# Clones over Finite Sets and Minor Conditions 

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Col tuo ordine discreto dentro il cuore

Fabrizio De André

# Acknowledgements 

( a letter to the reader)

Dear reader,
by the time you read this letter, my thesis will have been submitted, irrevocably.
It is customary to begin a dissertation with a section - of arbitrary length - that includes a list of people you wish to thank as you consider them an integral part of the journey. Who am I to infringe such a tradition?

I do not deny that while writing this part I thought back to what I have experienced over these wonderful years. For some of the results I am going to present to you, I even remember the exact day I obtained them and the people I had around me or in my thoughts. So, I only ask you for a little patience before you start Chapter 1 I am about to proceed to list some people whom I consider to be pivotal over the past few years. If your name, dear reader, does not turn out to be among them, do not despair: I am sure that if you are here reading these pages, you have played a role in this story. But in case you don't know me at all and are here solely because you are looking for a result that you think might be here, then skip this part and go directly to the next chapter. There is no time to waste! We don't even know half of the mysteries of this world. We are wanderers in darkness.

Remark 0.0.1. Yes, this is the emotional part of the dissertation.
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[^0]A Silvia. Here, the acknowledgements were fragmented into seven different parts. The interested reader therefore has only to set out in search of them, but please note that - just like a Horcrux - these may be anywhere and anything. One of these fragments consists of exactly the lines you are reading in this paragraph, including this word. The second one can be found in the booklet given to Silvia to celebrate the defence of her PhD . The remaining ones? Look for them, if you want.

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## Chapter 1

## Introduction

The main goal of this dissertation is to initiate and lay the basis for a systematic study of the so-called pp-constructability poset, the study of which is motivated by the serendipitous connection between Constraint Satisfaction Problems (CSPs) and universal algebra. In particular, it has been relatively recently proved that the complexity of the CSP of a fixed finite relational structure depends only on particular identities (universally quantified equations) satisfied by its polymorphism clone (see [18, 38]).

## The origins

Clones have undoubtedly played a prominent role as an object of study in universal algebra; researchers have used different approaches in order to study clones, making use of many diverse techniques from fields such as combinatorics, set theory or topology. However, there is one approach that towers above the rest as it is the cord of many fundamental results in clone theory: it is the Galois connection Pol-Inv between operations and relations, based on the notion of preservation [30, 53, 90].

We could historically locate the initiation of the systematic study of clones already in the 1940s when Post completely described the lattice of Boolean clones ordered up to inclusion [91]: the so-called Post's lattice is countably infinite and enjoys a mirror symmetry given by clones that are dual to each other, see Figure 5.1. Prompted by the enthusiasm of this result, researchers tried to obtain a similar description for clones on sets with more than two elements. This wave of research reached its peak in 1959 when Janov and Mučnik [61] presented an equally famous result in clone theory: there exists a continuum of clones over a $k$-element set for $k \geq 3$. The goal to achieve a result à la Post seemed to falter, subsequent research in universal algebra therefore focused on understanding particular aspects of clone lattices on finite domains, for example on the description of maximal clones [60, 92] or minimal clones [42, 43, 93].

One might still hope to classify all operation clones on finite domains up to some

[^1]equivalence relation so that equivalent clones share many of the properties that are of interest in universal algebra. Perhaps the most important equivalence relation on clones is homomorphic equivalence: a function from $\mathcal{C}$ to $\mathcal{D}$ is called a clone homomorphism if it preserves the arity of the operations, composition, and maps the $i$-th projection of arity $n$ in $\mathcal{C}$ to the $i$-th projection of arity $n$ in $\mathcal{D}$; two clones $\mathcal{C}$ and $\mathcal{D}$ are called homomorphically equivalent if there exists a clone homomorphism from $\mathcal{C}$ to $\mathcal{D}$ and vice versa. An attractive feature of homomorphic equivalence is that it also relates clones on different domains.

A research strand that has developed in parallel with clone theory is the study of varieties and strong Mal'cev conditions, i.e., primitive positive sentences in the language of clones. In 1974, Neumann [80] defined the notion of interpretability between varieties, and introduced the lattice of interpretability types of varieties formed by the class of all varieties under the quasi-ordering $\mathcal{V} \leq_{\iota} \mathcal{W}$ if and only if $\mathcal{V}$ is interpretable in $\mathcal{W}$. Indeed, such a quasi-order generates a lattice $\mathfrak{L}_{\text {Var }}$ that is in a sense the natural setting in which to attack open problems related to Mal'cev conditions and has therefore been the subject of study by many researchers both in the 1980s (see, e.g., [51, 58]) and more recently (see, e.g., [69, 96]).

There is a rather natural link between the notions of clone homomorphism and interpretability of types of varieties: a variety $\mathcal{V}$ is interpretable in a variety $\mathcal{W}$ if and only if there exists a clone homomorphism from $\operatorname{Clo}(\mathcal{V})$ to $\operatorname{Clo}(\mathcal{W})$; in particular, $\mathcal{W}$ satisfies all the identities that hold in $\mathcal{V}$ (see Chapter 4). The origin of the research on Mal'cev conditions is closely linked to the study of congruence properties, and one reason might be traced back to the following: in the light of Birkhoff's theorem [21], it became clear that in order to understand varieties, it was necessary to study products, subalgebras, and factors of algebras. While products and subalgebras are relatively easy to handle, the same cannot be said of algebras obtained by factoring by congruences. Thus, algebraists began to study the properties of congruences in order to understand factors of algebras. Concerning the case of locally finite varieties, one can say that the peak of the systematic study of Mal'cev conditions was reached with the foundation of Tame Congruence Theory by Hobby and McKenzie [58]; surprisingly, Kearnes and Kiss [69] generalized many of the results of Hobby and McKenzie without the assumption of local finiteness. Over the recent past, it turned out that Mal'cev conditions play a key-role also in a different context, not related to the study of congruence properties, but rather to the complexity of some specific decision problems known as Constraint Satisfaction Problems (CSPs).

## The Constraint Satisfaction Problem

Constraint Satisfaction Problems (CSPs) provide a common framework for expressing a vast rang $\epsilon^{2}$ ] of both theoretical and real-life combinatorial problems [94]. As a matter

[^2]of fact, the satisfiability problem for systems of linear equations over a Galois field or the colouring-problem for graphs can be expressed as CSPs of suitable finite relational structures; similarly, CSPs can also express widespread problems in theoretical computer science or artificial intelligence such as scheduling, optimisation, and computational biology, just to mention a few. In short, a CSP is a problem where as an input we get a finite set of variables and a finite set of constraints that are imposed on the variables, and the task is to assign values to the variables such that all the given constraints are satisfied. In its logical formulation, the Constraint Satisfaction Problem of a given finite relational structure $\mathbb{A}$, in short $\operatorname{CSP}(\mathbb{A})$, is the computational problem of deciding whether a given conjunction of atomic formulae over the signature of $\mathbb{A}$ holds in $\mathbb{A}$. The question that naturally arises is: for a given finite relational structure $\mathbb{A}$, what is the computational complexity of $\operatorname{CSP}(\mathbb{A})$ ? What properties enjoyed by $\mathbb{A}$ ensure that $\operatorname{CSP}(\mathbb{A})$ is tractable (also, "in P"), i.e., solvable in polynomial time? The first result we can report in this vein and which provided a complete answer for relational structures on a two-elements set is due to Schaefer [95]: for every Boolean structure $\mathbb{A}$, either $\operatorname{CSP}(\mathbb{A})$ is in $P$ or it is NP-complete. Additionally, a concrete criterion was presented to distinguish which of the two cases applies depending on which operations are polymorphisms of $\mathbb{A}$, that is, operations preserving all the relations in $\mathbb{A}$. Subsequently, Hell and Nešetřil [57] proved a similar result for undirected graphs. Again, only two different complexity classes (assuming $\mathrm{P} \neq \mathrm{NP}$ ) show up: for every finite undirected graph $\mathbb{G}$ either $\operatorname{CSP}(\mathbb{G})$ is in P or NP-complete. In the light of these results, Feder and Vardi [49] conjectured that such a dichotomy is enjoyed by all finite relational structures. Note that this would mean that, despite the wide range of problems that CSP covers, the NP-intermediate problems à la Ladner [74] do not occur in CSP. In the literature, the Feder and Vardi conjecture is often referred to as the Dichotomy Conjecture or the Tractability Conjecture.

In the last two decades, research in theoretical computer science has been moving towards this research line and yielded quite a number of results: the conjecture was indeed confirmed for structures with three elements [34] (extending Schaefer's result), for the class of smooth digraphs [15] (extending the result of Hell and Nešetřil), and for the class of finite relational structures containing all unary relations [6, 36].

Universal algebra turned out to be a somewhat unexpected "ally" that catalysed the process which eventually led to a series of results culminating in the complete resolution of the Dichotomy Conjecture [37, 106, 107] and made this research field fertile.

## Back to the future: the algebraic approach

It follows from a result of Bulatov, Jeavons, and Krokhin [38] that the complexity of $\operatorname{CSP}(\mathbb{A})$ depends only on set of identities of a particular form that are satisfied by the polymorphisms of $\mathbb{A}$. The latter result, together with the fact that the set $\operatorname{Pol}(\mathbb{A})$ of all polymorphisms of any finite relational structure $\mathbb{A}$ is a clone, suggested that in order to solve the Dichotomy Conjecture it would be possible to make use of the rich theory of universal algebra that focused on studying the connection between identities and "good
algebraic properties", thus giving a new impetus to the Pol-Inv Galois connection. This marked the beginning of the so-called algebraic approach to the CSP which culminated in an article by Barto, Opršal, and Pinsker [18]: the authors introduced the notion of $p p$-constructability which gathered together, in a unique reduction, all the previous unrelated methods that allow to reduce the CSP of a given structure to the CSP of another structure. More precisely, if a finite structure $\mathbb{A} p p$-constructs a finite structure $\mathbb{B}$, then there is a $\log$-space reduction from $\operatorname{CSP}(\mathbb{B})$ to $\operatorname{CSP}(\mathbb{A})$. Interestingly, the notion of pp-constructability is connected to a weakening of the notion of clone homomorphism, known in the literature as minor-preserving map or minion homomorphism. In fact, it was proved in [18] that, for every pair of finite relational structures $\mathbb{A}$ and $\mathbb{B}$, it holds that $\mathbb{A}$ pp-constructs $\mathbb{B}$ if and only if there exists a minor-preserving map from $\operatorname{Pol}(\mathbb{A})$ to $\operatorname{Pol}(\mathbb{B})$ [18]. Two clones $\mathcal{C}$ and $\mathcal{D}$ such that there exists a minor-preserving map from $\mathcal{C}$ to $\mathcal{D}$ and vice-versa are called minor-equivalent, and we write $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{D}$. As it turned out, minor-equivalent clones need not satisfy the same Mal'cev conditions, with the exception of those identities where exactly one operation symbol appears in both sides of the equality. Such conditions are known in the literature as minor conditions or height 1 conditions and are tightly related to what is often referred to in the literature as linear Mal'cev condition: the only difference is that, in minor conditions, identities of the form $f\left(x_{1}, \ldots, x_{n}\right)=y$ are not allowed since there is no operation symbol occurring on the right-hand side of the equality; given this, one could say that a linear Mal'cev identity is an identity of height at most 1. Again in [18], the authors provided an analogue of Birkhoff's HSP theorem for classes of algebras described by minor conditions. Thus, the algebraic approach led to a reformulation of the Dichotomy Conjecture in terms of a purely algebraic statement, and this indeed turned out to be a key step. In 2017, Bulatov and Zhuk independently provided a positive resolution to the Dichotomy Conjecture which we can now finally call the CSP Dichotomy Theorem. In the literature, there are several equivalent ways of describing the exact borderline between tractability and hardness; below we present one of them:

Theorem 1.0.1 (CSP Dichotomy Theorem, [37, 106, 107]). If a finite relational structure $\mathbb{A}$ has a weak near-unanimity polymorphism, i.e., there exists an n-ary operation $w$ in $\operatorname{Pol}(\mathbb{A})$, for some $n \geq 3$, satisfying

$$
w(x, \ldots, x, y) \approx w(x, \ldots, x, y, x) \approx \cdots \approx w(y, x, \ldots, x)
$$

then $\operatorname{CSP}(\mathbb{A})$ is solvable in polynomial time; otherwise, it is NP-complete.
Besides the prestigious achievement of having finally solved a conjecture that had resisted for twenty years, the proof of Theorem 1.0 .1 not only gives us a concrete borderline but also provides an algorithm for solving $\operatorname{CSP}(\mathbb{A})$ when this is tractable. Moreover, one of the greatest points in favour of this theorem is that the achievement of this result has led as a byproduct to the development of new algebraic theories that aim to better understand finite algebras. Theories such as Absorption Theory [14], Edge-coloring [37],

Strong Subalgebras [108], and Minimal Taylor Algebras [9] go in this direction and add to a list of already established theories with the same purpose such as Commutator Theory (see, e.g., [50, 69]) or Tame Congruence Theory [58].

It is indeed in this fruitful context that the main object of study of this dissertation, i.e., the pp-constructability poset is rooted. Since pp-constructability is a reflexive and transitive relation, it is a quasi-order on the class of all finite relational structures [18], hence it makes sense to write $\mathbb{A} \leq_{\text {Con }} \mathbb{B}$ if $\mathbb{A}$ pp-constructs $\mathbb{B}$ and to consider the induced equivalence relation $\mathbb{A} \equiv_{\text {Con }} \mathbb{B}$ if and only if $\mathbb{A} \leq_{\text {Con }} \mathbb{B}$ and $\mathbb{B} \leq_{\text {Con }} \mathbb{A}$. The poset that arises by considering the $\equiv_{\text {Con }}$-classes, also known as pp-constructability types, of finite relational structures ordered up to $\leq_{\text {Con }}$ is called the pp-constructability poset and denoted by $\mathfrak{P}_{\mathrm{fin}}$. Note that, via the Pol-Inv Galois connection and the main result from [18], it follows that $\mathfrak{P}_{\text {fin }}$ is isomorphic to the poset that one obtains considering $\equiv_{\mathrm{m}}$-classes of clones over a finite set, ordered by the existence of minor-preserving maps. As we have already mentioned, the study of this poset finds motivation both from CSP and universal algebra. The homomorphism order on clones has been studied intensively by Garcia and Taylor [51] and it is, in a certain sense, the natural context in which Mal'cev conditions and primeness of Mal'cev conditions can be rigorously described (see Chapter 4. Analogously, the pp-constructability poset is the natural environment to the study of minor conditions.

