This version is slightly more expanded in order to demonstrate the properties small separating path systems should have.

Lemma 8.1 (Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [10]). For the complete graph on $n$ vertices, the minimum size of a path separating system is at least $n-1$. That is, $f\left(K_{n}\right) \geq n-1$.

Proof. Let $n \in \mathbb{N}$ and suppose for a contradiction that $\mathcal{S}$ is a separating path system for $K_{n}$ such that $|\mathcal{S}| \leq n-2$.

Each $P \in \mathcal{S}$ has at most 1 unique edge, therefore at most $n-2$ edges of $K_{n}$ appear exactly once in $\mathcal{S}$. Recall that there is at most 1 edge that appears exactly 0 times in $\mathcal{S}$. We use this to count the number of edges that appear at least twice in $\mathcal{S}$, the number of such edges is at least

$$
\binom{n}{2}-(n-2)-1=\frac{1}{2}(n-2)(n-1) .
$$

On the other hand, as the maximum length of any $P \in \mathcal{S}$ is $n-1$, we have that the total number of edges used (with multiplicity) is at most $(n-1)(n-2)$. The number of edges in $K_{n}$ appearing in $\mathcal{S}$ is at least $\binom{n}{2}-1$. We can count the edges that appear twice using these values, this is at most

$$
(n-2)(n-1)-\left(\binom{n}{2}-1\right)=\frac{1}{2}(n-2)(n-3)
$$

This is a contradiction since $\mathcal{S}$ cannot satisfy both conditions, hence $f\left(K_{n}\right) \geq n-1$.

This proof demonstrates the importance of long paths in separating systems. To get an upper bound matching this lower bound of $n-1$, we must construct families with full length paths. In fact any separating path system for $K_{n}$ with size $n-1$ must have the following properties:

- Each path has length $n-1$,
- Every path in the system has one unique edge,
- All other edges appear in exactly two paths.

With this in mind it is easy to construct separating path systems of size $n-1$ by hand for small values of $n$, showing this lower bound is tight for small $n$.






Figure 8.1: Example of a separating path system of size $n-1$

It should also be noted that the only property of paths used in the proof of Lemma 8.1 is the fact that a path contains at most $n-1$ edges. Since this is true of many graphs it can be used in other separation contexts. For example, any tree which is a subgraph of $K_{n}$ has at most $n-1$ edges, so this lower bound holds for trees in general. In fact, by taking a family of $n-1$ stars, each centred around a different vertex of $K_{n}$, we create a separating tree system for $K_{n}$ of size exactly $n-1$. So the lower bound of $n-1$ is tight for separating tree systems.

### 8.2 Symmetries and generator paths

Thinking of the vertices of $K_{n}$ as vertices of a regular polygon, we may take any path $P$ and create a new path $P^{\prime}$ by rotating each edge of $P$ one vertex clockwise. In this way we can find $n$ paths that are isomorphic copies of each other. If our initial path $P$ has some carefully chosen properties forcing each rotation to share exactly one edge with $P$,
then we may have a family of separating paths generated by a single path. This reduces the problem to looking for one path rather than a whole system of paths. We need to be careful when choosing our path so that the rotations overlap each other in the right way.

To set it up, label the vertices $1, \ldots, n$ and arrange them clockwise on a regular polygon. It is helpful to think of our edges by the distance they travel rather than endpoints here. We call an edge at vertex $v$ an $\boldsymbol{x}$-type edge if its other endpoint is $v+x$ for $1 \leq x \leq\left\lfloor\frac{n}{2}\right\rfloor$, where all calculations are done modulo $n$. Note that there are precisely $n$ edges in $K_{n}$ that have type $x$ for $1 \leq x<\frac{n}{2}$, and in the case where $n$ is even there are precisely $\frac{n}{2}$ edges that have type $\frac{n}{2}$.

In Figure 8.2 we have an example of a path $P=(4,1,2,5)$


Figure 8.2: Rotations in $K_{5}$ in $K_{5}$. This path contains two 2-type edges, at vertices 4 and 5 , and one 1 -type edge at vertex 1 . Consider the path $P^{\prime}=(5,2,3,1)$, it also contains two 2-type edges, this time at vertices 5 and 1, and one 1-type edge at vertex 2 . This new path is simply $P$ rotated once clockwise. Note that the edge types appear in the same order and at the same distance from each other, this is because edge type is unchanged by rotation. If we take the family consisting of all 5 rotations of $P$, we get a separating path system for $K_{5}$.

Naturally we wish to pin down the properties a path must have in order to generate a separating path system entirely on its own. But first we clarify some terminology and notation. For any $x$-type edge $e=(v, v+x)$ we call $v$ the starting vertex of $e$. Note that every edge has a unique starting vertex unless it is of type $\frac{n}{2}$, in which case either endpoint can be considered as the starting vertex. We define the clockwise distance (on $\boldsymbol{K}_{n}$ ) between vertices $v$ and $u$ to be the value $\min (|v-u|, n-|v-u|$ ) and write $c d(v, u)$ for this. Similarly, we say the clockwise distance (on $\boldsymbol{K}_{n}$ ) between two edges $e$ and $e^{\prime}$ to mean the clockwise distance between the starting vertices of $e$ and $e^{\prime}$ and we write $c d\left(e, e^{\prime}\right)$ for this value. In particular, if $e=(v, v+x)$ has type $x$ and $e^{\prime}=(u, u+y)$ has type $y$, then $c d\left(e, e^{\prime}\right)=c d(v, u)=\min (|v-u|, n-|v-u|)$.

Firstly, note that if any $x$-type edge is included in $P$, then all $n x$-type edges of $K_{n}$ appear among the $n$ rotations of $P$. Similarly, if there are no edges of type $x$ in $P$, then none of the rotations of $P$ will contain an $x$-type edge and therefore none of the $n x$-type edges of $K_{n}$ will be covered by the family of rotations of $P$. We are permitted a single edge
of $K_{n}$ to stay uncovered by the family, but clearly this cannot happen in a family of $n$ rotations of a single path, therefore we must cover all the edges of $K_{n}$ with our rotations. This means that there must be at least one edge of each of the $\left\lfloor\frac{n}{2}\right\rfloor$ types in $P$.

It quickly becomes obvious at this point that we have to treat $K_{n}$ differently depending on whether $n$ is odd or even. If $n$ is even then we have that an $\frac{n}{2}$-type edge at any vertex $v$ is the same as an $\frac{n}{2}$-type edge at $v+\frac{n}{2}$. This, along with some other properties of even $n$, means more care is required when dealing with even $n$ and the two cases will have slightly different properties for the path $P$.

Now, we want the edges of our path to be separated from each other by the family of rotations, and recall that at most one edge can be unique to $P$. Therefore, we need all edges (except at most one) in the original path $P$ to appear separately in the subsequent rotations. Note that if there is exactly one edge of type $x$ in $P$, then this edge is unique to $P$ and in turn each rotation of $P$ has a unique $x$-type edge. Clearly then we require that at most one edge type appears exactly once in $P$, the remaining types must all have at least two edges in $P$.

The last thing to ensure is that no two edges in $P$ appear together later on in a different rotation, otherwise they will not be separated. This means if some rotation $P_{r}$ of $P$ covers two edges of $P$ we have a problem. In terms of edge types, we need to look at the pairs of edges in $P$ with the same type. So in order to avoid this double copying we must have that the distance between two edges of type $x$ must not appear at the same clockwise distance as two edges of type $y$ in $P$.

These are the necessary conditions for the family of rotations of $P$ to be a separating path system of $K_{n}$. We formalise these below (with a few more conditions for ease) in the definition of a generator path.

Definition 8.2. A path $P$ on the complete graph $K_{n}$ is called a generator path for $n$ if it satisfies the following conditions. For odd $n$ :
(GP1) $P$ contains at least one edge of each of the $\frac{n-1}{2}$ types.
(GP2) There is at most one edge type that appears exactly once, and there are no edge types that appear more than twice in $P$.
(GP3) Let e and $e^{\prime}$ be x-type edges in $P$, and let $c d\left(e, e^{\prime}\right)=d$. If there are two other edges $h, h^{\prime} \in P$, such that both have type $y \neq x$, then $c d\left(h, h^{\prime}\right) \neq d$.

For even n:
(GP1) $P$ contains at least one edge of each of the $\frac{n}{2}$ types.
(GP2) There is at most one edge type from $\left[\frac{n}{2}-1\right]$ that appears exactly once, and there are no edge types that appear more than twice in $P$.
(GP3) Let $e$ and $e^{\prime}$ be $x$-type edges in $P$, and let $c d\left(e, e^{\prime}\right)=d$. If there are two other edges $h, h^{\prime} \in P$, such that both have type $y \neq x$, then $c d\left(h, h^{\prime}\right) \neq d$. Additionally, no pair of edges in $P$ with the same type are at distance $\frac{n}{2}$.

Example 8.3. Consider $K_{7}$, and the paths $P=(1,3,4,2,5,6)$ and $Q=(1,2,7,3,5,4)$. Let $\mathcal{P}$ be the family of rotations of $P$, and similarly $\mathcal{Q}$ the rotations of $Q$.

$P$ is a generator path for $K_{7}$, it has two 1-type edges (3,4) and (5,6), two 2-type edges $(1,3)$ and $(2,4)$, and a single 3 -type edge $(2,5)$. The clockwise distance between 1 -type edges is 2 and the distance between 2 -types is 1 . The edge $(2,5)$ is unique to $P$ in the system $\mathcal{P}$, and similarly all the 3 -type edges will be unique in their paths. $\mathcal{P}$ is a separating path system of $K_{7}$.

Like $P, Q$ has two 1-type edges $(1,2)$ and $(5,4)$, two 2-type edges $(2,7)$ and $(3,5)$, and a single 3 -type edge (7,3). However, $Q$ is not a generator path since the clockwise distance between the 1-type edges and the distance between the 2-type edges is 3 in both cases, thereby not obeying (GP3), Let $Q_{i}=(1+i, 2+i, 7+i, 3+i, 5+i, 4+i)$ be the $i$ th clockwise rotation of $Q$ (so $Q=Q_{0}$ ), and observe that the edges $(5,4)$ and $(3,5)$ both appear in $Q_{3}$. Since $(5,4)$ and $(3,5)$ only appear in two rotations of $Q, Q_{0}$ and $Q_{3}$, it is clear that they are not separated by $\mathcal{Q}$. Therefore $\mathcal{Q}$ is not a separating path system of $K_{7}$.

Now consider an even case, the graph $K_{8}$. This time let $P=(1,4,3,5,6,2,8)$ and $Q=(1,2,7,5,6,4,8)$, and again let $\mathcal{P}$ be the family of rotations of $P$ and similarly $\mathcal{Q}$ for $Q$.


In this example we can see why even values of $n$ are slightly more complex. $P$ is a generator path with two 1 -type edges $(3,4)$ and $(5,6)$, two 2 -type edges $(8,2)$ and $(3,5)$, one 3 -type edge ( 1,4 ), and 4 -type edge ( 2,6 ). The edge ( 1,4 ) is unique to $P$ in $\mathcal{P}$ since there is only one edge of this type and the type is not $\frac{n}{2}$. Since $n=8$ is even we have that the 4 -type edge $(2,6)$ will appear in the path $P_{4}=(1+4,4+4,3+4,5+4,6+4,2+4,8+4)$ despite there only being one 4 -type edge in $P$. We can think of the distance between the 4 -type edges in $P$ as being 4 since both vertices of $(2,6)$ are starting vertices. Then the distance between the pairs of edges with the same type are all distinct, 1-types at distance 2 , 3 -types at distance 2 , 4 -types at distance $4 . \mathcal{P}$ is a separating path system for $K_{8}$.
$Q$ has two 1-type edges $(1,2)$ and $(5,6)$, two 2 -type edges $(4,6)$ and $(5,7)$, one 3 -type edge $(7,2)$, and 4 -type edge $(4,8)$. But the distances between the 1 -type edges is $\frac{n}{2}=4$, therefore $Q$ is not a generator path for failing (GP3). Note that the edges $(5,6)$ and $(4,8)$ both appear in the path $Q_{4}=(1+4,2+4,7+4,5+4,6+4,4+4,8+4)$ and do not appear in any other path of $\mathcal{Q}$. Hence, $\mathcal{Q}$ is not a separating path system of $K_{8}$.

We will now see formally that taking the family of rotations of a generator path will always give a separating path system.

Theorem 8.4. If $P$ is a generator path for $n$, then the family of all $n$ rotations of $P$ is a separating path system for $K_{n}$. Hence, if such a path $P$ exists then $f\left(K_{n}\right) \leq n$.

Proof. Let $P$ be a generator path for $n$, and $\mathcal{S}$ be the family generated by taking all the rotations of $P$. In particular, let $P=P_{0}$ and $P_{i}=\{(u+i, v+i):(u, v) \in P\}$ be the path generated by rotating $P$ clockwise by $i$. Since $P$ is a path in $K_{n}$ we have that every $P_{i} \in \mathcal{S}$ is a path in $K_{n}$. It is left only to show that every pair of edges in $K_{n}$ is separated by some path in $\mathcal{S}$.

Consider edges $e=(v, v+x)$ and $e^{\prime}=\left(v^{\prime}, v^{\prime}+y\right)$ in $K_{n}$, where $e$ is an $x$-type edge and $e^{\prime}$
is a $y$-type edge.
By (GP1), $P$ contains at least one edge of type $x$, therefore there is some $i \in[0, n-1]$ such that $e \in P_{i}$. If $e^{\prime} \notin P_{i}$ then the edges are separated by $P_{i}$, so assume otherwise.

Suppose first that $y=x$. Let $c d\left(e, e^{\prime}\right)=d$, without loss of generality we can assume that $v^{\prime}=v+d$. Consider the path $P_{i+d}$, where calculations are modulo $n$, we must have $e^{\prime} \in P_{i+d}$. Note that $v^{\prime}+d=v+2 d \neq v$. Indeed, this is only true when $2 d=n$, since $P$ is a generator path we have $d \neq \frac{n}{2}$ by (GP3). Therefore, $e \notin P_{i+d}$ since $P_{i}$ does not contain any additional $x$-type edges (by (GP2). Hence, $e$ and $e^{\prime}$ are separated by $P_{i+d}$.

Now suppose that $y \neq x$. We first consider the case where $n$ is even and $x=\frac{n}{2}$. If $x=\frac{n}{2}$, then we have that $e \in P_{i+\frac{n}{2}}$. Since $e^{\prime} \in P_{i}$ in order for $e^{\prime}$ to be in $P_{i+\frac{n}{2}}$ we must have that $P_{i}$ contains another $y$-type edge $h^{\prime}=\left(u^{\prime}, u^{\prime}+y\right)$, such that $u^{\prime}+\frac{n}{2}=v^{\prime}$ or $u^{\prime}-\frac{n}{2}=v^{\prime}$. In other words we must have $c d\left(e^{\prime}, h^{\prime}\right)=\frac{n}{2}$, this cannot happen by (GP3). Similarly, if $y=\frac{n}{2}$ then $e$ and $e^{\prime}$ are separated by $\mathcal{S}$.

Finally we suppose that $y \neq x$ and $x, y \neq \frac{n}{2}$. Then by (GP1), $P$ contains two edges of one of the types $x$ or $y$. Without loss of generality assume that $P$ contains two $x$-type edges. Let $h=(u, u+x) \in P_{i}$ be the other $x$-type edge, and let $c d(e, h)=d$. We have that $e \in P_{j}$ for one of $j=i+d$ or $j=i-d$. The only way for $e^{\prime} \in P_{j}$ is if there is another $y$-type edge $h^{\prime}=\left(u^{\prime}, u^{\prime}+y\right) \in P_{i}$ such that $c d\left(e^{\prime}, h^{\prime}\right)=d$. This clearly cannot happen by (GP3). So we have that $e$ and $e^{\prime}$ are separated by $P_{j}$.

So if we can find a generator path, then we can find a separating path system of $K_{n}$ with size $n$. The next question is therefore whether generator paths exist in general. It is fairly straightforward to find generator paths for small values of $n$, in fact we can find generator paths for all $n \leq 20$ by hand. Examples of generator paths for $n=2,3,4$ are trivial, examples of generator paths $P(n)$ for other values of $n \leq 20$ are given below:

$$
\begin{aligned}
& P(5)=(1,3,2,5) \\
& P(6)=(1,5,4,3,6), \\
& P(7)=(1,2,3,5,7,4) \\
& P(8)=(1,3,5,2,6,7,8) \\
& P(9)=(1,5,9,3,4,6,8,2) \\
& P(10)=(1,4,7,6,5,9,3,8,10) \\
& P(11)=(1,3,5,10,4,11,7,8,9,6), \\
& P(12)=(1,2,11,9,10,3,7,4,8,6,12,5) \\
& P(13)=(1,3,4,13,11,6,10,7,12,5,8,9)
\end{aligned}
$$

$$
\begin{aligned}
& P(14)=(1,3,6,9,10,11,2,7,13,5,12,8,4), \\
& P(15)=(1,14,15,5,10,3,12,6,9,13,2,4,11,8,7), \\
& P(16)=(1,11,13,15,14,3,8,12,16,9,2,10,7,4,5), \\
& P(17)=(1,3,5,16,10,11,12,9,6,15,7,14,4,17,13,8), \\
& P(18)=(1,15,10,5,13,3,12,9,6,7,8,2,14,16,18,11,4), \\
& P(19)=(1,3,5,18,12,11,10,13,16,7,17,6,14,9,4,19,15,8), \\
& P(20)=(1,5,10,15,18,8,17,6,20,14,7,19,2,4,16,9,13,12,11) .
\end{aligned}
$$

It is worth noting that there is no reason for a generator path to be unique. In fact for many of the above examples there are a number of suitable generator paths to choose from.

It should also be noted that there is an easy adaptation to the definition of a generator path which would allow for a strongly separating path system to be produced via rotations. Indeed, if we combine and strengthen (GP1) and (GP2) to say that $P$ must contain exactly two edges of every type (except $\frac{n}{2}$ in the even case), and keep (GP3) the same in both cases, then the path will generate a strong separating system. This can be seen with an easy variant of the argument in the proof of Theorem 8.4. Then if a strong generator path can be found it would also give an upper bound of $n$ for the strong version. In fact, $P(12)$ and $P(15)$ above are both generators of strongly separating path systems.

Definition 8.5. A path $P$ on the complete graph $K_{n}$ is called a strong generator path for $n$ if it satisfies the following conditions.
(SGP:1) P contains exactly two edges of each type from $\left[\frac{n-1}{2}\right]$, and exactly one edge of type $\frac{n}{2}$ when $n$ is even.
(SGP:2) Let $e$ and $e^{\prime}$ be $x$-type edges in $P$, and let $c d\left(e, e^{\prime}\right)=d$. If there two other edges $h, h^{\prime} \in P$, such that both have type $y \neq x$, then $c d\left(h, h^{\prime}\right) \neq d$. Additionally, no pair of edges in $P$ with the same type are at distance $\frac{n}{2}$.

It is also worth noting that the definition given in 8.2 is a little more strict than needed for generating a separating path system. For $n$ odd, let $P$ be a path in $K_{n}$ that contains 3 edges of type $x$ such that $P \backslash\{e\}$ (which is not necessarily a path) satisfies (GP1), (GP2), and (GP3), where $e$ is one of the $x$-types in $P$. Then the rotations of $P$ give a separating path system for $K_{n}$ as long as $c d\left(e, e^{\prime}\right)$ and $c d\left(e, e^{\prime \prime}\right)$ are not both equal to $\frac{n}{3}$, where $e^{\prime}$ and $e^{\prime \prime}$ are the other $x$-type edges in $P$. There is also an equivalent condition for even values of $n$, with more care taken when $x=\frac{n}{2}$. Since the conditions for these paths are slightly more relaxed they are possibly easier to find, but we do not make use of them here and
the simplified version given allows for cleaner notation and arguments.

### 8.3 The case where $\boldsymbol{n}$ is prime

As well as the small examples, we can also find generator paths for prime values of $n$. The construction uses the properties of primitive roots, an integer $g$ is a primitive root modulo $n$ if for every integer $h$ which is co-prime to $n$ there is some integer $i$ such that $h \equiv g^{i} \bmod n$.

Theorem 8.6. There exists a generator path for $n$ whenever $n$ is an odd prime.

Proof. Let $p$ be an odd prime, then it is known that there exists a primitive root $g$ modulo $p$. We can therefore write every integer in $[p-1]$ in the form $g^{i}$ for $i \in[p-1]$.

Consider the path on $K_{p}$ given by $P=\left(p, g, g+g^{2}, g+g^{2}+g^{3}, \ldots, \sum_{i=1}^{p-2} g^{i}\right)$, which is the path starting at vertex $p$ and taking a $g$ length edge, followed by a $g^{2}$ length edge, followed by a $g^{3}$ length edge, and so on until there are $p-2$ edges in the path. Here we use 'length' to avoid confusion, an $x$-type edge may have length $x$ or $n-x$. We claim that this path is a generator path for $n=p$.

We must check that $P$ is indeed a path in $K_{p}$, and that it satisfies (GP1), (GP2), and (GP3)

First we show that $P$ is indeed a path. To see this note that $P$ is a path as long as none of the vertices $p, g, g+g^{2}, g+g^{2}+g^{3}, \ldots, \sum_{i=1}^{j} g^{i}, \ldots, \sum_{i=1}^{p-2} g^{i}$ are congruent modulo $p$. Suppose that

$$
\sum_{i=1}^{m} g^{i} \equiv \sum_{i=1}^{j} g^{i} \quad \bmod p
$$

for some $m, j \in[p-2]$. Then we must have that

$$
\frac{g^{m+1}-g}{g-1} \equiv \frac{g^{j+1}-g}{g-1} \quad \bmod p
$$

and hence

$$
g^{m-j} \equiv 1 \quad \bmod p .
$$

Since $g$ is a primitive root we know that $g^{p-1} \equiv 1 \bmod p$ and $g^{i} \not \equiv 1 \bmod p$ whenever $i<p-1$. Therefore we must have

$$
m \equiv j \quad \bmod p-1
$$

Clearly this means $m=j$ since $m, j \in[p-2]$. Now we must also check that the vertex $p$ is distinct from the others. For a contradiction suppose that

$$
\sum_{i=1}^{m} g^{i} \equiv p \quad \bmod p
$$

for some $m \in[p-2]$. This gives

$$
g^{m} \equiv 1 \quad \bmod p
$$

so we must have $m \equiv 0 \bmod p-1$, a contradiction of $m \in[p-2]$.
We conclude that $P$ is indeed a path in $K_{p}$. It remains to show that $P$ satisfies the three conditions.

Let $k=\frac{p-1}{2}$ and note that since $g$ is a primitive root and $p$ is prime, we have that $g^{k} \equiv-1$ $\bmod p$ and $g^{i} \not \equiv-1 \bmod p$ whenever $i<k$. This means that $g^{i}+g^{k+i} \equiv 0 \bmod p$. In other words an edge in $P$ with length $g^{i}$ and an edge with length $g^{k+i}$ have the same edge type.

Observe that the first $k$ edges of $P$ each have unique edge type, and the $(k+i)$ th edge of $P$ has the same type as edge $i$. Thus $P$ contains two edges of every type except type 1, since the $k$ th edge in $P$ is the unique 1-type edge. Hence, $P$ satisfies (GP1) and (GP2).

