BIRATIONAL GEOMETRY OF QUIVER VARIETIES AND OTHER GIT QUOTIENTS

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In memory of Tom Nevins

Abstract. We introduce a sufficient condition for the Geometric Invariant Theory (GIT) quotient of an affine variety V by the action of a reductive group G to be a relative Mori Dream Space. When the condition holds, we show that the linearisation map identifies a region of the GIT fan with the Mori chamber decomposition of the relative movable cone of $V/\!\!/_\theta G$. If $V/\!\!/_\theta G$ is a crepant resolution of $Y := V/\!\!/_0 G$, then every projective crepant resolution of Y is obtained by varying θ. Under suitable conditions, we show that this is the case for Nakajima quiver varieties; in particular, all projective partial crepant resolutions of the affine quiver variety Y are quiver varieties. Similarly, for any finite subgroup $\Gamma \subset \mathrm{SL}(3, \Bbbk)$ whose nontrivial conjugacy classes are all junior, we obtain a simple geometric proof of the fact that every projective crepant resolution of $\mathbb{A}^3/\Gamma$ is a fine moduli space of θ-stable Γ-constellations. Our methods apply equally well to nonsingular hypertoric varieties.

1. Introduction

Nakajima quiver varieties [42, 43] provide a rich source of examples illustrating many beautiful phenomena in algebraic geometry and geometric representation theory. To recall the construction, consider a finite graph with vertex set I, and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$. This combinatorial data determines a Hamiltonian action of the group $G := \prod_{i\in I} \mathrm{GL}(v_i)$ on a symplectic vector space $\mathbf{M}(\mathbf{v}, \mathbf{w})$, giving rise to a moment map $\mu \colon \mathbf{M}(\mathbf{v}, \mathbf{w}) \to \mathfrak{g}^*$. For any character $\theta \in G^\vee$, the Nakajima *quiver variety* is defined to be the Geometric Invariant Theory (GIT) quotient

$$\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) := \mu^{-1}(0)/\!\!/_\theta\, G. \tag{1.1}$$

Under suitable conditions on $\mathbf{v}, \mathbf{w}$, and for any sufficiently general θ, the projective morphism $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ obtained by variation of GIT quotient is a crepant resolution of singularities. It follows from the work of Birkar, Cascini, Hacon and McKernan [8, Corollary 1.3.2] that $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ is a relative Mori Dream Space (see Namikawa [47] or [2, Lemma 5.3]). Put simply, the birational geometry of $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ is especially well-behaved. It is therefore natural to ask for a concrete description of the relative movable cone and the set of all projective crepant resolutions of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$.

Here, we answer these questions in full by proving directly that quiver varieties are relative Mori Dream Spaces, and in doing so, we establish that every projective crepant resolution of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ is itself a quiver variety. Our proof does not use results from [8], nor does it apply the relative version of the sufficient condition to be a Mori Dream Space given by Hu and Keel [28, Theorem 2.3], because that condition does not apply even to the simplest quiver variety, namely, the minimal resolution of the A_1 surface singularity.

1.1. **The main result for GIT quotients.** In fact, our approach is much more general. Consider the action of a reductive group G on an affine variety V. We do not assume that V is normal. The vector space $G^\vee_{\mathbb{Q}} = G^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of rational characters decomposes into a polyhedral wall-and-chamber structure called the GIT fan. The set of generic stability parameters θ in $G^\vee_{\mathbb{Q}}$ decomposes as the union of finitely many GIT chambers, each of which is the interior of a top-dimensional cone in the GIT fan. Our main result introduces a new sufficient condition guaranteeing that, for any chamber C and any $\theta \in C$, the GIT quotient $X_\theta := V /\!\!/_\theta G$ is a relative Mori Dream Space over the affine quotient $Y := V /\!\!/_0 G$; in fact, we describe a region of the GIT fan that captures completely the birational geometry of X_θ over Y.

Before stating our main result for GIT quotients, we describe our sufficient condition in general terms (see Condition 3.4 for details). Recall that each character $\zeta \in G^\vee$ determines a G-linearisation of the trivial bundle on V that descends to a line bundle L_ζ on X_θ for generic $\theta \in G^\vee_{\mathbb{Q}}$. Let C be the chamber containing θ. The *linearisation map* for C is the map of rational vector spaces

$$L_C \colon G^\vee_{\mathbb{Q}} \longrightarrow \operatorname{Pic}(X_\theta/Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

defined by $L_C(\zeta) = L_\zeta$. One of the sufficient conditions from [28, Lemma 2.2(4)] requires that L_C is an isomorphism, and we impose this as part of the first criterion in our sufficient condition.

The novel aspect in our Condition 3.4 is that the second and third criteria are phrased in terms of wall-crossing. For each GIT chamber C, we define a closed cone R_C in $G^\vee_{\mathbb{Q}}$ to be the union of the closures of a collection of chambers (see Definition 3.1), and our second and third criteria guarantee that (i) variation of GIT quotient across each wall in the interior of R_C induces a flip $V /\!\!/_{\theta_-} G \dashrightarrow V /\!\!/_{\theta_+} G$; and that (ii) the morphisms induced by variation of GIT quotient into each boundary wall from the interior of R_C contract a divisor. We provide examples to show that even when the linearisation map L_C is an isomorphism, it can happen that an interior GIT wall of R_C does not induce a flip (see Example 3.7), and moreover, that even when L_C is an isomorphism and all interior walls induce flips, it can happen that a boundary wall induces a morphism that does not contract a divisor (see Example 3.3). Thus, all three criteria from Condition 3.4 must be imposed to obtain the geometric behaviour that we seek.

The importance of our sufficient condition is illustrated by our main result for GIT quotients that can be stated as follows (see Theorem 3.12 and Corollaries 3.14-3.15):

Theorem 1.1. *For the action of a reductive group G on an affine variety V, suppose that a GIT chamber C in $G^\vee_{\mathbb{Q}}$ satisfies Condition 3.4. For $\theta \in C$, write $X_\theta := V /\!\!/_\theta G$. Then:*

(i) *the linearisation map is an isomorphism that identifies the GIT wall-and-chamber structure in R_C with the decomposition of the movable cone $\operatorname{Mov}(X_\theta/Y)$ into Mori chambers;*
(ii) *for every $\zeta \in R_C$, the GIT quotient $V /\!\!/_\zeta G$ is the birational model of X_θ determined by the line bundle $L_C(\zeta)$; and*
(iii) *the GIT quotient X_θ is a Mori Dream Space over Y.*

In particular, every small, $\mathbb{Q}$-factorial, birational model of X_θ is a GIT quotient of the form $V /\!\!/_\zeta G$ for some $\zeta \in R_C$ lying in a GIT chamber.

Thus, when Condition 3.4 applies, Theorem 1.1 shows that the birational geometry of X_θ over Y is determined completely by variation of GIT quotient within the cone R_C. In fact, Theorem 1.1(iii) implies that Condition 3.4 is a new sufficient condition for a GIT problem to define a relative Mori Dream Space. Our approach does not use in any way the deep geometric results in the minimal model programme from [8], relying instead only on elementary arguments from GIT.

The work of Hu and Keel [28] shows that for any Mori Dream Space X, Theorem 1.1 applies for the action of an algebraic torus on $\operatorname{Spec}\operatorname{Cox}(X)$. However, we are particularly interested in examples where the reductive group G need not be an algebraic torus, and where the affine variety V is not the spectrum of $\operatorname{Cox}(X)$. In short, the Cox ring does not have a monopoly on finitely generated $\Bbbk$-algebras that encode perfectly the birational geometry of a Mori Dream Space.

1.2. **Application to Nakajima quiver varieties.** While our Condition 3.4 is strong enough to establish Theorem 1.1, it is also weak enough to apply in a number of interesting situations. The case of primary interest to us is the group action that defines a quiver variety.

As above, for any graph with vertex set I, choose dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$ with $\mathbf{w} \neq 0$ and $v_i \neq 0$ for all $i \in I$. For any $\theta \in G^\vee$, the Nakajima quiver variety $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ is the GIT quotient from (1.1). We assume throughout that there exists a simple representation of the doubled quiver in $\mu^{-1}(0)$, or equivalently, the vector $\alpha := (1, \mathbf{v}) \in \mathbb{N} \times \mathbb{N}^I$ satisfies Crawley-Boevey's condition $\alpha \in \Sigma_0$ (see Definition 4.4). It follows that the zero fibre of the moment map $V := \mu^{-1}(0)$ is an affine variety [18, Theorem 1.2], and moreover, if θ is generic, then the projective morphism $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ to the affine GIT quotient is a crepant resolution of singularities.

For any GIT chamber C and for $\theta \in C$, the quiver variety $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ is nonsingular and hence the linearisation map L_C is surjective by the work of McGerty and Nevins [41, Theorem 1.2]. The assumption $\alpha \in \Sigma_0$ implies that $\mathfrak{M}_{\theta_0}(\mathbf{v}, \mathbf{w})$ is singular for any non-generic $\theta_0 \in G^\vee_{\mathbb{Q}}$, so the morphism

$$\tau\colon \mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) \longrightarrow \mathfrak{M}_{\theta_0}(\mathbf{v}, \mathbf{w}) \tag{1.2}$$

obtained by varying θ into the boundary of the chamber C necessarily contracts at least one curve. This allows us to prove that L_C is actually an isomorphism; we provide examples to show that L_C need not be injective when $\alpha \notin \Sigma_0$ (see Remark 4.3). The second and third criteria in our Condition 3.4 are phrased in terms of wall crossing for quiver varieties, and for these, we control the dimension of the unstable locus of the morphism τ from (1.2) by analysing the singular locus of $\mathfrak{M}_{\theta_0}(\mathbf{v}, \mathbf{w})$ for non-generic $\theta_0 \in G^\vee_{\mathbb{Q}}$. We distinguish flipping and divisorial contractions using the fact that τ is semi-small, a result due to Kaledin [32].

This leads to our main result for quiver varieties (see Theorem 4.6 and Proposition 4.11):

Theorem 1.2. *Under the above assumptions, the following hold:*

(i) *every GIT chamber C satisfies Condition 3.4, so Theorem 1.1 holds for the Nakajima quiver variety $X_\theta := \mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ with $\theta \in C$; and*
(ii) *for any chamber C, the GIT region R_C is a simplicial cone that provides a fundamental domain for the action of the Namikawa–Weyl group on $G^\vee_{\mathbb{Q}}$.*

Thus, projective crepant resolutions $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w})$, taken up to isomorphism over $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ are in bijection with GIT chambers in $G^\vee_\mathbb{Q}$ modulo the action of the Namikawa–Weyl group.

This theorem provides a broad generalisation of the geometric interpretation by Kronheimer [37] of the McKay correspondence, in which, for any finite subgroup $\Gamma \subset \mathrm{SL}(2,\Bbbk)$, the minimal resolution of the Kleinian singularity $\mathfrak{M}_0(\mathbf{v},\mathbf{w}) \cong \mathbb{A}^2/\Gamma$ is constructed by variation of GIT (or hyperkähler) quotient as a quiver variety $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ for generic θ, and moreover, any Weyl chamber of finite type ADE can be identified with the ample cone of the minimal resolution.

Theorem 1.2 provides a direct, geometric proof of the fact that every quiver variety $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is a relative Mori Dream Space over $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$. In fact, we go further for quiver varieties by describing explicitly the hyperplane arrangement that determines the GIT chamber decomposition appearing in Theorem 1.2 (see Theorem 4.17).

Corollary 1.3. *Under the above assumptions, every projective crepant resolution of the affine quotient $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ is itself a quiver variety $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ for some generic $\theta \in G^\vee$*

This result implies that every relative minimal model of a quiver variety is itself a quiver variety. This generalises and unifies the results from Yamagishi [56, Section 5], and a pair of results of the authors [1, Theorem 1.2], [2, Theorem 1.2].

1.3. **Application to threefold quotient singularities.** The holomorphic symplectic nature of Nakajima quiver varieties plays a role in our proof of Theorem 1.2, since we use the fact that the VGIT morphism τ from (1.2) is semi-small. However, to emphasise that this is not an essential feature when applying Theorem 1.1, we also study a class of examples in odd dimension.

Consider any threefold quotient singularity of the form $\mathbb{A}^3/\Gamma$, where $\Gamma \subset \mathrm{SL}(3,\Bbbk)$ is a finite subgroup for which every nontrivial conjugacy class is *junior* in the sense of Ito and Reid [30]. This condition is equivalent to requiring that any projective crepant resolution $f\colon X \to Y := \mathbb{A}^3/\Gamma$ has all fibres of dimension at most one. One such resolution is provided by $X := \Gamma\text{-Hilb}(\mathbb{A}^3)$, the fine moduli space of Γ-clusters in $\mathbb{A}^3$, as in [11], for which there is a natural GIT quotient construction $X \cong X_\theta = V /\!\!/_\theta\, G$ for some generic θ. The fact that X contains no proper surfaces allows us to show that Condition 3.4 holds in this setting, so the conclusions of Theorem 1.1 hold for the given GIT quotient description of X_θ (see Theorem 5.9). Thus, we obtain:

Theorem 1.4. *Let $\Gamma \subset \mathrm{SL}(3,\Bbbk)$ be a finite subgroup such that every non-trivial conjugacy class of Γ is junior. Then every projective crepant resolution of $\mathbb{A}^3/\Gamma$ is a fine moduli space $\mathcal{M}_\theta$ of θ-stable Γ-constellations for some generic $\theta \in \Theta$.*

Our direct and simple geometric proof of this result bypasses the algebraic approach via mutation from Nolla de Celis and Sekiya [49, Corollaries 1.3 and 1.5] that was pioneered by Wemyss and later generalised in his beautiful paper [55, Theorem 6.2]. Our Theorem 1.1(i) also provides a direct GIT description of the relative movable cone $\mathrm{Mov}(X/Y)$ in this setting.

Very recently, Yamagishi [57, Theorem 1.1] announced that the conclusion of Theorem 1.4 holds for any finite subgroup Γ of $\mathrm{SL}(3,\Bbbk)$. While the scope of our Theorem 1.4 is much more limited, our approach is elementary: we show that there are no GIT walls of 'type 0', and also, we do not require the deep results from [8]. Put simply, those Γ for which every nontrivial conjugacy class is junior provide an especially simple family of examples that is amenable to our geometric approach.

1.4. **Birational geometry of smooth Hamiltonian reductions.** The statement of Theorem 1.2 realises the vision set out by Braden, Proudfoot and Webster [10, Remarks 2.20-2.21], at least in the case of quiver varieties. In the more general setting of representations of reductive group actions, the expectation of *op. cit.* (namely, that Theorem 1.1(i) holds) would be a consequence of the following conjecture:

Conjecture 1.5. *Suppose that U is a finite-dimensional representation of a reductive group G and $V = \mu^{-1}(0) \subseteq T^*U$ is the zero fibre of the moment map $T^*U \to \mathfrak{g}^* = (\operatorname{Lie} G)^*$. If, for some GIT chamber C, the morphism $X_\theta \to X_0$ is a crepant resolution for $\theta \in C$, and L_C is an isomorphism, then Condition 3.4 is satisfied.*

In other words, in the setting of smooth Hamiltonian reductions of vector spaces by reductive groups, we conjecture that the first criterion from Condition 3.4 implies that the second and third criteria from Condition 3.4 hold. Thus, when L_C is an isomorphism, we anticipate that Theorem 1.1 applies. As explained above, such an implication does not hold for general GIT quotients.

As further evidence for this conjecture, we consider nonsingular hypertoric varieties. Here, as defined in Section 6, a hypertoric variety means a Hamiltonian reduction of a vector space by an algebraic torus. In this case, the verification of our Condition 3.4 for the standard GIT construction of a nonsingular hypertoric variety X was largely carried out by Konno [36, Theorem 6.4], though we also use the tilting bundle on X constructed Špenko and Van den Bergh [54] (see also [40]) to deduce that the linearisation map is surjective. Our main result for nonsingular hypertoric varieties, given in Theorem 6.1, establishes the following result.

Theorem 1.6. *Conjecture 1.5 holds whenever G is an algebraic torus.*

In this context, Theorem 1.1 implies in particular that every projective crepant resolution of a hypertoric cone is itself a hypertoric variety.

Notation. Let $\Bbbk$ be an algebraic closed field of characteristic zero. Throughout the paper, a variety is an integral separated scheme of finite type over $\Bbbk$.

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2. Background

2.1. **Birational geometry.** Consider a projective morphism $f\colon X \to Y$ of normal varieties over $\Bbbk$, where Y is affine. The relative Picard group is $\operatorname{Pic}(X/Y) := \operatorname{Pic}(X)/f^*\operatorname{Pic}(Y)$, and we set $\operatorname{Pic}(X/Y)_{\mathbb{Q}} := \operatorname{Pic}(X/Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. A line bundle $L \in \operatorname{Pic}(X/Y)_{\mathbb{Q}}$ is *nef* (over Y) if $\deg L|_{\ell} \geq 0$ for every proper curve ℓ in X, and it is *semiample* (over Y) if L^m is basepoint-free for some $m \geq 1$. The stable base locus of L is defined to be the intersection of the base loci of the linear series $|L^m|$ for all $m \geq 1$, and we say that L is *movable* if its stable base locus is of codimension at least two in X. Every semiample line bundle is nef, but the converse is not true in general.

The *nef cone* of X over Y is the closed convex cone $\operatorname{Nef}(X/Y)$ in $\operatorname{Pic}(X/Y)_{\mathbb{Q}}$ generated by line bundles on X that are nef over Y. The relative version of Kleiman's ampleness criterion [34, IV, §4] implies that the relative *ample cone* $\operatorname{Amp}(X/Y)$ is the interior of $\operatorname{Nef}(X/Y)$. The *movable cone* $\operatorname{Mov}(X/Y)$ is the closed convex cone in $\operatorname{Pic}(X/Y)_{\mathbb{Q}}$ obtained as the closure of the cone generated by all movable divisor classes. Note that the nef cone is contained in the movable cone.