## Contributions and outline of the dissertation

The results we have already achieved suggest that this area of research may be fruitful and may lead to a classifications of a flavour similar to Post's result [91] of the two principal objects of study of this dissertation: $\mathfrak{P}_{\mathrm{fin}}$ and $\mathfrak{P}_{n}$, where $\mathfrak{P}_{n}$ is the subposet of $\mathfrak{P}_{\text {fin }}$ arising from considering only clones over an $n$-element set.

In Chapter 2 we fix notation and present the mathematical background that is necessary for an understanding of the results presented throughout the dissertation; this chapter is purely intended to give an overview of classic and more recent results in universal algebra and CSP.

Chapter 3 is more goal-oriented and focuses on the pp-constructability poset $\mathfrak{P}_{\text {fin }}$. First, following the article The wonderland of reflections [18], we provide a formal definition of the poset. Later, we prove results on the "general shape" of $\mathfrak{P}_{\mathrm{fin}}$; in particular, we describe the top element, the bottom element, and prove that $\mathfrak{P}_{\mathrm{fin}}$ is a semilattice which has a unique coatom and it has no atoms. On the other hand, it turns out that for every $n$, the poset $\mathfrak{P}_{n}$ has atoms and every atom is the pp-constructability type of some minimal Taylor clone [9] over $\{0, \ldots, n-1\}$. We conclude Chapter 3 presenting several open problems which are both directly and intrinsically related to $\mathfrak{P}_{\text {fin }}$.

Chapter 4] is a collection of results from the literature (mainly from [58] and [69]) concerning Mal'cev conditions. Therefore, this chapter has an almost independent interest from the rest of the dissertation and, in principle, could be skipped without compromising the reader's understanding of the subsequent chapters significantly. However, in this
chapter we define most of the Mal'cev conditions which will then be used in Chapter 5 and Chapter 6 to prove the separations in $\mathfrak{P}_{2}$ and in $\mathfrak{P}_{3}$. In addition, the author is not aware of a survey in the literature covering such a large number of Mal'cev conditions ordered by strength, thus Chapter 4 might be useful to researchers from different areas. The impatient reader could then directly read Chapter 5 and Chapter 6 and return to consult Chapter 4 only when some definition is needed. The author wishes that the "map" illustrated in Figure 4.2 will be useful for the reader's orientation, too.

In Chapter 5 we deal with the Boolean case of the pp-constructability poset, that is, we provide a full description of $\mathfrak{P}_{2}$. Furthermore, we use $\mathfrak{P}_{2}$ to present the various complexity regimes of Boolean constraint satisfaction problems that were described by Allender, Bauland, Immerman, Schnoor, and Vollmer [3].

Chapter 6] is entirely devoted to the three-element case. We prove that $\mathfrak{P}_{3}$ has exactly three submaximal elements. The proof we present is of independent interest in universal algebra. Indeed, we prove that if $\mathcal{C}$ is a clone over a three-element set such that $\mathcal{C}$ has a binary symmetric operation, a 3 -cyclic operation, and a Mal'cev operation, then $\mathcal{C}$ also has: a fully symmetric majority operation, totally symmetric operations of every arity $n \geq 2$ and an oddition, i.e., a generalization of the minority operation over $\{0,1\}$, for every odd arity $l \geq 3$. Moreover, we present a complete description of the downset generated by one of the three aforementioned submaximal elements. More precisely, we present a full description of the lattice that arises by considering clones of self-dual operations ordered by the existence of minor-preserving maps. It follows from our description that the latter lattice is countable, a result that is auspicious towards obtaining a complete description of $\mathfrak{P}_{3}$.

Finally, in Chapter 7, we present a list of open problems. The aim is to suggest a possible direction of research that could be beneficial for a better comprehension of clones over finite sets and, as a byproduct, of the pp-constructability poset.

## Chapter 2

## Preliminaries

In this chapter we present notation, definitions and some basic results from the literature which we are going to use throughout the dissertation. The main aim is to introduce the reader to several concepts of universal algebra that have many applications in theoretical computer science, in particular in the classification of the complexity of CSPs. The link between CSPs and universal algebra revolves around two celebrated Galois connections: the Mod-Th Galois connection [21] between varieties and sets of identities (Section 2.1), and the Pol-Inv Galois connection [30, 53] between finite relational structures and finite algebras (Section 2.3).

### 2.1 Universal algebra

We assume the reader to be familiar with naive set theory and the elementary theory of functions. This section is supposed to provide a short introduction that can hopefully help the reader become familiar with all the ingredients needed in order to introduce the Mod-Th Galois connection and the Inv-Pol Galois connection. For a more detailed introduction to universal algebra we refer to the textbooks [20, 40].

### 2.1.1 Algebras

A type of algebras is a set $\tau$ of function symbols, where each symbol is associated with a positive natural number, called the arity of the symbol.

Definition 2.1.1. An algebra $\mathbf{A}$ of type $\tau$ is an ordered pair $\mathbf{A}:=\left(A ; F^{\mathbf{A}}\right)$ where $A$ is a non-empty set called the universe of $\mathbf{A}$, and $F^{\mathbf{A}}$ is a set of finitary operations on $A$ indexed by symbols in $\tau$ such that, for every $n$-ary function symbol $f \in \tau$, there is an $n$-ary operation $f^{\mathbf{A}}$ on $A$. We call the elements of $F^{\mathbf{A}}$ basic operations of $\mathbf{A}$.

We follow the convention that an algebra is denoted by a capital letter in bold-style, while its universe is denoted by the same capital letter in italics. Moreover, if $A$ is a finite set, we say that the algebra $\mathbf{A}$ is finite. Throughout the whole dissertation we
work with finite algebras. If $F=\left\{f_{1}, \ldots, f_{n}\right\}$, we write $\mathbf{A}=\left(A ; f_{1}^{\mathbf{A}}, \ldots, f_{n}^{\mathbf{A}}\right)$ instead of $\mathbf{A}=\left(A ; F^{\mathbf{A}}\right)$. If $f$ is a function symbol of arity $n$, we say that $f^{\mathbf{A}}$ is an $n$-ary function; if $n=1, n=2$, or $n=3$ we say that $f^{\mathbf{A}}$ is unary, binary, or ternary, respectively. Also, when $\mathbf{A}$ is clear from the contest, we prefer to write $f$ instead of $f$.

Another important notational convention in this dissertation is that if $f: A^{k} \rightarrow A$ is an operation on $A$ and $t_{1}, \ldots, t_{k} \in A^{m}$, then $f\left(t_{1}, \ldots, t_{k}\right)$ denotes the tuple in $A^{m}$ obtained from applying $f$ componentwise. We also use the convention that if $f: A \rightarrow B$ is a function and $S \subseteq A$, then $f(S)$ denotes the set $\{f(a) \mid a \in S\}$ and $f^{-1}(S)$ denotes the set $\left\{f^{-1}(a) \mid a \in S\right\}$. Note that the two conventions can be combined.

Example 2.1.2. Let us provide some examples of algebras. In particular, we define the notions of semilattice and lattice; also, we establish some terminology that we will use extensively throughout the dissertation.

- Let $\tau$ be a type of algebras that consists of a unique binary function symbol $\wedge$. A semilattice is an algebra $\mathbf{S}=\left(S ; \wedge^{\mathbf{S}}\right)$ of type $\tau$ such that for all $a, b, c \in S$

$$
\begin{aligned}
a \wedge^{\mathbf{S}} a & =a ; \\
a \wedge^{\mathbf{S}} b & =b \wedge^{\mathbf{S}} a ; \\
a \wedge^{\mathbf{S}}\left(b \wedge^{\mathbf{S}} c\right) & =\left(a \wedge^{\mathbf{S}} b\right) \wedge^{\mathbf{S}} c .
\end{aligned}
$$

Note that a semilattice naturally carries a partial order. As a matter of fact, by defining $R:=\left\{(a, b) \in S^{2} \mid a \wedge^{\mathbf{S}} b=a\right\}$, we get that $(S ; R)$ is a poset.

- Let $\tau$ be a type of algebras that consists of two binary function symbols $\wedge$ and $\vee$. A lattice is a $\tau$-algebra $\mathbf{L}=\left(L ; \wedge^{\mathbf{L}}, \vee^{\mathbf{L}}\right)$, such that $\left(L ; \wedge^{\mathbf{L}}\right)$ and $\left(L ; \vee^{\mathbf{L}}\right)$ are semilattices and

$$
a \vee^{\mathbf{L}}\left(a \wedge^{\mathbf{L}} b\right)=a, \quad a \wedge^{\mathbf{L}}\left(a \vee^{\mathbf{L}} b\right)=a
$$

A bounded lattice $\mathbf{L}$ is a lattice that has a top element 1 (also called maximum or supremum) and a bottom element 0 (also called minimum) such that

$$
0 \leq a \leq 1, \text { for every } a \in L
$$

Let $a, b$ be elements of a poset $(L ; \leq)$; by $\operatorname{Int}[a, b]$ we denote the interval between $a$ and $b$, i.e., $\operatorname{Int}[a, b]:=\{c \in L \mid a \leq c \leq b\}$. We say that $b$ covers $a$ (or $b$ is covered by $a$ ), written $a \prec b$, if $\operatorname{Int}[a, b]=\{a, b\}$. If $\mathbf{L}$ be a bounded lattice, we say that an element $a \in L$ is an atom if $0 \prec a$; analogously, we say that an element $a \in L$ is a coatom if $a \prec 1$. We say that $s$ is a submaximal element of $\mathbf{L}$ if $s$ is covered by a coatom.

Example 2.1.3. Let $\tau$ be a type of algebras that consists of a unique ternary function
symbol. Consider the following ternary operation $d_{3}$ on the (Boolean) set $\{0,1\}$ :

$$
\begin{aligned}
d_{3}(0,0,0) & =d_{3}(0,0,1)=d_{3}(0,1,0)=d_{3}(1,0,0)=0 ; \\
d_{3}(1,1,1) & =d_{3}(1,1,0)=d_{3}(1,0,1)=d_{3}(0,1,1)=1 .
\end{aligned}
$$

This operation is usually called the majority operation on $\{0,1\}$ and the reason for this name should be clear from the definition of $d_{3}$. Then $\mathbf{A}:=\left(\{0,1\} ; d_{3}\right)$ is an algebra of type $\tau$.

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of the same type. A map $\alpha: A \rightarrow B$ is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ if for every $i$, and $a_{1}, \ldots, a_{n} \in A$, we have

$$
\alpha\left(f_{i}^{\mathbf{A}}\left(a_{1}, \ldots, a_{n}\right)\right)=f_{i}^{\mathbf{B}}\left(\alpha\left(a_{1}\right), \ldots, \alpha\left(a_{n}\right)\right) .
$$

If additionally $\alpha$ is bijective, we say that $\mathbf{A}$ is isomorphic to $\mathbf{B}$ and write $\mathbf{A} \simeq \mathbf{B}$.
Let $\mathbf{A}=\left(A ; F^{\mathbf{A}}\right)$ be an algebra and $B \subseteq A$. We say that $B$ is a subuniverse of $\mathbf{A}$ if for every $n$-ary $f^{\mathbf{A}} \in F^{\mathbf{A}}$ and for all $b_{1}, \ldots, b_{n} \in B$, it holds that $f^{\mathbf{A}}\left(b_{1}, \ldots, b_{n}\right) \in B$. We say that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ if $B$ is a subuniverse of $\mathbf{A}$ and every basic operation of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$ to the set $B$. Note that it is implicit from the definition that $\mathbf{A}$ and $\mathbf{B}$ are of the same type.

Let $\left\{\mathbf{A}_{i} \mid i \in I\right\}$ be a family of algebras of the same type $\tau$. The direct product of $\left\{\mathbf{A}_{i} \mid i \in I\right\}$, denoted by $\prod_{i \in I} \mathbf{A}_{i}$, is the algebra of type $\tau$, whose universe is the Cartesian product $\prod_{i \in I} A_{i}$, where a basic operation $f \prod_{i \in I} \mathbf{A}_{i}$ of arity $n \geq 0$ is defined as

$$
f \prod_{i \in I} \mathbf{A}_{i}\left(\left(a_{0, i} \mid i \in I\right), \ldots,\left(a_{n-1, i} \mid i \in I\right)\right)=\left(f^{\mathbf{A}_{i}}\left(a_{0, i}, \ldots, a_{n-1, i}\right) \mid i \in I\right),
$$

for every function symbol $f$ and $\left(a_{0, i} \mid i \in I\right), \ldots,\left(a_{n-1, i} \mid i \in I\right) \in \prod_{i \in I} A_{i}$.
Definition 2.1.4. Let $K$ be a class of algebras of the same type. We define the following operators:

- $\boldsymbol{H}(K)$ denotes the class of all homomorphic images of algebras from $K$;
- $\boldsymbol{S}(K)$ denotes the class of all subalgebras of algebras from $K$;
- $\boldsymbol{P}(K)$ denotes the class of all finite products of algebras from $K$;
- $\boldsymbol{P}_{\text {fin }}(K)$ denotes the class of all finite products of algebras from $K$.

If $K$ is such that $\boldsymbol{H}(K) \subseteq K, \boldsymbol{S}(K) \subseteq K$, and $\boldsymbol{P}(K) \subseteq K\left(\boldsymbol{P}_{\text {fin }}(K) \subseteq K\right)$, then we say that $K$ is closed under homomorphic images, closed under subalgebras, and closed under (finite) direct products, respectively. A class of algebras of the same type which is closed under homomorphic images, subalgebras and direct products is called a variety. If $K$ is a class of similar algebras, one might consider a new operator $\boldsymbol{V}(K)$ which denotes the smallest variety containing $K$, also referred to as the variety generated by $K$. A
celebrated result in the literature, also known as the HSP Lemma, characterizes the operator $\boldsymbol{V}$ via the operators $\boldsymbol{H}, \boldsymbol{S}$, and $\boldsymbol{P}$ that we have already defined.