For any pair of same type edges in $P$ one edge will be length $g^{j}$ and the other will be length $g^{k+j}$ for $j \in[k]$. Since $g^{i} \not \equiv g^{k+i} \bmod p$ this means we have one of two cases, either

1. the starting vertex of the $g^{j}$ edge will be $\sum_{i=1}^{j-1} g^{i}$ and the starting vertex of the $g^{k+j}$ edge will be $\sum_{i=1}^{k+j} g^{i}$, or
2. the starting vertex of the $g^{j}$ edge will be $\sum_{i=1}^{j} g^{i}$ and the starting vertex of the $g^{k+j}$ edge will be $\sum_{i=1}^{k+j-1} g^{i}$.

Note that the clockwise distance between the edges for the first case is given by

$$
\sum_{i=1}^{k+j} g^{i}-\sum_{i=1}^{j-1} g^{i} \equiv \frac{g^{k+j+1}-g^{j}}{g-1} \quad \bmod p \quad \text { or } \quad-\frac{g^{k+j+1}-g^{j}}{g-1} \bmod p
$$

whichever is in $[k]$, and in the second case it is given by

$$
\sum_{i=1}^{k+j-1} g^{i}-\sum_{i=1}^{j} g^{i} \equiv \frac{g^{k+j}-g^{j+1}}{g-1} \bmod p \quad \text { or } \quad-\frac{g^{k+j}-g^{j+1}}{g-1} \bmod p
$$

Since we know the additive inverse of $g^{j}$ is $-g^{j} \equiv g^{k+j} \bmod p$ we can see that the distances in the two cases are equivalent. So we assume we are in case 1.

We must now show that the clockwise distance between pairs of same type edges are not repeated, (GP3).

Consider edges in $P$ given by $g^{j}, g^{k+j}, g^{m}$, and $g^{k+m}$. Suppose that

$$
\frac{g^{k+j+1}-g^{j}}{g-1} \equiv \frac{g^{k+m+1}-g^{m}}{g-1} \quad \bmod p .
$$

Then we have that

$$
g^{k+j+1}\left(1-g^{m-j}\right) \equiv g^{j}\left(1-g^{m-j}\right) \quad \bmod p
$$

and hence

$$
g^{k+1} \equiv 1 \quad \bmod p .
$$

This is a contradiction as we know that $g^{p-1} \equiv 1 \bmod p$ and $g^{i} \not \equiv 1 \bmod p$ whenever $i<p-1$. The proof when we take one or both of the clockwise distances to be $-\frac{g^{k+i+1}-g^{j} i}{g-1}$ $\bmod p$ is equivalent.

Therefore $P$ is indeed a generator path for $n=p$.

This gives us an upper bound for $f\left(K_{n}\right)$ when $n$ is a prime number.

Corollary 8.7. We have $f\left(K_{p}\right) \leq p$ whenever $p$ is prime.
We can also use the structure of the generator path in Theorem 8.6 along with the properties of primes to give an upper bound for $f\left(K_{n}\right)$ when $n=p+1$.

Theorem 8.8. Let $p$ be an odd prime, then we have $f\left(K_{p+1}\right) \leq p+1$.

Proof. Let $K^{\prime}$ be the complete graph on $p$ vertices given by removing the vertex $p+1$ from $K_{p+1}$. Let $P$ be the generator path for $p$ given in the proof of Theorem 8.6, and let $\mathcal{P}=\left\{P_{i}: 0 \leq i \leq p-1\right\}$ be the family of rotations of $P$, where $P_{i}$ is $P$ rotated clockwise by $i$. Recall that the edge $h=\left(\sum_{i=1}^{k-1} g^{i}, \sum_{i=1}^{k} g^{i}\right)$ where $k=\frac{p-1}{2}$ is the unique 1 -type edge in $P$.

Let $T=\{(1,2),(2,3), \ldots,(p-1, p),(p, 1)\}$ be the set of all 1-type edges in $K^{\prime}$, and note that $T \backslash\{h\}$ is a path in $K^{\prime}$ and also in $K_{p+1}$.

Let $P_{i}^{\prime}=P_{i} \cup\{(i, p+1)\}$ for every $1 \leq i \leq p-1$. Since each $P_{i}$ has the vertex $i$ as an endpoint, each $P_{i}^{\prime}$ is a path in $K_{p+1}$ and contains $P_{i}$ as a sub-path.

Define $\mathcal{S}=\left\{P_{0}, P_{1}^{\prime}, P_{2}^{\prime}, \ldots, P_{p-1}^{\prime}, T \backslash\{h\}\right\}$. We claim that $\mathcal{S}$ is a separating path system for $K_{p+1}$.

Clearly all elements of $\mathcal{S}$ are paths in $K_{p+1}$, so it remains to check that any two edges are separated by some path in $\mathcal{S}$. Let $e, e^{\prime} \in E\left(K_{p+1}\right)$, suppose first that $e, e^{\prime} \in E\left(K^{\prime}\right)$. Then $e$ and $e^{\prime}$ are separated by $\left\{P_{0}, P_{1}^{\prime}, P_{2}^{\prime}, \ldots, P_{p-1}^{\prime}\right\}$ since we know they are separated by $\mathcal{P}$.

Suppose instead that $e^{\prime} \in E\left(K^{\prime}\right)$ and $e \notin E\left(K^{\prime}\right)$. Then $e$ must be of the form $(v, p+1)$ for some $v \in[p]$. Let $x \in\left[\frac{p-1}{2}\right]$ be the edge type of $e^{\prime}$ in $K^{\prime}$, and note that if $x \neq 1$ there exists $0 \leq i, j \leq p-1$ such that $e^{\prime} \in P_{i}, P_{j}$. Therefore we have $e^{\prime} \in P_{i}^{\prime}, P_{j}^{\prime}$. Clearly $e=(v, p+1)$ cannot be in both of these paths, therefore the edges $e$ and $e^{\prime}$ must be separated by $\mathcal{S}$. Now, if $x=1$ note that either $e^{\prime} \in P_{0}$ or $e^{\prime} \in T \backslash\{h\}$, and since $e$ cannot be in either of these paths, the edges are again separated by $\mathcal{S}$.

Finally, suppose $e, e^{\prime} \notin E\left(K^{\prime}\right)$. Then we can write them in the form $e=(v, p+1)$ and $e^{\prime}=(u, p+1)$ where $u, v \in[p]$. Since $u \neq v$ we must have that at least one of $u$ and $v$ lies in $[p-1]$, without loss of generality assume $v \in[p-1]$. Then we have that $e \in P_{v}^{\prime}$, and clearly $u \notin P_{v}^{\prime}$. Hence the edges are separated by $\mathcal{S}$.

It is worth noting here that the above proof highlights one of the major differences when working in the weak case as opposed to the strong version. It is reasonably easy to adapt a path system based on rotations of a path to work in additional cases, as we just saw. This is usually takes the form of using a path system based on a generator path and changing the paths slightly while adding some additional 'fixing' paths. This technique works well for the weak variation of the problem, however, attempting something similar in the strong case often increases the size of the new path system significantly.

Take for example the above argument of Theorem 8.8. We used a one step induction by removing a vertex from $K_{n}$ and adding all the incident edges to the end of paths in $K_{n-1}$. This works because we only need to cover all the edges, and ensure that every edge from $K_{n-1}$ appears in two paths. In the strong version we would need to not only cover all the edges at the removed vertex, but also ensure they appear in two paths that share no other edges. Since the edges incident form a star, this very path-intensive to achieve directly. This leads to even more complicated adaptation of the original paths which cannot always be achieved.

We will discuss adapting Theorem 8.6 to the strong version later on in Section 8.6.2.

### 8.4 General upper bound

Our aim in this section is to prove the following upper bound.

Theorem 8.9. For $n \geq 44$, there exists a separating path system for $K_{n}$ with size at most

$$
\frac{21 n+16 \log _{2} n+232}{16}
$$

The general framework for achieving this is to construct path which is almost a generator path but fails some conditions in a small number of cases, then take the family of rotations of this not-quite generator. We can then examine the edges which are not separated by this, which will hopefully be small in number, and construct some additional paths we can add into our system to fix or patch up these problem edges.

Note that, as the conditions for a generator path are all based on edge types, any path $P$ which is not a generator path fails the conditions for at least one type. In other words, we can separate the edge types of $P$ into two classifications, those that follow (GP1), (GP2), and (GP3), $F$, and those that do not, $D$. Formally we have the following definition.

Definition 8.10. $A$ set of edges $A$ is an $F$-separator for $K_{n}$ if we can partition the edge types of $K_{n}$ into sets $F$ and $D$ such that the following holds.

1. A contains at least one $y$-type edge for every $y \in F \cup D$.
2. A contains exactly two $x$-type edges for every $x \in F$.
3. $A_{F}$ satisfies (GP3), where $A_{F}=\{e \in A$ : the edge type of $e$ is in $F\}$.

If $A$ is a path in $K_{n}$, we call it an $F$-separator path. Whenever we have an $F$-separator we will use $D:=\left[\frac{n}{2}\right] \backslash F$ to mean the other half of the partition.

Note that a generator path is an $F$-separator path, where $F=\left[\frac{n-1}{2}\right] \backslash\{x\}$ with $x$ the edge type that appears exactly once (of which there is at most one).

It is plain to see by the argument in the proof of Theorem 8.4, the family of paths constructed by taking all the rotations of an $F$-separator path $P$ will separate all edges with type in $F$ from each other. This means if we can find some $P$ which is an $F$-separator path for some large set $F$, then we can find a family of $n$ paths that separates most of the edges in $K_{n}$. We are then able to use a number of 'fixing' paths that will separate edges with type in $D$. The family containing the two types of path gives a separating path system of $K_{n}$.

Our aim for the rest of this chapter is as follows. We first define these fixing paths, giving a separating path system based on any path (Theorem 8.12). We then construct an $F$ separator path $P$ with large $F$ (and hence small $D$ ) to use as the base (Theorem 8.13). Our construction only works when $n \equiv 3 \bmod 6$ or $n \equiv 5 \bmod 6$. The final step is to use the construction of $P$ and the family based on it to give separating path systems for other values of $n$ (Theorem 8.9).

Note that the rotation construction method naturally forces edges of the same type to be separated from each other. Indeed, looking at the proof of Theorem 8.4 we see that for the case $x=y$ the only condition needed was that $c d\left(e, e^{\prime}\right) \neq \frac{n}{2}$. This is always the case when $n$ is odd since $\frac{n}{2}$ is not an integer. So as long as there is no edge type appearing more than twice in $P$, and $n$ is odd, we have that all edges of the same type are separated from each other. In general, if $P$ contains exactly $m$ edges of the same type $e_{1}, \ldots, e_{m}$, then they are separated from each other by the rotations of $P$ unless $c d\left(e_{i}, e_{i+1}\right)=\frac{n}{m}$ for all $i \in[m-1]$. We say that any $P$ with such edges of type $x$ has equally spaced $\boldsymbol{x}$-type edges. This leads us to Theorem 8.12, but before we state this we need the following result.

Lemma 8.11. For all $n$ and each edge type $x$ except $x=\frac{n}{2}$, there exist two paths in $K_{n}$, $Q_{x}$ and $Q_{x}^{\prime}$, such that $Q_{x} \cup Q_{x}^{\prime}$ covers all x-type edges in $K_{n}$ and all edges in $Q_{x} \cup Q_{x}^{\prime}$ have type from $\{1, x\}$.

Proof. Let $f$ be the highest common factor of $n$ and $x$, and let $a \in \mathbb{N}$ be such that $a f=n$.
Consider the subgraph of $K_{n}$ containing only $x$-type edges, we have exactly $f$ isomorphic cycles of length $a$. Let $C_{1}, C_{2}, \ldots, C_{f}$ be the cycles labelled so that a vertex $i$ will be contained in cycle $C_{i} \bmod f$. We now describe a path in $K_{n}$ using sections of each $C_{i}$ and some linking 1-type edges.

The path $Q_{x}$ starts at vertex 1, and follows $C_{1}$ for $a-1$ edges, before moving to $C_{2}$ with a 1-type edge. It then follows $C_{2}$ for $a-1$ edges. This continues until $a-1$ edges from each of the cycles $C_{1}, C_{2}, \ldots, C_{f}$ have been followed. Formally, set $v_{1}=1$ and recursively define $v_{i}^{\prime}=v_{i}-x \bmod n$ and $v_{i+1}=v_{i}^{\prime}+1$ for $i \in[f]$. Note that for each $i \in[f]$ the edge $\left(v_{i}, v_{i}^{\prime}\right) \in C_{i}$ and therefore has type $x$. Also note that the edges $\left(v_{i}^{\prime}, v_{i+1}\right)$ are 1-type edges for all $i \in[f]$. We define the path $Q_{x}$ as follows,

$$
Q_{x}:=\left(\bigcup_{i=1}^{f} C_{i} \backslash\left\{\left(v_{i}, v_{i}^{\prime}\right)\right\}\right) \cup\left\{\left(v_{i}^{\prime}, v_{i+1}\right): i \in[f-1]\right\} .
$$

Note this is indeed a path since the cycles $C_{1}, C_{2}, \ldots, C_{f}$ are pairwise vertex disjoint, and each edge $\left(v_{i}^{\prime}, v_{i+1}\right)$ links $C_{i}$ to $C_{i+1}$.

Then we define $Q_{x}^{\prime}$ to be

$$
Q_{x}^{\prime}:=\left\{\left(v_{i}, v_{i}^{\prime}\right): i \in[f]\right\} \cup\left\{\left(v_{i}^{\prime}, v_{i+1}\right): i \in[f-1]\right\} .
$$

Thus all the $x$-type edges are covered in two paths using only $x$-type and 1 -type edges.

Now we can give our result.

Theorem 8.12. Let $n \in \mathbb{N}$ be odd, and $P$ an $F$-separator path for $K_{n}$ with no equally spaced $x$-type edges. Then $\mathcal{P} \cup \mathcal{D}$ is a separating path system for $K_{n}$ where $\mathcal{D}=\left\{Q_{x}, Q_{x}^{\prime}\right.$ : $x \in D \cup\{1\}\}$ (with $Q_{x}$ and $Q_{x}^{\prime}$ as in Lemma 8.11 and $D=\left[\frac{n}{2}\right] \backslash F$ ), and $\mathcal{P}$ is the family of $n$ rotations of $P$. In particular $f\left(K_{n}\right) \leq n+2|D \cup\{1\}|$.

Proof. Consider a pair of edges $e, e^{\prime} \in E\left(K_{n}\right)$. Suppose first that $e$ and $e^{\prime}$ have the same type, $x$. Then $\mathcal{P}$ separates $e$ and $e^{\prime}$. Indeed, if $e, e^{\prime} \in P_{i}$ (the rotation of $P$ by $i$ vertices clockwise), and $c d\left(e, e^{\prime}\right)=d$ then one of $P_{i+d}, P_{i+2 d}, P_{i+3 d}, \ldots$ contains $e$ but not $e^{\prime}$. Otherwise we would have a collection of $\frac{n}{d}$ edges $e_{1}, \ldots, e_{\frac{n}{d}}$ (which includes $e$ and $e^{\prime}$ ), all with type $x$ and such that $c d\left(e_{i}, e_{i+1}\right)=d$. This cannot be, since we have no equally spaced $x$-type edges in $P$.

Suppose instead that $e$ has type $x$ and $e^{\prime}$ has type $y$. If $x, y \in F$ then $e$ and $e^{\prime}$ are separated by $\mathcal{P}$ since $P$ is an $F$-separator path. The paths $Q_{1}, Q_{1}^{\prime} \in \mathcal{D}$ contain only 1-type edges and together cover all 1-types in $K_{n}$. Therefore if $x=1$ then one of these two paths separates $e$ and $e^{\prime}$. So assume $x, y \neq 1$, and that $x \notin F$. Then $Q_{x}$ and $Q_{x}^{\prime}$ contain only $x$-type and 1 -type edges and together cover all $x$-types in $K_{n}$. One of these paths separates $e$ and $e^{\prime}$.

To prove Theorem 8.9 it is left to find some path $P$ with small $D$ and no equally spaced edges of the same type.

Theorem 8.13. When $n \equiv 3 \bmod 6$ or $n \equiv 5 \bmod 6$, there is an $F$-separator path $P$ for $K_{n}$, with no equally spaced $x$-type edges and with $|D \cup\{1\}| \leq \frac{1}{32}\left(5 n+16 \log _{2} n+167\right)$. Where $D=\left[\frac{n}{2}\right] \backslash F$.

The construction of this path is rather involved and makes up the bulk of the argument, the proof can be found in Section 8.5. We first show how to use this result to get separating
path systems for all sufficiently large values of $n$. The methods for extending paths that are used here are similar to those used in the proof of Theorem 8.8.

Theorem 8.9. For $n \geq 44$, there exists a separating path system for $K_{n}$ with size at most

$$
\frac{21 n+16 \log _{2} n+232}{16}
$$

Proof. Case 1: $n$ is odd and $\frac{n-1}{2}$ is not a multiple of 3 .
Let $P$ be the path from Theorem 8.13, and $D$ its associated set of badly behaved edge types. Then by 8.12 there is a separating path system for $K_{n}$ with size at most

$$
n+2 \cdot \frac{5 n+16 \log _{2} n+167}{32}
$$

Case 2: $n-1$ is odd and $\frac{n-2}{2}$ is not a multiple of 3 .
Consider the complete graph on $n-1$ vertices formed by removing vertex $v$ from $K_{n}$. Then $K_{n-1}$ fits the conditions for Case 1 , let $P$ and $D$ be as in Case 1. Use 8.12 to give a separating path system for $K_{n-1}$ of the form $\mathcal{P} \cup \mathcal{D}$, where $\mathcal{P}$ is the rotations of $P$ and $\mathcal{D}=\left\{Q_{x}, Q_{x}^{\prime}: x \in D \cup\{1\}\right\}$. Then $|\mathcal{P} \cup \mathcal{D}| \leq n-1+2|D \cup\{1\}|$ where

$$
|D \cup\{1\}| \leq \frac{5(n-1)+16 \log _{2}(n-1)+167}{32}
$$

Let $P_{i}$ denote the rotation of $P$ by $i$ vertices clockwise on $K_{n-1}$, and similarly $w_{i}$ for the vertex of $K_{n-1}$ which is $i$ clockwise vertices on from $w$. Let $u$ be an endpoint of the path $P=P_{0}$. Then the family $\mathcal{P}^{\prime} \cup \mathcal{D}$ is a separating path system for $K_{n}$ where $\mathcal{P}^{\prime}=\left\{P_{i} \cup\left\{\left(u_{i}, v\right)\right\}: i \in[0, n-2]\right\}$.

Indeed, any pair of edges from $E\left(K_{n-1}\right)$ are separated since they are separated by $\mathcal{P} \cup \mathcal{D}$. So let $e=(w, v)$ for some $w \in V\left(K_{n-1}\right)$, and consider any edge $e^{\prime} \in E\left(K_{n-1}\right)$. The edge $e$ only appears in one path from $\mathcal{P}^{\prime} \cup \mathcal{D}$, namely the path $P_{i} \cup\left\{\left(u_{i}, v\right)\right\}$ where $u_{i}=w$. Therefore if $e^{\prime}$ appears in any other path, then we are done. Clearly this is the case if $e^{\prime}$ has type $x \in D$. Suppose that $e^{\prime}$ has type $x \in F$, then by definition $P$ must have two $x$-type edges, therefore $e^{\prime}$ appears in two rotations of $P$.

Finally if $e^{\prime}$ is also an edge at $v$, then clearly it is not contained in the path $P_{i} \cup\left\{\left(u_{i}, v\right)\right\}$, and the two edges must be separated.

Note that $\left|\mathcal{P}^{\prime}\right|=|\mathcal{P}|=n-1$, and $|\mathcal{D}|=2|D \cup\{1\}|$.
Case 3: $n$ is odd and $\frac{n-1}{2}$ is a multiple of 3 .
Consider the complete graph on $n-2$ vertices given by removing vertices $v$ and $v^{\prime}$ from
$K_{n}$. Clearly $K_{n-2}$ satisfies the conditions for Case 1 , so let $\mathcal{P} \cup \mathcal{D}$ be the corresponding separating path system for $K_{n-2}$ given by Theorem 8.12. We will take each path from $\mathcal{P}$ and adapt it to a path on $K_{n}$, we will also add some edge types to the set $D$ and create additional paths for our family this way.

Let $u$ be any endpoint of $P$. Select some edge $g \in P$ such that if $x_{g}$ is the edge type of $g$, then $P$ contains at least one other edge with type $x_{g}$. Set $g=(p, q)$ for $p, q \in V\left(K_{n-2}\right)$, and let $g_{i}=\left(p_{i}, q_{i}\right)$ be the rotation of $g$ in $K_{n-2}$ by $i$ vertices. We fix the new paths $P_{i}^{\prime}=\left(P_{i} \backslash\left\{g_{i}\right\}\right) \cup\left\{\left(p_{i}, v\right),\left(v, q_{i}\right),\left(u_{i}, v^{\prime}\right)\right\}$, and let $\mathcal{P}^{\prime}=\left\{P_{i}^{\prime}: i \in[0, n-3]\right\}$.

If there are edges $h, h^{\prime} \in P$ with the same edge type $x_{h} \in F$, such that $c d\left(h, h^{\prime}\right)=x_{g}$ on $K_{n-2}$, then define $D^{\prime}=D \cup\left\{x_{g}, x_{h}\right\}$. Note that there can be at most one such pair since $x_{h} \in F$. Otherwise define $D^{\prime}=D \cup\left\{x_{g}\right\}$. Let $\mathcal{D}^{\prime}=\left\{Q_{x}, Q_{x}^{\prime}: x \in D^{\prime} \cup\{1\}\right\}$ be the family of fixing paths on each type in $D^{\prime}$ (such paths described in 8.11). Then $\mathcal{P}^{\prime} \cup \mathcal{D}^{\prime}$ is a separating path system for $K_{n}$.

Let $e, e^{\prime}$ be any two edges in $K_{n}$. If $e, e^{\prime} \in E\left(K_{n-2}\right) \backslash\left\{g_{i}: i \in[n-2]\right\}$ then the edges are separated by $\mathcal{P} \cup \mathcal{D}$ and hence by $\mathcal{P}^{\prime} \cup \mathcal{D}^{\prime}$. Suppose $e^{\prime}=g_{i}$ for some $i \in[n-2]$, and let $x_{e}$ be the edge type of $e$. Since $x_{g} \in D^{\prime}$ we have that the path $Q \in\left\{Q_{x_{g}}, Q_{x_{g}}^{\prime}\right\} \subseteq \mathcal{D}^{\prime}$ contains the edge $g_{i}$. The path $Q$ separates $g_{i}$ from $e$ unless $x_{e}=x_{g}$ or $x_{e}=1$. If $x_{e}=1$ then one of $Q_{1}, Q_{1}^{\prime} \in \mathcal{D}^{\prime}$ separates the pair. If $x_{e}=x_{g}$ then, since there is still an $x_{g}$-type edge in $P^{\prime}$ and there are no equally spaced edge types, $\mathcal{P}^{\prime}$ separates the pair.