Let $\tau\colon X \to X_0$ be a projective morphism over Y. After replacing τ by its Stein factorisation if necessary, we may assume that τ is surjective, X_0 is normal, and $\tau_*(\mathcal{O}_X) = \mathcal{O}_{X_0}$. We say that τ is of *fibre type* if $\dim X_0 < \dim X$. Otherwise, τ is birational, and there are two cases: either the exceptional locus of τ, denoted $\operatorname{Exc}(\tau)$, contains a divisor, in which case τ is a *divisorial contraction*; or it does not, in which case τ is a *small contraction*. In the latter case, let L be a line bundle on X such that L^{-1} is τ-ample. The *flip of τ with respect to L* is a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\tau'} & \\ & X_0 & \end{array} \tag{2.1}$$

where τ' is a small contraction, ψ is an isomorphism in codimension-one, and the strict transform of L along ψ is τ'-ample. If, in addition, the canonical class K_X satisfies $K_X \cdot \ell = 0$ for each curve ℓ contracted by τ, then (2.1) is the *flop* of the curve class ℓ [35, Definition 6.10].

Let $L \in \operatorname{Pic}(X/Y)_{\mathbb{Q}}$ be such that the section ring

$$R(X, L) := \bigoplus_{m \geq 0} f_* L^m$$

is a finitely generated $\mathcal{O}_Y$-algebra. Then $X(L) := \operatorname{Proj}_Y R(X, L)$ fits into a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi_L}{\dashrightarrow} & X(L) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f_L} & \\ & Y & \end{array} \tag{2.2}$$

where ψ_L is regular on the complement of the stable base locus of L in X. We do not assume in general that f is birational, nor do we assume that L is *big*, i.e. ψ_L need not be birational either. However, if L is movable, then the rational map ψ_L is an isomorphism in codimension-one, in which case we call $X(L)$ a *small birational model* of X over Y. In this case, we identify $\operatorname{Pic}(X(L)/Y)$

with $\operatorname{Pic}(X/Y)$ by taking the strict transform along the birational map ψ_L; this in turn identifies $\operatorname{Mov}(X(L)/Y)$ with $\operatorname{Mov}(X/Y)$. Let $\psi_L^*\operatorname{Amp}(X(L)/Y)$ and $\psi_L^*\operatorname{Nef}(X(L)/Y)$ denote the cones in $\operatorname{Pic}(X/Y)_{\mathbb{Q}}$ obtained by taking the strict transform along ψ_L of all classes on $X(L)$ that are relatively ample and nef respectively.

Given $L, L' \in \operatorname{Pic}(X/Y)_{\mathbb{Q}}$ with finitely generated section rings, we say that L is *Mori equivalent* to L' if there is an isomorphism $\varphi\colon X(L) \to X(L')$ such that the rational maps $\psi_L, \psi_{L'}$ satisfy $\varphi\circ\psi_L = \psi_{L'}$. A *Mori chamber* is a Mori equivalence class whose interior is open in $\operatorname{Pic}(X/Y)_{\mathbb{Q}}$. These chambers are typically studied under the additional assumption that $\operatorname{Pic}(X/Y)_{\mathbb{Q}}$ is isomorphic to the Néron–Severi space $N^1(X/Y) := \operatorname{Pic}(X/Y)_{\mathbb{Q}}/\equiv$ of *numerical equivalence* classes, where $L \equiv L'$ if and only if $\deg(L|_\ell) = \deg(L'|_\ell)$ for every proper curve ℓ in X.

To see how the isomorphism $\operatorname{Pic}(X/Y)_{\mathbb{Q}} \cong N^1(X/Y)$ arises in the case of interest to us, recall first the following fundamental and well-known result.

Proposition 2.1. *If $L \in \operatorname{Pic}(X/Y)$ is semi-ample over Y, then it is nef over Y. Moreover, the section ring $R(X, L)$ is a finitely generated $\mathcal{O}_Y$-algebra, and the morphism from X to $\operatorname{Proj}_Y R(X, L)$ determined by any power of L contracts a proper curve ℓ in X if and only if $L \cdot \ell = 0$.*

Proof. Suppose that $L^m \in \operatorname{Pic}(X/Y)$ is a basepoint-free line bundle over Y. The induced morphism $h\colon X \to |L^m| \cong \mathbb{P}^N_Y$ satisfies $h^*(\mathcal{O}(1)) \cong L^m$. For a proper curve ℓ in X, we have

$$L \cdot \ell := \frac{1}{m}\deg\left(h^*(\mathcal{O}(1))|_\ell\right) = \frac{1}{m}\deg\left(\mathcal{O}(1)|_{h_*[\ell]}\right)$$

where $h_*[\ell]$ is the pushforward of the curve class of ℓ. Thus, L is nef over Y, and ℓ is contracted by h if and only if $L \cdot \ell = 0$. Finite generation of $R(X, L)$ is the relative version of a theorem of Zariski (see [50, Lemma 6.11]), and the image of h is $\operatorname{Proj}_Y R(X, L)$. □

Corollary 2.2. *If each $L \in \operatorname{Pic}(X/Y)$ that is nef over Y is actually semiample over Y, then there is an isomorphism $\operatorname{Pic}(X/Y)_{\mathbb{Q}} \cong N^1(X/Y)$.*

Proof. The quotient map $\operatorname{Pic}(X/Y)_{\mathbb{Q}} \to N^1(X/Y)$ is injective (see [50, Proposition 3.2]). □

2.2. **GIT quotients and the linearisation map.** Let G be a reductive algebraic group acting on an affine variety V with coordinate ring $\Bbbk[V]$. Let $G^\vee$ denote the character group of G. For $\theta \in G^\vee$, we say that $f \in \Bbbk[V]$ is *θ-semi-invariant* if $f(g.v) = \theta(g)f(v)$ for all $v \in V$ and $g \in G$, and we write $\Bbbk[V]_\theta$ for the space of θ-semi-invariant functions. A point $v \in V$ is *θ-semistable* if there exists $j > 0$ and $f \in \Bbbk[V]_{j\theta}$ such that $f(v) \neq 0$. The *θ-semistable locus* $V^\theta \subseteq V$ is the G-invariant, open subset of θ-semistable points. A point $v \in V^\theta$ is *θ-stable* if the stabiliser G_v is finite and the orbit $G \cdot v$ is closed in V^θ. A character $\theta \in G^\vee$ is *effective* if V^θ is non-empty, and an effective character θ is *generic* if every θ-semistable point of V is θ-stable. The θ-semistable locus is unchanged if we replace θ by a positive multiple, so the definitions extend to any fractional character $\theta \in G^\vee_{\mathbb{Q}} := G^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$.

For any effective $\theta \in G^\vee_{\mathbb{Q}}$, the GIT quotient

$$X_\theta := V /\!\!/_\theta\, G := \operatorname{Proj}\Big(\bigoplus_{j\geq 0} \Bbbk[V]_{j\theta}\Big)$$

is the categorical quotient of the θ-semistable locus V^θ by the action of G. Note that X_θ is projective over the affine quotient

$$Y := V /\!\!/_0\, G = \operatorname{Spec} \Bbbk[V]^G.$$

If θ is generic, then X_θ is the geometric quotient of V^θ by G.

The set of effective fractional characters is a closed, convex cone in $G^\vee_{\mathbb{Q}}$ that admits a wall-and-chamber structure as follows. Fractional characters $\theta, \theta' \in G^\vee_{\mathbb{Q}}$ are *GIT-equivalent* if $V^\theta = V^{\theta'}$. The GIT-equivalence classes form the relative interiors of a finite collection of rational polyhedral cones in $G^\vee_{\mathbb{Q}}$, and the collection of all such cones, called *GIT cones*, forms a fan, called the *GIT fan*, whose support is the convex cone of effective fractional characters in $G^\vee_{\mathbb{Q}}$. The set of generic stability parameters θ in $G^\vee_{\mathbb{Q}}$ decomposes as the union of (GIT) *chambers*, each of which is the interior of a top-dimensional cone in the GIT fan. As shown by Ressayre [51], it can happen that the interior of a top-dimensional GIT cone is not a chamber. However, in this paper we work only with stability parameters θ lying in the closure of the union of all GIT chambers, and we reserve the phrase *GIT wall* for any codimension-one face of the closure $\overline{C}$ of some GIT chamber C. The characterisation of the GIT fan via GIT-equivalence was established by Ressayre [52] (see Halic [24] for affine V), building on the earlier work of Dolgachev and Hu [19], and Thaddeus [53]. Those papers assume that V is normal, but in fact, GIT-equivalence is unaffected by passing to the normalisation of V; explicitly, if $\nu\colon \widetilde{V} \to V$ is the normalisation, then $\widetilde{V}^\theta = \nu^{-1}(V^\theta)$ for any $\theta \in G^\vee_{\mathbb{Q}}$.

Let C be a GIT chamber and fix $\theta \in C$, so θ is generic. For $\chi \in G^\vee$, consider the G-equivariant line bundle $\chi \otimes \mathcal{O}_{V^\theta}$ on the θ-stable locus in V given by equipping the trivial line bundle with the action of G on each fibre given by χ; explicitly, the action of G on V^θ lifts to the action on $V^\theta \times \mathbb{A}^1$ such that the dual action on functions is $g \cdot (f, t) = (g \cdot f, \chi^{-1}(g)t)$. It follows that the space of sections is isomorphic to the space $\Bbbk[V^\theta]_\chi$ of χ-semi-invariant functions on V^θ. By descent [48], $\chi \otimes \mathcal{O}_{V^\theta}$ descends to a line bundle on X_θ if the stabiliser of each $x \in V^\theta$ is in the kernel of χ. Since all stabilisers are finite, and there are only finitely many conjugacy classes of such stabilisers by [39, Corollaire 3], there is some multiple $j\chi$ of χ that descends. We define $L_\chi := \frac{1}{j} L_{j\chi} \in \operatorname{Pic}(X_\theta/Y)_{\mathbb{Q}}$.

Lemma 2.3. *Let $\theta \in C$. Then $H^0(X_\theta, L_\chi) \cong H^0(V^\theta, \chi \otimes \mathcal{O}_{V^\theta})$ for all $\chi \in G^\vee$.*

Proof. The line bundle L_χ descends from $\chi \otimes \mathcal{O}_{V^\theta}$, so $L_\chi \cong \pi_*(\pi^*(L_\chi))^G \cong \pi_*(\chi \otimes \mathcal{O}_{V^\theta})^G$ and hence $H^0(X_\theta, L_\chi) \cong \operatorname{Hom}(\mathcal{O}_{X_\theta}, \pi_*(\chi \otimes \mathcal{O}_{V^\theta})^G)$. Adjunction for $\pi_*(-)^G$ and $\pi^*(-)$ [48, Lemma 2.5] implies that this is isomorphic to $\operatorname{Hom}_G(\pi^*(\mathcal{O}_{X_\theta}), \chi \otimes \mathcal{O}_{V^\theta}) \cong H^0(V^\theta, \chi \otimes \mathcal{O}_{V^\theta})$. □

Definition 2.4. Let C be a GIT chamber. For $\theta \in C$ and $X_\theta = V /\!\!/_\theta\, G$, the *linearisation map* for C is the $\mathbb{Q}$-linear map

$$L_C\colon G^\vee_{\mathbb{Q}} \longrightarrow \operatorname{Pic}(X_\theta/Y)_{\mathbb{Q}} \tag{2.3}$$

determined by setting $L_C(\chi) := L_\chi$ for all $\chi \in G^\vee$.

2.3. **Variation of GIT quotient.** Let C be a GIT chamber and let $\theta \in C$. In addition, let θ_0 be a general point in any face of the closure $\overline{C}$. The G-equivariant inclusion of the θ-stable locus into the θ_0-semistable locus of V fits into a commutative diagram of varieties

$$\begin{array}{ccc} V^{\theta} & \hookrightarrow & V^{\theta_0} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\ X_\theta & \xrightarrow{\tau} & X_{\theta_0} \end{array} \tag{2.4}$$

where π_0 is a good categorical quotient, π is a geometric quotient and τ is a projective morphism; the morphism τ is said to be induced by *variation of GIT quotient* (VGIT). The G-equivariant line bundle $\theta_0 \otimes \mathcal{O}$ on V^{θ_0} descends to the polarising ample bundle $\mathcal{O}(1)$ on X_{θ_0}, and its restriction to V^θ descends to the line bundle $L_C(\theta_0)$ on X_θ. Commutativity of diagram (2.4) gives

$$L_C(\theta_0) = \tau^*\big(\mathcal{O}(1)\big). \tag{2.5}$$

Let C_- and C_+ be adjacent GIT chambers separated by a wall. Let $\theta_- \in C_-$, $\theta_+ \in C_+$ and let θ_0 be a general point in the wall $\overline{C_-} \cap \overline{C_+}$. The morphisms $\tau_- \colon X_{\theta_-} \to X_{\theta_0}$ and $\tau_+ \colon X_{\theta_+} \to X_{\theta_0}$ obtained by VGIT as in (2.4) fit into a commutative diagram

$$\begin{array}{ccccc} X_{\theta_-} & & \overset{\psi}{\dashrightarrow} & & X_{\theta_+} \\ & {\scriptstyle \tau_-}\searrow & & \swarrow{\scriptstyle \tau_+} & \\ & & X_{\theta_0} & & \end{array} \tag{2.6}$$

of varieties over $Y = V /\!\!/_0\, G$. Replacing X_{θ_0} by its normalisation and $\tau_\pm$ by their Stein factorisations allows us to assume that both $\tau_\pm$ have connected fibres.

Lemma 2.5. (i) *The θ_0-stable locus in V is the intersection $V^{\theta_0\text{-st}} := V^{\theta_+} \cap V^{\theta_-}$.*
(ii) *Each map in diagram (2.6) is an isomorphism over the subset $V^{\theta_0\text{-st}}/G \subseteq X_{\theta_0}$.*
(iii) *The subset $\tau_-^{-1}\big(\pi_0(V^{\theta_0} \setminus V^{\theta_0\text{-st}})\big)$ is Zariski-closed in X_{θ_-} (and the same with $+$ for $-$).*

Proof. Part (i) follows by combining two results from Thaddeus [53, Proposition 1.3, Lemma 3.2]; alternatively, the proof by Dolgachev–Hu [19, Proposition 3.4.7, Lemma 4.1.5] can be applied under our assumptions on V. Part (ii) follows from the description of the open set $V^{\theta_0\text{-st}}$ given in part (i). For part (iii), the semistable locus for any character of G is open and G-invariant in V, so $V^{\theta_0} \setminus V^{\theta_+}$ is a closed and G-invariant subset of V^{θ_0}. The image $\pi_0(V^{\theta_0} \setminus V^{\theta_+})$ is closed because π_0 is a good quotient, so the statement follows from continuity of τ_+ in the Zariski topology. □

Definition 2.6. The *unstable locus* for τ_- is the subset in X_{θ_-} parametrising strictly θ_0-semistable points, $\operatorname{Uns}(\tau_-) := \tau_-^{-1}\big(\pi_0(V^{\theta_0} \setminus V^{\theta_0\text{-st}})\big)$. The locus $\operatorname{Uns}(\tau_+)$ is the same with $+$ replacing $-$.

Remarks 2.7. (1) Lemma 2.5 implies that the exceptional locus of τ_-, denoted $\operatorname{Exc}(\tau_-)$, is a subset of the unstable locus $\operatorname{Uns}(\tau_-)$. This inclusion may be strict.
(2) The unstable locus is closed by Lemma 2.5(iii). We give it the reduced scheme structure.

The next result generalises a result of Thaddeus [53, Theorem 3.3], though the necessary assumption on the dimension of the unstable locus is missing from that statement.

Proposition 2.8. *Consider the set-up from diagram (2.6). Assume that X_{θ_+} and X_{θ_-} are normal, that $\mathrm{Uns}(\tau_+) \subseteq X_{\theta_+}$ and $\mathrm{Uns}(\tau_-) \subseteq X_{\theta_-}$ have codimension at least two, and that τ_+ and τ_- contract at least one curve. Then (2.6) is a flip with respect to the line bundle $L_{C_-}(\theta_+)$ on X_{θ_-}, and*

$$L_{C_-}(\eta) \cong \psi^*\big(L_{C_+}(\eta)\big) \text{ for all } \eta \in G^\vee_{\mathbb{Q}}. \tag{2.7}$$

If, in addition, X_{θ_-} is $\mathbb{Q}$-factorial and L_{C_-} is surjective, then X_{θ_+} is $\mathbb{Q}$-factorial.

Proof. The exceptional loci $\mathrm{Exc}(\tau_+)$ and $\mathrm{Exc}(\tau_-)$ are contained in the unstable loci $\mathrm{Uns}(\tau_+)$ and $\mathrm{Uns}(\tau_-)$ respectively by Remark 2.7(2), so our codimension assumption shows that both τ_+ and τ_- are small contractions. It follows that ψ is an isomorphism in codimension-one, so the pullback $\psi^*\colon \mathrm{Pic}(X_{\theta_+}/Y) \to \mathrm{Pic}(X_{\theta_-}/Y)$ is an isomorphism.

Combining Lemma 2.5(ii) with our assumption on the unstable locus implies that the complement of $V^{\theta_0-\mathrm{st}}/G$ is of codimension at least two in both X_{θ_+} and X_{θ_-}. Since X_{θ_+} and X_{θ_-} are normal, line bundles on both X_{θ_+} and X_{θ_-} are uniquely determined, up to isomorphism, by their restriction to $V^{\theta_0-\mathrm{st}}/G$. By restricting all three maps from (2.6) to the isomorphisms over this locus, we see that both $L_{C_+}(\eta)$ on X_{θ_+} and $L_{C_-}(\eta)$ on X_{θ_-} are obtained by descent from $\eta \otimes \mathcal{O}$ on $V^{\theta_0-\mathrm{st}}$. Therefore isomorphism (2.7) holds. In particular, the strict transform of $L_{C_-}(\theta_+)$ along ψ is the line bundle $L_{C_+}(\theta_+)$ on X_{θ_+}.