Lemma 2.1.5 ([100]). Let $K$ be a class of algebras of the same type. It holds that

$$
\boldsymbol{V}(K)=\boldsymbol{H S P}(K)
$$

Note that for every variety $\mathcal{V}$ there is an algebra $\mathbf{A}$ such that $\mathcal{V}=\boldsymbol{H S P}(\mathbf{A})$; in this case we say that $\mathbf{A}$ is a generator of $\mathcal{V}$. A variety $\mathcal{V}$ is finitely generated if $\mathcal{V}=\boldsymbol{V}(K)$ for some finite $K$ containing finite algebras. A variety $\mathcal{V}$ is locally finite if and only if for all $\mathbf{A} \in \mathcal{V}$ and for all finite subsets $\left\{a_{1}, \ldots, a_{n}\right\} \subseteq A$, the subalgebra of $\mathbf{A}$ generated by $a_{1}, \ldots, a_{n}$ is finite.

The latter lemma provides a syntactic characterization of the operator $\boldsymbol{V}$. A semantic characterization can be provided as well: every variety can be described by the identities satisfied by its members. Let $X$ be a set of variables and let $\tau$ be a set of function symbols. The set of $\tau$-terms over $X$ is the smallest set containing expressions of the form $t\left(x_{1}, \ldots, x_{n}\right)$, with $x_{1}, \ldots, x_{n} \in X$, such that:

- every variable $x_{i} \in X$ is a $\tau$-term;
- if $t_{1}\left(x_{1}, \ldots, x_{n}\right), \ldots, t_{k}\left(x_{1}, \ldots, x_{n}\right)$ are $\tau$-terms and $f \in \tau$ is a symbol of arity $k$, then $\left(f\left(t_{1}, \ldots, t_{k}\right)\right)\left(x_{1}, \ldots, x_{n}\right)$ is a $\tau$-term.

Definition 2.1.6. An identity (over $\tau$ ) is a $\tau$-sentence of the form

$$
\begin{equation*}
\forall x_{1}, \ldots, x_{n}: p\left(x_{1}, \ldots, x_{n}\right)=q\left(x_{1}, \ldots, x_{n}\right) \tag{2.1}
\end{equation*}
$$

where $p$ and $q$ are $\tau$-terms. We often write $p \approx q$ as a shortcut for (2.1).
Definition 2.1.7. Let $\Sigma$ be a set of identities over $\tau$. We say that

- an algebra $\mathbf{A}$ satisfies $\Sigma$, denoted by $\mathbf{A} \models \Sigma$, if $p^{\mathbf{A}}=q^{\mathbf{A}}$ for all $p \approx q \in \Sigma$.
- a class $K$ of algebras of the same type $\tau$ satisfies $\Sigma$, denoted by $K \models \Sigma$, if $\mathbf{A} \models \Sigma$ for every $\mathbf{A} \in K$.

Let $\Sigma$ be a set of identities and $K$ be a class of similar algebras. We define

$$
\operatorname{Mod}(\Sigma):=\{\mathbf{A} \mid \mathbf{A} \models \Sigma\}
$$

called the class of models of $\Sigma$ and

$$
\operatorname{Th}(K):=\{p \approx q \mid K \models p \approx q\}
$$

called the equational theory of $K$. We are now ready to present a result due to G. Birkhoff which states that any variety is the class of models of a certain equational theory, and viceversa.

Theorem 2.1.8 ([21]). For every class $K$ of algebras of the same type, we have

$$
\boldsymbol{V}(K)=\operatorname{Mod}(\operatorname{Th}(K))
$$

We say that a variety $\mathcal{V}$ is finitely presentable if it has finitely many basic operation symbols and $\mathcal{V}=\operatorname{Mod}(\Sigma)$ for some finite set of identities $\Sigma$. We conclude this section with an example.

Example 2.1.9. Note that every non-empty set $A$ can be seen as an algebra $\mathbf{A}:=(A ; \emptyset)$ having no basic operations. Let us denote by $\mathcal{S e t s}$ the class of non-empty sets. Observe that, if we look at $\mathcal{S}$ ets as the class of models of $\{x \approx x\}$, with no basic operation symbols, it follows that such a class is a variety. Moreover, let $\mathbf{E}_{2}$ be the algebra $(\{0,1\} ; \emptyset)$; it is well-known that $\operatorname{Sets}=\boldsymbol{H S P}\left(\mathbf{E}_{2}\right)$.

### 2.1.2 Clones

The notion of clone is a fundamental concept in universal algebra as an abstraction of an algebra. A clone consists of operations which are defined by terms in a given algebra; clones reappear in multi-valued logic in the form of a collection of truth functions definable by formulae on a given set of connectives. Although they constitute research of independent interest in universal algebra and have been used in the study of varieties, more recently, clones are used in computer science where they appear as a generalization of "symmetries" in the study of CSPs. In case the reader is interested in a more in-depth introduction to clone theory, we strongly recommend the textbooks [76, 90, 99].

Definition 2.1.10. A (function) clone $\mathcal{C}$ on is a set of operations on a set $A$ such that

- $\mathcal{C}$ contains all the projections, i.e., for all $1 \leq i \leq n$ it contains the $n$-ary operation $\operatorname{pr}_{i}^{n}$ on $A$ defined by $\operatorname{pr}_{i}^{n}\left(x_{1}, \ldots, x_{n}\right)=x_{i}$;
- $\mathcal{C}$ is closed under composition, i.e., for every $n$-ary $f \in \mathcal{C}$ and for all $m$-ary operations $g_{1}, \ldots, g_{n} \in \mathcal{C}$ it holds that the operation $f\left(g_{1}, \ldots, g_{n}\right) \in \mathcal{C}$; where $f\left(g_{1}, \ldots, g_{n}\right)$ is defined by

$$
x_{1}, \ldots, x_{m} \mapsto f\left(g_{1}\left(x_{1}, \ldots, x_{m}\right), \ldots, g_{n}\left(x_{1}, \ldots, x_{m}\right)\right)
$$

Throughout the dissertation, we denote by $E_{n}$ be the set $E_{n}:=\{0,1, \ldots, n-1\}$. We define:

$$
\mathcal{O}_{n}^{(k)}=\left\{f \mid f: E_{n}^{k} \rightarrow E_{n}\right\}, \quad \quad \mathcal{O}_{n}=\bigcup_{k \geq 1} \mathcal{O}_{n}^{(k)}
$$

By $\mathcal{P}_{n}$ we denote the set of all projections on the set $E_{n}$; it is immediate to see that $\mathcal{P}_{n}$ is a clone for every $n \geq 2$. Let $F \subseteq \mathcal{O}_{n}$, we define $\langle F\rangle$ to be the clone generated by $F$, i.e., the smallest clone which contains $F$. If $F=\{f\}$, we simply write $\langle f\rangle$. Moreover, if $\mathcal{C}$ is a clone, then $\mathcal{C}^{(n)}$ denotes the set of all at most $n$-ary operations in $\mathcal{C}$.

Example 2.1.11. Consider the universe $E_{2}=\{0,1\}$. Then

- $\mathcal{O}_{2}$, i.e., the set of all operations on $E_{2}$ is a clone.
- The set $\mathcal{P}_{2}$ of all projections on $E_{2}$ is a clone and clearly $\mathcal{P}_{2}=\langle\emptyset\rangle$.

The following result can be considered folklore, we will obtain it as a byproduct of the Inv-Pol Galois connection that we are going to present in Section 2.3

Theorem 2.1.12. All clones of operations on the finite set $E_{n}$ form an algebraic lattice $\mathfrak{L}_{n}$ under set inclusion. The lattice operations are defined as follows:

$$
\mathcal{C} \wedge \mathcal{D}:=\mathcal{C} \cap \mathcal{D} \quad \text { and } \quad \mathcal{C} \vee \mathcal{D}:=\langle\mathcal{C} \cup \mathcal{D}\rangle
$$

The least element of the lattice is $\mathcal{P}_{n}$; the greatest element is $\mathcal{O}_{n}$.

Remark 2.1.13. Let $\mathcal{C}$ be a clone on a finite set $A$, we consider the type of algebras $\tau_{\mathcal{C}}$, having an operation symbol $f$ for every $f^{\mathcal{C}} \in \mathcal{C}$; in this way, $\mathcal{C}$ can be viewed as an algebra of type $\tau_{\mathcal{C}}$ over the universe $A$. Therefore, one can easily adapt the notions already defined for algebras to clones. For instance, we can apply the operators from Definition 2.1.4 also to clones.

Definition 2.1.14. Let $\mathcal{C}$ and $\mathcal{D}$ be clones, and let $\xi: \mathcal{C} \rightarrow \mathcal{D}$ be a mapping that preserves arities. We say that $\xi$ is a clone homomorphism if

$$
\xi\left(\operatorname{pr}_{i}^{n}\right)=\operatorname{pr}_{i}^{n} \quad \text { and } \quad \xi\left(f\left(g_{1}, \ldots, g_{n}\right)\right)=\xi(f)\left(\xi\left(g_{1}\right), \ldots, \xi\left(g_{n}\right)\right)
$$

Note that, if $\xi$ is a clone homomorphism from $\mathcal{C}$ to $\mathcal{D}$ then $\mathcal{D}$ can be considered as an algebra of type $\tau_{\mathcal{C}}$ : if $f$ is an operation symbol in $\tau_{\mathcal{C}}$, we define $f^{\mathcal{D}}:=\xi\left(f^{\mathcal{C}}\right)$.

Definition 2.1.15. A set of operations (for instance a clone) $F$ satisfies a set of identities $\Sigma$, denoted $F \models \Sigma$, if there is a map $\xi$ assigning to each function symbol occurring in $\Sigma$ an operation in $F$ of the same arity, such that if $p \approx q$ is in $\Sigma$, then $\xi(p)=\xi(q)$. Additionally, we say that an operation $f$ satisfies $\Sigma$, and write $f \models \Sigma$, if $\{f\} \models \Sigma$.

We are finally ready to present Birkhoff's famous HSP theorem:
Theorem 2.1.16 ([21). Let $\mathcal{A}$ and $\mathcal{B}$ be clones, and let $\xi: \mathcal{A} \rightarrow \mathcal{B}$ be a mapping that preserves arities. The following are equivalent:
(1) every identity satisfied by $\mathcal{A}$ is also satisfied by $\mathcal{B}$;
(2) $\mathcal{B} \in \boldsymbol{H S P}_{\text {fin }}(\mathcal{A})$;
(3) $\xi$ is a surjective clone homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

### 2.1.3 Reducts and expansions

Let $\mathbf{A}:=\left(A ; F^{\mathbf{A}}\right)$ be an algebra, we define $\operatorname{Clo}(\mathbf{A})$ to be the clone generated by the basic operations of $\mathbf{A}$, i.e., $\operatorname{Clo}(\mathbf{A})=\left\langle F^{\mathbf{A}}\right\rangle$. We refer to $\operatorname{Clo}(\mathbf{A})$ as the clone of term operations of $\mathbf{A}$. We denote the clone of all $n$-ary term operations by $\operatorname{Clo}_{n}(\mathbf{A})$. We say that $\mathbf{A}$ and $\mathbf{B}$ are term-equivalent if $\operatorname{Clo}(\mathbf{A})=\operatorname{Clo}(\mathbf{B})$.

Definition 2.1.17. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras such that $A=B$. We say that $\mathbf{B}$ is a reduct of $\mathbf{A}$ if every basic operation of $\mathbf{B}$ is in $\operatorname{Clo}(\mathbf{A})$; in this case we also say that $\mathbf{A}$ is an expansion of $\mathbf{B}$.

If $K$ is a class of algebras of the same type, we denote by $\boldsymbol{E}(K)$ the class of all expansions of algebras from $K$. Following Remark 2.1.13, we extend the definition of expansion to clones, hence $\boldsymbol{E}(\mathcal{C})$ denotes the class of all clones obtained from the clone $\mathcal{C}$ by adding operations to it. Recall that an algebra $\mathbf{A}:=(A ; F)$ is idempotent if every operation of $\mathbf{A}$ is idempotent, that is, for every $f \in F$ it holds that $f(x, \ldots, x) \approx x$. By $\mathbf{A}^{\text {id }}$ we denote the idempotent reduct of $\mathbf{A}:=(A ; F)$, i.e., the algebra $\mathbf{A}^{\text {id }}:=\left(A ; F^{\text {id }}\right)$ where $F^{\mathrm{id}}$ is the set of all idempotent operations $f \in\langle F\rangle$. Likewise, the idempotent reduct of a clone $\mathcal{C}$ is the clone $\mathcal{C}^{\text {id }}$ which consists of all idempotent operations in $\mathcal{C}$. Conservative operations are a particular case of idempotent operations; an operation $f$ is conservative if $f\left(x_{1}, \ldots, x_{n}\right) \in\left\{x_{1}, \ldots, x_{n}\right\}$.

### 2.2 Structures and relational clones

A relational signature $\tau$ is a set of relational symbols where each symbol is associated with a natural number called its arity.
Definition 2.2.1. A $\tau$-structure $\mathbb{A}$ is a tuple $\left(A ;\left(R^{\mathbb{A}}\right)_{R \in \tau}\right)$ where $A$ is a (finite) set called the universe of $\mathbb{A}$ and $\left(R^{\mathbb{A}}\right)_{R \in \tau}$ is a list of relations on $A$ where every relation $R^{\mathbb{A}}$ has the opportune arity specified by $\tau$.