Suppose then that $e=(v, w)$ for some $w \in V\left(K_{n-2}\right)$, and $e^{\prime} \in E\left(K_{n-2}\right)$, then the edges are separated if $e^{\prime}$ is in some $\mathcal{D}^{\prime}$ path, otherwise there is some other edge in $P_{i}$ with the same type as $e^{\prime}$. The distance between these edges cannot be the same as the distance between the two edges at $v$ since $x_{h} \in D^{\prime}$, therefore the edges must be separated by the rotations (as in 8.4). If $e=(v, w)$ and $e^{\prime}=\left(v, w^{\prime}\right)$, then let $i$ be such that $w_{i}=w^{\prime}$. We have that if $e, e^{\prime} \in P_{j}^{\prime}$ then $e^{\prime} \in P_{j+i}$, the only other edge at $v$ in $P_{j+i}$ is $\left(v, w_{i}^{\prime}\right)$. Therefore the edges are separated. Finally, if $e=\left(v^{\prime}, w\right)$ then note that it is the unique edge a path in $\mathcal{P}^{\prime}$, since every other edge type either appears twice in $P^{\prime}$ or in $D^{\prime}$, there are no other unique edges. Therefore the edges are separated.

Note that $\left|\mathcal{P}^{\prime}\right|=|\mathcal{P}|=n-2$, and $\left|\mathcal{D}^{\prime}\right|=2\left|D^{\prime} \cup\{1\}\right|=2(|D \cup\{1\}|+2)$.
Case 4:n-1 is odd and $\frac{n-2}{2}$ is a multiple of 3 .
This case is very similar to Case 3 . We consider the family $\mathcal{P} \cup \mathcal{D}$ from 8.12 on the $K_{n-3}$ obtained by removing the vertices $v, v^{\prime}$ and $\bar{v}$ from $K_{n}$. Then we adapt the paths $P$ and add to the set $D$ to obtain a new family.

Let $u$ be any endpoint of $P$, and let $g=(p, q), g^{\prime}=\left(p^{\prime}, q^{\prime}\right) \in P$ be edges with type $x_{g}$ and $x_{g}^{\prime}$ respectively such that $x_{g} \neq x_{g}^{\prime}$ and $P \backslash\left\{g, g^{\prime}\right\}$ still contains an edge of type $x_{g}$ and and edge of type $x_{g}^{\prime}$. Set the new paths to be $P_{i}^{\prime}=\left(P_{i} \backslash\left\{g_{i}, g_{i}^{\prime}\right\}\right) \cup$ $\left\{\left(p_{i}, v\right),\left(v, q_{i}\right),\left(p_{i}^{\prime}, v^{\prime}\right),\left(v^{\prime}, q_{i}^{\prime}\right),(w, \bar{v})\right\}$ and $\mathcal{P}^{\prime}=\left\{P_{i}^{\prime}: i \in[0, n-4]\right\}$.

If there are edges $h, h^{\prime} \in P$ both with edge type $x_{h} \in F$ such that the clockwise distance between them on $K_{n-3}$ is equal to $x_{g}$, then set $D^{\prime}=D \cup\left\{x_{g}, x_{g}^{\prime}, x_{h}\right\}$. If there is also a pair of edges with type $x_{h}^{\prime} \in F$ with clockwise distance equal to $x_{g}^{\prime}$ then set $D^{\prime}=D \cup$ $\left\{x_{g}, x_{g}^{\prime}, x_{h}, x_{h}^{\prime}\right\}$. Note that there can be at most one of each since $x_{h}, x_{h}^{\prime} \in F$. Otherwise, set $D^{\prime}=D \cup\left\{x_{g}, x_{g}^{\prime}\right\}$. Then by the same reasoning as for Case 3 , the family $\mathcal{P}^{\prime} \cup \mathcal{D}^{\prime}$ is a separating path system for $K_{n}$.

Note that $\left|\mathcal{P}^{\prime}\right|=|\mathcal{P}|=n-3$, and $\left|\mathcal{D}^{\prime}\right|=2\left|D^{\prime} \cup\{1\}\right|=2(|D \cup\{1\}|+4)$.

### 8.5 Proof of Theorem 8.13

The aim now is to construct an $F$-separator path with many edge types in $F$. We will start by defining three sets of edges $\left(M_{0}, R\right.$, and $\left.B\right)$ which together form a linear forest. This linear forest will almost follow (GP1), (GP2), and (GP3). We will then add certain joining edges to connect our forest into a single path. The types associated to the special edges in the linear forest will end up in $F$ and the types associated to the joining edges will end up in $D$.

## Step 1 : Defining the linear forest

Let $n \in \mathbb{N}$ be such that $n \equiv 3 \bmod 6$ or $n \equiv 5 \bmod 6$. For the remainder of this section we label the vertices of $K_{n}$ slightly differently for ease of notation. First label any vertex 0 , from here label vertices clockwise following the order $1,2, \ldots, \frac{n-1}{2},-\frac{n-1}{2},-\left(\frac{n-1}{2}-\right.$ $1), \ldots,-2,-1$. Now we define three sets of edges from $K_{n}$ which we will combine to make an approximation of a generator path.

First we define a set containing one edge of each type,

$$
M_{0}=\left\{(-i, i): i \in\left[\frac{n-1}{2}\right]\right\} .
$$

Note that $M_{0}$ is a maximal matching in $K_{n}$ and that the vertex 0 is the only vertex which is not an endpoint of an edge in $M_{0}$. We also use $M_{k}$ to denote the rotation of $M_{0}$ in which vertex $k$ has no incident edges. See Figure 8.3.


Figure 8.3: An example with $n=7$ showing the vertex labels and matchings $M_{k}$

Next we define an edge set containing only the largest edge types. Let $R=R_{1} \cup R_{2}$ where

$$
R_{1}=\left\{\left(1,-\frac{n-3}{2}\right),\left(-1,-\frac{n-1}{2}\right)\right\},
$$

and

$$
R_{2}=\left\{\left(-3-2 k, \frac{n-1}{2}-k\right): 0 \leq k \leq r-1\right\}
$$

where $r=\frac{n-7}{4}$ for when $\frac{n-1}{2}$ is odd, and $r=\frac{n-9}{4}$ otherwise.
Note that $R_{1}$ consists of a $\frac{n-1}{2}$-type edge and a $\frac{n-3}{2}$-type edge. Further, note that $\left(-3, \frac{n-1}{2}\right)$ has type $\frac{n-5}{2}$, and that if an edge $\left(-3-2 k, \frac{n-1}{2}-k\right)$ is $x$-type, then $\left(-3-2(k+1), \frac{n-1}{2}-\right.$ $(k+1))$ is an $(x-1)$-type edge. Thus, $R$ contains the largest $r+2$ edge types.
Define $r(i)=\frac{n+2+i}{2}$ for every $i=-3-2 k$ where $0 \leq k \leq r-1$, and $r(i)=-\frac{n-2-i}{2}$ for $i=-1,1$. Then we can write each edge in $R$ as $(i, r(i))$. Similarly let $r^{-1}(i)=2 i-n-2$ for $i=\frac{n-1}{2}-k$ where $0 \leq k \leq r-1$, and $r^{-1}(i)=2 i+n-2$ for $i=-\frac{n-3}{2},-\frac{n-1}{2}$. Then we can also write $R$ edges in the form $\left(r^{-1}(i), i\right)$.

The contents of the final edge set depend on the edges in $R$ as well as the properties of $\frac{n-1}{2}$. We define this set so that it continues with the large edge types roughly where $R$ left off, containing approximately types $\frac{n}{4}$ down to $\frac{n}{8}$. Choose $i_{b}$ even and as large as possible such that $\left(-i_{b}, i_{b}+3\right)$ is an edge with odd type at most $\frac{n-1}{4}$. Set $b=\left(-i_{b}, i_{b}+3\right)$ and denote $x_{b}=2 i_{b}+3$ as the edge type of $b$. Note that $b \in M_{-\frac{n-3}{2}}$ and

$$
x_{b} \in\left\{\left\lfloor\frac{n-1}{4}\right\rfloor,\left\lfloor\frac{n-1}{4}\right\rfloor-1,\left\lfloor\frac{n-1}{4}\right\rfloor-2,\left\lfloor\frac{n-1}{4}\right\rfloor-3\right\} .
$$

Then we can define the final edge set as

$$
B=\left\{\left(-\frac{x_{b}-3}{2}+2 k, \frac{x_{b}+3}{2}+k\right): 0 \leq k \leq t-1\right\}
$$

where $t=\frac{x_{b}+1}{2}$.
Let $b(i)=\frac{3 x_{b}+3+2 i}{4}$ for each $i=-\frac{x_{b}-3}{2}+2 k$ where $0 \leq k \leq t-1$. Also, let $b^{-1}(i)=$ $-\frac{3 x_{b}+3-4 i}{2}$ for every $i=\frac{x_{b}+3}{2}+k$ where $0 \leq k \leq t-1$. Then we can write $B$ edges in the form $(i, b(i))$ and $\left(b^{-1}(i), i\right)$.

See Figure 8.4 for an example of the edge sets $M_{0}, R$, and $B$.
Let $L=M_{0} \cup R \cup B$ denote this collection of edges. Our task now is to extend $L$ to a path. In order to do this we must have that $L$ is acyclic, and that the maximum degree of any vertex is 2 .


Figure 8.4: The linear forest $L$ for $n=35$

Claim 8.14. L is a linear forest.

Proof. First we consider the degree condition. Observe that $M_{0}$ is a maximal matching and therefore no $M_{0}$ edges share vertices. It is clear from the constructions of $R$ that no two $R$ edges share a vertex, similarly no two $B$ edges share a vertex. This means all edges
of high degree must be an endpoint of an edge in each of $M_{0}, R$, and $B$. As usual we write $[a, b]=\{c \in \mathbb{N}: a \leq c \leq b\}$ where $a$ and $b$ are non-negative integers with $a \leq b$. We also write $[-a, b]=\{c \in \mathbb{Z}:-a \leq c \leq b\}$ where $a$ and $b$ are non-negative integers, and $[-a,-b]=\{c \in \mathbb{Z}:-a \leq c \leq-b\}$ where $a$ and $b$ are non-negative integers and $a \geq b$.

Let $I_{1}=[-2,1]=\{-2,-1,0,1\}$ and $I_{2}=\left[-\frac{n-5}{2},-3\right]$, then each $R$ edge has one vertex in the set $I=I_{1} \cup I_{2}$, moreover this vertex is odd. Let $I_{1}^{\prime}=\left[-\frac{n-1}{2},-\frac{n-3}{2}\right]$ and $I_{2}^{\prime}=\left[\frac{n+9}{4}, \frac{n-1}{2}\right]$, then each $R$ edge has one vertex in the set $I^{\prime}=I_{1}^{\prime} \cup I_{2}^{\prime}$. Similarly, let $J=\left[-\frac{x_{b}-3}{2}, \frac{x_{b}+1}{2}\right]$ and $J^{\prime}=\left[\frac{x_{b}+3}{2}, x_{b}+1\right]$, each $B$ edge has exactly one vertex in $J$, which is even, and one vertex in $J^{\prime}$.

Using the fact that all $R$ endpoints in $I$ are odd and every $B$ endpoint in $J$ is even together with the fact that $I \cap J^{\prime}=\emptyset$, we have that no vertex in $I$ is the endpoint of both a $B$ and an $R$ edge. Then the only candidates for a vertex of high degree must be found in $I^{\prime}$. Note first that $I_{1}^{\prime} \cap\left(J \cup J^{\prime}\right)=\emptyset$, so any high degree vertex must come from $I_{2}^{\prime}$. The largest vertex which is also an endpoint in $B$ is $x_{b}+1$. Recall that $x_{b} \leq \frac{n-1}{4}$, therefore $x_{b}+1 \leq \frac{n+3}{4}<\frac{n-1}{2}-k$ for all $0 \leq k \leq r-1$. Therefore there are no vertices of degree greater than 2 in $L$.

It is left to show that $L$ is acyclic. Suppose for a contradiction that $C$ is a cycle in $M_{0} \cup R$, clearly $C$ must alternate between $M_{0}$ edges and $R$ edges. Let $e=(-i, i)$ be the edge in $C \cap M_{0}$ such that $i \in\left[\frac{n-1}{2}\right]$ is maximal. Let $r^{+}, r^{-} \in C \cap R$ be edges with an endpoint at $i$ and $-i$ respectively. Observe that if $i=\frac{n-1}{2}$ then $r^{+}=\left(-3, \frac{n-1}{2}\right) \in R_{2}$ and so $(-3,3) \in C$. Since $3 \notin I \cup I^{\prime}$ there is only one edge in $M_{0} \cup R$ at 3 , meaning $C$ cannot be a cycle. Further note that the edge $\left(-\frac{n-3}{2}, \frac{n-3}{2}\right)$ is in the same path as $\left(-\frac{n-1}{2}, \frac{n-1}{2}\right)$ by edges in $R_{1}$, therefore $i<\frac{n-3}{2}$. This means that $r^{+}, r^{-} \in R_{2}$, so let $r^{+}=\left(r^{-1}(i), i\right)$ and $r^{-}=(-i, r(-i))$. Suppose that $\left|r^{-1}(i)\right|<r(-i)$, then after $r^{+}$the cycle $C$ must follow an $M_{0}$ edge to $\ell=\left|r^{-1}(i)\right|$ where there must be another $R$ edge $\left(r^{-1}(\ell), \ell\right)$ for $C$ to continue. Since $\ell<r(-i)$, the construction of $R_{2}$ means we have $r^{-1}(\ell)<-i$. Since all $R$ vertices in $I_{2}$ are negative this means $\left|r^{-1}(\ell)\right|>i$. The next edge in $C$ after $\left(r^{-1}(\ell), \ell\right)$ must be in $M_{0}$, so we have that $\left(r^{-1}(\ell),\left|r^{-1}(\ell)\right|\right) \in C$ where $\left|r^{-1}(\ell)\right|>i$ contradicting the maximality of $i$. The case where $\left|r^{-1}(i)\right|>r(-i)$ is analogous.

So every cycle must contain a $B$ edge. Again, for a contradiction, let $C$ be a cycle in $L$, then $C$ must alternate between $M_{0}$ edges and $R \cup B$ edges. Let $(i, b(i)) \in B \cap C$ and note that by definition of $B$ we have $|i|<b(i)$. The edge $(-b(i), b(i))$ must appear in $C$. Since $b(i) \in J^{\prime}$ we have that $-b(i) \notin J \cup J^{\prime}$, this means there is no $B$ edge at $-b(i)$. So there must be an $R$ edge at $-b(i)$ to continue the cycle, in particular we must have $-b(i) \in I$.

In order to have $C$ be a cycle, we must be able to continue in this direction along $C$ and end up back at the vertex $i$.

Let $\ell=r(-b(i))$, then the edge $(-b(i), \ell)$ must be in $C$. Note that $\ell \in I^{\prime}$ and therefore $\ell>|i|$ since $i \in J . C$ must continue after $(-b(i), \ell) \in R$ with the edge $(-\ell, \ell) \in M_{0}$. Since $-\ell \neq i$ we have not connected $C$. The next edge must be from $R \cup B$, and since $\ell \notin J \cup J^{\prime}$ this must be another $R$ edge. In particular $(-\ell, r(-\ell))$. Note that as $r(-\ell) \in I^{\prime}$ we must have that $r(-\ell)>|i|$. Again, the next edge in $C$ must be $(-r(-\ell), r(-\ell)) \in M_{0}$. Since $r(-\ell)>|i|$ we have $-r(-\ell) \neq i$ and we have not completed $C$.

We can continue with this same argument, but at each stage we go through a vertex in $I^{\prime}$ then to a vertex in $-I^{\prime}$ via $M_{0}$, then back to $I^{\prime}$ with an $R$ edge. This means we can never get back to the vertex $i$ since $|i|$ is smaller than all vertices in $I^{\prime}$.

Claim 8.15. L satisfies (GP1) and (GP3). Moreover, the sub-forest of $L$ which contains all edges of type matching those in $R \cup B$ satisfies (GP2).

Proof. For (GP1), note that $n$ is odd and $M_{0}$ contains exactly one edge of every type. Clearly then, $L$ contains at least one edge of each type.

Recall that each edge in $R$ has a different type from [ $\left.\left[\frac{n-1}{4}\right\rceil, \frac{n-1}{2}\right]$, and that each $B$ edge has a different type from $\left[\frac{x_{b}+3}{2}, x_{b}\right]$ where $x_{b} \leq\left\lfloor\frac{n-1}{4}\right\rfloor$. This means that there is no edge type that appears more than twice in $L$. Moreover, for any edges $e, e^{\prime} \in L$ that have the same type, we have $e \in M_{0}$ and $e^{\prime} \in R \cup B$. This means the restriction of $L$ to edges with type found in $R \cup B$ contains exactly two edges of each type, hence satisfies (GP2).

Since $M_{0}$ contains exactly one of each edge type, the family of rotations of $M_{0}$ must cover the edges of $K_{n}$. In other words, any edge in $R \cup B$ is found in $M_{i}$ for some $i \in\left[-\frac{n-1}{2}, \frac{n-1}{2}\right]$. Let $e \in M_{0}$ and $e^{\prime} \in R \cup B$ both be $x$-type edges. Then $e^{\prime} \in M_{i}$ for some $i$, and since $M_{i}$ is a rotation of $M_{0}$ by $|i|$ we must have that $c d\left(e, e^{\prime}\right)=|i|$. This means that (GP3) is equivalent to the conditions

$$
M_{0} \cap(R \cup B)=\emptyset \quad \text { and } \quad\left|\left(M_{i} \cup M_{-i}\right) \cap(L)\right| \leq 1
$$

for every $i \in\left[\frac{n-1}{2}\right]$.
We associate to the edges of $K_{n}$ a number from $\left[0, \frac{n-1}{2}\right]$ called the crossing number. Each $M_{i}$ (except $M_{0}$ ) contains a single edge $e_{i}$ at the vertex 0 , define the crossing number of all edges in $M_{i}$ to be the type of the edge $e_{i}$. We define the crossing number of edges in $M_{0}$ to be 0 . Note that by this definition the crossing number of edges in $M_{i}$ is the same as the
crossing number of $M_{-i}$ edges, we also have that any pair $M_{i}, M_{j}$ with $|i| \neq|j|$ contain edges with different crossing numbers. Therefore, to show the conditions above hold, it is sufficient to show that no two edges from $R \cup B$ have the same crossing number, and that no edge in $R \cup B$ has crossing number 0 .

Consider the edge $e=\left(1,-\frac{n-3}{2}\right) \in R_{1}$, let $M_{\ell}$ be the matching containing $e$. Then $M_{\ell}$ must also contain the edge $\left(0,-\frac{n-5}{2}\right)$ and hence $e$ has crossing number $\frac{n-5}{2}$. Similarly we see that $\left(-1,-\frac{n-1}{2}\right)$ has crossing number $\frac{n-1}{2}$. Now, for edges in $R_{2}$ we first note that $\left(-3, \frac{n-1}{2}\right)$ has crossing number $m=\frac{n-7}{2}$. Then observe that $\left(-3-2, \frac{n-1}{2}-1\right)$ has crossing number $m-3$, and similarly $\left(-3-2 k, \frac{n-1}{2}-k\right)$ has crossing number $|m-3 k|$. Note that $|m-3 k| \neq\left|m-3 k^{\prime}\right|$ for any $k \neq k^{\prime}$, so no edges in $R_{2}$ have the same crossing number. Further note that $m$ is not a multiple of 3 since $\frac{n-1}{2}$ is not a multiple of 3 , this means no $R_{2}$ edge has crossing number 0 .

Since $r \leq \frac{n-7}{4}$ we have that $|m-3 k|<\frac{n-5}{2}$ for all $0 \leq k \leq r-1$, meaning that there are no edges in $R_{2}$ with the same crossing number as an edge in $R_{1}$. In particular this means that no two edges in $R$ have the same crossing number, and no edge in $R$ has a crossing number which is 0 or a multiple of 3 .


Figure 8.5: Examples of $R$ when $\frac{n-1}{2}$ is and is not a multiple of 3

We now consider edges in $B$. Recall that the edge $b=\left(-\frac{x_{b}-3}{2}, \frac{x_{b}+3}{2}\right) \in M_{-\frac{n-3}{2}}$ and therefore has crossing number 3 . Note also that $\left(-\frac{x_{b}-3}{2}+2, \frac{x_{b}+3}{2}+1\right)$ has crossing number 6 and in general the edge $\left(-\frac{x_{b}-3}{2}+2 k, \frac{x_{b}+3}{2}+k\right)$ has crossing number $3(k+1)$ for $0 \leq k \leq t-1$. Recall also that $t \leq \frac{n+3}{8}$, therefore $3(k+1) \leq \frac{n-1}{2}$ for all $n \geq 13$. In particular we have that no edges in $B$ have crossing number 0 , and clearly no two edges from $B$ have the
same crossing number. We saw earlier that no edges in $R$ have a crossing number which is a multiple of 3 , since edges in $B$ only have crossing numbers which are multiples of 3 we see that there are no two edges in $R \cup B$ with the same crossing number.

The above tells us that the only edge types that fail the conditions of Definition 8.2 are those that appear only once (and hence fail (GP2)). These are exactly the types that do not appear in $R \cup B$. In other words, let

$$
F_{L}=\{x: \text { there is an } e \in R \cup B \text { with edge type } x\}
$$

be the set of edge types from edges in $R \cup B$, then $L$ is an $F_{L}$-separator. Note that $F_{L}$ is close to the set $\left[\frac{n}{8}, \frac{n-1}{2}\right]$, possibly with a small number of elements missing.

We now must connect our $L$ into a single path without using too many edges with type in $F_{L}$.

Step 2: Finding connecting edges to join the linear forest into a path


Figure 8.6: The segments of $L$ for $n=35$

We can do this by adding edges between endpoints of paths in $L$, but it is in our interests to keep these joining edges as short as possible to avoid using edges from $F_{L}$. In this step we will find a set of edges $C$ such that $L \cup C$ is a path and where the edge types in $C$ do not overlap too much with those in $F_{L}$. To do this, we need to know where the endpoints of paths in $L$ appear, and a little more about the behaviour of each path in $L$. We partition the non-zero vertices of $K_{n}$ into sets depending on their label. First set $T^{+}=\left[x_{b}+1\right], M^{+}=\left[x_{b}+2, \frac{n-1}{2}-r\right]$ and $U^{+}=\left[\frac{n-1}{2}-r+1, \frac{n-1}{2}\right]$, then $T^{-}, M^{-}$and $U^{-}$ contain the respective negative vertices (see Figure 8.6).

First consider vertices in $U^{+}$, note that each vertex in $U^{+}$is an $I^{\prime}$ vertex of an $R_{2}$ edge. Since these vertices also all have an incident $M_{0}$ edge we conclude that there are no endpoints in $U^{+}$.

Next $M^{+}$, since all vertices in $M^{+}$are larger than $x_{b}+1$ and smaller than $\frac{n-1}{2}-r+1$ there can be no incident $B$ or $R$ edges here. Hence every vertex in $M^{+}$is an endpoint of a path in $L$. Note that $\left|M^{+}\right|=\frac{n-1}{2}-r-x_{b}-1$, then due to the values of $x_{b}$ and $r$, we must have that $1 \leq\left|M^{+}\right| \leq 4$.