The polarising ample bundle $L_{C_+}(\theta_+)$ on X_{θ_+} is τ_+-ample, so to prove that (2.6) is a flip, we need only show that $L_{C_-}(\theta_+)^{-1}$ is τ_--ample. For this, the ample bundle $L_0 := \mathcal{O}(1)$ on X_{θ_0} satisfies $L_{C_-}(\theta_0) = \tau_-^*(L_0)$ by (2.5). By choosing alternative characters $\theta_+ \in C_+$ and $\theta_- \in C_-$ if necessary, we may assume that $\theta_0 = \frac{1}{2}(\theta_+ + \theta_-)$, in which case

$$L_{C_-}(\theta_+) \otimes L_{C_-}(\theta_-) = L_{C_-}(\theta_+ + \theta_-) = L_{C_-}(2\theta_0) = \tau_-^*(L_0)^2.$$

The set of curve classes contracted by τ_- is non-empty by assumption. The line bundles $L_{C_-}(\theta_-)$ and $\tau_-^*(L_0)$ have positive and zero degree respectively on all such curves. It follows that

$$L_{C_-}(\theta_+)^{-1} = L_{C_-}(\theta_-) \otimes \tau_-^*(L_0)^{-2}$$

has positive degree on all such curves, so it is τ_--ample as required.

For the final statement, let D_+ be a Weil divisor on X_{θ_+}. Then $(\psi^{-1})_*D_+$ is a Weil divisor on X_{θ_-}, and since X_{θ_-} is $\mathbb{Q}$-factorial, the divisor $m(\psi^{-1})_*D_+$ is Cartier for some $m > 0$. Since L_{C_-} is surjective, there exists $\eta \in G^\vee \otimes \mathbb{Q}$ such that $L_{C_-}(\eta) \cong \mathcal{O}_{X_{\theta_-}}(m(\psi^{-1})_*D_+)$. Now (2.7) gives

$$\mathcal{O}_{X_{\theta_+}}(mD_+) \cong (\psi^{-1})^*\mathcal{O}_{X_{\theta_-}}(m(\psi^{-1})_*D_+) \cong (\psi^{-1})^*L_{C_-}(\eta) \cong L_{C_+}(\eta)$$

which lies in $\mathrm{Pic}(X_{\theta_+}/Y)$, so mD_+ is Cartier. □

3. Reconstructing relative Mori Dream Spaces by GIT

3.1. **GIT regions.** As before, let G denote a reductive algebraic group acting on an affine variety V. For $\theta \in G^\vee_{\mathbb{Q}}$, write $X_\theta := V /\!\!/_\theta G$, and let $f\colon X_\theta \to Y := X_0$ denote the projective morphism obtained by VGIT. If necessary, replace f by its Stein factorisation, in which case Y is replaced by its normalisation.

To formulate our key condition, let C_- and C_+ be GIT chambers separated by a wall $\overline{C_-} \cap \overline{C_+}$. We delete this separating wall if and only if the morphisms τ_- and τ_+ from diagram (2.6) are both small. The result is an a priori coarser wall-and-chamber decomposition of the support of the GIT fan.

Definition 3.1. A *GIT region* in $G^\vee_\mathbb{Q}$ is any top-dimensional cone of the coarse fan defined above. By construction, every GIT region that contains a chamber is the union of the closures of a collection of GIT chambers. For any chamber C, let R_C denote the unique GIT region containing C.

Example 3.2. The Cox construction of the first Hirzebruch surface $\mathbb{F}_1 := \mathbb{P}_\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ passes via the action of the torus $G = (\Bbbk^\times)^2$ on $\mathbb{A}^4$ with weights $\left[\begin{smallmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right]$. There are two GIT chambers

$$C_- := \operatorname{Amp}(\mathbb{F}_1) = \left\{\alpha\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \beta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \mid \alpha, \beta > 0\right\} \quad \text{and} \quad C_+ := \left\{\alpha\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right] + \beta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \mid \alpha, \beta > 0\right\}$$

For $\theta_- \in C_-$ and $\theta_+ \in C_+$, we have $X_{\theta_-} \cong \mathbb{F}_1$ and $X_{\theta_+} \cong \mathbb{P}^2$. The VGIT morphism $\tau_- \colon \mathbb{F}_1 \to \mathbb{P}^2$ contracts the (-1)-curve, whereas τ_+ is an isomorphism. Thus, $R_{C_-} = \overline{C_-}$ and $R_{C_+} = \overline{C_+}$.

The linearisation map L_{C_-} in Example 3.2 is an isomorphism that identifies R_{C_-} with the movable (in fact, the nef) cone of X_{θ_-} for $\theta_- \in C_-$. The next example illustrates that even when L_{C_-} is an isomorphism, it need not identify R_{C_-} with $\operatorname{Mov}(X_{\theta_-})$ for $\theta \in C_-$.

Example 3.3. Consider the action of the algebraic torus $(\Bbbk^\times)^2$ on $\mathbb{A}^5$ with weights $\left[\begin{smallmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 \end{smallmatrix}\right]$. For the character $L_- := \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, the Cox construction gives $X_- = \mathbb{A}^5 /\!\!/_{L_-} (\Bbbk^\times)^2$, where we identify the character lattice of $(\Bbbk^\times)^2$ with $\operatorname{Pic}(X_-)$. For $L_+ := \left[\begin{smallmatrix}2\\-1\end{smallmatrix}\right]$, variation of GIT quotient determines a flop $\psi \colon X_- \dashrightarrow X_+ := \mathbb{A}^5 /\!\!/_{L_+} (\Bbbk^\times)^2$ of smooth projective toric threefolds. The ample cones of X_- and X_+ satisfy

$$\operatorname{Amp}(X_-) = \left\{\alpha\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \beta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \mid \alpha, \beta > 0\right\} \quad \text{and} \quad \psi^*\operatorname{Amp}(X_+) = \left\{\alpha\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \beta\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right] \mid \alpha, \beta > 0\right\}.$$

Our interest lies with an alternative GIT construction introduced in [16]. For this, consider the globally generated line bundles $L_0 := \mathcal{O}_{X_-} = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $L_1 := \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$, $L_2 := \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \in \operatorname{Pic}(X_-)$, and set $\mathcal{L} := \{L_0, L_1, L_2\}$. Let $\mathbb{N}(\mathcal{L})$ denote the semigroup generated by the columns of the matrix

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \end{bmatrix},$$

where for $0 \leq i, j \leq 2$, each column with -1 in row i and $+1$ in row j corresponds to a given torus-invariant divisor defining a section of $L_j \otimes L_i^{-1}$; see [16, Section 3]. The semigroup algebra of $\mathbb{N}(\mathcal{L})$ defines the toric variety $V := \operatorname{Spec} \Bbbk[\mathbb{N}(\mathcal{L})]$, and the top three rows of the above matrix encode the weights of an action on V by the algebraic torus G of rank two whose character lattice is $G^\vee = \{(\theta_i) \in \mathbb{Z}^3 \mid \theta_0 + \theta_1 + \theta_2 = 0\}$. There are two GIT chambers

$$C_- := \{\theta \in G^\vee_\mathbb{Q} \mid \theta_1 > 0, \theta_2 > 0\} \quad \text{and} \quad C_+ := \{\theta \in G^\vee_\mathbb{Q} \mid \theta_1 + \theta_2 > 0, \theta_2 < 0\}.$$

Observe that the GIT quotient X_η is non-empty if and only if $\eta \in \overline{C_-} \cup \overline{C_+}$.

We claim that the GIT regions satisfy $R_{C_-} = \overline{C_-} \cup \overline{C_+} = R_{C_+}$. To see this, fix $\theta_- = (-2, 1, 1) \in C_-$ and $\theta_+ := (-1, 2, -1) \in C_+$. Applying [16, Corollary 4.10, Theorem 4.15] shows that $X_- \cong X_{\theta_-}$, and moreover, that L_{C_-} identifies $G^\vee$ with the index 2 sublattice of $\mathrm{Pic}(X_-)$ spanned by L_1 and L_2. It follows that L_{C_-} is an isomorphism of rational vector spaces that identifies C_- with $\mathrm{Amp}(X_-)$. More generally, for $\eta \in \overline{C_-} \cup \overline{C_+}$, the columns of the matrix are chosen to ensure that the η-graded piece $\Bbbk[V]_\eta$ of the coordinate ring of V is isomorphic to the vector space $H^0(X_-, L_{C_-}(\eta))$, so X_η is isomorphic to Proj of the section ring of $L_{C_-}(\eta)$. In particular, $X_{\theta_+} \cong X_+$ and the rational map $X_{\theta_-} \dashrightarrow X_{\theta_+}$ induced by crossing the wall separating C_- and C_+ is the flop $\psi\colon X_- \dashrightarrow X_+$. Thus, both VGIT morphisms $\tau_\pm$ are small, so the claim follows.

Note, however, that while L_{C_-} is an isomorphism, it fails to identify C_+ with $\mathrm{Amp}(X_+)$, so L_{C_-} does not identify R_{C_-} with $\mathrm{Mov}(X_{\theta_-})$. In fact,

$$L_{C_-}(C_+) = \left\{ L_1^\alpha \otimes (L_2 \otimes L_1^{-1})^\beta \mid \alpha, \beta > 0 \right\} = \left\{ \alpha\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right] + \beta\left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right] \mid \alpha, \beta > 0 \right\}.$$

It follows that L_{C_-} identifies the character $\eta = (0, 1, -1)$ in the boundary of R_{C_-} with the ample bundle $\left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$ on X_+, so the induced VGIT morphism $X_{\theta_+} \to X_\eta$ is an isomorphism rather than a divisorial or fibre type contraction.

3.2. **The new GIT condition.** Define a GIT wall to be a *flipping wall* if it satisfies the assumptions of Proposition 2.8, i.e. the wall separates two GIT chambers $C_\pm$ such that, in the notation of diagram (2.6), both X_{θ_-} and X_{θ_+} are normal, the unstable loci $\mathrm{Uns}(\tau_+) \subseteq X_{\theta_+}$ and $\mathrm{Uns}(\tau_-) \subseteq X_{\theta_-}$ have codimension at least two, and the morphisms τ_+ and τ_- both contract at least one curve. For any wall in the boundary of R_C, let C_- denote the unique chamber in R_C that contains the wall in its closure. Let $\theta_- \in C_-$ and let θ_0 be general in the wall. We say that the wall, when approached from the chamber C_- in R_C, is *small, divisorial* or *of fibre type* if the (Stein factorisation of the) induced morphism $\tau_-\colon X_{\theta_-} \to X_{\theta_0}$ is of the same type.

Condition 3.4. *There exists a GIT chamber C such that:*

(1) for $\theta \in C$, the GIT quotient $X := X_\theta$ is a $\mathbb{Q}$-factorial, normal variety and the linearisation map

$$L_C\colon G^\vee_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X/Y)_{\mathbb{Q}}$$

is an isomorphism of rational vector spaces;

(2) each wall in the interior of the GIT region R_C containing C is a flipping wall; and

(3) each boundary wall of R_C is either divisorial or of fibre type.

Remark 3.5. Condition 3.4 is required for the statement and proof of Theorem 3.12. We present several examples to shed light on the three different parts of this assumption as follows:

- Condition 3.4 holds for the chamber C_- in Example 3.2, while Condition 3.4(1) fails for C_+. Many similar examples are described in Example 3.6 below.
- If Condition 3.4(1) holds, it can happen that (2) fails to hold; see Example 3.7 below.
- If Condition 3.4(1) and (2) both hold, it can happen that (3) fails to hold; see Example 3.3.

Note in addition that Condition 3.4(2) implies that X_θ is normal for every generic $\theta \in R_C$.

Example 3.6 (Mori Dream Spaces via the Cox ring). Generalising Example 3.2, let X be any Mori Dream Space in the sense of Hu and Keel [28]. That is, X is a $\mathbb{Q}$-factorial normal projective variety with $\operatorname{Pic}(X)_\mathbb{Q} \cong N^1(X)$, such that the Cox ring of X, denoted $\operatorname{Cox}(X)$, is a finitely generated $\Bbbk$-algebra. For simplicity, assume that $\operatorname{Pic}(X)$ is free. The $\operatorname{Pic}(X)$-grading of $\operatorname{Cox}(X)$ defines an action of the algebraic torus $G := \operatorname{Hom}(\operatorname{Pic}(X), \Bbbk^\times)$ on the affine variety $V := \operatorname{Spec} \operatorname{Cox}(X)$.

(i) For the chamber $C = \operatorname{Amp}(X)$, the linearisation map $L_C \colon G^\vee \to \operatorname{Pic}(X)_\mathbb{Q}$ is an isomorphism by [28, Proof of Proposition 2.11], so Condition 3.4(1) holds, whilst [28, Proposition 1.11] shows that conditions (2) and (3) also hold.

(ii) For any chamber C' that does not lie in $\operatorname{Mov}(X)$, the kernel of $L_{C'}$ has dimension at least one because the rank of $\operatorname{Pic}(X_{\theta'})_\mathbb{Q}$ for $\theta' \in C'$ drops by one as we cross each boundary wall of the movable cone. In particular, Condition 3.4 fails for C'.

In fact, these statements hold for the action induced by the $\operatorname{Pic}(X)$-grading on the Cox ring for any relative Mori Dream Space; see Grab [23] or Ohta [50].

Example 3.7 (A local del Pezzo by quiver GIT). Let Z be the two-point blow-up of $\mathbb{P}^2$. The total space $X := \operatorname{tot}(\omega_Z)$ of the canonical bundle on Z is smooth with trivial canonical class, the anticanonical ring $R := \bigoplus_{k\geq 0} H^0(Z, \omega_Z^{-\otimes k})$ is Gorenstein [22, Example 5.1.13], and the morphism

$$f \colon X = \operatorname{tot}(\omega_Z) \longrightarrow Y := \operatorname{Spec} R$$

that contracts the zero section is a projective crepant resolution. In fact, f is a morphism of toric varieties: for the lattice $M = \mathbb{Z}^3$, we have that $R \cong \mathbb{C}[\sigma^\vee \cap M]$, where $\sigma \subseteq N \otimes_\mathbb{Z} \mathbb{Q}$ is the strongly convex rational polyhedral cone obtained as the cone over the pentagon in Figure 1(a); the basic triangulation of the pentagon that determines the fan Σ of X is also shown in Figure 1(a). The

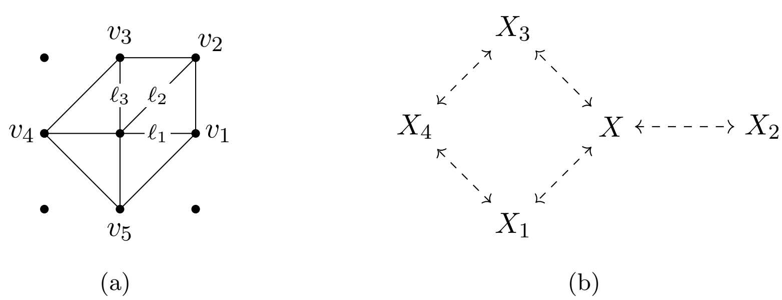
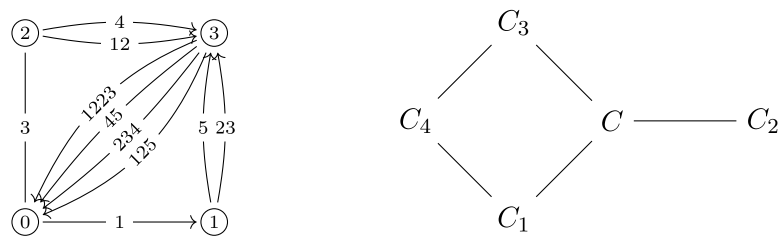


FIGURE 1. (a) slice of the fan defining X; (b) flops linking crepant resolutions

cones $\ell_1, \ell_2, \ell_3 \in \Sigma$ each determine a $(-1,-1)$-curve in X, and for $1 \leq i \leq 3$, flopping the curve defined by ℓ_i produces a projective crepant resolution $X_i \to Y$. A curve in each of X_1 and X_3 can be flopped to produce the projective crepant resolution $X_4 \to Y$ as shown in Figure 1(b).

The morphism $f \colon X \to Y$ is a relative Mori Dream Space, but our interest here lies with a GIT quotient construction that differs from the construction via the Cox ring of X as in Example 3.6.

For this, list the torus-invariant prime Weil divisors $D_1, \dots, D_5$ on Y, one for each lattice point $v_1, \dots, v_5$ on the boundary of the pentagon as above. Define four reflexive sheaves of rank one on Y, namely $E_0 := \mathcal{O}_Y$ $E_1 := \mathcal{O}_Y(D_1)$, $E_2 := \mathcal{O}_Y(D_3)$ and $E_3 := \mathcal{O}_Y(D_1 + D_5)$, and set $\mathcal{E} := \{E_0, E_1, E_2, E_3\}$. Following [15, Definition 2.2], the *quiver of sections* of $\mathcal{E}$, denoted Q, is shown in Figure 2(a): the vertex set corresponds to the collection $\mathcal{E}$; and each arrow is labelled by a Weil divisor where, for example, the label 12 is shorthand for the divisor $D_1 + D_2$. The algebra $\mathrm{End}_R(\bigoplus_{0 \leq i \leq 3} E_i)$ can be presented as the quotient of the path algebra of Q by a two-sided ideal of relations determined by the labelling of arrows by divisors [15, Lemma 2.5].