Without loss of generality, throughout the dissertation, we assume that every relational structure has universe $E_{n}:=\{0, \ldots, n-1\}$ for some $n \geq 1$, unless otherwise specified.
Example 2.2.2. Let $\tau$ be a relational signature consisting of two binary relational symbols and two unary relational symbols. Consider the relations $C_{2}$ and $\leq_{2}$ on the universe $\{0,1\}$ defined as follows:

$$
C_{2}:=\{(0,1),(1,0)\} \quad \leq_{2}:=\{(0,0),(0,1),(1,1)\}
$$

Then $\mathbb{C}_{2}^{\leq}:=\left(\{0,1\} ; C_{2}, \leq_{2},\{0\},\{1\}\right)$ is a $\tau$-structure.
Let $A$ be a set and $t \in A^{k}$. For $I=\left\{i_{1}, \ldots, i_{n}\right\} \subseteq\{1, \ldots, k\}$ with $i_{1}<\cdots<i_{k}$ we write $\operatorname{pr}_{I}(t)$ for the tuple $\left(t_{i_{1}}, \ldots, t_{i_{n}}\right)$. For $R \subseteq A^{k}$ we write $\operatorname{pr}_{I}(R):=\left\{\operatorname{pr}_{I}(t) \mid t \in R\right\}$ for the projection of $R$ to the indices from $I$. If $I=\{i\}$ for $i \in\{1, \ldots, k\}$ then $\operatorname{pr}_{i}(R)$ denotes $\operatorname{pr}_{I}(R)$. A relation $R \subseteq A^{k}$ is called subdirect if $\operatorname{pr}_{i}(R)=A$ for every $i \in\{1, \ldots, k\}$.

Definition 2.2.3. Let $\mathbb{A}$ be a $\tau$-structure and $\mathbb{B}$ be $\sigma$-structure, with $\tau \subseteq \sigma$. If $A=B$ and $R^{\mathbb{A}}=R^{\mathbb{B}}$ for every $R \in \tau$, then we say that $\mathbb{A}$ is a $\tau$-reduct (or a reduct) of $\mathbb{B}$.

Let $\mathbb{A}$ and $\mathbb{B}$ be two relational $\tau$-structures, a map $h: A \rightarrow B$ is a homomorphism if for every $R \in \tau$

$$
\begin{equation*}
\text { if }\left(a_{1}, \ldots, a_{n}\right) \in R^{\mathbb{A}}, \text { then }\left(h\left(a_{1}\right), \ldots, h\left(a_{n}\right)\right) \in R^{\mathbb{B}} \tag{2.2}
\end{equation*}
$$

We write $\mathbb{A} \rightarrow \mathbb{B}$ if there exists a homomorphism from $\mathbb{A}$ to $\mathbb{B}$, and we say that $\mathbb{A}$ and $\mathbb{B}$ are homomorphically equivalent if $\mathbb{A} \rightarrow \mathbb{B}$ and $\mathbb{B} \rightarrow \mathbb{A}$, written $\mathbb{A} \equiv \mathbb{B}$. An isomorphism of $\mathbb{A}$ and $\mathbb{B}$ is a bijective homomorphism $h$ such that the inverse mapping $h^{-1}: B \rightarrow A$ that maps $h(x)$ to $x$ is a homomorphism, too. In this case we say that $\mathbb{A}$ and $\mathbb{B}$ are isomorphic and write $\mathbb{A} \simeq \mathbb{B}$. An embedding of $\mathbb{A}$ into $\mathbb{B}$ is an injective homomorphism from $\mathbb{A}$ to $\mathbb{B}$ such that the implication in 2.2 is an equivalence. An endomorphism of $\mathbb{A}$ is a homomorphism from $\mathbb{A}$ to $\mathbb{A}$; an automorphism of $\mathbb{A}$ is a bijective embedding of $\mathbb{A}$ into $\mathbb{A}$.

Definition 2.2.4. A finite structure $\mathbb{A}$ is called a core if every endomorphism of $\mathbb{A}$ is an automorphism. We say that $\mathbb{C}$ is a core of $\mathbb{A}$ if $\mathbb{C}$ is a core and $\mathbb{C} \equiv \mathbb{A}$.

We can talk about the core of a (not necessarily finite) relational structure : it has in fact been proved that every relational structure has a unique core up to isomorphism [23].

Proposition 2.2.5 ([23]). Every finite structure $\mathbb{A}$ has a core, which is unique up to isomorphism.

For $n \geq 1$, we denote by $\mathbb{A}^{n}$ the structure with same signature $\tau$ as $\mathbb{A}$ whose domain is $A^{n}$ such that for any $k$-ary $R \in \tau$, a tuple $\left(\boldsymbol{a}_{1}, \ldots, \boldsymbol{a}_{k}\right)$ of $n$-tuples is contained in $R^{\mathbb{A}^{n}}$ if and only if it is contained in $R^{\mathbb{A}}$ componentwise, i.e., $\left(a_{1 j}, \ldots, a_{k j}\right) \in R^{\mathbb{A}}$ for all $1 \leq j \leq n$.

Definition 2.2.6. A polymorphism of a structure $\mathbb{A}$ is a homomorphism from $\mathbb{A}^{n}$ to $\mathbb{A}$.
We denote by $\operatorname{Pol}(\mathbb{A})$ the set of all the polymorphisms of a structure $\mathbb{A}$ and we call it the polymorphism clone of $\mathbb{A}$. Indeed, it is easy to verify that $\operatorname{Pol}(\mathbb{A})$ is a clone in the sense of Definition 2.1.10. When the domain is clear from the contest, we also write $\operatorname{Pol}(\Gamma)$ instead of $\operatorname{Pol}((A ; \Gamma))$.

Definition 2.2.7. A primitive positive formula (over $\tau$ ) is a first-order formula which only uses relation symbols in $\tau$, equality, conjunction and existential quantification.

When $\mathbb{A}$ is a $\tau$-structure and $\phi\left(x_{1}, \ldots, x_{n}\right)$ is a $\tau$-formula with $n$ free-variables $x_{1}, \ldots, x_{n}$ then $\left\{\left(a_{1}, \ldots, a_{n}\right) \mid \mathbb{A} \vDash \phi\left(a_{1}, \ldots, a_{n}\right)\right\}$ is called the the relation defined by $\phi$. If $\phi$ is primitive positive, then this relation is said to be $p p$-definable in $\mathbb{A}$.

Definition 2.2.8. A relational clone is a set of relations $\Gamma$ which is closed under defining new relations via primitive positive formulae.

Analogously to the case of clones: let $\Gamma$ be a set of relations on a universe $A$, we define $\langle\Gamma\rangle$ to be the relational clone generated by $\Gamma$, i.e., the smallest relational clone which contains $\Gamma$.

### 2.3 The Inv-Pol Galois connection

The Galois connection between relational clones and clones was originally proved by Bodnarčuk, Kalužnin, Kotov, and Romov [30], and indipendently by Geiger [53]. They proved that a relation $R$ has a primitive positive definition in a finite relational structure $\mathbb{A}$ if and only if $R$ is preserved by all polymorphisms of $\mathbb{A}$. This connection turned out to be a fertile research ground: Pöschel investigated the general case where the domain may be infinite [89]; more recently, Jeavons [62, 63] gave a new life to the Inv-Pol Galois connection by showing that it plays a crucial part in the study of the complexity of CSPs (see Section 2.5).
Definition 2.3.1. We say that an operation $f: A^{n} \rightarrow A$ preserves a $k$-ary relation $R$ on $A$, written $f \triangleright R$, if for every

$$
\left(\begin{array}{c}
a_{1,1} \\
a_{1,2} \\
\vdots \\
a_{1, k}
\end{array}\right), \ldots,\left(\begin{array}{c}
a_{n, 1} \\
a_{n, 2} \\
\vdots \\
a_{n, k}
\end{array}\right) \in R \text {, then }\left(\begin{array}{c}
f\left(a_{1,1}, \ldots, a_{n, 1}\right) \\
f\left(a_{1,2}, \ldots, a_{n, 2}\right) \\
\vdots \\
f\left(a_{1, k}, \ldots, a_{n, k}\right)
\end{array}\right) \in R .
$$

In this case we also say that the relation $R$ is invariant under $f$.
Note that we already introduced the notion of polymorphism in Definition 2.2.6 equivalently: a polymorphism of a relational structure $\mathbb{A}:=(A ; \Gamma)$ is an operation that preserves $R_{i}$ for each relation $R_{i} \in \Gamma$.
Example 2.3.2. Consider the structure $\mathbb{B}_{2}:=\left(\{0,1\} ; B_{2}\right)$ with a single binary relation $B_{2}:=\{0,1\}^{2} \backslash\{(0,0)\}$. Let us define the following operations

| $\vee$ | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 0 | 1 |
| 1 | 1 | 1 |

$$
m(x, y, z):=x \oplus y \oplus z
$$

where $\oplus$ is the usual addition modulo 2 . We claim that $\vee \in \operatorname{Pol}\left(\mathbb{B}_{2}\right)$ and $m \notin \operatorname{Pol}\left(\mathbb{B}_{2}\right)$. Suppose $\vee \notin \operatorname{Pol}\left(\mathbb{B}_{2}\right)$, then we would have the following

$$
\vee\left(\begin{array}{ll}
a_{1} & b_{1}  \tag{2.3}\\
a_{2} & b_{2}
\end{array}\right):=\binom{\vee\left(a_{1}, b_{1}\right)}{\vee\left(a_{2}, b_{2}\right)}=\binom{0}{0},
$$

where $\left(a_{1}, a_{2}\right),\left(b_{1}, b_{2}\right) \in B_{2}$. By definition, $\vee\left(a_{1}, b_{1}\right)=0$ if and only if $a_{1}=b_{1}=0$. Since $\left(a_{1}, a_{2}\right) \in B_{2}$, we have that $a_{2}=1$ and therefore $\vee\left(a_{2}, b_{2}\right)=1$. This is in contradiction
with 2.3 . On the other side, consider the following

$$
m\left(\begin{array}{lll}
0 & 1 & 1  \tag{2.4}\\
1 & 1 & 0
\end{array}\right):=\binom{m(0,1,1)}{m(1,1,0)}=\binom{0}{0}
$$

Every column of the matrix in 2.4 is a tuple in the relation $B_{2}$; by applying $m$ componentwise we get $(0,0) \notin B_{2}$.

Definition 2.3.3. Let $F$ be any set of operations on a universe $A$ and let $\Gamma$ be any set of relations on $A$. We define:

$$
\begin{aligned}
\operatorname{Pol}(\Gamma) & :=\{f \mid \forall R \in \Gamma, f \triangleright R\} \\
\operatorname{Inv}(F) & :=\{R \mid \forall f \in F, f \triangleright R\}
\end{aligned}
$$

We already stressed in Section 2.2 that for all $\Gamma$, the set $\operatorname{Pol}(\Gamma)$ is a clone; note that for every set of operations $F$, the set $\operatorname{Inv}(F)$ is a relational clone. It follows that $\langle F\rangle \subseteq \operatorname{Pol}(\operatorname{Inv}(F))$ and $\langle\Gamma\rangle \subseteq \operatorname{Inv}(\operatorname{Pol}(\Gamma))$. The next two theorems show that, in the finite case, the inclusion $\supseteq$ holds, too.

Theorem 2.3.4. Let $F$ be a set of operations on a finite universe, then it holds that $\operatorname{Pol}(\operatorname{Inv}(F))=\langle F\rangle$.

In particular, a relation is pp-definable in a relational structure $\mathbb{A}$ if and only if it is preserved by every polymorphism of $\mathbb{A}$.

Theorem 2.3.5 ([30, 53]). Let $\Gamma$ be a set of relations on a finite universe, then $\operatorname{Inv}(\operatorname{Pol}(\Gamma))=\langle\Gamma\rangle$.

Definition 2.3.6. Let $\mathbb{A}$ be a structure on a universe $A$. An algebra $\mathbf{A}$ with universe $A$ is called a polymorphism algebra of $\mathbb{A}$ if $\operatorname{Clo}(\mathbf{A})=\operatorname{Pol}(\mathbb{A})$.

It follows from the Galois connection [30, 53] that we presented in this section that every finite algebra $\mathbf{A}$ is the polymorphism algebra of some suitable relational structure $\mathbb{A}$ with the same universe.

Definition 2.3.7. An algebra $\mathbf{A}$ is called finitely related if finitely many relations suffice to determine $\operatorname{Clo}(\mathbf{A})$, i.e., if $\operatorname{Clo}(\mathbf{A})=\operatorname{Pol}(\mathbb{A})$ and $\mathbb{A}$ has finitely many relations.

Example 2.3.8. Consider the algebra $\mathbf{A}$ from Example 2.1.3 and the structure $\mathbb{C}_{2}^{\leq}$from Example 2.2.2. It is well known that $\operatorname{Clo}(\mathbf{A})=\operatorname{Pol}\left(\mathbb{C}_{2}^{\leq}\right)=\left\langle d_{3}\right\rangle$, where $d_{3}$ is the majority operation defined in Example 2.1.3 In particular, $\mathbf{A}$ is finitely related since $\mathbb{C}_{2}^{\leq}$has finite signature.

Throughout the dissertation, we denote clones in calligraphic-style with the following convention: we often $\operatorname{refer} \operatorname{to} \operatorname{Clo}(\mathbf{A})=\operatorname{Pol}(\mathbb{A})$ as $\mathcal{A}$, unless otherwise specified.

### 2.4 CSP in a nutshell

This dissertation does not focus on the Constraint Satisfaction Problem (CSP), which rather is a topic that runs through this dissertation only in a cross-cutting way; in this section we simply fix some terminology with the aim of setting the pp-constructability poset in a broader context, the study of which is also strongly motivated by questions about CSPs. There are different variants of CSPs and, depending on the research area, different approaches to it; in the present dissertation by $\operatorname{CSP}(\mathbb{A})$ we refer to the decision version of the CSP of $\mathbb{A}$, where $\mathbb{A}$ is a fixed finite relational structure and we are interested in what is now referred to as the algebraic approach to CSP [18, 38]. Here we try to give a general overview of CSP for finite structures, yet trying to keep the presentation concise. We strongly recommend the surveys [5] and [17] for the basics of the finite domain CSP; indeed, most of the examples we are going to present in this section are taken from [17]. For an introduction to complexity theory and for the definition of all the complexity classes that we are going to mention throughout the dissertation, we refer the reader to the textbook [85]. The reader who is particularly interested in CSP, can find a detailed discussion of the current state of research and the problems still open in this area in the book [24].

