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LINEAR STABILITY OF FERROFLUIDS IN A NON-UNIFORM MAGNETIC FIELD

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Statement of Originality

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Abstract

The linear stability of Newtonian ferrofluids subject to non-uniform magnetic fields is considered in different geometries. First, a ferrofluid column with constant magnetic susceptibility, centred on a current-carrying rigid wire and surrounded by another ferrofluid with a different susceptibility, is investigated. Ferrofluids with non-uniform susceptibilities are considered next. For a constant susceptibility the magnetic forcing is confined to the interface, but for a non-constant susceptibility the forcing is felt in the bulk of the fluid. It is postulated that a stationary state of a ferrofluid with a non-uniform susceptibility in the presence of a non-uniform field, such that regions of highest susceptibility do not coincide with regions of highest field, may be unstable. An instability could be driven by the release of magnetic energy, since a minimum energy configuration may be reached when ferrofluid regions of high susceptibility and regions of high field coincide. This is explored for equilibria in a cylindrical domain, a planar domain and in a general three-dimensional domain. In a cylindrical domain, a stability condition is determined for a ferrofluid surrounding a current-carrying rigid wire, whose susceptibility varies radially. In a planar configuration, a stationary state of ferrofluid between two channel walls is found. The susceptibility and field vary normal to the wall, such that the regions of highest field and susceptibility do not coincide, and it is proven to be unstable. Methods of stabilising both systems are determined. For the cylindrical system, a constant axial field suffices, but for the planar domain it is shown that a rapidly rotating field is necessary to dampen unstable modes. Lastly, a stability condition is obtained for a general volume of ferrofluid, whose susceptibility varies slowly with position, subject to a non-uniform field.

Acknowledgements

I would like to express my deepest gratitude to Jonathan Mestel. His constant guidance, patience and invaluable knowledge were fundamental in the success of this project. Many thanks to Demetrios Papageorgiou, who gave continuous insight and direction. Thank you to my examiners Mark Blyth and Eric Keaveny for their time, thoughts and suggestions. Lastly, I'd like to thank all my friends and family. A special mention to my parents and my sister Mary for their endless support.

To my family and friends.

Contents

Chapter 1

Introduction

1.1 Introduction

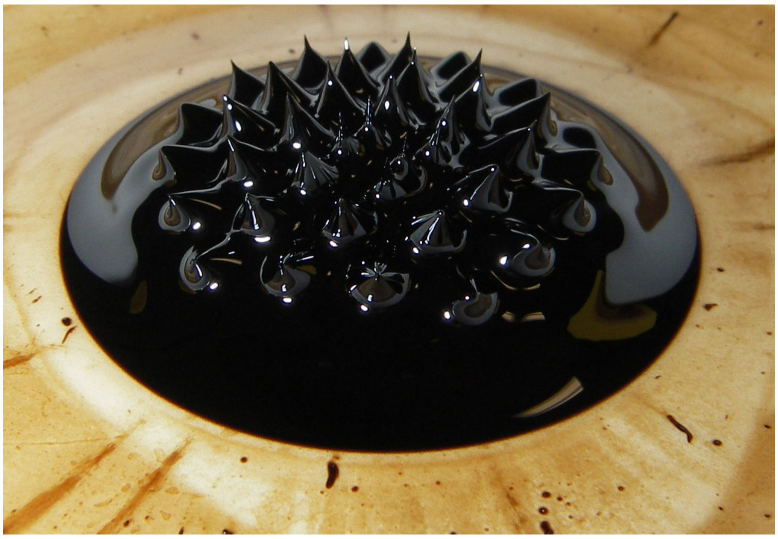

Figure 1.1: Image of a ferrofluid surface exhibiting peaks when subject to a magnetic field. Image taken from Arrighi et al. (2021).

This thesis investigates the linear stability of ferrofluids subject to non-uniform fields, in different configurations. Ferrofluids are stable colloidal fluids, consisting of magnetic solids suspended in a carrier solution (Cowley & Rosensweig, 1967). They are viscous fluids, and if no magnetic field is present, they obey the governing equations for a Newtonian fluid. Yet, when a magnetic field is applied to a ferrofluid, it becomes magnetised, whilst retaining its fluidity. Ferrofluids are current-free and have no free electric charge, so that the Lorentz force and electric forces are not present (Rosensweig, 1985). However, if the magnetisation or applied field exhibit discontinuities or gradients, a magnetic body force can act on the fluid (Rosensweig, 1985). Consequently, the magnetic energy stored in the magnetic field has the potential to drive instabilities. Figure 1.1 (Arrighi et al., 2021) shows an example of the Rosensweig normal-field instability, the first ferrofluid instability to be discovered (Cowley & Rosensweig, 1967). The instability occurs when a field is applied normal to the flat interface between a ferrofluid and a non-magnetic medium. The field generates a magnetic forcing localised to the interface, as a result of a jump in the magnetic susceptibility between the two mediums. The force is destabilising, counteracting the stabilising effects from surface tension and gravity (if the less dense fluid is the upper fluid). For a sufficiently strong field, the flat surface can no longer be supported and a new surface deformation of peaks is reached, in order to achieve the minimum energy configuration (Cowley & Rosensweig, 1967), as shown in Figure 1.2 (Andelman & Rosensweig, 2009). Investigating the stability of equilibria where the magnetic susceptibility is discontinuous, or continuous and non-uniform, forms this thesis.

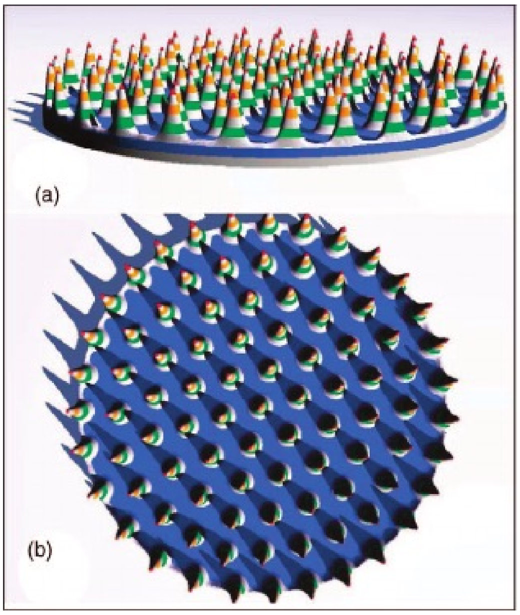

Figure 1.2: Figure taken from Andelman & Rosensweig (2009) showing X-ray images of a normal-field instability of a ferrofluid (a) oblique view (b) a plane view.

In this chapter, we will introduce ferrofluids to the reader. Their properties, synthesis, process of magnetisation, and applications are discussed. Complex processes occurring within the ferrofluid are highlighted, and their effects on the dynamics of the ferrofluid reviewed. The different approaches for theoretically modelling the magnetisation of ferrofluids are outlined, and the theory and assumptions used for the analysis in this thesis stated. Lastly, we give an outline of each chapter and the relevant hydrodynamic, electrohydrodynamic and ferrohydrodynamic literature is reviewed.

1.2 Ferrofluid properties, synthesis and magnetisation

Ferrofluids comprise of magnetic solids, typically magnetite, cobalt, hematite or other ferrites, suspended in a carrier solution, such as water, kerosene or oils (Bozhko & Suslov, 2018) (Rosensweig, 1985). They can be synthesised by chemical precipitation, thermal decomposition, emulsion technique and by wet-grinding ferrite powders in a carrier liquid with a stabilisation agent (Shliomis, 1974; Charles, 1987). The magnetic particles interact through spherically symmetric energy by steric repulsion, Van der Waals attraction, electrostratic repulsions and dipole-dipole interactions (Shliomis, 1974; Ivanov & Kuznetsoza, 2001). The particles must have a sufficiently high thermal energy in order to counteract magnetic forces which would otherwise cause agglomeration of the particles, and subsequent sedimentation from gravitational forces (Odenbach, 2002). Stabilisation of the particles, meaning the prevention of agglomeration, is achieved by either coating the magnetic grains in a surfactant (soaps, alcohols or fatty acids (Shliomis, 1974)), thereby producing an entropic repulsion, or by double electric layer formation, where the particle is surrounded with an electrically charged shell, producing an electrostatic repulsion (Bozhko & Suslov, 2018; Ivanov & Kuznetsoza, 2001; Charles, 1987).

The particle size ranges between $3-20nm$, and for this reason each particle is treated as a single magnetic domain in a permanent state of saturated magnetisation (Odenbachy, 2002; Charles, 1987). Each particle is therefore modelled as a small magnetic dipole in the carrier liquid with a net magnetic moment (Odenbach, 2002; Bozhko & Suslov, 2018; Ivanov & Kuznetsoza, 2001).

If no magnetic field is present their net magnetisation is zero and the particles are randomly orientated. Yet, upon the application of a magnetic field, the dipoles move to align with the field. For a sufficiently strong field, thermal agitation is overcome and the particles' moments will be aligned with the field, having reached the saturation magnetisation (Rosensweig, 1985). This is analogous to that of a paramagnetic gas or liquid. Paramagnetic substances are composed of atoms or ions with net magnetic moments, a result of unpaired electrons within the atoms (ions) (Cullity et al., 2015). They are weakly magnetised and exhibit a process called paramagnetism, where the individual particles align with an applied field, with no long-range order between the particles. Ferrofluids display superparamagnetism, a behaviour analogous to paramagnetism, but the magnetisation occurs for a much lower field strength (Rosensweig, 1985). In fact, a ferrofluid's degree of magnetisation is $10^3 - 10^4$ larger than those of paramagnetic materials, resulting in them being more receptive to ordinary permanent magnets or electromagnets (Bozhko & Suslov, 2018; Ivanov & Kuznetsoza, 2001).

The mechanism of alignment of the moment with an applied field and subsequent relaxation when the field is removed, depends on whether the particle is classified as either magnetically hard or weak. This is determined by the size of the particle, the specific anisotrophy energy of the ferromagnetic material, and the viscosity of the carrier liquid (Muller & Liu, 2002). In general, smaller particles are classified as magnetically weak particles, and larger particles classified as magnetically hard. The Neel process of relaxation is said to occur in magnetically weak particles, whereby the magnetic moment rotates within the particle, while a Brownian relaxation process occurs in magnetically hard particles, in which the whole particle rotates to align with the applied field. Specifically, it is found that the relaxation process becomes Brownian for particles with a diameter greater $\sim 13nm$ (Odenbach, 2004). In reality, both Brownian and Neel processes occur as a real ferrofluid has a variety of particle sizes (Muller & Liu, 2002; Bozhko & Suslov, 2018; Odenbach, 2002). Characterising the way in which the particles align with an applied field, and the subsequent behaviour of the magnetisation of the ferrofluid, determines the complexity of the governing equations. The magnetisation of ferrofluids has been studied extensively theoretically and experimentally, and models have been derived to best describe the magnetisation of a ferrofluid. We now outline some (but not all) methods of describing the magnetisation of ferrofluids.

It should be noted that when a ferrofluid is subject to temperature gradients, the magnetisation process will respond differently. The equations take a different form for a non-isothermal ferrofluid subject to temperature changes, and different processes occur, for example thermomagnetic convection. This is not considered here and the reader is advised to read Bozhko & Suslov (2018) for the theory on this.

1.2.1 Magnetisation characteristic diagrams

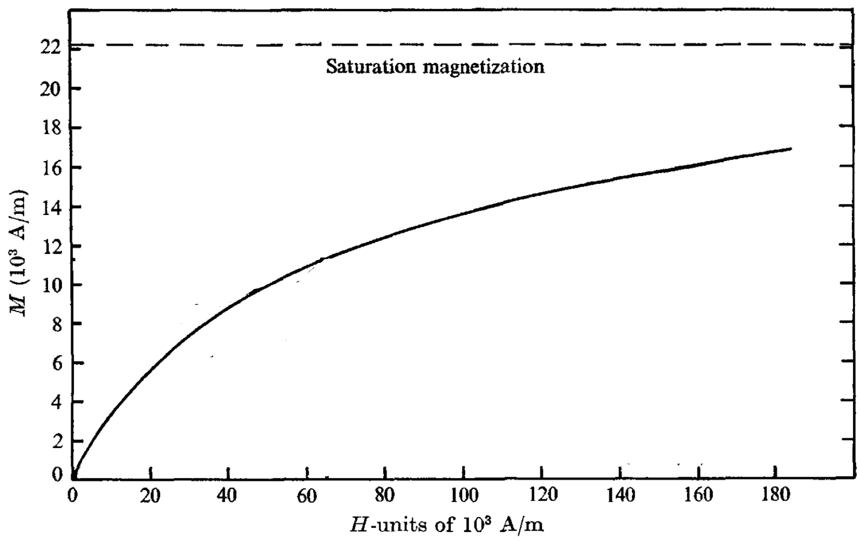

Figure 1.3: Diagram adapted from Figure 1 in Zelazo & Melcher (1969). The magnetisation density M is plotted against the magnetic field strength H. The inserts of the original figure have been removed.

For a homogeneous liquid sample, magnetisation characteristic diagrams can be produced, where the magnetisation density is plotted as a function of the field density, starting from an initial magnetisation, and reaching a saturation magnetisation as the magnitude of the field strength is increased. The initial susceptibility is often given by

$$\chi_0 = \frac{1}{\mu_0}\frac{\partial M}{\partial H}\bigg|_{H=0}, \tag{1.1}$$

where μ_0 is the magnetic permeability, while M and H are the magnitude of the magnetisation and applied field, respectively (Rosensweig, 2002). Figure 1.3 is an example of a magnetisation characteristic diagram, adapted from Figure 1 in Zelazo & Melcher (1969). If it is assumed that the magnetic susceptibility χ depends on H, you can assign relationships for M and H by fitting the data from such plots for specific χ in known ferrofluids (Zelazo & Melcher, 1969).

1.2.2 Quasi-stationary theory

For sufficiently low concentrations of magnetic particles, effects of magnetic inter-particle interaction can be neglected (Bozhko & Suslov, 2018; Jansons, 1983), and the particles can be treated as small thermally agitated permanent magnets in a carrier liquid. This is named quasi-stationary theory (Shliomis, 2002), and is used in many works; Rosensweig (1985); Zelazo & Melcher (1969); Yecko (2009, 2010), to name a few.

Although the initial susceptibility of a ferrofluid is significantly larger than a paramagnetic fluid, upon assuming quasi-stationary theory, the magnetisation can be described using Langevin's equation for paramagnetic systems;

$$M = M_s\left(\coth(\alpha) - \frac{1}{\alpha}\right), \quad \alpha = \mu_0 m H / k_B T \tag{1.2}$$

where $M_s = CM_0$ is the saturation magnetization of the ferrofluid, C the volume concentration of suspended magnetic material, M_0 the spontaneous magnetisation, α is the Langevin parameter, k_B the Boltzmann constant, temperature T and m the total magnetic moment (Odenbach, 2009; Jansons, 1983). (1.2) assumes that the magnetisation is independent of the flow, and the magnetisation relaxes instantaneously to an equilibrium value, but note that the flow may depend on the magnetisation (Shliomis, 2002).

Moreover, in the low-magnetic field limit, if the relaxation rate is instantaneous on the same time scale as the dynamic processes of interest, then the magnetisation $\boldsymbol{M}$ and applied field $\boldsymbol{H}$ can be treated as sensibly co-linear (Rosensweig, 2002; Muller & Liu, 2002). The proportionality factor is named the magnetic susceptibility χ (Erne et al., 2003). By considering $\boldsymbol{M}$ co-linear with $\boldsymbol{H}$, the magnetisation is then determined at a given moment by an instantaneous value, and thus the relaxation time is considered to be zero. Consequently, when the ferrofluid is subject to a field $\boldsymbol{H}$, it is magnetisable to a linear approximation $\sim \chi H$. If χ is non-constant, a force acts throughout the fluid often proportional to $H^2 \boldsymbol{\nabla} \chi$. However, for constant χ, the forcing does not act in the bulk of the fluid, and will only act at an interface where there exists a jump in χ, say between a ferrofluid and a different medium (e.g. the Rosensweig normal-field instability in Figure 1.1 (Arrighi et al., 2021)), or two ferrofluids with different magnetic susceptibilities (see Chapter 3).

Assuming the magnetisation is co-linear with the field at all times, is named the quasi-equilibrium theory and is used often in the literature; Cowley & Rosensweig (1967); Rosensweig (1985); Bashtovoi & Krakov (1978); Arkhipenko et al. (1980); Doak & Vanden-Broeck (2019); Rannacher & Engel (2007); Canu & Renoult (2021). However, Quasi-stationary theory is valid only for colloids of Neel particles and the magnetisation can not be hysteric or rate-dependent (Shliomis, 2002). Moreover, for sufficiently strong applied fields or for large volume concentrations of magnetic particles, one can no longer neglect interactions between magnetic moments of the particles (Shliomis, 1974). Since ferrofluids have a mix of particle sizes, and therefore will have magnetically hard particles, as well as weak particles, there is an importance in incorporating inter-particle interactions and internal rotation in the ferrohydrodynamic equations, due to the “magneto-viscous effect” and chain formation.

1.2.3 The magneto-viscous effect

It has been known since the work of Rosensweig et al. (1968) that a ferrofluid viscosity can change when subject to a magnetic field, and it has been shown experimentally to occur for both high (Rosensweig et al., 1968) and low (McTague, 1969) concentrated ferrofluids. This is now commonly known as the magneto-viscous effect. In the absence of a magnetic field, suspended particles in a vortical flow will rotate in accordance with the vorticity, generating a mechanical torque on the particle (Shliomis & Morozov, 1994). When the ferrofluid is then subject to a magnetic field, the particles move to align their individual moment with the field, a process achieved by either a Neel relaxation process or a Brownian relaxation process. The latter generates a magnetic torque on the particle, since the whole particle rotates to align. Consequently, if the vorticity and field are not parallel, there is a competition between the mechanical and magnetic torque, increasing flow resistance, and generating a viscous friction (Shliomis & Morozov, 1994; Odenbach, 2004).

Furthermore, an alternating field will induce "rotational swings" in the particles. For a stationary ferrofluid, these average out to give a zero mean angular velocity of the particles, but, once vorticity is present in the ferrofluid, the mean angular velocity becomes non-zero. For slow oscillations of the field, there is an increase in viscosity as the angular velocity of the magnetic particles is less than the angular velocity of the flow. The magnetic torque prevents the free particle rotation with the fluid, resulting in a dissipation of kinetic energy as the liquid has to flow past the particle. For sufficiently fast oscillations, the particles rotate faster than the local fluid rotation rate, creating a "spin up" of the flow. The energy from the field is converted to kinetic energy, and the viscosity reduces (Shliomis & Morozov, 1994; Shliomis, 2002).

Aggregation of particles can occur if the particles aren't spherical, by an increase of particle concentration, an increase in field strength, or a decrease in temperature (Bozhko & Suslov, 2018; Camp et al., 2021). Under certain conditions, aggregation can cause chains to be formed (Shliomis, 1974), which in turn produces large changes in viscosity (Odenbach, 2004). The onset of chain formation is governed by an "interaction parameter", which measures the ratio of the magnitude of the inter-particle interaction between two particles, to the thermal energy of the particles (Odenbach, 2004; Camp et al., 2021). The chains have high net magnetic moments, and the susceptibility of the ferrofluid is increased, invalidating the Langevin equation. The chains can then form rings if the "interaction parameter" exceeds a critical value. The rings have negligible net moments, and the Langevin relation is applicable again (Camp et al., 2021). Aggregation utilises both Brownian and Neel mechanisms. It is postulated by Odenbach & Raj

(2000) that although large particles are dominant in causing magnetoviscous effects, chains of smaller particles may be formed, and these can then produce effects similar to those of large particles. Odenbach (2004) supports this experimentally, showing that the magnetic hard particle concentration in a ferrofluid is not sufficient in producing the large changes in viscosity seen at high fields, and it is therefore assumed that this is a product of magnetic inter-particle interactions forming chains of both hard and weak particles.

1.2.4 Modelling a non-instant magnetisation relaxation and inter-particle interactions

Various models exist to describe the magnetisation of a ferrofluid when there is not an instant relaxation of magnetisation with the field, or when inter-particle interactions occur. Shliomis (2002) produced a model taking into account the "visible" (Brownian relaxation process) and "internal" (Neel relaxation process) rotations of the particles. Rosensweig (2002) employed an irreversible thermodynamic framework to an applied field in a conductive ferrofluid with internal rotation, to model a non-instantaneous alignment of the magnetic moments. Other, more complicated models, have been developed that expand on *Debye theory* (Debye, 1929) and *effective-field theory* (Shliomis, 2002; Muller & Liu, 2002). Namely, Ivanov et al. (2016) uses a modified mean-field approach to model inter-particle dipole-dipole interactions, introducing an additional term into the Fokker-Plank equations. To describe the dynamics caused from alternating fields, Yoshida & Enpuku (2009) uses a modification to *Debye theory* and utilises the Fokker-Planck equations, but does not take into account inter-particle dipole-dipole interactions. Rusanov et al. (2021) expands on Ivanov et al. (2016) and Yoshida & Enpuku (2009), and models the effects of field amplitude and inter-particle interactions by a modified mean-field approach. I have not gone into detail on the models here, as the complexity of the models and the computational analysis necessary to employ them, meant quasi-stationary theory was preferable for the analysis performed in this thesis.

1.2.5 Our approach

We use a quasi-equilibrium approach to model the magnetisation. Although it may not fully capture the ferrofluid dynamics on all scales, it is often assumed in the literature that the magnetisation is colinear with the field at all times. Despite the simplification of the equations, it can still produce accurate results, and many of the works discussed in Section 1.5 use quasi-stationary theory. Moreover, it can be assumed that the concentration of particles is sufficiently low to neglect particle interactions and magneto-viscous effects in the analysis. Furthermore, we assume the ferrofluid is isotropic and neglect thermal effects.

1.3 Applications

Magnetic fluids are advantageous to many disciplines. They can be directed to an area or held in place by a magnetic field, they can heat up an intended area by their ability to absorb electromagnetic energy, as well as having the properties of a viscous liquid (Scherer & Figueiredo Neto, 2005). Ferrofluids are preferable, as they are manipulated with a relatively low field strength in comparison to a paramagnetic fluid, minimising physical constraints for a desired application, as well as energy costs. Furthermore, a wide range of carrier liquids can be used for the synthesis of ferrofluids, and thus the desired viscosity, pressure and temperature of the liquids can be chosen accordingly for each individual application (Charles, 1987).

Ferrofluids are most commonly used for dynamic sealing and heat dissipation in industrial applications. Dynamic sealing with a ferrofluid, for example in the hard discs of computers, allows the interior to be protected without restricting movement. Ferrofluids conduct heat and they can be held in place by a magnetic field without the need for structural supports and allow for movement of the object in question, where a solid conductor could block the functionality of the equipment. For example, the coil of an audio speaker is commonly surrounded by a ferrofluid, and the set up is shown by a simple diagram in Figure 1.4 (Rene, 2019). High volumes can be reached, as the ferrofluid dissipates the thermal energy. Moreover, the ferrofluid also dampens unwanted resonances in the loudspeaker, and is often used in other systems for vibration dampening, such as stepper motors (Scherer & Figueiredo Neto, 2005). Due to their receptivity to a field, they are useful to ink-jet printing (Fattah et al., 2016; Charles, 1987) and 3-D printing (Löwa et al., 2019), where the jet disintegrates into drops, which are then directed by a magnetic field. Other industrial applications include actuators, modulators of laser radiations, sensors, power transformers and converters, and solar collectors (Bozhko & Suslov, 2018).

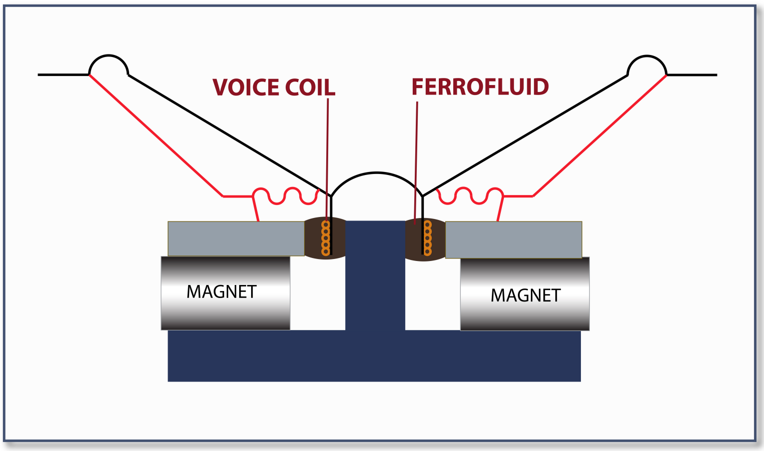

Figure 1.4: Schematic of the mechanism within an audio speaker. Image taken from Rene (2019).

Biological applications include hyperthermia treatment for cancer, and magnetic drug targeting. For the former, a ferrofluid is injected into a desired area of the body, for example at a tumour site, and by applying a high frequency, alternating current field, the ferrofluid will heat up and the heat released will weaken the malignant tissue (Asfer et al., 2017; Zhang et al., 2007; Scherer & Figueiredo Neto, 2005). Magnetic drug targeting with ferrofluids has been investigated experimentally (Asfer et al., 2017) and theoretically (Voltairas et al., 2002; Gonella et al., 2020), and allows for a localised and generally non-invasive treatment in various medical areas; early diagnosis, therapeutic treatments, and treatment for diseases, in particular cancer therapy (Gonella et al., 2020). Nanoparticles are bounded with the required drug, injected into the blood, and then directed to the target area by a field. The field is generated either by a permanent magnet, or an electromagnet positioned directly beneath the area (Asfer et al., 2017).

The geometry, configuration of the applied field and ferrofluid, and other parameters of the system will vary with each application. Theoretical analysis is therefore necessary in determining the stability of various geometrical configurations, and the possible parameter ranges for a chosen application.

1.4 Outline

The relevant governing equations for the analysis performed in this thesis are given in Chapter 2. Chapters 3-6 investigate the linear stability of ferrofluid equilibria subject to non-uniform fields. Chapter 3 determines the stability of a Newtonian ferrofluid centred on a current carrying wire, surrounded by another ferrofluid of a different magnetic susceptibility. Three-dimensional disturbances are considered, and an analytical solution given to the linearised Navier-Stokes equations. The associated growth rate of the disturbance is determined. Chapter 4 considers one ferrofluid with a continuous radially-varying susceptibility, centred on a current-carrying wire. A stability criterion is obtained for three-dimensional disturbances to the system. In both Chapter 3 and 4, it is shown an axial field can dampen unstable disturbances. Chapter 5 investigates the stability of an equilibrium in a planar configuration. A ferrofluid is between two channel walls, subject to a non-uniform field applied normal to the walls. The susceptibility of the ferrofluid and the field vary normal to the walls. The system is proven to be unstable, but it is shown that by applying a rapidly rotating field, the unstable modes can be dampened. In Chapter 6 the stability of a general volume of ferrofluid in a stationary state, whose magnetic susceptibility varies spatially, subject to a non-uniform magnetic field, is investigated. A stability condition is determined, provided the susceptibility varies slowly spatially within the volume. In Chapter 7 concluding remarks are given, as well as the potential applications of the analysis performed. Additionally, extensions and future work are discussed.

1.5 Relevant literature

If the magnetisation characteristics of a ferrofluid are linear, then there is a direct analogue between a ferrofluid subject to a magnetic field in ferro-hydrodynamics and a dielectric exposed to an electric field in electro-hydrodynamics (EHD) (Zelazo & Melcher, 1969; Rosensweig, 1985). Provided the dielectric is current-free, and the electric field potential, the electric stress is analogous to the ferrofluid stress tensor under quasi-equilibrium theory. The dielectric constant is replaced with $1 + \chi$, the electric permittivity of free space replaced with the magnetic permittivity of free space, and the electric field replaced with $\boldsymbol{H}$ (Stone et al., 1999). Consequently, certain ferro-hydrodynamic problems have been answered in the EHD literature, for example, drop behaviour of a ferrofluid and dielectric fluid is discussed by Basaran & Wohlhuter (1992) and Stone et al. (1999). Figure 1.5 (Andelman & Rosensweig, 2009) shows the similarity in pattern formation of a ferrofluid subject to a magnetic field and a dielectric subject to an electric field. The analogy has limitations in that the effects on a polarizable fluid from a electric field is of much lower magnitude than the effects of a magnetic field on a ferrofluid (Rosensweig, 1985). Cowley & Rosensweig (1967) argue that although the classic normal-field instability to the free surface between two fluids (one magnetic) could be analogous to dielectrics theoretically, the experiments for a ferrofluid interface show a stable interface is reached that exhibits a regular periodic structure of peaks, as shown in Figure 1.2 (Andelman & Rosensweig, 2009). Moreover, ferrofluids exhibit nonlinear magnetisation characteristics, and in general, EHD works consider linear polarizable material. Saying this, Basaran & Wohlhuter (1992) consider the effect of nonlinear polarization on drop deformation, describing the nonlinear polarization with a Langevin equation. In this case, the governing equations for the deformation of the drop are the same for a dielectric liquid in an applied electric field, and a ferrofluid in an applied magnetic field, and therefore the results of Basaran & Wohlhuter (1992) are applicable to both. Nevertheless, there is limited literature on nonlinear polarization mediums in EHD. To the best of my knowledge, the configurations considered in this thesis do not have a counterpart in the EHD literature.

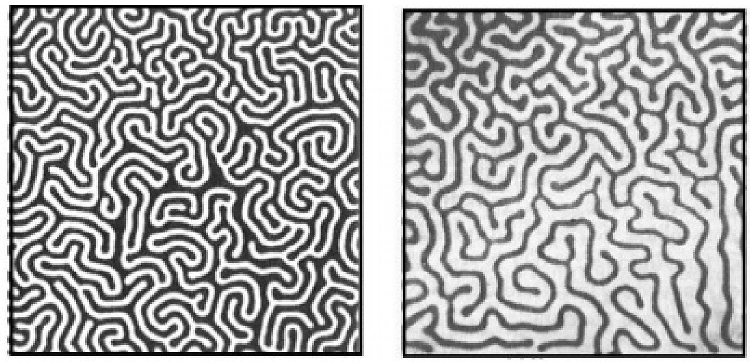

Figure 1.5: Figure taken from Andelman & Rosensweig (2009) showing photos ($7cm^2$ in size) of a labyrinth instability in (a) A ferrofluid subject to a magnetic field and (b) A dielectric oil subject to an electric field

In Chapter 3, the first configuration considered is a ferrofluid column centred on a current-carrying wire in a cylindrical domain, surrounded by another ferrofluid of a different susceptibility. The instability of liquid jets and cylindrical columns of ideal fluids has been rigorously researched for over a century, since the experimental work of Plateau (1873), which in turn motivated the theoretical investigation by Rayleigh (1878). Rayleigh (1878) used an energy argument to show that an axisymmetric liquid jet in a gas medium is unstable to axisymmetric disturbances of wavelengths larger than the radius of the jet. Rayleigh then analysed the effect of inertia (Rayleigh, 1892b) and viscosity (Rayleigh, 1892a) on the stability of a jet, by defining a stream function, perturbing the system and seeking separable solutions of the new surface, thereby producing a dispersion relation, supporting the result of the energy argument. This was followed by many works. Christiansen (1955) analysed the more general case of no axisymmetry with inertial effects and found that axisymmetric modes were the most unstable. Christiansen & Hixson (1957) investigated the stability of an inviscid column of fluid, surrounded by another inviscid fluid, theoretically and experimentally. Tomotika (1935) derived the equations for the stability of a viscous jet in a viscous medium, and obtained an explicit expression for the dispersion relation in the Stokes limit. Following Tomotika (1935), Meister & Scheele (1969) investigated numerically the growth rate for the full Navier-Stokes equations.

In the absence of a magnetic field, a ferrofluid jet behaves just as a Newtonian fluid jet, and is unstable due to surface tension. However, in the presence of a sufficiently strong magnetic field the system can be stable. Rosensweig (1985) shows an inviscid axisymmetric jet is linearly stable to axisymmetric disturbances in the presence of a uniform axial field, generated by positioning a solenoid co-axially to the jet. Bashtovoi & Krakov (1978) was the first to consider the effect of a multi-component field on an axisymmetric ferrofluid jet, surrounded by a vacuum. They assume from the outset that the potential field satisfies the governing equations and does not disturb the axial symmetry of the jet. An example given, has radial and azimuthal components of the field behaving as the reciprocal of the radius, and constant axial components. For solely a radial field, the axisymmetric disturbances are proven to be the most unstable. If a tangential field is applied, with components in both the azimuthal and axial directions, the critical values for the onset of instability vary with the form of the field, and interestingly, the most unstable modes are not perpendicular to the tangential field, as might be expected. Note that, the radial component of the field solution given by Bashtovoi & Krakov (1978) could be generated by a line source at $r = 0$, but physically this is challenging to produce. The azimuthal component is singular at $r = 0$, and in contrast to electrostatics, a current running along the axis of the cylinder can produce such a field. Thus, subsequent works considered a column of ferrofluid centred on a current carrying wire. The current produces an azimuthal field, resulting in a magnetic stress at the interface of the ferrofluid with the surrounding medium, which can stabilise the system. The critical parameter is the magnetic Bond number, B, which measures the ratio between the capillary pressure and the magnetic forcing from the current through the wire. The linear stability of a ferrofluid jet centred on a current-carrying wire has been studied in the literature but with limiting assumptions.

Arkhipenko et al. (1980) consider axisymmetric disturbances to an irrotational, inviscid ferrofluid jet, and show increasing the current such that $B > 1$ produces a stable system. Rannacher & Engel (2006) considers non-axisymmetric disturbances too, finding only axisymmetric modes are unstable for a non-magnetic fluid surrounding the ferrofluid. The analysis performed by Arkhipenko et al. (1980) and Rannacher & Engel (2006) is for an irrotational, inviscid system, but the experimental work performed by Arkhipenko et al. (1980) and Bourdin et al.

(2010), show the development of the instability is on a longer time-scale than the theoretical results, suggesting viscous effects are important. Cornish (2018) considers the highly viscous limit and Canu & Renoult (2021) studies a Newtonian ferrofluid jet, surrounded by a Newtonian non-magnetic fluid. Both investigate axisymmetric disturbances only and the results obtained by Canu & Renoult (2021) show that accounting for viscosity better agrees with the experimental results of Arkhipenko et al. (1980) and Bourdin et al. (2010), than the inviscid system. Canu & Renoult (2021), like Bashtovoi & Krakov (1978), consider fields in the axial and radial directions, as well as the azimuthal direction, finding an axial field can dampen unstable disturbances, but a radial field can destabilise the system. The analysis by Canu & Renoult (2021), although the most developed of the literature, is relevant only if the outer-fluid is non-magnetic. Arkhipenko et al. (1980) briefly discusses the outer-fluid as a magnetisable liquid, but the analysis and results obtained focus on the outer fluid being a non-magnetic gas. Korovin (2001) and Korovin (2004) do consider the surrounding liquid being a ferrofluid, but with a lower susceptibility than the inner fluid. Korovin (2001) considers the outer ferrofluid being inviscid and in an unbounded domain. Whereas, Korovin (2004) considers both ferrofluids as viscous but the outer fluid filling a cuvette, rather than an infinite domain. The dispersion relation for axisymmetric disturbances is derived using a modified equation of motion, much simpler than the full Newtonian problem.

Similar results are found in EHD. When the inner fluid is a ferrofluid and the outer fluid non-magnetic, the magnetic susceptibility could be treated as continuous radially such that $\sim d\chi/dr > 0$. Since the magnetic forcing behaves $\sim H^2\boldsymbol{\nabla}\chi$ (under quasi-equilibrium theory), the magnetic field gradient is directed to the center, and there is a compression of the column of ferrofluid similar to the "pinch effect" in plasmas (Bashtovoi & Krakov, 1978). In dielectrics, a jet of charged fluid will be unstable due to capillary and charge relaxation forces but a suitably strong electric field can dampen disturbances. Nayyar & Murty (1960) and Goldman et al. (1997) study, respectively, the stability of an inviscid and viscous dielectric liquid column, subject to a longitudinal electric field. The first uses an energy argument to show the electric field has a stabilising effect. The latter considers axisymmetric perturbations to the system and produces a dispersion relation, showing viscous dissipation and dielectric forces at the interface work to stabilise the system. A crucial difference between the dielectric work on jets and the ferrofluid work is the presence of a wire and the associated azimuthal field.

An extension to previous works is performed in Chapter 3. We determine the linear stability of a column of ferrofluid, centred on a current-carrying wire, surrounded by a ferrofluid with a different magnetic susceptibility, where both the susceptibilities are modelled as constant. For constant susceptibilities, the magnetic forcing acts as a stress at the interface and is not felt in the bulk of the fluids. We consider three-dimensional disturbances to the full Navier-Stokes equations, and obtain an analytic expression for the respective growth rate of the disturbance for arbitrary Reynolds number. In the inviscid and viscous limits the growth rate is given explicitly and a stability condition is determined. For arbitrary Reynolds number, the implicit expression for the growth rate is solved numerically using a root solver, and the results given for a range of different parameters of the system. It has been shown theoretically (Rosensweig, 1985; Canu & Renoult, 2021) and experimentally (Bashtovoi & Krakov, 1978) that an axial field will stabilise axisymmetric disturbances to a ferrofluid jet surrounded by a vacuum, and Kazhan & Korovin (2003) show an axial field will dampen unstable modes for a non-magnetic viscous thread surrounded by a ferrofluid. We prove an axial field stabilises all disturbances of our system, with finite wave length in the axial direction.

Some works have used non-linear theory to analyse the behaviour of a ferrofluid jet. Rannacher & Engel (2006) shows, in the long-wave limit, axisymmetric surface deformations can propagate at the surface, but non-axisymmetric disturbances always dissipate. They also investigate whether effects of non-linearity and dispersion can balance each other to give rise to axisymmetric soliton solutions at the interface. Blyth & Parau (2014) use a fully non-linear, numerical model to show axisymmetric solitary waves propagate at the surface of an inviscid column of ferrofluid and compare their results with the experimental work by Bourdin et al. (2010), who show the existence of axisymmetric periodic and solitary waves at the interface of a ferrofluid jet. Doak & Vanden-Broeck (2019), as well as studying the linear stability, use a numerical model to find stable travelling wave solutions on an inviscid ferrofluid jet. Cornish (2018) uses weakly non-linear stability theory and long wave theory for both a highly viscous and inviscid axisymmetric jet, studying the resultant drop formation. In this thesis, we focus solely on linear stability analysis.

In Chapter 3 we find that if the outer fluid is more magnetic than the inner fluid, the system is unstable as a result of magnetic forcing generated from the current in the wire, as well as capillary forces. Whereas, when the inner fluid is more magnetic, the magnetic forcing can dampen the unstable modes caused by surface tension. Zelazo & Melcher (1969) consider two ferrofluids, both whose susceptibilities have a non-linear dependence on the non-uniform applied field, between two current sheets. When the magnetic susceptibility is non-uniform, the magnetic forcing is felt in the bulk of the ferrofluid, but Zelazo & Melcher (1969) assume a form of the force density, such that the forcing remains confined to the interface. They implement a quasi-one-dimensional model, transforming a cylindrical geometry to a planar geometry, and use a Boussinesq approximation to obtain a dispersion relation, showing that when the field decreases upwards, and the stronger ferrofluid is the lower fluid, only capillary forces are destabilising. Yet, if the stronger ferrofluid is the upper fluid, both capillary and magnetic forcing are destabilising, implying that the magnetic force is destabilising when the highest region of susceptibility does not coincide with the highest region of field. Indeed, when a magnetic fluid is subject to an inhomogeneous field, the magnetic particles are attracted to regions of strongest field, obtaining a minimum energy configuration (Scherer & Figueiredo Neto, 2005). In the system in Chapter 3, the azimuthal field produced by the current in the wire decreases inversely with radius. Consequently, the magnetic forcing is destabilising when the ferrofluid with the highest susceptibility, is not in the region of strongest field. This motivates the stability analysis of Chapter 4, where a ferrofluid with a continuous radially-varying susceptibility, centred on a current-carrying wire is investigated. A stability condition is derived, depending on the sign of the gradient of the field strength with respect to the susceptibility, in particular $d\chi/dr > 0$ produces an instability. Moreover, it is shown that an axial field will suppress all disturbances of finite wavelength.

The results of Chapter 4 prompt the investigation of the linear stability of other configurations where the magnetic susceptibility and field are non-uniform. In Chapter 5 a planar configuration is considered, where a stationary state is found by considering equipotential surfaces of zero-mean curvature. For a planar domain, it is well-known that a layer of ferrofluid with a non-magnetizable fluid above it, experiences destabilising forces from a uniform field applied normal to the interface of the two fluids (Cowley & Rosensweig, 1967). In the EHD literature, Li et al. (2007) investigate the stability of a two-fluid interface between two walls, subject to a normal electric field, where the lower wall is grounded and the upper wall has a constant electric potential. They find the configuration is unstable for perfect dielectrics, a result that holds for the analogous system of two ferrofluids with linear magnetic susceptibilities, but not necessarily for non-linear susceptibilities. Yecko (2009) and Yecko (2010) consider a Poiseuille flow in an analogous set up to Li et al. (2007), but with the lower fluid a ferrofluid with a non-linear susceptibility, and a uniform field is applied at some orientation to the interface. They assume a Langevin relation for the magnetic susceptibility, and allow the susceptibility to depend on the applied field. Despite the susceptibility being non-linear, Yecko (2009) and Yecko (2010) assume a form of the magnetic stress tensor such that the forcing remains confined to the interface, rather than in the bulk of the fluids. Yecko (2009) initially considers linear material, and upon adding a non-linearity approximation to the susceptibility, different stability regions arise, with a greater impact to the viscous regime. For a linear susceptibility and a sufficiently large strength of field, there is an instability for all orientations of the field. If there exists an unstable mode, a tangential orientated field is more unstable than a normal field. Moreover, as the orientation of the field is varied, unstable modes are dampened, but modes that were stable are rendered unstable. They find an analogy of Squire's theorem holds, namely that two dimensional modes are the most unstable. Yecko (2009) varied the magnetic parameters only, and Yecko (2010) goes further by also varying thickness and viscosity of the layers. Moreover, Yecko (2010) consider a fully non-linear magnetic material but, in light of the results of Yecko (2009, 2010) only considers two-dimensional disturbances. Yecko (2010) considers normal or parallel fields applied to the interface and shows that both normal and parallel fields to the interface will produce an instability, but a parallel field will produce a smaller growth rate. The equilibrium found in Chapter 5 is such that the ferrofluid is between two parallel walls, subject to a non-uniform magnetic field acting normal to the walls, where the susceptibility and field vary normal to the channel walls. It is proven that this configuration is unstable to all three-dimensional disturbances, and a method of stabilising the system is investigated.