(a) (b)

FIGURE 2. (a) quiver of sections Q on Y; (b) graph indicating chambers that lie adjacent

To reconstruct the morphism f by quiver GIT, let $\mathbb{N}(Q)$ denote the semigroup generated by the columns of the matrix

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix};$$

each column corresponds to an arrow, where the top four entries record the head and tail, while the bottom five entries record the labelling divisor. Consider the toric variety $V := \operatorname{Spec} \Bbbk[\mathbb{N}(Q)]$ defined by the semigroup algebra of $\mathbb{N}(Q)$. The algebraic torus $G := \operatorname{Spec} \Bbbk[\mathrm{Wt}(Q)]$ with character lattice $\mathrm{Wt}(Q) := \{\theta \in \mathbb{Z}^4 \mid \sum_i \theta_i = 0\}$ acts on V, where the weights of the action are recorded in the top four rows of the matrix. Following [15, Proposition 2.14], the affine quotient satisfies $V /\!\!/_0 T \cong Y$, while for each $\theta \in \mathrm{Wt}(Q)$, variation of GIT gives a projective, birational, toric morphism $f_\theta \colon X_\theta := V /\!\!/_\theta G \to Y$. The GIT chamber

$$C := \left\{ \theta \in G^\vee_{\mathbb{Q}} \mid \theta_1 < 0;\ \theta_2 < 0,\ \theta_1 + \theta_2 + \theta_3 > 0 \right\}$$

gives $X \cong X_\theta$ and $f = f_\theta$ for $\theta \in C$. The linearisation map L_C is an isomorphism that identifies C with the ample cone of X, so Criterion 3.4(1) is satisfied.

The defining inequalities of the chambers C_1, C_2, C_3 that lie adjacent to C are shown below, together with the unique chamber C_4 that lies adjacent to both C_1 and C_3 as in Figure 2(b):

Chamber	Defining inequalities		
C_1	$\theta_1 < 0$	$\theta_2 > 0$	$\theta_1 + \theta_3 > 0$
C_2	$\theta_1 < 0$	$\theta_2 < 0$	$\theta_1 + \theta_2 + \theta_3 > 0$
C_3	$\theta_1 > 0$	$\theta_2 < 0$	$\theta_2 + \theta_3 > 0$
C_4	$\theta_1 > 0$	$\theta_2 > 0$	$\theta_3 > 0$

Crossing the wall from C to C_1 induces the flop $X \dashrightarrow X_1$, and symmetrically, crossing from C to C_3 induces the flop $X \dashrightarrow X_3$. In addition, for $i \in \{1, 3\}$, one can cross a wall from C_i to C_4 to induce the flop $X_i \dashrightarrow X_4$. However, crossing the wall separating $C_- := C$ from $C_+ := C_2$ does not induce the flop $X \dashrightarrow X_2$ as one might expect after comparing Figures 1(b) and 2(b). Rather, the morphism τ_- from diagram (2.6) contracts the curve in X determined by ℓ_2, whereas τ_+ is an isomorphism. In particular, $V /\!\!/_{\theta_2} G$ for $\theta_2 \in C_2$ is not $\mathbb{Q}$-factorial, and $X_2 \ncong V /\!\!/_{\theta_2} G$. Therefore the wall separating C from C_2 is not a flipping wall despite being an internal wall of R_C, so Condition 3.4(2) fails even though Condition 3.4(1) holds.

Remark 3.8. Example 3.7 is obtained by modifying that from Ishii–Ueda [29, Example 12.6], where for the additional reflexive sheaf $E_4 := \mathcal{O}_Y(2D_1 + 2D_5)$, a collection of GIT quotients associated to a quiver with relations defining the algebra $\mathrm{End}_R(\bigoplus_{0 \leq i \leq 4} E_i)$ are studied. The linearisation map is not injective for any chamber in that case. In Example 3.7, we omitted the summand E_4 which reduces the dimension of the space of stability conditions by one, making L_C an isomorphism.

3.3. **The main result.** We now establish the main geometric consequences of Condition 3.4.

Lemma 3.9. *Let C be a chamber satisfying Condition 3.4, and set $X = X_\theta$ for $\theta \in C$. Then:*

- (i) *the isomorphism L_C identifies C and $\overline{C}$ with $\mathrm{Amp}(X/Y)$ and $\mathrm{Nef}(X/Y)$ respectively;*
- (ii) *each $L \in \mathrm{Pic}(X/Y)$ that is nef over Y is also semiample over Y;*
- (iii) *for any $\eta \in \overline{C}$, the section ring $R(X, L_C(\eta))$ is a finitely generated $\mathcal{O}_Y$-algebra, and moreover, the GIT quotient X_η is isomorphic to the model $\mathrm{Proj}\, R(X, L_C(\eta))$; and*
- (iv) *$\mathrm{Pic}(X/Y)_{\mathbb{Q}} \cong N^1(X/Y)$.*

Proof. For part (i), the line bundle $L_C(\theta)$ is ample, so the isomorphism L_C from Condition 3.4(1) identifies C with the interior of a top-dimensional polyhedral cone in $\mathrm{Nef}(X/Y)$. Let θ_0 be general in a wall of $\overline{C}$. If this wall passes through the interior of R_C, then the morphism $\tau \colon X \to X_{\theta_0}$ induced by VGIT contracts at least one curve by Condition 3.4(2). Otherwise, it's a boundary wall of R_C, in which case $\tau \colon X \to X_{\theta_0}$ is of fibre type or it is a divisorial contraction. Thus, L_C sends the boundary of $\overline{C}$ into the boundary of $\mathrm{Nef}(X/Y)$, so in fact L_C identifies C with $\mathrm{Amp}(X/Y)$ and $\overline{C}$ with $\mathrm{Nef}(X/Y)$.

For part (ii), any class in the interior of $\mathrm{Nef}(X/Y)$ is ample and hence semiample. For L in the boundary of $\mathrm{Nef}(X/Y)$, it follows that $\eta := L_C^{-1}(L)$ lies in the boundary of C. After multiplying by some $m > 0$ if necessary, (2.5) shows that $L_C(m\eta) = \tau^*(\mathcal{O}(1))$ for the morphism $\tau \colon X \to X_{m\eta}$ over Y induced by VGIT. Thus, $L^m = \tau^*\big(\mathcal{O}(1)\big)$ is basepoint-free, so L is semiample over Y.

For part (iii), let $\eta \in \overline{C}$. Parts (i) and (ii) imply that $L_C(\eta)$ is semiample, so the section ring $R(X, L_C(\eta))$ is finitely generated by a theorem of Zariski [38, Example 2.1.30], and hence the model $X(L_C(\eta))$ is well-defined. Since η lies in a face of $\overline{C}$, the VGIT morphism $\tau\colon X \to X_\eta$ satisfies $L_C(\eta) = \tau^*(\mathcal{O}(1))$ by (2.4), so $X_\eta \cong X(L_C(\eta)) = \operatorname{Proj} R(X, L_C(\eta))$ as required.

Part (iv) follows by combining part (ii) with Corollary 2.2. ☐

Lemma 3.10. *If one GIT chamber C satisfies Condition 3.4, then every GIT chamber $C' \subset R_C$ satisfies Condition 3.4. In particular, the statement of Lemma 3.9 holds for each chamber in R_C.*

Proof. Conditions 3.4(ii) and (iii) are independent of the choice of chamber $C' \subset R_C$, so it suffices to prove that Condition 3.4(i) holds for C'. There are only finitely many GIT chambers [53, Theorem 2.4], so we proceed by induction. We know Condition 3.4(i) holds for C. For the induction step, let C_+ and C_- be adjacent chambers in R_C separated by a GIT wall, where Condition 3.4(i) holds for C_-. Let $\theta_+ \in C_+$ and $\theta_- \in C_-$. Since the wall separating C_+ and C_- lies in the interior of R_C, Condition 3.4(2) and Proposition 2.8 together imply that the map $\psi\colon X_{\theta_-} \dashrightarrow X_{\theta_+}$ from diagram (2.4) is an isomorphism in codimension-one such that

$$L_{C_-}(\eta) \cong \psi^* L_{C_+}(\eta) \quad \text{for all } \eta \in G^\vee_{\mathbb{Q}}. \tag{3.1}$$

That is, the linearisation maps L_{C_-} and L_{C_+} agree up to the identification of the rational Picard groups $\operatorname{Pic}(X/Y)_{\mathbb{Q}} \cong \operatorname{Pic}(X_{\theta_-}/Y)_{\mathbb{Q}} \cong \operatorname{Pic}(X_{\theta_+}/Y)_{\mathbb{Q}}$ via the isomorphism ψ^*. Now, L_{C_-} is an isomorphism, hence so is L_{C_+} by (3.1). Since X_{θ_-} is $\mathbb{Q}$-factorial and L_{C_-} is surjective, Proposition 2.8 implies that X_{θ_+} is $\mathbb{Q}$-factorial. This shows that Condition 3.4(i) holds for C_+, so it holds for each chamber in R_C by induction. This completes the proof of the first statement, while the second statement follows by applying the proof of Lemma 3.9 verbatim to each chamber C' in R_C. ☐

Remark 3.11. The proof of Lemma 3.10 shows that for each chamber $C' \subset R_C$ and any $\theta' \in C'$:

(i) the small birational model $X_{\theta'}$ is $\mathbb{Q}$-factorial; and

(ii) the linearisation map $L_{C'}$ is equal to L_C, up to the identification $\operatorname{Pic}(X/Y) \cong \operatorname{Pic}(X_{\theta'}/Y)$.

By taking the composition of the linearisation map L_C with the isomorphism from Lemma 3.9(iv), we may identify the target of L_C with $N^1(X/Y)$ whenever Condition 3.4 holds.

Theorem 3.12. *Suppose that Condition 3.4 holds. Then:*

(i) *the linearisation map $L_C\colon G^\vee_{\mathbb{Q}} \to N^1(X/Y)$ is an isomorphism that identifies the GIT decomposition of the region R_C with the Mori chamber decomposition of* $\operatorname{Mov}(X/Y)$*; and*

(ii) *for any $\eta \in R_C$, the section ring $R(X, L_C(\eta))$ is a finitely generated $\mathcal{O}_Y$-algebra and the GIT quotient X_η is isomorphic to the model* $\operatorname{Proj} R(X, L_C(\eta))$.

In particular, every small $\mathbb{Q}$-factorial birational model of X over Y can be obtained as a GIT quotient of the form X_η for some generic $\eta \in R_C$.

Remark 3.13. In particular, Theorem 3.12 establishes that the region R_C from Condition 3.4 is the convex polyhedral cone $L_C^{-1}\big(\operatorname{Mov}(X/Y)\big)$.

Conversely, for a given GIT set-up, suppose that there is a chamber C with $X = X_\theta$ for $\theta \in C$, such that L_C is an isomorphism of fans between R_C and $\mathrm{Mov}(X/Y)$. If, in addition, each VGIT morphism for an interior wall of R_C has unstable locus of codimension at least two, then all parts of Condition 3.4 are satisfied.

Proof. Suppose first that R_C contains a unique GIT chamber, i.e. $\overline{C} = R_C$. For every wall of C and any θ_0 that is general in the wall, Condition 3.4(3) implies that the induced VGIT morphism $\tau\colon X \to X_{\theta_0}$ is either of fibre type or it is a divisorial contraction, so $\mathrm{Nef}(X/Y) = \mathrm{Mov}(X/Y)$. The identification of $\overline{C}$ with $\mathrm{Nef}(X/Y)$ and the isomorphisms $X_\eta \cong X(L_C(\eta)) := \mathrm{Proj}\, R(X, L_C(\eta))$ for all $\eta \in \overline{C}$ were established in Lemma 3.9. This proves (i) and (ii) when $\overline{C} = R_C$.

For the general case, we noted in Remark 3.11(ii) that the linearisation maps $L_{C'}$ for all $C' \subset R_C$ are compatible with taking the strict transform along the appropriate birational map $\psi_{C'}$. Thus, for the chamber C from Condition 3.4 and for any other chamber $C' \subset R_C$ with $\theta' \in C'$, we obtain $L_C\big(\overline{C'}\big) = \psi_{C'}^* L_{C'}\big(\overline{C'}\big) = \psi_{C'}^*\big(\mathrm{Nef}(X_{\theta'}/Y)\big)$ by (3.1) and Lemma 3.10. Applying L_C to the obvious decomposition $R_C = \bigcup_{C' \subset R_C} \overline{C'}$ gives

$$L_C(R_C) = \bigcup_{C' \subset R_C} \psi_{C'}^*\big(\mathrm{Nef}(X_{\theta'}/Y)\big). \tag{3.2}$$

Any wall in the boundary of $L_C(R_C)$ therefore lies in the boundary of $\mathrm{Nef}(X_{\theta'}/Y)$ for some $C' \subset R_C$ and $\theta' \in C'$. Condition 3.4(3) ensures that if θ_0 is general in the corresponding boundary wall of R_C, then the induced VGIT morphism $\tau\colon X_{\theta'} \to X_{\theta_0}$ is either of fibre type or it is a divisorial contraction, so L_C sends the boundary of R_C into the boundary of $\mathrm{Mov}(X/Y)$. It follows from (3.2) that L_C identifies the GIT chamber decomposition of R_C with the decomposition of $\mathrm{Mov}(X/Y)$ into the nef cones of all small $\mathbb{Q}$-factorial birational models. This proves (i).

For (ii), let $\eta \in R_C$. Let $C' \subset R_C$ denote any chamber such that $\eta \in \overline{C'}$. Lemma 3.10 implies that $X_\eta \cong X(L_{C'}(\eta))$. Since both C and C' are contained in R_C, the linearisation maps $L_{C'}$ and L_C are compatible with taking the strict transform along the appropriate birational map ψ, and hence $X_\eta \cong X(L_{C'}(\eta)) \cong X(L_C(\eta))$ as required.

For the final statement, let X' be a small, $\mathbb{Q}$-factorial birational model of X over Y. Part (i) shows that $L_C^{-1}(\mathrm{Nef}(X'/Y))$ is the closure of a chamber C' in R_C. Then for any $\theta' \in C'$, part (ii) gives $X_{\theta'} \cong X(L_{C'}(\theta'))$ which is isomorphic to X' because $L_C(\theta') \in \mathrm{Amp}(X'/Y)$. $\square$

This result allows us to draw conclusions about the GIT quotients directly from known results in birational geometry as follows.

Corollary 3.14. *Suppose that Condition 3.4 holds, and let $\theta, \theta' \in R_C$. Then:*

- (i) *X_θ is isomorphic to $X_{\theta'}$ over Y if and only if θ, θ' lie in the same GIT cone;*
- (ii) *if θ, θ' lie in the interior of R_C, then $X_{\theta'}$ is a small birational model of X_θ over Y, and every small birational model of X_θ arises this way;*
- (iii) *if θ lies in the interior of R_C, then X_θ is $\mathbb{Q}$-factorial if and only if θ is generic;*
- (iv) *the dimension of $N^1(X_\theta/Y)$ equals the dimension of the minimal GIT cone containing θ.*

Proof. For (i), the models $X(L)$ and $X(L')$ associated to $L, L' \in \operatorname{Pic}(X/Y)$ are isomorphic over Y if and only if L, L' lie in the same face of the decomposition of $\operatorname{Mov}(X/Y)$ from (3.2). Thus, (i) follows from Theorem 3.12(i). For (ii), the interior of R_C is identified with the interior of $\operatorname{Mov}(X/Y)$, so the GIT quotients $X_{\theta'}$ and X_θ are isomorphic in codimension-one over Y. Conversely, if X' is a small birational model of $X = X_\theta$ over Y, then there is a line bundle L' in the interior of $\operatorname{Mov}(X/Y)$ satisfying $X(L') \cong X'$. The character $\theta' := L_C^{-1}(L')$ lies in the interior of R_C and satisfies $X_{\theta'} \cong X'$ as required. For (iii), one direction was noted in Remark 3.11, while for the other, it is well-known that the base of a flip is not $\mathbb{Q}$-factorial. Indeed, if X_{θ_0} were $\mathbb{Q}$-factorial, then in the notation of Proposition 2.8, any Cartier divisor D satisfying $\mathcal{O}_{X_{\theta_-}}(D) = L_{C_-}(\theta_+)$ would define a Weil divisor $(\tau_-)_*(D)$ on X_{θ_0}, making mD and $(\tau_-)^*(\tau_-)_*(mD))$ linearly equivalent for some $m > 0$. However, the intersection numbers of these divisors with respect to a curve contracted by τ_- are negative and zero respectively.

For (iv), let F be the minimal GIT cone containing θ, let $C' \subset R_C$ be a chamber containing F in its closure, and let $\theta' \in C'$. Lemma 3.10 implies that L_C identifies F with the minimal face of $\operatorname{Nef}(X_{\theta'}/Y)$ containing $L_C(\theta)$. By (2.3), every line bundle in the interior of this face is the pullback of an ample bundle via the VGIT morphism $\tau\colon X_{\theta'} \to X_\theta$, so $\dim L_C(F) \leq \dim \tau^* N^1(X_\theta/Y)$. On the other hand, $L_C(F)$ is dual via the intersection pairing to the face σ of the Mori cone of curves generated by the numerical classes of curves contracted by τ, so $\dim L_C(F) = \dim(\sigma^\perp)$ for $\sigma^\perp = \{L \in N^1(X_{\theta'}/Y) \mid \deg L|_\ell = 0 \ \forall\ \ell \in \sigma\}$. The pullback via τ of any class in $N^1(X_\theta/Y)$ has degree zero on each generator of σ, so $\tau^* N^1(X_\theta/Y) \subseteq \sigma^\perp$ and hence $\dim \tau^* N^1(X_\theta/Y) \leq \dim(\sigma^\perp) = \dim L_C(F)$. The map τ^* is injective, so $\dim F = \dim L_C(F) = \dim N^1(X_\theta/Y)$. ☐

3.4. **Relative Mori Dream Spaces.** Example 3.6 illustrates that the GIT construction of any Mori Dream Space via its Cox ring gives rise to a GIT chamber that satisfies our Condition 3.4. The next result provides a partial converse, but we emphasise that even for a Mori Dream Space, we are typically interested in applying our Condition 3.4 for new GIT descriptions that do not involve the Cox ring directly.