For a finite relational signature $\tau$ and a $\tau$-structure $\mathbb{A}$, the $\operatorname{CSP}(\mathbb{A})$ is the membership problem of the class
$\{\mathbb{S} \mid \mathbb{S}$ is a $\tau$-structure and there exists a homomorphism from $\mathbb{S}$ to $\mathbb{A}\}$.
Equivalently, $\operatorname{CSP}(\mathbb{A})$ can be expressed as the membership problem of the set of primitive positive sentences which hold in $\mathbb{A}$.

Definition 2.4.1. An instance of the CSP is a triple ( $V, D, C$ ) where

- $V$ is a finite set of variables,
- $D$ is a finite domain,
- $C$ is a finite set of constraints: each constraint is a pair $C=(\boldsymbol{x}, R)$ where $\boldsymbol{x}$ is a tuple of variables of length $n$ and $R$ is an $n$-ary relation on $D$.

An assignment is a mapping $f: V \rightarrow D$; we say that an assignment $f$ satisfies a constraint $C:=(\boldsymbol{x}, R)$ if $f(\boldsymbol{x}) \in R$. An assignment is a solution if it satisfies all constraints.

Let us provide some examples of computational problems that can be expressed as $\operatorname{CSP}(\mathbb{A})$ for suitable finite relational structures $\mathbb{A}$.

Example 2.4.2 ( $n$-colorability). Given a graph $\mathbb{G}$ and a fixed natural number $n$, the $n$-colorability problem is to decide whether it is possible to assign colors $\{0, \ldots, n-1\}$
to the vertices of $\mathbb{G}$ such that adjacent vertices receive different colors. This problem is equivalent to $\operatorname{CSP}\left(\mathbb{K}_{n}\right)$ where

$$
\mathbb{K}_{n}:=\left(E_{n} ; \not F_{n}\right) \quad \text { and } \quad \not F_{n}:=\left\{(a, b) \mid a, b \in E_{n} \text { and } a \neq b\right\} .
$$

Indeed, if we consider variables as vertices and draw an edge from $x$ to $y$ whenever the instance contains the constraint $x \not{ }_{n} y$, then we get a graph. It is easy to check that the original instance has a solution if and only if the obtained graph is $n$-colorable. The translation in the other direction is similar. The $n$-coloring problem is NP-complete for $k \geq 3$ (see e.g., 85 ); on the other hand, the 2-coloring problem is solvable in polynomial time, in fact, it is in the complexity class L. Throughout the dissertation we prefer to write $\mathbb{C}_{2}$ instead of $\mathbb{K}_{2}$.

Example 2.4.3 ( $n$-SAT). The problem $n$-SAT, for a natural number $n$, is the problem where the instance is a Boolean formula in conjunctive normal form with exactly $n$ literals per clause and the question is whether there is a Boolean assignment for the variables such that in each clause at least one literal is true. It is well known that 3-SAT is an NP-complete problem. The problem 2-SAT, instead, is solvable in polynomial time, and is in fact complete for the complexity class NL under log-space reductions [85]. Moreover, 2-SAT is equivalent to $\operatorname{CSP}\left(\mathbb{C}_{2}^{\leq}\right)$, where $\mathbb{C}_{2}^{\leq}$is the structure from Example 2.2.2,

Example 2.4.4 (HORN-SAT). The problem HORN-SAT is a version of 3-SAT where each clause may have at most one positive literal. This problem is known to be equivalent to $\operatorname{CSP}(\mathbb{H O R N})$ where

$$
\left.\mathbb{H O R N}:=\left(E_{2} ; R_{110}, R_{111},\{0\},\{1\}\right)\right) \quad \text { and } \quad R_{a b c}:=\{0,1\}^{3} \backslash\{(a, b, c)\} .
$$

HORN-SAT is solvable in polynomial time, and is in fact a P -complete problem under log-space reductions [85].

Example 2.4.5 (Linear equations). The problem $3-\operatorname{LIN}(p)$, for a prime number $p$, has as an input a system of linear equations over the Galois field over $E_{p}$ (usually denoted by $\mathbb{Z}_{p}$ ), where each equation has 3 variables; the question is whether the system has a solution. this problem is equivalent to $\operatorname{CSP}\left(3 \mathbb{L} \mathbb{N} \mathbb{N}_{p}\right)$ where

$$
\begin{aligned}
3 \mathbb{L I N}_{p} & :=\left(E_{p} ; \text { all affine subspaces } R_{a b c d} \text { of } \mathbb{Z}_{p}^{3} \text { of dimension } 2\right), \text { and } \\
R_{a b c d} & :=\left\{(x, y, z) \in \mathbb{Z}_{p}^{3} \mid a x+b y+c z=d\right\} .
\end{aligned}
$$

This problem is solvable in polynomial time, e.g., by Gaussian elimination and it is complete for the class $\operatorname{Mod}_{p} \mathrm{~L}$ [39].

Example 2.4.6 ( $s, t$-connectivity). The STCON problem has as an input a directed graph and two of its vertices, $s$ and $t$; the question is whether there exists a directed path from $s$ to $t$. The constraint satisfaction problem $\operatorname{CSP}(\mathbb{B} \leq)$, where $\mathbb{B} \leq:=\left(E_{2} ; \leq_{2},\{0\},\{1\}\right)$
and $\leq_{2}$ is the relation defined in Example 2.2 .2 is closely related to $s, t$-connectivity; indeed, from an instance of $\operatorname{CSP}(\mathbb{B} \leq)$ we get a graph in the same way as we did in Example 2.4 .2 and then label some vertices 0 or 1 according to the unary constraints. The original instance of $\operatorname{CSP}(\mathbb{B} \leq)$ has a solution if and only if there is no directed path from a vertex labeled 1 to a vertex labeled 0 . Thus, both $\operatorname{STCON}$ and $\operatorname{CSP}(\mathbb{B} \leq)$ can be solved in polynomial time and in particular are NL-complete under log-space reductions [59, 98 .

### 2.5 Reductions that preserve the complexity of CSPs

We already mentioned in Section 2.4 that the CSP over a fixed language can also be formulated as the homomorphism problem between relational structures with a fixed target structure [49, 62]. Hence, by definition, if $\mathbb{A}$ and $\mathbb{B}$ are homomorphically equivalent, then $\operatorname{CSP}(\mathbb{A})$ and $\operatorname{CSP}(\mathbb{B})$ are $\log$-space equivalent. In this section we briefly recap some reductions that preserve the complexity of CSPs. Most of the results that we are going to present in this section are directly taken over or reformulated from [30, 38, 53]. For more details we strongly recommend the survey [17].

Definition 2.5.1. A relational structure $\mathbb{A}$ is a rigid core if $\mathbb{A}$ has only the identity as an endomorphism.

For finite structures there is the following characterization of rigid cores.
Theorem 2.5.2 ([18). Let $\mathbb{A}:=(A ; \Gamma)$ be a finite relational signature. Then the following are equivalent:

- $\mathbb{A}$ is a rigid core;
- for every $a \in A$, the relation $\{a\} \in\langle\Gamma\rangle$.

Following the latter result, we denote by $\mathbb{A}^{c}$ the rigid core obtained by adding all singleton unary relations to $\mathbb{A}$.

Proposition 2.5.3 ([18]). Let $\mathbb{A}$ be a finite core with a finite relational signature. Then $\operatorname{CSP}(\mathbb{A})$ is equivalent to $\operatorname{CSP}\left(\mathbb{A}^{c}\right)$ under log-space reductions.

Moreover, the notion of rigid core is closely related to the notion of idempotent reduct (Section 2.1.3) as explained in the next proposition.

Proposition 2.5.4. Let $\mathbb{A}$ be a relational structure that is a core and let $\mathcal{A}:=\operatorname{Pol}(\mathbb{A})$. Then $\mathcal{A}^{\text {id }}=\operatorname{Pol}\left(\mathbb{A}^{c}\right)$.

We already defined, in Section 2.2, when a relation is pp-definable in a structure. We can easily extend this notion to structures: let $\mathbb{A}$ and $\mathbb{B}$ be structures on the same universe, we say that $\mathbb{A} p p$-defines $\mathbb{B}$, written $\mathbb{A} \leq_{\text {Def }} \mathbb{B}$, if every relation in $\mathbb{B}$ can be
pp-defined in $\mathbb{A}$. It is easy to check that the relation $\leq_{\text {Def }}$ is indeed a preorder on all the structures having the same universe; we denote by $\equiv_{\text {Def }}$ the associated equivalence relation, which we call interdefinability. The next result is an immediate consequence of the Galois connection from Section 2.3

Theorem 2.5.5 ([30, [53]). Let $\mathbb{A}$ and $\mathbb{B}$ be structures on the same finite universe $A$, and let $\mathcal{A}=\operatorname{Pol}(\mathbb{A})$ and $\mathcal{B}=\operatorname{Pol}(\mathbb{B})$. Then $\mathbb{A} \leq_{\text {Def }} \mathbb{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$.

In the early ' 40 s Post completely described $\mathfrak{L}_{2}$, i.e., the lattice that arises by considering all Boolean clones ordered by inclusion [91. It turns out that this lattice, which is named the Post's Lattice, is countable. More details about this lattice can be found in Section 5.1 The order that arises from pp-interdefinability allows us to compare CSPs of finite structures with the same universe.

Proposition 2.5.6 ([38]). Let $\mathbb{A}$ and $\mathbb{B}$ be relational structures. If $\mathbb{A}$ pp-defines $\mathbb{B}$ then $\operatorname{CSP}(\mathbb{B})$ is log-space reducible to $\operatorname{CSP}(\mathbb{A})$.

A more powerful tool, which can also be used to compare structures with different domains, is pp-interpretability.

Definition 2.5.7. Let $\mathbb{A}$ and $\mathbb{B}$ be structures with possibly different universes and signatures. We say that $\mathbb{A} p p$-interprets $\mathbb{B}$, written $\mathbb{A} \leq_{\operatorname{Int}} \mathbb{B}$, if there exists $d \geq 1$ and a surjective mapping $I: A^{d} \rightarrow B$ such that the following relations are pp-definable in $\mathbb{A}$ :

- the domain of $I$;
- the preimage of the equality relation in $\mathbb{B}$ under $I$;
- the preimage of every relation in $\mathbb{B}$ under $I$
where the preimage of every $n$-ary relation in $\mathbb{B}$ under $I$ is regarded as a $d n$-ary relation in $\mathbb{A}$.

Proposition 2.5.8 ( 38 ). Let $\mathbb{A}$ and $\mathbb{B}$ be relational structures. If $\mathbb{A} p p$-interprets $\mathbb{B}$ then $\operatorname{CSP}(\mathbb{B})$ is log-space reducible to $\operatorname{CSP}(\mathbb{A})$.

Again, via the Inv-Pol Galois connection, we obtain a similar result to Theorem 2.5.5 In Definition 2.1.14 we introduced the notion of clone homomorphism. Consider the following preorder: we write $\mathcal{A} \preceq_{\mathrm{h}} \mathcal{B}$ if and only if there exists a clone homomorphism from $\mathcal{A}$ to $\mathcal{B}$. The next theorem is obtained as a combination of results from [21] and [38].

Theorem 2.5.9 ([21, [38]). Let $\mathbb{A}$ and $\mathbb{B}$ be finite structures, and let $\mathcal{A}=\operatorname{Pol}(\mathbb{A})$ and $\mathcal{B}=\operatorname{Pol}(\mathbb{B})$. Then $\mathbb{A} \leq_{\text {Int }} \mathbb{B}$ if and only if $\mathcal{A} \preceq_{\mathrm{h}} \mathcal{B}$.

In the 1980's, Garcia, Taylor [51], and independently Neumann [80] studied the so called lattice of interpretability types of varieties. It turns out that such lattice is isomorphic to the lattice that arises from the order considered in Theorem 2.5.9 Let $\operatorname{Clo}(\mathcal{V})$
be equal to $\operatorname{Clo}(\mathbf{A})$ for some generator $\mathbf{A}$ of $\mathcal{V}$. The notion of interpretability between varieties [80] translates to clone homomorphism as follows: $\mathcal{V}$ has an interpretation in $\mathcal{W}$ if there exists a clone homomorphism from $\operatorname{Clo}(\mathcal{V})$ to $\operatorname{Clo}(\mathcal{W})$. The lattice of interpretability types of varieties was used to attack some open problems concerning Mal'cev conditions and it is, in a sense, the natural environment for studying Mal'cev conditions. Indeed, every Mal'cev condition determines a filter in the lattice of interpretability types of varieties, see Chapter 4 for more details.

In Chapter 3 we deal with a coarser poset which we will use to study minor conditions, a particular case of strong linear Mal'cev conditions. This poset arises by considering finite relational structures ordered by pp-constructability, a new reduction introduced by Barto, Opršal, and Pinsker [18] that combines all the reductions presented in this section.

## Chapter 3

## The pp-constructability poset

We concluded Chapter 2 presenting two posets that have been studied in universal algebra for reasons that are of independent interest: the study of clones, varieties, and Mal'cev conditions. In a recent turn of events, clones and Mal'cev conditions have been shown to be of great importance in studying the complexity of CSPs. Barto, Opršal, and Pinsker found unsatisfactory to have several uncorrelated methods of proving reductions between CSPs. In the celebrated article The wonderland of reflections [18 they introduced a coarser order than $\leq_{\text {Def }}$ and $\leq_{\text {Int }}$ which we presented in Section 2.5. All the mentioned posets preserve the complexity of the CSPs (see Proposition 2.5.6 and Proposition 2.5.8.