For $T^{+}$note that every vertex in $J^{\prime} \subseteq T^{+}$is an endpoint of a $B$ edge, therefore the only vertices in $T^{+}$with degree 1 must be in $\left[\frac{x_{b}+1}{2}\right]$. In particular, by the construction of $B$, these endpoints must all be odd vertices since $-\frac{x_{b}-3}{2}+2 k \in J$ is even for $0 \leq k \leq t-1$. Further note that there is an $R_{1}$ edge at vertex 1 , but this is the only $R$ edge with an endpoint in $T^{+}$. Together this means all odd vertices in $\left[3, \frac{x_{b}-1}{2}\right]$ are endpoints in $L$, this is $\frac{x_{b}-3}{4}$ vertices since $\left(i_{b}+3\right)-1=\frac{x_{b}+1}{2}$ is even.

For the negative side, we know that all odd vertices have an incident $R$ edge, so the only endpoints on this side must be even. For $U^{-}$note that $-\frac{n-3}{2}$ and $-\frac{n-1}{2}$ are both vertices with edges from $R_{1}$, so the endpoints in $U^{-}$are all even vertices in $\left[-\frac{n-5}{2},-\frac{n-1}{2}+r-1\right]$. For $M^{-}$this means that all even vertices are endpoints. In particular each endpoint is from a path consisting of one edge and terminating in $M^{+}$, there are $\left\lfloor\frac{\left\lfloor M^{+}\right\rfloor}{2}\right\rfloor$ such vertices (this is at most 2). Finally for $T^{-}$we have that even vertices in $J$ are endpoints of $B$ edges, therefore the only vertices with degree 1 are in $\left[-x_{b}-1,-\frac{x_{b}-3}{2}-2\right]$. This is $\frac{x_{b}+5}{4}$ vertices. This information is summarised below in Table 8.1.

In total we have $\frac{n-2 x_{b}+1}{2}$ vertices in $L$ with degree 1 when $\frac{n-1}{2}$ is even, and $\frac{n-2 x_{b}-1}{2}$ otherwise. This means $L$ consists of

$$
\frac{n-2 x_{b}+(-1)^{\frac{n-1}{2}}}{4}
$$

paths. We now look at the behaviour of each path.

| Set | Range in which endpoints lie | Endpoint type | Number of endpoints |
| :--- | :--- | :--- | :--- |
| $\{0\}$ | - | all | 1 |
| $U^{+}$ | - | none | 0 |
| $M^{+}$ | $\left[x_{b}+2, \frac{n-1}{2}-r\right]$ | all | 1 to 4 |
| $T^{+}$ | $\left[3, \frac{x_{b}-1}{2}\right]$ | odd | $\frac{x_{b}-3}{4}$ |
| $U^{-}$ | $\left[-\frac{n-5}{2},-\frac{n-1}{2}+r-1\right]$ | even | $\left\lceil\frac{n-17}{8}\right\rceil$ to $\left\lfloor\frac{n-15}{8}\right\rfloor$ |
| $M^{-}$ | $\left[-\frac{n-1}{2}-r,-x_{b}-3\right]$ | even | 0 to 2 |
| $T^{-}$ | $\left[-x_{b}-1,-\frac{x_{b}-3}{2}-2\right]$ | even | $\frac{x_{b}+5}{4}$ |

Table 8.1: Endpoints in each segment of $L$

Claim 8.16. Any path in $L$ with an endpoint in $T^{-}$must have its other endpoint in $T^{-} \cup\{0\}$.

Proof. Recall that all endpoints in $T^{-}$are in the set $\left[-x_{b}-1,-\frac{x_{b}-3}{2}-2\right]$, and that only even vertices are endpoints. Set $a_{1}=-x_{b}-1$ and $a_{2}=-\frac{x_{b}-3}{2}-2$, so our interval of endpoints can be written as $\left[a_{1}, a_{2}\right]$. We will work in stages, checking the largest endpoint in our interval at each step. We will show that the path from this largest vertex terminates at the smallest endpoint in our interval (or at 0 ). We then remove these endpoints from our interval and start again, checking the largest endpoint, continuing until we have found all the paths with endpoints in $T^{-}$. We will start with step 0 .


Figure 8.7: Example of endpoints in $T^{-}$
Step 0, our interval is [ $a_{1}, a_{2}$ ] and we check the path at the largest vertex, $a_{2}=-\frac{x_{b}-3}{2}-2$.

We know there is an $M_{0}$ edge at $a_{2}$, so our path must go through the vertex $-a_{2}=$ $\frac{x_{b}-3}{2}+2=-\frac{x_{b}-3}{2}+2(t-1)$. From the construction of $B$, we see that there is a $B$ edge at this vertex which our path must follow next. This takes the path to the vertex $b\left(-a_{2}\right)=x_{b}+1$, where we must again follow an $M_{0}$ edge to $-\left(x_{b}+1\right)=a_{1}$. We know that $a_{1}$ is even and is the smallest vertex in our interval $\left[a_{1}, a_{2}\right]$, so the path terminates here.

Every remaining endpoint is an even vertex in $\left[-x_{b}+1,-\frac{x_{b}-3}{2}-4\right]$. Indeed, our original interval was $\left[-x_{b}-1,-\frac{x_{b}-3}{2}-2\right]$ and we found $-\left(x_{b}+1\right)$ and $-\frac{x_{b}+1}{2}$ to be endpoints of the first path. Since $-\left(x_{b}+1\right)+1$ and $-\frac{x_{b}+1}{2}-1$ are odd, they cannot be endpoints of a path in $L$. Hence our new interval is $\left[a_{1}+2, a_{2}-2\right]$. This completes step 0 .

Step $k$, our interval is $\left[a_{1}+2 k, a_{2}-2 k\right]$, note that $k \leq \frac{x_{b}+1}{8}$. We check the largest vertex in our interval, $-v=a_{2}-2 k$. We know $-v$ has an edge to $v$ via $M_{0}$, and since $v>\frac{x_{b}+3}{2}$ we have that $v \in J^{\prime}$ has an incident $B$ edge. In particular this $B$ edge is $\left(-\frac{x_{b}-3}{2}+2(2 k-1), \frac{x_{b}+3}{2}+2 k-1\right)$ where $v=\frac{x_{b}+3}{2}+2 k-1$.

We know that $b^{-1}(v)$ is even, if $b^{-1}(v)=0$ then the path must end here since there is only one edge at vertex 0 . Also note that when $b^{-1}(v)=0$ we have $k=\frac{x_{b}+1}{8}$ and hence our interval consists of a single vertex, therefore we must be on the final step. Assume then that $b^{-1}(v) \neq 0$. In this case we have that $b^{-1}(v)<0$ since we are assuming $k<\frac{x_{b}+1}{8}$.

From $b^{-1}(v)$ we move along the path to $\left|b^{-1}(v)\right|$, since this is an even vertex and as $\left|b^{-1}(v)\right|<\left|-\frac{x_{b}-3}{2}\right|$, there must be another $B$ edge here. Since $b^{-1}(v)$ is even, there are exactly $\left|b^{-1}(v)\right|-1$ even vertices between $b^{-1}(v)$ to $\left|b^{-1}(v)\right|$ (not inclusive). This means there are $\left|b^{-1}(v)\right|-1 B$ edges between those at $b^{-1}(v)$ and $\left|b^{-1}(v)\right|$. In other words, the $B$ edge at $\left|b^{-1}(v)\right|$ must be $\left(-\frac{x_{b}-3}{2}+2(\ell-1), \frac{x_{b}+3}{2}+\ell-1\right)$ where $\frac{x_{b}+3}{2}+\ell-1=v+\left|b^{-1}(v)\right|$. Now, recall that $\left|b^{-1}(v)\right|=\frac{x_{b}-3}{2}-2(2 k-1)$, and that $v=\frac{x_{b}-3}{2}+2+2 k$. Using these we get $\frac{x_{b}+3}{2}+\ell-1=x_{b}+1-2 k$. We then follow the $M_{0}$ edge at $x_{b}+1-2 k$ to get to $-\left(x_{b}+1-2 k\right)=a_{1}+2 k$.

Clearly then $a_{1}+2 k$ is even and inside the interval $\left[a_{1}+2 k, a_{2}-2 k\right]$. So it must be an endpoint in $L$, and therefore our path ends in $T^{-}$as claimed. We remove vertices from our interval to give the new interval of $\left[a_{1}+2 k+2, a_{2}-2 k-2\right]$.

We can repeat the above process until we have seen that all paths with endpoints in $T^{-}$ have their other endpoint at 0 or in $T^{-}$.

Note that due to the structure shown in the above, it is easy to join these paths into a single path using short edges.

Claim 8.17. We can use only 2-type edges to connect the paths with endpoints in $T^{-}$ into a single path.

Proof. Use the connecting edges

$$
\begin{aligned}
& C_{B}=\left\{\left(-\frac{x_{b}-3}{2}-2,-\frac{x_{b}-3}{2}-4\right),\left(-x_{b}+1,-x_{b}+3\right), \ldots,\right. \\
& \left.\left(-\frac{x_{b}-3}{2}-2-4 k,-\frac{x_{b}-3}{2}-4-4 k\right),\left(-x_{b}-1+2+4 k,-x_{b}-1+4+4 k\right), \ldots\right\} .
\end{aligned}
$$

Note that this connected path has one end vertex at $\left(-x_{b}-1\right)$, which is the vertex with the smallest label in $T^{-}$, and the other end is either 0 or the vertex $-\frac{3 x_{b}+7}{4} \in T^{-}$.

By doing this, we have replaced $\left\lceil\frac{x_{b}+5}{8}\right\rceil$ paths with a single path.
Now we consider vertices in $T^{+}$. Firstly, for $n \geq 44$ we have that vertices 3 and 5 are endpoints of the same path, this path uses the all $R_{1}$ edges along with the first two edges of $R_{2}$. When $n<44$ there is a $B$ edge at vertex 5 meaning it cannot be an endpoint. This leaves us to check odd vertices in $\left[7, \frac{x_{b}-1}{2}\right]$.

Claim 8.18. Every odd vertex in $\left[7, \frac{x_{b}-1}{2}\right]$ is the beginning of a path in $L$ that terminates in $U^{-}$.

Proof. Every vertex $v \in\left[7, \frac{x_{b}-1}{2}\right]$ has an incident $M_{0}$ edge to $-v \in T^{-}$, since $v$ is odd $-v$ has an incident $R_{2}$ edge. By the definition of the $R_{2}$ edges and of $U^{+}$, this edge must be $(-v, r(-v))$ where $r(-v) \in U^{+}$. Again from $r(-v)$ the path must follow the $M_{0}$ edge to $-r(-v) \in U^{-}$. Now we have two cases, either $r(-v)$ is even and hence there is no $R$ edge at $-r(-v)$ (also clearly no $B$ in $U^{-}$), or $r(-v)$ is odd and there is another $R$ edge that leads to a vertex in $U^{+}$.

For the case where $r(-v)$ is even, since there are no other edges at $-r(-v)$ the path must terminate here. This follows the claim as $-r(-v) \in U^{-}$. So we assume $r(-v)$ is odd and that $(\ell, r(\ell))$ is the $R$ edge with $-r(-v)=\ell$. By definition of $U^{+}$we have that $r(\ell) \in U^{+}$, so the path must continue via $M_{0}$ to $-r(\ell) \in U^{-}$. Once again, if $r(\ell)$ is even the path terminates here and proves the claim. So we must assume $r(\ell)$ is odd, in which case we follow the path to another vertex in $U^{+}$via $R$ as before, returning to $U^{-}$via $M_{0}$. Here we are faced again with termination of the path at an even vertex, or continuing to $U^{+}$ and subsequently back to $U^{-}$at odd vertex. Since the graph is finite this process must end, clearly the only location the process finishes is at an even vertex in $U^{-}$.

Once again we can use this structure to join together the above paths into fewer longer paths. Indeed, using the edges $(5,7),(9,11),(13,15), \ldots$ for all odd vertices in $\left[7, \frac{x_{b}-1}{2}\right]$ immediately halves the number of paths. But we can do better than this.

Claim 8.19. We can use only 2-type edges to connect the paths with endpoints in $T^{+}$ into at most $\frac{1}{2}\left(\left\lceil\log _{2}\left(\frac{x_{b}-11}{4}\right)\right\rceil+1\right)$ paths.

Proof. Recall that we are considering $n \geq 44$.
Consider the path $P_{1}$ from vertex 7 , and $P_{2}$ from vertex 9. After three edges $\left(M_{0}, R\right.$, and $\left.M_{0}\right), P_{1}$ must go through vertex $-\frac{n-5}{2}$, and $P_{2}$ through $-\frac{n-7}{2}$. Clearly, one of $-\frac{n-5}{2}$ and $-\frac{n-7}{2}$ is even and therefore is the endpoint of the path. Let $v=7$ if $\frac{n-5}{2}$ is even, and $v=9$ otherwise, and let $-u$ be the other endpoint of the path at $v$. Then we must have that the path from $v+4$ terminates at $-u+2$, and in general the path at $v+4 k \in\left[7, \frac{x_{b}-1}{2}\right]$ terminates at $-u+2 k$ for $k \in \mathbb{N}$.

Similarly, consider the paths from vertices $v+2$ and $v+6, P_{3}$ and $P_{4}$. After 3 edges, both paths are at an odd vertex in $U^{-}$, in particular, $P_{3}$ is on $-u+1$ and $P_{4}$ is on $-u+3$. This means they are on odd vertices in $U^{-}$and hence the next edge in both paths is an $R$ edge to $U^{+}$. Let $w=r(-u+1)$, then we must have that $(-u+3, w+1)$ is the $R$ edge at $-u+3$. The paths $P_{3}$ and $P_{4}$ must then follow $M_{0}$ edges to $-w$ and $-w-1$ respectively. Once again, one of $-w$ and $-w-1$ is even and hence the path terminates here, the other must continue along another $R$ edge. Let $w^{\prime}$ be the even vertex out of $w$ and $w+1$, and let $v^{\prime}$ be the vertex in $T^{+}$on the path to $w^{\prime}$ (either $v+2$ or $v+6$ ). Then we have that the path from $v^{\prime}+8$ must terminate at $-w^{\prime}-2$, and similarly the path from vertex $v^{\prime}+8 k \in\left[7, \frac{x_{b}-1}{2}\right]$ terminates at vertex $-w^{\prime}-2 k$.

In the first stage we found that paths from every other vertex in $\left[7, \frac{x_{b}-1}{2}\right]$ terminate at consecutive endpoints in $U^{-}$. Let $T_{1} \subseteq\left[7, \frac{x_{b}-1}{2}\right]$ be those vertices. Then we looked at $\left[7, \frac{x_{b}-1}{2}\right] \backslash T_{1}$ and found that half of these vertices lead a path that terminates after 5 edges and have consecutive endpoints in $U^{-}$. Let $T_{2} \subseteq\left[7, \frac{x_{b}-1}{2}\right] \backslash T_{1}$ be those vertices found to be in paths of length 5 . Then we can continue and consider odd vertices in $\left[7, \frac{x_{b}-1}{2}\right] \backslash\left(T_{1} \cup T_{2}\right)$. By using the same argument as above we find again that half of these vertices are endpoints to paths that have 7 edges and with the $U^{-}$endpoints all consecutive.

By the end of this process we have at most $\left\lceil\log _{2}\left(\frac{x_{b}-11}{4}\right)\right\rceil+1$ sets $T_{i}$. For each set $T_{i}$, let $U_{i}$ be the set of vertices in $U^{-}$which are endpoints of paths starting in $T_{i}$.

We are now ready to define our connecting edges. First we use the edges

$$
C_{0}=\{(9,11),(13,15), \ldots\}
$$

for all odd vertices in $\left[9, \frac{x_{b}-1}{2}\right]$. Since every path starting in $T^{+}$terminates in $U^{-}$adding these edges does not create any cycles. In particular, these edges connect a vertex in $T_{1}$ with a vertex in $T_{i}$ where $i \neq 1$. This means that the new path has one endpoint in $U_{1}$ and the other in $U_{i}$. After adding these edges, the vertex 7 remains an endpoint in $T^{+}$, but the only other possible endpoint is the vertex $\frac{x_{b}+3}{2}-2=\frac{x_{b}-1}{2}$.

Let $T_{i}=\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{\ell_{i}}^{i}\right\}$ where $v_{j}^{i}<v_{k}^{i}$ whenever $j<k$. Similarly, we let $U_{i}=$ $\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{\ell_{i}}^{i}\right\}$ where the path from $v_{j}^{i}$ ends at $u_{j}^{i}$. The next connecting edges we add will be

$$
C_{1}=\left\{\left(u_{1}^{1}, u_{2}^{1}\right),\left(u_{3}^{1}, u_{4}^{1}\right), \ldots\right\}
$$

Again, this cannot create a cycle since no path begins and ends in $U_{1}$. Observe that the edge $\left(u_{2 i-1}^{1}, u_{2 i}^{1}\right)$ joins a path through vertex $v_{2 i-1}^{1}$ with a path through $v_{2 i}^{1}$. If $v_{2 i-1}^{1}=k$ then $v_{2 i}^{1}=k+4$, meaning that $k$ is joined to $k^{\prime} \in\{k-2, k+2\}$ by an edge in $C_{0}$ (or $k=7$ has degree 1 ) and $k+4$ is joined to $k^{\prime}+4$ by a $C_{0}$ edge (unless $k+4=\frac{x_{b}-1}{2}$ has degree 1). Since $k^{\prime}$ and $k^{\prime}+4$ differ by 4 and neither are in $T_{1}$ we must have that either $k^{\prime} \in T_{2}$ or $k^{\prime}+4 \in T_{2}$ but cannot have both. This means that the new path containing the edge ( $u_{2 i-1}^{1}, u_{2 i}^{1}$ ) cannot have both endpoints in $U_{2}$.

Again, we now do an analogous process for connectors in $T_{2}$. That is, we use connectors $C_{2}=\left\{\left(u_{1}^{2}, u_{2}^{2}\right),\left(u_{3}^{2}, u_{4}^{2}\right), \ldots\right\}$, noting that we have not created a cycle since no path has two endpoints in $U_{2}$. Using an analogous argument to the above we also have that no new path has two endpoints in $U_{3}$. We continue this way until we have defined connector sets $C_{0}, C_{1}, \ldots, C_{\ell}$ where $\ell \leq\left\lceil\log _{2}\left(\frac{x_{b}-11}{4}\right)\right\rceil+1$.

Finally we add the special connecting edge (5, 7).
It is left to count the number of paths we now have, as well as the number of paths we replaced. Recall that there were $\frac{x_{b}-11}{4}$ endpoints in $\left[7, \frac{x_{b}-1}{2}\right]$ and hence $\frac{x_{b}-11}{4}$ paths from $T^{+}$to $U^{-}$, along with the path from 3 to 5 . We count the new number of paths by again counting the number of endpoints. Note that the vertex 3 is an endpoint and $\frac{x_{b}-1}{2}$ may also be an endpoint (if it is $1 \bmod 4$ ). There are now no other endpoints in $T^{+}$. Since we connected vertices in $U_{i}$ by pairing them, there can be at most 1 endpoint in each $U_{i}$, and hence at most $\left\lceil\log _{2}\left(\frac{x_{b}-11}{4}\right)\right\rceil+1$ endpoints in $\cup_{1 \leq i \leq \ell} U_{i}$. Therefore, in total we have replaced $\frac{x_{b}-7}{4}$ paths with at most

$$
\frac{\left\lceil\log _{2}\left(\frac{x_{b}-11}{4}\right)\right\rceil+1}{2}
$$

new paths. It is left only to note that we have only used 2-type edges in this construction as we have connected consecutive odd vertices in $T^{+}$and consecutive even vertices in each $U_{i}$.

At this point, the total number of paths in $L \cup C_{B} \cup C_{0} \cup \cdots \cup C_{\ell} \cup\{(5,7)\}$ is given by

$$
\frac{n-2 x_{b}+(-1)^{\frac{n-1}{2}}}{4}-\left\lceil\frac{x_{b}+5}{8}\right\rceil+1-\frac{x_{b}-7}{4}+\frac{\left\lceil\log _{2}\left(\frac{x_{b}-11}{4}\right)\right\rceil+1}{2} .
$$

This is at most

$$
\frac{n+16 \log _{2} n+163}{32}
$$

Let $L^{\prime}=L \cup C_{B} \cup C_{0} \cup \cdots \cup C_{\ell} \cup\{(5,7)\}$ be the linear forest at this point. Finally we will connect $L^{\prime}$ using any suitable edges, call the set of these additional edges $C_{A}$. Let $P$ be the path $P=L \cup C$, where $C=C_{B} \cup C_{0} \cup \cdots \cup C_{\ell} \cup\{(5,7)\} \cup C_{A}$.

Note that $\left|C_{A}\right| \leq \frac{1}{32}\left(n+16 \log _{2} n+131\right)$ by the number of paths remaining in $L^{\prime}$. Furthermore, we are able to choose the edges of $C$ in such a way that for any $x$, if there are more than two $x$-type edges in $P$ then they are not spaced out evenly on $K_{n}$. This ensures that all edges of the same type are separated from each other by the rotations of $P$. All the edge types in $C_{A}$ will end up in $D$, so apart from ensuring the edges are not equally spaced there are no other conditions for these edges to follow.

Claim 8.20. We can join the paths in the linear forest $L^{\prime}$ into a single path $P$ such that there are no evenly spaced $x$-type edges in $P$.

Proof. Consider the linear forest $L^{\prime}$ and its endpoints (see summary in Table 8.2).

| Set | Endpoints in set | Other endpoint of path | Reference |
| :--- | :--- | :--- | :--- |
| $\{0\}$ | all | $-x_{b}-1$ or $U^{-}$ | Claim 8.17 |
| $U^{+}$ | none | - | See Table 8.1 |
| $M^{+}$ | all | $U^{-}$and possibly $M^{-}$ | See $M^{-}$ |
| $T^{+}$ | 3 and possibly $\frac{x_{b}-1}{2}$ | $U^{-}$or the pair $\left\{3, \frac{x_{b}-1}{2}\right\}$ | Claim 8.19 |
| $U^{-}$ | some evens | $U^{-}, M^{+}$, and possibly $T^{+}$and 0 | See $M^{+}, T^{+},\{0\}$ |
| $M^{-}$ | all evens | $M^{+}$ | Length 1 paths |
| $T^{-}$ | $-x_{b}-1$ and possibly $-\frac{3 x_{b}+7}{4}$ | 0 or the pair $\left\{-x_{b}-1,-\frac{3 x_{b}+7}{4}\right\}$ | Claim 8.17 |

Table 8.2: Path connections between each segment of $L^{\prime}$

Starting with $C_{A}$ empty, we add edges as follows. First add the edge ( 0,3 ), note that this is a 3 -type edge. Since $R$ and $B$ only contain edges of large type, there is only one 3 -type edge in $L^{\prime}$. In particular, since $n$ is odd, $L^{\prime} \cup\{(0,3)\}$ does not have evenly spaced 3 -type edges.