Corollary 3.15. *Suppose that Condition 3.4 holds, and let $\theta \in R_C$ be any generic stability parameter. Then the GIT quotient $X = X_\theta$ is a Mori Dream Space over Y, i.e.:*

(i) *X is $\mathbb{Q}$-factorial and normal;*
(ii) $\operatorname{Pic}(X/Y)_{\mathbb{Q}} \cong N^1(X/Y)$;
(iii) *the relative nef cone* $\operatorname{Nef}(X/Y)$ *is generated by finitely many semiample line bundles; and*
(iv) *there exists $k \geq 0$ and $\mathbb{Q}$-factorial varieties $X = X_0, X_1, \dots, X_k$, each projective over Y, as well as birational maps $\psi_i\colon X \dashrightarrow X_i$ over Y for $0 \leq i \leq k$ that are isomorphisms in codimension-one, such that*

$$\operatorname{Mov}(X/Y) = \bigcup_{0 \leq i \leq k} \psi_i^* \operatorname{Nef}(X_i/Y), \tag{3.3}$$

where each cone in this description is generated by finitely many semiample line bundles.

Proof. Let $C' \subset R_C$ be the chamber with $\theta \in C'$. Part (i) was observed in Remark 3.11, while part (ii) is Lemma 3.9(iv). For part (iii), the closure $\overline{C'}$ is a polyhedral cone [53, Theorems 2.3-2.4], and hence so is $\mathrm{Nef}(X/Y)$ by Lemma 3.10. Any choice of cone generators for $\mathrm{Nef}(X/Y)$ are semiample over Y by Lemma 3.10. Part (iv) follows from the decomposition (3.2) and the equality of cones $L_C(R_C) = \mathrm{Mov}(X/Y)$ from Theorem 3.12. □

Remark 3.16. (1) The birational maps that feature in Corollary 3.15 are all constructed by variation of GIT quotient, so we need not appeal to the existence of flips from [8].

(2) It is instructive to compare Condition 3.4 with the criteria for a GIT quotient to be a Mori Dream Space given by Hu and Keel [28, Lemma 2.2] (see also Ohta [50, Theorem 6.7]). While our Condition 3.4(1) is equivalent to their third and fourth criteria, our criteria (2) and (3) differ considerably from their first and second criteria. We show in Theorem 4.6 that quiver varieties satisfy our Condition 3.4, so they are Mori Dream Spaces by Corollary 3.15. However, the next example shows that even the simplest quiver varieties can fail to satisfy the Hu and Keel criteria.

Example 3.17. The minimal resolution of the A_1 singularity $Y = \mathbb{V}(uv - w^2) \subset \mathbb{A}^3$ is obtained by variation of GIT quotient for quiver varieties associated to the graph with one node, and vectors $\mathbf{v} = 1$, $\mathbf{w} = 2$. The quiver Q is the McKay quiver for the cyclic group of order two in $\mathrm{SL}(2, \Bbbk)$, and dimension vector $\alpha = (1, 1)$. In this case, $V = \mathbb{V}(ad - bc) \subset \mathbb{A}^4$ admits an action by $G = (\Bbbk^\times)^2/\Bbbk^\times$, and for $\theta = (-1, 1) \in G^\vee$, the θ-unstable locus is the intersection of V with $\mathbb{V}(a, b)$. This locus is of codimension one in V, so $X = V /\!\!/_\theta\, G$ fails to satisfy the Hu and Keel criterion [28, Lemma 2.2(1)]. However, this example satisfies Condition 3.4 (see Theorem 4.6), so it is a Mori Dream Space over Y by Corollary 3.15.

Remark 3.18. If Condition 3.4 holds, then combining Corollary 3.15 with the statement of Hu and Keel [28, Proposition 2.9] (see also Ohta [50, Proposition 6.9]) implies that the Cox ring of the variety X_θ is a finitely generated $\Bbbk$-algebra for $\theta \in C$. Our Theorem 3.12 allows for an alternative description of the Cox ring of X_θ, see [3].

3.5. **Strong convexity and injectivity.** In practice, injectivity of L_C can often follow from the following simple criterion.

A cone is called *strongly convex* if the origin is a face. Given a fan, the origin is a cone if and only if all cones are strongly convex.

Given a chamber C, let K_C be the intersection of all supporting hyperplanes of the walls of C. The following lemma is standard.

Lemma 3.19. *The chamber C is a product of K_C and a strongly convex cone. In particular, K_C is the largest linear subspace of C, the intersection of all faces of C, and the only face which is a linear subspace.*

Proof. Since all faces of C are defined by intersections of half-spaces supported on the hyperplanes defining K_C, we see that K_C is contained in all faces. This also shows that it is the largest linear subspace of C and the intersection of all faces. For the first statement, C/K_C is a strongly convex cone, and by taking a splitting of the ambient vector space, $C \cong C/K_C \times K_C$. □

In particular, K_C does not depend on the choice of chamber, and it is zero if and only if all cones are strongly convex. Let us simply call it K from now on.

Lemma 3.20. *Suppose that V has a G-fixed point. Then each GIT cone is strongly convex.*

Proof. By definition, if θ is a nontrivial effective character, then any G-fixed point is θ-unstable. So $V^\theta \neq V = V^0$, and hence $\{0\}$ is a GIT cone. □

In particular, if V is conical and the G action is equivariant, then the origin is a G-fixed point.

Proposition 3.21. *Suppose that C is a GIT chamber such that the VGIT morphism τ from (2.4) obtained by choosing general θ_0 in any wall of $\overline{C}$ contracts at least one curve. Then* $\ker(L_C) \subseteq K$.

Proof. We adapt the proof of [1, Proposition 6.1]. Suppose on the contrary that there is a vector $\eta \in \ker(L_C)$ not contained in K. Then η is not contained in every supporting hyperplane of $\overline{C}$. We may translate any $\theta \in C$ by a rational multiple of η to hit a wall of $\overline{C}$, say at θ_0. Since $\eta \in \ker(L_C)$, the line bundle $L_C(\theta) = L_C(\theta_0)$ is ample, so some multiple is very ample. However, this is a contradiction because the VGIT morphism $\tau\colon X_\theta \to X_{\theta_0}$ contracts a curve. □

Corollary 3.22. *Suppose that $\overline{C}$ is strongly convex. If, for every wall of $\overline{C}$, the VGIT morphism τ from (2.4) obtained by choosing θ_0 to be general in the wall contracts at least one curve, then L_C is injective.*

Remark 3.23. In the situation of diagram (2.6), suppose $\tau_\pm$ are both small contractions. If the image under both $\tau_\pm$ of the unstable locus $\mathrm{Uns}(\tau_\pm)$ is singular, and the singular locus of X_{θ_0} has codimension at least two, then each of $\mathrm{Uns}(\tau_\pm)$ has codimension at least two in $X_{\theta_\pm}$. This can be a convenient condition to check in practice.

Putting these together, we conclude the following useful criteria for Condition 3.4 to hold:

Corollary 3.24. *Suppose that there exists a GIT chamber C such that:*

(1) *the closed cone $\overline{C}$ is strongly convex, the map L_C is surjective, and for $\theta \in C$, the GIT quotient $X := X_\theta$ is a $\mathbb{Q}$-factorial, normal variety;*
(2) *for any GIT chambers $C_\pm$ in R_C sharing a wall, the VGIT morphisms $\tau_\pm$ from (2.6) both contract a curve and they each map their unstable locus to the singular locus of X_{θ_0} which is of codimension at least two; and*
(3) *each boundary wall of R_C is either divisorial or of fibre type.*

Then Condition 3.4 is satisfied.

3.6. **Minimal models of Gorenstein singularities.** In this section we assume, in addition, that Y has Gorenstein singularities and that $f\colon X \to Y$ is a projective crepant resolution, or more generally, a projective $\mathbb{Q}$-factorial terminalisation, i.e. crepant morphism from a terminal variety.

Suppose that there exists another projective $\mathbb{Q}$-factorial terminalisation $f'\colon X' \to Y$. Then X and X' are birational minimal models over Y, so by [35, Theorem 3.52] there is a movable line bundle L on X such that $X' \cong X(L)$, and the morphism $f' = f_L$ fits into a commutative diagram (2.2). In particular, if there is a GIT construction such that $X \cong X_\theta$ for $\theta \in C$ as in Condition 3.4, then Theorem 3.12 implies that there is a chamber C' in the GIT region R_C such that $X' \cong X_{\theta'}$ for $\theta' \in C'$. More generally, we have the following.

Corollary 3.25. *Suppose that Condition 3.4 holds and that $f_\theta\colon X_\theta \to Y$ is a projective $\mathbb{Q}$-factorial terminalisation for $\theta \in C$.*

(i) *If a projective, crepant morphism $g\colon Z \to Y$ is dominated by a projective, $\mathbb{Q}$-factorial terminalisation $f'\colon X' \to Y$, then there is a chamber C' in R_C and $\eta \in \overline{C'}$ such that $Z \cong X_\eta$ and $g = f_\eta$. Moreover, Z has terminal singularities if and only if η lies in the interior of R_C.*

(ii) *Conversely, for all $\eta \in R_C$, the morphism $f_\eta\colon X_\eta \to Y$ is projective and crepant.*

Proof. For (i), the choice of $C' \subseteq R_C$ is described in the paragraph preceding Corollary 3.25. Since f' factors via g, the morphism $h\colon X' \to Z$ satisfying $f' = g \circ h$ is obtained from a basepoint-free line bundle $L \in \operatorname{Nef}(X'/Y)$. The first statement follows from Theorem 3.12 by setting $\eta := L_C^{-1}(L)$. For the second statement, Z fails to be terminal if and only if the crepant morphism $h\colon X \to Z \cong X_\eta$ contracts a divisor. This holds if and only if η lies in the boundary of $\operatorname{Mov}(X'/Y)$, which is identified with $\operatorname{Mov}(X/Y)$ by pullback along the birational map ψ from (2.1).

For (ii), each $\eta \in R_C$ lies in the closure of some chamber $C' \subseteq R_C$, so for $\theta' \in C'$, the morphism $f_{\theta'}\colon X_{\theta'} \to Y$ factors via $f_\eta\colon X_\eta \to Y$. Since $f_{\theta'}$ is crepant, then so too is f_η. □

Remark 3.26. The hypothesis from Corollary 3.25(i), namely that a projective, crepant morphism $g\colon Z \to Y$ is dominated by a projective, $\mathbb{Q}$-factorial terminalisation $f'\colon X' \to Y$, is superfluous in light of results from [8]. However, we choose to leave this as an explicit assumption here to underline the fact that the results in the current paper do not rely on [8] in any way.

3.7. **Weakening the hypotheses.** We conclude this section by noting that our results hold in a slightly more general context.

Rather than assume that L_C is an isomorphism in Condition 3.4(1), suppose instead that there exists an affine subspace $\Lambda \subseteq G^\vee \otimes_\mathbb{Z} \mathbb{Q}$ that is not contained in the linear span of any GIT wall, such that the restriction of L_C to Λ is an affine isomorphism $\Lambda \cong \operatorname{Pic}(X/Y)_\mathbb{Q}$. If one replaces the study of chambers C in $G^\vee \otimes_\mathbb{Z} \mathbb{Q}$ by the open cones $C \cap \Lambda$ in Λ, and the closed GIT region R_C by $R_C \cap \Lambda$, then the proof of Theorem 3.12 applies verbatim if one assumes the analogue of Condition 3.4 in this context.

Example 3.27. If X_θ denotes the quiver variety for a framed extended Dynkin quiver of dimension vector $(1, \delta)$ where δ denotes the minimal imaginary root for an affine root system of type ADE as in [1, Proposition 7.11(i)], then any affine hyperplane Λ orthogonal to the kernel of L_C will do.

One might hope to weaken Condition 3.4(1) further to require only surjectivity of L_C. However, as Corollary 3.22 shows, if L_C is not injective, then there are two cases to consider. Firstly, if $\overline{C}$ is not strongly convex, then it is natural to study a transverse slice Λ to the maximal linear subspace in $\overline{C}$, as described above. The second case is where C has a wall for which the VGIT morphism does not contract a curve. In this latter case, one might hope to restrict attention to an affine subspace that intersects none of these walls (such walls are called 'fake' or 'type 0' in the literature); this is precisely the situation described in Example 3.27 above.

4. Nakajima quiver varieties

In this section we establish our main result for Nakajima quiver varieties, see Theorem 4.6.

4.1. **Quiver varieties.** Choose an arbitrary finite graph with vertices $0, \dots, r$ and let H be the set of pairs consisting of an edge, together with an orientation on it. Let $\mathrm{tl}(a)$ and $\mathrm{hd}(a)$ denote the tail and head respectively of the oriented edge $a \in H$. Let a^* denote the same edge, but with opposite orientation. We fix an orientation of the graph, that is, a subset $\Omega \subset H$ such that $\Omega \cup \Omega^* = H$ and $\Omega \cap \Omega^* = \emptyset$. Then $\epsilon : H \to \{\pm 1\}$ is defined to take value 1 on Ω and -1 on Ω^*.

Fix collections $V_0, \dots, V_r$ and $W_0, \dots, W_r$ of finite-dimensional vector spaces over $\Bbbk$ and set

$$\mathbf{v} = (\dim V_0, \dots, \dim V_r), \quad \mathbf{w} = (\dim W_0, \dots, \dim W_r).$$

The group $G(\mathbf{v}) := \prod_{k=0}^r \mathrm{GL}(V_k)$ acts naturally on the space

$$\mathbf{M}(\mathbf{v}, \mathbf{w}) := \left(\bigoplus_{a \in H} \mathrm{Hom}_\Bbbk(V_{\mathrm{tl}(a)}, V_{\mathrm{hd}(a)}) \right) \oplus \left(\bigoplus_{k=0}^{r} \big(\mathrm{Hom}_\Bbbk(V_k, W_k) \oplus \mathrm{Hom}_\Bbbk(W_k, V_k) \big) \right).$$

This action of $G(\mathbf{v})$ is Hamiltonian for the natural symplectic structure on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ and, after identifying the dual of $\mathfrak{g}(\mathbf{v}) := \mathrm{Lie}\, G(\mathbf{v})$ with $\mathfrak{g}(\mathbf{v})$ via the trace pairing, the corresponding moment map $\mu \colon \mathbf{M}(\mathbf{v}, \mathbf{w}) \to \mathfrak{g}(\mathbf{v})$ satisfies

$$\mu(B, i, j) = \left(\sum_{\mathrm{hd}(a) = k} \epsilon(a) B_a B_{a^*} + i_k j_k \right)_{k=0}^{r}$$

where $i_k \in \mathrm{Hom}_\Bbbk(W_k, V_k)$, $j_k \in \mathrm{Hom}_\Bbbk(V_k, W_k)$ and $B_a \in \mathrm{Hom}_\Bbbk(V_{\mathrm{tl}(a)}, V_{\mathrm{hd}(a)})$. Though one can talk about arbitrary stability conditions in this context, as was done in [45], it is easier in our case to apply the trick of Crawley-Boevey [18] and reduce to the case where each $W_k = 0$ by introducing a framing vertex.

The set H associated to the graph can be thought of as the arrow set of a quiver. We frame this quiver by adding an additional vertex ∞, as well as $\mathbf{w}_i$ arrows from vertex ∞ to vertex i and another $\mathbf{w}_i$ arrows from vertex i to vertex ∞. This framed (doubled) quiver is denoted $Q = (Q_0, Q_1)$, where $Q_0 = \{\infty, 0, \dots, r\}$. Each dimension vector $\mathbf{v} = (\dim V_0, \dots, \dim V_r)$ for the original graph determines a dimension vector $\alpha := (1, \mathbf{v}) = (1, \dim V_0, \dots, \dim V_r)$ for Q. We may identify $\mathbf{M}(\mathbf{v}, \mathbf{w})$ with the space

$$\operatorname{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \operatorname{Hom}_{\Bbbk}(\Bbbk^{\alpha_{\mathrm{tl}(a)}}, \Bbbk^{\alpha_{\mathrm{hd}(a)}})$$

of representations of Q of dimension vector α in such a way that the $G(\mathbf{v})$-action on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ corresponds to the action of the group $G(\alpha) := \big(\prod_{i \in Q_0} \mathrm{GL}(\alpha_i)\big)/\Bbbk^\times$ on $\operatorname{Rep}(Q, \alpha)$ by conjugation and, moreover, the above map μ corresponds to the moment map μ induced by this $G(\alpha)$-action on $\operatorname{Rep}(Q, \alpha)$.

From now on, we assume that $\mathbf{v}$ satisfies $v_i \neq 0$ for all $0 \leq i \leq r$, and $\mathbf{w} \neq 0$. This is equivalent to choosing a dimension vector α for Q with component $\alpha_\infty = 1$ and $\alpha_i \neq 0$ for all i. Then the rational vector space

$$\Theta := \big\{\theta \in \operatorname{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Q}) \mid \theta(\alpha) = 0\big\}$$

satisfies $G(\alpha)^\vee \otimes_{\mathbb{Z}} \mathbb{Q} = \Theta$, where $\chi_\theta(g) = \prod_{i \in Q_0} \det(g_i)^{\theta_i}$ for $g \in G(\alpha)$. For $\theta \in \Theta$, after replacing θ by a positive multiple if necessary, the (Nakajima) *quiver variety* associated to θ is the categorical quotient

$$\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) := \mu^{-1}(0) /\!\!/_{\chi_\theta} G(\alpha) = \mu^{-1}(0)^\theta /\!\!/ G(\alpha) = \operatorname{Proj} \bigoplus_{k \geq 0} \Bbbk[\mu^{-1}(0)]^{\chi_{k\theta}},$$

where $\mu^{-1}(0)^\theta$ denotes the locus of χ_θ-semistable points in $\mu^{-1}(0)$ and $\Bbbk[\mu^{-1}(0)]^{\chi_{k\theta}}$ is the $\chi_{k\theta}$-semi-invariant slice of the coordinate ring of the locus $\mu^{-1}(0)$. Note that $\Bbbk^\times$ acts on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ by scaling, and this action descends to an action on $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$.