It is known that a uniform field applied parallel to the plane interface between a ferrofluid and a non-magnetic fluid will suppress linearly unstable modes at the interface whose wave vector is parallel to the field, and Zelazo & Melcher (1969) shows this for a non-uniform field too. Yet, for a three-dimensional system, modes perpendicular to the field will not be dampened (Rannacher & Engel, 2007). Korovin (2014) considers the stability in the inviscid limit of the classical Rosensweig instability, subject to a tilted piecewise-constant magnetic field, and shows that in the presence of a horizontal field, a stronger vertical field is needed to produce the instability. Dorbolo & Falcon (2011) show experimentally and theoretically, a horizontal magnetic field acting on sinusoidal waves at a fixed frequency produces a monotonic dispersion relation, but observe that in the nonlinear regime wave turbulence occurs. In a channel system, Zelazo & Melcher (1969) and Yecko (2010) both show that adding a constant field down the channel does not stabilise all modes, and we demonstrate that a constant field across the channel is not sufficient in stabilising the system in Chapter 5 either. Rannacher & Engel (2007) employ a rotating field to stabilise the classical Rayleigh-Taylor instability, for a ferrofluid as the upper (more dense) fluid. By Floquet theory, they find a rotating field will stabilise the modes where the modulus of the wave-number exceeds a threshold value, but otherwise the modes remain unstable. In Chapter 5 it is proven that a constant field applied across the domain will not dampen all disturbances, and neither will an alternating-current field. Yet, a rapidly rotating field can stabilise the system for a sufficiently large field strength.

The analysis of Chapter 4 and 5 both agree that the stability of the system is determined by the sign of the gradient of the applied field strength with respect to the susceptibility, for a non-uniform field and susceptibility. Chapter 6 proves the stability of a stationary state of a general volume of ferrofluid, whose susceptibility varies slowly spatially, is indeed determined by the sign of the gradient of the magnitude of the applied field with respect to the susceptibility. Lastly, in Chapter 7 a stability condition for a general configuration of a ferrofluid subject to a magnetic field is discussed.

Chapter 2

Governing equations

2.1 Introduction

Here we outline the relevant equations for the analysis that follows in the subsequent chapters. Ferrofluids are generally modelled as non-conducting fluids and obey Maxwell's equations with free current density and Maxwell's displacement current deemed negligible (Rosensweig, 1985). The stress tensor for a ferrofluid consists of the hydrodynamic stress tensor for a viscous fluid, Maxwell's stress tensor and extra terms unique to ferrohydrodynamics. The full derivation, along with the derivation for the boundary conditions can be found in Rosensweig (1985).

2.2 Maxwell's equations

In ferrohydrodynamics the free charge and electric displacement are regarded as absent in Maxwell's equations. Relevant here is Ampere's law,

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{2.1}$$

Gauss' magnetisation law,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2.2}$$

and Faraday's law,

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \frac{-\partial \boldsymbol{B}}{\partial t}, \tag{2.3}$$

where $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{J}_f$ is the free current density, $\boldsymbol{D}$ the electric displacement field and $\boldsymbol{B}$ the induced magnetic field. For a non-conducting ferrofluid the electric conductivity σ and $\boldsymbol{J}_f$ are zero. Consequently, in ferrohydrodynamics (2.1) becomes

$$\boldsymbol{\nabla} \times \boldsymbol{H} = 0, \tag{2.4}$$

and (2.2) remains the same. For a magneto-static system (2.3) is irrelevant. However, in Chapter 5 a rapidly rotating magnetic field is employed and (2.3) should be considered in the governing equations. We address the effects of a time-varying magnetic field here, rather than in Chapter 5. It proves helpful to consider the magnetic induction equation for a small magnetic Reynolds number, namely

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \boldsymbol{\nabla}^2 \boldsymbol{B}. \tag{2.5}$$

Observe from (2.5), for a length scale L and a time scale T of a system, if $\mu_0 \sigma L^2 \ll T$ then the magneto-static limit of Maxwell's equations remains valid in governing the dynamics of the system. In Chapter 5, a time scale of $1/\omega$ is used, where $\omega \gg 1$, and therefore $T \ll 1$. Yet, $\sigma = 0$, resulting in $\omega \mu_0 \sigma L^2 \ll 1$. Consequently, the relevant Maxwell's equations are (2.2) and (2.4), with (2.3) deemed negligible.

(2.4) allows us to define a magnetic potential, ϕ, such that

$$\boldsymbol{H} = \boldsymbol{\nabla} \phi. \tag{2.6}$$

Moreover, $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \mu_0 (\boldsymbol{H} + \boldsymbol{M}), \tag{2.7}$$

where μ_0 is the permeability of free space, and $\boldsymbol{M}$ the magnetisation.

2.3 Magnetisation colinear with applied field

For a sufficiently low concentration of magnetic particles, or if the magnetic relaxation of the particles is slow in comparison with the other processes of interest, it can be assumed that $\boldsymbol{M}$ is colinear with $\boldsymbol{H}$, named quasi-equilibrium theory. Under this approach, it is assumed that

$$\boldsymbol{M} = \chi \boldsymbol{H}, \tag{2.8}$$

where χ is the magnetic susceptibility of the ferrofluid. χ can depend both on position and explicitly on the magnitude of the magnetic field H, that is

$$\chi \equiv \chi(\boldsymbol{x}, H). \tag{2.9}$$

χ may not depend explicitly on H, and therefore $\chi \equiv \chi(\boldsymbol{x})$. However, if H is a function of position, it may be such that χ can be written in terms of H, without explicitly depending on H.

(2.7) can now be written as

$$\boldsymbol{B} = \mu_0(1+\chi)\boldsymbol{H}. \tag{2.10}$$

It follows from (2.2) that

$$\boldsymbol{\nabla}\cdot((1+\chi)\boldsymbol{\nabla}\phi) = 0, \tag{2.11}$$

and for constant χ, ϕ satisfies

$$\boldsymbol{\nabla}^2\phi = 0. \tag{2.12}$$

2.4 Stress tensor for a ferrofluid

Rosensweig (1985) gives the most general form of the magnetic stress tensor for a ferrofluid as

$$\mathbf{T_m} = -\mu_0\left(\int_0^H \left(\frac{\partial(vM)}{\partial v}\right) dH + \frac{1}{2}H^2\right)\mathbf{I} + \boldsymbol{B}\boldsymbol{H}^{\boldsymbol{T}}, \tag{2.13}$$

where $M = |\boldsymbol{M}|$ the magnitude of $\boldsymbol{M}$, v is the specific volume (defined as the reciprocal of the density of the material), $\mathbf{I}$ is the identity matrix, and the partial differentiation is done holding H and temperature constant. Mv represents the magnetic moment per unit mass of the mixture. Observe that for a non-magnetisable material, the first term of (2.13) is zero, and we retrieve Maxwell's stress tensor in the absence of an electric field. $\mathbf{T_m}$ can be written as

$$\mathbf{T_m} = -\mu_0\bigg(\int_0^H \Big(v\frac{\partial M}{\partial v}\Big)dH + \int_0^H MdH + \frac{1}{2}H^2\bigg)\mathbf{I} + \boldsymbol{B}\boldsymbol{H}^{\boldsymbol{T}}. \tag{2.14}$$

The first integral, named the magnetostrictive pressure, can be neglected for incompressible fluids upon assuming no changes in the physical properties or chemical structure of the ferrofluid (Rosensweig, 1985). We do so hereon, giving

$$\mathbf{T_m} = -\mu_0\bigg(\int_0^H MdH + \frac{1}{2}H^2\bigg)\mathbf{I} + \boldsymbol{B}\boldsymbol{H}^{\boldsymbol{T}}. \tag{2.15}$$

The magnetic force density is obtained by

$$\begin{aligned}\boldsymbol{f}_m &= \boldsymbol{\nabla}\cdot\mathbf{T_m}\\ &= -\boldsymbol{\nabla}\bigg(\Big(\mu_0\int_0^H MdH + \frac{1}{2}\mu_0 H^2\Big)\mathbf{I}\bigg) + \boldsymbol{H}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{H},\end{aligned} \tag{2.16}$$

which from (2.2) results in

$$\boldsymbol{f}_m = -\boldsymbol{\nabla}\bigg(\Big(\mu_0\int_0^H MdH + \frac{1}{2}\mu_0 H^2\Big)\mathbf{I}\bigg) + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{H}. \tag{2.17}$$

For colinear field vectors, we can write

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{H} = \frac{B}{H}\boldsymbol{H}\cdot\boldsymbol{\nabla}\boldsymbol{H}, \tag{2.18}$$

where here $B = |\boldsymbol{B}|$, not the Bond number. Since $\boldsymbol{H}$ is curl free,

$$\boldsymbol{H} \cdot \boldsymbol{\nabla} \boldsymbol{H} = \frac{1}{2} \boldsymbol{\nabla} (\boldsymbol{H} \cdot \boldsymbol{H}), \tag{2.19}$$

and therefore,

$$\boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{H} = \frac{B}{H} \frac{1}{2} \boldsymbol{\nabla} H^2 = B \boldsymbol{\nabla} H. \tag{2.20}$$

Consequently, (2.17) simplifies to

$$\boldsymbol{f}_m = -\boldsymbol{\nabla} \left(\left(\mu_0 \int_0^H M dH \right) \mathbf{I} \right) + \mu_0 M \boldsymbol{\nabla} H. \tag{2.21}$$

Using an extension of the Leibniz formula such that

$$\boldsymbol{\nabla} \int_0^H M dH = M \boldsymbol{\nabla} H + \int_0^H \boldsymbol{\nabla}_H M dH, \tag{2.22}$$

where $\boldsymbol{\nabla}_H$ is $\boldsymbol{\nabla}$ performed at constant H, reduces (2.21) to

$$\boldsymbol{f}_m = -\mu_0 \int_0^H \boldsymbol{\nabla}_H M dH. \tag{2.23}$$

Furthermore, assuming (2.8) results in

$$\mathbf{T_m} = -\left(\mu_0 \int_0^H \chi H dH + \frac{1}{2} \mu_0 H^2 \right) \mathbf{I} + \mu_0 (1 + \chi) \boldsymbol{H} \boldsymbol{H}^{\boldsymbol{T}}, \tag{2.24}$$

and

$$\boldsymbol{f_m} = -\mu_0 \int_0^H H \boldsymbol{\nabla}_H \chi dH. \tag{2.25}$$

Note that since $\chi \equiv \chi(\boldsymbol{x}, H)$,

$$\boldsymbol{\nabla} \chi = \boldsymbol{\nabla}_H \chi + \frac{\partial \chi}{\partial H} \boldsymbol{\nabla} H. \tag{2.26}$$

If χ does not depend explicitly on the field, then the integral in (2.25) is performed for constant $\boldsymbol{\nabla}\chi$, to give

$$\boldsymbol{f_m} = -\frac{\mu_0 H^2}{2}\boldsymbol{\nabla}\chi. \tag{2.27}$$

If χ is constant,

$$\boldsymbol{f_m} = 0. \tag{2.28}$$

The stress tensor for an incompressible Newtonian fluid is given by

$$\mathbf{T_f} = \eta\big[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\big] - p\mathbf{I}, \tag{2.29}$$

where η is the viscosity, $\boldsymbol{u}$ is the velocity and p the pressure of the fluid. Consequently, the stress tensor for an incompressible, Newtonian ferrofluid is given by

$$\mathbf{T} = -\mu_0\bigg(\int_0^H MdH + \frac{1}{2}H^2\bigg)\mathbf{I} + \boldsymbol{BH^T} + \eta\big[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\big] - p\mathbf{I}, \tag{2.30}$$

and assuming (2.8),

$$\mathbf{T} = -\mu_0\bigg(\int_0^H \chi HdH + \frac{1}{2}H^2\bigg)\mathbf{I} - p\mathbf{I} + \mu_0(1+\chi)\boldsymbol{HH}^T + \eta(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T). \tag{2.31}$$

The force density is

$$\boldsymbol{f} = \boldsymbol{\nabla}\cdot\mathbf{T} = -\boldsymbol{\nabla}\bigg(\bigg(\mu_0\int_0^H MdH + \frac{1}{2}\mu_0 H^2\bigg)\mathbf{I}\bigg) + \boldsymbol{B}\cdot\boldsymbol{\nabla H} + \eta\nabla^2\boldsymbol{u} - \boldsymbol{\nabla}p, \tag{2.32}$$

and assuming (2.8) is

$$\boldsymbol{f} = -\mu_0\int_0^H H\boldsymbol{\nabla}_H\chi dH + \eta\nabla^2\boldsymbol{u} - \boldsymbol{\nabla}p. \tag{2.33}$$

2.5 Equations of motion

The Navier-Stokes equations for an incompressible fluid are

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{2.34}$$

$$\rho \frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\nabla} \cdot \mathbf{T} + \rho \boldsymbol{g}, \tag{2.35}$$

where ρ is the density of the fluid, $\boldsymbol{g}$ is the local acceleration due to gravity and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \tag{2.36}$$

is the material derivative. We neglect forces due to gravity hereon.

For $\mathbf{T}$ given by (2.31), $\boldsymbol{\nabla} \cdot \mathbf{T}$ is given by (2.25), and (2.35) is

$$\rho \frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla} p = \eta \boldsymbol{\nabla}^2 \boldsymbol{u} - \mu_0 \int_0^H H \boldsymbol{\nabla}_H \chi dH, \tag{2.37}$$

the equation of motion, for an incompressible, isothermal, Newtonian ferrofluid, assuming (2.8) holds.

For the analysis that follows in subsequent chapters, it will be useful to have the vorticity equation. Taking the curl of (2.37) gives

$$\rho \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \eta \boldsymbol{\nabla}^2 \boldsymbol{\omega} + \mu_0 H \boldsymbol{\nabla} \chi \times \boldsymbol{\nabla} H, \tag{2.38}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$. Note that (2.38) is true when $\boldsymbol{f}$ is given by either (2.25) or (2.27). It follows that for a stationary state $\chi \equiv \chi(H)$. Here we write $\chi \equiv \chi(H)$ to mean χ can be written in terms of H. Both H and χ may depend on position so that χ can be written in terms of H, but χ does not need to depend explicitly on H. If χ does depend explicitly on H we will state this.

2.5.1 Non-dimensionalising

For a given system, (2.35) can be non-dimensionalised to give

$$Re\frac{D\boldsymbol{u_*}}{Dt_*} + \boldsymbol{\nabla}_* p_* = \boldsymbol{\nabla}^2_{\boldsymbol{*}}\boldsymbol{u_*} + B\boldsymbol{(f_m)}_*, \tag{2.39}$$

and

$$\boldsymbol{\nabla}_* \cdot \boldsymbol{u_*} = 0, \tag{2.40}$$

where the subscript star denotes the dimensionless variables and operators,

$$Re = \frac{\rho U L}{\eta}, \quad B = \frac{\mu_0 L J^2}{\eta U}, \tag{2.41}$$

J is the chosen scaling for H, and U,L are the velocity and length scales of the system, respectively. The scaling for p is $P = L/(\eta U)$ and the scaling for time $T = L/U$. When $\boldsymbol{f_m}$ is given by (2.25),

$$\boldsymbol{(f_m)}_* = \int_0^{H_*} H_*(\boldsymbol{\nabla}_H)_* \chi dH_*, \tag{2.42}$$

and for $\boldsymbol{f_m}$ given by (2.27),

$$\boldsymbol{(f_m)}_* = \frac{1}{2}H_*^2\boldsymbol{\nabla}_*\chi. \tag{2.43}$$

Taking the curl of (2.39) gives the non-dimensionalised vorticity equation;

$$Re\frac{D\boldsymbol{\omega_*}}{Dt_*} = \boldsymbol{\nabla}^2_{\boldsymbol{*}}\boldsymbol{\omega_*} + BH\boldsymbol{\nabla_*}\chi \times \boldsymbol{\nabla_*}H_*. \tag{2.44}$$

The chosen scalings are determined by the individual problem and are therefore different in Chapters 3-6.

2.6 Magnetic susceptibility as a fluid property

As discussed in Section 1.2.4, there are many interpretations as how to best model the magnetic susceptibility of a ferrofluid. Since we assume the magnetisation is colinear with the applied field, $\boldsymbol{M} = \chi \boldsymbol{H}$, we neglect effects of inter-particle interactions, and assume the magnetic relaxation of the particles is instantaneous. Nevertheless, we allow χ to vary spatially, and depend explicitly on H, thereby allowing for non-linear magnetisation characteristics, and for the application of (1.2) if desired.

Zelazo & Melcher (1969) consider $\chi \equiv \chi(\alpha_i, H)$, where α_i are physical properties of the ferrofluid, and both α_i and H can depend on position. They require

$$\frac{D\alpha_i}{Dt} = 0, \tag{2.45}$$

but not the field dependent parts of χ, such that a fluid parcel retains its physical properties over time scales of interest. For simplicity, we impose

$$\frac{D\chi}{Dt} = 0. \tag{2.46}$$

If χ does not depend explicitly on H, but solely on position, (2.45) and (2.46) are analogous. However, for (2.46) to hold when χ depends explicitly on H, any change in χ due to its dependence on H, must happen on a slower time scale than time scales of interest. Under this assumption, we allow for an initial dependence of χ on H that remains fixed when the parcel is displaced.

2.7 Boundary conditions

At a boundary we require continuity of the normal component of $\boldsymbol{B}$ and continuity of the tangential component of $\boldsymbol{H}$, for which the full derivations can be found in Rosensweig (1985). In terms of $\boldsymbol{H}$,

$$\left[\mu_0(1+\chi)\boldsymbol{H} \cdot \boldsymbol{n}\right] = 0, \tag{2.47}$$

and

$$\left[\boldsymbol{H} \cdot \boldsymbol{\tau}\right] = 0, \tag{2.48}$$

where $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are respectively the unitary normal and tangential vectors to the boundary, and the square brackets denotes the jump across it. Thus ϕ satisfies

$$\big[\mu_0(1+\chi)\boldsymbol{\nabla}\phi\cdot\boldsymbol{n}\big] = 0, \tag{2.49}$$

and

$$\big[\boldsymbol{\nabla}\phi\cdot\boldsymbol{\tau}\big] = 0. \tag{2.50}$$

Furthermore, we require continuity of $\boldsymbol{u}$ in the normal and tangential directions;

$$\Big[\boldsymbol{u}\cdot\boldsymbol{n}\Big] = 0, \tag{2.51}$$

$$\Big[\boldsymbol{u}\cdot\boldsymbol{\tau}\Big] = 0, \tag{2.52}$$

and a normal and tangential stress balance across a boundary;

$$\Big[\boldsymbol{n}\cdot\mathbf{T}\cdot\boldsymbol{n}\Big] = \sigma\boldsymbol{\nabla}\cdot\boldsymbol{n}, \tag{2.53}$$

$$\Big[\boldsymbol{n}\cdot\mathbf{T}\cdot\boldsymbol{\tau}\Big] = 0, \tag{2.54}$$

where σ is the surface tension at the interface, $\mathbf{T}$ is given by (2.30), or (2.31) under the assumption of (2.8).

2.8 Summary

From here on we assume an isothermal, Newtonian flow, and assume the magnetisation and field are colinear. We allow the susceptibility to depend on H and position, but to be a fluid property. Under these assumptions, the dimensional governing equations are

$$\boldsymbol{\nabla}\cdot((1+\chi)\boldsymbol{\nabla}\phi) = 0, \tag{2.55}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{2.56}$$

$$\rho \frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla} p = \eta \boldsymbol{\nabla}^2 \boldsymbol{u} - \mu_0 \int_0^H H \boldsymbol{\nabla}_H \chi dH, \tag{2.57}$$

$$\rho \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \eta \boldsymbol{\nabla}^2 \boldsymbol{\omega} + \mu_0 H \boldsymbol{\nabla} \chi \times \boldsymbol{\nabla} H. \tag{2.58}$$

At a boundary

$$\left[\mu_0(1+\chi)\boldsymbol{\nabla}\phi \cdot \boldsymbol{n}\right] = 0, \tag{2.59}$$

$$\left[\boldsymbol{\nabla}\phi \cdot \boldsymbol{\tau}\right] = 0, \tag{2.60}$$

$$\left[\boldsymbol{u} \cdot \boldsymbol{n}\right] = 0, \tag{2.61}$$

$$\left[\boldsymbol{u} \cdot \boldsymbol{\tau}\right] = 0, \tag{2.62}$$

$$\left[\boldsymbol{n} \cdot \mathbf{T} \cdot \boldsymbol{n}\right] = \sigma \boldsymbol{\nabla} \cdot \boldsymbol{n}, \tag{2.63}$$

$$\left[\boldsymbol{n} \cdot \mathbf{T} \cdot \boldsymbol{\tau}\right] = 0, \tag{2.64}$$

where

$$\boldsymbol{T} = -\mu_0 \left(\int_0^H \chi H dH + \frac{1}{2} H^2 \right) \mathbf{I} - p\mathbf{I} + \mu_0(1+\chi)\boldsymbol{H}\boldsymbol{H}^{\boldsymbol{T}} + \eta(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathbf{T}}), \tag{2.65}$$

and finally,

$$\frac{D\chi}{Dt} = 0. \tag{2.66}$$

Chapter 3

Cylindrical column of ferrofluid, centred on a current-carrying wire, surrounded by another ferrofluid

3.1 Introduction

The work in this Chapter appears in Ferguson Briggs & Mestel (2022a), investigating the linear stability of a column of ferrofluid, with constant magnetic susceptibility, centred on a current-carrying wire, surrounded by a ferrofluid of a different magnetic susceptibility. Three-dimensional disturbances are considered and an analytic solution to the linearised governing equations is given. The corresponding growth rate is analysed for arbitrary Reynolds number, and a stability condition is obtained for the inviscid and Stokes regimes. Moreover, it is shown unstable modes can be dampened by an axial field.

3.2 Formulation

We consider a ferrofluid (fluid 1) with $\chi = \chi^{(1)}$, centred on a rigid conducting wire of radius a, surrounded by another ferrofluid (fluid 2) with $\chi = \chi^{(2)}$, where $\chi^{(1)}, \chi^{(2)}$ are constant. Both fluids have constant density ρ and viscosity η. We choose the cylindrical system (r, θ, z), such that r and θ are the radial and azimuthal co-ordinates, and z points along the wire. Fluid 1 occupies the region $a < r < R$ in the stationary state, and fluid 2 is unbounded, such that

$$\left.\begin{aligned} &\chi = 0 \quad \text{for} \quad r \leq a, \\ &\chi = \chi^{(1)} \quad \text{for} \quad a < r < R, \\ &\chi = \chi^{(2)} \quad \text{for} \quad r \geq R. \end{aligned}\right\} \tag{3.1}$$

A steady electric current runs through the wire, $\boldsymbol{J} = J_0 \boldsymbol{e_z}$ and produces an azimuthal magnetic field $\boldsymbol{H} = J_0/2\pi r \boldsymbol{e_\theta}$, satisfying (2.1), where $\boldsymbol{e_z}$ is the unit vector in the z direction and $\boldsymbol{e_\theta}$ the unit vector in the anti-clockwise, azimuthal direction. The set up is shown in Figure 3.1.

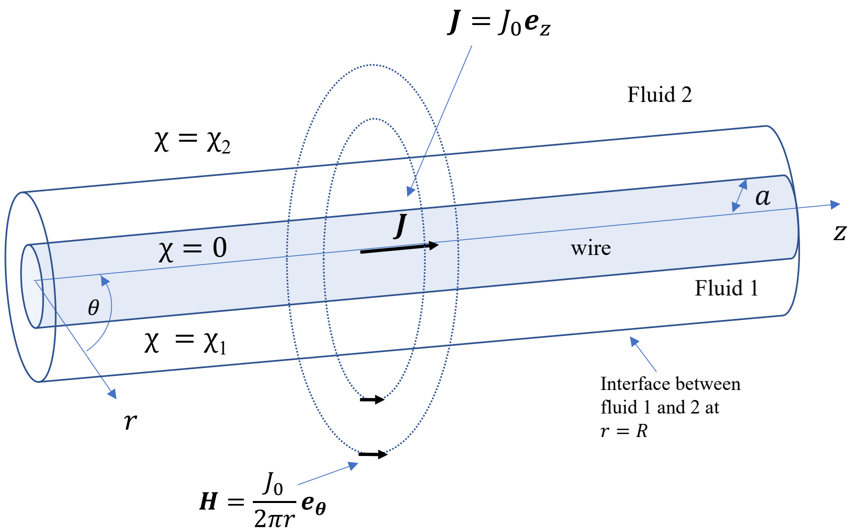

Figure 3.1: Schematic of the two-fluid system

Since χ and the pressure are piecewise constant in a stationary state, it follows that $\boldsymbol{f_m} = \boldsymbol{0}$ in (2.57). The magnetic forcing acts as a magnetic stress confined to the interface of the two fluids, which at rest is located at $r = R$. Take R as the length scale, and define

$a = a_* R$. We non-dimensionalise pressure as $p = \sigma p_*/R$, where σ is the surface tension, the field as $\boldsymbol{H} = J_0 \boldsymbol{H}_*/2\pi R$, and pick the time scale $T = \eta R/\sigma$, such that the velocity is non-dimensionalised as $\boldsymbol{u} = (\sigma/\eta)\boldsymbol{u}_*$. Consequently, (2.39) is

$$Re\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla} p = \nabla^2 \boldsymbol{u}, \tag{3.2}$$

where

$$Re = \frac{\rho\sigma R}{\eta^2} \tag{3.3}$$

and the subscripts have been dropped. The equilibrium is given by $p = p_0^{(\iota)}$, $\boldsymbol{u}^{(\iota)} = \boldsymbol{0}$, $\phi^{(\iota)} = \theta$, where $\iota = 0, 1, 2$ for the wire, inner and outer fluid respectively. In the next section, three-dimensional disturbances are considered to the stationary state and linear stability analysis is performed.

3.3 Stability Analysis

We consider perturbations to the equilibrium such that the interface is located at $r = S(\theta, z, t)$ and

$$S = 1 + \epsilon \mathcal{R}\big(\hat{S}\zeta\big), \tag{3.4}$$

where $\zeta = e^{\mathrm{i}(kz+m\theta)+st}$, $\epsilon \ll 1$, k, m are real and positive wave numbers, $\hat{S}$ may be a complex constant (or it could be unity) and s is the growth rate of the disturbance and could be complex. In (3.4) the real part of the perturbation is taken, and this is done here on (and in subsequent chapters) for the other variables, but it is not written explicitly. A schematic of the perturbed system is shown in Figure 3.2. Neglecting terms of $O(\epsilon^2)$, the normal vector to the interface becomes

$$\boldsymbol{n} = \left(1, -\frac{\epsilon \mathrm{i} m \hat{S}\zeta}{r}, -\epsilon \mathrm{i} k \hat{S}\zeta\right)^T, \tag{3.5}$$

and the tangential vectors,

$$\boldsymbol{\tau_1} = \left(\epsilon \mathrm{i} k \hat{S}\zeta, 0, 1\right)^T, \quad \boldsymbol{\tau_2} = \left(\frac{\epsilon \mathrm{i} m \hat{S}\zeta}{r}, 1, 0\right)^T. \tag{3.6}$$

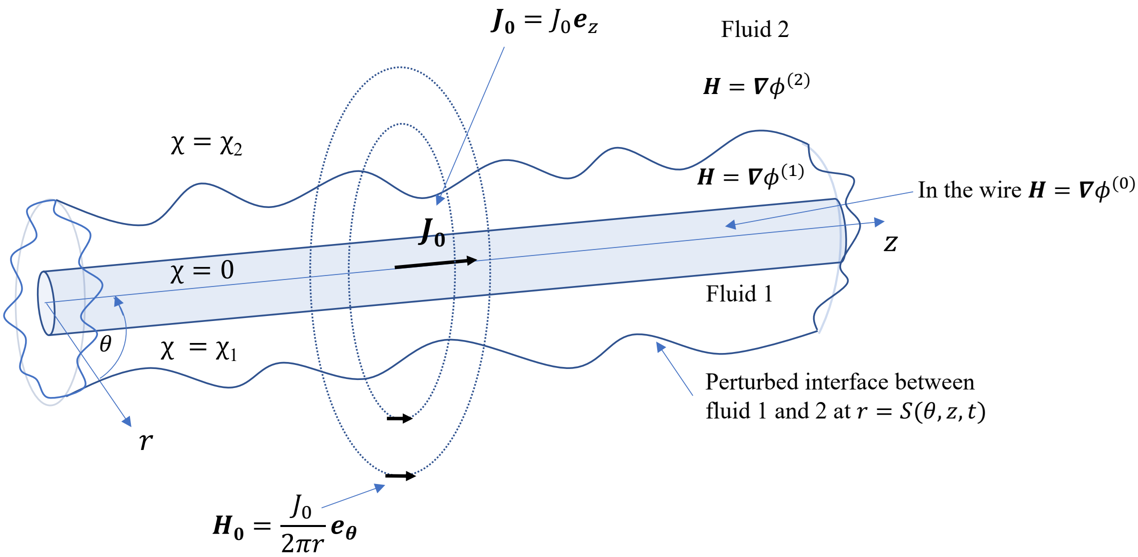

Figure 3.2: Schematic of the perturbed two-fluid system

We perturb the magnetic potential such that

$$\phi^{(\iota)} = \theta + \epsilon\hat{\phi}^{(\iota)}(r)\zeta, \tag{3.7}$$

and (2.55) gives

$$r^2\hat{\phi''}^{(\iota)} + \hat{\phi'}^{(\iota)} - (m^2 + k^2r^2)\hat{\phi}^{(\iota)} = 0, \tag{3.8}$$

where $'$ denotes the first derivative with respect to r, and (3.8) has general solution

$$\hat{\phi}(r)^{(\iota)} = q_1^{(\iota)}\mathrm{I}_m(kr) + q_2^{(\iota)}\mathrm{K}_m(kr), \tag{3.9}$$

for constants $q_1^{(\iota)}, q_2^{(\iota)}$, where $\mathrm{I}_n(\Phi)$ and $\mathrm{K}_n(\Phi)$ are the modified Bessel functions of the first and second kind, respectively, with argument Φ and order n. For $\hat{\phi}^{(0)}$ regular at $r = 0$,

$$\hat{\phi}^{(0)} = q_1^{(0)}\mathrm{I}_m(kr), \tag{3.10}$$

and imposing $\phi^{(2)} \to 0$, as $r \to \infty$ gives

$$\hat{\phi}^{(2)} = q_2^{(2)}\mathrm{K}_m(kr). \tag{3.11}$$

(2.59) and (2.60) give

$$\big(1+\chi^{(2)}\big)\hat{\phi}'^{(2)} - \big(1+\chi^{(1)}\big)\hat{\phi}'^{(1)} + \mathrm{i}m(\chi^{(1)}-\chi^{(2)})\hat{S} = 0, \tag{3.12}$$

$$\hat{\phi}^{(1)} = \hat{\phi}^{(2)} \tag{3.13}$$

at $r = 1$, and

$$\big(1+\chi^{(1)}\big)\hat{\phi}'^{(1)} - \hat{\phi}'^{(0)} = 0, \tag{3.14}$$

$$\hat{\phi}^{(1)} = \hat{\phi}^{(0)} \tag{3.15}$$

at $r = a$, determining the constants $q_1^{(0)}, q_1^{(1)}, q_2^{(1)}, q_2^{(2)}$, given in the Appendix (A.1)-(A.4).

The pressure $p^{(\iota)}$ and velocity $\boldsymbol{u}^{(\iota)}$, for $\iota = 1, 2$ after the perturbation are

$$p^{(\iota)} = p_0 + \epsilon\hat{p}^{(\iota)}(r)\zeta \quad \text{and} \quad \boldsymbol{u}^{(\iota)} = \epsilon\hat{\boldsymbol{u}}^{(\iota)}(r)\zeta, \tag{3.16}$$

and satisfy the linearised equations (2.56) and (3.2);

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}^{(\boldsymbol{\iota})} = 0 \qquad \text{and} \qquad sRe\hat{\boldsymbol{u}}^{(\iota)} + \boldsymbol{\nabla}\hat{p}^{(\iota)} = \boldsymbol{\nabla}^2\hat{\boldsymbol{u}}^{(\iota)}. \tag{3.17}$$

In component form we have

$$(r\hat{u}^{(\iota)})' + \mathrm{i}m\hat{v}^{(\iota)} + \mathrm{i}kr\hat{w}^{(\iota)} = 0, \tag{3.18}$$

$$r^2(\hat{p}^{(\iota)})' = -2\mathrm{i}m\hat{v}^{(\iota)} - \hat{u}^{(\iota)} + r^2(\hat{u}^{(\iota)})'' + r(\hat{u}^{(\iota)})' - \big(m^2+\bar{k}^2r^2\big)\hat{u}^{(\iota)}, \tag{3.19}$$

$$\mathrm{i}mr\hat{p}^{(\iota)} = -\hat{v} + 2\mathrm{i}m\hat{u}^{(\iota)} + r^2(\hat{v}^{(\iota)})'' + r(\hat{v}^{(\iota)})' - \big(m^2+\bar{k}^2r^2\big)\hat{v}^{(\iota)}, \tag{3.20}$$

$$\mathrm{i}kr^2\hat{p}^{(\iota)} = r^2(\hat{w}^{(\iota)})'' + r(\hat{w}^{(\iota)})' - \big(m^2+\bar{k}^2r^2\big)\hat{w}^{(\iota)}, \tag{3.21}$$

where $\bar{k} = \sqrt{k^2 + sRe}$. The general solution of (3.17) is modified from Saville (1971) and Mestel (1966) to account for the inner wire at $r = a$ and is given by

$$\hat{p}^{(\iota)} = c_1^{(\iota)}\mathrm{I}_m + c_2^{(\iota)}\mathrm{K}_m, \tag{3.22}$$

$$\hat{u}^{(\iota)} = -\frac{1}{(sRe)^2 r}\left(c_1^{(\iota)}(kr\mathrm{I}_{m+1} + m\mathrm{I}_m) + c_2^{(\iota)}(m\mathrm{K}_m - kr\mathrm{K}_{m+1})\right) - \frac{\mathrm{i}k}{\bar{k}}\left(c_3^{(\iota)}\bar{\mathrm{I}}_{m+1} + c_4^{(\iota)}\bar{\mathrm{K}}_{m+1}\right)$$
$$+ \frac{2m}{\bar{k}r}\left(c_5^{(\iota)}\bar{\mathrm{I}}_m + c_6^{(\iota)}\bar{\mathrm{K}}_m\right), \tag{3.23}$$

$$\hat{v}^{(\iota)} = \left(\frac{-c_3^{(\iota)}k}{\bar{k}} + 2\mathrm{i}c_5^{(\iota)}\right)\bar{\mathrm{I}}_{m+1} + \frac{2\mathrm{i}m}{\bar{k}r}\left(c_5^{(\iota)}\bar{\mathrm{I}}_m + c_6^{(\iota)}\bar{\mathrm{K}}_m\right) - \left(\frac{c_4^{(\iota)}k}{\bar{k}} + 2\mathrm{i}c_6^{(\iota)}\right)\bar{\mathrm{K}}_{m+1}$$
$$- \frac{\mathrm{i}m}{(sRe)^2 r}\left(c_2^{(\iota)}\mathrm{K}_m + c_1^{(\iota)}\mathrm{I}_m\right), \tag{3.24}$$

$$\hat{w}^{(\iota)} = \frac{-\mathrm{i}k}{(sRe)^2}\left(c_1^{(\iota)}\mathrm{I}_m + c_2^{(\iota)}\mathrm{K}_m\right) + c_3^{(\iota)}\bar{\mathrm{I}}_m - c_4^{(\iota)}\bar{\mathrm{K}}_m, \tag{3.25}$$

for constants $c_1^{(\iota)}...c_6^{(\iota)}$. We write $\mathrm{I}_n, \mathrm{K}_n$ when $\Phi = kr$ and $\bar{\mathrm{I}}_n, \bar{\mathrm{K}}_n$ when $\Phi = \bar{k}r$, for the modified Bessel functions $\mathrm{I}_n(\Phi), \mathrm{K}_n(\Phi)$, but give the argument otherwise. To satisfy $u^{(2)} \to 0$ as $r \to \infty$, $\mathbb{R}(\bar{k}) > 0$ and $c_1^{(2)} = c_3^{(2)} = c_5^{(2)} = 0$. The constants $c_1^{(1)}...c_6^{(1)}$, $c_2^{(2)}, c_4^{(2)}$ and $c_6^{(2)}$ are determined by (2.61)-(2.64). Firstly,

$$\hat{\boldsymbol{u}}^{(1)} = \hat{\boldsymbol{u}}^{(2)}, \tag{3.26}$$

at $r = 1$ and

$$\hat{\boldsymbol{u}}^{(1)} = 0 \tag{3.27}$$

at $r = a$. For piecewise constant χ,

$$\mathbf{T}^{(\iota)} = -\left(p^{(\iota)} + \frac{\mu_0}{2}(1+\chi^{(\iota)})(H^{(\iota)})^2\right)\mathbf{I} + \mu_0(1+\chi^{(\iota)})\boldsymbol{H}^{(\iota)}(\boldsymbol{H}^{(\iota)})^T + \eta^{(\iota)}(\boldsymbol{\nabla u}^{(\iota)} + (\boldsymbol{\nabla u}^{(\iota)})^T) \tag{3.28}$$

and substituting (3.28) into (2.63), non-dimensionalising, substituting the perturbed variables and linearising, we obtain

$$\left(m^2 + k^2 - 1 + B(\chi^{(1)} - \chi^{(2)})\right)\hat{S} = \mathrm{i}mB\left(\chi^{(1)} - \chi^{(2)}\right)\hat{\phi}^{(1)} + \hat{p}^{(1)} - \hat{p}^{(2)} - 2\hat{u}'^{(1)} + 2\hat{u}'^{(2)} \tag{3.29}$$

at $r = 1$, where

$$B = \frac{\mu_0 J_0^2}{4\pi^2 \sigma R}, \tag{3.30}$$

the magnetic Bond number. Here, B measures the ratio of magnetic forcing to capillary forcing. B appears in (3.29) after non-dimensionalising (2.63) with the chosen scalings of the problem given in Section 3.2. Moreover, substituting the chosen scalings into (2.41) also produces (3.30). Similarly, (2.64) gives

$$\hat{v}'^{(1)} - \hat{v}'^{(2)} = 0 \tag{3.31}$$

and

$$\hat{w}'^{(1)} - \hat{w}'^{(2)} = 0 \tag{3.32}$$

at $r = 1$. The full derivation for (3.29), (3.31) and (3.32) are given in the Appendix A.2 and A.3. (3.26), (3.27), (3.29)-(3.32) determine the constants $c_1^{(\iota)} ... c_6^{(\iota)}$, given in the Appendix (A.40)-(A.48).

The growth rate appears in a kinematic condition, namely

$$\frac{\partial S}{\partial t} + (\boldsymbol{u}^{(\iota)} \cdot \boldsymbol{\nabla})(r - S) = 0 \tag{3.33}$$

at $r = S$. Substituting the perturbed variables into (3.33) and linearising gives

$$s\hat{S} - \hat{u}^{(\iota)} = 0. \tag{3.34}$$

at $r = 1$. Consequently, substituting $\hat{u}^{(\iota)}$ into (3.34) we obtain

$$s = -gF\bigg(f_1\big(k^2 + m^2 - 1 + B(\chi^{(1)} - \chi^{(2)})\big) + f_2 B(\chi^{(1)} - \chi^{(2)})^2 m^2 \bigg), \tag{3.35}$$

where $g, f_1, f_2 > 0$. g, f_1, f_2 are functions of $m, k, a, \chi^{(1)}, \chi^{(2)}$, and F is a function of $\bar{k}, m, k, a, \chi^{(1)}, \chi^{(2)}$, all given in the Appendix (A.37)-(A.39). Since $\bar{k}$ is a function of s, F is also a function of s and therefore (3.35) is an implicit relation which must be solved numerically for an arbitrary Reynolds number. These results are outlined in Section 3.3.2. However, in the highly viscous and inviscid limits, the growth rate is given explicitly in 3.3.1 and an exact stability condition is obtained.

3.3.1 The highly viscous and inviscid limits

In the highly viscous limit, $Re \to 0$, $F \to F_v$ where F_v is a function no longer depending on s and $F_v > 0$, given in the Appendix (A.49). Consequently, we obtain the growth rate in the highly viscous regime, s_v, by taking the limit $Re \to 0$ in (3.35) to give

$$s_v = -gF_v\bigg(f_1\big(k^2 + m^2 - 1 + B(\chi^{(1)} - \chi^{(2)})\big) + f_2 B(\chi^{(1)} - \chi^{(2)})^2 m^2 \bigg). \tag{3.36}$$

In the inviscid limit, $\eta \to 0$, and a more appropriate scaling for time, T_I, is $T_I = \sqrt{R^3\rho/\sigma}$. Since $T_I = \sqrt{Re}T$ we substitute $s = s_I/\sqrt{Re}$ into (3.35), where s_I is the inviscid growth rate. Taking the limit as $Re \to \infty$ gives $F \to F_I/s_I$, where F_I is a function no longer depending on the growth rate and $F_I > 0$, given in the Appendix (A.50). We obtain

$$s_I^2 = -gf_I\bigg(f_1\big(k^2 + m^2 - 1 + B(\chi^{(1)} - \chi^{(2)})\big) + f_2 B(\chi^{(1)} - \chi^{(2)})^2 m^2 \bigg). \tag{3.37}$$

More simply, (3.37) can be obtained by taking the limit $\eta \to 0$ in the governing equations from the outset, and applying the boundary conditions for an inviscid system; (3.29) and (3.32) with $\eta = 0$ and

$$[\hat{u}] = 0 \tag{3.38}$$

at the wire and interface.