Next, suppose the vertex $v=-\frac{3 x_{b}+7}{4}$ is indeed and endpoint. This means that the distance from $v$ to some endpoint in $U^{-}$is at most $\frac{3 n-29}{16}<\frac{n-1}{4}$. Indeed, in $L$ there was an endpoint at every even vertex of $\left[-\frac{n-5}{2},-\frac{n-1}{2}+r-1\right]$ until we added $C_{0} \cup \cdots \cup C_{\ell} \cup\{(5,7)\}$. This left at least $\frac{n+3}{16}$ vertices in $U^{-}$. Further note that any edge from $v$ to an endpoint in $U^{-}$ must have even edge type since both endpoints are on even vertices. Let $e_{v}$ be any such edge, note that $L^{\prime} \cup\left\{(0,3), e_{v}\right\}$ is still a linear forest.

We now move on to the vertices in $M^{+}$. If $\left|M^{+}\right|=1$ leave this vertex as an endpoint. If $\left|M^{+}\right|=2$ then use a 1-type edge $e_{+}$to join them, noting that this creates a path with one vertex in $M^{-}$and the other in $U^{-}$. This cannot create a cycle since we have not added any edges to vertices in $M^{-}$. If $\left|M^{+}\right|=3$, join with $e_{+}$as in the previous case, and leave one as an endpoint. Finally, if $\left|M^{+}\right|=4$ join with two 1-type edges $e_{+}$and $e_{+}^{\prime}$, again this does not create a cycle. Denote by $E_{M}$ the set of edges added within $M^{+}$, so $E_{M}$ depends on $\left|M^{+}\right|$and is equal to one of $\emptyset,\left\{e_{+}\right\}$or $\left\{e_{+}, e_{+}^{\prime}\right\}$. Note that $L^{\prime} \cup E_{M}$ does not contain evenly spaced 1-type edges. Indeed, when $E_{M}=\emptyset,\left\{e_{+}\right\}$then $L^{\prime} \cup E_{M}$ contains at most 2 type 1 edges, and therefore they cannot be evenly spaced. When $E_{M}=\left\{e_{+}, e_{+}^{\prime}\right\}$ there are exactly three 1-type edges in $L^{\prime} \cup E_{M}$ and two of them, $e_{+}$and $e_{+}^{\prime}$, have clockwise distance 2 between them. Therefore there are no evenly spaced 1-type edges.

At this point $C_{A}=\left\{(0,3), e_{v}\right\} \cup E_{M}$.
Now, $L^{\prime} \cup C_{A}$ is a linear forest and has at most two endpoints outside of $U^{-} \cup M^{-} \cup$ $\left\{-x_{b}-1\right\}$, the vertices $\frac{x_{b}-1}{2}$, and some vertex $u \in M^{+}$. Since $u$ and $\frac{x_{b}-1}{2}$ are not endpoints of the same path (see construction in 8.19) , we simply need to join all endpoints within $U^{-} \cup M^{-} \cup\left\{-x_{b}-1\right\}$ to create our path $P$. Note that since all endpoints in this region are even vertices, this will only require edges of even edge type. Moreover it only requires edges of type $\frac{n-1}{4}$ or shorter. Indeed, the longest possible edge in this interval is $\left(-x_{b}-1,-\frac{n-5}{2}\right)$, which has type at most $\frac{n+1}{4}$, but since there are at least $\frac{n+3}{16}-1$ endpoints remaining in $U^{-}$(as we saw earlier in this proof) we do not need to use this longest edge. All other edges in the interval have type at most $\frac{n-1}{4}$ as required.

We add as many of these short even edges to $C_{A}$ as required, until $L^{\prime} \cup C_{A}$ is a path. Then we set $P=L^{\prime} \cup C_{A}$.

It is left to check that $P$ has no evenly spaced edge types. Note that $L$ clearly does not since $L$ contains at most 2 edges of each type. So we only need to check the edge types appearing in $C$. Note that among all the edges in $C$, the only edges that are not even are the edges in $E_{M}$ and the edge $(0,3)$. The remaining edges in $C$ are all even and have edge type at most $\frac{n-1}{4}$. We have seen that the 3 -type edge and the 1 -type edges do not create a problem, it is left to check the even edges of $C$.

Consider edges of even type in the linear forest $L$, with starting vertex in $\left[\frac{x_{b}+3}{2}, \frac{n-1}{2}\right] \cup$ $\left[-\frac{n-1}{2},-\frac{n-3}{2}\right]$. Such edges must all be from $R$. Indeed, even-type $M_{0}$ edges all start on a negative vertex, and in particular the $M_{0}$ edges at $-\frac{n-1}{2}$ and $-\frac{n-3}{2}$ both have odd type. Also all $B$ edges have their starting vertex in the interval $J$, which does not intersect $\left[\frac{x_{b}+3}{2}, \frac{n-1}{2}\right] \cup\left[-\frac{n-1}{2},-\frac{n-3}{2}\right]$. We also know that all edges in $R$ have a type from $\left[\left\lceil\frac{n-1}{4}\right\rceil, \frac{n-1}{2}\right]$.

The only edges added to this interval to create $P$ are the edges in $E_{M}$. We know that $E_{M}$ only contains edges with odd type. In other words, in $P$ no edges of even type from $\left[2, \frac{n-1}{4}\right]$ has a starting vertex in the interval $\left[\frac{x_{b}+3}{2}, \frac{n-1}{2}\right] \cup\left[-\frac{n-1}{2},-\frac{n-3}{2}\right]$. This means that if $e_{1}, \ldots, e_{m}$ are all the $x$-types in $P$ with even $x \in\left[2, \frac{n-1}{4}\right]$, and $c d\left(e_{i}, e_{i+1}\right)=\frac{n}{m}$ then $\frac{n}{m} \leq \frac{3 n+17}{8}$. This forces $m<3$, since $n$ is odd we cannot have any evenly spaced $x$-type edges when $m=2$.

Therefore there are no evenly spaced $x$-type edges in $P$.
We use $P$ as the base path for Theorem 8.12 to give a separating path system for $K_{n}$. The size of this family is dependent on the size of $D=\left[\frac{n}{2}\right] \backslash F$.

Claim 8.21. Let $P$ be the path defined in this section. Then $P$ is an $F$-separator path, where $D$ satisfies the following.

$$
|D \cup\{1\}| \leq \frac{5 n+16 \log _{2} n+167}{32}
$$

Proof. From 8.15 we know that in $L$ all edge types in $R \cup B$ appear exactly twice, and the remainder appear exactly once. This means there are at most

$$
\frac{n-1}{2}-\frac{n-9}{4}-2-\frac{x_{b}+1}{2} \leq \frac{n+9}{8}
$$

edge types in $L$ that appear exactly once. We put all of these edge types into $D$. Note that, by the construction of $R$ and $B$, these are the shortest $\approx \frac{n}{8}$ edge types. In particular $1,2 \in D$.

Moreover, we know by 8.15 that $L$ is an $F_{L}$-separator where $F_{L}=\{x$ : there is an $e \in$ $R \cup B$ with edge type $x\}$. To turn $L$ into $P$ we added various connecting edges $C$. The
only way an edge type can be in $F_{L}$ but not in $F$ is if some edge in $C$ has edge type from $F_{L}$. Therefore, for the path $P$, the types associated with these $C$ edges must also appear in $D$, and the remaining types in $F$.

All edges in $C_{B} \cup C_{0} \cup \cdots \cup C_{\ell} \cup\{(5,7)\}$ have type 2, which are already accounted for. We used a final $\frac{1}{32}\left(n+16 \log _{2} n+131\right)$ edges in $C_{A}$. This means $D$ contains at most

$$
\frac{n+9}{8}+\frac{n+16 \log _{2} n+131}{32}=\frac{5 n+16 \log _{2} n+167}{32}
$$

edge types.

We remark that the methods in this section do not adapt too well to the strong version of the problem. Indeed, the way to adapt the path system given in Theorem 8.9 using these methods would be to use more fixing paths (as in Lemma 8.11). However, this gives an upper bound of approximately $2 n$, which is not an improvement from the previously known result of [3] - although it would be constructive.

### 8.6 Speculations on generator path construction

The complete graphs $K_{n}$ with the smallest known separating path system all have values of $n$ for which there exists a generator path. While we are able to use and adapt the structure of generator paths to give a bound in general, it is still valuable to investigate when generator paths exist or even if they do not exist for some values of $n$.

The question is still open as to exactly which values of $n$ there is a generator path. Our motivation for this question comes from separating path systems, but the question of existence of paths containing specific arrangements of edge types seems tricky and interesting in its own right. For instance, McKay and Peters investigate which multisets of edge types are realisable as a path in [20].

In this section we will have a brief look at two directions for answering this question which arise from the methods used in the previous sections of this chapter.

### 8.6.1 Zig-Zag

When trying to construct a generator path, the most basic properties we need to ensure are (GP1) and (GP2). A very useful and symmetric path in any $K_{n}$, is one that zig-zags back and forth, each new edge being one type larger than the previous, before working back down. When $n$ is even this takes the form of the standard Hamilton path which,
when rotated, provides a Hamilton decomposition of $K_{n}$. These paths always satisfy (GP1) and (GP2), but fail at being generator paths through (GP3). In actual fact these paths put all same type edge pairs at the same distance from each other.

Formally, we call the zig-zag path for $n$ the path which follows

$$
P=\left(1,2, n, 3, n-1,4, n-2, \ldots,\left\lceil\frac{n}{2}\right\rceil+1\right)
$$

When we refer to the first half of this path, we mean the first $\left\lfloor\frac{n}{2}\right\rfloor$ edges of $P$. Note that $P$ has exactly two edges of each edge type except $\frac{n}{2}$ if $n$ is even, in which case there is exactly one such edge. Moreover, the first half of $P$ contains exactly one edge of each type. Note that this means the $n$ rotations of the the first half of $P$ cover every edge of $K_{n}$. Let $P^{\prime}$ denote the first half of a zig-zag path for $n$.

Clearly these paths are not generators, but it seems reasonable that we might be able to borrow some of the structure to build a path, ensuring we fulfil (GP1) and (GP2). In this section, we will show that


Figure 8.8: The zig-zag path $P$ on $K_{9}$, with the $P^{\prime}$ shown as dashed this is not the case, and in fact any generator path for $n$ cannot contain $P^{\prime}$ as a subgraph. Assume for ease that $n$ is odd, we will build our path in the following way which makes clear that a generator path cannot contain $P^{\prime}$.

Let $E$ be the set of edges we add to $P^{\prime}$ to get our generator path, meaning $P^{\prime} \cup E$ is a generator path for $n$. Then $P^{\prime} \cup E$ must be a path and also $P^{\prime} \cup E$ must follow (GP1), (GP2) and (GP3). Since $P^{\prime}$ contains one edge of each type we have satisfied (GP1) already. In order to satisfy (GP2), $E$ must contain no more than one edge of each type, and at most one edge type can be missing from $E$.

Note that when considering edges to add to $E$, each addition forms a pair of same type edges in $P^{\prime} \cup E$ and therefore the pair must be checked against (GP3),

Let $L$ be the subset of $E\left(K_{n}\right) \backslash P^{\prime}$ containing only legal extra edges, that is, for any $e \in L$ we have that $P^{\prime} \cup e$ is a linear forest. We will colour the edges of $L$ in such a way that any two edges which have different colours can be added to $E$ without breaching (GP3).

Let $e \in L$ and say $e$ has type $x$. Since the $n$ rotations of $P^{\prime}\left(P_{0}^{\prime}, \ldots, P_{n-1}^{\prime}\right)$ cover all the


Figure 8.9: Example of the complete colouring showing $P_{0}^{\prime}$ in black, with $n=11$ and $n=13$
edges of $K_{n}$, we have that there is an $i \in[n-1]$ such that $e \in P_{i}^{\prime}$. Note that $i \neq 0$ since $P_{0}^{\prime}=P^{\prime}$ and $e \in L$ cannot belong to $P^{\prime}$.

Any edge $e^{\prime} \in P_{i}^{\prime} \cap L$ must be the same colour as $e$. This is because the distance between $e$ and the $x$-type of $P^{\prime}$ is $i$, and clearly the distance between the $y$-type of $P^{\prime}$ and the $y$-type of $P_{i}^{\prime}$ is also $i$.

Similarly, all edges in $P_{n-i}^{\prime}$ must be the same colour as each other, moreover this must be the same colour as those in $P_{i}^{\prime}$. So for each $i \in\left[\frac{n-1}{2}\right]$ colour the edges of $P_{i}^{\prime}$ colour $c_{i}$, and the edges of $P_{n-i}^{\prime}$ colour $c_{i}$.

Now we need to choose the edges of $E$ from $L$ in such a way that we do not take more than one edge of any type or colour. As we can see in Figure 8.9, this is easily done when $n=11$ by taking $E=\{(4,5),(5,9),(9,7),(1,6)\}$, we have that $P^{\prime} \cup E$ is a generator path.

It is equally easy to see however, that this is not the case for $n=13$. There are five colours for this case and to have $P^{\prime} \cup E$ be a generator path for $n$ we require $E$ to contain an edge from every one of the 5 colours. Clearly this can not be done, by looking at vertex 1, two colours appear only at this vertex and therefore only one of these colours can be used to extend $P^{\prime}$.

Formally, let $n>13$ be odd. There are at most $\frac{n-1}{2}$ colours used as all edges in $P_{i}^{\prime}$ and $P_{n-i}^{\prime}$ have the same colour for each $i \in\left[\frac{n-1}{2}\right]$. $E$ must not contain more than one edge of
each colour, and must contain at least $\frac{n-1}{2}-1$ edge types. Therefore, edges from $\frac{n-1}{2}-1$ different colour classes must be selected.

First note that $\left(P_{1}^{\prime} \cup P_{n-1}^{\prime}\right) \cap L=\emptyset$, and hence the colour $c_{1}$ is not used. Indeed, note that there are no edges in $L$ at vertices $2, \ldots,\left\lfloor\frac{n-1}{4}\right\rfloor+1$ because these all have degree 2 in $P^{\prime}$. Similarly there are no $L$ edges with vertices at $n-\left(\left\lceil\frac{n-1}{4}\right\rceil-2\right), \ldots, n$. All edges in $P_{1}^{\prime} \cup P_{n-1}^{\prime}$ have at least one endpoint at these vertices and hence cannot belong to $L$.

In the same way, consider $P_{i}^{\prime} \cup P_{n-i}^{\prime}$. When $i \in\left[\left\lfloor\frac{n-1}{4}\right\rfloor+1, \frac{n-1}{2}\right], L$ definitely contains an edge with colour $c_{i}$ since the 1-type edge in $P_{i}^{\prime}$ is also in $L$. Let $k=\left\lfloor\frac{1}{2}\left(\left\lceil\frac{n-1}{4}\right\rceil+1\right)\right\rfloor$, then when $i \in\left[k,\left\lfloor\frac{n-1}{4}\right\rfloor\right\rfloor$ the colour $c_{i}$ is found in $L$ since long edges at vertex 1 belong to these rotations. No other colours are found in $L$.

This means the colouring contains only $\frac{n-1}{2}-(k-1) \approx \frac{7 n}{8}$ colours. We must choose at least $\frac{n-1}{2}-1$ edges of different colours to complete $E$ in the appropriate way. Clearly, there are not enough different colour edges to choose from, and therefore $E$ cannot be completed in such a way that $P^{\prime} \cup E$ is a generator path.

If we are able to add one edge from each colour class to $E$ so that $P^{\prime} \cup E$ is a path, then we can use the rotations of this along with any fixing paths (from Lemma 8.11) for the edge types missing from $E$. This would give a separating path system for $K_{n}$, but the best possible bound this method can give is

$$
f\left(K_{n}\right) \leq n+\frac{n-5}{2}=\frac{3 n-5}{2}
$$

for odd values of $n$.
For example, in the Figure 8.9 when $n=13$, we can extend the path by four edges, $E=\{(5,11),(5,6),(1,10),(8,10)\}$, getting a separating path system of size $n+2+2=17$ ( $n$ rotations, and two each for the missing edge types from the extension $E$ ).

### 8.6.2 Primitive roots

The only known values of $n$ which admit a generator path are $n \leq 20$ and $n=p$ for all odd primes $p$. Since we know that generator paths exist for all primes, we know there are infinitely many values of $n$ which admit a generator path, and also that we are able to find generator paths for arbitrarily large values of $n$. The proof of this (Theorem 8.6) is both constructive and concise, but crucially the method uses algebraic properties, unlike other constructions of separating path systems discussed in this thesis.

The proof of Theorem 8.6 relies on some of the properties of primitive roots, and constructs
a path using a single arbitrary primitive root of $n$. Upon first glance it appears that we only use the property that we can write the elements of $[n-1]$ in the form $g^{i}$ where $g$ is a primitive root modulo $n$ and $i \in[n-1]$. We then take the path given by $v, v+$ $g^{1}, v+g^{1}+g^{2}, \ldots$, where $v$ is some arbitrary vertex to begin the path, typically we choose $v=n$. It is therefore reasonable to ask whether any special integer $h$ which gives $[n-1]=\left\{h^{i}: i \in[n-1]\right\}$ can also give rise to a generator path. For example, if there exists a finite field on $n$ elements, then its multiplicative group is cyclic and generated by some $h$.

The trouble with this is because $h$ is not necessarily an integer. This would not be a problem if we were using the $h^{i}$ s to label the vertices, since these are only labels. The way the path is structured means that the elements $h^{i}$ are giving us the edge types of all our path edges, and not the vertices as we might assume. What it means to be an $h^{i}$-type edge, when $h^{i}$ is not an integer, is undefined. Choosing some equivalence between the integers and $h^{i} \mathrm{~S}$ is essentially the same as choosing the correct order of vertices to make a generator path, and we are no better off.

So the proof of Theorem 8.6 does rely on primitive root properties beyond having a generating element. But there are other known values of $n$ where a primitive root modulo $n$ exists, for example it is known that $p^{k}$ and $2 p^{k}$, where $p$ is an odd prime, both have primitive roots. What can be said about using the method of Theorem 8.6 for these other values of $n$.

Recall that $g$ is a primitive root modulo $n$ if for every integer $x$ which is co-prime to $n$ there is some integer $i$ such that $x \equiv g^{i} \bmod n$. When $n$ is not prime, those $x$ which are co-prime to $n$ do not cover the whole of $[n-1]$. This becomes a problem when considering (GP1) and (GP2). Indeed, the values of each $g^{i}$ give us the edge type of the $i$ th edge in our path $P$. Since edge types belong to $\left[\frac{n}{2}\right]$ and the $g^{i}$ s belong to $[n-1]$, we use the fact that an edge which travels $x$ many vertices has the same edge type as one which travels $n-x$ vertices. So we have that $g^{i}$ and $g^{j}$ such that $g^{i} \equiv n-g^{j}$ will give edges of the same type.

If we have some $x \in[n-1]$ which shares at least one factor with $n$, which we do in the case where $n$ is not prime, then there will be no $x$-type edges in $P$. This means $P$ does not satisfy (GP1).

One final remark on the primitive root strategy is that the paths constructed in Theorem 8.6 do not give a strongly separating path system for any of the prime values. Recall that for a generator path to be useful in the strong variation of the problem, we require
that each edge type appears exactly twice as well as (GP3). The path we take is $P=$ $\left(p, g, g+g^{2}, g+g^{2}+g^{3}, \ldots, \sum_{i=1}^{p-2} g^{i}\right)$, note that this path has $p-2$ edges and therefore cannot contain two of each edge type. In fact, it is a 1-type edge which is missing.

We note that the edge $e=\left(\sum_{i=1}^{p-2} g^{i}, \sum_{i=1}^{p-1} g^{i}\right)$ is a one type edge, but that $P \cup\{e\}$ is a cycle with exactly two edges of each type. This means that we are able to strongly separate the edges of $K_{n}$ by $n$ cycles whenever $n$ is an odd prime. But clearly this means that no path constructed in the proof of Theorem 8.6 will generate a strongly separating path system.

## Chapter 9

## Balanced complete bipartite graphs

### 9.1 Generalising generator paths

Taking multiple rotations of some structure within a graph to give a family of substructures is not unique to complete graphs. In fact the driving force behind the constructions in Chapter 8 and generator paths themselves is just the symmetry of the base graph. It is therefore reasonable to consider these techniques applied to other symmetric base graphs, and give a more generalised notion of a generator path.

It is particularly unwieldy to directly define a generator path in this generalised sense, along with all the associated notions such as edge types. The definition would be too complex to be helpful in understanding, so we do not attempt to formalise this idea in a thorough definition - but rather give a short explanation.

Observe that the rotation of a path as used in Chapter 8 is nothing more than an automorphism. That is, we can think of the rotation as a permutation of the vertices. We tend to consider all $n$ rotations of any path, which translates to the cyclic group generated by the chosen permutation (or automorphism). Therefore, any graph with an automorphism can use the technique. Simply choose a automorphism and consider the cyclic group generated by that permutation.

In order to understand which edges are separated by these path automorphisms, we must have some notion of edge types. These will be a class of edges which are closed under the automorphism group.

In this chapter we develop this approach for balanced complete bipartite graphs. The results of this provide the following bounds for $f\left(K_{n, n}\right)$.

Theorem 9.1. We have that $1.16 n-\frac{1}{2} \leq f\left(K_{n, n}\right) \leq \frac{5 n+5}{4}$.
We will first see the lower bound which will be given in Lemma 9.3. The upper bound from Lemma 9.4 will be constructive, and use a rotation technique. We begin by clarifying some notation and defining our rotation.

Let $K_{n, n}$ denote the complete bipartite graph on vertex classes $A=\left\{a_{1}, \ldots, a_{n}\right\}$ and $B=\left\{b_{1}, \ldots, b_{n}\right\}$. Let $Q$ be a path in $K_{n, n}$, we define a (clockwise) rotation in $K_{n, n}$ of $Q$ by $r$ as the path $Q_{r}$ which contains the edge $(a+r, b)$ if and only if $(a, b) \in Q$. In other words, only the vertices in class $A$ are shifted and all vertices in class $B$ are fixed.

In complete graphs, we have the concept of edge types, which are unchanged under rotation. The conditions of a generator path for $n$ are all related to the edge types and in particular the relationship between two edges of the same type. A similar notion arises in paths on $K_{n, n}$, and in fact we can adapt a path from $K_{n}$ to a path in $K_{n, n}$ in such a way that each class of edge types from $K_{n}$ gives rise to a class of similarly behaved edges in $K_{n, n}$.

Indeed, any path on $K_{n}$ may be adapted to a path on $K_{n, n}$ by associating each vertex of $K_{n}$ to a unique vertex in class $A$ of $K_{n, n}$, and associating each edge on the path to a unique vertex in class $B$. More precisely, if $P=\left(v_{1}, v_{2}, \ldots, v_{k}\right)$ is a path in $K_{n}$, then $P^{\prime}=\left(a_{1}, b_{1}, a_{2}, b_{2}, \ldots, a_{k}\right)$ is a path in $K_{n, n}$ where $a_{i} \in A$ and $b_{i} \in B$ for every $i$ and $a_{i}$ represents the vertex $v_{i}$ of $K_{n}$ and $b_{i}$ represents the edge $\left(v_{i}, v_{i+1}\right)$. Then the pair of edges $\left(a_{i}, b_{i}\right)$ and $\left(a_{i+1}, b_{i}\right)$ are descended from the edge $\left(v_{i}, v_{i+1}\right)$ of $K_{n}$, moreover they retain the information about the edge type and starting vertex of $\left(v_{i}, v_{i+1}\right)$.