### 3.1 The wonderland of reflections

### 3.1.1 Primitive positive constructions

Definition 3.1.1. Let $\mathbb{A}, \mathbb{B}$ be relational structures. We say that $\mathbb{A} p p$-constructs $\mathbb{B}$, in symbols $\mathbb{A} \leq_{\text {Con }} \mathbb{B}$, if there exists a sequence $\mathbb{A}=\mathbb{S}_{1}, \mathbb{S}_{2}, \ldots, \mathbb{S}_{k}=\mathbb{B}$ such that for every $1 \leq i<k$

- $\mathbb{S}_{i}$ pp-interprets $\mathbb{S}_{i+1}$, or
- $\mathbb{S}_{i+1}$ is homomorphically equivalent to $\mathbb{S}_{i}$, or
- $\mathbb{S}_{i}$ is a core and $\mathbb{S}_{i+1}$ is obtained from $\mathbb{S}_{i}$ by adding a singleton unary relation.

Barto, Opršal, and Pinsker introduced a weakening of pp-interpretations which together with homomorphic equivalence covers all reductions presented in Section [2.5 A relational structure $\mathbb{B}$ is a pp-power of $\mathbb{A}$ if it is isomorphic to a structure with domain $A^{n}$, where $n \geq 1$, whose relations are pp-definable from $\mathbb{A}$. Recall that a $k$-ary relation on $A^{n}$ is regarded as a $k n$-ary relation on $A$.

Theorem 3.1.2 ([18]). Let $\mathbb{A}$ and $\mathbb{B}$ be relational structures. Then $\mathbb{A}$ pp-constructs $\mathbb{B}$ if and only if $\mathbb{B}$ is homomorphically equivalent to a pp-power of $\mathbb{A}$.

The next result asserts that, analogously to pp-definability and pp-interpretability, pp-constructability preserves the complexity of CSPs:

Proposition 3.1.3 ([18]). Let $\mathbb{A}$ and $\mathbb{B}$ be relational structures. If $\mathbb{A} p p$-constructs $\mathbb{B}$ then $\operatorname{CSP}(\mathbb{B})$ is log-space reducible to $\operatorname{CSP}(\mathbb{A})$.

As an example, we would like to mention that every finite core $\mathbb{A} p p$-constructs its rigid core $\mathbb{A}^{c}$.

Lemma 3.1.4 ([18]). Let $\mathbb{A}$ be a finite core. Then $\mathbb{A} p p$-constructs $\mathbb{A}^{c}$. In particular, $\mathbb{A} \equiv{ }_{\text {Con }} \mathbb{A}^{c}$.

Since pp-constructability is a reflexive and transitive relation on the class of all finite relational structures [18], the equivalence relation $\equiv_{\text {Con }}$ can defined by as follows

$$
\mathbb{A} \equiv \equiv_{\text {Con }} \mathbb{B}: \Leftrightarrow \mathbb{B} \leq_{\text {Con }} \mathbb{A} \wedge \mathbb{A} \leq_{\text {Con }} \mathbb{B} \text {. }
$$

The equivalence classes of $\equiv_{\text {Con }}$ are called the pp-constructability types and we denote by $\overline{\mathbb{A}}$ the pp-constructability type of $\mathbb{A}$. For any two relational structures $\mathbb{A}$ and $\mathbb{B}$ we write $\overline{\mathbb{A}} \leq_{\text {Con }} \overline{\mathbb{B}}$ if and only if $\mathbb{A} \leq_{\text {Con }} \mathbb{B}$. We also write $\overline{\mathbb{A}}<_{\text {Con }} \overline{\mathbb{B}}$ if $\overline{\mathbb{A}} \leq_{\text {Con }} \overline{\mathbb{B}}$ and $\overline{\mathbb{B}} \not \leq_{\text {Con }} \overline{\mathbb{A}}$. The poset

$$
\mathfrak{P}_{\mathrm{fin}}:=\left(\{\overline{\mathbb{A}} \mid \mathbb{A} \text { is a finite structure }\} ; \leq_{\text {Con }}\right)
$$

is called the pp-constructability poset. We denote by $\mathfrak{P}_{n}$ the pp-constructability poset restricted to structures with domain $E_{n}$.

### 3.1.2 Minor-preserving maps

An alternative, yet equivalent, approach to the pp-constructability poset involves a weakening of the notion of clone homomorphism and particular identities called minor identities. In the literature minor-preserving maps are also known as minion homomorphisms and occupy a central part in the algebraic theory inherent to the study of promise CSPs [10]. However, we refrain from defining minions in this dissertation, thus we opt for the name minor-preserving map.

Definition 3.1.5. Let $\tau$ be a set function symbols. An identity is said to be a minor identity if it is of the form

$$
f\left(x_{\pi(0)}, \ldots, x_{\pi(n-1)}\right) \approx g\left(x_{\sigma(0)}, \ldots, x_{\sigma(m-1)}\right)
$$

where $f, g$ are function symbols in $\tau$, and $\pi: E_{n} \rightarrow E_{r}$ and $\sigma: E_{m} \rightarrow E_{r}$ are mappings.
In other words, we require that there is exactly one occurrence of a function symbol on both sides of the equality. The use of nested terms is forbidden. Identities of the form $f\left(x_{1}, \ldots, x_{n}\right) \approx y$ are forbidden as well. A minor condition is a finite set $\Sigma$ of minor identities. The reader might notice that the notion of minor condition is closely
related (see Remark 4.3.2) to the notion of strong linear Mal'cev condition which has been intensively studied in the literature (see ,e.g., [51, 69, 80, 96]). One difference is that, in strong linear Mal'cev conditions, identities of the form $f\left(x_{1}, \ldots, x_{n}\right) \approx y$ are allowed. There is a way to get around this little difference and still make use of several results on Mal'cev conditions from the literature; we will mention it later in this chapter. A minor condition $\Sigma$ is trivial if $\mathcal{P}_{n} \models \Sigma$ for some $n \geq 2$; it is non-trivial otherwise.

In Chapter 4 we deal with Mal'cev conditions in more detail, providing concrete examples and ordering them by strength.

Let $f$ be any $n$-ary operation, and let $\sigma$ be a map from $E_{n}$ to $E_{r}$. We denote by $f_{\sigma}$ the following $r$-ary operation

$$
f_{\sigma}\left(x_{0}, \ldots, x_{r-1}\right):=f\left(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}\right) .
$$

Any operation of the form $f_{\sigma}$, for some map $\sigma: E_{n} \rightarrow E_{r}$, is called a minor of $f$.
Definition 3.1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be clones and let $\xi: \mathcal{A} \rightarrow \mathcal{B}$ be a mapping that preserves arities. We say that $\xi$ is a minor-preserving map if

$$
\xi\left(f_{\sigma}\right)=\xi(f)_{\sigma}
$$

for any $n$-ary operation $f \in \mathcal{A}$ and $\sigma: E_{n} \rightarrow E_{r}$.
We write $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{B}$ if there exists a minor-preserving map $\xi: \mathcal{A} \rightarrow \mathcal{B}$, and we denote by $\equiv_{\mathrm{m}}$ the equivalence relation where $\mathcal{A} \equiv_{\mathrm{m}} \mathcal{B}$ if $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{B}$ and $\mathcal{B} \preceq_{\mathrm{m}} \mathcal{A}$; in this case we say that $\mathcal{A}$ and $\mathcal{B}$ are minor-equivalent or that $\mathcal{A}$ and $\mathcal{B}$ collapse. Analogously to pp-constructability types, we denote by $\overline{\mathcal{A}}$ the $\equiv_{\mathrm{m}}$-class of $\mathcal{A}$ and we write $\overline{\mathcal{A}} \preceq_{\mathrm{m}} \overline{\mathcal{B}}$ if and only if $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{B}$. We also write $\overline{\mathcal{A}} \prec_{\mathrm{m}} \overline{\mathcal{B}}$ if $\overline{\mathcal{A}} \preceq_{\mathrm{m}} \overline{\mathcal{B}}$ and $\overline{\mathcal{B}} \preceq_{\mathrm{m}} \overline{\mathcal{A}}$.

An example of collapse comes from Lemma 3.1.4 phrased in terms of clones, this means that every clone of the form $\mathcal{C}=\operatorname{Pol}(\mathbb{A})$, where $\mathbb{A}$ is a finite core, is minor-equivalent to its idempotent reduct.
Lemma 3.1.7 ([18]). Let $\mathbb{A}$ be a finite core and $\operatorname{let} \mathcal{C}=\operatorname{Pol}(\mathbb{A})$. It holds that $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{C}^{\mathrm{id}}$.

### 3.1.3 Reflections

We present another characterization of the pp-constructability poset. Barto, Opršal, and Pinsker [18] also characterized what happens on the algebraic side of the picture when one compares relational structures by a pp-power or a homomorphic equivalence. The notion of reflection, which we are going to present in this section, somehow is the algebraic counterpart of the notion of homomorphic equivalence. By defining a new operator $\boldsymbol{R}$ for reflections, they obtain a Birkhoff-like theorem, also known as linear Birkhoff or ERP-theorem.
Definition 3.1.8. Let $\mathbf{B}$ be an algebra of type $\tau$, let $A$ be a set, and let $g: A \rightarrow B$ and $h: B \rightarrow A$ be two maps. Then the reflection of $\mathbf{B}$ with respect to $g$ and $h$ is the algebra

A of type $\tau$ on the universe $A$ where for all $a_{1}, \ldots, a_{n} \in A$ and for every $n$-ary $f \in \tau$ we define

$$
f^{\mathbf{A}}\left(a_{1}, \ldots, a_{n}\right):=h\left(f^{\mathbf{B}}\left(g\left(a_{1}\right), \ldots, g\left(a_{n}\right)\right)\right)
$$

We denote by $\boldsymbol{R}(K)$ the class of all reflections of algebras in $K$, where $K$ is a class of algebras of the same type. Once again, as we did previously for the operators $\boldsymbol{E}, \boldsymbol{H}$, $\boldsymbol{S}$, and $\boldsymbol{P}$, we apply the operator $\boldsymbol{R}$ also to clones, with the obvious meaning. First, we present an analogous to the HSP Lemma (Lemma 2.1.5).
Lemma 3.1.9 ([18]). Let $K$ be a class of algebras of the same type $\tau$. Then $\boldsymbol{R} \boldsymbol{P}_{\mathrm{fin}}(K)$ is the smallest class of algebras of type $\tau$ which contains $K$ and is closed under the operators $\boldsymbol{R}, \boldsymbol{H}, \boldsymbol{S}$, and $\boldsymbol{P}_{\text {fin }}$.

Now we are ready to present the linear Birkhoff theorem (for finite structures) which is essentially the genesis of the pp-constructability poset.

Theorem 3.1.10 ([18]). Let $\mathbb{A}$ and $\mathbb{B}$ be finite structures, and let $\mathcal{A}=\operatorname{Pol}(\mathbb{A})$ and $\mathcal{B}=\operatorname{Pol}(\mathbb{B})$. Then the following are equivalent:
(1) $\mathbb{A} \leq_{\text {Con }} \mathbb{B}$ (i.e., $\mathbb{A} p p$-constructs $\mathbb{B}$ );
(2) every minor condition satisfied by $\mathcal{A}$ is also satisfied by $\mathcal{B}$;
(3) $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{B}$ (i.e., there exists a minor-preserving map from $\mathcal{A}$ to $\mathcal{B}$ );
(4) $\mathcal{B} \in \boldsymbol{E} \boldsymbol{R} \boldsymbol{P}_{\text {fin }}(\mathcal{A})$;

Remark 3.1.11. Note that the latter theorem provides an important tool to prove that two elements are distinct in $\mathfrak{P}_{\text {fin }}$ : a compactness argument shows that if $\mathbb{A} \not \mathbb{K}_{\text {Con }} \mathbb{B}$, then there is a minor condition $\Sigma$ which is satisfied in $\mathcal{A}$ but not in $\mathcal{B}$. In this case we also say that $\Sigma$ is a witness of $\mathbb{A} \not \leq_{\mathrm{Con}} \mathbb{B}$, equivalently of $\mathcal{A} \not \varliminf_{\mathrm{m}} \mathcal{B}$.

Corollary 3.1.12 ( 18$]$ ). Let $\mathbb{A}$ be a finite relational structure, let $\mathbb{C}$ be the core of $\mathbb{A}$ and let $\mathbb{C}^{c}$ be the expansion of $\mathbb{C}$ by all unary relations $\{a\}_{a \in C}$. Then:
(1) $\mathbb{A} \equiv_{\text {Con }} \mathbb{C} \equiv_{\text {Con }} \mathbb{C}^{c}$;
(2) for every minor condition $\Sigma, \operatorname{Pol}(\mathbb{A}) \models \Sigma$ if and only if $\operatorname{Pol}\left(\mathbb{C}^{c}\right) \models \Sigma$.

From Theorem 3.1.10 it also follows that the notions presented in the three sections of the current chapter are equivalent; also note that every clone $\mathcal{C}$ on a finite set is of the form $\mathcal{C}=\operatorname{Pol}(\mathbb{C})$, for some finite relational structure $\mathbb{C}$. In the light of this result, as anticipated at the end of Section 3.1 .2 the poset

$$
\left(\{\overline{\mathcal{C}} \mid \mathcal{C} \text { is a clone on a finite set }\} ; \preceq_{\mathrm{m}}\right)
$$

is isomorphic to $\mathfrak{P}_{\text {fin }}$. In fact, we call both posets $\mathfrak{P}_{\text {fin }}$.
Moreover, combining Theorem 3.1.10 and Lemma 3.1.4 it follows that, in order to prove that two elements are distinct in $\mathfrak{P}_{\text {fin }}$, it is sufficient to focus on idempotent clones.

Corollary 3.1.13 ([18]). Let $\mathbb{A}$ be a finite relational structure and let $\mathbb{C}$ be its core. Then, for every minor condition $\Sigma$ we have that $\operatorname{Pol}(\mathbb{A}) \vDash \Sigma$ if and only if $\operatorname{Pol}\left(\mathbb{C}^{c}\right) \models \Sigma$.

Remark 3.1.14. Unlike the case with the operators presented in Theorem 2.1.16, the reflection of a function clone does not necessarily have to be a function clone as it is not required to contain the projections or be closed under composition. Thus, we do not have a result of a flavour similar to Theorem 2.1 .12 and we do not know whether $\mathfrak{P}_{\text {fin }}$ is a lattice or not. In Section 3.2 we prove that $\mathfrak{P}_{\text {fin }}$ is a semilattice.