Since (3.36) and (3.37) are explicit expressions for the growth rate and $g, f_I, f_v > 0$, the system is stable (or neutrally stable), in both the inviscid and highly viscous regimes, if and only if,

$$f_1\big(k^2 + m^2 - 1 + B(\chi^{(1)} - \chi^{(2)})\big) + f_2 B(\chi^{(1)} - \chi^{(2)})^2 m^2 \geq 0. \tag{3.39}$$

Note that in the inviscid regime, if (3.39) holds, the system is neutrally stable as the growth rate is imaginary.

If the inner fluid has a higher susceptibility than the outer fluid, any instability is a result of capillary forces only. Solely axisymmetric modes can be unstable when $k < 1$, and increasing the current in the wire will stabilise the system provided $B(\chi^{(1)} - \chi^{(2)}) > 1$. Figures 3.3 and 3.4 show the growth rate of the modes being dampened as the current in the wire is increased for the viscous and inviscid regime, respectively. It is important to note that when comparing the inviscid regime with the viscous regime, s_I and s_v are on different time scales. Consequently, for $\chi^{(1)} > \chi^{(2)}$, the axisymmetric modes are the "most" (and only) unstable modes, supporting the assumptions made by previous works, where only axisymmetric disturbances are considered, on the grounds that they would be the most unstable.

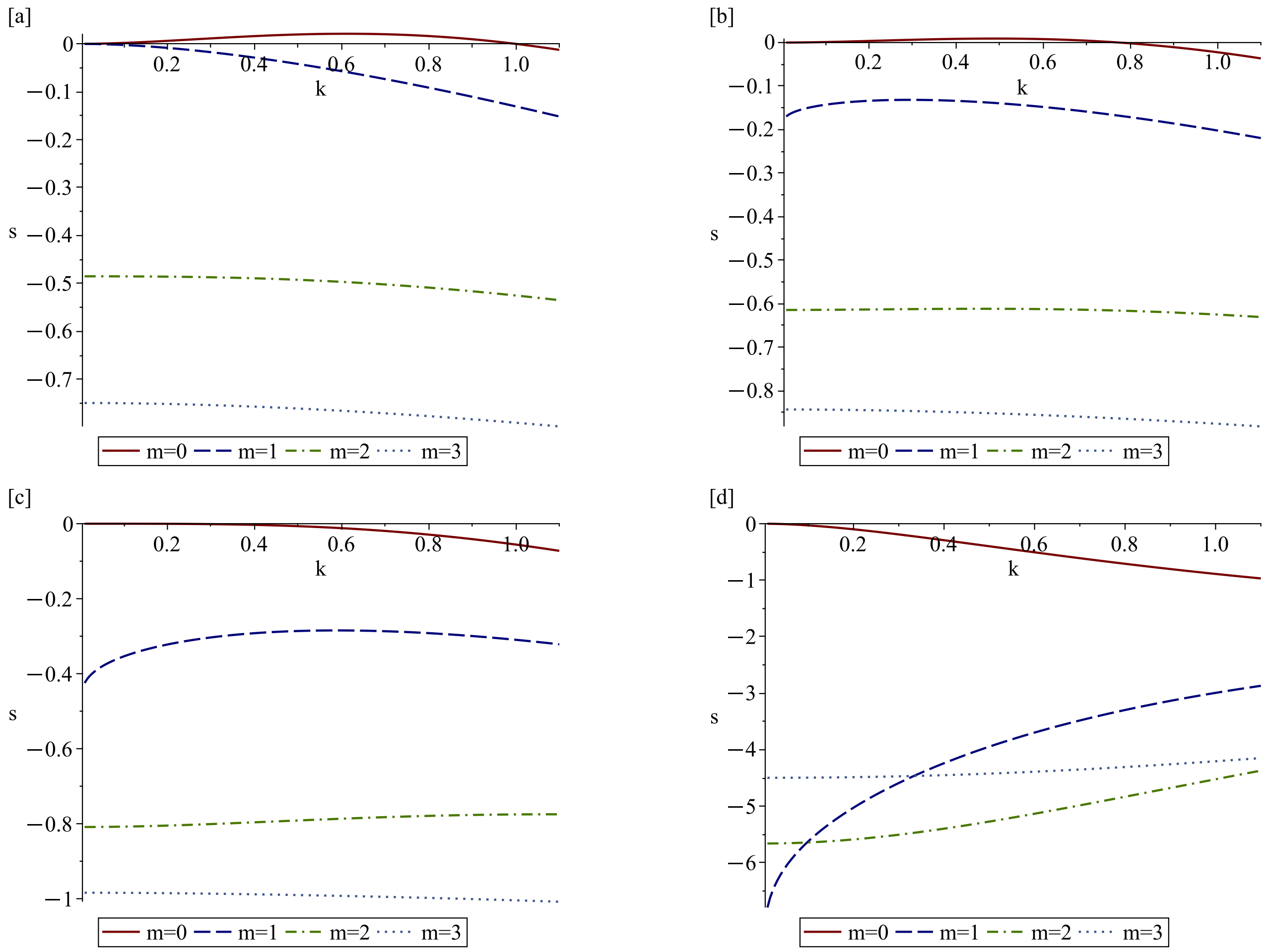

Figure 3.3: Viscous growth rate plotted when $a = 0.1$, $\chi_1 = 5$, $\chi_2 = 1$, [a] $B = 0$, [b] $B = 0.1$, [c] $B = 0.25$, [d] $B = 4$.

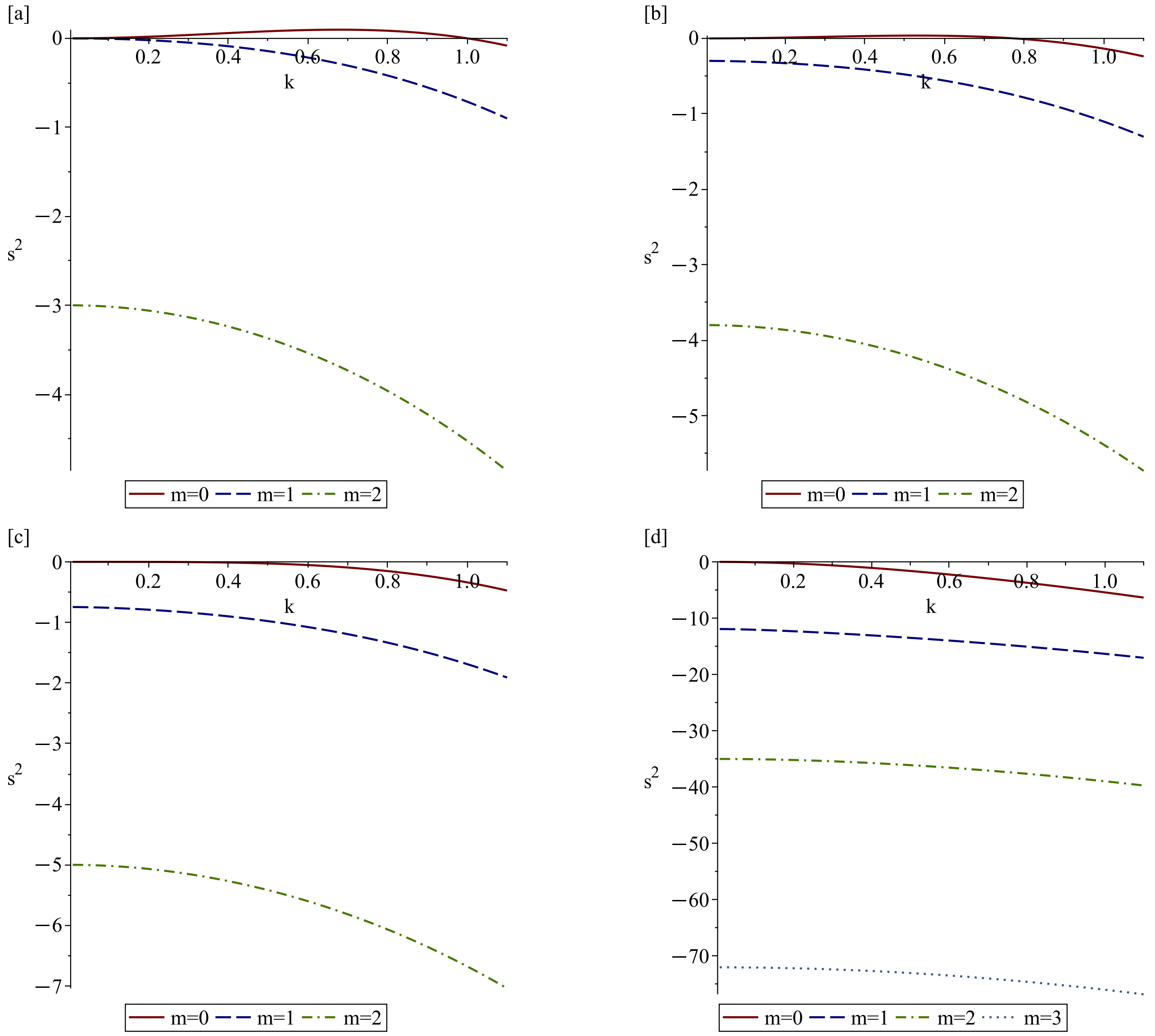

Figure 3.4: Inviscid growth rate plotted when $a = 0.1$, $\chi_1 = 5$, $\chi_2 = 1$, [a] $B = 0$, [b] $B = 0.1$, [c] $B = 0.25$, [d] $B = 4$.

On the other hand, when the outer fluid has a higher susceptibility, both capillary and magnetic forces may be destabilising. Increasing the current increases the magnetic forcing at the interface, and non-axisymmetric modes can be rendered unstable, as well as axisymmetric modes. In fact, for sufficiently large B all modes m can be rendered unstable. Figures 3.5, 3.6 and 3.7 show increasing B not only increases the magnitude of the growth rate of unstable modes, but also results in an increase in unstable modes.

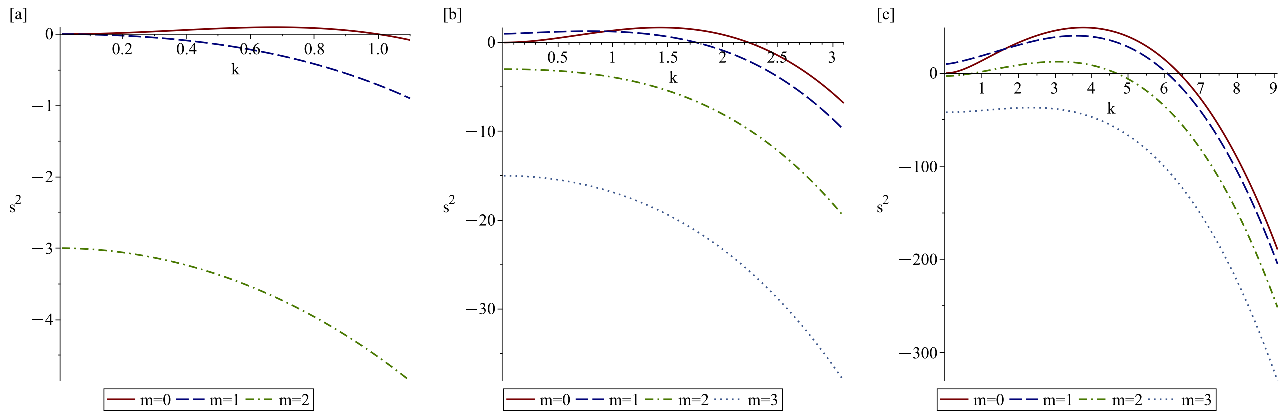

Figure 3.5: Inviscid growth rate plotted when $a = 0.1$, $\chi_1 = 1$, $\chi_2 = 5$, [a] $B = 0$, [b] $B = 1$, [c] $B = 10$.

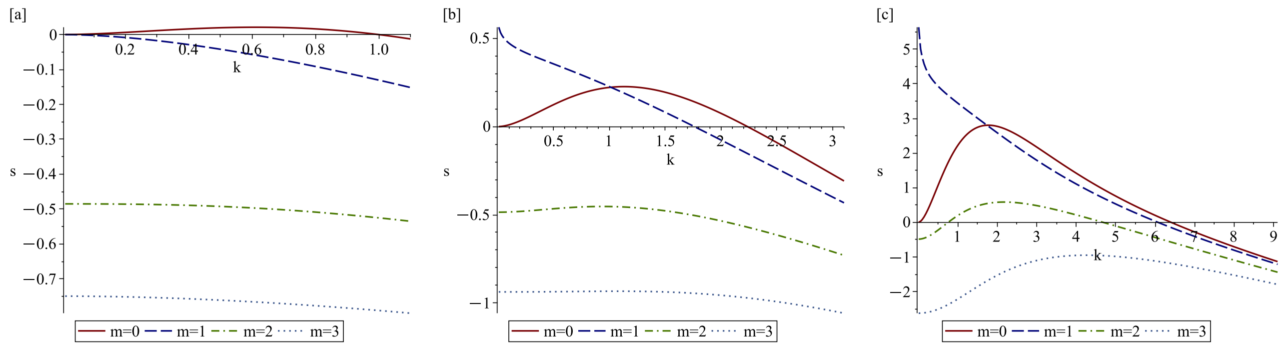

Figure 3.6: Viscous growth rate plotted when $a = 0.1$, $\chi_1 = 1$, $\chi_2 = 5$, [a] $B = 0$, [b] $B = 1$, [c] $B = 10$.

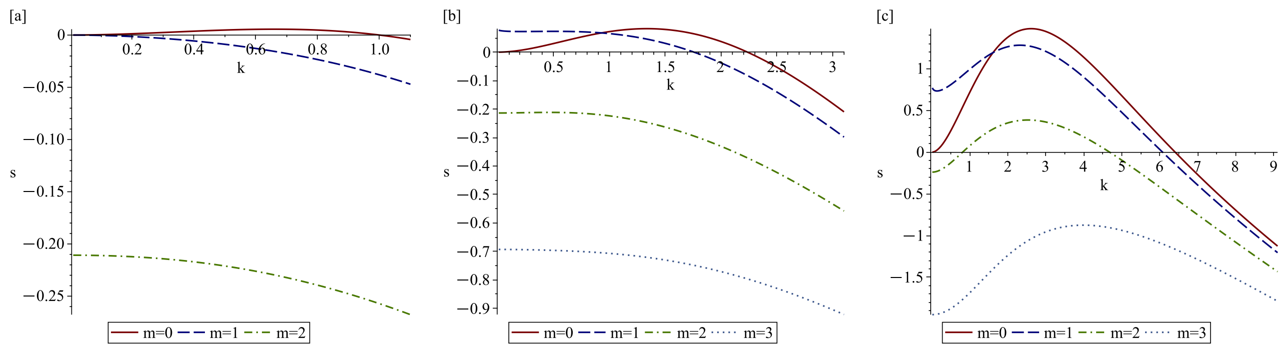

Figure 3.7: Viscous growth rate plotted when $a = 0.5$, $\chi_1 = 1$, $\chi_2 = 5$, [a] $B = 0$, [b] $B = 1$, [c] $B = 10$.

Figures 3.8-3.11 show that increasing a, decreases the magnitude of the growth rate of both unstable and stable modes, irrespective of whether $\chi^{(1)} > \chi^{(2)}$ or $\chi^{(2)} > \chi^{(1)}$, and this is true for all B. Thus, the smaller the ratio between the radius of the wire and the the radius of the inner fluid, the greater the magnitude of the growth rate. Moreover, when $\chi^{(2)} > \chi^{(1)}$, in the highly viscous regime, Figures 3.6 and 3.11 show that for $a = 0.1$, the most unstable mode is $m = 1$, $k \to 0$, yet when a is increased to $a = 0.5$ in Figure 3.7, $m = 0$ is the most unstable mode. Figure 3.6 shows increasing the current in the wire, increases the magnitude of the growth rate, and in fact $s \to \infty$ as $k \to 0$ for $a = 0.1$. On the other hand, in the inviscid regime, Figure 3.5 shows that although $m = 1$ is the most unstable mode when $k \to 0$ and $a = 0.1$, there exists a more unstable axisymmetric mode for other values of k. Performing a series expansion on the viscous growth rate as $a, k \to 0$, we find $s \sim -\ln(a)$ when $m = 1$, and thus $s \to \infty$ in the limit, but s converges to a constant when $m = 0$ or $m > 1$, a result seen in the context of electro-hydrodynamics too (Saville, 1971), (Mestel, 1966).

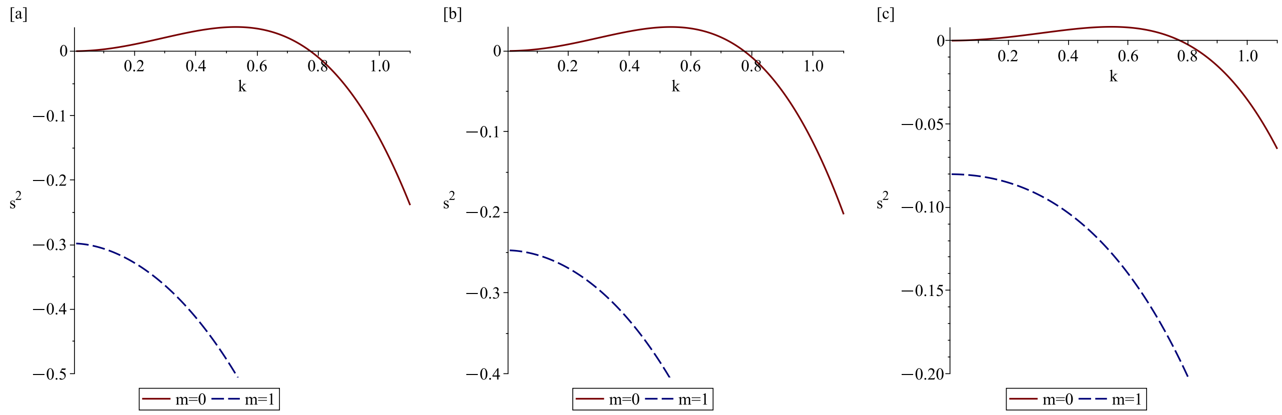

Figure 3.8: Inviscid growth rate plotted when $B = 0.1$, $\chi_1 = 5$, $\chi_2 = 1$, [a] $a = 0.1$, [b] $a = 0.5$, [c] $a = 0.9$.

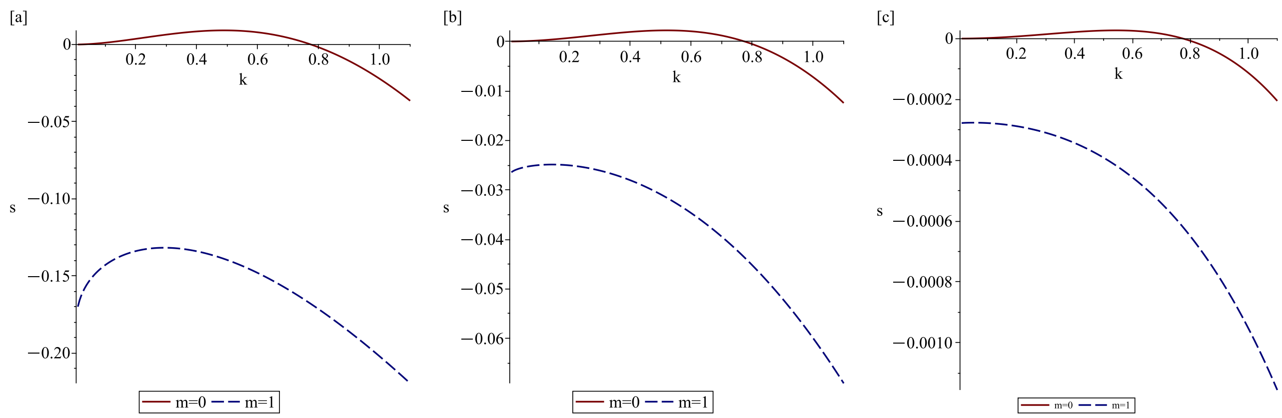

Figure 3.9: Viscous growth rate plotted when $B = 0.1$, $\chi_1 = 5$, $\chi_2 = 1$, [a] $a = 0.1$, [b] $a = 0.5$, [c] $a = 0.9$.

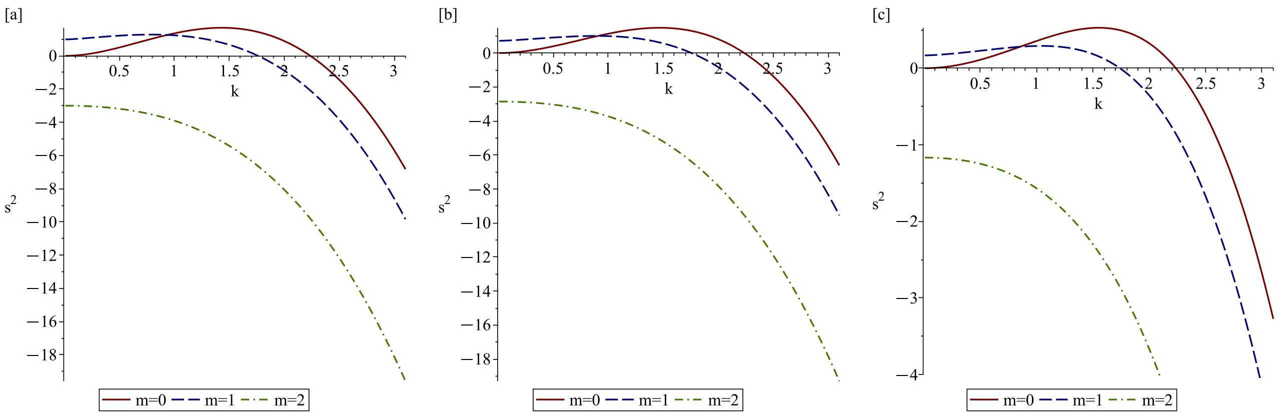

Figure 3.10: Inviscid growth rate plotted when $B = 1$, $\chi_1 = 1$, $\chi_2 = 5$, [a] $a = 0.1$, [b] $a = 0.5$, [c] $a = 0.9$.

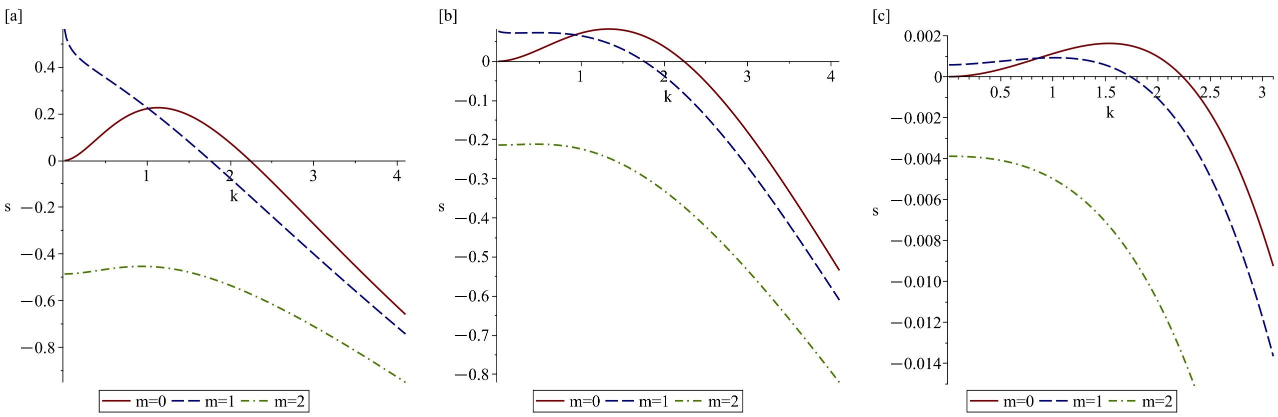

Figure 3.11: Viscous growth rate plotted when $B = 1$, $\chi_1 = 1$, $\chi_2 = 5$, [a] $a = 0.1$, [b] $a = 0.5$, [c] $a = 0.9$.

3.3.2 Arbitrary Reynolds number

We use a root solver in Maple on (3.35) for specific values of $k, a, m, \chi^{(1)}, \chi^{(2)}, B$ and Re, to find the associated growth rate of the mode. We find the stability condition (3.39) appears to hold for all Reynolds numbers.

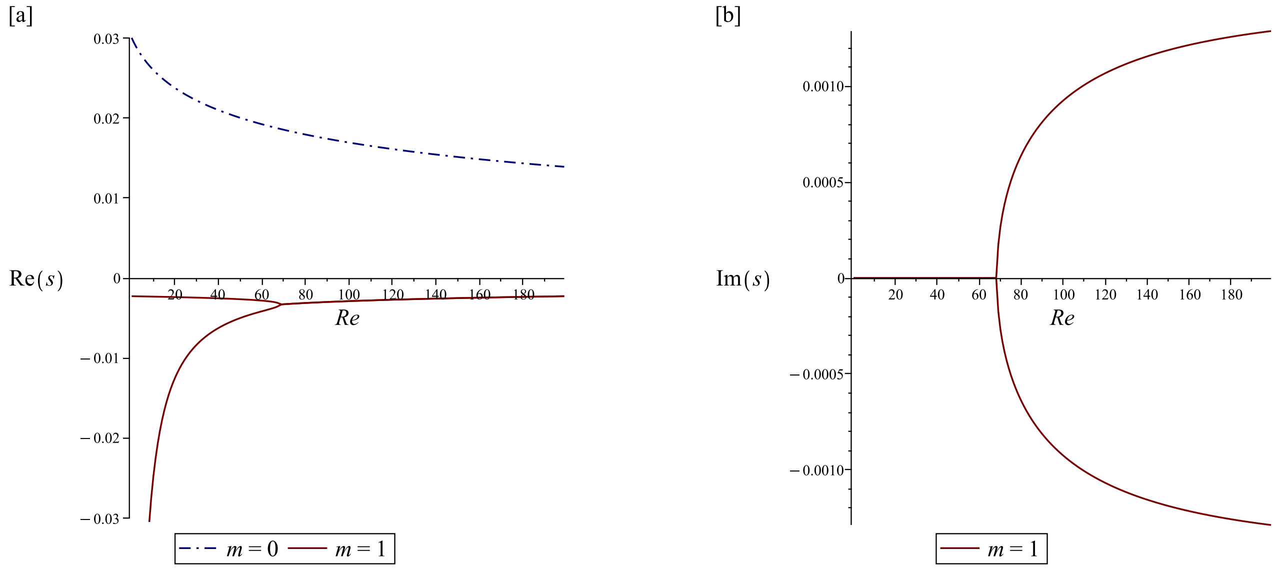

Figure 3.12: Growth rate plotted when $a = 0.1$, $k = 0.5$, $\chi_1 = 1$, $\chi_2 = 5$ and $B = 0.1$ for arbitrary Re. [a] and [b] are, respectively, the real and imaginary parts of s. The $m = 0$ branch is solely real, but the $m = 1$ branch starts as real for low Re and then becomes a complex conjugate pair.

Figure 3.12 is the growth rate plotted when $a = 0.1$, $k = 0.5$, $\chi^{(1)} = 1$, $\chi^{(2)} = 5$, $B = 0.1$, for a range of Reynolds numbers, showing two stable branches when $m = 1$ and an unstable branch for $m = 0$. Given a stable mode, as $Re \to 0$, there are two branches, both real, where one branch tends to s_v, and the other tends to $-\infty$, the latter a result of $Re \to 0$ for the chosen time scale. As Re is increased the two branches meet and then split, becoming complex conjugates of each other, tending towards $\pm s_I$ as $Re \to \infty$, where s_I is solely imaginary for a stable mode. Given an unstable mode, we get one branch, starting at s_v and tending to $|s_I|$. There exists a branch which tends to the negative inviscid root, $-|s_I|$, but this is invalid for finite Re, since the boundary conditions require $\mathcal{R}(\bar{k}) \geq 0$. Figure 3.12 is an example of this, but this behaviour of the stable and unstable modes happens for all other values of $a, k, \chi_1, \chi_2, B, m$ considered.

When $\chi^{(1)} > \chi^{(2)}$, only axisymmetric modes are unstable, and we find that increasing the current in the wire stabilises the system for arbitrary Reynolds number too. This is shown in Figure 3.13 where the current is increased from $B = 0.1$ to $B = 0.5$. Note that when $B = 0.5$, the branch is stable, and we see the bifurcation at Re ~ 10, where one branch will tend to $-\infty$ as Re $\to 0$, but we do not include this on the graph. Figures 3.12 and 3.14, where $\chi^{(2)} > \chi^{(1)}$, show increasing B from $B = 0.1$ to $B = 0.5$ renders the mode $m = 1$ unstable. We find for all Reynolds numbers, increasing the current does not stabilise the system if $\chi^{(2)} > \chi^{(1)}$, but renders more modes unstable. Consequently non-axisymmetric modes can be unstable, as well as axisymmetric modes. When $\chi^{(2)} > \chi^{(1)}$, and a, k are sufficiently small, $m = 1$ modes are more unstable than axisymmetric modes. Yet, for all k, $m = 1$ is only the most unstable mode for sufficiently small Re. This is shown in Figure 3.15 where the growth rate is plotted against k, for different Reynolds numbers; when $Re = 0.001, 0.1$, $m = 1$ is the most unstable mode, but this not the case for the other Reynolds numbers shown.

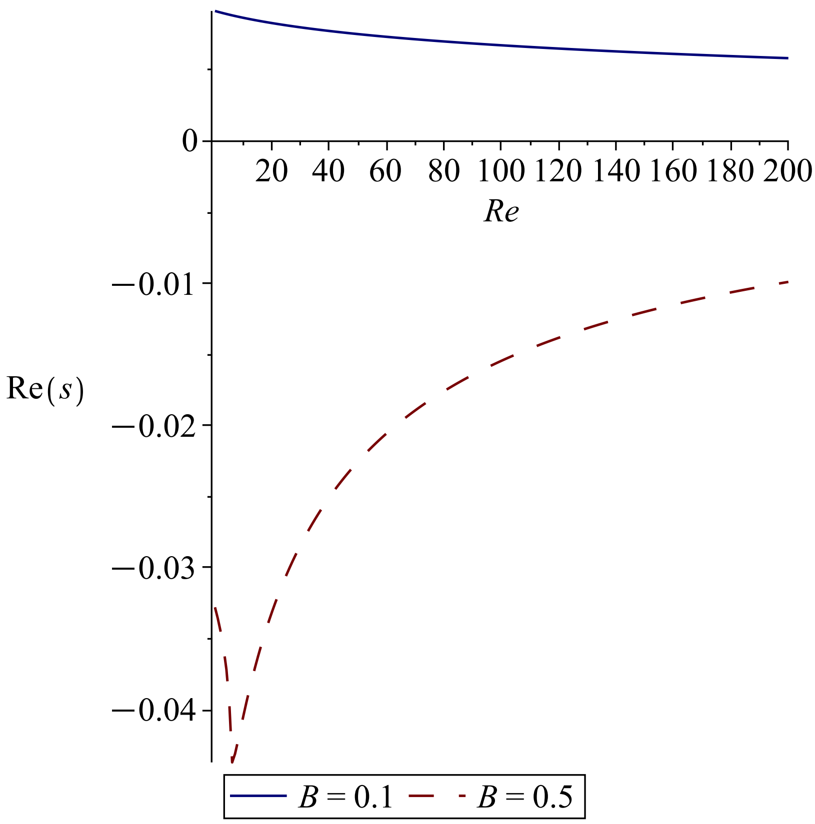

Figure 3.13: Growth rate plotted when $m = 0$, $a = 0.1$, $k = 0.5$, $\chi_1 = 5$ and $\chi_2 = 1$ for arbitrary Re

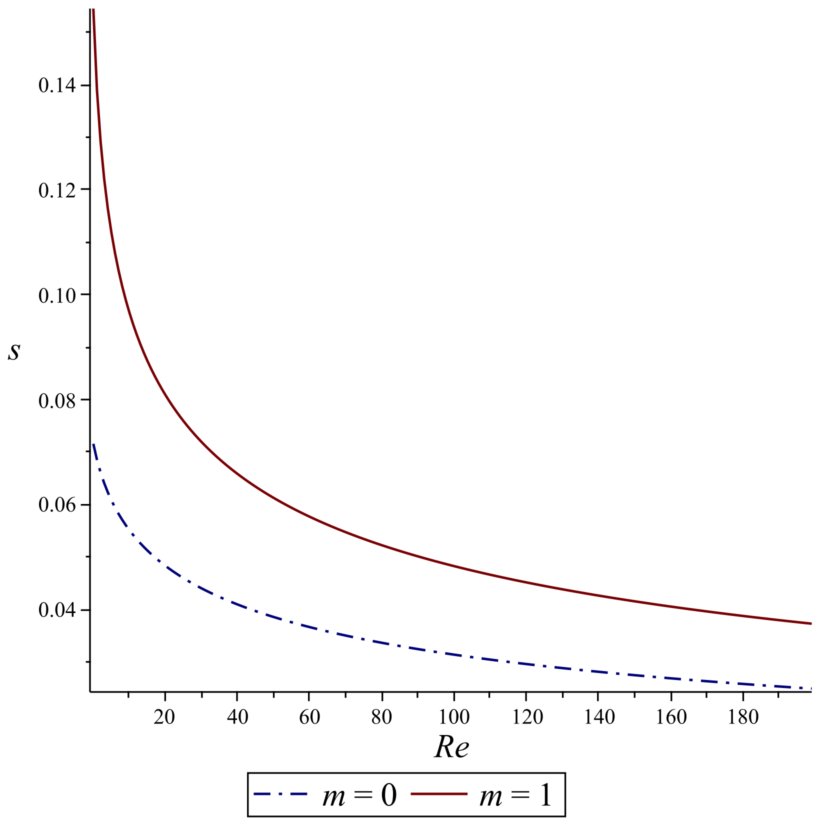

Figure 3.14: Growth rate plotted when $a = 0.1$, $k = 0.5$, $\chi_1 = 1$, $\chi_2 = 5$ and $B = 0.5$ for arbitrary Re

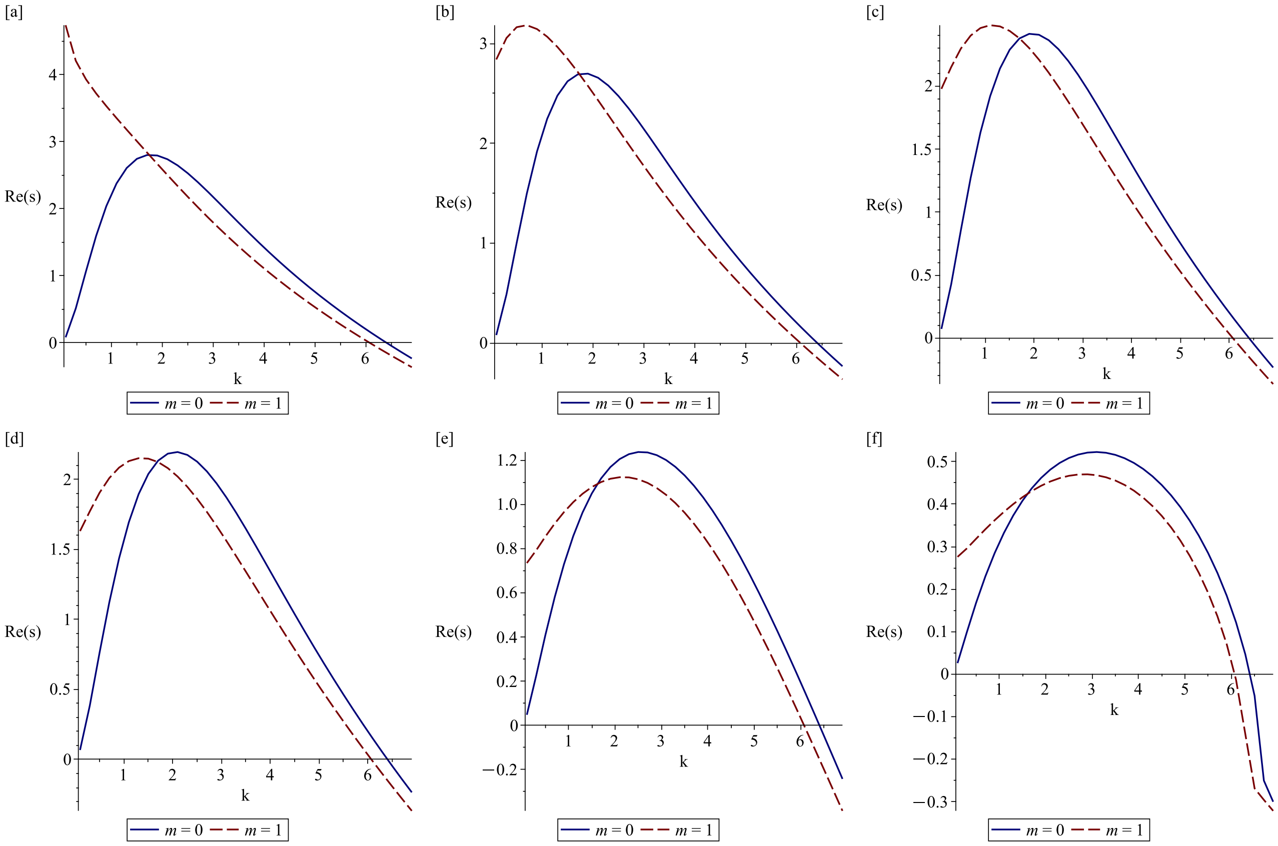

Figure 3.15: Growth rate plotted against k, when $a = 0.1$, $\chi_1 = 1$, $\chi_2 = 5$, $B = 10$, for Reynolds numbers: (a) Re = 0.0001 (b) Re = 0.1 (c) Re = 0.5 (d) Re = 1 (e) Re = 10 (f) Re = 100

3.3.3 Suppressing unstable modes with an axial field

To stabilise the system, irrespective of whether $\chi^{(2)} > \chi^{(1)}$ or $\chi^{(1)} > \chi^{(2)}$, we consider adding an axial field. Physically, an axial field can be produced by adding a solenoid positioned co-axially with the wire. Note, the radius of the coil must be sufficiently large so as not to alter the boundary conditions of the problem.

Now, let $\boldsymbol{H_0} = (0, 1/r, Z)^T$, Z constant, thereby adding an axial field. It follows that

$$\phi_0 = \theta + Zz$$

and we perform analogous analysis to Section 3.3. The general solutions (3.9), (3.22) still hold and $\hat{\phi}^{(0)}, \hat{\phi}^{(2)}$ are still given by (3.10) and (3.11), respectively. At $r = a$ we apply (3.14), (3.15) and (3.27). At $r = 1$ we apply (3.26), (3.31), (3.32), but (2.59), (2.60) and (2.63) now give

$$\big(1+\chi^{(2)}\big)\hat{\phi}'^{(2)} - \big(1+\chi^{(1)}\big)\hat{\phi}'^{(1)} + \mathrm{i}(m+kZ)(\chi^{(1)}-\chi^{(2)})\hat{S} = 0, \quad \hat{\phi}^{(1)} = \hat{\phi}^{(2)}, \tag{3.40}$$

and

$$\big(m^2+k^2-1+B(\chi^{(1)}-\chi^{(2)})\big)\hat{S} = 2\mathrm{i}(m+kZ)B\big(\chi^{(1)}-\chi^{(2)})\hat{\phi}^{(1)}\big)+\hat{p}^{(1)}-\hat{p}^{(2)}-2\hat{u}'^{(1)}+2\hat{u}'^{(2)}, \tag{3.41}$$

at $r = 1$. Consequently, the growth rates become

$$s = -gF\Bigg(f_1\big(k^2+m^2-1+B(\chi^{(1)}-\chi^{(2)})\big) + f_2B(\chi^{(1)}-\chi^{(2)})^2(Zk+m)^2\Bigg), \tag{3.42}$$

$$s_v = -gF_v\Bigg(f_1\big(k^2+m^2-1+B(\chi^{(1)}-\chi^{(2)})\big) + f_2B(\chi^{(1)}-\chi^{(2)})^2(Zk+m)^2\Bigg), \tag{3.43}$$

$$s_I^2 = -gF_I\Bigg(f_1\big(k^2+m^2-1+B(\chi^{(1)}-\chi^{(2)})\big) + f_2B(\chi^{(1)}-\chi^{(2)})^2(Zk+m)^2\Bigg), \tag{3.44}$$

in an analogous approach to obtaining (3.35), (3.36) and (3.37).

(3.43) and (3.44) show that a sufficiently large kZ will stabilise all modes in the inviscid and highly viscous regimes, irrespective of the sign of $(\chi^{(1)} - \chi^{(2)})$, provided $B \neq 0$. This result is found to hold for arbitrary Reynolds number too. Comparing Figure 3.12, where the system was unstable for $m = 0$, $a = 0.1$, $k = 0.5$, $\chi^{(1)} = 5$, $\chi^{(2)} = 1$, $B = 0.1$, with Figure 3.16, where the values are the same but with the addition of the axial field, shows that $kZ = 10$ is sufficient to dampen the unstable modes. We conclude, that a sufficiently large axial field will dampen unstable modes. Although extremely long waves in the z-direction, $k \to 0$, would remain unstable, k can be bounded away from zero by physical restrictions of the system.