For a more algebraic description of $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$, extend ϵ to Q by setting $\epsilon(a) = 1$ if $a \colon \infty \to i$ and $\epsilon(a) = -1$ if $a \colon i \to \infty$. The *preprojective algebra* Π is the quotient of the path algebra $\Bbbk Q$ by the relation

$$\sum_{a \in Q_1} \epsilon(a) a a^* = 0. \tag{4.1}$$

Given $\theta \in \Theta$, we say that a Π-module M of dimension vector α is θ-semistable if $\theta(N) \geq 0$ for all submodules $N \subseteq M$, and it is θ-stable if $\theta(N) > 0$ for all proper nonzero submodules. Two θ-semistable Π-modules are S-equivalent if their composition series agree in the abelian category of θ-semistable Π-modules. A finite dimensional Π-module is said to be θ-polystable if it is a direct sum of θ-stable Π-modules. King [33] proved that a Π-module M of dimension vector α is θ-semistable (resp. θ-stable) if and only if the corresponding point of $\mu^{-1}(0)$ is χ_θ-semistable (resp. χ_θ-stable) in the sense of GIT. In fact [33, Propositions 3.2, 5.2] establishes that the quiver variety $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ is the coarse moduli space of S-equivalence classes of θ-semistable Π-modules of dimension vector α, where the closed points of $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ are in bijection with the θ-polystable representations of Π of dimension α. We write $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})^s$ for the (possibly empty) open subset of $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ parametrising θ-stable representations.

4.2. **Kirwan surjectivity.** Recall that $\mathbf{w} \neq 0$ and $v_i \neq 0$ for all i, with $\alpha := (1, \mathbf{v}) \in \mathbb{N}^{Q_0}$. In this case, there exist θ that are non-degenerate in the sense of [41, Definition 3.1], but $\mu^{-1}(0)^\theta$ can be empty. Let us first recall the condition on α which excludes this.

Associated to the quiver Q is a root system R with positive roots $R^+ = R \cap \mathbb{Z}^{Q_0}_{\geq 0}$. We set $R^+_\theta = \{\gamma \in R^+ \mid \theta(\gamma) = 0\}$.

Lemma 4.1. *The following are equivalent:*

(i) $\alpha \in R^+$*;*

(ii) *each* $\theta \in \Theta$ *is effective, i.e.* $\mu^{-1}(0)^\theta \neq \emptyset$*;*

(iii) $G(\alpha)$ *acts freely on* $\mu^{-1}(0)^\theta$ *for* θ *in some dense open subset of* Θ*.*

Moreover, when these conditions hold, the GIT chambers are precisely the interiors of the top dimensional cones of the GIT fan.

Proof. The space $\mu^{-1}(0)^\theta$ is non-empty if and only if the quiver variety $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ is non-empty if and only if $\alpha \in \mathbb{N}R^+_\theta$. Since $\alpha_\infty = 1$, we have $\alpha \in \mathbb{N}R^+_\theta$ for all $\theta \in \Theta$ only if $\alpha \in R^+$. This shows (i) $\iff$ (ii). For a general $\theta \in \Theta$, the condition α indivisible implies that every θ-semistable representation is θ-stable; in other words, a general θ is generic. Therefore, $G(\alpha)$ will act freely if $\mu^{-1}(0)^\theta \neq \emptyset$, so (ii) implies (iii). Conversely, if $G(\alpha)$ acts freely on $\mu^{-1}(0)^\theta$ then the latter must be non-empty by definition. But $\mu^{-1}(0)^\theta \subset \mu^{-1}(0)^{\theta_0}$ for $\theta \in C$ and $\theta_0 \in \overline{C}$, so (iii) implies (ii).

In the preceding paragraph, we noted that each general $\theta \in \Theta$ is generic. This is precisely the final statement. ☐

By [5, Theorem 1.15], the smooth locus of $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ equals the *canonically polystable* locus, which is the locus where the decomposition of polystable representations into a sum of stable ones is of generic type. In particular, if the stable locus is nonempty, then the smooth locus equals the stable locus $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})^s$.

For $\theta \geq \theta_0$, the morphism $f_\theta \colon \mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\theta_0}(\mathbf{v}, \mathbf{w})$ need not be surjective. However, it is birational onto its image [1, Theorem A.1]. Hence it is not of fibre type. If the stable locus with respect to θ_0 is empty then it can happen that f_θ is an isomorphism onto (the normalisation of) its image, showing that 'fake' walls (also known as walls of type 0, see section 5.3) exist.

As a consequence of Kirwan surjectivity, established in [41], we note that:

Theorem 4.2 (McGerty–Nevins)**.** *Assume that* $\mathbf{w} \neq 0$ *and the equivalent conditions of Lemma 4.1 hold. For any chamber* C *and for* $\theta \in C$*, the restriction of the linearisation map defines a surjective map of lattices* $L_C \colon G^\vee \to \operatorname{Pic}(\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}))$*.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} G^\vee & \xrightarrow{\quad L_C \quad} & \operatorname{Pic}(\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})) \\ & \searrow \qquad \swarrow_{c_1} & \\ & H^2(\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}), \mathbb{Z}). & \end{array}$$

The main result of [41] says that the map $G^\vee \to H^2(\mathfrak{M}_\theta(\mathbf{v},\mathbf{w}),\mathbb{Z})$ is surjective. Since $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is smooth, [44, Theorem 7.3.5] says that the cycle map $\mathrm{Pic}(\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})) \to H^2(\mathfrak{M}_\theta(\mathbf{v},\mathbf{w}),\mathbb{Z})$ is an isomorphism. We note that the paper [44] assumed that the quiver Q has no loops. However, the proof of [44, Theorem 7.3.5] does not require this restriction. □

Remark 4.3. The map L_C will have a non-trivial kernel in general. A simple example is given by taking a framed affine Dynkin quiver and $\alpha = e_\infty + \delta$; see [1, Proposition 7.11]. In fact, one can choose suitable $(Q,\mathbf{v},\mathbf{w})$ and generic θ such that the quiver variety $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is a minimal resolution of the corresponding Kleinian singularity and L_C has kernel of arbitrarily large dimension.

4.3. **Applying the main result.** Let $\langle -,-\rangle$ denote the Ringel form on $\mathbb{Z}^{Q_0}$ and $(-,-)$ its symmetrisation. For any $\gamma \in R$, define $p(\gamma) := 1 - \frac{1}{2}(\gamma,\gamma)$.

Definition 4.4. Define Σ_θ to be the set of $\gamma \in R^+_\theta$ such that

$$p(\gamma) > p\big(\beta^{(1)}\big) + \cdots + p\big(\beta^{(k)}\big)$$

for every proper decomposition $\gamma = \beta^{(1)} + \cdots + \beta^{(k)}$ with $\beta^{(i)} \in R^+_\theta$.

Crawley-Boevey [18, Theorem 1.2] showed that $\alpha \in \Sigma_0$ if and only if there exists a simple (= 0-stable) Π-module of dimension vector α. More generally, it is shown in [5, Theorem 1.3] that there exists a θ-stable Π-module of dimension α if and only if $\alpha \in \Sigma_\theta$. Since $\alpha = (1,\mathbf{v})$ is indivisible, every θ-semistable representation will be θ-stable if $\theta(\beta) \neq 0$ for all roots $\beta < \alpha$. That is, the GIT walls are contained in the union of hyperplanes $\beta^\perp$. For a precise description of the GIT walls, see Section 4.5 below.

Lemma 4.5. *Assume $\alpha \in \Sigma_0$. For any chamber C, let $\theta \in C$ and choose $\theta_0 \in \overline{C} \setminus C$. Then the surjective, birational VGIT morphism $\tau\colon \mathfrak{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$ contracts at least one curve.*

Proof. Since $\alpha \in \Sigma_0$, the simple locus (= 0-stable locus) of $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ is nonempty, hence open and dense. Therefore, the simple locus of $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$ is non-empty for all $\zeta \in \Theta$. Since this locus is contained in the ζ-stable locus, the latter is also always non-empty. Now [5, Theorem 1.15] implies that the singular locus is precisely the strictly θ_0-polystable locus.

The morphism $\tau\colon \mathfrak{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$ is a surjective birational morphism because $\alpha \in \Sigma_0$. Moreover, since $\theta_0 \in \overline{C} \smallsetminus C$, there is a θ-stable Π-module M of dimension α that is not θ_0-stable. Therefore, the image under τ of the corresponding point $[M] \in \mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is strictly θ_0-polystable. The previous paragraph implies that $\tau[M]$ lies in the singular locus of $\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$, so τ contracts at least one curve by Zariski's Main Theorem [25, III, Corollary 11.4]. □

Theorem 4.6. *Let $\alpha \in \Sigma_0$. Then every GIT chamber C satisfies Condition 3.4.*

Proof. Since $\alpha \in R^+$, Lemma 4.1 implies that $G(\alpha)$ acts freely on $\mu^{-1}(0)^\theta$ for $\theta \in C$, so the quotient $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is nonsingular (see e.g [17, Lemma 10.3]) and Theorem 4.2 says that L_C is surjective. The origin in $\mu^{-1}(0)$ is a $G(\alpha)$-fixed point, so Lemma 3.20 says that $\overline{C}$ is strongly convex. For $\theta \in C$ and any $\theta_0 \in \overline{C} \setminus C$ general in a wall, the morphism $\tau\colon \mathfrak{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$ contracts at least one curve by Lemma 4.5. Therefore L_C is injective by Corollary 3.22, so Condition 3.4(1) holds. In addition, the unstable locus in $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is precisely the preimage under τ of the singular locus of $\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$; that is, the unstable locus is the exceptional locus of τ. Since $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is smooth, it follows from [5, Theorem 1.2] that τ is a symplectic resolution of singularities. Therefore, it is semi-small [32], so

$$\operatorname{codim}_{\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})} \operatorname{Uns}(\tau) \geq \frac{1}{2}\operatorname{codim}_{\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})} \operatorname{Sing}\big(\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})\big). \tag{4.2}$$

Since the singular locus of $\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$ is a union of symplectic leaves, its codimension is even. There are two cases:

(1) If this codimension is at least 4 then the codimension of the unstable locus is at least two. This depends only on θ_0 in the GIT wall rather than on the chamber whose closure contains the wall, so this analysis applies equally to both morphisms τ_+ and τ_- in diagram (2.6). Therefore, the GIT wall is a flipping wall.

(2) Otherwise, the codimension of the singular locus of $\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$ is two. Locally, the singularities of $\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$ transverse to a codimension two leaf are Kleinian, which implies that the inequality in (4.2) is an equality. In other words, τ is divisorial.

It remains to note that the GIT region R_C is defined in such a way that the walls in the interior of R_C cannot induce a divisorial contraction, so they are flipping by the above; boundary walls are not flipping, so they are divisorial by the above. Thus, Conditions 3.4(2) and (3) hold. □

Our main result (Theorem 3.12) therefore holds under the assumptions of Theorem 4.6, so we obtain Theorem 1.2. In fact, Corollary 3.25 implies the following stronger result:

Corollary 4.7. *Let $\alpha \in \Sigma_0$, and let $C \subset \Theta$ be a chamber with GIT region R_C.*

(i) *Projective partial crepant resolutions of $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$, taken up to isomorphism over $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$, are in bijection with the GIT cones in R_C;*

(ii) *Under this bijection, the crepant resolutions of $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ correspond to the GIT chambers.*

In particular, every projective partial crepant resolution of the affine quiver variety $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ is of the form $f_\theta\colon \mathfrak{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w})$, for some $\theta \in R_C$. We also deduce from Corollary 3.15 the following result, independent of [8, 47].

Corollary 4.8. *Let $\alpha \in \Sigma_0$. For any generic $\theta \in \Theta$, the quiver variety $\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})$ is a Mori Dream Space over $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$.*

Remark 4.9. One can consider quiver varieties associated to deformed preprojective algebras (at deformation parameter λ). The assumption $\lambda = 0$ is only required to deduce the surjectivity of L_C from [41]. This is also the only place where we require $\alpha_\infty = 1$ (or $\mathbf{w} \neq 0$). Using a detailed analysis of codimension 3 singularities in $\mu^{-1}(\lambda)$, we show in [6] that L_C is an isomorphism over $\mathbb{Q}$ for any $\alpha \in \Sigma_0$ except possibly when $(\alpha,\alpha) = -8, -12$.

4.4. **The Namikawa–Weyl group.** We now prove a general result about polyhedral cones and automorphisms of real vector spaces for which we could not find a suitable reference.

Lemma 4.10. *Let $C, C' \subset \mathbb{R}^n$ be the interiors of rational polyhedral cones such that $W := \overline{C} \cap \overline{C'}$ is a common codimension-one face. Let γ be an integral automorphism of $\mathbb{R}^n$ with $\gamma(C) = C'$ fixing W pointwise. Then $\gamma^2 = 1$ and $\mathrm{Fix}(\gamma)$ is the hyperplane spanned by W. If $\overline{C}$ is strongly convex, then such an automorphism is unique.*

Proof. If $H \subset \mathbb{R}^n$ is the hyperplane spanned by the vectors in W then γ is the identity on H. Write $H = \beta^\perp$ for some primitive vector $\beta \in \mathbb{Z}^n$. Then $\gamma(\beta) = -\beta + v$ for some $v \in H$, as γ is an integral automorphism sending C to C'. As a result, $\det(\gamma) = -1$, and γ must have an eigenvector β' of eigenvalue -1. This proves the first assertion.

For the second assertion, suppose that γ' is another integral automorphism fixing W pointwise and sending C to C'. Then $\varphi := \gamma \circ \gamma'$ is an integral automorphism fixing C and fixing W pointwise. Thus, for β as before, we have $\varphi(\beta) = \beta + u$ for some $u \in H$. Now if $\overline{C}$ is strongly convex, then for some other codimension-one face W' of $\overline{C}$, we have that u is not in the hyperplane spanned by W'. Therefore $\varphi(W')$ is either in the interior or the exterior of $\overline{C}$, which contradicts $\varphi(C) = C$. □

For symplectic resolutions of conical symplectic singularities, such as $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, Namikawa [46] has shown that there is a finite Weyl group that acts (as a reflection group) on $H^2(\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}), \mathbb{Q})$. We refer to this action as the *Namikawa–Weyl group*.

Proposition 4.11. *Let $\alpha \in \Sigma_0$. Each GIT region of the form R_C is a simplicial cone. Reflections about the boundary walls of R_C generate a group Γ isomorphic to the Namikawa–Weyl group, which acts simply transitively on the set of all GIT regions. The union of these regions is all of Θ. Given GIT chambers C, C', if $g \in \Gamma$ is the element satisfying $g(R_C) = R_{C'}$, then $L_{C'} = L_C \circ g$.*

Proof. Let $Y = \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ and $X_\theta = \mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$.

Let C be a GIT chamber with wall W for which $\tau_- \colon X_\theta \to X_{\theta_0}$, with $\theta \in C$ and a general $\theta_0 \in W$, is a divisorial contraction. If C' is the other chamber with wall W then τ_+ from (2.6) is also a divisorial contraction since X_{θ_0} has a codimension two leaf. Under the isomorphism L_C, the nef cone $L_C(\overline{C})$ contains $L_C(W)$. Since τ_- is a divisorial contraction, W is a boundary wall of R_C. Then Theorem 3.12(i) says that $L_C(W)$ must be a boundary wall of the movable cone. Fix $\theta_0 \in W, \theta \in C, \theta' \in C'$ and let $\mathcal{O}(\theta_0)$ denote the corresponding polarising ample line bundle on X_{θ_0}. Then (2.5) says that $L_C(\theta_0) = \tau_-^*(\mathcal{O}(\theta_0))$, But since $L_C(W)$ is a boundary wall of the movable cone of X_θ, the latter is the unique minimal model of Y dominating X_{θ_0}. This implies that $X_\theta \cong X_{\theta'}$ over Y. We deduce that $L_C(C) = L_{C'}(C')$ is the ample cone of X_θ over Y. Moreover,

$$L_C(\theta_0) = \tau_-^*(\mathcal{O}(\theta_0)) = \tau_+^*(\mathcal{O}(\theta_0)) = L_{C'}(\theta_0)$$

shows that $L_C|_W = L_{C'}|_W$. Hence, Theorem 4.2 says that $\gamma := L_{C'}^{-1} \circ L_C$ is an integral automorphism of Θ that maps C to C' and fixes W pointwise. Lemma 4.10 says that $\gamma^2 = 1$. Moreover, $L_{C'}(R_{C'}) = L_C(R_C) = \mathrm{Mov}(X_\theta/Y)$ implies that $R_{C'} = \gamma(R_C)$.