### 3.2 The shape of $\mathfrak{P}_{\mathrm{fin}}$

In this section, we provide general results on the pp-constructability poset when we do not impose any restrictions on either the domain or the signature. We therefore answer natural questions such as: is $\mathfrak{P}_{\text {fin }}$ a lattice? What is the top/bottom element of $\mathfrak{P}_{\text {fin }}$ ? Does $\mathfrak{P}_{\text {fin }}$ have atoms? Similar issues will be addressed again in Chapters 5 and 6 where we restrict the domain to two-element sets and three-element sets, respectively.

First, we show that $\mathfrak{P}_{\text {fin }}$ is a meet-semilattice. The question whether it is a lattice or not is still open (see Question 7.0.1). Let $\mathbb{A}$ and $\mathbb{B}$ be finite relational structures; for every $f \in \operatorname{Pol}(\mathbb{A})$ and $g \in \operatorname{Pol}(\mathbb{B})$ we define $h:=(f, g) \in \operatorname{Pol}(\mathbb{A}) \times \operatorname{Pol}(\mathbb{B})$ to be the operation on $A \times B$ defined as follows

$$
h\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{n}, b_{n}\right)\right):=\left(f\left(a_{1}, \ldots, a_{n}\right), g\left(b_{1}, \ldots, b_{n}\right)\right)
$$

where $a_{i} \in A$ and $b_{i} \in B$ for every $i \in\{1, \ldots, n\}$.
Let $\Gamma^{\mathbb{A} \otimes \mathbb{B}}:=\operatorname{Inv}(\{(f, g) \mid f \in \operatorname{Pol}(\mathbb{A}), g \in \operatorname{Pol}(\mathbb{B})\})$; we define

$$
\mathbb{A} \otimes \mathbb{B}:=\left(A \times B ; \Gamma^{\mathbb{A} \otimes \mathbb{B}}\right)
$$

Proposition 3.2.1. Let $\mathbb{A}$ and $\mathbb{B}$ be finite relational structures. Then $\overline{\mathbb{A} \otimes \mathbb{B}}$ is the greatest lower bound of $\overline{\mathbb{A}}$ and $\overline{\mathbb{B}}$.
Proof. From Theorem 2.3 .4 it follows that $\operatorname{Pol}(\mathbb{A} \otimes \mathbb{B})=\operatorname{Pol}(\mathbb{A}) \times \operatorname{Pol}(\mathbb{B})$. It is straightforward to check that, for every $f \in \operatorname{Pol}(\mathbb{A})$ and $g \in \operatorname{Pol}(\mathbb{B})$, the maps $(f, g) \mapsto f$ and $(f, g) \mapsto g$ are minor-preserving maps from $\operatorname{Pol}(\mathbb{A} \otimes \mathbb{B})$ to $\operatorname{Pol}(\mathbb{A})$ and to $\operatorname{Pol}(\mathbb{B})$, respectively. Hence, $\mathbb{A} \otimes \mathbb{B} \leq_{\text {Con }} \mathbb{A}, \mathbb{B}$ and $\overline{\mathbb{A} \otimes \mathbb{B}}$ is a lower bound for $\overline{\mathbb{A}}$ and $\overline{\mathbb{B}}$.

Now we show that $\overline{\mathbb{A} \otimes \mathbb{B}}$ is the greatest lower bound: suppose $\mathbb{S} \leq_{\text {Con }} \mathbb{A}, \mathbb{B}$. By Theorem 3.1.10 there exist minor-preserving maps $\xi_{1}: \operatorname{Pol}(\mathbb{S}) \rightarrow \operatorname{Pol}(\mathbb{A})$ and $\xi_{2}: \operatorname{Pol}(\mathbb{S}) \rightarrow$ $\operatorname{Pol}(\mathbb{B})$. Then, for every $h \in \operatorname{Pol}(\mathbb{S})$, the $\operatorname{map} \xi: h \mapsto\left(\xi_{1}(h), \xi_{2}(h)\right)$ is a minor-preserving map from $\operatorname{Pol}(\mathbb{S}) \rightarrow \operatorname{Pol}(\mathbb{A} \otimes \mathbb{B})$, and therefore $\mathbb{S} \leq_{\text {Con }} \mathbb{A} \otimes \mathbb{B}$.

### 3.2.1 The top element

In this section we prove that the pp-constructability poset has a maximum. In particular, we show that the top element of $\mathfrak{P}_{\mathrm{fin}}$ is the pp-constructability type of the loop graph,
i.e., of the structure $\mathbb{C}_{1}:=(\{0\} ;\{(0,0)\})$. Indeed, it is immediate to see that every finite structure pp-constructs $\mathbb{C}_{1}$.

Proposition 3.2.2. Every finite structure $\mathbb{A}$ pp-constructs $\mathbb{C}_{1}$.
Proof. Let $\mathbb{A}$ be a finite structure on the universe $E_{n}$ for some $n \geq 1$. Let $\mathbb{A}^{\prime}$ be the structure $\left(E_{n} ; R\right)$, where $R$ is defined by the primitive positive formula

$$
R(x, y):=\exists z(z=z)
$$

Clearly, $\mathbb{A}^{\prime}$ is homomorphically equivalent to $\mathbb{C}_{1}$.
It follows that $\overline{\mathbb{C}_{1}}$ is the top element of $\mathfrak{P}_{\text {fin }}$. Observe that $\overline{\mathbb{C}_{1}}$ corresponds to the class of finite relational structure whose CSP is trivial: every finite structure $\mathbb{F}$ homomorphically maps to some structure $\mathbb{S} \in \overline{\mathbb{C}_{1}}$. We now give a description of the top element of $\mathfrak{P}_{\text {fin }}$ from the perspective of clones. A constant operation (of arity $n$ ) is an operation $c^{n}$ defined as follows

$$
c^{n}\left(x_{1}, \ldots, x_{n}\right):=c
$$

where $c \in E_{m}$, for some $n, m \geq 1$; if $n=1$, we simply write $c$ to denote the unary constant operation $c^{1}$. We want to remark that if $\mathcal{C}$ has a constant operation $c^{n}$, for some $n$, then it has a constant operation for every arity.

Theorem 3.2.3. Let $\mathcal{A}$ be a clone on a finite set and let $\mathcal{B}$ be a clone on a finite set with a constant operation. Then $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{B}$.

Proof. Note that $\mathcal{B}$ contains a constant operation $c^{n}$ of arity $n$ for every $n \geq 1$. The map $\xi: \mathcal{A} \rightarrow \mathcal{B}$ that maps every $n$-ary operation to $c^{n}$ is minor-preserving.

An immediate consequence of Theorem 3.2 .3 is that all clones with a constant operation belong to the same $\equiv_{\mathrm{m}}$-class. Let $\langle 0\rangle$ be the clone generated by the unary constant operation 0 .

Corollary 3.2.4. The $\equiv_{\mathrm{m}}$-class $\overline{\langle 0\rangle}$ is the top element of $\mathfrak{P}_{\text {fin }}$.
Note that for every minor condition $\Sigma$, it holds that $\langle 0\rangle \models \Sigma$. In particular, the clone $\langle 0\rangle$ satisfies the identity $f(x) \approx f(y)$ which is not satisfied by any idempotent operation. This remark will be useful to us later in this dissertation, e.g., in Proposition 5.3.13.

### 3.2.2 The unique coatom

We continue our investigation of $\mathfrak{P}_{\text {fin }}$ by studying what lies immediately below the top element $\overline{\mathbb{C}_{1}}$. It turns out that $\mathfrak{P}_{\text {fin }}$ has a unique coatom, that is, there is exactly one element of $\mathfrak{P}_{\mathrm{fin}}$ that is covered by the top element $\overline{\mathbb{C}_{1}}$.

For every $n \geq 2$ we denote by $\mathbb{I}_{n}$ the structure $\mathbb{I}_{n}:=\left(E_{n} ;\{0\}, \ldots,\{n-1\}\right)$. We show that $\overline{\mathbb{I}_{2}}$ is the unique coatom of $\mathfrak{P}_{\text {fin }}$. Observe that $\mathcal{I}_{2}:=\operatorname{Pol}\left(\mathbb{I}_{2}\right)$ is the clone that consists
of all idempotent operations on $\{0,1\}$, see Proposition 2.5.4. Note that, for every $n \geq 2$, $\operatorname{Pol}\left(\mathbb{C}_{1}\right) \not \varliminf_{\mathrm{m}} \operatorname{Pol}\left(\mathbb{I}_{n}\right)$, since $\operatorname{Pol}\left(\mathbb{C}_{1}\right)$ satisfies the minor identity $f(x) \approx f(y)$, while $\operatorname{Pol}\left(\mathbb{I}_{n}\right)$ does not. Therefore, for every $n \geq 2$, the structure $\mathbb{C}_{1}$ does not pp-construct $\mathbb{I}_{n}$.

Proposition 3.2.5. For every finite relational structure $\mathbb{A}$ exactly one of the following holds: either $\mathbb{C}_{1} \leq_{\text {Con }} \mathbb{A}$ or $\mathbb{A} \leq_{\text {Con }} \mathbb{I}_{2}$.

Proof. Let $\mathbb{A}$ be a relational structure and let $\mathbb{B}$ be its core expanded by all unary relations. By Corollary 3.1 .12 it holds that $\mathbb{C}_{1}$ pp-constructs $\mathbb{A}$ if and only if $\mathbb{C}_{1}$ pp-constructs $\mathbb{B}$ and $\mathbb{A}$ pp-constructs $\mathbb{I}_{2}$ if and only if $\mathbb{B}$ pp-constructs $\mathbb{I}_{2}$. Thus, we are going to prove the claim for $\mathbb{B}$. Let $B=\left\{b_{0}, \ldots, b_{n-1}\right\}$ be the domain of $\mathbb{B}$. If $n=1$, then it is straightforward to see that $\mathbb{C}_{1} \leq_{\text {Con }} \mathbb{B}$. Let us assume that $n>1$, we need to show that $\mathbb{B} \leq$ Con $\mathbb{I}_{2}$. Consider the pp-power $\mathbb{S}:=\left(\left\{b_{0}, \ldots, b_{n-1}\right\} ; O, I\right)$ of $\mathbb{B}$, where $O$ and $I$ are the unary relations defined by the formulae $O(x):=\left(x=b_{0}\right)$ and $I(x):=\left(x=b_{1}\right)$, respectively. Let us define the maps $g: \mathbb{S} \rightarrow \mathbb{I}_{2}$ that maps $b_{0}$ to 0 and every other element to 1 and $h: \mathbb{I}_{2} \rightarrow \mathbb{S}$ that maps 0 to $b_{0}$ and 1 to $b_{1}$. It is straightforward to check that $g$ and $h$ are homomorphisms. Thus $\mathbb{I}_{2}$ and $\mathbb{S}$ are homomorphically equivalent and $\mathbb{B} \leq$ Con $\mathbb{I}_{2}$.

Proposition 3.2.6. For every $n \geq 2$ it holds that $\mathbb{I}_{2} \equiv{ }_{\text {Con }} \mathbb{I}_{n}$.
Proof. First, we show that $\mathbb{I}_{2} \leq_{\text {Con }} \mathbb{I}_{n}$. Consider the structure $\mathbb{S}:=\left(\{0,1\}^{n} ; \Phi_{0}, \ldots, \Phi_{n-1}\right)$ where each $\Phi_{i}$ is defined as follows:

$$
\Phi_{i}:=\left\{\left(x_{0}, \ldots, x_{n-1}\right) \mid\left(x_{i}=1\right) \wedge \bigwedge_{j \in E_{n} \backslash\{i\}}\left(x_{j}=0\right)\right\}
$$

Let us denote by $\boldsymbol{e}_{i} \in\{0,1\}^{n}$ the tuple that has a 1 in the $i$-th coordinate and 0 s elsewhere. The maps

$$
g: i \mapsto \boldsymbol{e}_{i} \quad h: \boldsymbol{x} \mapsto \begin{cases}i & \text { if } \boldsymbol{x}=\boldsymbol{e}_{i} \\ 0 & \text { otherwise }\end{cases}
$$

are respectively homomorphisms from $\mathbb{I}_{n}$ to $\mathbb{S}$ and from $\mathbb{S}$ to $\mathbb{I}_{n}$. This proves that $\mathbb{I}_{2} \leq$ Con $\mathbb{I}_{n}$. The other inclusion follows from Proposition 3.2.5.

As a consequence, one can think of $\overline{\mathcal{I}_{2}}$ as the class of clones on finite sets that satisfy every minor condition that does not imply the existence of a constant operation.

### 3.2.3 The bottom element

Some well-known problems, such as 3-SAT, 2-SAT, HORN-SAT, systems of linear equations over finite fields or the 3-colorability problem, can be formulated as CSPs over suitable finite relational structures (see Section 2.4. In fact, it was conjectured that finite structures enjoy the following dichotomy: for every finite relational structure $\mathbb{A}$,
either $\operatorname{CSP}(\mathbb{A})$ is NP-complete, or it is solvable in polynomial time [49]. Recently, the dichotomy conjecture was proved independently by Andrei Bulatov [37] and Dmitriy Zhuk [106, 107]. It follows from Proposition 3.1.3 that structures with a "hard" CSP lie at the bottom of $\mathfrak{P}_{\text {fin }}$. What is satisfactory in our setting is that, in a sense, there is actually only one reason of hardness: the class of all structures whose CSP is NP-complete coincides with $\overline{\mathbb{K}_{3}}$, i.e., the pp-constructability type of the complete graph over three vertexes, or equivalently, with $\overline{\mathcal{P}_{2}}$, where $\mathcal{P}_{2}$ is the clone of all projections on $\{0,1\}$. In this section we prove that $\overline{\mathcal{P}_{2}}$ is indeed the bottom element in $\mathfrak{P}_{\text {fin }}$. Furthermore, we provide examples of relational structures structures $\mathbb{A}$ such that $\overline{\mathbb{A}}$ is the bottom element in $\mathfrak{P}_{\text {fin }}$; also, in Theorem 3.2 .18 we provide a characterization of the bottom element in terms of pp -constructability, satisfiability of particular identities, and via minor-preserving maps.