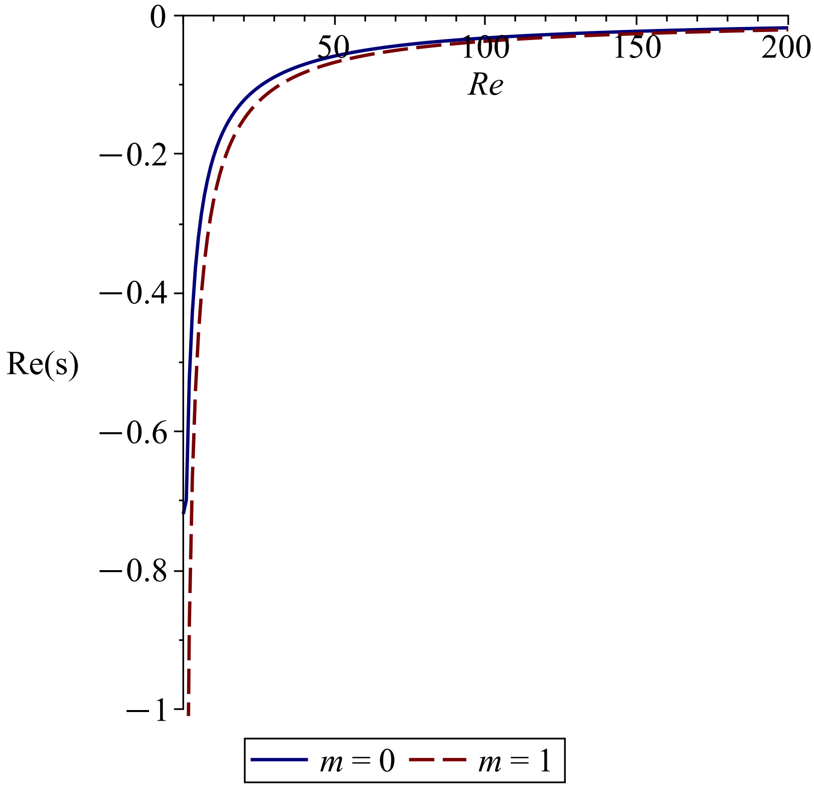

Figure 3.16: Real part of the growth rate plotted when $a = 0.1$, $k = 0.5$, $\chi_1 = 1$, $\chi_2 = 5$ and $B = 0.1$ for arbitrary Re, when $Z = 20$.

3.4 Concluding remarks

Three-dimensional disturbances to a ferrofluid column, surrounded by another ferrofluid of a different susceptibility, centred on a current-carrying wire have been analysed. An analytical solution was found to the linearised Navier-Stokes equations for two Newtonian ferrofluids, and an implicit expression for the growth rate obtained. In the highly viscous and inviscid regimes the growth rate is given explicitly, and a stability condition is determined. The greatest growth rate is found when the ratio between the radius of the wire and the radius of the inner fluid, a/R, is at its smallest. This agrees with Arkhipenko et al. (1980) and Korovin (2004), who find a decrease in the relative thickness of the jet to the wire slows down the decay of the column. When the inner fluid is more magnetic, only axisymmetric modes with $k < 1$ and $B(\chi^{(1)} - \chi^{(2)}) < 1$ are unstable, supporting the previous literature on the subject (if $\chi^{(2)} = 0$). Novel results are found when the outer fluid is more magnetic. When the outer fluid is more magnetic, both non-axisymmetric and axisymmetric modes can be unstable. Interestingly, for sufficiently small Reynolds numbers, the non-axisymmetric mode $m = 1$ is the most unstable, but otherwise axisymmetric modes are the most unstable.

Sufficient current in the wire suppresses instabilities due to surface tension, only if the inner fluid is more magnetic than the outer fluid. On the other hand, when the outer fluid is more magnetic, instabilities are not only due to capillary forcing from the surface tension, but also a result of magnetic forcing at the interface, produced from the current in the wire. Thus, when $\chi^{(2)} > \chi^{(1)}$, increasing the current in the wire will only increase the strength of the forcing at the interface, thereby increasing the growth rate of the perturbation and rendering more modes unstable. However, adding a large enough axial field will suppress all disturbances, irrespective of which fluid has a higher susceptibility, provided there exists some current in the wire.

Chapter 4

A ferrofluid column, with a radially varying magnetic susceptibility, centred on a current-carrying wire

4.1 Motivation

In the previous chapter, we saw that when the inner fluid had a higher susceptibility than the outer fluid, the system was unstable due to capillary forcing only. But, when the outer fluid was more magnetic than the inner fluid, the system was unstable as a result of both capillary and magnetic forcing. The latter may be a result of the highest region of magnetic susceptibility not being aligned with the highest region of field. When magnetic fluids are subject to non-uniform fields, the magnetic fluid is attracted to the region where the field intensity is maximum (Scherer & Figueiredo Neto, 2005). This motivates the investigation of the stability of one ferrofluid, whose susceptibility varies continuously with radius. The ferrofluid is centred on a current-carrying wire, with an associated azimuthal field decreasing as the reciprocal of the radius. The results of Chapter 3 suggest that the system may be stable if the susceptibility decreases radially, since the regions of highest strength of field and susceptibility will coincide, and there is no longer forcing from surface tension at an interface. Similarly, a stationary state where the susceptibility increases radially may be unstable. However, this is not trivial due to the global variation of the susceptibility and field. The magnetic forcing acts throughout the fluid, where as for a constant discontinuous susceptibility, the magnetic forcing is confined to the interface. This chapter rigorously proves that the stability is indeed determined by the sign of the gradient of the susceptibility. Moreover, it is shown that adding an axial field can suppress the instability. The analysis in this chapter has appeared in Ferguson Briggs & Mestel (2022a).

4.2 Formulation

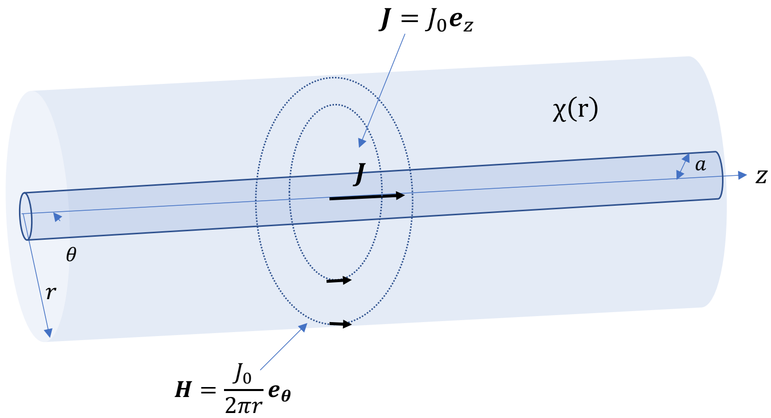

Figure 4.1: Schematic of the system

We consider one incompressible, isothermal, ferrofluid whose susceptibility depends on position and the field. Consequently the magnetic forcing acts throughout the fluid, and $\boldsymbol{f} \neq 0$ in (2.37). The ferrofluid is centred on a current-carrying wire, which produces an azimuthal field, $\boldsymbol{H} = J_0/2\pi r\boldsymbol{e_\theta}$, as shown in Figure 4.1. Retaining the scaling choice in Chapter 3 for non-dimensionalising the governing equations, (2.44) gives

$$Re\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}^2\boldsymbol{\omega} + BH\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}H. \tag{4.1}$$

where Re and B are given by (3.3) and (3.30), respectively.

Since $H_0 \equiv H_0(r)$, (4.1) is satisfied by $\boldsymbol{u} = 0$, $\chi = \chi_0(r)$, and to satisfy (2.39),

$$\boldsymbol{\nabla} p = -\frac{B(H_0(r))^2}{2} \boldsymbol{\nabla} \chi_0(r). \tag{4.2}$$

After non-dimensionalising, the stationary state is

$$H_0 = 1/r, \quad \boldsymbol{u} = 0, \quad \chi = \chi_0(r) \quad \text{and} \quad p = -\int_0^r \frac{B}{2r^2} \chi_0' dr, \tag{4.3}$$

where χ_0' is defined as

$$\chi_0' = \frac{d\chi_0}{dr}. \tag{4.4}$$

4.3 Linear stability analysis

We now perform linear stability analysis on the given equilibrium to derive a stability condition. Consider a perturbation to the equilibrium;

$$\chi = \chi_0(r) + \epsilon \chi_1 + O(\epsilon^2), \quad \boldsymbol{H} = \boldsymbol{H_0} + \epsilon \boldsymbol{H_1} + O(\epsilon^2), \quad \mathbf{u} = \epsilon \mathbf{u_1} + O(\epsilon^2), \tag{4.5a, b, c}$$

and

$$|\boldsymbol{H}| = H = H_0 + \epsilon H_1 + O(\epsilon^2), \quad \text{where} \quad H_0 = \sqrt{\boldsymbol{H_0} \cdot \boldsymbol{H_0}}, \quad H_1 = \frac{\boldsymbol{H_0} \cdot \boldsymbol{H_1}}{H_0}. \tag{4.6}$$

Note H_1 is defined by (4.6) and $H_1 \neq |\boldsymbol{H}_1|$. Substituting into (4.1) and linearising gives

$$Re \frac{\partial \boldsymbol{\omega_1}}{\partial t} = \boldsymbol{\nabla}^2 \boldsymbol{\omega_1} + B H_0 \big(\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} H_1 + \boldsymbol{\nabla} \chi_1 \times \boldsymbol{\nabla} H_0 \big), \tag{4.7}$$

where $\boldsymbol{\omega_1} = \boldsymbol{\nabla} \times \boldsymbol{u_1}$. We consider perturbations such that

$$\boldsymbol{\omega_1} = \boldsymbol{\nabla} \times (\hat{\boldsymbol{u}}(r)\zeta), \quad \chi_1 = \hat{\chi}(r)\zeta, \quad \boldsymbol{H_1} = \boldsymbol{\nabla}(\hat{\phi}(r)\zeta) \quad \text{and} \quad H_1 = \frac{\mathrm{i}m\hat{\phi}(r)\zeta}{r}, \tag{4.8}$$

where $\zeta = e^{\mathrm{i}(kz+m\theta)+st}$ and $\hat{\phi}$ were defined in Chapter 3; $\phi_1 = \hat{\phi}(r)\zeta$ is the magnetic potential perturbation, but here we write $\hat{\phi}$ for $\hat{\phi}$ in the fluid and $\hat{\phi}_0$ to denote $\hat{\phi}$ in the wire. Substituting (4.8) into (4.7), we obtain

$$(sRe - \mathcal{L})\hat{\omega}_r + \frac{2\mathrm{i}m}{r^2}\hat{\omega}_\theta = 0, \tag{4.9}$$

$$(sRe - \mathcal{L})\hat{\omega}_\theta - \frac{2\mathrm{i}m}{r^2}\hat{\omega}_r = B\left(\frac{mk\chi_0'\hat{\phi}}{r^2} - \frac{\mathrm{i}k\hat{\chi}}{r^3}\right), \tag{4.10}$$

$$(sRe - \mathcal{D})\hat{\omega}_z = B\left(\frac{-m^2\chi_0'\hat{\phi}}{r^3} + \frac{\mathrm{i}m\hat{\chi}}{r^4}\right), \tag{4.11}$$

where

$$\hat{\omega}_r = \frac{\mathrm{i}m\hat{w}}{r} - \mathrm{i}k\hat{v}, \quad \hat{\omega}_\theta = \mathrm{i}k\hat{u} - \hat{w}', \quad \hat{\omega}_z = \frac{1}{r}\Big((r\hat{v})' - \mathrm{i}m\hat{u}\Big), \tag{4.12}$$

$$\mathcal{L} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - k^2 - \frac{1}{r^2}, \tag{4.13}$$

$$\mathcal{D} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - k^2. \tag{4.14}$$

It follows from (2.56), that $\hat{\boldsymbol{u}}$ satisfies

$$(r\hat{u})' + \mathrm{i}m\hat{v} + \mathrm{i}kr\hat{w} = 0. \tag{4.15}$$

(2.61) and (2.62) result in $\hat{\boldsymbol{u}} = 0$ at $r = a$ and $\hat{\boldsymbol{u}} \to 0$ as $r \to \infty$. Substituting the perturbed variables into (2.66) and linearising, results in

$$s\hat{\chi} = -\chi_0'\hat{u}, \tag{4.16}$$

and therefore, $\hat{\chi} = 0$ at $r = a$ and $\hat{\chi} \to 0$ as $r \to \infty$. $\hat{\phi}$ in the fluid satisfies

$$r^2\big((1+\chi_0)\mathcal{D}\hat{\phi} + \chi_0'\hat{\phi}'\big) = -\mathrm{i}m\hat{\chi}. \tag{4.17}$$

and

$$\hat{\phi}^{(0)} = q_1^{(0)}\mathrm{I}_m. \tag{4.18}$$

(2.59) and (2.60) give

$$(\hat{\phi}^{(0)})' = (1+\chi_0)\hat{\phi}' \tag{4.19}$$

and

$$\hat{\phi}^{(0)} = \hat{\phi}, \tag{4.20}$$

respectively, at $r = a$. Consequently, (4.18)-(4.20) result in

$$\hat{\phi}' = \frac{\big(\mathrm{I}_m(kr)\big)'\hat{\phi}}{(1+\chi_0)\mathrm{I}_m(kr)} \tag{4.21}$$

on $r = a$, which evaluates to

$$\hat{\phi}' = \frac{\hat{\phi}}{1+\chi_0}\left(\frac{k\mathrm{I}_{m+1}(ka)}{\mathrm{I}_m(ka)} + \frac{m}{a}\right), \tag{4.22}$$

at $r = a$. Moreover, $\hat{\phi}, \hat{\phi}' \to 0$ as $r \to \infty$. Note that $'$ denotes the derivative with respect to r.

We now consider situations in which the equations are simplified, allowing us to produce an eigenvalue equation, and in turn, an expression for the growth rate.

4.3.1 Axisymmetric disturbances

Consider solely axisymmetric disturbances $m = 0$. For axisymmetric disturbances, (4.9)-(4.11) becomes

$$(sRe - \mathcal{L}_0)\hat{\omega}_r = 0, \tag{4.23}$$

$$(sRe - \mathcal{L}_0)\hat{\omega}_\theta = -\frac{\mathrm{i}kB\hat{\chi}}{r^3}, \tag{4.24}$$

$$(sRe - \mathcal{D}_0)\hat{\omega}_z = 0, \tag{4.25}$$

where

$$\mathcal{L}_0 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2 - \frac{1}{r^2}, \tag{4.26}$$

$$\mathcal{D}_0 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2. \tag{4.27}$$

Define a stream function Ψ, such that $\hat{\boldsymbol{u}} = \boldsymbol{\nabla} \times (0, \Psi/r, 0)$, and use the change of variables, $\Psi = r\psi$, to give

$$\hat{\boldsymbol{\omega}} = -\mathcal{L}_0\hat{\psi}\boldsymbol{e_\theta}, \quad \boldsymbol{\nabla}^2\hat{\boldsymbol{\omega}} = -\mathcal{L}_0^2\hat{\psi}\boldsymbol{e_\theta}, \tag{4.28}$$

where $\psi = \hat{\psi}(r)e^{ikz+st}$. It follows from the boundary conditions for $\boldsymbol{u}$ that $\hat{\psi}, \hat{\psi}' = 0$ at $r = a$ and as $r \to \infty$. (4.16) and (4.23)-(4.25) give the eigenvalue equation

$$\left(s^2 Re\mathcal{L}_0 - s\mathcal{L}_0^2\right)\hat{\psi} = -\frac{k^2 B\chi_0'\hat{\psi}}{r^3}. \tag{4.29}$$

Rather than find the eigenvalues of (4.29) numerically, we prove a stability condition. Multiply (4.29) by $r\hat{\psi}^*$, where $\hat{\psi}^*$ is the complex conjugate of $\hat{\psi}$ and integrate over the domain to give

$$\int_a^\infty r\hat{\psi}^*\left(s^2 Re\mathcal{L}_0 - s\mathcal{L}_0^2\right)\hat{\psi}dr = -\int_a^\infty \frac{k^2 B\chi_0'|\hat{\psi}|^2}{r^2}dr. \tag{4.30}$$

Due to the self-adjoint properties of $\mathcal{L}_0$, we use integration by parts and the boundary conditions, to obtain

$$s^2 Re\int_a^\infty \left(|\hat{\psi}'|^2 + \left(\frac{1}{r^2} + k^2\right)|\hat{\psi}|^2\right)rdr + s\int_a^\infty |\mathcal{L}_0\hat{\psi}|^2 rdr - k^2 B\int_a^\infty \frac{\chi_0'|\hat{\psi}|^2}{r^2}dr = 0, \tag{4.31}$$

an equation for s. (4.31) is of the form $as^2 + bs + c = 0$, where a,b,c all depend on $\hat{\psi}$ and therefore s too, but a,b,c are real, as well as a, b positive and bounded from zero. We conclude that if $\chi_0' > 0$, $c < 0$ and there exists a root with $\mathcal{R}(s) > 0$. In fact, in this case s and $\hat{\psi}$ are real. Whereas, when $\chi_0' < 0$, $c > 0$, $\mathcal{R}(s) < 0$ and $s, \hat{\psi}$ could be complex. Thus, if $\chi_0' > 0$ everywhere, there exists an unstable mode, while if $\chi_0' \leq 0$ everywhere, all axisymmetric modes are stable.

We prove a stronger stability condition using variational methods. Crucially, we have shown that if the flow is unstable then s must be real and therefore $\hat{\psi}$ is real, thus it suffices to only consider y real in the following argument. Consider the functional

$$F(y) = \frac{-\int_a^\infty (\mathcal{L}_0 y)^2 r dr + \sqrt{\left(\int_a^\infty (\mathcal{L}_0 y)^2 r dr\right)^2 + 4k^2 ReB \int_a^\infty \left((y')^2 + \left(\frac{1}{r^2} + k^2\right)y^2\right) r dr \int_a^\infty \frac{\chi_0' y^2}{r^2} dr}}{2Re \int_a^\infty \left((y')^2 + \left(\frac{1}{r^2} + k^2\right)y^2\right) r dr}, \tag{4.32}$$

for all real functions $y(r)$ satisfying $y, y' = 0$ at $r = a$, $r \to \infty$, and proceed in a manner similar to the Rayleigh-Ritz argument.

First, we argue $F(y)$ is bounded above. Since $F(y)$ is homogeneous in y, we could normalise the denominator such that the denominator is bounded from zero. If y'' is bounded, then the numerator is bounded, but suppose y is highly oscillatory and y'' is large. By a series expansion for $y'' \gg 1$,

$$F(y) \sim \frac{-\int_a^\infty (y'')^2 r dr + \sqrt{\left(\int_a^\infty (y'')^2 r dr\right)^2}}{2Re \int_a^\infty \left((y')^2 + \left(\frac{1}{r^2} + k^2\right)y^2\right) r dr} + \frac{k^2 B \int_a^\infty \frac{\chi_0' y^2}{r^2} dr}{\int_a^\infty (y'')^2 r dr}, \tag{4.33}$$

and

$$F(y) \to \frac{k^2 B \int_a^\infty \frac{\chi_0' y^2}{r^2} dr}{\int_a^\infty (y'')^2 r dr}, \tag{4.34}$$

a small departure from zero, where the sign is dependant on χ_0'. Thus for $\chi_0' > 0$, $F(y)$ is positive, and is bounded above. It follows that $F(y)$ has a global maximum.

We now prove that stationary points of $F(y)$ are real eigenvalues of (4.29). Suppose $y = y_0$ is a stationary point of F and $F(y_0) = F_0$. Consider $y = y_0 + \epsilon y_1$, where $\epsilon \ll 1$, and y_1 satisfies the boundary conditions for y. Taylor expanding

$$F(y_0 + \epsilon y_1) = \frac{-(C_3 + \epsilon C_4) + \sqrt{C_3^2 + 4k^2 ReBC_1C_5 + \epsilon\left(2C_3C_4 + 4k^2 ReB\left(C_2C_5 + C_1C_6\right)\right)}}{C_1 + \epsilon C_2} + O(\epsilon^2), \tag{4.35}$$

we obtain

$$F(y_0 + \epsilon y_1) = F_0 + \frac{\epsilon}{C_1}\left(-C_4 - F_0C_2 + \frac{4k^2 ReB(C_1C_6 + C_2C_5) + 2C_3C_4}{2\sqrt{4k^2 ReBC_1C_5 + C_3^2}}\right) + O(\epsilon^2), \tag{4.36}$$

where

$$
\begin{aligned}
C_1 &= 2Re\int_a^\infty \left((y_0')^2 + \left(\frac{1}{r^2}+k^2\right)y_0^2\right)rdr,\\
C_2 &= 4Re\int_a^\infty \left(y_0'y_1' + \left(\frac{1}{r^2}+k^2\right)y_0y_1\right)rdr,\\
C_3 &= \int_a^\infty (\mathcal{L}_0 y_0)^2 rdr,\\
C_4 &= 2\int_a^\infty (\mathcal{L}_0 y_0)(\mathcal{L}_0 y_1)rdr,\\
C_5 &= \int_a^\infty \frac{\chi_0' y_0^2}{r^2}dr,\\
C_6 &= 2\int_a^\infty \frac{\chi_0' y_0 y_1}{r^2}dr.
\end{aligned} \tag{4.37}
$$

Note that

$$
F_0 = F(y_0) = \frac{-C_3+\sqrt{C_3^2+4k^2ReBC_1C_5}}{C_1}. \tag{4.38}
$$

F_0 consists of y_0 values, and since y_0 is a stationary point of $F(y)$, in the ϵ neighbourhood of y_0 the first variation must be zero, and therefore

$$
-C_4 - F_0C_2 + \frac{4k^2ReB(C_1C_6+C_2C_5)+2C_3C_4}{2\sqrt{C_3^2+4k^2ReBC_1C_5}} = 0. \tag{4.39}
$$

After some algebra, we write this as

$$
-ReF_0^2\int_a^\infty \left(y_0'y_1' + \left(\frac{1}{r^2}+k^2\right)y_0y_1\right)rdr - F_0\int_a^\infty \mathcal{L}_0y_0\mathcal{L}_0y_1 rdr = k^2B\int_a^\infty \frac{\chi_0'y_0y_1}{r^2}dr. \tag{4.40}
$$

Invoking the self adjoint property of $\mathcal{L}_0$, using integration by parts, and the boundary conditions for y_0, y_1, write (4.40) as

$$
ReF_0^2\int_a^\infty y_1 r\mathcal{L}_0y_0 dr - F_0\int_a^\infty ry_1\mathcal{L}_0^2y_0dr = k^2B\int_a^\infty \frac{\chi_0'y_0y_1}{r^2}dr. \tag{4.41}
$$

(4.41) is valid for any y_1, and therefore

$$
ReF_0^2\mathcal{L}_0y_0 - F_0\mathcal{L}_0^2y_0 = \frac{k^2B\chi_0'y_0}{r^3}, \tag{4.42}
$$

which is (4.29) for $F_0 = s$. It follows that the stationary points of $F(y)$ satisfy (4.29) with real eigenvalues $s = F_0$. Thus, the stationary points of $F(y)$ correspond to the real eigenvalues of (4.29), and crucially, the global maximum of $F(y)$ is an eigenvalue of (4.29). We now seek a function y such that $F(y) > 0$ and therefore the global maximum must be positive (and real), proving the existence of a positive real eigenvalue, and therefore an unstable mode.

Suppose

$$\chi_0' > 0 \quad \text{for} \quad r_1 \leq r \leq r_2, \tag{4.43}$$

and pick an arbitrary, real function, $\hat{y}(r)$, that satisfies the boundary conditions of y, such that

$$\left.\begin{aligned} \hat{y}(r) &\neq 0 \quad \text{for} \quad r_1 \leq r \leq r_2, \\ \hat{y}(r) &= 0 \quad \text{for} \quad r \notin [r_1, r_2]. \end{aligned}\right\} \tag{4.44}$$

Substitute $y = \hat{y}(r)$ into (4.32) and define $\hat{\xi} = F(\hat{y})$, where $\hat{\xi}$ must be a real value. It follows that $\hat{\xi} > 0$, and either $\hat{\xi}$ is the global maximum stationary point of $F(y)$, or the global maximum of $F(y)$ is greater than $\hat{\xi}$, since $F(y)$ is bounded above. Thus there exists a positive real stationary point of $F(y)$ and therefore a positive real eigenvalue of (4.29), resulting in an unstable mode. We conclude that, if, and only if, $\chi_0' > 0$ anywhere in the domain, every axisymmetric mode is unstable.

4.3.2 Two-dimensional modes

An alternative way to simplify the equations is to consider any m, but two-dimensional modes such that $k = 0$. By considering $k = 0$ in (4.9)-(4.17), we obtain an eigenvalue equation,

$$(s^2 Re\mathcal{L}_m - s\mathcal{L}_m^2)\mathcal{M}_m\hat{\phi} = -m^2 B\left(\frac{\chi_0'\mathcal{M}_m\hat{\phi}}{r^5} + \frac{m^2\chi_0'\hat{\phi}}{r^3}\right), \tag{4.45}$$

where

$$\mathcal{L}_m = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{m^2}{r^2}, \quad \mathcal{M}_m = \frac{r^3}{\chi_0'}\left((1+\chi_0)\mathcal{L}_m + \chi_0'\frac{d}{dr}\right), \tag{4.46}$$

and $\phi, \phi' \to 0$ as $r \to \infty$. Taking the limit $k \to 0$ in (4.22) gives

$$\hat{\phi}' = \frac{m}{r(1+\chi_0)}\hat{\phi}(r) \tag{4.47}$$

at $r = a$. Since, for small z

$$\mathrm{I}_m(z) \sim \frac{\left(\frac{z}{2}\right)^m}{m!}, \tag{4.48}$$

where $m!$ is the factorial of m (Abramowitz et al., 1988), and therefore

$$\frac{k\mathrm{I}_{m+1}(ka)}{\mathrm{I}_m(ka)} + \frac{m}{r} \sim \frac{k^2a^2 + 2m(m+1)}{2(m+1)a} \tag{4.49}$$

and

$$\frac{k\mathrm{I}_{m+1}(ka)}{\mathrm{I}_m(ka)} + \frac{m}{r} \to \frac{m}{a} \tag{4.50}$$

in the limit $k \to 0$.

We multiply (4.45) by $r\mathcal{M}_m\hat{\phi}^*$, where $\hat{\phi}^*$ is the complex conjugate of $\hat{\phi}$, and integrate over the domain to obtain

$$\int_a^\infty \left(\mathcal{M}_m\hat{\phi}^*(s^2Re\mathcal{L}_m - s\mathcal{L}_m^2)\mathcal{M}_m\hat{\phi}\right)rdr = -Bm^2\int_a^\infty \left(\frac{\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4} + \frac{m^2\chi_0'}{r^2}\hat{\phi}\mathcal{M}_m\hat{\phi}^*\right)dr. \tag{4.51}$$

Now,

$$\int_a^\infty \frac{1}{r^2}\chi_0'\hat{\phi}\mathcal{M}_m\hat{\phi}^*dr = \int_a^\infty \left((1+\chi_0)\hat{\phi}(r(\hat{\phi}^*)')' + r\chi_0'\hat{\phi}(\hat{\phi}^*)' - \frac{m^2(1+\chi_0)|\hat{\phi}|^2}{r}\right)dr, \tag{4.52}$$

and integration by parts gives

$$\int_a^\infty \frac{1}{r^2}\chi_0'\hat{\phi}\mathcal{M}_m\hat{\phi}^*dr = -m|\hat{\phi}(a)|^2 - \int_a^\infty r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2}\right)dr. \tag{4.53}$$

(4.53) and the self-adjoint property of $\mathcal{L}_m$ allows (4.51) to be written as

$$s^2 Re\int_a^\infty\left(\left|(\mathcal{M}_m\hat{\phi})'\right|^2+\frac{m^2|\mathcal{M}_m\hat{\phi}|^2}{r^2}\right)rdr+s\int_a^\infty\left|\mathcal{L}_m(\mathcal{M}_m\hat{\phi})\right|^2rdr+Bm^2\Bigg(m^3|\hat{\phi}(a)|^2$$
$$+\int_a^\infty\left(\frac{-\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4}+m^2r(1+\chi_0)\left(|\hat{\phi}'|^2+\frac{m^2|\hat{\phi}|^2}{r^2}\right)\right)dr\Bigg)=0. \tag{4.54}$$

It follows that if $\chi_0'\leq 0$ everywhere, two-dimensional modes are stable. Yet, if

$$\int_a^\infty\frac{\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4}rdr>m^3|\hat{\phi}(a)|^2+m^2\int_a^\infty r(1+\chi_0)\left(|\hat{\phi}'|^2+\frac{m^2|\hat{\phi}|^2}{r^2}\right)dr \tag{4.55}$$

holds for an eigenfunction $\hat{\phi}$, then there exists a growing mode ($s>0$) and therefore a two-dimensional unstable mode. Moreover, s and therefore $\hat{\phi}$ are real if (4.55) holds, otherwise $\mathcal{R}(s)<0$ and s could be complex.

Furthermore, consider the functional

$$F(y)=\frac{-\int_a^\infty\left(\mathcal{L}_m(\mathcal{M}_my)\right)^2rdr+\sqrt{\left(\int_a^\infty\left(\mathcal{L}_m(\mathcal{M}_my)\right)^2rdr\right)^2-W_1}}{2Re\int_a^\infty\left((\mathcal{M}_my')^2+\frac{m^2}{r^2}(\mathcal{M}_my)^2\right)rdr}, \tag{4.56}$$

where

$$W_1 = 4m^2BRe\left(\int_a^\infty\left((\mathcal{M}_my')^2+\frac{m^2}{r^2}(\mathcal{M}_my)^2\right)rdr\right)\Bigg(m^3|\hat{\phi}(a)|^2$$
$$+\int_a^\infty\left(\frac{-\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4}+m^2r(1+\chi_0)\left(|\hat{\phi}'|^2+\frac{m^2|\hat{\phi}|^2}{r^2}\right)\right)dr\Bigg), \tag{4.57}$$

for all real functions y satisfying the boundary conditions of $\hat{\phi}$. By an analogous argument to Section 4.3.1, $F(y)$ has a global maximum, and we can prove the stationary points of $F(y)$ correspond to the real eigenvalues of (4.45). Again, if (4.43) is true, then by picking a $y=\hat{y}(r)$, where $\hat{y}$ satisfies (4.44), with highly oscillatory behaviour in the interval $r\in[r_1,r_2]$, then $F(\hat{y})>0$ and (4.55) holds as $(\mathcal{M}_m\hat{y})^2\gg\hat{y}'^2\gg\hat{y}^2$. Consequently, if, and only if, $\chi_0'>0$ anywhere in the fluid, every mode where $k=0$ is unstable.

4.3.3 The inviscid limit

To deduce a stability condition for all wave numbers, we consider the inviscid limit, $Re \to \infty$, of (4.9)-(4.17) to obtain an eigenvalue equation valid for three-dimensional disturbances. In the inviscid limit (4.9)-(4.11) becomes

$$s_I \hat{\omega}_r = 0, \tag{4.58}$$

$$s_I \hat{\omega}_\theta = B\left(\frac{mk\chi_0'\hat{\phi}}{r^2} - \frac{\mathrm{i}k\hat{\chi}}{r^3}\right), \tag{4.59}$$

$$s_I \hat{\omega}_z = B\left(\frac{-m^2\chi_0'\hat{\phi}}{r^3} + \frac{\mathrm{i}m\hat{\chi}}{r^4}\right), \tag{4.60}$$

and consequently

$$m\hat{w} = kr\hat{v}. \tag{4.61}$$

(4.15) and (4.61) result in

$$\hat{v} = \frac{\mathrm{i}m\big(r\hat{u}\big)'}{m^2 + k^2r^2}, \tag{4.62}$$

and substituting (4.62) into (4.58) provides

$$s_I\left(\left(\frac{r\big(r\hat{u}\big)'}{m^2 + k^2r^2}\right)' - \hat{u}\right) = B\left(\frac{m\chi_0'\hat{\phi}}{r^2} + \frac{\hat{\chi}}{r^3}\right). \tag{4.63}$$

Substituting (4.16) into (4.17) results in an expression for $\hat{u}$ in terms of $\hat{\phi}$;

$$\hat{u} = \frac{s_I r^2}{\mathrm{i}m\chi_0'}\left((1+\chi_0)\mathcal{D}\hat{\phi} + \chi_0'\hat{\phi}'\right). \tag{4.64}$$

Finally, by substituting (4.16) and (4.64) into (4.63), we obtain an eigenvalue equation for $\hat{\phi}$;

$$-\left(\frac{r}{m^2+k^2r^2}\big(\mathcal{M}\hat{\phi}\big)'\right)' + \frac{\mathcal{M}\hat{\phi}}{r} = \lambda\left(\frac{m^2}{r^2}\chi_0'\hat{\phi} + \frac{\chi_0'\mathcal{M}\hat{\phi}}{r^4}\right), \tag{4.65}$$

where

$$\mathcal{M}\hat{\phi} = \frac{r^3}{\chi_0'}\left((1+\chi_0)\mathcal{D}\hat{\phi} + \chi_0'\hat{\phi}'\right), \tag{4.66}$$

$\lambda = B/(s_I^2)$. For an inviscid system, $\hat{u} = 0$ at $r = a$, $r \to \infty$, and it follows from (4.16) and (4.17) that $\mathcal{M}\hat{\phi} = 0$ at $r = a$, $r \to \infty$.

Multiply (4.65) by $\mathcal{M}\hat{\phi}^*$, the complex conjugate of $\mathcal{M}\hat{\phi}$, and integrate over the domain to obtain

$$
\begin{aligned}
&-\int_a^\infty \left(\mathcal{M}\hat{\phi}^*\left(\frac{r}{m^2+k^2r^2}(\mathcal{M}\hat{\phi})'\right)' + \frac{|\mathcal{M}\hat{\phi}|^2}{r}\right)dr \\
&= \lambda\int_a^\infty \left(m^2 r\hat{\phi}\left((1+\chi_0)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-k^2\right)\hat{\phi}^* + \chi_0'(\hat{\phi}^*)'\right) + \frac{\chi_0'|\mathcal{M}\hat{\phi}|^2}{r^4}\right)dr. \qquad (4.67)
\end{aligned}
$$

Integration by parts gives

$$
\begin{aligned}
&\int_a^\infty \left(\frac{r}{m^2+k^2r^2}\left|(\mathcal{M}\hat{\phi})'\right|^2 + \frac{\left|\mathcal{M}\hat{\phi}\right|^2}{r}\right)dr \\
&= \lambda\left(\left[m^2 r(1+\chi_0)\hat{\phi}(\hat{\phi}^*)'\right]_a^\infty + \int_a^\infty \left(\frac{\chi_0'}{r^4}\left|\mathcal{M}\hat{\phi}\right|^2 - m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \left(\frac{m^2}{r^2}+k^2\right)|\hat{\phi}|^2\right)\right)dr\right),
\end{aligned}
\qquad (4.68)
$$

and imposing the boundary conditions gives

$$
\begin{aligned}
&\int_a^\infty \left(\frac{r}{m^2+k^2r^2}\left|(\mathcal{M}\hat{\phi})'\right|^2 + \frac{\left|\mathcal{M}\hat{\phi}\right|^2}{r}\right)dr \\
&= \lambda\left(C_1 + \int_a^\infty \left(\frac{\chi_0'}{r^4}\left|\mathcal{M}\hat{\phi}\right|^2 - m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \left(\frac{m^2}{r^2}+k^2\right)|\hat{\phi}|^2\right)\right)dr\right)m \qquad (4.69)
\end{aligned}
$$

where

$$
C_1 = -m^2 a|\hat{\phi}|^2\left(\frac{k\mathrm{I}_{m+1}(ka)}{\mathrm{I}_m(ka)} + \frac{m}{a}\right). \qquad (4.70)
$$

Note that $C_1 < 0$ since the modified Bessel functions of the first kind are positive increasing functions. Writing this in the form

$$
s_I^2 = B\frac{-C_1 + \int_a^\infty \left(\frac{\chi_0'}{r^4}\left|\mathcal{M}\hat{\phi}\right|^2 - m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \left(\frac{m^2}{r^2}+k^2\right)|\hat{\phi}|^2\right)\right)dr}{\int_a^\infty \frac{r}{m^2+k^2r^2}\left|(\mathcal{M}\hat{\phi})'\right|^2 + \frac{1}{r}\left|\mathcal{M}\hat{\phi}\right|^2 dr}, \qquad (4.71)
$$

we observe that if there exists an eigenfunction $\hat{\phi}$ such that

$$\int_a^\infty \frac{\chi_0'}{r^4}\left|\mathcal{M}\hat{\phi}\right|^2 dr > \int_a^\infty m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \left(\frac{m^2}{r^2}+k^2\right)|\hat{\phi}|^2\right)dr + m^2 a|\hat{\phi}|^2\left(\frac{k\mathrm{I}_{m+1}(ka)}{\mathrm{I}_m(ka)}+\frac{m}{a}\right), \tag{4.72}$$

then $s_I^2 > 0$, and there exists a positive real eigenvalue, and therefore an unstable mode. Furthermore, when (4.72) holds, s_I^2 and $\hat{\phi}$ are real, and therefore s_I is either real or imaginary, but not complex.

To prove s_I^2 is always real, multiply the complex conjugate of (4.65) by $\mathcal{M}\hat{\phi}$, integrate over the domain, and perform integration by parts, to obtain

$$\begin{aligned}&\int_a^\infty \left(\frac{r}{m^2+k^2r^2}\left|(\mathcal{M}\hat{\phi})'\right|^2 + \frac{\left|\mathcal{M}\hat{\phi}\right|^2}{r}\right)dr \\ &= \lambda^*\left(C_1 + \int_a^\infty \frac{\chi_0'}{r^4}\left|\mathcal{M}\hat{\phi}\right|^2 - m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \left(\frac{m^2}{r^2}+k^2\right)|\hat{\phi}|^2\right)dr\right),\end{aligned} \tag{4.73}$$

where λ^* is the complex conjugate of λ. Subtracting (4.69) from (4.73) gives

$$(\lambda^*-\lambda)\left(C_1 + \int_a^\infty \left(\frac{\chi_0'}{r^4}\left|\mathcal{M}\hat{\phi}\right|^2 - m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \left(\frac{m^2}{r^2}+k^2\right)|\hat{\phi}|^2\right)\right)dr\right) = 0. \tag{4.74}$$

For a non-trivial solution,

$$C_1 + \int_a^\infty \left(\frac{\chi_0'}{r^4}\left|\mathcal{M}\hat{\phi}\right|^2 - m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \left(\frac{m^2}{r^2}+k^2\right)|\hat{\phi}|^2\right)\right)dr = 0, \tag{4.75}$$

or λ is real. If (4.75) holds $s_I^2 = 0$ and is real, and therefore λ is real. Hence ${s_I}^2$ and $\hat{\phi}$ are real in the inviscid limit, irrespective of the sign of χ_0'. Furthermore, if $\chi_0' \leq 0$ everywhere, observe from (4.71) that $s_I^2 \leq 0$ and all three-dimensional disturbances are stable. Since s_I^2 is real, s_I is imaginary (or zero) for stable modes.

By considering the functional

$$F(y) = B\frac{C_1 + \int_a^\infty \left(\frac{\chi_0'}{r^4}\left(\mathcal{M}y\right)^2 - m^2 r(1+\chi_0)\left((y')^2 + \left(\frac{m^2}{r^2}+k^2\right)y^2\right)\right)dr}{\int_a^\infty \frac{r}{m^2+k^2r^2}\left(\left(\mathcal{M}y\right)'\right)^2 + \frac{1}{r}\left(\mathcal{M}y\right)^2 dr} \tag{4.76}$$

for all real functions y satisfying the same boundary conditions as $\hat{\phi}$, with analogous reasoning to Sections 4.3.1 and 4.3.2, it follows that, if $\chi_0' > 0$ somewhere in the domain, then there exists an unstable mode. Consequently, in the inviscid limit, if, and only if, $\chi_0' > 0$ somewhere in the ferrofluid, the system is unstable to all three-dimensional disturbances.

4.3.4 The Stokes regime

In the highly viscous limit the method and reasoning is analogous to Sections 4.3.1 and 4.3.2, with $Re \to 0$, to prove that axisymmetric and $k = 0$ modes are unstable if, and only if, $\chi_0' > 0$ somewhere in the domain. For arbitrary Reynolds number, the eigenvalues (and eigenfunctions) are proven to be real for only unstable axisymmetric and two-dimensional modes. In the highly viscous limit however, we can prove the eigenvalues are real for all axisymmetric and two-dimensional disturbances.

(4.29) and (4.45) in the highly viscous limit become

$$\mathcal{L}_0^2\hat{\psi} = \frac{k^2\lambda_0\chi_0'\hat{\psi}}{r^3} \tag{4.77}$$

and

$$\mathcal{L}_m^2\mathcal{M}_m\hat{\phi} = m^2\lambda_1\left(\frac{\chi_0'\mathcal{M}_m\hat{\phi}}{r^5} + \frac{m^2\chi_0'\hat{\phi}}{r^3}\right), \tag{4.78}$$

respectively, where $\lambda_0 = B/s_v$ for $m = 0$ modes, and $\lambda_1 = B/s_v$ for $k = 0$ modes. Multiply (4.77) by $\hat{\psi}^*$, and multiply the complex conjugate of (4.77) by $\hat{\psi}$, to obtain

$$\int_a^\infty |\mathcal{L}_0\hat{\psi}|^2 r dr = k^2\lambda_0\int_a^\infty \frac{\chi_0'|\hat{\psi}|^2}{r^2}dr, \tag{4.79}$$

and

$$\int_a^\infty |\mathcal{L}_0\hat{\psi}|^2 r dr = k^2\lambda_0^* \int_a^\infty \frac{\chi_0'|\hat{\psi}|^2}{r^2} dr, \tag{4.80}$$

respectively, where λ_0^* is the complex conjugate of λ_0. It follows that

$$(\lambda_0 - \lambda_0^*) \int_a^\infty \frac{\chi_0'|\hat{\psi}|^2}{r^2} dr = 0, \tag{4.81}$$

and λ_0 and $\hat{\psi}$ are real. Note if the integral is zero, then $s_v = 0$ and the growth rate is real. Similarly for $k = 0$ modes, we obtain

$$\begin{aligned}&\int_a^\infty \left|\mathcal{L}_m(\mathcal{M}_m\hat{\phi})\right|^2 r dr + \lambda_1 m^2 \Bigg(m^3|\hat{\phi}(a)|^2 \\ &+ \int_a^\infty \left(\frac{-\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4} + m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2} \right)\right) dr \Bigg) = 0\end{aligned} \tag{4.82}$$

and

$$\begin{aligned}&\int_a^\infty \left|\mathcal{L}_m(\mathcal{M}_m\hat{\phi})\right|^2 r dr + \lambda_1^* m^2 \Bigg(m^3|\hat{\phi}(a)|^2 \\ &+ \int_a^\infty \left(\frac{-\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4} + m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2} \right)\right) dr \Bigg) = 0.\end{aligned} \tag{4.83}$$

Consequently,

$$(\lambda_1^* - \lambda_1)\left(m^3|\hat{\phi}(a)|^2 + \int_a^\infty \left(\frac{-\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4} + m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2} \right)\right) dr \right) = 0, \tag{4.84}$$

and either

$$m^3|\hat{\phi}(a)|^2 + \int_a^\infty \left(\frac{-\chi_0'|\mathcal{M}_m\hat{\phi}|^2}{r^4} + m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2} \right)\right) dr = 0 \tag{4.85}$$

or λ_1 is real. If (4.85) is true, then $s_v = 0$ (and real), proving that λ_1 is always real. Thus for axisymmetric and $k = 0$ modes in the highly viscous regime, the growth rate and eigenfunctions are real irrespective of the sign of χ_0'. By considering $Re \to 0$ in Sections 4.3.1 and 4.3.2, we can prove that axisymmetric and $k = 0$ modes are unstable if, and only if, $\chi_0' > 0$ somewhere in the domain.