Next, inside $N^1(X_\theta/Y)$, [9, Proposition 2.17] shows that $\mathrm{Mov}(X_\theta/Y)$ is a fundamental domain for the Namikawa–Weyl group, generated by reflections about the boundary walls of $\mathrm{Mov}(X_\theta/Y)$. Since the Namikawa–Weyl group is a Weyl group [46] acting on the reflection representation, its fundamental regions are simplicial cones. Pulling this back via L_C shows that R_C is a simplicial cone. If s is the reflection in the Namikawa–Weyl group about the wall $L_C(W)$ then the uniqueness statement of Lemma 4.10 implies that $L_C \circ \gamma = s \circ L_C$. Hence the reflections about the boundary walls of R_C generate a group $\Gamma \subset \mathrm{GL}(\Theta)$ isomorphic to the Namikawa–Weyl group. This group acts with R_C as a fundamental region. In particular, $\Theta = \bigcup_{g\in\Gamma} g(R_C)$.

We claim $g(R_C) = R_{C'}$ for some GIT chamber C'. This is done by induction on the length of g, the case $\ell(g) = 1$ having been done already. If $g = \gamma h$ with $\ell(h) < \ell(g)$ then $h(R_C) = R_{h(C)}$ and applying the previous argument with C replaced by $h(C)$ shows that $g(R_C) = \gamma(R_{h(C)}) = R_{C'}$ for some C'. A similar induction shows that if $C'' \subset R_{C'}$ and $g(R_C) = R_{C'}$ then $L_{C''} \circ g = L_C$. □

4.5. **Combinatorics and hyperplane arrangements.** The results of this final section require that the dimension vector α for the quiver Q is indivisible, but they do not require α to come from a nonzero framing. Thus, we write $\mathfrak{M}_\theta(\alpha) = \mu^{-1}(0)^\theta /\!\!/ G(\alpha)$ for the quiver variety. Note that the indivisibility assumption on α ensures that $\mathfrak{M}_\theta(\alpha)$ is nonsingular for general θ.

For any root $\gamma \in R$, consider the hyperplane $\gamma^\perp := \{\theta \in \mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Q}) \mid \theta(\gamma) = 0\}$. Note that $\alpha^\perp = \Theta$.

Definition 4.12. Consider the hyperplane arrangement in Θ given by

$$\mathcal{A}_\alpha = \left\{\beta^\perp \cap \alpha^\perp \mid \alpha = \beta + (\alpha - \beta) \text{ is a decomposition into two roots in } R^+\right\}.$$

The hyperplane arrangement $\mathcal{A}_\alpha$ determines a polyhedral wall-and-chamber decomposition of Θ, and the resulting (closed) cones form a complete fan in Θ. The interior of each top-dimensional cone in the fan of $\mathcal{A}_\alpha$ is the intersection of Θ with a connected component of the locus

$$\Theta_{\mathbb{R}} \setminus \bigcup_{\gamma^\perp \in \mathcal{A}_\alpha} \gamma^\perp.$$

The goal of this section is to show that the above fan is precisely the GIT fan.

Lemma 4.13. *If $\beta, \gamma \in R^+$ and $\langle \beta, \gamma \rangle < 0$, then $\beta + \gamma \in R^+$ as well.*

Proof. If either β or γ is real then this is true thanks to [31, Proposition 3.6(a)] (see also Proposition 5.1(c) of *op. cit.*): if β is a real root then the restriction of an integral representation to $\mathfrak{g}(\beta) \cong \mathfrak{sl}_2$ is a sum of finite-dimensional modules, and the adjoint representation is integrable. We give a purely combinatorial proof of the general case.

Let $\langle \beta, \gamma \rangle = -m < 0$. If γ is real and $m = -1$, then $\beta + \gamma$ is a reflection of β, so also a root; the same is true swapping β and γ. We may assume therefore that either $m \neq -1$ or β, γ are both imaginary.

Let $\eta^{(0)} := \beta + \gamma$, $\beta^{(0)} := \beta$, $\gamma^{(0)} := \gamma$. Inductively, let us apply a maximal sequence of simple reflections so that $\eta^{(j)} = s_{i_j}(\eta^{(j-1)})$, with $\eta^{(j-1)} < \eta^{(j)}$; this means that $\langle \eta^{(j-1)}, e_{i_j} \rangle > 0$.

Let $\beta^{(j)} := s_{i_j}(\beta^{(j-1)})$ and $\gamma^{(j)} := s_{i_j}(\gamma^{(j-1)})$. We claim that under this sequence $\beta^{(j)}, \gamma^{(j)}$ always remain positive. If, at some stage, $\beta^{(j)}$ is negative, then $\beta^{(j-1)} = e_{i_j}$. Then $-m = \langle \beta^{(j-1)}, \gamma^{(j-1)} \rangle = \langle e_{i_j}, \gamma^{(j-1)} \rangle$. But $\langle \eta^{(j-1)}, e_{i_j} \rangle \geq 1$ by assumption, so $\langle e_{i_j}, \gamma^{(j-1)} \rangle \geq -1$. Thus $m = 1$. In this case β and γ are both imaginary, which contradicts $\beta^{(j-1)} = e_{i_j}$.

Since $\beta^{(j)}$ and $\gamma^{(j)}$ are always positive roots with nonzero pairing, their sum is always connected and positive. So $\eta^{(j)}$ remains connected and positive. Eventually, this sequence must terminate (say at $\eta^{(k)}$). Then $\langle \eta^{(k)}, e_i \rangle \leq 0$ for all loop free vertices i, implying that $\eta^{(k)}$ is in the fundamental domain. This implies that $\beta + \gamma$ is an imaginary root. □

Given a tuple $D := (\alpha^{(1)}, \dots, \alpha^{(m)})$ of roots in R^+ we associate the quiver Q_D whose vertices are $1, \dots, m$, with $-\langle \alpha^{(i)}, \alpha^{(j)} \rangle$ arrows from i to j for $i \neq j$. Given a decomposition $\alpha = \alpha^{(1)} + \cdots + \alpha^{(m)}$, we associate this tuple and hence the quiver.

Lemma 4.14. *Suppose that $\alpha \in \Sigma_\theta$ has a decomposition $D : \alpha = \alpha^{(1)} + \cdots + \alpha^{(m)}$ into roots in R^+_θ. Then the associated quiver Q_D is connected.*

Proof. If $\{1, ..., m\} = I \cup J$ with I and J disconnected from each other in Q_D, then we get a decomposition $\alpha = \alpha_I + \alpha_J$, where $\langle \alpha_I, \alpha_J \rangle = 0$. This implies that $p(\alpha) < p(\alpha_I) + p(\alpha_J)$. Taking canonical decompositions of α_I and α_J, and applying [5, Lemma 7.3], we get a contradiction to the fact that $\alpha \in \Sigma_\theta$. □

Proposition 4.15. *For every decomposition $D : \alpha = \alpha^{(1)} + \cdots + \alpha^{(m)}$, with connected quiver Q_D, the intersection $\cap_i (\alpha^{(i)})^\perp$ equals an intersection of hyperplanes in $\mathcal{A}_\alpha$.*

The proof of this is based on an easy, purely combinatorial statement:

Lemma 4.16. *Let Q be a connected (undirected) graph with vertex set Q_0. For $J \subseteq Q_0$ define $e_J := \sum_{j \in J} e_j$, where $e_j \in \mathbb{Z}Q_0$ is the trivial path at vertex j. Then $\mathbb{Z}Q_0$ is spanned by the set*

$$S_Q := \{e_J \mid J \subseteq Q_0 \text{ is such that } J \text{ and } Q_0 \setminus J \text{ are connected}\}.$$

Proof. By induction on $|Q_0|$. Note that in a connected graph there is always a vertex $j \in Q_0$ which can be removed leaving a connected graph (this is obvious for a tree, and every connected graph has a spanning tree). Then e_j is in the set S_Q above. Let $j_0 \in Q_0$ be such a vertex. Define Q' to be the graph obtained from Q by deleting the vertex j_0 and all edges that have an endpoint at j_0. By induction $\mathbb{Z}Q'_0$ is spanned by $S_{Q'}$. But for each $J \subseteq Q'_0$ such that both J and $Q'_0 \setminus J$ are connected, either $J \cup \{j_0\}$ or $Q_0 \setminus J$ is connected, so e_J or $e_J + e_{j_0}$ is in $S_{Q'}$. Thus $\mathrm{Span}(S_Q)$ contains $\mathbb{Z} \cdot e_{j_0}$, while the quotient $\mathrm{Span}(S_Q)/\mathbb{Z} \cdot e_{j_0}$ contains $\mathrm{Span}(S_{Q'}) = \mathbb{Z}Q'_0$, so S_Q spans $\mathbb{Z}Q_0$. □

Proof of Proposition 4.15. Lemma 4.13 implies that for every $J \subset (Q_D)_0$ connected, the sum $\beta_J := \sum_{j \in J} \alpha^{(j)}$ is in R^+. In the case $e_J \in S_{Q_D}$ as in Lemma 4.16, we get that both β_J and $\alpha - \beta_J$ belong to R^+. Thus, β_J is the perpendicular vector to a hyperplane in $\mathcal{A}_\alpha$. Lemma 4.16, applied to Q_D, then says that the intersection of these hyperplanes $\beta_J^\perp$, for $e_J \in S_{Q_D}$, equals the intersection of the hyperplanes $(\alpha^{(i)})^\perp$ for $1 \leq i \leq m$, since intersecting hyperplanes produces the linear subspace perpendicular to the span of the normal vectors. □

Theorem 4.17. *Assume $\alpha \in \Sigma_0$ is indivisible. The GIT fan equals the fan given by the arrangement $\mathcal{A}_\alpha$.*

Proof. It suffices to show that the GIT walls are precisely the union of the hyperplanes in $\mathcal{A}_\alpha$. In other words, $\mathfrak{M}_\theta(\alpha)^s = \mathfrak{M}_\theta(\alpha)$ if and only if θ lies in the complement to the hyperplanes in $\mathcal{A}_\alpha$.

Assume that θ is a general element of $\gamma^\perp \in \mathcal{A}_\alpha$. Then there exists a positive root β such that $\alpha - \beta \in R^+$ and $\theta(\beta) = 0$. It is a consequence of [5, Theorem 1.3] that there exists a θ-polystable representation of dimension vector η for any $\eta \in \mathbb{N}R_\theta^+$. In particular, this implies that there exist θ-polystable representations M, N of dimension vector β and $\alpha - \beta$ respectively. The point $[M \oplus N] \in \mathfrak{M}_{\theta_0}(\alpha)$ is strictly θ-polystable. Hence $\gamma^\perp$ is a GIT wall.

Conversely, if $\theta \in \Theta$ lies on some GIT wall then, by definition, there exists a properly θ-polystable representation $M = M_1^{\oplus n_1} \oplus \cdots \oplus M_k^{\oplus n_k}$ with $\alpha^{(i)} := \dim M_i$ belonging to Σ_θ. Counting the $\alpha^{(i)}$ with multiplicity gives a decomposition D of α. Since $\alpha \in \Sigma_0$, the associated quiver Q_D is connected by Lemma 4.14. Then Proposition 4.15 implies that θ lies on some hyperplane in $\mathcal{A}_\alpha$. □

We note the following useful consequence of the proof of Theorem 4.17.

Corollary 4.18. *If $\alpha \in \Sigma_0$ is indivisible, then the quiver variety $\mathfrak{M}_\theta(\alpha)$ is nonsingular if and only if θ does not lie on any hyperplane in $\mathcal{A}_\alpha$.*

We may describe the GIT regions R_C more explicitly. By [5, Theorem 1.20], the walls in the boundary of R_C all lie in the hyperplanes $\beta^\perp$ where β is a *codimension two root*, meaning that there is a codimension two stratum $\mathfrak{M}_\theta(\alpha)_\tau$ where the dimension vectors $\beta^{(i)}$ appearing in the representation type τ are all rational combinations of α and β. In fact, in [6], we will show that this is equivalent to the condition that β and $\alpha - \beta$ are both roots, and $(\beta, \alpha - \beta) = -2$ (but we do not need this fact here). Then R_C is the closure of one of the complementary regions of these hyperplanes, namely, the one containing C. Conversely every such region can be used as R_C, and C can be taken to be any GIT chamber inside it.

5. Crepant resolutions of some threefold quotient singularities

We now show that our main results apply to projective, crepant resolutions of certain Gorenstein, threefold quotient singularities, including all polyhedral singularities. That our methods can be applied to threefolds emphasises the fact that our results do not in any way rely on the holomorphic symplectic structure of Nakajima quiver varieties.

5.1. **McKay quiver moduli spaces.** Let $\Gamma \subset \mathrm{SL}(3, \Bbbk)$ be a finite subgroup. The affine quotient singularity $\mathbb{A}^3/\Gamma := \operatorname{Spec} \Bbbk[\mathbb{A}^3]^\Gamma$ is a normal, Gorenstein threefold that admits a projective, crepant resolution. Rather than recall the construction of Bridgeland, King and Reid [11], it is convenient for our purpose to recall the more general construction appearing in [14, Section 2].

Let $\mathrm{Irr}(\Gamma)$ denote the set of isomorphism classes of irreducible representations of Γ, and write $R(\Gamma) = \bigoplus_{\rho \in \mathrm{Irr}(\Gamma)} \mathbb{Z}\rho$ for the representation ring of Γ. A Γ-*constellation* is a Γ-equivariant coherent sheaf F on $\mathbb{A}^3$ such that $H^0(F)$ is isomorphic to the regular representation $R = \bigoplus_{\rho \in \mathrm{Irr}(\Gamma)} R_\rho \otimes \rho$ as a

$\Bbbk[\Gamma]$-module. Note that $H^0(F)$ is a module over the skew group algebra $\Bbbk[\mathbb{A}^3]\rtimes\Gamma$ of dimension vector $(\dim\rho)_{\rho\in\mathrm{Irr}(\Gamma)}$, and conversely, the sheaf on $\mathbb{A}^3$ associated to any such module is a Γ-constellation. Consider the rational vector space

$$\Theta := \big\{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(\Gamma),\mathbb{Q}) \mid \theta(R) = 0\big\}.$$

For $\theta \in \Theta$, a Γ-constellation F is *θ-semistable* if every proper nonzero Γ-equivariant coherent subsheaf F' of F satisfies $\theta(F') := \theta(H^0(F')) \geq 0$; it is *$\theta$-stable* if these inequalities are strict. Two θ-semistable Γ-constellations are S-equivalent if their composition series agree in the abelian category of θ-semistable Γ-constellations. The space Θ supports a polyhedral fan characterised by the following property: $\theta \in \Theta$ lies in the interior of a top-dimensional cone if and only if every θ-semistable Γ-constellation is θ-stable, in which case we say θ is *generic* [14, Lemma 3.1]. In particular, the interior of every top-dimensional cone in the GIT fan is a *chamber*, so a *wall* is a codimension-one face of the closure of any chamber.

Let W denote the given three-dimensional representation of Γ, and consider the affine scheme $V = \{B \in \mathrm{Hom}_{\Bbbk[\Gamma]}(R, W\otimes R) \mid B\wedge B = 0\}$ parametrising Γ-constellations. Isomorphism classes of Γ-constellations correspond to orbits in V under the action of the group $G_\Gamma = \prod_{\rho\in\mathrm{Irr}(\Gamma)}\mathrm{GL}(\dim\rho)$ by change of basis on the summands of R. For any integer-valued $\theta \in \Theta$, consider the character $\chi_\theta \in {G_\Gamma}^\vee$ satisfying $\chi_\theta(g) = \prod_{\rho\in\mathrm{Irr}(\Gamma)}\det(g)^{\theta(\rho)}$ for $g \in G_\Gamma$. As in the construction by King [33], the GIT quotient

$$\mathcal{M}_\theta = V /\!\!/_{\chi_\theta} G_\Gamma$$

is the coarse moduli space of S-equivalence classes of θ-semistable Γ-constellations. The dimension vector $(\dim\rho)_{\rho\in\mathrm{Irr}(\Gamma)}$ is indivisible, so for any generic $\theta\in\Theta$, the GIT quotient $\mathcal{M}_\theta$ is the fine moduli space of Γ-constellations up to isomorphism.

The tautological family on $\mathcal{M}_\theta$ is a locally-free sheaf $\mathcal{R} = \bigoplus_{\rho\in\mathrm{Irr}(\Gamma)}\mathcal{R}_\rho\otimes\rho$ and a tautological Γ-equivariant homomorphism $\mathcal{R}\to W\otimes\mathcal{R}$, where $\mathcal{R}_\rho$ has rank $\dim(\rho)$. We normalise the family so that the summand indexed by the trivial representation is the trivial bundle; see [14, Section 2] for details.

Proposition 5.1. *Let $C \subseteq \Theta$ be a chamber and let $\theta \in C$. Then*

(i) *variation of GIT quotient given by sending $\theta \rightsquigarrow 0$ induces a projective crepant resolution $f_\theta\colon \mathcal{M}_\theta \to \mathbb{A}^3/\Gamma$ that sends each Γ-constellation to its supporting Γ-orbit; and*

(ii) *the linearisation map L_C is surjective.*

Proof. Part (i) is due to [11], though it appears in this form only in [14, Proposition 2.2, Theorem 2.5]; note that $\mathbb{A}^3/\Gamma$ is only an irreducible component of the affine quotient in general. Part (ii) appears in [14, Section 3.2], or more explicitly, as [13, Corollary 3.9]. □

5.2. **The linearisation map.** Our interest lies with those quotient singularities for which the linearisation map is an isomorphism. This property can be characterised in several ways as follows.