For ease of notation we will write all edges in $K_{n, n}$ in the form $(a, b)$ with first coordinate $a \in A$ and $b \in B$ second. We will denote a path $Q=\left(a_{1}, b_{1}, a_{2}, b_{2}, \ldots, a_{k}, b_{k}\right)$ in $K_{n, n}$ by listing alternate edges starting with the first edge, that is $Q=\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right), \ldots,\left(a_{k}, b_{k}\right)$. This is both to clearly distinguish paths in $K_{n, n}$ from paths in $K_{n}$, and to avoid confusion over vertices in $A$ with the same labels as those in $B$.

In adapting $P$ to $P^{\prime}$ we effectively take each edge and replace it with two edges which divert the path to $B$ before returning to $A$. This means that any $x$-type edge will be transformed into a pair of edges with a common vertex in $B$ and with their $A$ vertices at a distance of $x$. Meaning that, for a path $P^{\prime}=\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right), \ldots,\left(a_{k}\right)$ where $\left(v_{i}, v_{i+1}\right)$
is an $x$-type edge of $P$, the path $P^{\prime}$ rotated by $x$ will contain the edge $\left(a_{i+1}, b_{i}\right)$ if the starting vertex (in $K_{n}$ ) was $v_{i}$, or the edge $\left(a_{i}, b_{i}\right)$ if the starting vertex was $v_{i+1}$. We call the edge $\left(a_{i}, b_{i}\right)$ a leading edge and $\left(a_{i+1}, b_{i}\right)$ the trailing edge if $v_{i}$ was the starting vertex, similarly we call $\left(a_{i+1}, b_{i}\right)$ the leading edge and $\left(a_{i}, b_{i}\right)$ the trailing edge if $v_{i+1}$ is the starting vertex.

### 9.2 Lower bound

Just as for complete graphs, to get a lower bound for $f\left(K_{n, n}\right)$ we use a counting argument. Recall that the argument of Lemma 8.1, providing the lower bound for $f\left(K_{n}\right)$, only used two properties of $K_{n}$ : the number of edges, and the length of the longest path. We can use the same argument to give a lower bound in terms of these to properties for any graph $G$.

Lemma 9.2. Let $G$ be a graph with e edges and where the length of the longest path is $p$. Then we have that $f(G) \geq \frac{2(e-1)}{p+1}$.

The proof of this is identical to the proof of Lemma 8.1.
There are $n^{2}$ edges in $K_{n, n}$, and the length of the longest path is $2 n-1$. Applying Lemma 9.2 with these values gives us a lower bound of $f\left(K_{n, n}\right) \geq n$. However, we can adapt the ideas from Lemma 9.2 to give a better bound in this case. For $K_{n, n}$ we can push the steps a little further due to the difference in number of edges and path lengths in $K_{n, n}$ compared to those of $K_{n}$.

Lemma 9.3. For the complete bipartite graph $K_{n, n}$ with vertex classes $A$ and $B$ each of size $n$, the minimum size of separating path system is at least $(\sqrt{10}-2) n-\frac{1}{2}$. Giving $f\left(K_{n, n}\right) \geq(\sqrt{10}-2) n-\frac{1}{2} \geq 1.16 n-\frac{1}{2}$.

Proof. Let $\mathcal{S}$ be a family of paths on $K_{n, n}$ where $|\mathcal{S}|=s$. In order for $\mathcal{S}$ to be a separating path system, $\mathcal{S}$ must cover $n^{2}-1$ edges of $K_{n, n}$. In other words, there is at most one edge of $K_{n, n}$ which appears 0 times in $\mathcal{S}$.

Since there can be at most one unique edge per path, we must have that at most $s$ edges in $K_{n, n}$ appear exactly 1 time in $\mathcal{S}$.

Note that the maximum length of any $P \in \mathcal{S}$ is $2 n-1$. Of these $2 n-1$ edges, at most one may be unique to $P$, the remaining $2 n-2$ edges must all appear in at least one additional
path from $\mathcal{S}$. In other words, we have $2 n-2$ edges distributed across $s-1$ paths. Note that if $k$ of these edges are found in the same additional path, then at least $k-1$ of the $k$ must be found in yet another path of $\mathcal{S}$ in order them to be separated. Clearly then, of these $2 n-2$ edges, at most $s-1$ appear in exactly one additional path and hence appear exactly twice in $\mathcal{S}$. Since this is true of all paths $P \in \mathcal{S}$, there are at most $\frac{(s-1) s}{2}$ edges in $K_{n, n}$ which appear exactly twice in $\mathcal{S}$.

We can use this to calculate the minimum number of edges which must appear in $\mathcal{S}$ three or more times. Indeed, the number of edges covered by $\mathcal{S}$ is at least $n^{2}-1$. Of these at most $s$ appear once and at most $\frac{(s-1) s}{2}$ appear twice. Therefore,

$$
n^{2}-1-s-\frac{(s-1) s}{2}
$$

edges of $K_{n, n}$ must appear at least three times.
We need to have enough paths in $\mathcal{S}$ to ensure that these edges get covered as many times as required. Observe then

$$
\begin{aligned}
\underbrace{(2 n-1) s}_{\begin{array}{c}
\text { edges used } \\
\text { with multiplicity }
\end{array}}-\underbrace{\left(n^{2}-1\right)}_{\text {first copy of edges }}-\underbrace{\left(n^{2}-1-s\right)}_{\text {second copy of edges }} & \geq n^{2}-1-s-\frac{(s-1) s}{2} \\
s^{2}+s(4 n+1)-6\left(n^{2}-1\right) & \geq 0 \\
s & \geq \sqrt{10 n^{2}+2 n-\frac{23}{4}}-2 n-\frac{1}{2}
\end{aligned}
$$

Which gives $|\mathcal{S}|=s \geq(\sqrt{10}-2) n-\frac{1}{2}$ as required.
In the lower bound above, we consider the edges which appear exactly 0,1 , and 2 times in $\mathcal{S}$ and count these edges in two different ways. It is reasonable to ask if this method can be extended by considering more edges, for example edges which appear 3 times in $\mathcal{S}$.

In the case of $K_{n, n}$, it turns out that that considering these edges does not help. The above method works by counting the minimum number of edges which need to appear exactly 3 times in order for a system of size $s$ to be separating. We then balance this against the amount of 'space' we have available by considering the length of the paths and multiplicity of the edges. When all paths have length $2 n-1$ there is enough available space to have 1 edge appear 0 times, $s$ edges to appear once, $\frac{(s-1) s}{2}$ edges to appear twice, and the remaining edges to all appear three times exactly. Since there is no requirement for any edge to appear 4 times (or more), we do not gain anything from trying to consider these.

To see this, recall that in any $P \in \mathcal{S}$ there is at most one edge unique to $P$ and at most $s-1$ edges which appear in exactly one other path of $\mathcal{S}$. Additionally, there are at most $\frac{(s-1)(s-2)}{2}$ edges of $P$ which can appear in exactly two other paths. Indeed, consider edges $e, e^{\prime} \in P$ which each appear in exactly two other paths, if the pair of extra paths that contains $e$ is the same as the pair of paths containing $e^{\prime}$, then $e$ and $e^{\prime}$ are not separated by $\mathcal{S}$. Therefore the pair of paths from $\mathcal{S} \backslash\{P\}$ which contain $e$ must be distinct from the pair that contain $e^{\prime}$. In other words, the number of edges of $P$ which can appear in two other paths is at most the number of different pairs from $\mathcal{S} \backslash\{P\}$, which is $\frac{(s-1)(s-2)}{2}$. Observe that

$$
1+s-1+\frac{(s-1)(s-2)}{2} \geq 2 n-1
$$

holds if $s \geq n$. Since the maximum length of a path is $2 n-1$, this shows that no path needs to contain any edges which appear 4 or more times in $\mathcal{S}$.

### 9.3 Upper bound

We will use a construction based on rotations of paths in $K_{n, n}$ which have been adapted from paths in $K_{n}$ in the manner discussed earlier in this chapter. This will provide the upper bound for $f\left(K_{n, n}\right)$.

Lemma 9.4. For the complete bipartite graph $K_{n, n}$, there exists a separating path system of size at most $\frac{5 n+5}{4}$. This gives $f\left(K_{n, n}\right) \leq \frac{5 n+5}{4}$.

Proof. Consider the zig-zag paths of $K_{n}$ (see Section 8.6.1) of the form $P=(1,2, n, 3, n-$ $\left.1,4, n-2, \ldots,\left\lceil\frac{n}{2}\right\rceil+1\right)$. Recall that these paths contain exactly one $\frac{n}{2}$-type if $n$ is even, and exactly two of each other edge type. Moreover, each pair of same type edges appear at a set distance from each other, distance $\frac{n}{2}$ in all cases when $n$ is even, and otherwise $\frac{n-1}{2}$ for odd edge types and $\frac{n+1}{2}$ for even.

We consider the path $P_{0}$ on $K_{n, n}$, which is adapted from $P$ using the above method, and where the first $x$-type edge in $P$ is bisected in $P_{0}$ by the $B$ vertex $2 x-1$ and the second $x$-type edge is bisected by $2 x$. We also include in $P_{0}$ an additional final edge between the vertex in $A$ representing the final vertex of $P$, and the vertex $n$ in $B$.

See Figure 9.1 for an example of the path $P_{0}$.
Let $\mathcal{P}=\left\{P_{i}: i \in[n-1]\right\}$ be all $n$ rotations of $P_{0}=\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right), \ldots,\left(a_{n}, b_{n}\right)$, where $P_{i}=\left(a_{1}+i, b_{1}\right),\left(a_{2}+i, b_{2}\right), \ldots,\left(a_{n}+i, b_{n}\right)$ is the rotation of $P_{0}$ by $i$. Note that $\mathcal{P}$ covers every edge of $K_{n, n}$. Indeed, each vertex in $B$ has at least one incident edge in $P_{0}$, the


Figure 9.1: The zig-zag path of $K_{7}$ and the path $P_{0}$ adapted from it in $K_{7,7}$
rotations fix $B$ vertices and cycle through $n$ different endpoints for edges at $B$ vertices, therefore each $B$ vertex has $n$ incident edges covered by $\mathcal{P}$.

Consider two edges $e_{1}$ and $e_{2}$ in $K_{n, n}$, we make the following claim.
Claim : $e_{1}$ and $e_{2}$ are separated by $\mathcal{P}$ unless they originate from two different edges of the same type in $P$, and there is some $P_{i}$ in which both are leading edges.

To see this, let $e_{1} \in P_{i}$ and note that if $e_{2} \notin P_{i}$ then the edges are separated and we are done. Suppose then that $e_{1}, e_{2} \in P_{i}$, and let $e_{1}=\left(a_{i}, b_{j}\right)$ and $e_{2}=\left(a_{r}, b_{s}\right)$. Suppose first that $b_{j}=n$ then $e_{1}$ is the unique edge in $P_{i}$ (i.e. $e_{1} \notin P_{j}$ whenever $j \neq i$ ), since there is only one incident edge at $n$ in $P_{0}$. Note that every other $B$ vertex has two incident edges in $P_{0}$, therefore all edges at these vertices appear in more than one rotation of $P_{0}$, which means there is a $j \in[0, n-1]$ with $j \neq i$ such that $e_{2} \in P_{j}$.

We may now suppose that $b_{j} \neq n$ and similarly $b_{s} \neq n$. Then we must have that $e_{1}$ originated from an $x$-type edge in $K_{n}$, and $e_{2}$ from a $y$-type edge. As we saw earlier, this means that $P_{i-x}$ contains $e_{1}$ if it is a leading edge in $P_{i}$, and $P_{i+x}$ contains $e_{1}$ if it is a trailing edge in $P_{i}$, where $i \pm x$ is calculated modulo $n$. Similarly, $e_{2}$ is found in either $P_{i-y}$ or $P_{i+y}$ depending on whether it is a leading edge in $P_{i}$.

Since there are two incident edges at both $b_{j}$ and $b_{s}$ in $P_{0}$ we know that every edge incident to $b_{j}$ or $b_{s}$ in $K_{n, n}$ appears in exactly two rotations of $P_{0}$. In other words $e_{2}$ cannot be found in $P_{i-x}$ or $P_{i+x}$ unless $x=y$, meaning that $e_{1}$ and $e_{2}$ are separated if they originate from edges of different types. In fact, in order for the pair of edges to not be separated by $\mathcal{P}$ we must have that they are both contained in $P_{i-x}$ or both in $P_{i+x}$, that is, they must both be leading or both be trailing in $P_{i}$. Note that if both are trailing in $P_{i}$ then we have
that both are leading in $P_{i+x}$. Further note that in all rotations of $P_{0}$ the vertices $b_{j}$ and $b_{s}$ have exactly one incident leading edge and one trailing edge, therefore $b_{j} \neq b_{s}$. During the adaptation of $P$ to $P_{0}$ we bisect each $P$ edge with a unique vertex from $B$, therefore any edges in $P_{0}$ (or its rotations) which are incident to different $B$ vertices must originate from different edges. This proves the claim.


Figure 9.2: Two trailing 1-type edges in $P_{0}$ appearing together as leading 1-type edges in $P_{1}$, note that these are the only edges common to both paths

The remaining paths will separate pairs of edges which originate from two different edges of the same type in $P$ which are both leading edges in some $P_{i}$. Let $\mathcal{Q}=\left\{Q_{i}: i \in\left[\left\lceil\frac{n}{4}\right\rceil\right]\right\}$ be the family of paths with

$$
Q_{i}=(2 i-1,1),(2 i, 2),(2 i+1,3), \ldots,(2 i+n-3, n-1),(2 i+n-2, n) .
$$

The family $\mathcal{Q}$ can also be thought of as the first $\left\lceil\frac{n}{4}\right\rceil$ rotations by an even integer of the path $Q_{1}=(1,1),(2,2),(3,3), \ldots,(n-1, n-1),(n, n)$.


Figure 9.3: The path $Q_{1}$

Now consider two edges of $K_{n, n}, e_{1}$ and $e_{2}$, which originate from distinct $x$-type edges of $P$ and are both leading edges in $P_{i}$. Note that, by the construction of $P_{0}$, one must have its $B$ endpoint at $2 x-1$ and the other at $2 x$. Without loss of generality we may assume that $e_{1}=(a, 2 x-1)$, and therefore $e_{2}=\left(a^{\prime}, 2 x\right)$ where

$$
a^{\prime}= \begin{cases}a+\frac{n}{2} & \text { when } n \text { even } \\ a+\frac{n+1}{2} & \text { when } n \text { odd and } x \text { even } \\ a+\frac{n-1}{2} & \text { when } n \text { odd and } x \text { odd }\end{cases}
$$

Recall that leading edges in $K_{n, n}$ are descended from $x$-type edges in $K_{n}$ and correspond to the starting vertex the $x$-type edge. Therefore, the distance in $A$ between leading edges is the same as the clockwise distance on $K_{n}$ between the two $x$-type edges from which the leading edges originate. This distance has nothing to do with the placement of edges within any path, or the order in which vertices appear in a path.

We first claim that if there exists $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$ such that $a$ can be written in the form

$$
a=2 x-1+2 i-2 \quad \text { or } \quad a=2 x-1+2 i-1,
$$

then $e_{1} \in Q_{i}$ and $e_{2} \notin Q_{i}$.
It is easy to see that $e_{1} \in Q_{i}$, by noting that all edges in $Q_{i}$ have the form ( $b+2 i-2, b$ ) or $(b+2 i-1, b)$, since the $B$ vertex of $e_{1}$ is $2 x-1$ the statement follows. To see that $e_{2} \notin Q_{i}$, consider the $B$ edges at $2 x$ in $Q_{i}$, they must be of the form $(2 x+2 i-2,2 x)$ or $(2 x+2 i-1,2 x)$. Hence, if $e_{2} \in Q_{i}$ we must have either

$$
2 x+2 i-2= \begin{cases}a+\frac{n}{2} & \text { when } n \text { even } \\ a+\frac{n+1}{2} & \text { when } n \text { odd and } x \text { even } \\ a+\frac{n-1}{2} & \text { when } n \text { odd and } x \text { odd }\end{cases}
$$

or

$$
2 x+2 i-1= \begin{cases}a+\frac{n}{2} & \text { when } n \text { even } \\ a+\frac{n+1}{2} & \text { when } n \text { odd and } x \text { even } \\ a+\frac{n-1}{2} & \text { when } n \text { odd and } x \text { odd. }\end{cases}
$$

Rearranging gives this

$$
a= \begin{cases}2 x+2 i-2-\frac{n}{2} & \text { when } n \text { even } \\ 2 x+2 i-2-\frac{n+1}{2} & \text { when } n \text { odd and } x \text { even } \\ 2 x+2 i-2-\frac{n-1}{2} & \text { when } n \text { odd and } x \text { odd }\end{cases}
$$

which are clearly all contradictions, or this

$$
a= \begin{cases}2 x+2 i-1-\frac{n}{2} & \text { when } n \text { even } \\ 2 x+2 i-1-\frac{n+1}{2} & \text { when } n \text { odd and } x \text { even } \\ 2 x+2 i-1-\frac{n-1}{2} & \text { when } n \text { odd and } x \text { odd }\end{cases}
$$

which are also contradictory to the known possible values of $a$. Therefore, we know that $e_{2} \notin Q_{i}$ and hence the two edges are separated. It remains to show that the edges are
separated when $a$ cannot be written in this form. Since we have just seen that $e_{1} \in Q_{i}$ implies $e_{2} \notin Q_{i}$, it is sufficient to show that $e_{2} \in Q_{i}$ for some $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$.

Note first that if $a$ cannot be written in the appropriate form we must have that $a \in$ $\left[2 x-1+2\left\lceil\frac{n}{4}\right\rceil, 2 x-1+n-1\right]$. It is clear that $a$ may therefore be written as $a=2 x-1+k$ where $k \in\left[2\left\lceil\frac{n}{4}\right\rceil, n-1\right]$. Note that the value of $2\left\lceil\frac{n}{4}\right\rceil$ varies with the divisibility of $n$, but is always one of the four cases: $\frac{n}{2}, \frac{n}{2}+1, \frac{n-1}{2}+1, \frac{n-1}{2}+2$.

Case 1: $2\left\lceil\frac{n}{4}\right\rceil=\frac{n}{2}$.
This means that $a=2 x-1+k$ for $k \in\left[\frac{n}{2}, \frac{n}{2}+\frac{n}{2}-1\right]$. Also note that $n$ must be even and therefore $a^{\prime}=a+\frac{n}{2}$. This gives

$$
\begin{aligned}
a^{\prime} & =2 x-1+k+\frac{n}{2} \\
& =2 x-1+\frac{n}{2}-1+j+\frac{n}{2} \quad \text { where } j \in\left[1, \frac{n}{2}\right] \\
& =2 x+j-2 .
\end{aligned}
$$

Then there exists an $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$ such that $a^{\prime}$ can be written in the form

$$
a^{\prime}=2 x+2 i-2 \quad \text { or } \quad a=2 x+2 i-1,
$$

whenever $j \in\left[2, \frac{n}{2}\right]$. This means that as long as $a^{\prime} \neq 2 x-1$, we have that $e_{2} \in Q_{i}$. This means that $e_{1}$ and $e_{2}$ are separated by $Q_{i}$ unless $a^{\prime}=2 x-1$. To separate these final edges we consider a single path $Q^{+}=(1,2),(3,4), \ldots,(2 x-1,2 x), \ldots,(n-1, n)$. Note that $Q^{+}$ clearly contains $e_{2}=\left(a^{\prime}, 2 x\right)$ when $a^{\prime}=2 x-1$, and cannot contain $e_{1}$ since there are no edges at odd $B$ vertices in $Q^{+}$.

Then the collection $\mathcal{P} \cup \mathcal{Q} \cup\left\{Q^{+}\right\}$clearly separates all edges of $K_{n, n}$. Moreover, there are

$$
|\mathcal{P}|+|\mathcal{Q}|+1=n+\left\lceil\frac{n}{4}\right\rceil+1=\frac{5 n+4}{4}
$$

paths.
Case 2: $2\left\lceil\frac{n}{4}\right\rceil=\frac{n}{2}+1$.
This means that $a=2 x-1+k$ for $k \in\left[\frac{n}{2}+1, \frac{n}{2}+\frac{n}{2}-1\right]$. Again $n$ must be even and therefore $a^{\prime}=a+\frac{n}{2}$. This gives

$$
\begin{aligned}
a^{\prime} & =2 x-1+k+\frac{n}{2} \\
& =2 x-1+\frac{n}{2}+j+\frac{n}{2} \quad \text { where } j \in\left[1, \frac{n}{2}-1\right] \\
& =2 x+j-1 .
\end{aligned}
$$

Then there exists an $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$ such that $a^{\prime}$ can be written in the form

$$
a^{\prime}=2 x+2 i-2 \quad \text { or } \quad a=2 x+2 i-1,
$$

whenever $j \in\left[1, \frac{n}{2}-1\right]$. Therefore, we have that $e_{2} \in Q_{i}$ and hence the edges $e_{1}$ and $e_{2}$ are separated by $Q_{i}$.

Clearly the collection $\mathcal{P} \cup \mathcal{Q}$ separates all edges of $K_{n, n}$. Observe that this collection has

$$
|\mathcal{P}|+|\mathcal{Q}|=n+\left\lceil\frac{n}{4}\right\rceil=\frac{5 n+2}{4}
$$

paths.
Case 3: $2\left\lceil\frac{n}{4}\right\rceil=\frac{n-1}{2}+1$.
This means that $a=2 x-1+k$ for $k \in\left[\frac{n-1}{2}+1, \frac{n-1}{2}+\frac{n-1}{2}\right]$. This time $n$ must be odd and therefore $a^{\prime}=a+\frac{n+1}{2}$ if $x$ is even and $a^{\prime}=a+\frac{n-1}{2}$ when $x$ is odd. First suppose $x$ is even, then

$$
\begin{aligned}
a^{\prime} & =2 x-1+k+\frac{n+1}{2} \\
& =2 x-1+\frac{n-1}{2}+j+\frac{n+1}{2} \quad \text { where } j \in\left[1, \frac{n-1}{2}\right] \\
& =2 x+j-1 .
\end{aligned}
$$

As in Case 2, this means that there is an $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$ such that $e_{2} \in Q_{i}$ and the pair of edges is separated. Now suppose that $x$ is odd, we have

$$
\begin{aligned}
a^{\prime} & =2 x-1+k+\frac{n-1}{2} \\
& =2 x-1+\frac{n-1}{2}+j+\frac{n-1}{2} \quad \text { where } j \in\left[1, \frac{n-1}{2}\right] \\
& =2 x+j-2 .
\end{aligned}
$$

As in Case 1, there is an $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$ such that $e_{2} \in Q_{i}$ except when $a^{\prime}=2 x-1$. To remedy this we again use the path $Q^{+}$as in Case 1. The collection $\mathcal{P} \cup \mathcal{Q} \cup\left\{Q^{+}\right\}$separates all edges of $K^{n, n}$ with

$$
|\mathcal{P}|+|\mathcal{Q}|+1=n+\left\lceil\frac{n}{4}\right\rceil+1=\frac{5 n+5}{4}
$$

paths.
Case 4: $2\left\lceil\frac{n}{4}\right\rceil=\frac{n-1}{2}+2$.