4.4 Suppressing unstable modes with an axial field

We now show that by adding an axial field,

$$\boldsymbol{H_0} = \left(0, \frac{1}{r}, Z\right), \tag{4.86}$$

we can suppress unstable disturbances. It follows that

$$H_0 = \left(\frac{1}{r^2} + Z^2\right)^{\frac{1}{2}}, \quad H_1 = \frac{\mathrm{i}\,(m + kZr^2)}{r^2 H_0}\hat{\phi}(r)\zeta, \tag{4.87}$$

and (4.7) in component form is

$$(sRe - \mathcal{L})\hat{\omega}_r + \frac{2\mathrm{i}m}{r^2}\hat{\omega}_\theta = 0, \tag{4.88}$$

$$(sRe - \mathcal{L})\hat{\omega}_\theta - \frac{2\mathrm{i}m}{r^2}\hat{\omega}_r = B\left(\frac{k(m + kZr^2)\chi_0'\hat{\phi}}{r^2} + \mathrm{i}kH_0H_0'\hat{\chi}\right), \tag{4.89}$$

$$(sRe - \mathcal{D})\hat{\omega}_z = B\left(-\frac{m(m + kZr^2)\chi_0'\hat{\phi}}{r^3} - \frac{imH_0H_0'\hat{\chi}}{r}\right). \tag{4.90}$$

4.4.1 Axisymmetric disturbances

For solely axisymmetric disturbances

$$H_1 = \frac{\mathrm{i}kZ}{H_0}\hat{\phi}(r)e^{\mathrm{i}(kz)+st} \tag{4.91}$$

and (4.88)-(4.90) become

$$(sRe - \mathcal{L}_0)\hat{\omega}_r = 0, \tag{4.92}$$

$$(sRe - \mathcal{L}_0)\hat{\omega}_\theta = B\big(k^2Z\chi_0'\hat{\phi} + \mathrm{i}kH_0H_0'\hat{\chi}\big), \tag{4.93}$$

$$(sRe - \mathcal{D}_0)\hat{\omega}_z = 0. \tag{4.94}$$

(2.55) gives

$$\hat{\chi} = \frac{\mathrm{i}((1+\chi_0)\boldsymbol{\nabla}^2\hat{\phi} + \chi_0'\hat{\phi}')}{kZ}, \tag{4.95}$$

and it follows from (4.16), for axisymmetric disturbances, that

$$\hat{\psi} = \frac{-\mathrm{i}s\hat{\chi}}{k\chi_0'}, \tag{4.96}$$

where $\hat{\psi}$ was defined in Section 4.3.1.

(4.92)-(4.96) give an eigenvalue equation for $\hat{\phi}$,

$$(s^2 Re\mathcal{L}_0 - s\mathcal{L}_0^2)\mathcal{P}_0\hat{\phi} = k^2 B\chi_0'\big(H_0H_0'\mathcal{P}_0\hat{\phi} - k^2Z^2\hat{\phi}\big), \tag{4.97}$$

where

$$\mathcal{P}_0\hat{\phi} = \frac{(1+\chi_0)}{\chi_0'}\left(\frac{1}{r}(r\hat{\phi}')' - k^2\hat{\phi}\right) + \hat{\phi}'. \tag{4.98}$$

When $m=0$, (4.22) becomes

$$\hat{\phi}' = \frac{kI_1\hat{\phi}}{(1+\chi_0)I_0} \tag{4.99}$$

at $r=a$ and $\boldsymbol{\nabla}\hat{\phi} \to 0$ as $r\to\infty$. Moreover, due to the boundary conditions for $\hat{\chi}$, $\mathcal{P}_0\hat{\phi}=0$ at $r=a$ and as $r\to\infty$.

Multiply (4.97) by $r\mathcal{P}_0\hat{\phi}^*$, and integrate over the domain to give

$$\int_a^\infty \big(s^2 Re\mathcal{P}_0\hat{\phi}^*\mathcal{L}_0\mathcal{P}_0\hat{\phi} - s\mathcal{P}_0\hat{\phi}^*\mathcal{L}_0^2\mathcal{P}_0\hat{\phi}\big)rdr = k^2B\int_a^\infty r\chi_0'\big(H_0H_0'|\mathcal{P}_0\hat{\phi}|^2 - k^2Z^2\hat{\phi}\mathcal{P}_0\hat{\phi}^*\big)dr. \tag{4.100}$$

Invoking the self-adjoint property of $\mathcal{L}_0$ and the boundary conditions for $\hat{\phi}$ we can write (4.100) as

$$\begin{aligned}&\int_a^\infty \left(s^2 Re\left(\big|(\mathcal{P}_0\hat{\phi})'\big|^2 + \left(k^2+\frac{1}{r^2}\right)\big|\mathcal{P}_0\hat{\phi}\big|^2\right) + s\big|\mathcal{L}_0\mathcal{P}_0\hat{\phi}\big|^2\right)rdr\\ &+k^2B\int_a^\infty \left(H_0H_0'\chi_0'|\mathcal{P}_0\hat{\phi}|^2 + k^2Z^2(1+\chi_0)\big(|\hat{\phi}'|^2+k^2|\hat{\phi}|^2\big)\right)rdr = 0.\end{aligned} \tag{4.101}$$

Thus, for sufficiently large Z, $\mathcal{R}(s) < 0$, provided long waves are omitted and $B \neq 0$. Note that if $\mathcal{P}_0\hat{\phi}$ was very large, this may not hold. However, for $\mathcal{P}_0\hat{\phi} \gg \hat{\phi}$, either $\chi_0' \to 0$ (which is irrelevant to our investigation), or $\hat{\phi}'' \gg \hat{\phi}$ ($\hat{\phi}' \gg \hat{\phi}$). For the latter, $\hat{\phi}$ must be highly oscillatory, and $\mathcal{L}_0\mathcal{P}_0\hat{\phi} \gg \mathcal{P}_0\hat{\phi}$ and $\mathcal{L}_0^2\mathcal{P}_0\hat{\phi} \gg \mathcal{P}_0\hat{\phi}$. Consequently, (4.101) could be approximated as

$$\int_a^\infty \left(s^2 Re\left(\left|(\mathcal{P}_0\hat{\phi})'\right|^2 + \left(k^2 + \frac{1}{r^2}\right)\left|\mathcal{P}_0\hat{\phi}\right|^2\right) + s\left|\mathcal{L}_0\mathcal{P}_0\hat{\phi}\right|^2\right) r dr = 0. \tag{4.102}$$

In which case $s = 0$ or

$$s = -\frac{\int_a^\infty (|\mathcal{L}_0\mathcal{P}_0\hat{\phi}|^2) r dr}{\int_a^\infty (Re(|(\mathcal{P}_0\hat{\phi})'|^2 + (k^2 + \frac{1}{r^2})|\mathcal{P}_0\hat{\phi}|^2)) r dr}, \tag{4.103}$$

and therefore $s \leq 0$ (and real). Hence, it remains that applying a sufficiently large axial field will suppress axisymmetric unstable modes.

4.4.2 The inviscid limit

Similarly, in the inviscid limit, for all three-dimensional disturbances it can be proved that s_I^2 and $\hat{\phi}$ are real when an axial field is present, by an analogous approach to Section 4.3.3. s_I^2 is given by

$$s_I^2 = \frac{C_1 + B\int_a^\infty \frac{1}{r^2}\chi_0'|\mathcal{P}\hat{\phi}|^2 - (1+\chi_0) r\left((\hat{\phi}')^2 + (\frac{m^2}{r^2} + k^2)\hat{\phi}^2\right) dr}{\int_a^\infty \frac{r}{m^2 + k^2 r^2}|(r\mathcal{P}\hat{\phi})'|^2 + r|\mathcal{P}\hat{\phi}|^2 dr} \tag{4.104}$$

where

$$\mathcal{P}\hat{\phi} = \frac{r^2}{\chi_0'(m + Zkr^2)}\Big((1+\chi_0)\nabla^2\hat{\phi}_1 + \chi_0'\hat{\phi}_1'\Big). \tag{4.105}$$

It follows from (4.16) and (4.17) that $\mathcal{P}\hat{\phi} = 0$ at $r = a$ and $\mathcal{P}\hat{\phi} \to 0$ as $r \to \infty$. Moreover, (4.22) holds at $r = a$. We observe from (4.104), that applying a sufficiently large Z will ensure $s_I^2 < 0$, provided $B, k \neq 0$.

4.5 Concluding remarks

Although we have proven that if $\chi_0' > 0$ anywhere, all axisymmetric or two-dimensional disturbances are unstable, I was unable to prove, for arbitrary Reynolds number, that if $\chi_0' \leq 0$ everywhere, all modes are stable, only that axisymmetric or two-dimensional disturbances are stable. However in the inviscid regime every three-dimensional disturbance will be stable if $\chi_0' < 0$ everywhere. Consequently, in the inviscid limit the rigorous stability condition holds; if, and only if, $\chi_0' > 0$ somewhere in the fluid, the system is unstable. I conjecture this holds for arbitrary Reynolds number too.

As in Chapter 3, we find an axial field can dampen unstable modes. For arbitrary Reynolds number this is only proven for axisymmetric modes, yet in the inviscid regime all unstable modes are proven to be suppressed by a sufficiently large axial field, provided the axial wavenumber is bounded from zero.

Moreover, we prove that the square of the eigenvalues is real in the inviscid limit, the eigenvalues are real in the Stokes regime, and for arbitrary Reynolds number they are real for unstable modes. This was true for the eigenvalues in Chapter 3, where for arbitrary Reynolds number, a stable branch started as real when $Re \sim 0$, as Re increased it became complex, and as $Re \to \infty$ the real part of the growth rate tended to zero. On the other hand, an unstable branch remained real for all Reynolds number.

Physically, for an instability to occur, a source of energy is needed, enabling a perturbation to grow. Rosensweig (1985) gives the energy change, ΔE, when introducing a volume V of ferrofluid into a magneto-static field in free space as

$$\Delta E \sim -\int_V \chi H^2 dV. \tag{4.106}$$

For the system here,

$$\Delta E \sim -\int_a^\infty \frac{\chi(r)}{r} dr. \tag{4.107}$$

Observe from (4.107) that the integral is maximised, and therefore ΔE is minimised, if χ is largest where r is smallest. In this chapter we proved an instability occurs if ever $d\chi/dr > 0$. Since $H_0 = 1/r$, when $d\chi_0/dr > 0$, $dH_0/d\chi_0 < 0$, implying that an instability may occur to achieve a minimum energy configuration with $dH_0/d\chi_0 > 0$ everywhere. Although (4.107) suggests the minimal energy configuration would have χ_0 compacted as close to the wire as possible, there exist other stable χ_0 distributions such that $dH_0/d\chi_0 > 0$.

Similar energy arguments occur in other areas of fluid dynamics. An example is the Rayleigh-Taylor stability condition for the density of a fluid under gravity. The varying magnetic force over the ferrofluid as χ changes, plays the part of gravity and the resulting condition for stability is that χ must decrease continuously radially, just as the density must decrease continuously with height for a stable equilibrium. However, the stability criterion only applies to inviscid fluids. An analogous argument to Rayleigh's stability argument for centrifugal instability would be when one considers the change in energy when two parcels of ferrofluid at different radii are interchanged while conserving $\chi(r)$, where the resulting condition for stability is that χ must decrease continuously radially ($d\chi/dr < 0$). Yet, this argument would not account for viscous forces or three-dimensional disturbances.

Interestingly, $d\chi_0/dr > 0$ is a local condition, yet a global instability occurs, suggesting that when $dH_0/d\chi_0 < 0$ somewhere in the fluid, a release of energy locally suffices to drive a global instability. We surmise that given a more general geometry where the equilibrium satisfies $H_0 \equiv H_0(\chi_0)$, if $dH_0/d\chi_0 > 0$ everywhere the system would be stable, whereas there may be an instability if $dH_0/d\chi_0 < 0$ somewhere, a result that could be used to determine the stability of a stationary state in a more complicated geometry. This hypothesis motivates the subsequent chapters.

Chapter 5

An inhomogeneous ferrofluid in a channel, subject to a normal field

5.1 Introduction

Motivated by Chapter 4, we look for another equilibrium where the susceptibility and applied field are non-uniform, in order to test the hypothesis that $dH/d\chi < 0$ induces an instability. However, finding equilibria to the governing equations when the susceptibility is non-constant is not trivial. Nevertheless, in this chapter, by considering equipotential surfaces of zero mean curvature we find a family of solutions to the governing equations, and hence a family of equilibria, if the boundary conditions are satisfied. The stability of a specific configuration in a planar domain with this feature is determined. The equilibrium is proven unstable for all three-dimensional disturbances, for arbitrary Reynolds number. Yet, we prove a rapidly rotating field applied across the plane can suppress the instability, even though a constant or alternating field will not. The work in this chapter appears in Ferguson Briggs & Mestel (2022b).

5.2 Equipotential surfaces with zero-mean curvature

Stationary equilibria must satisfy both $\chi \equiv \chi(H)$ and (2.55). The simplest solutions have χ constant throughout the fluid, but non-constant solutions are hard to find. If χ takes an arbitrary but specific distribution in space, a field applied to the ferrofluid will adapt spatially to satisfy (2.55) and (2.58). We find (2.58) is satisfied by a set of solutions where

$$\chi = \frac{A}{H} - 1, \tag{5.1}$$

for a positive constant A, such that $A/H \geqslant 1$ throughout the fluid, ensuring $\chi \geqslant 0$. For χ given by (5.1), (2.55) becomes

$$\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla}\phi}{|\boldsymbol{\nabla}\phi|} \right) = 0. \tag{5.2}$$

Since $\boldsymbol{\nabla}\phi/|\boldsymbol{\nabla}\phi|$ is a unit normal to the surface of constant ϕ, (5.2) implies that the field adapts to produce equipotential surfaces with zero mean curvature (Goldman, 2005). Consequently, the governing equations are satisfied (subject to boundary conditions) for a stationary fluid where the spatial distributions of χ and H satisfy (5.1), and equipotential surfaces have zero mean curvature. It should be noted that (5.1) is not a physical or chemical relationship between H and χ, and is not valid for all H (for example $H \to 0$ and $H \to \infty$). χ does not need to depend explicitly on H for (5.1) to be satisfied.

Thus, in theory, we have a set of stationary equilibria where χ and H are related by (5.1) and ϕ satisfies (5.2), applicable to a general geometry, so long as the boundary conditions are satisfied. Moreover, observe from (5.1) that $dH/d\chi < 0$ and therefore we postulate they are unstable, since the regions of highest susceptibility and regions of strongest field do not coincide.

Consequently, we have produced a family of equilibria that satisfy (2.55)-(2.58), such that the equipotential surfaces have zero mean curvature. However, for the chosen geometry they need to satisfy the required boundary conditions (2.59)-(2.64). In this chapter, we consider the simplest zero-curvature surface; a planar domain.

5.3 Formulation

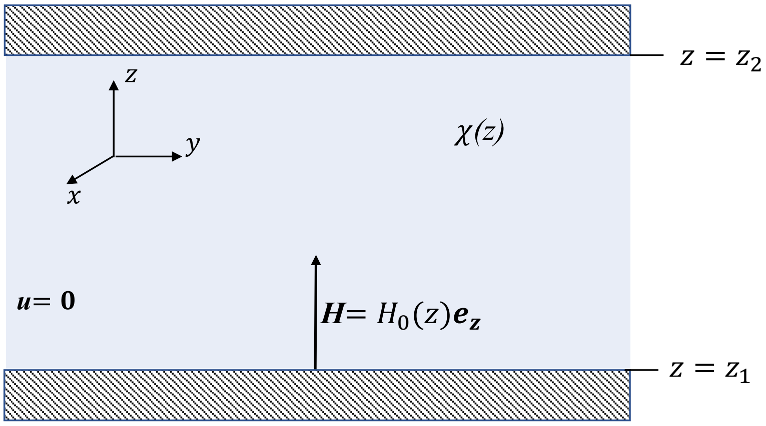

Figure 5.1: Schematic of the planar system

We consider a three-dimensional, Cartesian co-ordinate system, where a ferrofluid of spatially varying χ is between two solid plates at $z = z_1$ and $z = z_2$, where z points normal to the plates and x, y perpendicular to each other along the plates. An applied field varies normal to the plates in the region $z_1 < z < z_2$ such that $\boldsymbol{H} = H_0(z)\boldsymbol{e_z}$, where $\boldsymbol{e_z}$ is the unit vector in the z direction. The ferrofluid is incompressible and isothermal with constant ρ, η and μ_0. The fluid is initially at rest, and the susceptibility of the fluid satisfies

$$\chi(z) = \frac{A}{H_0(z)} - 1 \quad \text{for} \quad z_1 < z < z_2, \tag{5.3}$$

where $A/H_0 \geqslant 1$, but elsewhere,

$$\left.\begin{aligned} \chi = 0 \quad &\text{for} \quad z \leq z_1, \\ \chi = 0 \quad &\text{for} \quad z \geq z_2. \end{aligned}\right\} \tag{5.4}$$

Thus, in the plates the magnetic potential must satisfy,

$$\boldsymbol{\nabla}^2 \phi^{(l)} = 0, \tag{5.5}$$

where $l = 1, 2$ for $z < z_1, z > z_2$, respectively, with the boundary conditions

$$\frac{\partial \phi^{(l)}}{\partial z} = A, \tag{5.6}$$

$$\frac{\partial \phi^{(l)}}{\partial x} = 0, \quad \frac{\partial \phi^{(l)}}{\partial y} = 0 \tag{5.7}$$

at $z = z_l$. For simplicity, choose $\phi^{(l)} = Az$. A schematic of the problem is shown in Figure 5.1.

Now, given (5.1), it follows that

$$\frac{d\chi}{dH} = -\frac{1}{A}(1+\chi)^2 < 0, \tag{5.8}$$

and we expect that a disturbance to this equilibrium will result in a release of magnetic energy, driving an instability. For this equilibrium, an energy stability argument is easily applied, by assuming a parcel of fluid will retain its value of χ, and we illustrate this argument in Figure 5.2. Figure 5.2.a shows the parcels of fluid with highest susceptibility do not coincide with the regions of strongest field. Two parcels are swapped, resulting in the location of each parcel, and its corresponding value of susceptibility, "coinciding more" with the strength of the field, shown in Figure 5.2.b . This may result in a release of energy, as less energy is required for this formation, and the perturbation can use this energy to grow. The lowest energy formation is shown in figure 5.2.c . This argument is analogous to the energy arguments discussed in 4.5 in other areas of fluid dynamics.

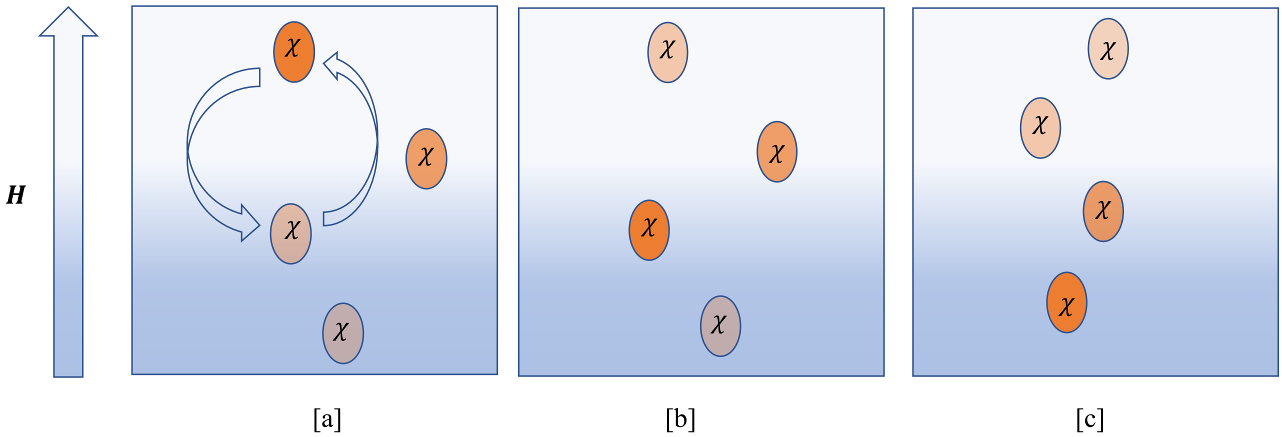

Figure 5.2: An illustration of an energy argument: The field acts in the vertical direction, where the darker the blue colour, the stronger the field. The ferrofluid is illustrated as parcels with individual values of susceptibility, where the darker the orange colour, the higher the susceptibility. Energy is released moving from [a] to [b], and [c] shows the formation of minimum energy.

Suspecting the system is unstable, we prove this using linear stability analysis. We consider perturbations to the stationary state such that

$$
\begin{aligned}
\boldsymbol{\omega} &= \epsilon \boldsymbol{\nabla} \times (\mathcal{R}(\hat{\boldsymbol{u}}(z)\delta)) + O(\epsilon^2), \\
\chi &= \frac{A}{H_0(z)} - 1 + \epsilon \mathcal{R}(\hat{\chi}(z)\delta) + O(\epsilon^2), \\
p &= p_0(z) + \epsilon \mathcal{R}(\hat{p}(z)\delta) + O(\epsilon^2), \\
H &= H_0(z) + \epsilon \mathcal{R}(\hat{\phi}'(z)\delta) + O(\epsilon^2),
\end{aligned}
\tag{5.9}
$$

where $'$ indicates the derivative w.r.t. z, $\hat{\boldsymbol{u}}(z) = (\hat{u}(z), \hat{v}(z), \hat{w}(z))^T$, $\delta = e^{\mathrm{i}(\alpha x + \beta y) + st}$, $\epsilon \ll 1$, α, β are real and positive wave numbers, the hat variables could be complex, s is the growth rate of the disturbance and could be complex, and $\mathcal{R}$ denotes the real part. From now on, the real part will not be written explicitly.

After substituting (5.9) and linearising, (2.58) in component form is

$$
(s\rho - \eta \nabla^2)(\mathrm{i}\beta \hat{w} - \hat{v}') \;=\; \mathrm{i}\mu_0 \beta \left(\frac{A H_0' \hat{\phi}'}{H_0} + H_0 H_0' \hat{\chi} \right), \tag{5.10}
$$

$$(s\rho - \eta\nabla^2)(\hat{u}' - \mathrm{i}\alpha\hat{w}) \;\; = \;\; -\mathrm{i}\mu_0\alpha\left(\frac{AH_0'}{H_0}\hat{\phi}' + H_0H_0'\hat{\chi}\right), \tag{5.11}$$

$$(s\rho - \eta\nabla^2)(\alpha\hat{v} - \beta\hat{u}) \;\; = \;\; 0. \tag{5.12}$$

(2.56) and (2.66) give

$$\mathrm{i}\alpha\hat{u} + \mathrm{i}\beta\hat{v} + \hat{w}' = 0 \tag{5.13}$$

and

$$\hat{w} = \frac{sH_0^2\hat{\chi}}{AH_0'}, \tag{5.14}$$

respectively, where

$$\nabla^2 = \frac{d^2}{dz^2} - (\alpha^2 + \beta^2). \tag{5.15}$$

We require $\hat{w} = 0$ at $z = z_{1,2}$, and it follows that $\hat{\chi} = 0$ at $z = z_{1,2}$. Note that by considering the sum of (5.10) multiplied by α and (5.11) multiplied by β, we find the equations are satisfied by $\alpha v = \beta u$. The direction of the wave-vector $(\alpha,\, \beta)$ is arbitrary, and we could have chosen $\beta = 0$ without loss of generality.

$\hat{\phi}^{(l)}$ has general solution

$$\hat{\phi}^{(l)} = q_1^{(l)} e^{\sqrt{\alpha^2+\beta^2}z} + q_2^{(l)} e^{-\sqrt{\alpha^2+\beta^2}z}, \tag{5.16}$$

for constants $q_1^{(l)}, q_2^{(l)}$. Imposing $\boldsymbol{\nabla}\phi^{(l)}$ regular as $z \to \pm\infty$ gives

$$\hat{\phi}^{(1)} = q_1^{(1)} e^{\sqrt{\alpha^2+\beta^2}z} \quad \text{and} \quad \hat{\phi}^{(2)} = q_2^{(2)} e^{-\sqrt{\alpha^2+\beta^2}z}. \tag{5.17}$$

Substituting (5.9) into (2.55) and linearising gives

$$\frac{A\nabla^2\hat{\phi}}{H_0} - \frac{AH_0'\hat{\phi}'}{H_0^2} = -(H_0\hat{\chi})'. \tag{5.18}$$

Write (5.18) as

$$(\hat{\chi}H_0)' = -\left(\frac{A\hat{\phi}'}{H_0}\right)' + \frac{A(\alpha^2 + \beta^2)\hat{\phi}}{H_0} \tag{5.19}$$

and integrate to give

$$\hat{\chi} = \frac{AH_0'\mathcal{H}\hat{\phi}}{H_0^2}, \tag{5.20}$$

where

$$\mathcal{H}\hat{\phi} = -\frac{1}{H_0'}\left(\hat{\phi}' - H_0(\alpha^2+\beta^2)\int_0^z \frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\right). \tag{5.21}$$

(2.59) and (2.60) applied at $z = z_{1,2}$ gives

$$\frac{A}{H_0}\hat{\phi}' = (\hat{\phi}^{(1,2)})' \tag{5.22}$$

and

$$\hat{\phi} = \hat{\phi}^{(1,2)}, \tag{5.23}$$

respectively. Substituting (5.17) into (5.22) and (5.23) results in

$$\frac{A\hat{\phi}'}{H_0} + \sqrt{\alpha^2+\beta^2}\hat{\phi} = 0 \tag{5.24}$$

at $z = z_2$, and

$$\frac{A\hat{\phi}'}{H_0} - \sqrt{\alpha^2+\beta^2}\hat{\phi} = 0 \tag{5.25}$$

at $z = z_1$.

Apply the operator

$$-\mathrm{i}\alpha(s\rho - \eta\nabla^2) \tag{5.26}$$

to (5.13) to give

$$\alpha^2(s\rho-\eta\nabla^2)\hat{u} + \alpha\beta(s\rho-\eta\nabla^2)\hat{v} = \mathrm{i}\alpha(s\rho-\eta\nabla^2)\hat{w}'. \tag{5.27}$$

Substitute (5.12) into (5.27) to give

$$(\alpha^2+\beta^2)(s\rho-\eta\nabla^2)\hat{u} = \mathrm{i}\alpha(s\rho-\eta\nabla^2)\hat{w}', \tag{5.28}$$

and take the derivative w.r.t z to give

$$(\alpha^2+\beta^2)(s\rho-\eta\nabla^2)\hat{u}' = \mathrm{i}\alpha(s\rho-\eta\nabla^2)\hat{w}''. \tag{5.29}$$

Substitute (5.29) into (5.11) to give

$$(s\rho-\eta\nabla^2)\nabla^2\hat{w} = -\mu_0(\alpha^2+\beta^2)\left(H_0H_0'\hat{\chi}+A\frac{H_0'}{H_0}\hat{\phi}'\right) \tag{5.30}$$

and substitute (5.14) and (5.20) to give

$$(s^2\rho-\eta s\nabla^2)\nabla^2\mathcal{H}\hat{\phi} = \frac{-\mu_0(\alpha^2+\beta^2)AH_0'\big(\hat{\phi}'+H_0'\mathcal{H}\hat{\phi}\big)}{H_0}. \tag{5.31}$$

Substituting (5.21) into (5.31) produces the eigenvalue equation

$$(s^2\rho-\eta s\nabla^2)\nabla^2\mathcal{H}\hat{\phi} = -\mu_0(\alpha^2+\beta^2)^2AH_0'\int_0^z\frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma. \tag{5.32}$$

(5.20) and (5.14) give $\mathcal{H}\hat{\phi}=0$ and $(\mathcal{H}\hat{\phi})'=0$ at $z=z_{1,2}$, as well as $\hat{\phi}$ satisfying (5.24) and (5.25).

For a specific H_0 we can solve (5.32) to determine the eigenmodes, $\hat{\phi}$, and the associated eigenvalues, s, but here we determine the stability of the system for general H_0. Multiply (5.32) by $\mathcal{H}\hat{\phi}^*$, where $\hat{\phi}^*$ is the complex conjugate of $\hat{\phi}$, to obtain

$$\begin{aligned}
&s^2\rho\int_{z_1}^{z_2}\big(\mathcal{H}\hat{\phi}^*\nabla^2\mathcal{H}\hat{\phi}\big)dz - s\eta\int_{z_1}^{z_2}\big(\mathcal{H}\hat{\phi}^*\nabla^4\mathcal{H}\hat{\phi}\big)dz \\
&+\mu_0A(\alpha^2+\beta^2)^2\int_{z_1}^{z_2}\left(H_0'\mathcal{H}\hat{\phi}^*\left(\int_0^z\frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\right)\right)dz = 0,
\end{aligned} \tag{5.33}$$

Using (5.21),

$$\begin{aligned}
&\int_{z_1}^{z_2}\left(H_0'\mathcal{H}\hat{\phi}^*\left(\int_0^z\frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\right)\right)dz \\
&= -\int_{z_1}^{z_2}\left(\hat{\phi}^{*\prime}\left(\int_0^z\frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\right)-H_0(\alpha^2+\beta^2)\left|\int_0^z\frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\right|^2\right)dz.
\end{aligned} \tag{5.34}$$

Integration by parts on (5.34) and invoking the boundary conditions gives

$$\int_{z_1}^{z_2}\left(H_0'\mathcal{H}\hat{\phi}^*\Big(\int_0^z \frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\Big)\right)dz = \int_{z_1}^{z_2}\left(\frac{|\hat{\phi}'|^2}{H_0} + H_0(\alpha^2+\beta^2)\Big|\int_0^z \frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\Big|^2\right)dz. \tag{5.35}$$

Substituting (5.35) into (5.33) and performing integration by parts on the remaining terms terms we obtain

$$\begin{aligned} &s^2\rho\int_{z_1}^{z_2}\left(|(\mathcal{H}\hat{\phi})'|^2 + (\alpha^2+\beta^2)|\mathcal{H}\hat{\phi}|^2\right)dz + s\eta\int_{z_1}^{z_2}|\nabla^2\mathcal{H}\hat{\phi}|^2 dz \\ &-A\mu_0(\alpha^2+\beta^2)^2\int_{z_1}^{z_2}\left(\frac{|\hat{\phi}'|^2}{H_0} + H_0(\alpha^2+\beta^2)\Big|\int_0^z \frac{\hat{\phi}(\gamma)}{H_0(\gamma)}d\gamma\Big|^2\right)dz = 0. \end{aligned} \tag{5.36}$$

(5.36) is of the form $as^2 + bs + c = 0$, with a, b, c real and $a, b > 0$, $c < 0$. It follows that s is real and therefore $\hat{\phi}$, $\hat{\chi}$ and $\hat{w}$ are real, but $\hat{u}$ and $\hat{v}$ are imaginary. Every three-dimensional disturbance possesses a growing mode ($s > 0$) and is unstable. This result and the stability condition in chapter 4, both support the theory that $dH/d\chi < 0$ results in an unstable configuration.

5.4 Adding a horizontal field

In an effort to stabilise the system, we apply a constant field in the x and y directions, such that $\boldsymbol{H}_0 = (D, E, H_0(z)))$. The equilibrium still requires $\chi_0 = A/H_0(z) - 1$, $\boldsymbol{u}_0 = 0$, and $p = p_0(z)$. Consequently the perturbed variables are now

$$\begin{aligned} \boldsymbol{\omega} &= \epsilon\boldsymbol{\nabla}\times(\mathcal{R}(\hat{\boldsymbol{u}}(z)\delta)) + O(\epsilon^2), \\ \chi &= \frac{A}{H_0(z)} - 1 + \epsilon\mathcal{R}(\hat{\chi}(z)\delta) + O(\epsilon^2), \\ p &= p_0(z) + \epsilon\mathcal{R}(\hat{p}(z)\delta) + O(\epsilon^2), \\ H &= \sqrt{D^2+E^2+(H_0(z))^2} + \epsilon\mathcal{R}\left(\frac{\mathrm{i}\alpha D\hat{\phi} + \mathrm{i}\beta E\hat{\phi} + H_0(z)\hat{\phi}'(z)}{\sqrt{D^2+E^2+(H_0(z))^2}}\delta\right) + O(\epsilon^2). \end{aligned} \tag{5.37}$$

and (2.58) in component form becomes

$$(s\rho - \eta\nabla^2)(\mathrm{i}\beta\hat{w} - \hat{v}') \;=\; -\frac{\mu_0 A H_0'}{H_0^2}\left(D\alpha\beta + E\beta^2\right)\hat{\phi} + \mathrm{i}\mu_0\beta\left(\frac{A H_0'\hat{\phi}'}{H_0} + H_0 H_0'\hat{\chi}\right), \quad (5.38)$$

$$(s\rho - \eta\nabla^2)(\hat{u}' - \mathrm{i}\alpha\hat{w}) \;=\; \frac{\mu_0 A H_0'}{H_0^2}\left(\alpha^2 D + \alpha\beta E\right)\hat{\phi} - \mathrm{i}\mu_0\alpha\left(\frac{A H_0'}{H_0}\hat{\phi}' + H_0 H_0'\hat{\chi}\right), \quad (5.39)$$

$$(s\rho - \eta\nabla^2)(\alpha\hat{v} - \beta\hat{u}) \;=\; 0. \quad (5.40)$$

(3.26) and (2.66) remain the same, but substituting (5.37) into (2.55) and linearising gives

$$\frac{A\nabla^2\hat{\phi}}{H_0} - \frac{A H_0'\hat{\phi}'}{H_0^2} = -(H_0\hat{\chi})' - \mathrm{i}\big(\alpha D + \beta E\big)\hat{\chi}. \quad (5.41)$$

Consequently we obtain two simultaneous equations

$$\left(s^2\rho\nabla^2 - s\eta\nabla^4\right)\frac{H_0^2\hat{\chi}}{H_0'} = -\mu_0 A(\alpha^2+\beta^2)H_0'\left(H_0\hat{\chi} + \frac{A}{H_0^2}(\mathrm{i}D\alpha + \mathrm{i}E\beta)\hat{\phi} + \frac{A}{H_0}\hat{\phi}'\right) \quad (5.42)$$

and

$$\left(\frac{A\hat{\phi}'}{H_0}\right)' - \frac{A}{H_0}(\alpha^2+\beta^2)\hat{\phi} = -\big((H_0\hat{\chi})' + \mathrm{i}(D\alpha + E\beta)\hat{\chi}\big). \quad (5.43)$$

Observe that modes satisfying $D\alpha + E\beta = 0$ remain unstable, since the equations return to that of Section 5.3. Thus, applying a horizontal field cannot suppress all unstable modes.

5.5 Stabilisation with a rotating field

We now investigate whether a rapidly rotating field is sufficient to stabilise the system. Applying an alternating field is analogous to applying a constant field, in that there will exist modes perpendicular to the field which remain unstable. But, for a rapidly rotating field, the field direction changes sufficiently often that an unstable mode perpendicular to the field may not have time to grow before it is no longer perpendicular to the field, and becomes dampened by the field.

Let $\boldsymbol{H} = (D\cos(\omega t), E\sin(\omega t), H_0(z))$, and for an equilibrium, it remains that $\chi = -1 + A/H_0(z)$, $\boldsymbol{u} = 0$, and $p = p_0(z)$ in the fluid. Note that $H_0(z) \neq |\boldsymbol{H}|$, but is the z-component of $\boldsymbol{H}$. In the plates let $\boldsymbol{H} = (D\cos(\omega t), E\sin(\omega t), A)$. Now, perturb the equilibrium such that

$$\begin{aligned}
\boldsymbol{\omega} &= \epsilon \boldsymbol{\nabla} \times (\mathcal{R}(\hat{\boldsymbol{u}}(z,t)e^{i\alpha x + i\beta y}) + O(\epsilon^2), \\
\chi &= \frac{A}{H_0(z)} - 1 + \epsilon\mathcal{R}(\hat{\chi}(z,t)e^{i\alpha x+i\beta y}) + O(\epsilon^2), \\
p &= p_0(z) + \epsilon\mathcal{R}(\hat{p}(z,t)e^{i\alpha x+i\beta y}\delta) + O(\epsilon^2), \\
H &= \sqrt{(D\cos(\omega t))^2 + (E\sin(\omega t))^2 + H_0^2} \\
&\quad + \epsilon\mathcal{R}\left(\frac{\mathrm{i}\alpha D\cos(\omega t)\hat{\phi} + \mathrm{i}\beta E\sin(\omega t)\hat{\phi} + H_0\hat{\phi}'(z)}{\sqrt{(D\cos(\omega t))^2 + (E\sin(\omega t))^2 + H_0^2}}\delta\right) + O(\epsilon^2).
\end{aligned} \tag{5.44}$$

Substituting the perturbed variables into equations (2.55)-(2.58) and (2.66), linearising and manipulating the equations by an analogous method to Section 5.4, with D replaced by $D\cos(\omega t)$ and E replaced with $E\sin(\omega t)$, we obtain two simultaneous equations;

$$\begin{aligned}
\left(\rho\partial_{tt}\nabla^2 - \eta\partial_t\nabla^4\right)\left(\frac{H_0^2\hat{\chi}}{H_0'}\right) =& - A\mu_0(\alpha^2+\beta^2)H_0'\Big(H_0\hat{\chi} \\
&- \frac{A}{H_0^2}\big(\mathrm{i}(D\alpha\cos(\omega t) + E\beta\sin(\omega t))\hat{\phi} + H_0\partial_z\hat{\phi}\big)\Big)
\end{aligned} \tag{5.45}$$

and

$$\partial_z\left(\frac{A}{H_0}\partial_z\hat{\phi}\right) - \frac{A}{H_0}(\alpha^2+\beta^2)\hat{\phi} = -\big(\partial_z(\hat{\chi}H_0) + \mathrm{i}(D\cos(\omega t)\alpha + E\sin(\omega t)\beta)\hat{\chi}\big). \tag{5.46}$$

The solution in the walls remains as (5.17), but (2.59) and (2.60) give

$$\frac{A}{H_0}\partial_z\hat{\phi} + \hat{\chi}H_0 = \partial_z\hat{\phi}^{(1,2)} \tag{5.47}$$

and

$$\hat{\phi} = \hat{\phi}^{(1,2)} \tag{5.48}$$

respectively, at $z = z_{1,2}$. Consequently,

$$\frac{A\partial_z\hat{\phi}}{H_0} + \hat{\chi}H_0 + \left(\sqrt{\alpha^2+\beta^2}\right)\hat{\phi} = 0 \tag{5.49}$$

at $z = z_2$ and

$$\frac{A\partial_z\hat{\phi}}{H_0} + \hat{\chi}H_0 - \left(\sqrt{\alpha^2+\beta^2}\right)\hat{\phi} = 0 \tag{5.50}$$

at $z = z_1$. By requiring $\boldsymbol{u} = 0$ at $z = z_{1,2}$, we infer $\partial_t\hat{\chi} = 0$ at $z = z_{1,2}$. Thus χ is constant at the walls, which we define to be zero, and (5.49) and (5.50) simplify to

$$\frac{A\partial_z\hat{\phi}}{H_0} + \left(\sqrt{\alpha^2+\beta^2}\right)\hat{\phi} = 0 \tag{5.51}$$

at $z = z_2$ and

$$\frac{A\partial_z\hat{\phi}}{H_0} - \left(\sqrt{\alpha^2+\beta^2}\right)\hat{\phi} = 0 \tag{5.52}$$

at $z = z_1$, respectively. We now set $D = E$ for simplicity, but the analysis is analogous for $D \neq E$.

We suppose there are two time scales; one for the growth rate of the instability and the other for the rotation speed such that

$$\hat{\phi} = \phi_c(z,t) + A_1(z,t)\cos(\omega t) + A_2(z,t)\sin(\omega t), \tag{5.53}$$

$$\hat{\chi} = \chi_c(z,t) + \frac{A_3(z,t)\cos(\omega t) + A_4(z,t)\sin(\omega t)}{\omega}. \tag{5.54}$$

The scaling in (5.54) anticipates that $\chi_t \sim \boldsymbol{u}\cdot\boldsymbol{\nabla}\chi$ in (2.66). The boundary conditions for χ, give $\chi_c(z,t) = A_{3,4} = 0$ at $z = z_{1,2}$. (5.51) and (5.52) give

$$\frac{A\partial_z\phi_c}{H_0\sqrt{\alpha^2+\beta^2}} + \phi_c = 0, \quad \frac{A\partial_z A_{1,2}}{H_0\sqrt{\alpha^2+\beta^2}} + A_{1,2} = 0 \tag{5.55a, b}$$

at $z=z_2$, and

$$\frac{A\partial_z\phi_c}{H_0\sqrt{\alpha^2+\beta^2}}-\phi_c=0,\qquad \frac{A\partial_z A_{1,2}}{H_0\sqrt{\alpha^2+\beta^2}}-A_{1,2}=0 \tag{5.56a,b}$$

at $z=z_1$, respectively.