Lemma 5.2. *Let $\Gamma \subset \mathrm{SL}(3,\Bbbk)$ be a finite subgroup. The following statements are equivalent:*

(i) *every nontrivial conjugacy class of* Γ *is 'junior' in the sense of Ito and Reid* [30]*;*
(ii) *some (and hence any) projective crepant resolution* $f\colon X \to \mathbb{A}^3/\Gamma$ *has all fibres of dimension at most one;*
(iii) *for any GIT chamber* $C \subset \Theta$ *and* $\theta \in C$*, the moduli space* $\mathcal{M}_\theta$ *contains no proper surfaces;*
(iv) *for any GIT chamber* $C \subset \Theta$*, the linearisation map* L_C *is an isomorphism.*

Proof. Since $\mathbb{A}^3/\Gamma$ admits a projective crepant resolution $f\colon X \to \mathbb{A}^3/\Gamma$, condition (i) is equivalent by [30, Theorem 1.6] to the statement that X contains no proper f-exceptional prime divisors, which is equivalent to f having all fibres of dimension at most one. This holds for one crepant resolution if and only if it holds for all such [35, Corollary 3.54], so (i) and (ii) are equivalent. For any chamber $C \subset \Theta$ and $\theta \in C$, the morphism $f_\theta\colon \mathcal{M}_\theta \to \mathbb{A}^3/\Gamma$ is a projective crepant resolution by Proposition 5.1, so (ii) is equivalent to (iii). Finally, [13, Lemma 4.2] shows that the kernel of L_C is dual to a vector space spanned by the numerical classes of proper surfaces in $\mathcal{M}_\theta$. Thus, there are no such surfaces if and only if $\ker(L_C) = 0$. The result follows from Proposition 5.1(ii). $\square$

Example 5.3. A simple and much-studied example is that of the subgroup $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ in $\mathrm{SL}(3, \Bbbk)$ generated by the diagonal matrices $\mathrm{diag}(1, -1, -1)$ and $\mathrm{diag}(-1, -1, 1)$. The toric threefold $\mathbb{A}^3/\Gamma$ admits four projective, crepant toric resolutions, one of which has exceptional locus comprising three $(-1, -1)$-curves meeting at a point; the remaining three such resolutions are obtained by flopping one of these curves. All four of these resolutions can be constructed as fine moduli spaces of θ-stable Γ-constellations for some generic θ; see [12, Chapter 5] or [55, Example 3.4, Remark 7.5].

Example 5.4 (Polyhedral singularities). It is classical that every finite subgroup $\Gamma \subset \mathrm{SO}(3, \mathbb{R})$ is a cyclic group, a dihedral group, or the rotational symmetry group of either the tetrahedron, the octahedron or the icosahedron. The quotient singularity $\mathbb{A}^3/\Gamma$ is called *polyhedral singularity*. For each of these groups, Gomi, Nakamura and Shinoda [20, 21] showed that the Hilbert–Chow morphism for the Γ-Hilbert scheme (this is f_θ from Proposition 5.1(i) for θ as in [12, Proposition 5.19]) satisfies the condition from Lemma 5.2(ii). The crepant resolution is unique when Γ is cyclic, but for the dihedral and tetrahedral cases, Nolla de Celis and Sekiya [49] subsequently proved that every projective, crepant resolution of $\mathbb{A}^3/\Gamma$ is of the form $\mathcal{M}_\theta$ for some generic θ. Compare Remark 5.10.

5.3. **On GIT walls.** We now turn our attention to the GIT walls in Θ. For adjacent chambers C_+, C_- separated by a wall, variation of GIT quotient as in (2.6) gives a diagram

$$\begin{array}{ccc} \mathcal{M}_{\theta_-} & \overset{\psi}{\dashrightarrow} & \mathcal{M}_{\theta_+} \\ & {\scriptstyle \tau_-}\searrow \quad \swarrow{\scriptstyle \tau_+} & \\ & \overline{\mathcal{M}_{\theta_0}} & \end{array} \tag{5.1}$$

of schemes over $Y = \mathbb{A}^3/\Gamma$, where $\overline{\mathcal{M}_{\theta_0}}$ is the normalisation of $\mathcal{M}_{\theta_0}$ and where both τ_-, τ_+ have connected fibres. The proof of the next result builds on that of [14, Proposition 4.4].

Lemma 5.5. *Let* Γ *satisfy the equivalent conditions from Lemma 5.2. Suppose that* $\mathrm{Uns}(\tau_-)$ *has an irreducible component* D *of codimension-one. Then* D *is contracted by* τ_- *onto a curve.*

Proof. Lemma 5.2 implies that D is not proper, so the resolution $f_{\theta_-}\colon \mathcal{M}_{\theta_-} \to \mathbb{A}^3/\Gamma$ that sends each θ_--stable Γ-constellation to its supporting Γ-orbit contracts D onto a curve ℓ in $\mathbb{A}^3/\Gamma$. Let $\pi\colon \mathbb{A}^3 \to \mathbb{A}^3/\Gamma$ be the quotient map and consider a nonzero $x \in \pi^{-1}(\ell)$. Then $x \in \mathbb{A}^3$ has a non-trivial stabiliser Γ'. As in the proof of [11, Lemma 8.1], the restriction functor provides an equivalence from the category of Γ-constellations supported on the orbit $\Gamma\cdot x$ to the category of Γ'-constellations supported at $x \in \mathbb{A}^3$, and moreover, the restriction map that sends a character χ_θ of G_Γ to the character $\chi_{\theta'} := \operatorname{res}^{G_\Gamma}_{G_{\Gamma'}}(\chi_\theta)$ of $G_{\Gamma'}$ determines the $\mathbb{Q}$-linear map $\Theta \to \Theta_{\Gamma'}$ between the spaces of stability parameters for Γ- and Γ'-constellations. This compatibility implies in particular that the restriction of a θ-stable Γ-constellation supported on $\Gamma\cdot x$ is a θ'-stable Γ'-constellation supported on x. Thus, if we write $f_{\theta'}\colon \mathcal{M}_{\theta'}(\Gamma') \to \mathbb{A}^3/\Gamma'$ for the morphism sending each Γ'-constellation to its supporting Γ'-orbit, and $E_\ell := (f_{\theta'})^{-1}(\ell)$ for the preimage of ℓ, then the restriction functor identifies $f_{\theta_-}|_D\colon D \to \ell$ with $f_{\theta'}|_{E_\ell}\colon E_\ell \to \ell$.

This description of $f_{\theta_-}|_D$ allows us to study $\tau_-|_D$. Indeed, the action of Γ' fixes $x \in \mathbb{A}^3 \smallsetminus \{0\}$, so we may choose coordinates with $\Gamma' \subset \mathrm{SL}(2,\Bbbk)\times \mathsf{id} \subset \mathrm{SL}(3,\Bbbk)$. Since $f_{\theta'}$ is a crepant resolution, we have that $f_{\theta'} = f \times \mathsf{id}_{\mathbb{A}^1}$ where f is the minimal resolution of an ADE singularity. The morphism f_{θ_-} is obtained by varying the stability parameter to zero, so f_{θ_-} factors via τ_-. The restriction functor identifies $\tau_-|_D$ with the restriction of $\tau'\colon \mathcal{M}_{\theta'}(\Gamma') \to \mathcal{M}_{\theta'_0}(\Gamma')$ to E_ℓ, where $\theta'_0 \in \Theta_{\Gamma'}$ is determined by the character $\chi_{\theta'_0} := \operatorname{res}^{G_\Gamma}_{G_{\Gamma'}}(\chi_{\theta_0})$ of $G_{\Gamma'}$. The parameter θ'_0 is generic in a wall of the chamber containing θ' since the same is true for $\theta_0 \in \Theta$ by assumption, so by Kronheimer [37], τ' is the product of $\mathsf{id}_{\mathbb{A}^1}$ with the contraction of at least one (-1)-curve. It particular, $\tau'|_{E_\ell}$ contracts a divisor to a curve, and hence so too does $\tau_-|_D$. □

The diagram (5.1) allows us to classify GIT walls into four types. Recall from (2.5) that $L_{C_-}(\theta_0)$ is the semi-ample line bundle that determines the morphism τ_-. Then either $L_{C_-}(\theta_0)$:

- is ample, in which case τ_- is an isomorphism and we say that the wall is of *type* 0; or it
- defines a class on the boundary of the ample cone of $\mathcal{M}_{\theta_-}$, and since θ_0 is general in the wall, this class lies in the interior of a codimension-one face of $\mathrm{Amp}(\mathcal{M}_{\theta_-}/Y)$ and hence τ_- is a primitive contraction. In this case, we say that the wall is:
  - of *type* I if τ_- contracts a curve to a point;
  - of *type* III if τ_- contracts a surface to a curve.

In principle, the morphism τ_- might contract a surface to a point - a *type* II contraction - but that surface would necessarily be proper, thereby contradicting Lemma 5.2.

Since $\mathcal{M}_{\theta_-}$ and $\mathcal{M}_{\theta_+}$ are both crepant resolutions of $\mathbb{A}^3/\Gamma$, the type of a wall is independent of whether we replace τ_- by τ_+ throughout the above. In short, the type is independent of the side from which we approach the wall.

Lemma 5.6. *Let Γ satisfy the conditions of Lemma 5.2. For any wall of type* I*, the unstable loci* $\mathrm{Uns}(\tau_-) \subseteq \mathcal{M}_{\theta_-}$ *and* $\mathrm{Uns}(\tau_+) \subseteq \mathcal{M}_{\theta_+}$ *each have codimension at least two.*

Proof. If $\mathrm{Uns}(\tau_-)$ had an irreducible component D of codimension-one, then Lemma 5.5 shows that τ_- contracts D. However, τ_- contracts only a curve, a contradiction. The τ_+ case is identical. □

Proposition 5.7. *Let Γ satisfy the conditions of Lemma 5.2. There are no GIT walls of type* 0*.*

Proof. Suppose for a contradiction that chambers $C_-, C_+ \subset \Theta$ are separated by a type 0 wall. For $\theta_- \in C_-$ and $\theta_+ \in C_+$, both τ_- and τ_+ from (2.6) are isomorphisms, and hence so is the rational map ψ from (2.6), but the tautological families agree only on the locus $\mathcal{M}_{\theta_-} \setminus \mathrm{Uns}(\tau_-) \cong \mathcal{M}_{\theta_+} \setminus \mathrm{Uns}(\tau_+)$. Since $\mathcal{M}_{\theta_-} \cong \mathcal{M}_{\theta_+}$ is normal, the locus $\mathrm{Uns}(\tau_-) \cong \mathrm{Uns}(\tau_+)$ where the tautological families differ cannot have an irreducible component of codimension at least two, otherwise these tautological families would extend uniquely over that component [26, Proposition 1.6], forcing them to agree beyond $\mathcal{M}_{\theta_-} \setminus \mathrm{Uns}(\tau_-) \cong \mathcal{M}_{\theta_+} \setminus \mathrm{Uns}(\tau_+)$. Thus, every irreducible component of $\mathrm{Uns}(\tau_-) \cong \mathrm{Uns}(\tau_+)$ is of codimension-one. However, if there were such a component, Lemma 5.5 shows that it would be contracted by τ_-, a contradiction. □

In passing, we record the following fact for groups Γ that do not satisfy Lemma 5.2.

Lemma 5.8. *If a finite subgroup $\Gamma \subset \mathrm{SL}(3, \Bbbk)$ fails to satisfy the conditions from Lemma 5.2, then every chamber C whose closure is strongly convex has a wall of type* 0*.*

Proof. The linearisation map L_C is surjective by Proposition 5.1, so the kernel of L_C must be nonzero by Lemma 5.2. Since $\overline{C}$ is strongly convex, Corollary 3.22 implies that C has a wall such that the VGIT morphism $\tau\colon X_\theta \to X_{\theta_0}$ into the wall is an isomorphism. This wall is of type 0. □

5.4. **Birational geometry.** We can now state and prove the main result of this section.

Theorem 5.9. *Let $\Gamma \subset \mathrm{SL}(3, \Bbbk)$ satisfy the equivalent conditions from Lemma 5.2. The conclusions of Theorem 3.12 hold for any chamber C and any projective crepant resolution $f\colon X \to Y = \mathbb{A}^3/\Gamma$.*

Proof. Let $C \subset \Theta$ be any GIT chamber. For $\theta \in C$, we know $\mathcal{M}_\theta$ is smooth by Proposition 5.1, and the linearisation map L_C is an isomorphism by Lemma 5.2, so Condition 3.4(1) holds for C. Next, consider any wall in the interior of the GIT region R_C containing C. The wall cannot be of type 0 or II by Proposition 5.7, nor can it be of type III because interior walls of R_C must induce small contractions. Therefore, the wall must be of type I, so τ_- and τ_+ each contract a curve to a point. Lemma 5.6 shows that every such wall satisfies the assumptions of Proposition 2.8. It follows that every interior wall of R_C is flipping, so Condition 3.4(2) holds. Finally, given a boundary wall of R_C, the only possibility left is that the wall is of type III. In particular, the morphism τ_- for that wall contracts a (necessarily nonproper) divisor to a curve, so the wall is of divisorial type. Thus, Condition 3.4(3) holds for the chamber C, so the conclusions of Theorem 3.12 hold for the specific projective crepant resolution $f_\theta\colon \mathcal{M}_\theta \to \mathbb{A}^3/\Gamma$, where $\theta \in C$. These same conclusions must therefore also hold for the chamber $C_X := L_C^{-1}(\mathrm{Amp}(X/Y))$ that defines $X \cong \mathcal{M}_\theta$ for $\theta \in C_X$. □

Remark 5.10. Theorem 5.9 implies in particular that every projective crepant resolution of $\mathbb{A}^3/\Gamma$ is of the form $\mathcal{M}_\theta$ for some generic θ. As noted in the introduction, this statement follows from the work of Wemyss [55, Theorem 6.2], which generalised the study of dihedral and trihedral singularities by Nolla de Celis and Sekiya [49, Corollaries 1.3 and 1.5]. Our direct, geometric proof bypasses the algebraic approach via mutation introduced in [55], while our description of the relative movable cone $\mathrm{Mov}(X/Y)$ follows from Theorem 3.12. In fact, our approach shows that for any chamber C, it is not hard to say which should be the next wall to crash through to induce any given flop of $\mathcal{M}_\theta$ for $\theta \in C$: one simply chooses the wall of $\overline{C}$ that's identified by L_C with the given flopping wall of the nef cone of $\mathcal{M}_\theta$ for $\theta \in C$.

6. Hypertoric varieties

We now show that our main results also apply to the class of nonsingular hypertoric varieties, leading to a proof of Theorem 1.6.

6.1. GIT construction.

Hypertoric varieties were originally constructed as hyperkähler quotients by Bielawski and Dancer [7], when they were known as toric hyperkähler varieties. Here we recall their construction as holomorphic symplectic varieties by GIT following Hausel and Sturmfels [27] (see also Konno [36]).

For $n, r \in \mathbb{N}$ with $r < n$, consider the action of the algebraic torus $G := (\Bbbk^\times)^r$ on the complex symplectic vector space $T^*\Bbbk^n = \Bbbk^n \times (\Bbbk^n)^*$, where the matrix that records the weights of the action is of the form $(A, -A)$, where A is an $r \times n$ integer-valued matrix whose columns $a_1, \dots, a_n$ span $\mathbb{Z}^r$. Note that this forces the $r \times r$-minors of A to be relatively prime. The G-action is Hamiltonian for the natural symplectic structure on $T^*\Bbbk^n$, and the induced moment map $\mu\colon T^*\Bbbk^n \to \mathfrak{g}^*$ satisfies

$$\mu(z, w) = \sum_{i=1}^{n} z_i w_i \cdot a_i.$$

Choose an integer $n \times (n - r)$ matrix B forming the short exact sequence

$$0 \to \mathbb{Z}^{n-r} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^r \to 0.$$

If no row of the matrix B is zero (equivalently, when the torus G contains no dilations along a single axis) then the locus $\mu^{-1}(0)$ is an affine variety by [4, Lemma 4.7], and for any character $\theta \in G^\vee$, the corresponding *hypertoric variety* is defined to be

$$X_\theta := \mu^{-1}(0) /\!\!/_\theta\, G.$$

Recall that a matrix A is said to be *unimodular* if all of the non-zero $r \times r$ minors of A belong to $\{-1, 0, 1\}$ (equivalently, the $(n - r) \times (n - r)$-minors of B belong to $\{-1, 0, 1\}$). Under the assumption that no row of the matrix B is zero, it is shown in [27, Proposition 6.2] that X_θ is nonsingular for general θ if and only if A is unimodular. Note in addition, that the interior of every top-dimensional cone in the GIT fan is a chamber, because G is a torus [19, Corollary 4.1.10].

6.2. Applying the main result.

In order to apply our Theorem 3.12 to nonsingular hypertoric varieties, we show that Condition 3.4 holds. Much of the heavy lifting was done by Konno [36].

Theorem 6.1. *Assume that A is unimodular and no row of the matrix B is zero, so the hypertoric variety X_θ is nonsingular for general θ. Then Condition 3.4 holds for every chamber C, and hence*

Theorem 3.12 applies. In particular, every projective crepant resolution of X_0 is a hypertoric variety X_θ for some generic θ.

Proof. Let C be a chamber and $\theta \in C$. Since X_θ is nonsingular, [54, Theorem 1.1] shows that there exist $\eta_1, \dots, \eta_k \in \Theta$ such that $\mathcal{T} = \bigoplus_{i=1}^k L_C(\eta_i)$ is a tilting bundle on X_θ. This implies that the line bundles $L_C(\eta_i)$ for $1 \leq i \leq k$ span the Grothendieck group $K_0(X_\theta)$. Since $\det\colon K_0(X_\theta) \to \operatorname{Pic}(X_\theta)$ is surjective, we deduce that L_C is surjective. We note that the closed cone $\overline{C}$ is strongly convex by Lemma 3.20 because $\mu^{-1}(0)$ has a G-fixed point. Thus, Corollary 3.24(1) holds.

Parts (2) and (3) of Corollary 3.24 follow directly from the work of Konno [36, Theorem 6.4] since condition (3.6) of [36] is equivalent to A being unimodular. In particular, it is shown there that every boundary wall is divisorial. We deduce from Corollary 3.24 that Condition 3.4 holds. □

Proof of Theorem 1.6. This is immediate from Theorem 6.1. □

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