This means that $a=2 x-1+k$ for $k \in\left[\frac{n-1}{2}+2, \frac{n-1}{2}+\frac{n-1}{2}\right]$. Again $n$ is odd and therefore $a^{\prime}=a+\frac{n+1}{2}$ if $x$ is even and $a^{\prime}=a+\frac{n-1}{2}$ when $x$ is odd. First suppose $x$ is even, then

$$
\begin{aligned}
a^{\prime} & =2 x-1+k+\frac{n+1}{2} \\
& =2 x-1+\frac{n+1}{2}+j+\frac{n+1}{2} \quad \text { where } j \in\left[1, \frac{n-1}{2}-1\right] \\
& =2 x+j .
\end{aligned}
$$

Then there exists an $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$ such that $a^{\prime}$ can be written in the form

$$
a^{\prime}=2 x+2 i-2 \quad \text { or } \quad a=2 x+2 i-1,
$$

whenever $j \in\left[1, \frac{n-1}{2}-1\right]$. This means that $e_{2} \in Q_{i}$ and hence the edges $e_{1}$ and $e_{2}$ are separated by $Q_{i}$. Now suppose $x$ is odd, and observe

$$
\begin{aligned}
a^{\prime} & =2 x-1+k+\frac{n-1}{2} \\
& =2 x-1+\frac{n+1}{2}+j+\frac{n-1}{2} \quad \text { where } j \in\left[1, \frac{n-1}{2}\right] \\
& =2 x+j-1 .
\end{aligned}
$$

As in Case 2, this means there exists an $i \in\left[1,\left\lceil\frac{n}{4}\right\rceil\right]$ such that $e_{2} \in Q_{i}$ and hence the edges $e_{1}$ and $e_{2}$ are separated. In this case we have the collection $\mathcal{P} \cup \mathcal{Q}$ separating all edges with

$$
|\mathcal{P}|+|\mathcal{Q}|=n+\left\lceil\frac{n}{4}\right\rceil=\frac{5 n+3}{4}
$$

paths.

The bound of Lemma 9.4 is best possible for the generator path method using the definition of rotation we have used. Meaning that any separating path system of $K_{n, n}$ constructed by using $n$ rotations of a single path $P$ (any path) must always have at least $\frac{n}{4}$ additional paths which are not rotations of $P$.

Lemma 9.5. Let $P_{0}$ be a path in $K_{n, n}$ and let $\mathcal{S}$ be a separating path system of $K_{n, n}$. If $P_{0}, \ldots, P_{n-1} \in \mathcal{S}$, where $P_{i}$ is the rotation of $P_{0}$ by $i$, then we have that $|\mathcal{S}| \geq \frac{5 n}{4}$.

Proof. Consider any path $P_{0}$ on $K_{n, n}$ and its $n$ rotations $P_{0}, \ldots, P_{n-1}$. We will consider the edges that are separated by $P_{0}, \ldots, P_{n-1}$. Since $P_{0}$ is a path, at most two vertices in $K_{n, n}$ have exactly one incident edge in $P_{0}$, the remaining vertices all have exactly two incident edges or none at all.

Vertices of $B$ with no incident edges in $P_{0}$ will have none of their $n$ incident edges of $K_{n, n}$ covered by $P_{0}, \ldots, P_{n-1}$. If there is any vertex $b \in B$ with only one incident edge in $P_{0}$, then all edges incident to $b$ in $K_{n, n}$ are covered by $P_{0}, \ldots, P_{n-1}$ and each edge will appear in exactly one unique path from the collection. If $b \in B$ has two edges in $P_{0}$, then every edge incident to $b$ in $K_{n, n}$ will appear in exactly two of the paths from $P_{0}, \ldots, P_{n-1}$.

Any vertex of $B$ which has no incident edges in $P_{0}$ contributes $n$ edges which are not separated from each other. Since these edges all share a vertex, it will require at least $\left\lfloor\frac{n}{2}\right\rfloor$ additional paths to cover these for separation. So we can assume that there are no vertices of $B$ which have no incident edges in $P_{0}$.

Suppose the edges $\left(a_{1}, b_{1}\right)$ and $\left(a_{2}, b_{2}\right)$ are in $P_{0}$ and that $b_{1}$ and $b_{2}$ are the endpoints of $P_{0}$. Then the edges $\left(a_{1}, b_{1}\right)$ and $\left(a_{2}, b_{2}\right)$ are both unique to $P_{0}$, they do not appear in any $P_{i}$ where $i \in[1, n-1]$. In particular, the edges $\left(a_{1}, b_{1}\right)$ and $\left(a_{2}, b_{2}\right)$ are not separated from each other. Similarly, the edges $\left(a_{1}+1, b_{1}\right)$ and $\left(a_{2}+1, b_{2}\right)$ are both unique to $P_{1}$ and hence not separated, and in general $\left(a_{1}+i, b_{1}\right)$ and $\left(a_{2}+i, b_{2}\right)$ are unique to $P_{i}$ and not separated.

The minimum number of additional paths required to separate these edges is $\left\lceil\frac{n}{4}\right\rceil$. Indeed, there are $n$ pairs of edges that require separation, each pair has one edge at $b_{1}$ and the other at $b_{2}$. At most two edges at each vertex can be covered with a single path, we require one edge from each pair to be covered by the additional paths in order to separate. So at most four pairs may be separated by a single path by choosing two edges at $b_{1}$ and two edges from different pairs at $b_{2}$.

Finally, suppose that $b_{n} \in B$ is the only vertex of $B$ with a single incident edge in $P_{0}$, and all other vertices in $B$ have two incident edges in $P_{0}$. Then $P_{0}$ must be a maximum length path, and all edges of $K_{n, n}$ are covered by $P_{0}, \ldots, P_{n-1}$. Note that every edge, except those at $b_{n}$, appears in exactly two paths from $P_{0}, \ldots, P_{n-1}$. This means that each edge at $b_{n}$ a unique edge in its path $P_{i}$, and is therefore separated from everything.

Let $P_{0}=\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right), \ldots,\left(a_{n-1}, b_{n-1}\right),\left(a_{n}, b_{n}\right)$, and consider the path on $K_{n}$ given by $P=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$. Just as before, edges are not separated by $P_{0}, \ldots, P_{n-1}$ if and only if they both originate from edges of the same type in $P$ and are both leading in some $P_{i}$. Since $P$ is a maximum length path in $K_{n}$, there must be at least two edges in $P$ that have the same edge type, $x$. Therefore, there are two vertices in $B, b_{i}$ and $b_{j}$, that are only incident to edges originating from $x$-types. This means that for every edge at $b_{i}$, there is an edge at $b_{j}$ which it is not separated from. This is the same situation as in the previous case, that is, there are $n$ pairs of edges that require separation, each pair has one edge at
$b_{i}$ and the other at $b_{j}$. We will require an additional $\left\lceil\frac{n}{4}\right\rceil$ paths to separate all of these pairs by the same argument as above.

## Chapter 10

## Ladders and grids

Recall that $f(G)$ lies between $\log |E(G)|$ and $19 n$ for any graph $G$ on $n$ vertices. Indeed, the upper bound comes from Theorem 7.10 which is the recent result of [6], moreover this is a strongly separating path system. The lower bound of $\log |E(G)|$ comes directly from the trivial bound for the base version of the problem, that is the problem where the ground set and separators are just sets with no additional structure or constraints.

Recall also that the conjectured upper bound for all graphs $G$ on $n$ vertices is $f(G) \leq n$. So far, the graphs we have considered have had $f(G)$ much closer to this top upper bound. It is entirely possible that the complete graph $K_{n}$ is an extremal example, making it the least efficient graph to have a separating path system for. In any case, it is certainly true that any separating path system for $K_{n}$ has the same order of magnitude as the worst case graph.

While these extremes are clearly important areas of study, it is also interesting and worthwhile to investigate the other extreme - graphs which can be efficiently separated with a path system as small as possible. The authors of [10] remark that ladders are such a class of graph, where a ladder can admit a separating path system of size $O(\log |E(G)|)$.

In this chapter, we will discuss these ladder graphs and other graph classifications which admit separating path systems of size close to $\log |E(G)|$, with the aim of extending the known graphs with small $f(G)$.

### 10.1 Ladders

A ladder graph is defined as follows.

Definition 10.1. Let $L_{n}$ be a graph with $2 n$ vertices and $3 n-2$ edges. The vertex set is given by $\left\{1_{a}, 2_{a}, \ldots, n_{a}, 1_{b}, 2_{b}, \ldots, n_{b}\right\}$. The edge set consists of the edges $\left(i_{a}, i_{b}\right)$ for every $i \in[n]$, and for every $i \in[n-1]$ the edges $\left(i_{a},(i+1)_{a}\right)$ and $\left(i_{b},(i+1)_{b}\right)$. We call this the ladder of order $n$.

Note that this is the cartesian product of a path of length $n-1$ and a single edge. Alternatively, it can be thought of as the graph on $2 n$ vertices and $3 n-2$ edges where $n$ edges make a perfect matching, splitting the vertices into two classes $A$ and $B$ each class containing one vertex from every matching edge. The remaining edges form an $n-1$ length path in $A$ and and $n-1$ length path in $B$, where the order of the vertices corresponding to matching edges is the same in both paths.

In [10], the authors point out that it is easy to show that $f\left(L_{n}\right) \leq 3 \log n+1$ by equating a subset of $[n-1]$ to a path in $L_{n}$. Using this method with a little more care we are able to get a slightly improved bound.

Lemma 10.2. We have that $f\left(L_{n}\right) \leq 2(\log n+2)$.

Proof. Let us view the ladder as two paths $A$ and $B$, each of length $n-1$, and a perfect matching as the 'rungs' of the ladder. Then $1_{a}, \ldots, n_{a}$ are the vertices of path $A$ and $1_{b}, \ldots, n_{b}$ are the vertices of $B$. The matching edges are those of the form $\left(i_{a}, i_{b}\right)$ for $i=1, \ldots, n$.

Given any subset $F$ of $[n-1]$, we construct a path $P^{1}(F)$ by taking edges $\left(i_{a},(i+1)_{a}\right)$ for all $i \notin F$ and taking $\left(j_{b},(j+1)_{b}\right)$ for all $j \in F$. We also take any edge $\left(i_{a}, i_{b}\right)$ if both $i-1 \notin F$ and $i \in F$, similarly we take $\left(i_{a}, i_{b}\right)$ if both $i-1 \in F$ and $i \notin F$. This does indeed make a path in $L_{n}$, see Figure 10.1 for an example of this.


Figure 10.1: Example on $L_{8}$ with the path $P^{1}(F)$ where $F=\{2,3,5,7\}$

Let $\mathcal{F}$ be a separating system for $[n-1]$ of size $|\mathcal{F}|=\lceil\log (n-1)\rceil$. We know such a system exists as this is the trivial set version of the separating problem. Let $\mathcal{P}^{1}(\mathcal{F})=$ $\left\{P^{1}(F): F \in \mathcal{F}\right\}$ be the family of paths given by constructing paths as described above for each set in $\mathcal{F}$.

It is plain that the edges of $A$ are separated from each other by $\mathcal{P}^{1}(\mathcal{F})$. Indeed, consider edges $\left(i_{a},(i+1)_{a}\right)$ and $\left(j_{a},(j+1)_{a}\right)$ where $i, j \in[n-1]$. Since $\mathcal{F}$ separated all pairs from $[n-1]$, we have that there is some $F \in \mathcal{F}$ where, without loss of generality, $i \in F$ and $j \notin F$. Then $P^{1}(F) \in \mathcal{P}^{1}(\mathcal{F})$, and by the construction of $P^{1}(F)$ we have that $\left(i_{a},(i+1)_{a}\right) \notin P^{1}(F)$ and $\left(j_{a},(j+1)_{a}\right) \in P^{1}(F)$. Similarly, the edges of $B$ are separated from each other by $\mathcal{P}^{1}(\mathcal{F})$.

We can use a similar method to equate subsets of $[n]$ to paths in $L_{n}$. Let $F^{\prime} \subseteq[n]$, we construct the path $P^{2}\left(F^{\prime}\right)$ by taking edges $\left(i_{a}, i_{b}\right)$ for all $i \in F^{\prime}$, we make it a path by using any required edges from $A$ and $B$. See Figure 10.2 for an example.


Figure 10.2: Example on $L_{8}$ with the path $P^{2}\left(F^{\prime}\right)$ where $F^{\prime}=\{2,3,5,7\}$

Let $\mathcal{F}^{\prime}$ be a separating system for $[n]$ of size $|\mathcal{F}|=\lceil\log n\rceil$. Let $\mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right)=\left\{P^{2}\left(F^{\prime}\right): F^{\prime} \in\right.$ $\left.\mathcal{F}^{\prime}\right\}$ be the family of paths given by constructing paths as described above for each set in $\mathcal{F}^{\prime}$.

Then $\mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right)$ separates all matching edges from each other. Indeed, consider $\left(i_{a}, i_{b}\right)$ and $\left(j_{a}, j_{b}\right)$ and note that there is some $F^{\prime} \in \mathcal{F}^{\prime}$ such that (without loss of generality) $i \in F^{\prime}$ and $j \notin F^{\prime}$. Then clearly we have that $\left(i_{a}, i_{b}\right) \in P^{2}\left(F^{\prime}\right)$ and $\left(j_{a}, j_{b}\right) \notin P^{2}\left(F^{\prime}\right)$ and $P^{2}\left(F^{\prime}\right) \in \mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right)$.

It is left only to separate edges in $A$ from those in $B$ and the matching, as well as separate edges in $B$ from the matching edges. This can easily be done with the two paths $A$ and $B$. Since any two edges $e \in A$ and $e^{\prime} \notin A$ are clearly separated by $A$, and similarly any two edges $e \in B$ and $e^{\prime} \notin B$ are separated by $B$.

Hence, the total number of paths used to separate the ladder $L_{n}$ is at most

$$
\left|\mathcal{P}^{1}(\mathcal{F})\right|+\left|\mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right)\right|+2=\lceil\log (n-1)\rceil+\lceil\log n\rceil+2 .
$$

This gives the result.

Given that a Ladder has $3 n-2$ edges, the set theoretic lower bound for $f\left(L_{n}\right)$ is $\lceil\log (3 n-$ $2)\rceil=\log n+c$, for some small constant $c$. Comparing this to the upper bound obtained in Lemma 10.2, we see that there is only a factor of 2 difference. There is a certain amount of wastage in the method of 10.2 . By considering the matching edges and $A / B$ edges
separately, we do not gain anything from the matching edges which appear in $\mathcal{P}^{1}(\mathcal{F})$ nor the $A / B$ edges which appear in $\mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right)$. It is therefore reasonable to expect the true value of $f\left(L_{n}\right)$ to be less than $2 \log n$, although a different construction method would be required.

We remark that, using the above construction, we may assume that a separating path system for $L_{n}$ covers all edges of the graph. Recall that there is an allowance in the definition of a separating path system for a single edge of the graph to not be contained in any path in the system. For these ladders and the method above, note that the uncovered edge must be one of the matching edges since the paths $A$ and $B$ appear in the system. Further note that we can assume from a relabelling of the edges, that the uncovered edge is $\left(1_{a}, 1_{b}\right)$ inherited from the fact that the element 1 does not appear in any sets of $\mathcal{F}^{\prime}$. Then we can always extend any single path $P^{1}(F) \in \mathcal{P}^{1}(\mathcal{F})$ with the edge $\left(1_{a}, 1_{b}\right)$ without compromising the separation or the fact that $P^{1}(F)$ is a path.

### 10.2 Ladder-Tubes

We have seen that ladders are approximately optimal graphs in terms of size of a separating path system. We extend the idea behind ladders to something we call a ladder-tube, which attempts to preserve the structure and ability to equate sets to paths, but increases the proportion of edges to vertices.

Definition 10.3. Let $L_{n}^{2}$ be a graph with $4 n$ vertices and $8 n-4$ edges. The vertex set is $\left\{1_{a}, \ldots, n_{a}, 1_{b}, \ldots, n_{b}, 1_{c}, \ldots, n_{c}, 1_{d}, \ldots, n_{d}\right\}$. We define $4 n$ of the edges by $\left(i_{a}, i_{b}\right),\left(i_{b}, i_{c}\right)$, $\left(i_{c}, i_{d}\right)$ and $\left(i_{d}, i_{a}\right)$ for each $i \in[n]$. The remaining edges are of the form $\left(i_{a},(i+1)_{a}\right)$, $\left(i_{b},(i+1)_{b}\right),\left(i_{c},(i+1)_{c}\right),\left(i_{d},(i+1)_{d}\right)$ for all $i \in[n-1]$. We call this the ladder tube of order $n$.

See Figure 10.3 for an example of this graph.


Figure 10.3: Example of the graph $L_{8}^{2}$

When it comes to determining a bound for $f\left(L_{n}^{2}\right)$ we first note that $L_{n}^{2}$ can be thought of
as two copies of $L_{n}$ joined by a perfect matching, one copy between $A$ and $B$ the other between $C$ and $D$. We know from Lemma 10.2 that we can cover and separate edges in ladders with $2(\log n+2)$ paths. If we consider the perfect matching to be split into two parts, one between $A$ and $C$, the other between $B$ and $D$, then we can use $\log n$ paths of the form $P^{2}\left(F^{\prime}\right)$ to separate edges in the $A C$ matching and similarly in the $B D$ matching. This gives us

$$
4(\log n+2)+2 \log n=6 \log n+8
$$

But we can do better than this.

Lemma 10.4. We have that $f\left(L_{n}^{2}\right) \leq 4 \log _{2}(n)+10$.

Proof. Consider the family of paths, $\mathcal{S}$, made up of the following. The paths $A, B, C$ and $D$ respectively, along with two paths $R_{a}$ and $R_{d}$ where

$$
R_{a}=\left\{\left(1_{a}, 1_{b}\right),\left(1_{b}, 1_{c}\right),\left(1_{c}, 2_{c}\right),\left(2_{c}, 2_{b}\right),\left(2_{b}, 2_{a}\right),\left(2_{a}, 3_{a}\right),\left(3_{a}, 3_{b}\right), \ldots,\left(n_{a}, n_{b}\right),\left(n_{b}, n_{c}\right)\right\}
$$

and

$$
R_{d}=\left\{\left(1_{d}, 1_{c}\right),\left(1_{c}, 1_{b}\right),\left(1_{b}, 2_{b}\right),\left(2_{b}, 2_{c}\right),\left(2_{c}, 2_{d}\right),\left(2_{d}, 3_{d}\right),\left(3_{d}, 3_{c}\right), \ldots,\left(n_{d}, n_{c}\right),\left(n_{c}, n_{b}\right)\right\}
$$



Figure 10.4: An example of $L_{8}^{2}$ and the paths $R_{a}$ in blue and $R_{d}$ in red

Given any subset $F$ of $[n-1]$, let $P_{A B}^{1}(F)$ be the path containing edges $\left(i_{a},(i+1)_{a}\right)$ for all $i \notin F$ and $\left(j_{b},(j+1)_{b}\right)$ for all $j \in F$, as well as edges $\left(i_{a}, i_{b}\right)$ if both $i-1 \notin F$ and $i \in F$ and edges $\left(i_{a}, i_{b}\right)$ if both $i-1 \in F$ and $i \notin F$. Such paths are equivalent to those in Lemma 10.2 and an example can be seen in Figure 10.1.

Define paths $P_{C D}^{1}(F)$ equivalently, and let $\mathcal{P}_{A B}^{1}(\mathcal{F})=\left\{P_{A B}^{1}(F): F \in \mathcal{F}\right\}$ and equivalently for $\mathcal{P}_{C D}^{1}(\mathcal{F})$ where $\mathcal{F}$ is a separating system for $[n-1]$ of size $|\mathcal{F}|=\lceil\log (n-1)\rceil$.

We also want $\mathcal{S}$ to contain the following $\lceil\log n\rceil$ paths. Let $\mathcal{F}^{\prime}$ be a separating system for $[n]$ of size $\lceil\log n\rceil$. For each $F^{\prime} \in \mathcal{F}^{\prime}$ define paths $P^{2}\left(F^{\prime}\right)$ to contain the edges $\left(i_{a}, i_{b}\right)$, $\left(i_{b}, i_{c}\right),\left(i_{c}, i_{d}\right)$ for each $i \in F^{\prime}$ along with any required edges from $A$ and $D$. Call this collection of paths $\mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right)$.


Figure 10.5: An example of $L_{8}^{2}$ and the path $P^{2}\left(F^{\prime}\right)$ where $F^{\prime}=\{2,4,5\}$

Finally, we want $\mathcal{S}$ to further contain the paths $\mathcal{P}_{A D}^{2}\left(\mathcal{F}^{\prime}\right)=\left\{P_{A D}^{2}\left(F^{\prime}\right): F^{\prime} \in \mathcal{F}^{\prime}\right\}$ where $P_{A D}^{2}\left(F^{\prime}\right)$ contains edges $\left(i_{d}, i_{a}\right)$ for every $i \in F^{\prime}$ along with any required edges of $A$ and $D$.

Then we can define

$$
\mathcal{S}=\left\{A, B, C, D, R_{a}, R_{d}\right\} \cup \mathcal{P}_{A B}^{1}(\mathcal{F}) \cup \mathcal{P}_{C D}^{1}(\mathcal{F}) \cup \mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right) \cup \mathcal{P}_{A D}^{2}\left(\mathcal{F}^{\prime}\right)
$$

In total the number of paths we have is

$$
|\mathcal{S}|=6+2\lceil\log (n-1)\rceil+2\lceil\log n\rceil \leq 4 \log _{2}(n)+10
$$

It remains to check that these paths do in fact separate the edges of $L_{n}^{2}$. Take any two edges $e$ and $e^{\prime}$. Let $X \in\{A, B, C, D\}$, without loss of generality we have three cases: $e \in X$ and $e^{\prime} \notin X ; e, e^{\prime} \in X ; e, e^{\prime} \notin A, B, C, D$.

First consider $e \in X$ and $e^{\prime} \notin X$. Clearly the path $X \in \mathcal{S}$ separates $e$ and $e^{\prime}$.
Next assume $e, e^{\prime} \in X$ and first consider $X=A$, then we can write $e=\left(i_{a},(i+1)_{a}\right)$ and $e^{\prime}=\left(j_{a},(j+1)_{a}\right)$. Since $\mathcal{F}$ is a separating system for $[n-1]$, there is some $F \in \mathcal{F}$ with $i \notin F$ and $j \in F$. Then $P_{A B}^{1}(F) \in \mathcal{S}$ is such that $e \in P$ and $e^{\prime} \notin P$. The case is identical when $X=B$, and using $P_{C D}^{1}(F)$ instead when $X=C, D$.