Substituting (5.53) and (5.54) into (5.45) gives

$$\begin{aligned}
&\omega\rho\nabla^2\bigg(\frac{H_0^2}{H_0'}\bigg(\partial_t^2\chi_c+\bigg(\frac{\partial_{tt}A_3}{\omega^2}+2\frac{\partial_tA_4}{\omega}-A_3\bigg)\cos(\omega t)+\bigg(\frac{\partial_{tt}A_4}{\omega^2}-A_4-2\frac{\partial_tA_3}{\omega}\bigg)\sin(\omega t)\bigg)\bigg)\\
&-\eta\nabla^4\bigg(\frac{H_0^2}{H_0'}\bigg(\partial_t\chi_c+\bigg(A_4+\frac{\partial_tA_3}{\omega}\bigg)\cos(\omega t)+\bigg(-A_3+\frac{\partial_tA_4}{\omega}\bigg)\sin(\omega t)\bigg)\bigg)\\
&=-A\mu_0(\alpha^2+\beta^2)\bigg(H_0H_0'\bigg(\chi_c+\frac{A_3}{\omega}\cos(\omega t)+\frac{A_4}{\omega}\sin(\omega t)\bigg)\\
&+\frac{AH_0'}{H_0^2}\big(iD(\alpha\cos(\omega t)+\beta\sin(\omega t))+H_0\partial_z\big)\big(\phi_c+A_1\cos(\omega t)+A_2\sin(\omega t)\big)\bigg),
\end{aligned} \tag{5.57}$$

and into (5.46) gives

$$\begin{aligned}
&\bigg(\partial_z\bigg(\frac{A}{H_0}\partial_z\bigg)-\frac{A}{H_0}(\alpha^2+\beta^2)\bigg)(\phi_c+A_1\cos(\omega t)+A_2\sin(\omega t))\\
&=-\mathrm{i}D\bigg(\alpha\cos(\omega t)+\beta\sin(\omega t)\bigg)\bigg(\chi_c+\frac{A_3}{\omega}\cos(\omega t)+\frac{A_4}{\omega}\sin(\omega t)\bigg)\\
&-\partial_z\bigg(H_0\bigg(\chi_c+\frac{A_3}{\omega}\cos(\omega t)+\frac{A_4}{\omega}\sin(\omega t)\bigg)\bigg).
\end{aligned} \tag{5.58}$$

Assume $O(1/\omega^2)$ and $O(1/\omega)$ terms are negligible, as a result of the rapid rotation. Equating terms in (5.57) gives

$$\rho\nabla^2\bigg(\frac{H_0^2A_3}{H_0'}\bigg)+\eta\nabla^4\bigg(\frac{H_0^2A_4}{H_0'}\bigg)=\frac{\mu_0A^2(\alpha^2+\beta^2)H_0'\big(\mathrm{i}D\alpha\phi_c+H_0\partial_zA_1\big)}{H_0^2} \tag{5.59}$$

and

$$\rho\nabla^2\bigg(\frac{H_0^2A_4}{AH_0'}\bigg)-\eta\nabla^4\bigg(\frac{H_0^2A_3}{AH_0'}\bigg)=\frac{AH_0'\mu_0(\alpha^2+\beta^2)\big(\mathrm{i}D\beta\phi+H_0\partial_zA_2\big)}{H_0^2}. \tag{5.60}$$

Similarly, (5.58) gives

$$\mathcal{M}A_1=-\mathrm{i}D\alpha\chi_c \tag{5.61}$$

and

$$\mathcal{M}A_2 = -\mathrm{i}D\beta\chi_c, \tag{5.62}$$

where

$$\mathcal{M} = \partial_z\left(\frac{A}{H_0}\partial_z\right) - \frac{A}{H_0}(\alpha^2+\beta^2). \tag{5.63}$$

Time averaging over the fast time scale in (5.57) and (5.58) gives

$$\begin{aligned}\rho\nabla^2\left(\frac{H_0^2\partial_t^2\chi_c}{H_0'}\right) - \eta\nabla^4\left(\frac{H_0^2\partial_t\chi_c}{H_0'}\right) = &-\mu_0 A(\alpha^2+\beta^2)\Bigg(H_0H_0'\chi_c \\ &+ \frac{AH_0'(2H_0\partial_z\phi_c + \mathrm{i}D(\alpha A_1+\beta A_2))}{2H_0^2}\Bigg)\end{aligned} \tag{5.64}$$

and

$$\mathcal{M}\phi_c = -\partial_z(H_0\chi_c). \tag{5.65}$$

Since the coefficients do not depend on t we assume $\partial_t\chi_c = s\chi(z)e^{st}$, $\partial_t\phi_c = s\phi(z)e^{st}$ to give

$$\left(\rho s^2\nabla^2 - \eta s\nabla^4\right)\left(\frac{H_0^2\chi}{AH_0'}\right) = -\mu_0(\alpha^2+\beta^2)\left(H_0H_0'\chi + \frac{AH_0'(2H_0\phi' + \mathrm{i}D(\alpha A_1+\beta A_2))}{2H_0^2}\right) \tag{5.66}$$

and

$$\mathcal{M}\phi = -(H_0\chi)'. \tag{5.67}$$

Integrating (5.67) gives

$$\chi = \frac{AH_0'\mathcal{H}\phi}{H_0^2}, \tag{5.68}$$

and substituting (5.68) into (5.66) results in

$$\left(\rho s^2\nabla^2 - \eta s\nabla^4\right)\left(\mathcal{H}\phi\right) = -\mu_0 A(\alpha^2+\beta^2)H_0'\left((\alpha^2+\beta^2)\int_0^z \frac{\phi(\delta)}{H_0(\delta)}d\delta + \frac{\mathrm{i}D(\alpha A_1+\beta A_2)}{2H_0^2}\right). \tag{5.69}$$

Apply the operator

$$\mathcal{M}\frac{H_0^2}{H_0'} \tag{5.70}$$

to (5.69), substitute for $\mathcal{M}A_1$, $\mathcal{M}A_2$, and use (5.68), to obtain the eigenvalue equation,

$$\mathcal{M}\left(\frac{H_0^2}{H_0'}(s^2\rho\boldsymbol{\nabla}^2-s\eta\boldsymbol{\nabla}^4)\mathcal{H}\phi\right) = -\mu_0 A(\alpha^2+\beta^2)^2\left(\mathcal{M}\left(H_0^2\int_0^z\frac{\phi(\delta)}{H_0(\delta)}d\delta\right)+\frac{AD^2H_0'\mathcal{H}\phi}{H_0^2}\right). \quad (5.71)$$

Coupled with the boundary conditions

$$\mathcal{H}\phi = 0, \quad (\mathcal{H}\phi)' = 0, \quad A\phi' - H_0\left(\sqrt{\alpha^2+\beta^2}\right)\phi = 0 \quad (5.72)$$

st $z = z_1$, and

$$\mathcal{H}\phi = 0, \quad (\mathcal{H}\phi)' = 0, \quad A\phi' + H_0\left(\sqrt{\alpha^2+\beta^2}\right)\phi = 0 \quad (5.73)$$

at $z = z_2$.

Next, suppose D is sufficiently large, such that

$$\frac{AD^2H_0'\mathcal{H}\phi}{H_0^2} \gg \mathcal{M}\left(H_0^2\int_0^z\frac{\phi(\delta)}{H_0(\delta)}d\delta\right), \quad (5.74)$$

so that (5.71) approximates as

$$\mathcal{M}\left(\frac{H_0^2}{H_0'}(s^2\rho\boldsymbol{\nabla}^2 - s\eta\boldsymbol{\nabla}^4)\mathcal{H}\phi\right) = -\frac{\mu_0(\alpha^2+\beta^2)^2A^2D^2H_0'\mathcal{H}\phi}{H_0^2}. \quad (5.75)$$

Rather than solve (5.75), we determine whether a sufficiently large D dampens all disturbances. Multiply (5.75) by

$$\frac{H_0^2}{H_0'}\nabla^2\mathcal{H}\phi^* \quad (5.76)$$

and integrate over the domain to give

$$s^2\rho\int_{z_1}^{z_2}\frac{H_0^2}{H_0'}(\nabla^2\mathcal{H}\phi^*)\mathcal{M}\left(\frac{H_0^2}{H_0'}\nabla^2\mathcal{H}\phi\right)dz - s\eta\int_{z_1}^{z_2}\frac{H_0^2}{H_0'}(\nabla^2\mathcal{H}\phi^*)\mathcal{M}\left(\frac{H_0^2}{H_0'}\nabla^4\mathcal{H}\phi\right)dz$$
$$= -\mu_0(\alpha^2+\beta^2)A^2D^2\int_{z_1}^{z_2}\mathcal{H}\phi\nabla^2\mathcal{H}\phi^*dz. \quad (5.77)$$

Integration by parts gives

$$s^2\rho\int_{z_1}^{z_2}\frac{A}{H_0}\left(\left|\left(\frac{H_0^2}{H_0'}\nabla^2\mathcal{H}\phi\right)'\right|^2+(\alpha^2+\beta^2)\left|\frac{H_0^2}{H_0'}\nabla^2\mathcal{H}\phi\right|^2\right)dz$$
$$+s\eta\int_{z_1}^{z_2}\frac{H_0^2}{H_0'}(\nabla^2\mathcal{H}\phi^*)\mathcal{M}\left(\frac{H_0^2}{H_0'}\nabla^4\mathcal{H}\phi\right)dz$$
$$+\mu_0(\alpha^2+\beta^2)^2A^2D^2\int_{z_1}^{z_2}\left(|(\mathcal{H}\phi)'|^2+(\alpha^2+\beta^2)|(\mathcal{H}\phi)|^2\right)dz=0. \tag{5.78}$$

To obtain a second equation multiply (5.75) by

$$\frac{H_0^2}{H_0'}\nabla^4\mathcal{H}\phi^*, \tag{5.79}$$

integrate over the domain and use integration by parts to obtain

$$s^2\rho\int_{z_1}^{z_2}\frac{H_0^2}{H_0'}\nabla^4(\mathcal{H}\phi^*)\mathcal{M}\left(\frac{H_0^2}{H_0'}\nabla^2\mathcal{H}\phi\right)dz$$
$$+s\eta\int_{z_1}^{z_2}\frac{A}{H_0}\left(\left|\left(\frac{H_0^2}{H_0'}\nabla^4\mathcal{H}\phi\right)'\right|^2+(\alpha^2+\beta^2)\left|\frac{H_0^2}{H_0'}\nabla^4\mathcal{H}\phi\right|^2\right)dz$$
$$+\mu_0(\alpha^2+\beta^2)^2A^2D^2\int_{z_1}^{z_2}|\nabla^2\mathcal{H}\phi|^2dz=0. \tag{5.80}$$

Although ϕ is an eigenfunction that depends implicitly on s, we express (5.78) and (5.80) in the form of a quadratic equation. We write $a_js_j^2+b_js_j+D^2c_j=0$, where $j=1,2$ for (5.78) and (5.80) respectively, and $a_1>0$, $b_2>0$, $c_j>0$. It follows that if $a_2>0$ and $b_1>0$, $\mathcal{R}(s_{1,2})<0$, and therefore $\mathcal{R}(s)<0$ for all disturbances, since s must satisfy (5.78) and (5.80) simultaneously.

We now prove $a_2>0$ and $b_1>0$. Using the quadratic formula and expanding s_j for $D\gg 1$ gives

$$s_1=\frac{-b_1}{2|a_1|}\pm\frac{\sqrt{D}}{2|a_1|}\left(2\sqrt{-|a_1||c_1|}-\frac{{b_1}^2\sqrt{-|a_1||c_1|}}{8D|a_1||c_1|}+O(D^{-2})\right) \tag{5.81}$$

and

$$s_2 = \frac{-|b_2|}{2a_2} \pm \frac{\sqrt{D}}{2a_2}\left(2\sqrt{-a_2|c_2|} - \frac{{b_2}^2\sqrt{-a_2|c_2|}}{8Da_2|c_2|} + O(D^{-2})\right), \tag{5.82}$$

where we use the modulus sign to denote that the variable is positive. Thus to highest order

$$s_1 = \pm \mathrm{i}\sqrt{\frac{D|c_1|}{|a_1|}}, \quad s_2 = \pm\sqrt{-\frac{D|c_2|}{a_2}}. \tag{5.83}$$

Since s_1 is imaginary and to leading order s must equal both s_1 and s_2, it follows that s_2 must be imaginary, and therefore $a_2 > 0$. To next order

$$s_1 = \frac{-b_1}{2|a_1|} \pm i\sqrt{\frac{D|c_1|}{|a_1|}}, \quad s_2 = \frac{-|b_2|}{2a_2} \pm \sqrt{-\frac{D|c_2|}{a_2}}. \tag{5.84}$$

Since $a_2 > 0$, it follows that $b_1 > 0$, and therefore $Re(s_j) < 0$. We conclude that for sufficiently large D, all modes are stable. We have thus demonstrated that adding a sufficiently strong rapidly rotating field in the (x, y) plane stabilises the system.

5.6 Concluding remarks

A planar equilibrium where the susceptibility and field vary in the normal direction such that $dH/d\chi < 0$, has been proven to be unstable. Adding a sufficiently large, rapidly rotating magnetic field will dampen all unstable modes, where a constant, or even an alternating field, cannot. Here we have considered rigid boundaries, but we could have no plates and impose the appropriate boundary conditions as $z \to \pm\infty$ and follow analogous analysis.

We acknowledge that a rapidly rotating field may introduce thermal effects on the ferrofluid (Beković et al., 2014) but we assume the system to be isothermal throughout. Arguably, a rotating field may also cause a magneto-viscosity in the ferrofluid. However it is found that for weakly non-equilibrium situations (and therefore in quasi-equilibrium situations), a rotating field does not induce a “spin up” like an alternating field (Shliomis, 2002). Moreover, under the quasi-equilibrium approach the magneto-viscous effects are ignored.

More generally, when the field is a function of the susceptibility, equipotential surfaces of zero mean curvature give a set of solutions to the governing equations. The planar geometry is a special case of this, satisfying the appropriate boundary conditions. Thus, in theory, for a general geometry there exist stationary states where $d\chi/dH < 0$, if the boundary conditions can be satisfied. We would expect these to be unstable, but there may be methods of stabilising the system. A rotating field may not be sufficient in more complicated configurations, and the addition of a different field may be needed to dampen all modes.

Chapter 6

Stability condition for a general volume of ferrofluid with non-uniform susceptibility, in the presence of a non-uniform field

6.1 Introduction

The stability of two equilibria where χ and H are both non-uniform, one in a cylindrical and the other in a planar geometry, has been determined. The "simplicity" of the equilibria and the resulting equations, allowed us to use linear stability analysis to prove that the equilibria were unstable when $dH/d\chi < 0$, supporting our hypothesis. However, linear stability analysis of more complicated equilibria where χ and H are both non-constant, may not be easily achieved, and a stability condition for a general geometry would therefore be useful, although this is not trivial to determine.

In this chapter, we consider a general volume of ferrofluid, whose susceptibility varies with position, in the presence of a non-uniform field. To simplify the equations so as to allow for the derivation of a stability condition, we consider the susceptibility varying slowly in space. By performing linear stability analysis on a stationary state, we obtain a stability condition for a general volume of ferrofluid, dependent on the sign of the gradient of the magnitude of the field with respect to the susceptibility of the ferrofluid.

6.2 Equilibrium and the linearised equations

Suppose we have a stationary equilibrium of a volume of ferrofluid in the presence of a field with magnitude $H = H_0$, where $\boldsymbol{u} = \boldsymbol{0}$ is imposed on the boundaries. χ varies with position throughout the ferrofluid, and it may depend explicitly on H. For a stationary state, the initial susceptibility is such that $\chi_0 \equiv \chi_0(H_0)$ and

$$\boldsymbol{\nabla} \cdot ((1 + \chi_0)\boldsymbol{H_0}) = 0. \tag{6.1}$$

From (2.58) it follows that

$$\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} H_0 = 0. \tag{6.2}$$

Moreover, the initial pressure, $p = p_0$, must satisfy

$$\boldsymbol{\nabla} p_0 = -\mu_0 \int_0^{H_0} H_0 \boldsymbol{\nabla}_{H_0}\chi_0 dH_0 \tag{6.3}$$

such that (2.57) is satisfied.

Consider perturbations to the stationary state such that

$$\boldsymbol{H} = \boldsymbol{H_0} + \epsilon \boldsymbol{H_1} e^{st}, \tag{6.4}$$

$$\chi = \chi_0 + \epsilon \chi_1 e^{st}, \tag{6.5}$$

$$\boldsymbol{u} = \epsilon \boldsymbol{u_1} e^{st}, \tag{6.6}$$

$$p = p_0 + \epsilon p_1 e^{st}, \tag{6.7}$$

where the variables with subscript 1 are functions of position. Substituting (6.4)-(6.7) into (2.55), (2.56), (2.44), (2.66), and linearising gives

$$(1 + \chi_0)\boldsymbol{\nabla} \cdot \boldsymbol{H_1} + \boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{H_1} + \chi_1\boldsymbol{\nabla} \cdot \boldsymbol{H_0} + \boldsymbol{\nabla}\chi_1 \cdot \boldsymbol{H_0} = 0, \tag{6.8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0, \tag{6.9}$$

$$sRe\boldsymbol{\omega}_1 = \nabla^2\boldsymbol{\omega_1} + BH_0(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}H_1 + \boldsymbol{\nabla}\chi_1 \times H_0), \tag{6.10}$$

$$s\chi_1 + (\boldsymbol{u}_1 \cdot \boldsymbol{\nabla})\chi_0 = 0, \tag{6.11}$$

where $\boldsymbol{\omega_1} = \boldsymbol{\nabla} \times \boldsymbol{u_1}$, and Re and B are defined in Section 2.5.1 and are determined by the appropriate scales for the system under investigation. Rather than solving the eigenvalue problem for a specific equilibrium, we seek a generic stability condition.

6.3 Formulation for a stationary state with a slowly spatially-varying magnetic susceptibility

To allow progress in determining a stability condition, we simplify the equations by considering χ_0 varying slowly spatially. Namely, assume that $\boldsymbol{\nabla}\chi_0 = O(\epsilon_2)$, where $\epsilon_2 \ll 1$. It follows from (6.3) that $\boldsymbol{\nabla}p_0 = O(\epsilon_2)$ and (6.1) to highest order becomes

$$\boldsymbol{\nabla} \cdot \boldsymbol{H_0} = 0. \tag{6.12}$$

In the perturbed system, (6.8)-(6.11) become

$$(1 + \chi_0)\boldsymbol{\nabla} \cdot \boldsymbol{H_1} + \epsilon_2\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{H_1} + \chi_1\boldsymbol{\nabla} \cdot \boldsymbol{H_0} + \boldsymbol{\nabla}\chi_1 \cdot \boldsymbol{H_0} = 0, \tag{6.13}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0, \tag{6.14}$$

$$sRe\boldsymbol{\omega}_1 = \nabla^2\boldsymbol{\omega_1} + B(\epsilon_2\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}H_1 + \boldsymbol{\nabla}\chi_1 \times H_0), \tag{6.15}$$

$$s\chi_1 + \epsilon_2\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\chi_0 = 0. \tag{6.16}$$

It follows from (6.16) that $s = O(\epsilon_2)$ to retain the time-dependence, and as Re is scaled with time in (6.15), $Re = O(1/\epsilon_2)$. Define $s^* = s/\epsilon_2$ and $Re^* = \epsilon_2 Re$, substitute these into (6.15) and (6.16) and drop the stars. Consequently, to highest order;

$$(1 + \chi_0)\boldsymbol{\nabla} \cdot \boldsymbol{H_1} + \boldsymbol{\nabla}\chi_1 \cdot \boldsymbol{H_0} = 0, \tag{6.17}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0, \tag{6.18}$$

$$sRe\boldsymbol{\omega}_1 = \boldsymbol{\nabla}^2\boldsymbol{\omega_1} + BH_0(\boldsymbol{\nabla}\chi_1 \times \boldsymbol{\nabla}H_0), \tag{6.19}$$

$$s\chi_1 + \boldsymbol{u} \cdot \boldsymbol{\nabla}\chi_0 = 0. \tag{6.20}$$

Conveniently, the magnetic perturbation no longer appears in (6.18)-(6.20) and we no longer need (6.17) in the analysis.

Substituting (6.20) into (6.19) results in

$$s^2Re\boldsymbol{\omega}_1 - s\nabla^2\boldsymbol{\omega_1} = BH_0\boldsymbol{\nabla}H_0 \times \boldsymbol{\nabla}(\boldsymbol{u_1} \cdot \boldsymbol{\nabla}\chi_0). \tag{6.21}$$

Since χ_0 is a function of H_0, define

$$H_0' = \frac{dH_0}{d\chi_0}, \tag{6.22}$$

to write

$$H_0\boldsymbol{\nabla}H_0 = H_0H_0'\boldsymbol{\nabla}\chi_0. \tag{6.23}$$

Re-write (6.21) as

$$s^2Re\boldsymbol{\omega}_1 - s\nabla^2\boldsymbol{\omega_1} = BH_0H_0'\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(\boldsymbol{u_1} \cdot \boldsymbol{\nabla}\chi_0), \tag{6.24}$$

In the next three sections we use (6.18), (6.24), and impose $\boldsymbol{u_1} = 0$ on the boundary of the volume of ferrofluid, to deduce a stability condition for a volume of ferrofluid with a slowly spatially-varying χ. We drop the subscripts hereon. The vector identities and theorems that we use in subsequent sections are given in Appendix A.6 for ease of reference.

6.4 Cartesian system

For simplicity, first consider a volume of ferrofluid V in a Cartesian co-ordinate system x, y, z, where the vectors $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$ are orthogonal unit vectors in the x and y directions, respectively, and $\boldsymbol{e_z}$ is the unit normal to the x, y plane. Suppose that the system is invariant in z. We impose $\boldsymbol{u} = 0$ and $\chi_0 = 0$ at the boundary of V, and define a cross-section of the volume as the surface S in the x, y plane, such that $\boldsymbol{e_z}$ is normal to S. It follows that $\boldsymbol{u}$ and χ_0 are zero on the boundary of S, which we define as ∂S.

Define a stream-function ψ such that

$$\boldsymbol{u} = \boldsymbol{\nabla} \times \psi \boldsymbol{e_z}, \tag{6.25}$$

and therefore

$$\boldsymbol{\nabla} \psi = 0, \quad \text{on} \quad \partial S. \tag{6.26}$$

Given $\boldsymbol{\omega} = \omega \boldsymbol{e_z}$ in Cartesian co-ordinates, it follows that

$$\nabla^2 \psi = -\omega. \tag{6.27}$$

Consequently (6.24) is written as

$$\big[- s^2 Re \nabla^2 \psi + s \nabla^4 \psi \big] \boldsymbol{e_z} = B H_0 H_0' \boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla}((\boldsymbol{\nabla} \times \psi \boldsymbol{e_z}) \cdot \boldsymbol{\nabla} \chi_0). \tag{6.28}$$

Using (A.55), (6.28) can be written as

$$\big[- s^2 Re \nabla^2 \psi + s \nabla^4 \psi \big] \boldsymbol{e_z} = B H_0 H_0' \boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla}((\boldsymbol{\nabla} \psi \times \boldsymbol{e_z}) \cdot \boldsymbol{\nabla} \chi_0), \tag{6.29}$$

since $\boldsymbol{\nabla} \times \boldsymbol{e_z} = 0$, and using (A.51),

$$\big[- s^2 Re \nabla^2 \psi + s \nabla^4 \psi \big] \boldsymbol{e_z} = B H_0 H_0' \boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \big((\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z} \big). \tag{6.30}$$

Dot (6.30) with $\boldsymbol{e_z}$, multiply by ψ^*, and integrate over the surface S to give

$$-s^2 Re \iint_S \psi^* \nabla^2 \psi dS + s \iint_S \psi^* \nabla^4 \psi dS = B \iint_S H_0 H_0' \psi^* \big(\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \big((\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z} \big) \big) \cdot \boldsymbol{e_z} dS, \tag{6.31}$$

where ψ^* is the complex conjugate of ψ.

Define the integral

$$I_{R1} = \iint_S H_0 H_0' \psi^* \big(\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \big((\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z} \big) \big) \cdot \boldsymbol{e_z} dS. \tag{6.32}$$

Now,

$$\begin{aligned} \boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \Big(\psi^* \big((\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z} \big) \Big) =& \psi^* \boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \big((\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z} \big) \\ &+ \big((\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z} \big) \boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi^*, \end{aligned} \tag{6.33}$$

thus

$$I_{R1} = \iint_S H_0 H_0' \Big(\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \nu - (\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi^*) \big((\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z} \big) \Big) \cdot \boldsymbol{e_z} dS, \tag{6.34}$$

where $\nu = \psi^* (\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \psi) \cdot \boldsymbol{e_z}$. Now,

$$\iint_S H_0 H_0' (\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla} \nu) \cdot \boldsymbol{e_z} dS = \iint_S (H_0 \boldsymbol{\nabla} H_0 \times \boldsymbol{\nabla} \nu) \cdot \boldsymbol{e_z} dS \tag{6.35}$$

by definition, and

$$\iint_S (H_0 \boldsymbol{\nabla} H_0 \times \boldsymbol{\nabla}\nu) \cdot \boldsymbol{e_z} dS = \frac{1}{2}\iint_S \boldsymbol{\nabla} \times (H_0^2 \boldsymbol{\nabla}\nu) \cdot \boldsymbol{e_z} dS. \tag{6.36}$$

Consequently,

$$\begin{aligned}\iint_S H_0 H_0'(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\nu) \cdot \boldsymbol{e_z} dS &= \iint_S \boldsymbol{\nabla} \times (H_0^2 \boldsymbol{\nabla}\nu) \cdot \boldsymbol{e_z} dS \\ &= 0 \end{aligned} \tag{6.37}$$

by Stokes theorem, since $H_0 \equiv H_0(\chi_0)$ and $\chi_0 = 0$ on the boundaries of the surface. Hence,

$$I_{R1} = -\iint_S H_0 H_0' |\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\psi|^2 dS. \tag{6.38}$$

Substituting (6.38) into (6.31) gives

$$s^2 Re \iint_S \psi^* \nabla^2 \psi dS - s \iint_S \psi^* \nabla^4 \psi dS = B \int_{\partial S} H_0 H_0' |\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\psi|^2 dS. \tag{6.39}$$

Integration by parts on the first term, twice on the second term, and invoking $\boldsymbol{u} = 0$ at ∂S, we obtain

$$s^2 Re \iint_S |\boldsymbol{\nabla}\psi|^2 dS + s \iint_S |\nabla^2\psi|^2 dS + B \iint_S H_0 H_0' |\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\psi|^2 dS = 0. \tag{6.40}$$

It follows that

$$s = \frac{-\iint_S |\nabla^2\psi|^2 dS \pm \sqrt{(\iint_S |\nabla^2\psi|^2 dS)^2 - 4BRe \iint_S |\boldsymbol{\nabla}\psi|^2 dS (\iint_S H_0 H_0' |\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\psi|^2 dS)}}{2Re \iint_S |\boldsymbol{\nabla}\psi|^2 dS}, \tag{6.41}$$

and when

$$\iint_S H_0 H_0' |\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\psi|^2 dS < 0, \tag{6.42}$$

s is real and $s > 0$. (6.42) is true when $H_0' < 0$, thus if $H_0' < 0$ in all of S, the disturbances are unstable modes. However, if $H_0' \geq 0$ in all of S, s can be complex and $\mathcal{R}(s) \leq 0$, and the system is stable.

By a similar method to Chapter 4, we can consider the functional

$$F(f) = \frac{-\iint_S |\nabla^2 f|^2 dS + \sqrt{(\iint_S |\nabla^2 f|^2 dS)^2 - 4BRe\iint_S |\boldsymbol{\nabla} f|^2 dS \iint_S H_0 H_0' |\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f|^2 dS}}{2Re\iint_S |\boldsymbol{\nabla} f|^2 dS}. \tag{6.43}$$

for all real functions $f(x, y)$ satisfying the boundary conditions of ψ, to prove that the stationary points of F correspond to the real eigenvalues of (6.30). Analogously to Section 4.3.1, we can argue that $F(f)$ is bounded above, since $F(f)$ is homogeneous in f and

$$F(f) \sim \frac{-\iint_S |\nabla^2 f|^2 dS + \sqrt{(\iint_S |\nabla^2 f|^2 dS)}}{2Re\iint_S |\boldsymbol{\nabla} f|^2} - \frac{B\iint_S H_0 H_0' |\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f|^2 dS}{\iint_S |\nabla^2 f|^2 dS} \tag{6.44}$$

for $|\nabla^2 f|^2 \gg 1$. We conclude that for $H_0' < 0$, $F(f)$ is positive and bounded above. Importantly, there exists a positive maximum point of F.

We now prove stationary points of F correspond to real eigenvalues of (6.30). Suppose $f = f_0$ is a stationary point of F, such that $F(f_0) = F_0$ and consider $f = f_0 + \epsilon f_1$, where f_0 and f_1 satisfy the same boundary conditions as f, and $\epsilon \ll 1$. Since $f = f_0$ is a stationary point of $F(f)$, the first variation of the Taylor expansion of $F(f_0 + \epsilon f_1)$ is zero, and we obtain

$$\begin{aligned} &- F_0^2 Re \iint_S \boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla} f_1 dS - F_0 \iint_S \nabla^2 f_0 \nabla^2 f_1 dS \\ &= B \iint_S H_0 H_0' (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_0) \cdot (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1) dS. \end{aligned} \tag{6.45}$$

Repeated integration by parts on the left-hand side of (6.45), and invoking the boundary conditions for f gives

$$F_0^2 Re \iint_S f_0 \nabla^2 f_1 dS - F_0 \iint_S f_0 \nabla^4 f_1 dS = B \iint_S H_0 H_0' (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_0) \cdot (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1) dS. \tag{6.46}$$

Concerning the right-hand side of (6.46), we can write

$$\iint_S H_0 H_0' (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_0) \cdot (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1) dS$$
$$= \iint_S H_0 H_0' \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(f_0(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1)\cdot \boldsymbol{e_z}) - (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_0)\big((\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\hat{f}_1)\cdot \boldsymbol{e_z}\big)\Big) \cdot \boldsymbol{e_z} dS. \tag{6.47}$$

Since we can add any integral that evaluates to zero to (6.47) and

$$\iint_S \big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1)\big) \cdot \boldsymbol{e_z} dS = \iint_S \boldsymbol{\nabla} \times (\chi_0 \boldsymbol{\nabla}(((\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1)) \cdot \boldsymbol{e_z} dS$$
$$= 0 \tag{6.48}$$

by Stokes' theorem. Using (A.53), we write

$$\iint_S H_0 H_0' \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(f_0(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1)\cdot \boldsymbol{e_z}) - (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_0)\big((\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\hat{f}_1)\cdot \boldsymbol{e_z}\big)\Big) \cdot \boldsymbol{e_z} dS$$
$$= \iint_S f_0 \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\big((\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1)\cdot \boldsymbol{e_z}\big)\Big) \cdot \boldsymbol{e_z} dS. \tag{6.49}$$

Consequently (6.45) can be written as

$$- F_0^2 Re \iint_S f_0 \nabla^2 f_1 dS + F_0 \iint_S f_0 \nabla^4 f_1 dS$$
$$= B \iint_S H_0 H_0' f_0 \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\big((\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1)\cdot \boldsymbol{e_z}\big)\Big) \cdot \boldsymbol{e_z} dS. \tag{6.50}$$

(6.50) is valid for any function f_1, thus

$$-F_0^2 Re \nabla^2 f_1 + F_0 \nabla^4 f_1 = B H_0 H_0' \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\big((\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla} f_1)\cdot \boldsymbol{e_z}\big)\Big) \cdot \boldsymbol{e_z}. \tag{6.51}$$

Since $F(\psi)$ is the same expression as the positive root of 6.41, we substitute $F_0 = s$ into (6.51), giving (6.30). We deduce that stationary points of F correspond to real eigenvalues of (6.30).

If $H_0' < 0$ in part of the domain, there exists an arbitrary function $\hat{f}(x, y)$, satisfying the boundary conditions of ψ, where $\hat{f}$ is zero everywhere, apart from in the region where $H_0' < 0$.

Suppose $F(\hat{f}) = \hat{\xi}$, it follows that $\hat{\xi} > 0$, and $\hat{\xi}$ is either the (positive) global maximum point of F or there exists a global maximum larger than $\hat{\xi}$. Thus, there exists a positive real eigenvalue of (6.30), and therefore an unstable mode.

Consequently, we obtain a stability condition for a volume of ferrofluid in a three-dimensional Cartesian domain, where the system is invariant in z. Namely, if and only if, $dH_0/d\chi_0 < 0$ somewhere in the ferrofluid, the system is unstable.

6.5 Axisymmetric domain

Alternatively, consider a volume of ferrofluid in a cylindrical co-ordinate system, r, θ, z, where r and θ are the radial and azimuthal co-ordinates, and z the longitudinal, with co-ordinate unit vectors $\boldsymbol{e_r}$, $\boldsymbol{e_\theta}$ and $\boldsymbol{e_z}$. Suppose the system is invariant in θ. We impose $\boldsymbol{u} = 0$ and $\chi_0 = 0$ on the boundary of the volume, and define S as a cross-section of the volume, occupying the r, z plane, such that $\boldsymbol{e_\theta}$ is normal to the surface S. We now determine an eigenvalue equation for an axisymmetric domain using (6.18) and (6.24). However, the subsequent method and reasoning to obtain a stability condition follows that of Section 6.4 and is not given here.

Define Stokes' stream function ψ such that

$$\boldsymbol{u} = -\boldsymbol{\nabla} \times \psi \boldsymbol{e_\theta} \tag{6.52}$$

and

$$\boldsymbol{\omega} = \left(\nabla^2 - \frac{1}{r^2}\right)\psi \boldsymbol{e_\theta}, \quad \nabla^2 \boldsymbol{\omega} = \left(\nabla^2 - \frac{1}{r^2}\right)^2 \psi \boldsymbol{e_\theta}. \tag{6.53}$$

Consequently, (6.24) becomes

$$s^2 Re\left(\nabla^2 - \frac{1}{r^2}\right)\psi \boldsymbol{e_\theta} - s\left(\nabla^2 - \frac{1}{r^2}\right)^2 \psi \boldsymbol{e_\theta} = -BH_0H_0' \boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\left(\left(\frac{1}{r}\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(r\psi)\right) \cdot \boldsymbol{e_\theta}\right). \tag{6.54}$$

Multiply (6.54) by $r\psi^*$, dot with $\boldsymbol{e_\theta}$, and integrate over S, to give

$$\iint_S r\psi^*\left(s^2 Re\left(\nabla^2 - \frac{1}{r^2}\right) - s\left(\nabla^2 - \frac{1}{r^2}\right)^2\right)\psi dS = -B\iint_S H_0 H_0' r^2 \psi^* \mathcal{O}^2(r\psi)dS, \qquad (6.55)$$

where

$$\mathcal{O} = \frac{\boldsymbol{e_\theta}}{r}\cdot\left(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\right). \qquad (6.56)$$

Integration by parts on the first term on the left-hand side, repeated integration by parts on the second, and invoking boundary conditions on ψ, gives

$$s^2 Re\iint_S\left(|\boldsymbol{\nabla}\psi|^2 + \frac{|\psi|^2}{r^2}\right) r dS + s\iint_S\left|\nabla^2\psi - \frac{\psi}{r^2}\right|^2 r dS = -B\iint_S H_0 H_0' r^2\psi^*\mathcal{O}^2(r\psi)dS. \ (6.57)$$

Now,

$$\iint_S H_0 H_0' r^2 \psi^* \mathcal{O}^2(r\psi)dS = \iint_S H_0 H_0' r\big(\mathcal{O}(r\psi^*\mathcal{O}(r\psi)) - |\mathcal{O}(r\psi)|^2\big)dS \qquad (6.58)$$

and

$$\iint_S H_0 H_0' r\big(\mathcal{O}(r\psi^*\mathcal{O}(r\psi))\big)dS = \iint_S H_0 H_0'\left(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\big(r\psi^*\mathcal{O}(r\psi)\big)\right)\cdot \boldsymbol{e_\theta} dS. \qquad (6.59)$$

But, using (A.55) and Stokes' theorem with $\chi_0 = 0$ on ∂S, gives

$$\iint_S H_0 H_0' r\big(\mathcal{O}(r\psi^*\mathcal{O}(r\psi))\big)dS = \iint_S H_0 H_0'\left(\boldsymbol{\nabla}\times\big(\chi_0\boldsymbol{\nabla}\big(r\psi^*\mathcal{O}(r\psi)\big)\big)\right)\cdot\boldsymbol{e_\theta} dS \quad = 0. \quad (6.60)$$

Resulting in

$$\iint_S H_0 H_0' r^2\psi^*\mathcal{O}^2(r\psi)dS = -\iint_S H_0 H_0' r|\mathcal{O}(r\psi)|^2 dS. \qquad (6.61)$$

Substituting (6.61) into (6.57) results in

$$s^2 Re \iint_S \left(|\boldsymbol{\nabla}\psi|^2 + \frac{|\psi|^2}{r^2} \right) r dS + s \int_{\partial S} \left(\left| \nabla^2 \psi - \frac{\psi}{r^2} \right|^2 \right) r dS + 2B \int_{\partial S} H_0 H_0' |\mathcal{O}(r\psi)|^2 r dS = 0 \quad (6.62)$$

and therefore

$$s = \frac{-\iint_S |(\nabla^2 - \frac{1}{r^2})\psi|^2 r dS \pm \sqrt{W_0}}{2Re \iint_S (|\boldsymbol{\nabla}\psi|^2 + \frac{1}{r^2}|\psi|^2) r dS}, \quad (6.63)$$

where

$$W_0 = \left(\int_{\partial S} \left| \left(\nabla^2 - \frac{1}{r^2} \right) \psi \right|^2 r dS \right)^2 - 8B Re \iint_S \left(|\boldsymbol{\nabla}\psi|^2 + \frac{|\psi|^2}{r^2} \right) r dS \iint_S H_0 H_0' |\mathcal{O}(r\psi)|^2 r dS. \quad (6.64)$$

By analogous reasoning to Section 6.4, we prove that if, and only if, $H_0' < 0$ somewhere in the domain, the system is unstable.

6.6 General three-dimensional domain

We now seek to obtain a stability condition for a three-dimensional configuration where there is variance in all co-ordinates. By requiring an invariance in one co-ordinate, we could determine a stability condition for arbitrary Reynolds number when χ depended explicitly on H. However, upon removing the invariance, the equations become more complicated. When χ depends explicitly on H and position, $\boldsymbol{f_m}$ is given by (2.42), and proving a definite sign of the forcing term and the viscous term in (6.24) is not trivial. We give a rigorous proof in the inviscid limit in Section 6.6.1. Yet, as a result of the complicated nature of the viscous term, we cannot prove it for the highly viscous limit nor arbitrary Reynolds number.

Nevertheless, by assuming $\boldsymbol{f_m}$ is given by (2.43), such that χ does not depend explicitly on H, a stability condition can be proven for arbitrary Reynolds number. This is outlined in Section 6.6.2. It is convenient to use (2.39) rather than the vorticity equation, and we multiply by $\boldsymbol{u}^*$ instead of $\boldsymbol{\omega}^*$. After integrating over the domain, the integrals of each term can be manipulated into an integral of definite sign (dependent on the sign of H_0'). As a consequence, a stability condition follows, by similar reasoning to previous sections.

6.6.1 Magnetic susceptibility depending explicitly on the field

As a result of the integral form of (2.42) in (2.39), we continue using the linearised vorticity equation obtained in Section 6.3. We suppose we have a volume of ferrofluid, where χ varies slowly spatially, and can depend on both position and H. There is variance in all co-ordinates of the three-dimensional system. Moreover, the stationary state is such that $\chi_0 \equiv \chi_0(H_0)$, and we impose $\boldsymbol{u} = 0$ and $\chi_0 = 0$ on the boundary of the ferrofluid volume, ∂V. Using (6.24) we now determine a stability condition.

To manipulate the right-hand side of (6.24), it proves useful to define a potential $\bar{\boldsymbol{A}}$, such that $\boldsymbol{u} = \boldsymbol{\nabla} \times \bar{\boldsymbol{A}}$. We then add a potential γ to $\bar{\boldsymbol{A}}$ and define

$$\boldsymbol{A} = \bar{\boldsymbol{A}} + \boldsymbol{\nabla}\gamma. \tag{6.65}$$

It follows that

$$\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{6.66}$$

and we can argue that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ by choosing $\bar{\boldsymbol{A}}$ and γ such that

$$\boldsymbol{\nabla} \cdot \bar{\boldsymbol{A}} = -\nabla^2\gamma. \tag{6.67}$$

Consequently,

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \\ &= -\nabla^2 \boldsymbol{A}. \end{aligned} \tag{6.68}$$

(6.20) and (6.24) in terms of $\boldsymbol{A}$ are

$$s\chi_1 = -\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) \tag{6.69}$$

and

$$-s^2 Re\nabla^2 \boldsymbol{A} + s\nabla^2(\nabla^2 \boldsymbol{A}) = BH_0H_0'\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A})), \tag{6.70}$$

respectively.

In an effort to determine a stability condition, dot (6.70) with $\boldsymbol{A}^*$, the complex conjugate of $\boldsymbol{A}$, and integrate over the volume, to give

$$\begin{aligned} &- s^2 Re \iiint_V \boldsymbol{A}^* \cdot \nabla^2 \boldsymbol{A} dV + s \iiint_V \boldsymbol{A}^* \cdot \nabla^2(\nabla^2 \boldsymbol{A}) dV \\ &= B \iiint_V H_0 H_0' \boldsymbol{A}^* \cdot (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}))) dV. \end{aligned} \tag{6.71}$$

We consider the inviscid limit of (6.71), since we cannot prove the viscous term, namely

$$\iiint_V \boldsymbol{A}^* \cdot \nabla^2(\nabla^2 \boldsymbol{A}) dV, \tag{6.72}$$

is of definite sign. An attempt to determine the sign of (6.72) is shown in Appendix (A.7). It should be noted that if we were to dot (6.24) with $\boldsymbol{u}^*$ or $\boldsymbol{\omega}^*$, in an effort to make the viscous term of definite sign, the forcing term becomes

$$B \iiint_V H_0 H_0' \boldsymbol{u}^* \cdot (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u})) dV, \tag{6.73}$$

or

$$B \iiint_V H_0 H_0' \boldsymbol{\omega}^* \cdot (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u})) dV, \tag{6.74}$$

respectively. Disappointingly, as it stands, I cannot prove (6.73) or (6.74) are of definite sign (when H_0' is of definite sign).

(6.71) in the inviscid limit ($\eta \to 0$) becomes

$$-s_I^2 \iiint_V \boldsymbol{A}^* \cdot \nabla^2 \boldsymbol{A} dV = \bar{B} \iiint_V H_0 H_0' \boldsymbol{A}^* \cdot (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}))) dV, \tag{6.75}$$

where $\bar{B} \neq B$ in (6.70), and $\bar{B}$ is determined by an appropriate inviscid scaling for the given system, but we now drop the bar.

Consider

$$
\begin{aligned}
I_{L1} &= -\iiint_V \boldsymbol{A}^* \cdot (\nabla^2 \boldsymbol{A}) dV \\
&= \iiint_V \boldsymbol{A}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) dV,
\end{aligned}
\tag{6.76}
$$

and use (A.56) to write

$$
I_{L1} = \iiint_V \left(|\boldsymbol{u}|^2 + \boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{A}^*) \right) dV,
\tag{6.77}
$$

and therefore

$$
\begin{aligned}
I_{L1} &= \iiint_V |\boldsymbol{u}|^2 dV + \iint_S (\boldsymbol{u} \times \boldsymbol{A}^*) \cdot \hat{\boldsymbol{n}} dS \\
&= \iiint_V |\boldsymbol{u}|^2 dV,
\end{aligned}
\tag{6.78}
$$

by the divergence theorem. As a result,

$$
-\iiint_V (\boldsymbol{A}^* \cdot (\nabla^2 \boldsymbol{A}) dV = \iiint_V |\boldsymbol{u}|^2 dV.
\tag{6.79}
$$

Consider the integral on the right-hand side of (6.75),

$$
I_R = \iiint_V H_0 H_0' \boldsymbol{A}^* \cdot (\boldsymbol{\nabla} \chi_0 \times \boldsymbol{\nabla})(\boldsymbol{\nabla} \chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A})) dV,
\tag{6.80}
$$

and define

$$
\mu = \boldsymbol{\nabla} \chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}),
\tag{6.81}
$$

to obtain

$$I_R = \iiint_V H_0 H_0' (\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla}\mu \times \boldsymbol{A}^*))dV, \tag{6.82}$$

by use of (A.51). Next, use (A.55) to write

$$\begin{aligned} I_R &= \iiint_V \Big(H_0 H_0' \boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times (\mu \boldsymbol{A}^*) - \mu(\boldsymbol{\nabla} \times \boldsymbol{A}^*)) \Big) dV \\ &= \iiint_V H_0 H_0' (\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times (\mu \boldsymbol{A}^*)))dV - \iiint_V H_0 H_0' |\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A})|^2 dV. \end{aligned} \tag{6.83}$$

The first term of (6.83) can be shown to be zero using (A.54) and the divergence theorem, coupled with $\chi_0 = 0$ (and therefore $H_0 = 0$) on the boundary. Hence,

$$I_R = -\iiint_V H_0 H_0' |\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A})|^2 dV. \tag{6.84}$$

Substituting (6.79) and (6.84) into (6.75) we obtain

$$s_I^2 = -\frac{B \iiint_V H_0 H_0' |\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u}|^2 dV}{\iiint_V |\boldsymbol{u}|^2 dV}. \tag{6.85}$$

By analogous reasoning to Section 6.4 we can prove that if, and only if, $H_0' < 0$ somewhere in the volume, the system is unstable in the inviscid limit. The proof is given in Appendix A.8.

6.6.2 Magnetic susceptibility not depending explicitly on the field

When χ does not depend explicitly on H, $\boldsymbol{f_m}$ takes the simpler form of (2.43), and it proves easier to use (2.39) to obtain a stability condition. Substituting the perturbed variables, (6.4)-(6.7), into (2.39), linearising and assuming $\boldsymbol{\nabla}\chi_0 = O(\epsilon_2)$ we obtain

$$sRe\boldsymbol{u_1} + \boldsymbol{\nabla} p_1 = \nabla^2 \boldsymbol{u_1} - BH_0^2 \boldsymbol{\nabla}\chi_1, \tag{6.86}$$

Substituting (6.20) into (6.86) results in

$$s^2 Re\boldsymbol{u} - s\nabla^2\boldsymbol{u} = -\boldsymbol{\nabla} p + BH_0^2\boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u}), \tag{6.87}$$

where the subscripts 1 have been dropped.

We suppose we have a volume V of ferrofluid subject to a field, such that $\chi_0 \equiv \chi_0(H_0)$, but χ does not depend explicitly on H, only on position. We impose $\boldsymbol{u} = 0$ and $\chi_0 = 0$ at the boundaries of the volume ∂V. To determine a stability condition dot (6.87) with $\boldsymbol{u}^*$ and integrate over the volume, to give

$$s^2 \iiint_V Re|\boldsymbol{u}|^2 dV - s\iiint_V \boldsymbol{u}^* \cdot \nabla^2\boldsymbol{u} = -s\iiint_V \boldsymbol{u}^* \cdot \boldsymbol{\nabla} p dV + B\iiint_V H_0^2\boldsymbol{u}^* \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u})dV. \tag{6.88}$$

(6.88) can be written as

$$\begin{aligned} s^2 \iiint_V Re|\boldsymbol{u}|^2 dV - s\iiint_V \boldsymbol{u}^* \cdot \nabla^2\boldsymbol{u} dV &= -s\iiint_V \boldsymbol{\nabla} \cdot (p\boldsymbol{u}^*)dV \\ &+ B\iiint_V \Big(\boldsymbol{\nabla} \cdot \big(H_0^2(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u})\boldsymbol{u}^*\big) - (\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u})(\boldsymbol{\nabla} H_0^2 \cdot \boldsymbol{u}^*)\Big)dV, \end{aligned} \tag{6.89}$$

using (A.54) and (A.53). Moreover, by the divergence theorem and invoking $\boldsymbol{u}^* = 0$ at the boundary of the volume, (6.89) simplifies to

$$s^2 \iiint_V Re|\boldsymbol{u}|^2 dV - s\iiint_V \boldsymbol{u}^* \cdot \nabla^2\boldsymbol{u} = -B\iiint_V (\boldsymbol{\nabla} H_0^2 \cdot \boldsymbol{u}^*)(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u})dV. \tag{6.90}$$

Expressing $\boldsymbol{\nabla}(H_0^2)$ as $2H_0\boldsymbol{\nabla} H_0$, we substitute (6.23) into (6.90) to obtain

$$s^2 \iiint_V Re|\boldsymbol{u}|^2 dV - s\iiint_V \boldsymbol{u}^* \cdot \nabla^2\boldsymbol{u} dV = -2B\iiint_V H_0 H_0'|\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u}|^2 dV. \tag{6.91}$$

Now,

$$\iiint_V \boldsymbol{u}^* \cdot \nabla^2 \boldsymbol{u} dV = -\iiint_V \boldsymbol{u}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{\omega}) dV$$
$$= \iiint_V \big(\boldsymbol{\nabla} \cdot \big(\boldsymbol{u}^* \times \boldsymbol{\omega}\big) - |\boldsymbol{\omega}|^2\big) dV \tag{6.92}$$

But, by the divergence theorem,

$$\iiint_V \boldsymbol{\nabla} \cdot \big(\boldsymbol{u}^* \times \boldsymbol{\omega}\big) dV = \iint_{\partial V} (\boldsymbol{u}^* \times \boldsymbol{\omega}) \cdot \hat{\boldsymbol{n}} dS$$
$$= \iint_{\partial V} (\boldsymbol{\omega} \times \hat{\boldsymbol{n}}) \cdot \boldsymbol{u}^* dS$$
$$= 0. \tag{6.93}$$

Thus,

$$\iiint_V \boldsymbol{u}^* \cdot \nabla^2 \boldsymbol{u} dV = -\iiint_V |\boldsymbol{\omega}|^2 dV. \tag{6.94}$$

Consequently, (6.91) can be written as

$$s^2 \iiint_V Re|\boldsymbol{u}|^2 dV + s \iiint_V |\boldsymbol{\omega}|^2 dV + 2B \iiint_V H_0 H_0' |\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u}|^2 dV = 0, \tag{6.95}$$

and therefore

$$s = \frac{-\iiint_V |\boldsymbol{\omega}|^2 dV \pm \sqrt{(\iiint_V |\boldsymbol{\omega}|^2 dV)^2 - 8BRe \iiint_V |\boldsymbol{u}|^2 dV (\iiint_V H_0 H_0' |\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{u}|^2 dV)}}{2Re \iiint_V |\boldsymbol{u}|^2 dV}. \tag{6.96}$$

We conclude that, if, and only if, $H_0' < 0$ somewhere in the volume, the system is unstable. The proof follows that of Section 6.4 and is given in Appendix A.9.

6.7 Conclusions

To simplify the governing equations for a general volume of ferrofluid subject to a non-uniform magnetic field, we assume that the susceptibility of the ferrofluid varies slowly with position. Assuming we have a stationary state satisfying the governing equations, such that χ can be written as a function of H, and imposing $\boldsymbol{u} = 0$ on the boundaries of the ferrofluid, we can prove a stability condition. By linear stability analysis we prove that if, and only if, $dH/d\chi < 0$ somewhere in the ferrofluid, the system will be unstable. If χ does not depend explicitly on H, it can be proved rigorously for a system with arbitrary Reynolds number in three-dimensions. When χ depends explicitly on H, it can be proven in the inviscid limit in three-dimensions. But, for arbitrary Reynolds number it can be proven when the system is invariant in one co-ordinate. At moderate Reynolds number the viscous term is unlikely to effect an instability, and I suspect the result is true in general, but I have not been able to demonstrate this rigorously.

Chapter 7

Conclusion

7.1 Summary and concluding remarks

Ferrofluids with both constant and non-uniform susceptibilities have been considered in different domains in the presence of non-uniform fields, and their respective stability determined. Initially, a two-fluid cylindrical system centred on a current-carrying wire was considered. The susceptibility was modelled as constant but discontinuous, with a jump at the interface of the two ferrofluids. Consequently, the magnetic forcing acts at the interface only. When the inner fluid was more magnetic than the outer fluid, the magnetic forcing produced from the current in the wire acted inwards, stabilising the disturbed interface. Only axisymmetric modes were found to be unstable and a sufficiently strong current in the wire suppressed any instabilities due to capillary forcing. The results found agreed with the literature. Novel results were found when the outer fluid was more magnetic than the inner fluid, in which case, the magnetic forcing generated from the current in the wire acted to destabilise the column. Axisymmetric and non-axisymmetric modes could be unstable, and increasing the current in the wire increased the magnitude of the growth rate of the disturbances, as well as rendering more modes unstable. Moreover, as the ratio between the radius of the wire and the radius of the inner fluid shrunk, non-axisymmetric modes became the most unstable at low Reynolds number. Although increasing the current in the wire couldn't dampen unstable modes when the outer fluid had a higher susceptibility, it was found that adding a sufficiently large axial field will suppress unstable modes, irrespective of which fluid had a higher susceptibility.

Next, configurations where the susceptibility is continuous and non-uniform were investigated. When a ferrofluid with a non-constant susceptibility is subject to a magnetic field, the magnetic forcing is felt throughout the bulk of the fluid, even for a uniform field. For a non-uniform field, it was proven for two equilibria, that if the gradient of the magnitude of the field with respect to the susceptibility ($dH/d\chi$) was negative, the system was unstable. First, a column of ferrofluid centred on a current-carrying wire, such that the susceptibility of the ferrofluid varied continuously radially, and the azimuthal field decreased as the reciprocal of the radius, was proven unstable if ever the susceptibility increased radially. That is, if ever $dH/d\chi < 0$ somewhere in the fluid. Secondly, a ferrofluid in a channel, where the susceptibility and field varied normal to the channel walls, such that $dH/d\chi < 0$ everywhere, was proven to be linearly unstable to all three-dimensional disturbances. For an unstable configuration with a non-uniform field and susceptibility, we found applying an additional field could suppress unstable modes. In the cylindrical configuration, a constant axial field was sufficient, yet in the planar domain, a rapidly rotating field applied across the channel was needed.

The stability analysis of both equilibria support the hypothesis that when the regions of fluid with highest susceptibility do not coincide with the regions of strongest field, an instability occurs to achieve a minimum energy configuration. Analogies with energy stability arguments seen in other areas of fluid dynamics can be made, yet these would only hold for one-dimensional disturbances, and do not take into account viscous effects. However, we have proven rigorously, for a stationary state of a volume of Newtonian ferrofluid in the presence of a non-uniform field, where the magnetic susceptibility varies slowly with position, that the system will be unstable, if and only if, $dH/d\chi < 0$ somewhere in the volume. We postulate that this condition holds when there is no restriction on the variance of the susceptibility, although it has not been proven analytically here. This condition allows for the stability of a general configuration to be determined without the need for an in depth analysis of the governing equations.

Moreover, a set of solutions to the governing equations have been found such that the field is a function of the susceptibility and equipotential surfaces have zero mean curvature. In theory, this gives a set of equilibria, if configurations can be found that satisfy the boundary conditions, and we would expect the equilibria to be unstable, since the gradient of the susceptibility with respect to the magnitude of the field is negative for these solutions.

In Chapter 4-6 the magnetic susceptibility depends on position and field, allowing for non-linear magnetisation characteristics to be applied to the results outlined. In particular, (1.2) would be valid. Furthermore, due to the analogy of ferro-hydrodynamics with EHD, the results could be applied to EHD for non-linear polarizable material.

7.2 Applications

Theoretical analysis of ferrofluids in different configurations is necessary for their use in all disciplines. Knowing the thresholds of parameters for a stable system is vital for the functionality of each application, and it is advantageous to know the minimum strength of field needed in a given system to minimise costs and energy usage. Moreover, the ability to destabilise and stabilise a system at will is appealing. The ferrous particles can be held in place using a field configuration known to produce a stable system, and released at will by turning off the field or re-orientating it. Allowing unstable modes to grow can induce mixing and dispersion, or release a solid/liquid interior that the ferrofluid is surrounding.

Theoretical understanding of ferrofluids in cylindrical configurations is relevant to cylindrical systems seen in ferrofluid applications, in particular ink-jet printing, 3D-printing and magnetic drug targeting. We have expanded on the current literature by investigating the stability of a two-fluid cylindrical interface with very few limitations to the applicability of the results, since both fluids are Newtonian and can be magnetic, as well having considered three-dimensional disturbances to the system. Many industrial applications will involve apparatus where the ferrofluid is in a channel, and we have proved the instability of a particular planar configuration in a channel and shown stability can be maintained by an additional rotating field. Furthermore, we have determined a stability condition for a general volume of ferrofluid, which can be used to gauge the stability of a potential system for an application, before doing an in depth analysis which may be computationally expensive or not possible. Moreover, we hope that the work could inspire novel applications where the ferrofluid susceptibility distribution is not uniform.

7.3 Future work

Establishing other equilibria and ensuring their stability extends the possible applications of ferrofluids in different configurations. At present, we have two approaches to find new equilibria. The first approach to find new equilibria is to perturb a known 2-D or axisymmetric equilibrium three-dimensionally. This generates another equilibrium family, subject to solving a second order eigenvalue problem with given boundary conditions. For some susceptibility distributions the equation has analytic solutions, however for the desired susceptibility, it must be solved numerically to find the new stationary state. Continually perturbing each new equilibrium, leads eventually to a state significantly different from the original equilibrium. The second approach is to find equipotential surfaces of zero mean curvature, which we have found form a set of solutions to the governing equations when the field is a function of the susceptibility. The equilibrium found for the planar domain was the simplest example of this. This forms a set of equilibria if the boundary conditions are satisfied, that theoretically could exist in more complex configurations than planar and cylindrical geometries. Moreover, we expect them to be unstable, since they are such that $d\chi/dH < 0$.

Finding new equilibria will be of interest in itself, even if they prove unstable. Unstable equilibria are of limited practical importance. However, we demonstrated how equilibria can be stabilised by the addition of a further magnetic field. It was proven that a constant axial field was sufficient in stabilising the cylindrical configuration, but a rapidly rotating field was necessary in stabilising the planar configuration. In other geometries, it may be that a constant field nor one rotating field is sufficient in stabilising the system. Two rotating fields may be needed to dampen all modes. It may be that the geometry may not support a rotating field, and a more complicated high frequency field may be required, and this would need to be determined. Additionally, in the analysis for the cylindrical system and for the general volume, a local condition gave rise to a global instability, something which would be interesting to explore.

The stability analysis could be taken further by investigating a nonlinear regime. Previous works have considered the nonlinear stability of axisymmetric disturbances for a ferrofluid jet surrounded by a non-magnetic fluid, and that same analysis could be applied for non-axisymmetric disturbances, with a magnetic fluid being the outer fluid. Travelling wave solutions may exist at the interface of the two ferrofluids and studying the full non-linear regime, as Doak & Vanden-Broeck (2019) and Blyth & Parau (2014) have done, could show this. The study of the resultant drop formations by methods similar to Cornish (2018) could also be of interest. Additionally, investigating the non-axisymmetric disturbances at low Reynolds number when the outer fluid is more magnetic is of particular interest. It was found that in the long wave limit, for a sufficiently small ratio between the radius of the wire and the radius of the inner fluid, that the non-axisymmetric mode was the most unstable. This warrants more investigation in the non-linear regime, and in a similar manner to Rannacher & Engel (2006) and Doak & Vanden-Broeck (2019), one could use long wave theory to analyse this further.

In conclusion, the options for extensions are vast, and we hope the novel results outlined in this thesis inspire further research and applications.

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Appendix A

A.1 Constants in the solution to the perturbation of the magnetic potential

The constants $q_{1,2}^{(\iota)}$ in (3.9) are as follows,

$$
\begin{aligned}
q_2^{(2)} \;=\; & \Big(\mathrm{i}m\hat{S}(\chi^{(1)}-\chi^{(2)})(ak\mathrm{I}_m(k)\mathrm{I}_m(ka)(1+\chi^{(1)})\mathrm{K}_{m+1}(ka)+ka(\chi^{(1)}\mathrm{K}_m(k)\mathrm{I}_m(ka) \\
& +\mathrm{K}_m(ka)\mathrm{I}_m(k))\mathrm{I}_{m+1}(ka)-m\chi^{(1)}\mathrm{I}_m(ka)(\mathrm{K}_m(ka)\mathrm{I}_m(k)-\mathrm{K}_m(k)\mathrm{I}_m(ka)))\Big) \\
& \times\Big(\mathrm{I}_m(ka)a(1+\chi^{(1)})k(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)+(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k) \\
& +m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))\mathrm{K}_m(k))\mathrm{K}_{m+1}(ka)+a(-\chi^{(1)}\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)}) \\
& \times(\mathrm{K}_{m+1}(k)k-m\mathrm{K}_m(k))\mathrm{I}_m(ka)+\mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k) \\
& +(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k)+m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))\mathrm{K}_m(k)))k\mathrm{I}_{m+1}(ka) \\
& -\mathrm{I}_m(ka)(\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)})(\mathrm{K}_{m+1}(k)k-m\mathrm{K}_m(k))\mathrm{I}_m(ka) \\
& +\mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)+(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k) \\
& +m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))\mathrm{K}_m(k)))m\chi^{(1)}\Big)^{-1}, \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
q_1^{(0)} \quad = \quad & \left(\frac{\mathrm{i}}{a}\hat{S}\mathrm{K}_m(k)(1+\chi^{(1)})ma(\chi^{(1)}-\chi^{(2)})\right) \\
& \times \left(k(1+\chi^{(1)})a\mathrm{I}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k)\right. \\
& + m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)})))\mathrm{K}_{m+1}(ka)+k(-\chi^{(1)}\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)})(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k))\mathrm{I}_m(ka)+\mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k) \\
& + \mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k)+m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))))a\mathrm{I}_{m+1}(ka) \\
& - m(\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)})(\mathrm{K}_{m+1}(k)k-m\mathrm{K}_m(k))\mathrm{I}_m(ka) \\
& + \mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k) \\
& \left. + m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))))\mathrm{I}_m(ka)\chi^{(1)}\right)^{-1},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
q_1^{(1)} \quad = \quad & \left(\mathrm{i}\hat{S}\mathrm{K}_m(k)(ak\mathrm{I}_m(ka)(1+\chi^{(1)})\mathrm{K}_{m+1}(ka)+\mathrm{K}_m(ka)(ak\mathrm{I}_{m+1}(ka)\right. \\
& \left. - m\chi^{(1)}\mathrm{I}_m(ka)))m(\chi^{(1)}-\chi^{(2)})\right)\left(k(1+\chi^{(1)})a\mathrm{I}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)\right. \\
& + \mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k)+m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)})))\mathrm{K}_{m+1}(ka) \\
& + k(-\chi^{(1)}\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)})(\mathrm{K}_{m+1}(k)k-m\mathrm{K}_m(k))\mathrm{I}_m(ka) \\
& + \mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k) \\
& + m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))))a\mathrm{I}_{m+1}(ka)-m(\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)})(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k))\mathrm{I}_m(ka)+\mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k) \\
& \left. + \mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k)+m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))))\mathrm{I}_m(ka)\chi^{(1)}\right)^{-1}
\end{aligned}
\tag{A.3}
$$

and

$$
\begin{aligned}
q_2^{(1)} \quad = \quad & \Big(\mathrm{i}\hat{S}\mathrm{I}_m(ka)\chi^{(1)}\mathrm{K}_m(k)m(\chi^{(1)}-\chi^{(2)})\big(ak\mathrm{I}_{m+1}(ka)+m\mathrm{I}_m(ka)\big)\Big) \\
& \times \Big(k(1+\chi^{(1)})a\mathrm{I}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k) \\
& + m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)})))\mathrm{K}_{m+1}(ka)+k(-\chi^{(1)}\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)})(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k))\mathrm{I}_m(ka)+\mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k) \\
& + \mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k)+m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))))a\mathrm{I}_{m+1}(ka) \\
& - m(\mathrm{K}_m(k)(\chi^{(1)}-\chi^{(2)})(\mathrm{K}_{m+1}(k)k-m\mathrm{K}_m(k))\mathrm{I}_m(ka) \\
& + \mathrm{K}_m(ka)(k\mathrm{I}_m(k)(1+\chi^{(2)})\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)(k(1+\chi^{(1)})\mathrm{I}_{m+1}(k) \\
& + m\mathrm{I}_m(k)(\chi^{(1)}-\chi^{(2)}))))\mathrm{I}_m(ka)\chi^{(1)}\Big)^{-1}. \qquad \text{(A.4)}
\end{aligned}
$$

A.2 Derivation of normal stress condition

After the disturbance to the stationary state

$$
\boldsymbol{H}^{(\iota)} = \left(\epsilon(\hat{\phi}^{(\iota)})'\zeta, \frac{1}{r}\left(1+\epsilon i m\hat{\phi}^{(\iota)}\zeta\right), \epsilon i k\hat{\phi}^{(\iota)}\zeta\right)^T, \qquad \text{(A.5)}
$$

and it follows that

$$
H^{(\iota)} = \sqrt{\frac{1}{r^2}+2\epsilon\frac{im\hat{\phi}^{(\iota)}\zeta}{r^2}}. \qquad \text{(A.6)}
$$

First, consider

$$
\boldsymbol{n}\cdot\Big(\big(1+\chi^{(1)}\big)(H^{(1)})^2-\big(1+\chi^{(2)}\big)(H^{(2)})^2\Big)\mathbf{I}\cdot\boldsymbol{n} = \big(1+\chi^{(1)}\big)(H^{(1)})^2-\big(1+\chi^{(2)}\big)(H^{(2)})^2 \quad \text{(A.7)}
$$

and substitute the perturbed variables and (3.5), into (A.7) to obtain

$$
\begin{aligned}
& \boldsymbol{n}\cdot\Big(\big(1+\chi^{(1)}\big)(H^{(1)})^2-\big(1+\chi^{(2)}\big)(H^{(2)})^2\Big)\mathbf{I}\cdot\boldsymbol{n} \\
& = \frac{\chi^{(1)}-\chi^{(2)}}{r^2}+\frac{2\epsilon i m\zeta}{r^2}\Big(\big(1+\chi^{(1)}\big)\hat{\phi}^{(1)}-\big(1+\chi^{(2)}\big)\hat{\phi}^{(2)}\Big)+O(\epsilon^2). \qquad \text{(A.8)}
\end{aligned}
$$

(A.8) at $r = S$ gives

$$\boldsymbol{n}\cdot\left(\big(1+\chi^{(1)}\big)(H^{(1)})^2-\big(1+\chi^{(2)}\big)(H^{(2)})^2\right)\mathbf{I}\cdot\boldsymbol{n}=\frac{\chi^{(1)}-\chi^{(2)}}{(1+\epsilon\hat{S}\zeta)^2}$$
$$+\frac{2\epsilon im\zeta}{(1+\epsilon\hat{S}\zeta)^2}\left(\big(1+\chi^{(1)}\big)\hat{\phi}^{(1)}-\big(1+\chi^{(2)}\big)\hat{\phi}^{(2)}\right)+O(\epsilon^2). \tag{A.9}$$

For small ϵ

$$\frac{1}{(1+\epsilon\hat{S}\zeta)^2}=1-2\epsilon\hat{S}\zeta+O(\epsilon^2), \tag{A.10}$$

and therefore, to $O(\epsilon)$,

$$\boldsymbol{n}\cdot\left(\big(1+\chi^{(1)}\big)(H^{(1)})^2-\big(1+\chi^{(2)}\big)(H^{(2)})^2\right)\mathbf{I}\cdot\boldsymbol{n}=2(\chi^{(2)}-\chi^{(1)})\hat{S}\zeta$$
$$+2im\zeta\big(\big(1+\chi^{(1)}\big)\hat{\phi}^{(1)}-\big(1+\chi^{(2)}\big)\hat{\phi}^{(2)}\big). \tag{A.11}$$

Substituting (3.13) into (A.11) gives

$$\boldsymbol{n}\cdot\left(\big(1+\chi^{(1)}\big)(H^{(1)})^2-\big(1+\chi^{(2)}\big)(H^{(2)})^2\right)\mathbf{I}\cdot\boldsymbol{n}=2\big(\chi^{(2)}-\chi^{(1)}\big)\hat{S}+2im\big(\chi^{(1)}-\chi^{(2)}\big)\hat{\phi}^{(1)} \tag{A.12}$$

For a cylindrical domain,

$$(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathbf{T}})= \tag{A.13}$$

$$\begin{bmatrix} 2\frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial\theta}-\frac{v}{r}+\frac{\partial v}{\partial r} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} \\ \frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}-\frac{v}{r} & \frac{2}{r}\frac{\partial v}{\partial\theta}+\frac{2u}{r} & \frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial\theta} \\ \frac{\partial w}{\partial r}+\frac{\partial u}{\partial z} & \frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{bmatrix}$$

and

$$\boldsymbol{n}\cdot\left(\boldsymbol{\nabla}\boldsymbol{u}^{(\iota)}+(\boldsymbol{\nabla}\boldsymbol{u}^{(\iota)})^T\right)\cdot\boldsymbol{n}=\frac{1}{1+\frac{1}{r^2}S_\theta^2+S_z^2}\left(2\frac{\partial u^{(\iota)}}{\partial r}-\frac{S_\theta}{r}\left(2\frac{\partial v^{(\iota)}}{\partial r}+\frac{2}{r}\frac{\partial u^{(\iota)}}{\partial\theta}-\frac{2v^{(\iota)}}{r}\right.\right.$$
$$-\frac{S_\theta}{r}\left(\frac{2}{r}\frac{\partial v^{(\iota)}}{\partial\theta}+\frac{2u^{(\iota)}}{r}\right)-S_z\left(\frac{\partial v^{(\iota)}}{\partial z}+\frac{1}{r}\frac{\partial w^{(\iota)}}{\partial\theta}\right)\Bigg)-S_z\left(2\left(\frac{\partial u^{(\iota)}}{\partial z}+\frac{\partial w^{(\iota)}}{\partial r}\right)\right.$$
$$\left.\left.-\frac{S_\theta}{r^2}\left(\frac{\partial w^{(\iota)}}{\partial\theta}+r\frac{\partial v^{(\iota)}}{\partial z}\right)-2S_z\frac{\partial w^{(\iota)}}{\partial z}\right)\right), \tag{A.14}$$

where $\boldsymbol{n}$ is given by (3.5). Substituting the perturbed variables and linearising (A.14) gives

$$\boldsymbol{n}\cdot\left[\boldsymbol{\nabla u}+(\boldsymbol{\nabla u})^T\right]\cdot\boldsymbol{n}=2\bigg(\big(\hat{u}^{(2)}\big)'-\big(\hat{u}^{(1)}\big)'\bigg)\zeta \tag{A.15}$$

Then

$$\begin{aligned}\boldsymbol{n}\cdot\left[\boldsymbol{H}\boldsymbol{H}^T\right]\cdot\boldsymbol{n}&=\boldsymbol{n}\cdot\bigg(\boldsymbol{H}^{(2)}(\boldsymbol{H}^{(2)})^T-\boldsymbol{H}^{(1)}(\boldsymbol{H}^{(1)})^T\bigg)\cdot\boldsymbol{n}\\&=(\boldsymbol{n}\cdot\boldsymbol{H}^{(2)})(\boldsymbol{H}^{(2)}\cdot\boldsymbol{n})-(\boldsymbol{n}\cdot\boldsymbol{H}^{(1)})(\boldsymbol{H}^{(1)}\cdot\boldsymbol{n})\end{aligned} \tag{A.16}$$

and after substituting (A.5) and (3.5) we find

$$\boldsymbol{n}\cdot\left[\boldsymbol{H}\boldsymbol{H}^T\right]\cdot\boldsymbol{n}=O(\epsilon^2). \tag{A.17}$$

(3.28) non-dimensionalised with the chosen scalings becomes

$$\mathbf{T}^{(\iota)}=-\bigg(p^{(\iota)}+\bar{B}(1+\chi^{(\iota)})(H^{(\iota)})^2\bigg)\mathbf{I}+\bar{B}(1+\chi^{(\iota)})\boldsymbol{H}^{(\iota)}(\boldsymbol{H}^{(\iota)})^T+\boldsymbol{\nabla u}^{(\iota)}+(\boldsymbol{\nabla u}^{(\iota)})^T \tag{A.18}$$

where

$$\bar{B}=\frac{J_0^2}{4\pi^2\sigma R}. \tag{A.19}$$

and we have used $\eta^{(1)}=\eta^{(2)}$. Substituting (A.12), (A.15) and (A.17) gives, to $O(\epsilon)$,

$$\boldsymbol{n}\cdot\big[\mathbf{T}\big]\cdot\boldsymbol{n}=p^{(1)}-p^{(2)}+B\big(\chi^{(2)}-\chi^{(1)}\big)\big(\hat{S}-im\hat{\phi}^{(1)}\big), \tag{A.20}$$

where $B=2\bar{B}$.

Next, in the disturbed system,

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{n}=\frac{1}{r}+\epsilon\bigg(\frac{m^2}{r^2}+k^2\bigg)\hat{S}\zeta+O(\epsilon^2), \tag{A.21}$$

and at $r = S$,

$$\boldsymbol{\nabla} \cdot \boldsymbol{n} = \frac{1}{1+\epsilon\hat{S}\zeta} + \left(\frac{\epsilon m^2}{(1+\epsilon\hat{S}\zeta)^2} + k^2\epsilon\right)\hat{S}\zeta + O(\epsilon^2) \tag{A.22}$$

Taylor expanding, gives, to $O(\epsilon)$,

$$\boldsymbol{\nabla} \cdot \boldsymbol{n} = 1 + \epsilon\big(m^2 + k^2 - 1\big)\hat{S}\zeta \tag{A.23}$$

Consequently (2.63) after non-dimensionalising, substituting the perturbed variables and linearising, becomes

$$(m^2+k^2-1+B(\chi^{(1)}-\chi^{(2)}))\hat{S} = 2imB(\chi^{(1)}-\chi^{(2)})\hat{\phi}^{(1)}+p^{(1)}-p^{(2)}+2\Big(\big(\hat{u}^{(2)}\big)'-\big(\hat{u}^{(1)}\big)'\Big). \tag{A.24}$$

A.3 Derivation of the tangential stress conditions

Since $\boldsymbol{n}$ and $\boldsymbol{\tau}_{1,2}$ are orthogonal, all diagonal terms in $\mathbf{T}^{(\iota)}$ can be neglected. Moreover,

$$\begin{aligned}
\boldsymbol{n} \cdot \big[(1+\chi)\boldsymbol{H}\boldsymbol{H}^T\big]_{1,2} \cdot \boldsymbol{\tau}_{1,2} &= \boldsymbol{n} \cdot \Big(\big(1+\chi^{(2)}\big)\boldsymbol{H}^{(2)}\big(\boldsymbol{H}^{(2)}\big)^T\big(1+\chi^{(1)}\big)\boldsymbol{H}^{(1)}\big(\boldsymbol{H}^{(1)}\big)^T\Big) \cdot \boldsymbol{\tau}_{1,2} \\
&= \big(1+\chi^{(2)}\big)(\boldsymbol{n} \cdot \boldsymbol{H}^{(2)})(\boldsymbol{H}^{(2)} \cdot \boldsymbol{\tau}_{1,2}) \\
&\quad - \big(1+\chi^{(1)}\big)\big(\boldsymbol{n} \cdot \boldsymbol{H}^{(1)}\big)\big(\boldsymbol{H}^{(1)} \cdot \boldsymbol{\tau}_{1,2}\big) \\
&= \big(1+\chi^{(2)}\big)\big(\boldsymbol{n} \cdot \boldsymbol{H}^{(2)}\big)\Big(\big(\boldsymbol{H}^{(2)} - \boldsymbol{H}^{(2)}\big) \cdot \boldsymbol{\tau}_{1,2}\Big),
\end{aligned} \tag{A.25}$$

and thus by (2.60),

$$\boldsymbol{n} \cdot [(1+\chi)\boldsymbol{H}\boldsymbol{H}^T]_{1,2} \cdot \boldsymbol{\tau}_{1,2} = 0. \tag{A.26}$$

Using (A.13), (3.5) and (3.6),

$$\boldsymbol{n}\cdot\left[\boldsymbol{\nabla u}^{(\iota)}+(\boldsymbol{\nabla u}^{(\iota)})^T\right]\cdot\boldsymbol{\tau}_1=\frac{1}{\sqrt{1+\frac{1}{r^2}S_\theta^2+S_z^2}\sqrt{1+\frac{1}{r^2}S_\theta^2}}\Bigg(\frac{1}{r}\frac{\partial u^{(\iota)}}{\partial\theta}-\frac{v^{(\iota)}}{r}+\frac{\partial v^{(\iota)}}{\partial r}$$
$$+\frac{S_\theta}{r}\left(2\frac{\partial u^{(\iota)}}{\partial r}-\frac{S_\theta}{r}\left(\frac{\partial v^{(\iota)}}{\partial r}+\frac{1}{r}\frac{\partial u^{(\iota)}}{\partial\theta}-\frac{v^{(\iota)}}{r}\right)-\frac{2}{r}\frac{\partial v^{(\iota)}}{\partial\theta}-\frac{2u^{(\iota)}}{r}\right)$$
$$-S_z\left(\frac{S_\theta}{r}\left(\frac{\partial w^{(\iota)}}{\partial r}+\frac{\partial u^{(\iota)}}{\partial z}\right)+\frac{1}{r}\frac{\partial w^{(\iota)}}{\partial\theta}+\frac{\partial v^{(\iota)}}{\partial z}\right)\Bigg) \tag{A.27}$$

and

$$\boldsymbol{n}\cdot\left[\boldsymbol{\nabla u}^{(\iota)}+(\boldsymbol{\nabla u}^{(\iota)})^T\right]\cdot\boldsymbol{\tau}_2=\frac{1}{\sqrt{1+\frac{1}{r^2}S_\theta^2+S_z^2}\sqrt{1+S_z^2}}\Bigg(2\frac{\partial u^{(\iota)}}{\partial r}S_z+\frac{\partial u^{(\iota)}}{\partial z}$$
$$+\frac{\partial w^{(\iota)}}{\partial r}-\frac{S_\theta}{r}\left(S_z\left(\frac{\partial v^{(\iota)}}{\partial r}+\frac{1}{r}\frac{\partial u^{(\iota)}}{\partial\theta}-\frac{v^{(\iota)}}{r}\right)+\frac{\partial v^{(\iota)}}{\partial z}+\frac{1}{r}\frac{\partial w^{(\iota)}}{\partial\theta}\right)$$
$$-S_z\left(S_z\left(\frac{\partial w^{(\iota)}}{\partial r}+\frac{\partial u^{(\iota)}}{\partial z}\right)+2\frac{\partial w^{(\iota)}}{\partial z}\right)\Bigg). \tag{A.28}$$

Non-dimensionalising (A.27) and (A.28), substituting the perturbed variables and linearising results in

$$\left[im\hat{u}-\hat{v}+\hat{v}'\right]_{1,2}=0 \tag{A.29}$$

and

$$\left[ik\hat{u}+\hat{w}'\right]_{1,2}=0 \tag{A.30}$$

at $r=1$, respectively. But, invoking (3.26) results in

$$\left[\hat{v}'\right]_{1,2}=0 \quad\text{and}\quad \left[\hat{w}'\right]_{1,2}=0 \tag{A.31}$$

at $r=1$.