Finally then, assume $e, e^{\prime} \notin A, B, C, D$. Then we may write $e=\left(i_{k}, i_{\ell}\right)$ and $e^{\prime}=\left(j_{r}, j_{s}\right)$ where $k, \ell, r, s \in\{a, b, c, d\}$. First note that if $i \neq j$ then the edges are separated by $\mathcal{P}^{2}\left(\mathcal{F}^{\prime}\right) \cup \mathcal{P}_{A D}^{2}\left(\mathcal{F}^{\prime}\right)$. Indeed, since $\mathcal{F}^{\prime}$ is a separating system for $[n]$, there is some $F^{\prime} \in \mathcal{F}^{\prime}$ such that $i \in F^{\prime}$ and $j \notin F^{\prime}$. If $\{k, \ell\}=\{a, d\}$ then $e \in P_{A D}^{2}\left(F^{\prime}\right)$ and $e^{\prime} \notin P_{A D}^{2}\left(F^{\prime}\right)$, otherwise $e \in P^{2}\left(F^{\prime}\right)$ and $e^{\prime} \notin P^{2}\left(F^{\prime}\right)$.

Assume then that $e=\left(i_{k}, i_{\ell}\right)$ and $e^{\prime}=\left(i_{r}, i_{s}\right)$, in particular this means that $\{k, \ell\} \neq\{r, s\}$ since $e$ and $e^{\prime}$ are distinct. If $\{k, \ell\}=\{a, d\}$ then $e \notin R_{a} \cup R_{d}$, but clearly $e^{\prime} \in R_{a} \cup R_{d}$. We may suppose that $\{k, \ell\}=\{a, b\}$ without loss of generality since the case is equivalent when $\{k, \ell\}=\{c, d\}$, and one of $\{k, \ell\}$ and $\{r, s\}$ must be equal to $\{a, b\}$ or $\{c, d\}$. So, $e=\left(i_{a}, i_{b}\right)$ therefore $e \in R_{a}$. If $e^{\prime} \in R_{a}$ then we must have that $e^{\prime}=\left(i_{b}, i_{c}\right)$ and hence $e^{\prime} \in R_{d}$, but clearly $e \notin R_{d}$.

### 10.3 Grids

The final class of graph we will consider is the grid.

Definition 10.5. Let $L(m, n)$ be the graph with a vertex at each coordinate $(i, j)$ where $i \in[n]$ and $j \in[m]$. There is an edge between vertices $(i, j)$ and $(r, s)$ if either $|i-r|=1$ or $|j-s|=1$, but not both. We call $L(m, n)$ the $\boldsymbol{m} \times \boldsymbol{n}$ grid.

Let $L(m, n)$ be the grid, we denote the edge between vertices $(i, j)$ and $(i+1, j)$ by $(i, j: \rightarrow)$ and the edge between $(i, j)$ and $(i, j+1)$ by $(i, j: \uparrow)$.

Define for each $i \in[n]$ the path $N_{i}=\{(i, 1: \uparrow),(i, 2: \uparrow), \ldots,(i, m-1: \uparrow)\}$, which is the vertical path containing every edge in vertex column $i$. Similarly, we define the path $M_{j}=\{(1, j: \rightarrow),(2, j: \rightarrow), \ldots,(n-1, j: \rightarrow)\}$ for each $j \in[m]$ to be the horizontal path consisting of all the edges in vertex row $j$.

The aim will again be to define a system for equating subsets of $[n]$ and $[m]$ to paths on $L(m, n)$.

Lemma 10.6. We have that $f(L(m, n)) \leq 3 \log (n m)+7$.

Proof. First consider the case where $n$ and $m$ are both even.

We are able to equate a subset of $\left[\frac{m}{2}(n-1)\right]$ with a path in $L(m, n)$ in the following way. Let $F \subseteq\left[\frac{m}{2}(n-1)\right]$, and observe that each element of $F$ can be written in the form $x(n-1)+y$ where $y \in[n-1]$ and $x \in\left[0, \frac{m}{2}\right]$. We will define a path $P(F)$, where the elements of $\left[\frac{m}{2}(n-1)\right]$ are represented by squares on the grid between an odd bottom row and an even top row, then inclusion in $F$ will be represented by $P(F)$ travelling along the even row over the corresponding section, and exclusion represented by travelling the odd route. Various vertical edges will be used to connect these sections.

To see an example of such a path, see Figure 10.6 which shows that for each $i \in F, P(F)$ travels along the even top row of the square corresponding to $i$, and for $j \notin F$ the path travels below on the odd row. The only vertical edges used in $P(F)$ are the necessary links between the odd and even rows (e.g. where $i \notin F$ and $i+1 \in F$ such as 8 and 9 ), and the edges in $N_{1}$ and $N_{n}$ linking the next pair of odd and even rows.


Figure 10.6: Example of the grid $L(8,8)$ and the path $P(F)$ where $F \subseteq[28]$ given by $F=\{6,9,10,11,14,15,16,18,20,23,25,27,28\}$.

Formally, let $P(F)$ be a path containing the following edges for each $x(n-1)+y \notin F$

$$
\begin{cases}(y, 2 x+1: \rightarrow) & \text { when } i-1 \in F \text { and } y>1 \\ (y, 2 x+1: \uparrow) & \text { when } y=n-1 \text { and } x \text { even } \\ (n, 2 x+1: \uparrow),(n, 2 x+2: \uparrow) & \text { when } y=n-1 \text { and } x \text { even and } i+n-1 \in F \\ (n, 2 x+3: \uparrow) & \text { when } y=1 \text { and } x \text { odd } \\ (1,2 x+1: \uparrow),(1,2 x+2: \uparrow) & \text { when } y=1 \text { and } x \text { odd and } i+n-1 \in F . \\ (1,2 x+3: \uparrow) & \end{cases}
$$

And for each $x(n-1)+y \in F$, the edges

$$
\begin{cases}(y, 2 x+2: \rightarrow) & \\ (y, 2 x+1: \uparrow) & \text { when } i-1 \notin F \text { and } y>1 \\ (n, 2 x+2: \uparrow) & \text { when } y=n-1 \text { and } x \text { even } \\ (n, 2 x+3: \uparrow) & \text { when } y=n-1 \text { and } x \text { even and } i+n-1 \in F \\ (1,2 x+2: \uparrow) & \text { when } y=1 \text { and } x \text { odd } \\ (1,2 x+3: \uparrow) & \text { when } y=1 \text { and } x \text { odd and } i+n-1 \in F\end{cases}
$$

Let $\mathcal{F}$ be a separating system for $\left[\frac{m}{2}(n-1)\right]$, and consider $\mathcal{P}=\{P(F): F \in \mathcal{F}\}$. Notice that any pair of edges of the form $e=(a, b: \rightarrow)$ and $e^{\prime}=(c, d: \rightarrow)$, are separated by
$\mathcal{P}$ as long as $b$ and $d$ have the same parity. Indeed, suppose $b$ and $d$ are both odd and $b \leq d$, and consider integers $i=\frac{b-1}{2}(n-1)+a$ and $j=\frac{d-1}{2}(n-1)+c$, both of which belong to $\left[\frac{m}{2}(n-1)\right]$ and are distinct if and only if $e$ and $e^{\prime}$ are distinct. We have that there is some $F \in \mathcal{F}$ which separates $i$ and $j$, assume that $i \in F$. Then observe that the path $P(F)$ contains $e$ but not $e^{\prime}$, and therefore separates them. An equivalent argument setting $i=\frac{b-2}{2}(n-1)+a$ and $j=\frac{d-2}{2}(n-1)+c$ shows separation when both $b$ and $d$ are even.

A single additional path $R=M_{2} \cup M_{4} \cup \cdots M_{n} \cup\{(n, 2: \uparrow),(n, 3: \uparrow),(1,4: \uparrow),(1,5: \uparrow), \ldots\}$, separates $e$ and $e^{\prime}$ when $b$ and $d$ have differing parity. So the family of paths $\mathcal{P} \cup\{R\}$ separates all horizontal edges from each other.

To separate all vertical edges from each other we can apply an analogous argument to get a path family $\mathcal{Q}$ from a separating system for $\left[\frac{n}{2}(m-1)\right]$. Let $\mathcal{F}^{\prime}$ be a separating system for $\left[\frac{n}{2}(m-1)\right]$, and paths $Q\left(F^{\prime}\right)$ for each $F^{\prime} \in \mathcal{F}^{\prime}$ defined as for $P(F)$ with the coordinates inverted. Just as in the horizontal case we will require an additional path $R^{\prime}$ made up of the paths $N_{i}$ for even $i$ and connecting edges in the top and bottom row.

The family $\mathcal{P} \cup \mathcal{Q} \cup\left\{R, R^{\prime}\right\}$ separates all horizontal edges from each other, and all vertical edges from each other and has size $|\mathcal{F}|+\left|\mathcal{F}^{\prime}\right|+2$. It is left to separate horizontal edges from vertical edges. Note that $R$ contains half of all the horizontal edges, and only contains vertical edges from $N_{1}$ and $N_{n}$. Let $O=M_{1} \cup M_{3} \cup \cdots M_{n-1} \cup\{(n, 1: \uparrow),(n, 2: \uparrow),(1,3: \uparrow$ $),(1,4: \uparrow), \ldots\}$ be a path that covers all remaining horizontal edges and only contains vertical edges from $N_{1}$ and $N_{n}$. Then note that all vertical edges are separated from horizontal edges by the paths $R$ and $O$, unless the vertical edges lie in $N_{1}$ and $N_{n}$. An easy way to separate these edges from all horizontal ones is by using the two paths $N_{1}$ and $N_{n}$.

In other words the family $\mathcal{P} \cup \mathcal{Q} \cup\left\{R, R^{\prime}, O, N_{1}, N_{n}\right\}$ separates all edges of $L(m, n)$. The size of this family is

$$
|\mathcal{F}|+\left|\mathcal{F}^{\prime}\right|+5=\left\lceil\log \left(\frac{m(n-1)}{2}\right)\right\rceil+\left\lceil\log \left(\frac{n(m-1)}{2}\right)\right\rceil+5 \leq 2 \log (n m)+5
$$

Suppose now that $m$ is odd. This causes a problem when trying to define the paths $P(F)$, for every odd row of the grid the horizontal edges represent elements that are not included in $F$ and the even rows represent those that are included in $F$. This means each odd row is paired with an even row with one row indicating 'out' and the other indicating 'in'. When $n$ is odd, we have an additional row with no partnering row to indicate 'in'.

In order to use the same technique as in the even case, we construct $\mathcal{Q}$ and $R^{\prime}$ as in the even case, and produce $\mathcal{P}$ and $R$ for the sub-grid $L(m-1, n)$. Note that $\mathcal{Q}$ and $R^{\prime}$ separate all vertical edges of $L(m, n)$ from each other, just as they did before. The majority of horizontal edges are separated from each other by $\mathcal{P}$ and $R$, just the edges of $M_{m}$ need to be separated from the other edges of $M_{m}$. This can be done by taking paths of the form $\mathcal{P}^{1}(\mathcal{H})$ from Lemma 10.2 , considering the ladder with $A=M_{m-1}$ and $B=M_{m}$, and where $\mathcal{H}$ is a small separating system for $[n-1]$. This is $|\mathcal{H}|=\lceil\log (n-1)\rceil$ additional paths.

With $\mathcal{Q}, \mathcal{P}, \mathcal{P}^{1}(\mathcal{H})$, and $\left\{R, R^{\prime}\right\}$ all horizontal edges are separated from other horizontal edges, and vertical edges are separated from other verticals. We separate horizontal from vertical edges in an analogous way to the even case. We define the path $O$ to contain all edges from $M_{i}$ for odd $i$ (including $i=m$ ), this ensures that all horizontal edges are covered by $R$ and $O$, which only contain vertical edges from $N_{1}$ and $N_{n}$. Then the family $\mathcal{Q} \cup \mathcal{P} \cup \mathcal{P}^{1}(\mathcal{H}) \cup\left\{R, R^{\prime}, O, N_{1}, N_{n}\right\}$ separate all edges of $L(m, n)$. The size of this family is

$$
\left|\mathcal{F}^{\prime}\right|+|\mathcal{F}|+|\mathcal{H}|+5 \leq 2 \log (n m)+\log n+6 .
$$

It is plain that we can we can achieve an equivalent family when $m$ is even and $n$ is odd, with this case giving us a family of size at most $2 \log (n m)+\log m+6$.

And of course we can apply the same adaptation to the case where both $n$ and $m$ are odd, giving us a separating path family of size at most

$$
2 \log (n m)+\log n+\log m+7=3 \log (n m)+7 .
$$

We remark that if we consider the torus, $T$, formed from $L(m, n)$ by equating $N_{1}$ with $N_{n}$ and $M_{1}$ with $M_{m}$, then the method of Lemma 10.6 may be used to provide a separating path system for $T$ with size at most $3 \log (n m)+7$. That is, the problem does not change significantly when considering the torus

## Chapter 11

## Summary and open problems

We have investigated the value of $f(G)$ in the cases where $G$ is the complete graph and the complete bipartite graph. Both of these are interesting classifications of graph to consider in their own right, but also they are good candidates for extremal graphs which require the largest separating path system of any graph on $n$ vertices. The symmetry of these graphs also allow us to consider the use of generator paths for both, and we are able to compare this technique in the two cases.

We also considered the value of $f(G)$ for graphs which admit a very small separating path system. The ladders, ladder-tubes, and grids all allow for very efficient separating path systems due to the ability to equate unstructured subsets to the edges of $G$. These graphs are interesting because they admit such small separating path systems.

There are still many open problems within this topic, we finish by discussing these areas of further study.

First recall Conjectures 7.5 and 7.7, which assert that there is some universal constant $C$ such that any graph on $n$ vertices has a (weakly or strongly) separating path system of size $C n$. These conjectures were proved by Bonamy, Botler, Dross, Naia, and Skokan in Theorem 7.10 with a constant of $C=19$. However, in their paper [6] the authors state that the value of 19 is likely far from best possible. In fact, the authors of [10] suggest that (in the weak setting) a constant as low as $C=1+o(1)$ may be enough. There are certainly no counterexamples known that show otherwise.

Question 11.1. Is it true that for every graph $G$ on $n$ vertices $f(G) \leq(1+o(1)) n$ ?
Observe that, to answer this question in the negative, it suffices to find a single graph $G$
on $k$ vertices with $f(G)=k+1$. Then simply taking $\frac{n}{k}$ disjoint copies of $G$ provides a graph on $n$ vertices which requires at least $\left(1+\frac{1}{k}\right) n$ paths in any separating path system. Indeed, since no connected path can cover edges in more than one copy of $G$, we must be able to partition the family into $\frac{n}{k}$ separating path systems each for a distinct copy of $G$.

We note that, if Question 11.1 is true, the complete graph $K_{n}$ is certainly an extremal example since $f\left(K_{n}\right) \geq n-1$ (Lemma 8.1). We have also seen that $f^{\prime}\left(K_{n}\right)=n$ for some values of $n$ and it is likely that $f^{\prime}\left(K_{n}\right)$ may be close to $n$ in general. To understand both of the problems better, it would be useful to pin down the exact values of $f\left(K_{n}\right)$ and $f^{\prime}\left(K_{n}\right)$.

Question 11.2. Is it true that $f\left(K_{n}\right) \leq n$ for all values of $n$ ? Could it be the case that $f^{\prime}\left(K_{n}\right) \leq n$ for all $n$ ?

Finding generator paths and strong generator paths is one way of answering this. We know from Theorem 8.4 that existence of a generator path for $n$ gives us $f\left(K_{n}\right) \leq n$, and also that existence of a strong generator path for $n$ gives $f^{\prime}\left(K_{n}\right) \leq n$.

Question 11.3. For which values of $n$ do generator paths exist? For which values of $n$ do strong generator paths exist?

We have shown in Theorem 8.6 that generator paths exist whenever $n$ is prime. This means that we know that $f\left(K_{n}\right) \leq n$ for infinitely many values of $n$. It would be very interesting to have an analogous result in the case of strong separation.

Question 11.4. Do we have $f^{\prime}\left(K_{n}\right) \leq n$ for infinitely many $n$ ?
Using generator paths and rotations in general naturally give an upper bound of $n$ rather than $n-1$ to match the lower bound of Lemma 8.1. There are constructions of separating path systems of $K_{n}$ for small values of $n$ which have $n-1$ paths, but a different approach is needed to have a general construction of this size.

Question 11.5. Is there a different approach which gives $f\left(K_{n}\right) \leq n-1$ for many values of $n$ (e.g. even/odd $n$, or sufficiently large $n$ )?

For bipartite graphs, we saw that $1.16 n-\frac{1}{2} \leq f\left(K_{n, n}\right) \leq \frac{5 n+5}{4}$ in Theorem 9.1. We also saw that an upper bound of $\frac{5 n}{4}$ is best possible for the method of construction we used
(Lemma 9.5). Interestingly, we get a similar result when we change the automorphism and define rotation of path $Q$ by $r$ as $Q_{r}$ where $(a+r, b+r) \in Q_{r}$ if and only if $(a, b) \in Q$. That is, a family containing all $n$ rotations (as defined above) of a path $Q$ which is a separating path system for $K_{n, n}$ must have size at least $\frac{5 n}{4}$. We omit the details as they are similar to the proof of Lemma 9.5. This points to the upper bound of Theorem 9.1 being closer to the truth, opening up the following questions.

Question 11.6. How can we improve the lower bound from $f\left(K_{n, n}\right) \geq(\sqrt{10}-2) n-\frac{1}{2}$ ?

Question 11.7. Is it the case that $f\left(K_{n, n}\right) \geq \frac{5 n}{4}$ ?
In the opposite direction, since using a different automorphism provided a similar bound it is reasonable to think that any serious improvements to the upper bound must come from a method which does not use rotations at all.

Question 11.8. Are there non rotation based methods that match or improve the upper bound of $f\left(K_{n, n}\right) \leq \frac{5 n+5}{4}$ ?

Finally, we saw that there is a graph which admits a separating path system with size within a factor of 2 of the optimal set theoretic bound of $f(G) \geq \log (|E(G)|)$ - the Ladder $L_{n}$. But is there a different graph for which the exact optimal bound is achieved.

Question 11.9. Is there a family of graphs such that $f(G)=(1+o(1)) \log (|E(G)|)$ for every $G$ in the family?

Of course it is still possible that $G=L_{n}$, as improving the lower bound of $f\left(L_{n}\right)$ seems difficult. In general we do not have many good ways of finding lower bounds for path separation problems, only the set theoretic bound or simple counting arguments. In the case of ladders, the counting arguments do not give good bounds at all since the number of edges in the graph is small compared to the length of the longest path. In this case then, there might be a better set-theoretical method to give a lower bound. The upper bound given in Lemma 10.2 also has room for improvement, by considering the horizontal and vertical edges separately we end up needing to use two different styles of separating system which gives us the factor of 2 in the result. An improved method may be able to make use of the edges which are wasted by considering them separately.

Question 11.10. What is the exact value of $f\left(L_{n}\right)$ ?

## References

[1] N. Alon. Explicit construction of exponential sized families of $k$-independent sets. Discrete Mathematics 58.2 (1986): 191-193.
[2] F. Arrepol, P. Asenjo, R. Astete, V. Cartes, A. Gajardo, V. Henríquez, C. Opazo, N. Sanhueza-Matamala, C. Thraves Caro. Separating path systems in trees. arXiv:2306.00843 [math.CO] (2023+).
[3] J. Balogh, B. Csaba, R. Martin, and A. Pluhár. On the path separation number of graphs. Discrete Applied Mathematics, 213 (2016): 26-33.
[4] B. Bollobás, A. Scott. Separating systems and oriented graphs of diameter two. Journal of Combinatorial Theory, Series B 97.2 (2007): 193-203.
[5] M. Bóna. Combinatorics of Permutations. Discrete mathematics and its applications 3rd edition (2022). Chapman and Hall/CRC.
[6] M. Bonamy, F. Botler, F. Dross, T. Naia, J. Skokan. Separating the edges of a graph by a linear number of paths. arXiv:2301.08707 [math.CO] (2023+).
[7] M. Cai. On separating systems of graphs. Discrete Mathematics 49 (1984): 15-20.
[8] F. Chung, R. Graham, and P. Winkler. On the Addressing Problem for Directed Graphs. Graphs and Combinatorics 1 (1985): 41-50.
[9] P. Erdős, and G. Szekeres. A combinatorial problem in geometry. Compositio Mathematica 2 (1935): 463-470.
[10] V. Falgas-Ravry, T. Kittipassorn, D. Korándi, S. Letzter, and B. Narayanan. Separating path systems. Journal of Combinatorics, 5.3 (2014): 335-354.
[11] F. Foucaud, M. Kovše. Identifying path covers in graphs. Journal of Discrete Algorithms 23 (2013): 21-34.
[12] Z. Füredi. Scrambling permutations and entropy of hypergraphs. Random Structures and Algorithms 8.2 (1996): 97-104.
[13] G. O. H. Katona. On separating systems of a finite set. Journal of Combinatorial Theory, Series A 1 (1966): 174-194.
[14] D. Kleitman, and J. Spencer. Families of $k$-independent sets. Discrete Mathematics 6.3 (1973): 255-262.
[15] D. Kuhn, J. Higdon, J. Lawrence, R. Kacker, and Y. Lei. Combinatorial methods for event sequence testing. IEEE Fifth International Conference on Software Testing, Verification and Validation (2012): 601-609.
[16] A. Kündgen, D. Mubayi, and P. Tetali. Minimal completely separating systems of $k$-sets. Journal of Combinatorial Theory, Series A 93 (2001): 192-198.
[17] S. Letzter. Separating paths systems of almost linear size. arXiv:2211.07732 [math.CO] (2022+).
[18] V. Levenshtein. Perfect codes in the metric of deletions and insertions. Diskretnaya Matematika 3.1 (1991): 3-20. (English Translation: Discrete Mathematics and Applications 2.3 (1992): 241-258.)
[19] R. Mathon, T. Van Trung. Directed $t$-Packings and Directed $t$-Steiner Systems. Designs, Codes and Cryptography 18 (1999): 187-198.
[20] B. D. McKay, and T. Peters. Paths through equally spaced points on a circle. Preprint 2022. arXiv:2205.06004 [math.CO]
[21] J. Radhakrishnan. A note on scrambling permutations. Random Structures and Algorithms 22.4 (2003): 435-439.
[22] C. Ramsay and I. T. Roberts. Minimal completely separating systems of sets. Australasian Journal of Combinatorics 13 (1996): 129-150.
[23] R. Raz. VC-Dimension of Sets of Permutations. Combinatorica 20 (2000): 241-255.
[24] A. Rényi. On random generating elements of a finite Boolean algebra. Acta Sci. Math. Szeged 22 (1961): 75-81.
[25] N. Sauer. On the Density of Families of Sets. Journal of Combinatorial Theory Series A, 13 (1972): 145-147.
[26] S. Shelah. A Combinatorial Problem; Stability and Order for Models and Theories in Infinitary Languages. Pacific Journal of Mathematics 41 (1972): 247-261.
[27] J. Spencer. Minimal scrambling sets of simple orders. Acta Mathematica Academiae Scientiarum Hungaricae 22 (1971): 349-353.
[28] J. Tarui. On the minimum number of completely 3 -scrambling permutations. Discrete Mathematics 308.8 (2008): 1350-1354.
[29] V. Vapnik, and A. Chervonenkis On the Uniform Convergence of Relative Frequencies of Events to Their Probabilities. Theory of Probability and its Applications 16.2 (1971): 264-280.
[30] I. Wegener. On separating systems whose elements are sets of at most $k$ elements. Discrete Mathematics 28.2 (1979): 219-222.
[31] R. Yuster. Perfect sequence covering arrays. Designs, Codes and Cryptography 88 (2020): 585-593.