A.4 Functions in the expression for the growth rate

F in (3.35) is given by

$$F = \frac{\hat{F}_1}{\hat{F}_2}, \tag{A.32}$$

where

$$\begin{aligned}
\hat{F}_1 = {} & k^2a^2\bar{k}((-\bar{k}^2k(\mathrm{I}_m(k)m + \mathrm{I}_{m+1}(k)k)\mathrm{K}_{m+1}(k) + \bar{k}k^2(\bar{k}\mathrm{I}_{m+1}(\bar{k}) + m\mathrm{I}_m(\bar{k}))\mathrm{K}_{m+1}(\bar{k}) \\
& + m(\bar{k}^2\mathrm{I}_{m+1}(k)kK_m(k) - \bar{k}k^2\mathrm{K}_m(\bar{k})\mathrm{I}_{m+1}(\bar{k}) + m(\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2)\mathrm{K}_m(\bar{k}) \\
& + \bar{k}^2\mathrm{I}_m(k)\mathrm{K}_m(k))))\mathrm{K}_m(ka) + \bar{k}^2(-\mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k))^2k^2 + k(\bar{k}\mathrm{K}_{m+1}(\bar{k})\mathrm{I}_m(\bar{k}a) \\
& + 2(\frac{1}{2}\mathrm{K}_m(\bar{k})(\mathrm{I}_m(k)m\mathrm{K}_m(k) - 1)\mathrm{I}_m(\bar{k}a) + \mathrm{I}_m(ka)\mathrm{K}_m(k))m)\mathrm{K}_{m+1}(k) \\
& - (\bar{k}\mathrm{K}_{m+1}(\bar{k})\mathrm{I}_m(\bar{k}a) + m\mathrm{K}_m(k)(-\mathrm{I}_{m+1}(k)k\mathrm{K}_m(\bar{k})\mathrm{I}_m(\bar{k}a) \\
& + \mathrm{I}_m(ka)))m\mathrm{K}_m(k)))(\mathrm{K}_{m+1}(\bar{k}a))^2 - a(k((-\bar{k}^2k(\mathrm{I}_m(k)m + \mathrm{I}_{m+1}(k)k)\mathrm{K}_{m+1}(k) \\
& + \bar{k}k^2(\bar{k}\mathrm{I}_{m+1}(\bar{k}) + m\mathrm{I}_m(\bar{k}))\mathrm{K}_{m+1}(\bar{k}) + m(\bar{k}^2\mathrm{I}_{m+1}(k)k\mathrm{K}_m(k) - \bar{k}k^2\mathrm{K}_m(\bar{k})\mathrm{I}_{m+1}(\bar{k}) \\
& + m(\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2)\mathrm{K}_m(\bar{k}) + \bar{k}^2\mathrm{I}_m(k)\mathrm{K}_m(k))))\mathrm{K}_m(\bar{k}a) + k^2\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) \\
& - \mathrm{K}_m(\bar{k})m)^2)a\bar{k}^2\mathrm{K}_{m+1}(ka) - \bar{k}k^2a(\bar{k}^2(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))(\bar{k}\mathrm{K}_{m+1}(\bar{k}) \\
& - \mathrm{K}_m(\bar{k})m)\mathrm{K}_m(\bar{k}a) - (\bar{k}^2(\mathrm{K}_{m+1}(\bar{k}))^2k^2 - 2\bar{k}\mathrm{K}_m(\bar{k})\mathrm{K}_{m+1}(\bar{k})k^2m \\
& - m^2(\mathrm{K}_m(\bar{k}))^2(\bar{k}^2 - 2k^2))\mathrm{K}_m(ka))\mathrm{I}_{m+1}(\bar{k}a) + \bar{k}^4\mathrm{K}_m(\bar{k}a)ak(k\mathrm{K}_{m+1}(k) \\
& - m\mathrm{K}_m(k))^2\mathrm{I}_{m+1}(ka) + ((k(-2\bar{k}k^2m\mathrm{I}_m(\bar{k})\mathrm{I}_m(k)\mathrm{K}_{m+1}(\bar{k}) + (\bar{k}^2k - 2k^3)\mathrm{I}_{m+1}(k) \\
& + \mathrm{I}_m(k)m(\bar{k} - k)(\bar{k} + k))\bar{k}^2\mathrm{K}_{m+1}(k) + ((-\bar{k}^4k^2 + 2\bar{k}^2k^4)\mathrm{I}_{m+1}(\bar{k}) \\
& + 2\bar{k}k^4m\mathrm{I}_m(\bar{k}))\mathrm{K}_{m+1}(\bar{k}) - (\bar{k}^2k\mathrm{K}_m(k)(-2\bar{k}k^2\mathrm{K}_m(\bar{k})\mathrm{I}_{m+1}(\bar{k}) + \bar{k}^2 - k^2)I_{m+1}(k) \\
& + 2\bar{k}k^4\mathrm{K}_m(\bar{k})I_{m+1}(\bar{k}) + m(\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2)\mathrm{K}_m(\bar{k}) + \bar{k}^2\mathrm{I}_m(k)\mathrm{K}_m(k))(\bar{k}^2 \\
& - 2k^2))m)\mathrm{K}_m(ka) + (\mathrm{I}_m(ka)k^2(\bar{k}^2 - 2k^2)(\mathrm{K}_{m+1}(k))^2 - 2k\bigg(- \frac{1}{2}\bar{k}\mathrm{K}_{m+1}(\bar{k})k^2\mathrm{I}_m(\bar{k}a) \\
& + m\bigg(- \frac{\bar{k}^2\mathrm{K}_m(\bar{k})}{2}(m\mathrm{I}_m(k)\mathrm{K}_m(k) - 1)\mathrm{I}_m(\bar{k}a) + \mathrm{I}_m(ka)\mathrm{K}_m(k)(\bar{k}^2 - 2k^2)\bigg)\bigg)\mathrm{K}_{m+1}(k) \\
& + (-\bar{k}\mathrm{K}_{m+1}(\bar{k})k^2\mathrm{I}_m(\bar{k}a) + (\bar{k}^2\mathrm{I}_{m+1}(k)k\mathrm{K}_m(\bar{k})\mathrm{I}_m(\bar{k}a) + \mathrm{I}_m(ka)(\bar{k}^2
\end{aligned}$$

$$
\begin{aligned}
&-2k^2))m\mathrm{K}_m(k))m\mathrm{K}_m(k))\bar{k}^2)m\mathrm{K}_m(\bar{k}a) - k^2(m\mathrm{I}_m(\bar{k}a)(\bar{k}-k)(\bar{k}+k)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) \\
&- 2\mathrm{K}_m(\bar{k})m)^2\mathrm{K}_m(ka) + \bar{k}^2(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - \mathrm{K}_m(\bar{k})m)))\mathrm{K}_{m+1}(\bar{k}a) \\
&+ \mathrm{K}_m(\bar{k}a)m(-k(\bar{k}(\mathrm{K}_m(\bar{k}))^2am(\bar{k}-k)(\bar{k}+k)\mathrm{I}_{m+1}(\bar{k}a) \\
&+ (\bar{k}^2k(\mathrm{I}_m(k)m + \mathrm{I}_{m+1}(k)k)\mathrm{K}_{m+1}(k) - \bar{k}k^2(\bar{k}\mathrm{I}_{m+1}(\bar{k}) + m\mathrm{I}_m(\bar{k}))\mathrm{K}_{m+1}(\bar{k}) \\
&- m(\bar{k}^2\mathrm{I}_{m+1}(k)k\mathrm{K}_m(k) - \bar{k}k^2\mathrm{K}_m(\bar{k})\mathrm{I}_{m+1}(\bar{k}) + m(\mathrm{I}_m(\bar{k})(\bar{k}^2-2k^2)\mathrm{K}_m(\bar{k}) \\
&+ \bar{k}^2\mathrm{I}_m(k)\mathrm{K}_m(k))))\mathrm{K}_m(\bar{k}a) - \bar{k}^2\mathrm{I}_m(ka)\mathrm{K}_{m+1}(k)km\mathrm{K}_m(\bar{k}) - \mathrm{I}_m(\bar{k}a)(\bar{k}^2(\mathrm{K}_{m+1}(\bar{k}))^2k^2 \\
&- 2\bar{k}\mathrm{K}_m(\bar{k})\mathrm{K}_{m+1}(\bar{k})k^2m - m^2(\mathrm{K}_m(\bar{k}))^2(\bar{k}^2-2k^2)))a\mathrm{K}_{m+1}(ka) + \bar{k}((\mathrm{K}_m(\bar{k})m\bar{k} \\
&- \mathrm{K}_{m+1}(\bar{k})k^2)(\bar{k}(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))\mathrm{K}_m(\bar{k}a) + (\mathrm{K}_m(\bar{k})m\bar{k} \\
&- \mathrm{K}_{m+1}(\bar{k})k^2)\mathrm{K}_m(ka))a\mathrm{I}_{m+1}(\bar{k}a) + k((k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))^2\mathrm{K}_m(\bar{k}a) \\
&+ \mathrm{K}_m(ka)\mathrm{K}_{m+1}(k)km\mathrm{K}_m(\bar{k}))a\bar{k}\mathrm{I}_{m+1}(ka) - \mathrm{K}_{m+1}(\bar{k})\mathrm{K}_{m+1}(k)k^3 \\
&- m\mathrm{K}_m(k)(\mathrm{K}_m(\bar{k})m\bar{k} - \mathrm{K}_{m+1}(\bar{k})k^2)))\bar{k} \qquad \text{(A.33)}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{F}_2 \;=\; & -\bar{k}^2\Big((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + \mathrm{K}_m(\bar{k}a)(-a\bar{k}^2k\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)m(\bar{k}^2 \\
& - 2k^2))\mathrm{K}_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km\Big)(\bar{k}+k)(\bar{k}-k)a. \qquad \text{(A.34)}
\end{aligned}
$$

Let

$$\bar{F} = f_1(k^2 + m^2 - 1 + B(\chi^{(1)} - \chi^{(2)})) + f_2B(\chi^{(1)} - \chi^{(2)})^2m^2, \qquad \text{(A.35)}$$

such that (3.35) is

$$s = -gF\bar{F}, \qquad \text{(A.36)}$$

where

$$
\begin{aligned}
f_1 \;=\;& (a\mathrm{I}_m(ka)\chi^{(1)}k(\mathrm{I}_m(k)\chi^{(2)}\mathrm{K}_{m+1}(k)k + \chi^{(1)}\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1 \\
& + m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi^{(1)} - \chi^{(2)}))\mathrm{K}_{m+1}(ka) - ak\chi^{(1)}\mathrm{K}_m(k)\mathrm{I}_m(ka)(\chi^{(1)} \\
& - \chi^{(2)})(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) - \chi^{(1)}\mathrm{K}_m(k)m(\chi^{(1)} - \chi^{(2)})(k\mathrm{K}_{m+1}(k) \\
& - m\mathrm{K}_m(k))(\mathrm{I}_m(ka))^2 - \mathrm{K}_m(ka)\chi^{(1)}m(\mathrm{I}_m(k)\chi^{(2)}\mathrm{K}_{m+1}(k)k \\
& + \chi^{(1)}\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1 + m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi^{(1)} - \chi^{(2)}))\mathrm{I}_m(ka) \\
& + \mathrm{I}_m(k)\chi^{(2)}\mathrm{K}_{m+1}(k)k + \chi^{(1)}\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1 \\
& + m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi^{(1)} - \chi^{(2)})),
\end{aligned}
\tag{A.37}
$$

$$
\begin{aligned}
f_2 \;=\;& 2\Big(\mathrm{K}_m(k)(\mathrm{I}_m(ka)\mathrm{K}_m(k)\mathrm{I}_{m+1}(ka)a\chi^{(1)}k + \mathrm{I}_m(ka)\mathrm{I}_m(k)\mathrm{K}_{m+1}(ka)a\chi^{(1)}k \\
& + (\mathrm{I}_m(ka))^2\mathrm{K}_m(k)\chi^{(1)}m - \mathrm{I}_m(ka)\mathrm{I}_m(k)\mathrm{K}_m(ka)\chi^{(1)}m + \mathrm{I}_m(k))\Big),
\end{aligned}
\tag{A.38}
$$

$$
\begin{aligned}
g \;=\;& -\Big(k\Big(- a\mathrm{I}_m(ka)(\mathrm{I}_m(k)\chi^{(2)}\mathrm{K}_{m+1}(k)k + k\mathrm{K}_m(k)(1 + \chi^{(1)})\mathrm{I}_{m+1}(k) + 1 \\
& + \mathrm{I}_m(k)m(\chi^{(1)} - \chi^{(2)} + 1)\mathrm{K}_m(k))\chi^{(1)}k\mathrm{K}_{m+1}(ka) - a(-(\chi^{(1)} - \chi^{(2)})(k\mathrm{K}_{m+1}(k) \\
& - m\mathrm{K}_m(k))\mathrm{I}_m(ka) + \mathrm{K}_m(ka)(\mathrm{I}_m(k)m + \mathrm{I}_{m+1}(k)k))\chi^{(1)}k\mathrm{K}_m(k)\mathrm{I}_{m+1}(ka) \\
& + \chi^{(1)}\mathrm{K}_m(k)m(\chi^{(1)} - \chi^{(2)})(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))(\mathrm{I}_m(ka))^2 \\
& + \mathrm{K}_m(ka)\chi^{(1)}m(\mathrm{I}_m(k)\chi^{(2)}\mathrm{K}_{m+1}(k)k + \chi^{(1)}\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1 \\
& + m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi^{(1)} - \chi^{(2)}))\mathrm{I}_m(ka) - \mathrm{I}_m(k)\chi^{(2)}\mathrm{K}_{m+1}(k)k \\
& + m\mathrm{K}_m(k)\mathrm{I}_m(k)\chi^{(2)} - 1\Big)\Big)^{-1}.
\end{aligned}
\tag{A.39}
$$

A.5 Constants in solution of the governing equations

The constants $c_1^{(\iota)}...c_6^{(\iota)}$ in (3.22) are as follows, $c_{1,3,5}^{(2)} = 0$,

$$
c_1^{(1)} = g\bar{F}\hat{S}(kK_{(m+1)}(k) - m\mathrm{K}_m(k)), \tag{A.40}
$$

$$
\begin{aligned}
c_2^{(1)} \;=\; & -g\bar{F}\hat{S}\Bigg(- a\bar{k}k^2\big((-\mathrm{K}_{m+1}(k)k + m\mathrm{K}_m(k))\mathrm{I}_m(ka) + \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) \\
& - m\mathrm{K}_m(\bar{k}))\big)(\mathrm{K}_{m+1}(\bar{k}a))^2 + (-\bar{k}ak^2(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) \\
& + \bar{k}^2ak(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + m((\bar{k}^2 - 2k^2)(K_{m+1}(k)k \\
& - m\mathrm{K}_m(k))I_m(ka) - \bar{k}\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a)
\end{aligned}
$$

$$
\begin{aligned}
& -m(\mathrm{K}_m(\bar{k}a))^2\bar{k}((\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) + \mathrm{I}_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k)))\Bigg) \times \Bigg((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2ak\mathrm{K}_{m+1}(ka) \\
& + \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a) + \\
& (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km\Bigg)^{-1},
\end{aligned}
\tag{A.41}
$$

$$
\begin{aligned}
c_2^{(2)} \;=\; & -g\bar{F}\hat{S}\Bigg(- a\bar{k}((-\mathrm{K}_{m+1}(k)k + m\mathrm{K}_m(k))\mathrm{I}_m(ka) + (-k\mathrm{I}_{m+1}(k) - m\mathrm{I}_m(k))\mathrm{K}_m(ka) \\
& + \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})))k^2(\mathrm{K}_{m+1}(\bar{k}a))^2 + \mathrm{K}_m(\bar{k}a)(-\bar{k}^2ak(k\mathrm{I}_{m+1}(k) \\
& + m\mathrm{I}_m(k))\mathrm{K}_{m+1}(ka) - \bar{k}ak^2(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) + \bar{k}^2ak(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + m((\bar{k}^2 - 2k^2)(k\mathrm{I}_{m+1}(k) + mI_m(k))\mathrm{K}_m(ka) \\
& + (\bar{k}^2 - 2k^2)(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_m(ka) - \bar{k}\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_m(\bar{k})m \\
& - k^2K_{m+1}(\bar{k}))))\mathrm{K}_{m+1}(\bar{k}a) - m(\mathrm{K}_m(\bar{k}a))^2\bar{k}((-\mathrm{I}_{m+1}(k)k^2 - km\mathrm{I}_m(k))\mathrm{K}_{m+1}(ka) \\
& + (\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))I_{m+1}(\bar{k}a) + \mathrm{I}_{m+1}(ka)k(K_{m+1}(k)k - m\mathrm{K}_m(k)))\Bigg) \\
& \times \Bigg((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2ak\mathrm{K}_{m+1}(ka) \\
& + \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2K_{m+1}(ka)\bar{k}km\Bigg),
\end{aligned}
\tag{A.42}
$$

$$
c_3^{(1)} = \frac{\bar{k}g\bar{F}\hat{S}(\mathrm{i}\bar{k}\mathrm{K}_{m+1}(\bar{k}) + m\mathrm{K}_m)}{(\bar{k}^2 - k^2)},
\tag{A.43}
$$

$$
c_5^{(1)} = g\bar{F}\hat{S}\frac{\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2)}{2\bar{k}^3 - 2\bar{k}k^2},
\tag{A.44}
$$

$$
\begin{aligned}
c_4^{(1)} \;=\; & g\bar{F}\hat{S}\Big(\mathrm{i}((-a(\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k)))\bar{k}^2 k\mathrm{K}_{m+1}(ka) + (-\bar{k}ak^2(\bar{k}K_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) \\
& + \bar{k}^2 ak(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + \mathrm{I}_m(\bar{k}a)m(\bar{k}-k)(\bar{k}+k) \\
& \times (\bar{k}\mathrm{K}_{m+1}(\bar{k}) - 2m\mathrm{K}_m(\bar{k})))\mathrm{K}_m(ka))\mathrm{K}_{m+1}(\bar{k}a) - m\mathrm{K}_m(\bar{k}a)\bar{k}(-(\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) \\
& - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))k\mathrm{K}_{m+1}(ka) \\
& + \mathrm{K}_m(ka)((\bar{k}\mathrm{K}_m(\bar{k})m - k^2 K_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) \\
& + I_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))))k\Big) \\
& \times \Big((\bar{k}+k)((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2 ak\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)m(\bar{k}^2 \\
& - 2k^2))\mathrm{K}_m(\bar{k}a)K_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k}-k)\Big)^{-1},
\end{aligned}
\tag{A.45}
$$

$$
\begin{aligned}
c_4^{(2)} \;=\; & g\bar{F}\hat{S}\Big(\mathrm{i}k(\bar{k}\mathrm{K}_m(ka)ak^2(\bar{k}\mathrm{I}_{m+1}(\bar{k}) + m\mathrm{I}_m(\bar{k}))(\mathrm{K}_{m+1}(\bar{k}a))^2 + (-a\bar{k}^2((\bar{k}\mathrm{I}_{m+1}(\bar{k}) \\
& + m\mathrm{I}_m(\bar{k}))\mathrm{K}_m(\bar{k}a) + \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k)))k\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)(-\bar{k}ak^2(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) \\
& + \bar{k}^2 ak(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + ((\bar{k}^2 - 2k^2)(\bar{k}\mathrm{I}_{m+1}(\bar{k}) \\
& + +m\mathrm{I}_m(\bar{k}))\mathrm{K}_m(\bar{k}a) + \mathrm{I}_m(\bar{k}a)(\bar{k}-k)(\bar{k}+k)(\bar{k}K_{m+1}(\bar{k}) \\
& - 2m\mathrm{K}_m(\bar{k})))m))\mathrm{K}_{m+1}(\bar{k}a) + m\mathrm{K}_m(\bar{k}a)\bar{k}(((\bar{k}\mathrm{I}_{m+1}(\bar{k})m\mathrm{I}_m(\bar{k}))\mathrm{K}_m(\bar{k}a) \\
& + \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k)))k\mathrm{K}_{m+1}(ka) - \mathrm{K}_m(ka)((\bar{k}\mathrm{K}_m(\bar{k})m \\
& - k^2 K_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) + I_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))))\Big) \\
& \times \Big((\bar{k}+k)((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 \\
& + (-\bar{k}^2 ak\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)K_{m+1}(\bar{k}a) \\
& + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k}-k)\Big)^{-1},
\end{aligned}
\tag{A.46}
$$

$$
\begin{aligned}
c_6^{(1)} \;=\;& g\bar{F}\hat{S}\bigg(-2(-\frac{a}{2}(\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k)))\bar{k}^2k\mathrm{K}_{m+1}(ka) + (-\frac{a\bar{k}}{2}(\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))\mathrm{I}_{m+1}(\bar{k}a) \\
& + \frac{\bar{k}^2ak}{2}(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + \mathrm{I}_m(\bar{k}a)m(\bar{k} - k)(\bar{k} + k)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) \\
& - 2m\mathrm{K}_m(\bar{k})))\mathrm{K}_m(ka))k^2K_{m+1}(\bar{k}a) + m\mathrm{K}_m(\bar{k}a)(-(\bar{k}\mathrm{K}_m(\bar{k})a(\bar{k} - k) \\
& \times (\bar{k} + k)\mathrm{I}_{m+1}(\bar{k}a) + (\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))\mathrm{I}_m(\bar{k}a) - \bar{k}^2(\mathrm{K}_{m+1}(k)k \\
& - m\mathrm{K}_m(k))\mathrm{I}_m(ka))k\mathrm{K}_{m+1}(ka) + \bar{k}^2\mathrm{K}_m(ka)((\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) \\
& + \mathrm{I}_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))))\bar{k}\bigg) \\
& \times \bigg((2\bar{k} + 2k)\bar{k}((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2ak\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)m(\bar{k}^2 \\
& - 2k^2))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k} - k)\bigg)^{-1}, \qquad \text{(A.47)}
\end{aligned}
$$

$$
\begin{aligned}
c_6^{(2)} \;=\;& g\bar{F}\hat{S}\bigg(a\bar{k}(-\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 + m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))K_m(ka)k^2(\mathrm{K}_{m+1}(\bar{k}a))^2 \\
& + (-a\bar{k}^2k((-\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 + m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a) - (\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) \\
& - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(K_{m+1}(k)k - m\mathrm{K}_m(k)))k^2)\mathrm{K}_{m+1}(ka) \\
& + \mathrm{K}_m(ka)(a\bar{k}(\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))k^2\mathrm{I}_{m+1}(\bar{k}a) \\
& - \bar{k}^2ak^3(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) \\
& + m((\bar{k}^2 - 2k^2)(-\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 + m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))K_m(\bar{k}a) \\
& - 2\mathrm{I}_m(\bar{k}a)k^2(\bar{k} - k)(\bar{k} + k)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - 2m\mathrm{K}_m(\bar{k})))))\mathrm{K}_{m+1}(\bar{k}a) + \\
& m\mathrm{K}_m(\bar{k}a)\bar{k}(-(\bar{k}\mathrm{K}_m(\bar{k})a(\bar{k} - k)(\bar{k} + k)\mathrm{I}_{m+1}(\bar{k}a) + (\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 \\
& - m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a) + (\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))\mathrm{I}_m(\bar{k}a) \\
& - \bar{k}^2(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_m(ka))k\mathrm{K}_{m+1}(ka) \\
& + \bar{k}^2\mathrm{K}_m(ka)((\bar{k}K_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) \\
& + \mathrm{I}_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))))\bigg) \times \bigg((2\bar{k} + 2k)\bar{k}((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 \\
& + (-\bar{k}^2ak\mathrm{K}_{m+1}(ka) + K_m(ka)m(\bar{k}^2 - 2k^2))K_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a) \\
& + (K_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k} - k)\bigg)^{-1}. \qquad \text{(A.48)}
\end{aligned}
$$

In the highly viscous limit, s_v is given by (3.36) where

$$
\begin{aligned}
F_v \quad = \quad & -\frac{1}{2k^2}\Bigg(2((k\mathrm{I}_{m+1}(k) - \frac{1}{2}(k^2 + m^2 - 2m)\mathrm{I}_m(k))k\mathrm{K}_{m+1}(k) + \frac{1}{2}K_m(k)(k^2 + m^2 \\
& - 2m)(k\mathrm{I}_{m+1}(k) + 2\mathrm{I}_m(k)m))a^2k^2(\mathrm{K}_{m+1}(ka))^3 + a(-2ka((\mathrm{K}_{m+1}(k))^2k^2 \\
& + k\mathrm{K}_m(k)(k^2 + m^2 - 2m)\mathrm{K}_{m+1}(k) - m(\mathrm{K}_m(k))^2(k^2 + m^2 - 2m))\mathrm{I}_{m+1}(ka) + \\
& (k^2(a^2k^2 + m^2 - 2m)(\mathrm{K}_{m+1}(k))^2 - 2m\mathrm{K}_m(k)((a^2 + 1)k^2 + 2m^2 - 4m)k\mathrm{K}_{m+1}(k) \\
& + (\mathrm{K}_m(k))^2(k^4 + m^2(a^2 + 4)k^2 + 4m^4 - 8m^3))\mathrm{I}_m(ka) - 6((k\mathrm{I}_{m+1}(k) \\
& - \frac{1}{2}\mathrm{I}_m(k)(k^2 + m^2 - 2m))k\mathrm{K}_{m+1}(k) + \frac{1}{2}\mathrm{K}_m(k)(k^2 + m^2 - 2m)(k\mathrm{I}_{m+1}(k) \\
& + 2\mathrm{I}_m(k)m))(m + 2/3)\mathrm{K}_m(ka))k(\mathrm{K}_{m+1}(ka))^2 + \mathrm{K}_m(ka)(((a^2k^2 + \\
& (m + 2)^2)k^2(\mathrm{K}_{m+1}(k))^2 - 2\mathrm{K}_m(k)((a^2m - 2m - 2)k^2 - m^3 + 4m)k\mathrm{K}_{m+1}(k) \\
& + (\mathrm{K}_m(k))^2(k^4 + ((a^2 - 2)m^2 - 4m)k^2 - 2m^4 + 8m^2))ak\mathrm{I}_{m+1}(ka) \\
& + (-k^2(m + 2)(a^2k^2 + m^2 - 2m)(\mathrm{K}_{m+1}(k))^2 - 2\mathrm{K}_m(k)(a^2k^4 - (2a^2m + m^2 + 2m)k^2 \\
& - 2m^4 + 8m^2)k\mathrm{K}_{m+1}(k) + 2m((a^2 - \frac{1}{2})k^4 + (\frac{1}{2}a^2m^2 - a^2m - 2m^2 - 4m)k^2 - 2m^4 \\
& + 8m^2)(\mathrm{K}_m(k))^2)\mathrm{I}_m(ka) - 2((kI_{m+1}(k) - \frac{1}{2}\mathrm{I}_m(k)(k^2 + m^2 - 2m))k\mathrm{K}_{m+1}(k) \\
& + \frac{1}{2}K_m(k)(k^2 + m^2 - 2m)(k\mathrm{I}_{m+1}(k) + 2\mathrm{I}_m(k)m))(a^2k^2 - 2m^2 - 4m)\mathrm{K}_m(ka))\mathrm{K}_{m+1}(ka) \\
& - m(((a^2k^2 + (m + 2)^2)(\mathrm{K}_{m+1}(k))^2 - 2k\mathrm{K}_m(k)(a^2m - m - 2)\mathrm{K}_{m+1}(k) \\
& + (k^2 + a^2m(m - 2))(\mathrm{K}_m(k))^2)k\mathrm{I}_{m+1}(ka) - 2a(((\mathrm{K}_{m+1}(k))^2k^2 + k\mathrm{K}_m(k)(k^2 + m^2 \\
& - 2m)\mathrm{K}_{m+1}(k) - m(\mathrm{K}_m(k))^2(k^2 + m^2 - 2m))\mathrm{I}_m(ka) + ((k\mathrm{I}_{m+1}(k) - \frac{1}{2}\mathrm{I}_m(k)(k^2 + m^2 \\
& - 2m))k\mathrm{K}_{m+1}(k) + \frac{1}{2}\mathrm{K}_m(k)(k^2 + m^2 - 2m)(k\mathrm{I}_{m+1}(k) \\
& + 2\mathrm{I}_m(k)m))\mathrm{K}_m(ka)))(\mathrm{K}_m(ka))^2k\Bigg) \times \Bigg(- k^2a^2(\mathrm{K}_{m+1}(ka))^3 \\
& + 3(m + 2/3)\mathrm{K}_m(ka)ak(\mathrm{K}_{m+1}(ka))^2 + (\mathrm{K}_m(ka))^2(a^2k^2 - 2m^2 \\
& - 4m)\mathrm{K}_{m+1}(ka) - ka(\mathrm{K}_m(ka))^3m\Bigg)^{-1}. \qquad \text{(A.49)}
\end{aligned}
$$

In this inviscid limit, s_I is given by (3.37), where

$$F_I = \Big((ka(k\mathrm{I}_{m+1}(k) + \mathrm{I}_m(k)m)\mathrm{K}_{m+1}(ka) - ka(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) - m((k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))\mathrm{I}_m(ka) + \mathrm{K}_m(ka)(k\mathrm{I}_{m+1}(k) + \mathrm{I}_m(k)m)))(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k)) \Big) \times \frac{1}{(\mathrm{K}_{m+1}(ka)ak - \mathrm{K}_m(ka)m)}. \tag{A.50}$$

A.6 Vector identities and theorems

The following vector calculus identities and theorems are used in Chapter 6 and are given here for ease of reference. Given vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, the scalar triple product is

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}), \tag{A.51}$$

and the vector triple product is

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}. \tag{A.52}$$

Given scalars κ_1, κ_2, the product rule for scalars is

$$\boldsymbol{\nabla}(\kappa_1\kappa_2) = \kappa_1\boldsymbol{\nabla}\kappa_2 + \kappa_2\boldsymbol{\nabla}\kappa_1. \tag{A.53}$$

Moreover,

$$\boldsymbol{\nabla} \cdot (\kappa_1\boldsymbol{a}) = \kappa_1(\boldsymbol{\nabla} \cdot \boldsymbol{a}) + \boldsymbol{\nabla}\kappa_1 \cdot \boldsymbol{a}, \tag{A.54}$$

$$\boldsymbol{\nabla} \times (\kappa_1\boldsymbol{a}) = \kappa_1(\boldsymbol{\nabla} \times \boldsymbol{a}) + \boldsymbol{\nabla}\kappa_1 \times \boldsymbol{a}, \tag{A.55}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \boldsymbol{b} - (\boldsymbol{\nabla} \times \boldsymbol{b}) \cdot \boldsymbol{a}, \tag{A.56}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \nabla^2\boldsymbol{a}, \tag{A.57}$$

$$\boldsymbol{\nabla} \cdot (\nabla^2\boldsymbol{a}) = \nabla^2(\boldsymbol{\nabla} \cdot \boldsymbol{a}), \tag{A.58}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})=0. \tag{A.59}$$

The divergence theorem, states that for a volume V with boundary ∂V

$$\iint_{\partial V}\boldsymbol{a}\cdot d\boldsymbol{S}=\iiint_V(\boldsymbol{\nabla}\cdot\boldsymbol{a})dV. \tag{A.60}$$

Stokes theorem for a surface S in three-dimensions, with a boundary curve ∂S, states

$$\oint_{\partial S}\boldsymbol{a}\cdot d\boldsymbol{l}=\iint_S(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot d\boldsymbol{S}. \tag{A.61}$$

A.7 The viscous term of (6.71)

Here, we show an attempt at proving

$$I_v=\iiint_V\boldsymbol{A}^*\cdot\nabla^2(\nabla^2\boldsymbol{A})dV \tag{A.62}$$

is of definite sign. I_v can be written as

$$I_v=\iiint_V\boldsymbol{A}^*\cdot\nabla^2(\nabla^2\boldsymbol{A})dV-\iiint_V\boldsymbol{A}^*\cdot\big(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\nabla^2\boldsymbol{A}))\big)dV, \tag{A.63}$$

and use (A.56) to write

$$I_v=\iiint_V\boldsymbol{\nabla}\cdot\big(\boldsymbol{A}^*\times(\boldsymbol{\nabla}\times(\nabla^2\boldsymbol{A}))\big)-(\boldsymbol{\nabla}\times\boldsymbol{A}^*)\cdot(\boldsymbol{\nabla}\times(\nabla^2\boldsymbol{A}))dV. \tag{A.64}$$

Using (A.56) again,

$$I_v = \iiint_V \Big(\boldsymbol{\nabla} \cdot \Big(\boldsymbol{A}^* \times \big(\boldsymbol{\nabla} \times (\nabla^2 \boldsymbol{A})\big)\Big) \\ - \boldsymbol{\nabla} \cdot \Big((\boldsymbol{\nabla} \times \boldsymbol{A}^*) \times (\nabla^2 \boldsymbol{A})\Big) + \big(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}^*)\big) \cdot (\nabla^2 \boldsymbol{A})\Big) dV. \tag{A.65}$$

Since $\nabla^2 \boldsymbol{A} = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$,

$$I_v = \iiint_V \Big(\boldsymbol{\nabla} \cdot \Big(\boldsymbol{A}^* \times \big(\boldsymbol{\nabla} \times (\nabla^2 \boldsymbol{A})\big)\Big) - \boldsymbol{\nabla} \cdot \Big((\boldsymbol{\nabla} \times \boldsymbol{A}^*) \times (\nabla^2 \boldsymbol{A})\Big) - |\nabla^2 \boldsymbol{A}|^2\Big) dV, \tag{A.66}$$

where the second term is zero by the divergence theorem. Now,

$$\iiint_V \boldsymbol{\nabla} \cdot \Big(\boldsymbol{A}^* \times (\boldsymbol{\nabla} \times (\nabla^2 \boldsymbol{A}))\Big) dV = \iint_{\partial V} \Big(\boldsymbol{A}^* \times \big(\boldsymbol{\nabla} \times (\nabla^2 \boldsymbol{A})\big)\Big) \cdot \hat{\boldsymbol{n}} dS \\ = \iint_{\partial V} \Big(\boldsymbol{A}^* \times \big(\nabla^2 (\boldsymbol{\nabla} \times \boldsymbol{A})\big)\Big) \cdot \hat{\boldsymbol{n}} dS, \tag{A.67}$$

by the divergence theorem. Ideally, either $\boldsymbol{A} = \boldsymbol{0}$ or $\nabla^2(\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$ on the boundaries of the volume, in which case (A.67) is zero and

$$I_v = s \iiint_V |\boldsymbol{\omega}|^2 dV. \tag{A.68}$$

We know for certain that $\boldsymbol{\nabla} \times \boldsymbol{A} = 0$ at the boundary, but cannot say otherwise. Alternatively, if by vector identities we can show

$$\iiint_V \boldsymbol{\nabla} \cdot \Big(\boldsymbol{A}^* \times \big(\boldsymbol{\nabla} \times (\nabla^2 \boldsymbol{A})\big)\Big) dV \tag{A.69}$$

is of definite sign, then I_v will be of definite sign and we can proceed. However, at present, we cannot do this and the integral in the term due to viscous forces in (6.71) remains of indefinite sign.

A.8 Proof of stability condition in Section 6.6.1

From (6.85) it follows that if $H_0' < 0$ throughout V, $s_I^2 > 0$ and there exists a mode such that $\mathcal{R}(s) > 0$, and the system will be unstable. In contrast, if $H_0' > 0$ throughout V the system will be stable. Here we prove a stronger stability condition in a similar manner to previous sections.

Consider the functional

$$F(\boldsymbol{y}) = \sqrt{-\frac{B\iiint_V H_0 H_0' |\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y}|^2 dV}{\iiint_V |\boldsymbol{y}|^2 dV}}. \tag{A.70}$$

for all real vector functions $\boldsymbol{y}$ satisfying the boundary conditions of $\boldsymbol{u}$. We now prove that stationary points of the functional (A.70) correspond to real eigenvalues satisfying (6.70) in the inviscid limit, namely

$$s_I^2 \nabla^2 \boldsymbol{A} + B H_0 H_0' (\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla})(\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \boldsymbol{A})) = 0. \tag{A.71}$$

Note that $F(\boldsymbol{y})$ is bounded above and for $H_0' > 0$ has a global maximum. Let $\boldsymbol{y} = \boldsymbol{y_0}$ be a stationary point of F, such that $F(\boldsymbol{y_0}) = F_0$ and consider $\boldsymbol{y} = \boldsymbol{y_0} + \epsilon \boldsymbol{y_1}$, where $\boldsymbol{y_0}$ and $\boldsymbol{y_1}$ are divergence free and satisfy the same boundary conditions as $\boldsymbol{u}$. Since $\boldsymbol{y} = \boldsymbol{y_0}$ is a stationary point of $F(\boldsymbol{y})$, the first variation from Taylor expanding $F(\boldsymbol{y_0} + \epsilon \boldsymbol{y_1})$ will be zero, and we obtain

$$F_0^2 \iiint_V \boldsymbol{y_0} \cdot \boldsymbol{y_1} dV + B \iiint_V H_0 H_0' (\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y_0})(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y_1}) dV = 0, \tag{A.72}$$

after some algebra.

Now, before we proceed, we must define vector potentials $\tilde{\boldsymbol{A}}_{\boldsymbol{0,1}}$ such that

$$\boldsymbol{y_{0,1}} = \boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\boldsymbol{0,1}} \tag{A.73}$$

and

$$\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{A}}_{\mathbf{0,1}} = 0. \tag{A.74}$$

Consider the first term of (A.72),

$$I_1 = \iiint_V \boldsymbol{y_0} \cdot \boldsymbol{y_1} dV. \tag{A.75}$$

We write

$$\begin{aligned} I_1 &= \iiint_V (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{0}}) \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{1}}) dV \\ &= \iiint_V \Big(\boldsymbol{\nabla} \cdot ((\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{1}}) \times \tilde{\boldsymbol{A}}_0) + \tilde{\boldsymbol{A}}_{\mathbf{0}} \cdot (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{1}})) \Big) dV. \end{aligned} \tag{A.76}$$

But, by the divergence theorem and boundary conditions for $\tilde{\boldsymbol{y}}_{\mathbf{1}}$,

$$\begin{aligned} I_1 &= \iiint_V \tilde{\boldsymbol{A}}_{\mathbf{0}} \cdot (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{1}})) dV \\ &= - \iiint_V \tilde{\boldsymbol{A}}_{\mathbf{0}} \cdot \nabla^2 \tilde{\boldsymbol{A}}_{\mathbf{1}} dV. \end{aligned} \tag{A.77}$$

For the second term of (A.72), namely

$$I_2 = \iiint_V H_0 H_0' (\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_0})(\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_1}) dV, \tag{A.78}$$

we write

$$I_2 = \iiint_V H_0 H_0' (\boldsymbol{\nabla} \chi_0 \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{0}}))(\boldsymbol{\nabla} \chi_0 \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{1}})) dV. \tag{A.79}$$

It follows that

$$I_2 = \iiint_V H_0 H_0' \boldsymbol{\nabla} \chi_0 \cdot \big(\boldsymbol{\nabla} \times (\tilde{\mu} \tilde{\boldsymbol{A}}_{\mathbf{0}}) \big) dV - \iiint_V H_0 H_0' \boldsymbol{\nabla} \chi_0 \cdot (\boldsymbol{\nabla} \tilde{\mu} \times \tilde{\boldsymbol{A}}_{\mathbf{0}}) dV, \tag{A.80}$$

where

$$\tilde{\mu} = \boldsymbol{\nabla} \chi_0 \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\mathbf{1}}). \tag{A.81}$$

But, by (A.54) and the divergence theorem the first time of (A.80) is zero. Hence,

$$I_2 = -\iiint_V H_0 H_0' \tilde{\boldsymbol{A}}_{\boldsymbol{0}} \cdot \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\boldsymbol{1}})\big)\Big) dV, \tag{A.82}$$

by (A.51).

Substituting (A.77) and (A.82) into (A.72) we obtain

$$F_0^2 \iiint_V \tilde{\boldsymbol{A}}_{\boldsymbol{0}} \cdot \nabla^2 \tilde{\boldsymbol{A}}_{\boldsymbol{1}} dV + B \iiint_V H_0 H_0' \tilde{\boldsymbol{A}}_{\boldsymbol{0}} \cdot \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\boldsymbol{1}})\big)\Big) dV = 0. \tag{A.83}$$

Substitute $F_0 = s_I$ and observe that (A.83) must be true for any vector $\tilde{\boldsymbol{A}}_{\boldsymbol{0}}$, to give

$$s_I^2 \nabla^2 \tilde{\boldsymbol{A}}_{\boldsymbol{1}} dV + B \iiint_V H_0 H_0' \Big(\boldsymbol{\nabla}\chi_0 \times \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\chi_0 \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{A}}_{\boldsymbol{1}})\big)\Big) dV = 0. \tag{A.84}$$

Hence, stationary points of the functional A.70 correspond to real eigenvalues of A.71. It follows that if, and only if, $H_0' < 0$ somewhere in the volume, the system is unstable in the inviscid limit. The reasoning is analogous to Section 6.4, and is not given here.

A.9 Proof of stability condition in Section 6.6.2

Observe from 6.96 that $s > 0$ when $H_0' < 0$, thus if $H_0' < 0$ throughout the volume the flow is unstable. Alternatively, if $H_0' \geq 0$ everywhere, $\mathcal{R}(s) \leq 0$ and the flow is stable. To prove a stronger stability condition, we consider the functional

$$F(\boldsymbol{y}) = \frac{-\iiint_V |\boldsymbol{\nabla} \times \boldsymbol{y}|^2 dV + \sqrt{\bar{S}}}{2Re \iiint_V |\boldsymbol{y}|^2 dV}, \tag{A.85}$$

where

$$\bar{S} = (\iiint_V |\boldsymbol{\nabla} \times \boldsymbol{y}|^2 dV)^2 - 8BRe \iiint_V |\boldsymbol{y}|^2 dV (\iiint_V H_0 H_0' |\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y}|^2 dV), \tag{A.86}$$

for all real functions $\boldsymbol{y}$ satisfying the boundary conditions of $\boldsymbol{u}$. We can argue analogously to Section 4.3.1 that $F(\boldsymbol{y})$ is bounded above, and therefore $F(y)$ has a positive maximum when $H_0' < 0$. We now prove stationary points of F correspond to real eigenvalues of (6.87). Let $\boldsymbol{y} = \boldsymbol{y_0}$ be a stationary point of F, such that $F(\boldsymbol{y_0}) = F_0$ and consider $\boldsymbol{y} = \boldsymbol{y_0} + \epsilon \boldsymbol{y_1}$, where $\boldsymbol{y_0}$ and $\boldsymbol{y_1}$ are divergence free and satisfy the same boundary conditions as $\boldsymbol{u}$. Since $\boldsymbol{y} = \boldsymbol{y_0}$ is a stationary point of $F(\boldsymbol{y})$, the first variation from Taylor expanding $F(\boldsymbol{y_0} + \epsilon \boldsymbol{y_1})$ will be zero, and we obtain

$$-F_0^2 Re \iiint_V \boldsymbol{y_0} \cdot \boldsymbol{y_1} dV - F_0 \iiint_V (\boldsymbol{\nabla} \times \boldsymbol{y_0}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{y_1}) dV = 2B \iiint_V H_0 H_0' (\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_0})(\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_1}) dV. \tag{A.87}$$

We write the second term on the left-hand side of (A.87) as

$$\iiint_V (\boldsymbol{\nabla} \times \boldsymbol{y_0}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{y_1}) dV = \iiint_V \Big(\boldsymbol{\nabla} \cdot ((\boldsymbol{\nabla} \times \boldsymbol{y_1}) \times \boldsymbol{y}_0) + \boldsymbol{y_0} \cdot (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{y_1})) \Big) dV. \tag{A.88}$$

But, by the divergence theorem and boundary conditions for $\boldsymbol{y_0}$, the first term of (A.88) is zero, and therefore

$$\begin{aligned} \iiint_V (\boldsymbol{\nabla} \times \boldsymbol{y}_0) \cdot (\boldsymbol{\nabla} \times \boldsymbol{y_1}) dV &= \iiint_V \boldsymbol{y_0} \cdot (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{y_1})) dV \\ &= - \iiint_V \boldsymbol{y_0} \cdot \nabla^2 \boldsymbol{y_1} dV. \end{aligned} \tag{A.89}$$

Now,

$$\begin{aligned} \iiint_V 2 H_0 H_0' (\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_0})(\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_1}) dV &= \iiint_V (\boldsymbol{\nabla} H_0^2 \cdot \boldsymbol{y_0})(\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_1}) dV \\ &= \iiint_V \Big(\boldsymbol{\nabla} \cdot \big(H_0^2 (\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_1}) \boldsymbol{y_0} \big) \\ &\quad - H_0^2 \boldsymbol{y_0} \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \chi_0 \cdot \boldsymbol{y_1}) \Big) dV, \end{aligned} \tag{A.90}$$

and the first term is zero by the divergence theorem. Therefore, (A.87) becomes

$$\begin{aligned}
&- F_0^2 Re \iiint_V \boldsymbol{y_0} \cdot \boldsymbol{y_1} dV + F_0 \iiint_V \boldsymbol{y_0} \cdot \nabla^2 \boldsymbol{y_1} dV \\
&= -B \iiint_V H_0^2 \boldsymbol{y_0} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y_1}) dV.
\end{aligned} \tag{A.91}$$

Since we can add any integral which evaluates to zero to (A.91), we add

$$- \iiint_V \boldsymbol{\nabla} \cdot (p\boldsymbol{y_0}) dV, \tag{A.92}$$

which is zero due to the divergence theorem, to give

$$\begin{aligned}
&- F_0^2 Re \iiint_V \boldsymbol{y_0} \cdot \boldsymbol{y_1} dV + F_0 \iiint_V \boldsymbol{y_0} \cdot \nabla^2 \boldsymbol{y_1} dV \\
&= - \iiint_V \boldsymbol{\nabla} \cdot (p\boldsymbol{y_0}) dV - B \iiint_V H_0^2 \boldsymbol{y_0} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y_1}) dV.
\end{aligned} \tag{A.93}$$

Since, $\boldsymbol{\nabla} \cdot \boldsymbol{y_0} = 0$, write (A.93) as

$$\begin{aligned}
&- F_0^2 Re \iiint_V \boldsymbol{y_0} \cdot \boldsymbol{y_1} dV + F_0 \int_V \boldsymbol{y_0} \cdot \nabla^2 \boldsymbol{y_1} dV \\
&= - \int_V \boldsymbol{\hat{y}_0} \cdot \boldsymbol{\nabla} p dV - B \iiint_V H_0^2 \boldsymbol{y_0} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y_1}) dV.
\end{aligned} \tag{A.94}$$

(A.94) is valid for any vector $\boldsymbol{y_1}$ and therefore

$$F_0^2 Re \boldsymbol{y_1} - F_0 \nabla^2 \boldsymbol{y_1} = -\boldsymbol{\nabla} p + B H_0^2 \boldsymbol{\nabla}(\boldsymbol{\nabla}\chi_0 \cdot \boldsymbol{y_1}). \tag{A.95}$$

Since $F_0 = s$, this yields (6.87) and stationary points of F are eigenvalues of (6.87). Following analogous reasoning to Section 6.4, it holds that if, and only if, $H_0' < 0$ somewhere in the volume, the system is unstable.