Let $\mathbb{K}_{3}$ be the complete graph over $\{0,1,2\}$, i.e.,

$$
\mathbb{K}_{3}:=\left(\{0,1,2\} ; R_{\neq}\right) \quad \text { where } \quad R_{\neq}:=\left\{(x, y) \in\{0,1,2\}^{2} \mid x \neq y\right\}
$$

The following result can be considered folklore (see, e.g., [24]).
Theorem 3.2.7. The structure $\mathbb{K}_{3} p p$-interprets every finite relational structure.
Corollary 3.2.8. For every finite structure $\mathbb{A}$ it holds that $\mathbb{K}_{3} \leq_{\text {Con }} \mathbb{A}$ and therefore $\overline{\mathbb{K}_{3}}$ is the bottom element in $\mathfrak{P}_{\text {fin }}$.

As we already pointed out in Example $2.4 .2, \operatorname{CSP}\left(\mathbb{K}_{3}\right)$ is equivalent to the 3-colorability problem which is known to be NP-complete (see e.g. [52]). There are other NP-complete problems which can naturally be expressed as $\operatorname{CSP}(\mathbb{S})$ for some suitable finite structure $\mathbb{S}$; here we present two relational structures whose CSP expresses variants of 3-SAT. Let us define the following Boolean relational structures

$$
\begin{aligned}
& 1 \mathbb{N} 3:=(\{0,1\} ;\{(0,0,1),(0,1,0),(1,0,0)\}), \text { and } \\
& \mathbb{N A} \mathbb{E}:=\left(\{0,1\} ;\{0,1\}^{3} \backslash\{(0,0,0),(1,1,1)\}\right) .
\end{aligned}
$$

Here the relations of $1 \mathbb{I N} 3$ and of $\mathbb{N A E}$ should be read as one-in-three and not-all-equal, respectively, two names that rather faithfully describe the tuples that are in the two relations. It is known that $\operatorname{CSP}(1 \mathbb{N} 3)$ and $\operatorname{CSP}(\mathbb{N A E})$ correspond to the well-known problems (1-IN-3)-SAT and NAE-SAT, respectively; both of these problems are known to be NP-complete. Combining these results with Proposition 2.5.8 we obtain the following corollary.

Corollary 3.2.9. Let $\mathbb{A}$ be a finite structure and let $\mathbb{S} \in\left\{\mathbb{K}_{3}, 1 \mathbb{N} 3, \mathbb{N} \mathbb{E}\right\}$. If $\mathbb{A} \leq \operatorname{Int} \mathbb{S}$, then there exist a reduct $\mathbb{A}^{\prime}$ of $\mathbb{A}$ with a finite relational signature and such that $\operatorname{CSP}\left(\mathbb{A}^{\prime}\right)$ is NP-complete.

Now we move to the other side of the Galois connection. The next result can be considered folklore; for a detailed illustration, we recommend Section 2 in [14].

Proposition 3.2.10. The relational structure $\mathbb{K}_{3}^{c}$ has no other polymorphisms than the projections. That is $\operatorname{Pol}\left(\mathbb{K}_{3}^{c}\right)=\mathcal{P}_{3}$.

The next result basically follows from Theorem 2.1.16 and from the fact that the variety $\mathcal{S}$ ets of all sets is generated by the two element set $\{0,1\}$.

Proposition 3.2.11. For every $n \geq 2$ we have that $\mathcal{P}_{2} \equiv$ Con $\mathcal{P}_{n}$.
Proof. Let $\mathbf{E}_{n}:=\left(E_{n} ; \emptyset\right)$; it is known that $\boldsymbol{H S P}\left(\mathbf{E}_{n}\right)=\boldsymbol{H S P}\left(\mathbf{E}_{2}\right)$ and $\operatorname{Clo}\left(\mathbf{E}_{n}\right)=\mathcal{P}_{n}$, for every $n \geq 2$. The claim follows from Theorem 2.1.16.

The next result follows from a straightforward combination of the previous propositions and Theorem 3.1.10

Proposition 3.2.12. There is a clone homomorphism from $\mathcal{P}_{2}$ to every clone on a finite set. Therefore $\overline{\mathcal{P}_{2}}$ is the bottom element in $\mathfrak{P}_{\text {fin }}$.

We could have achieved the same result using the fact that the variety $\mathcal{S}$ ets interprets in every variety $\mathcal{V}$ and by using the correspondence between the notion of interpretability of varieties and notion of clone homomorphism, see Section 2.5 (c.f. Example 4.1.2). In Figure 3.1 we show a draft of the shape of the semilattice $\mathfrak{P}_{\text {fin }}$, in its two facets; on the left: every element of $\mathfrak{P}_{\text {fin }}$ is the $\equiv_{\text {Con }}$-class of some suitable finite relational structure. On the right: every element of $\mathfrak{P}_{\text {fin }}$ is the $\equiv_{\mathrm{m}}$-class of some suitable clone on a finite set.


Figure 3.1: The meet-semilattice $\mathfrak{P}_{\text {fin }}$.
As a next step we want to point out that, for any finite relational structure $\mathbb{A}$, having an NP-complete CSP depends on the fact that $\operatorname{Pol}(\mathbb{A})$ satisfies identities that are too weak in a precise sense that we will specify shortly. With this perspective, we introduce the notion of Taylor identity whose satisfiability or not, as we will see, was proved to be the boundary between finite relational structures whose CSP is in P and those with an NP-complete CSP. First, using Theorem 2.1.16 we can get the following criterion for NP-completeness.

Theorem 3.2.13. Let $\mathbb{A}$ be a finite structure. If $\operatorname{CSP}(\mathbb{A})$ is not $N P$-complete, then there is a finite set of identities $\Sigma$ such that $\operatorname{Pol}(\mathbb{A})$ satisfies $\Sigma$ and $\mathcal{P}_{2}$ does not satisfy $\Sigma$.

Definition 3.2.14. We call Taylor identity a set of identities of the form

$$
t\left(\left[\begin{array}{cccc}
\mathrm{x} & ? & \cdots & ? \\
? & \mathrm{x} & \cdots & ? \\
\vdots & \vdots & \ddots & \vdots \\
? & ? & \cdots & \mathrm{x}
\end{array}\right]\right) \approx t\left(\left[\begin{array}{cccc}
\mathrm{y} & ? & \cdots & ? \\
? & \mathrm{y} & \cdots & ? \\
\vdots & \vdots & \ddots & \vdots \\
? & ? & \cdots & \mathrm{y}
\end{array}\right]\right)
$$

where each $? \in\{x, y\}$. We say that a structure $\mathbb{A}$ has a Taylor polymorphism if $\operatorname{Pol}(\mathbb{A})$ satisfies a Taylor identity.

Definition 3.2.15. An algebra is a Taylor algebra if it has an idempotent term $t$ that satisfies a Taylor identity. Such an operation $t$ is called a Taylor term, and a clone with a Taylor term is called a Taylor clone.

Note that, by definition, any Taylor term cannot be a projection unless the considered algebra has only one element. The next result was originally proved in [101].

Theorem 3.2.16 ([101]). Let $\mathbf{A}$ be an idempotent algebra. If $\operatorname{Pol}(\mathbf{A})$ satisfies a set of identities that cannot be satisfied by $\mathcal{P}_{2}$, then $\operatorname{Pol}(\mathbf{A})$ has a Taylor term.

Taylor's identities may seem obscure and difficult to digest, but they are equivalent to intuitively simpler conditions which we are going to introduce now.

Definition 3.2.17. We define the following minor conditions:

- We call cyclic identity of length $p$, with $p \geq 2$, the following identity

$$
\begin{equation*}
c\left(x_{1}, x_{2}, \ldots, x_{p}\right) \approx c\left(x_{2}, \ldots, x_{p}, x_{1}\right) \tag{p}
\end{equation*}
$$

- We call weak near-unanimity condition of arity $n \geq 3$ the following set of identities

$$
\begin{equation*}
w(x, \ldots, x, y) \approx w(x, \ldots, x, y, x) \approx \ldots \approx w(y, x, \ldots, x) \tag{WNU}
\end{equation*}
$$

The following theorem provides a list of properties that are equivalent to a clone satisfying a Taylor identity and it can be derived combining [12], [38], [79] and [101]. Other items that could be added to Theorem $\sqrt[3.2 .18]{ }$ can be find in the literature, see e.g., [16], [58], [70], [81], and [97]. We will come back to the condition presented in [97] in Section 3.3.1.

Theorem 3.2.18 ([12, 38, [79, 101]). Let $\mathbb{A}$ be a finite relational structure. The following are equivalent:
(1) $\mathbb{A}$ does not pp-construct $\mathbb{K}_{3}$;
(2) There is no minor-preserving map from $\operatorname{Pol}(\mathbb{A})$ to $\mathcal{P}_{2}$;
(3) $\overline{\mathbb{A}} \neq \overline{\mathbb{K}_{3}}$;
(4) $\operatorname{Pol}(\mathbb{A})$ satisfies a Taylor identity;
(5) $\operatorname{Pol}(\mathbb{A})$ satisfies $\mathrm{WNU}(n)$, for some $n \geq 3$;
(6) $\operatorname{Pol}(\mathbb{A})$ satisfies $\Sigma_{p}$, for some prime $p$;
(7) $\operatorname{Pol}(\mathbb{A})$ satisfies $\Sigma_{p}$, for every prime $p>|A|$.

We can then rephrase the so-called CSP Dichotomy Theorem (see Theorem 1.0.1) proved independently by Bulatov and Zhuk as follows:

Theorem 3.2.19 ( $37,106,107)$. Let $\mathbb{A}$ be a finite relational structure such that $\operatorname{Pol}(\mathbb{A})$ does not pp-construct $\mathbb{K}_{3}$, then $\operatorname{CSP}(\mathbb{A})$ is in P. Otherwise, $\operatorname{CSP}(\mathbb{A})$ is NP-complete.

Thus, it follows that the pp-constructability type $\overline{\mathbb{K}_{3}}$ of the complete graph over three vertexes coincides with the class of all the finite structures with a NP-complete CSP.

### 3.2.4 No atoms

In this section we deal with the following question, which arises rather naturally at this point: are there atoms in $\mathfrak{P}_{\text {fin }}$ ? Equivalently, does there exist a finite relational structure $\mathbb{A}$ such that $\overline{\mathbb{A}}$ covers $\overline{\mathbb{K}_{3}}$ ? We prove that the answer to this question is negative.

The following family of directed graphs turns out to be of particular interest in the study of the poset $\mathfrak{P}_{\mathrm{fin}}$. A directed cycle of length $n$ (for some $n \geq 1$ ) is defined as follows

$$
\mathbb{C}_{n}:=\left(E_{n} ;\left\{(i, i+1 \bmod n) \mid i \in E_{n}\right\}\right) .
$$

In the present dissertation, we make use of directed cycles to prove that $\mathfrak{P}_{\text {fin }}$ has an infinite antichain and that is has no atoms. Note that the poset

$$
\mathfrak{P}_{\mathrm{SD}}:=\left(\{\overline{\mathbb{C}} \mid \mathbb{C} \text { is a disjoint union of directed cycles }\} ; \leq_{\text {Con }}\right)
$$

was completely described by Bodirsky, Starke, and the author of the dissertation in [27].
Recall that two elements $a$ and $b$ of a poset $(P, \leq)$ are incomparable if $a \not \leq b$ and $b \not \leq a$. An infinite antichain is an infinite set of pairwise incomparable elements.

Lemma 3.2.20. Let $p, q$ be primes. Then $\operatorname{Pol}\left(\mathbb{C}_{q}\right) \models \Sigma_{p}$ if and only if $p \neq q$.
Proof. If $p \neq q$, then there is an $n \in \mathbb{N}^{+}$such that $p \cdot n=1 \bmod q$. The map

$$
f\left(x_{1}, \ldots, x_{p}\right)=n \cdot\left(x_{1}+\ldots+x_{p}\right) \quad(\bmod q)
$$

is a polymorphism of $\mathbb{C}_{q}$ satisfying $\Sigma_{p}$.

Assume that $f$ is a polymorphism of $\mathbb{C}_{p}$ satisfying $\Sigma_{p}$, then

$$
f(0, \ldots, p-2, p-1)=a=f(1, \ldots, p-1,0)
$$

and $(a, a)$ is a loop, a contradiction.
Corollary $\mathbf{3 . 2} \mathbf{2 1}$. There is an infinite antichain in $\mathfrak{P}_{\mathrm{fin}}$.
Proof. From Lemma 3.2 .20 if follows that, for every pair of distinct primes $p$ and $q, \Sigma_{p}$ is a witness for $\mathbb{C}_{q} \not \leq$ Con $\mathbb{C}_{p}$ and $\Sigma_{q}$ is a witness for $\mathbb{C}_{p} \not \leq$ Con $\mathbb{C}_{q}$. Therefore, $\left\{\overline{\mathbb{C}_{p}} \mid p\right.$ is prime $\}$ is an infinite antichain in $\mathfrak{P}_{\text {fin }}$.

Despite the fact that $\mathfrak{P}_{\text {fin }}$ has an infinite antichain, this does not imply that $\mathfrak{P}_{\text {fin }}$ has the cardinality of the continuum, see Proposition 3.3.23.

Proposition 3.2.22. For every finite structure $\mathbb{A}$, such that $\overline{\mathbb{A}} \neq \overline{\mathbb{K}_{3}}$, there is a finite structure $\mathbb{B}$ such that $\overline{\mathbb{B}}<_{\text {Con }} \overline{\mathbb{A}}$ and $\overline{\mathbb{B}} \neq \overline{\mathbb{K}_{3}}$.

Proof. Let $\mathbb{A}$ be a finite relational structure such that $\overline{\mathbb{A}} \neq \overline{\mathbb{K}_{3}}$; by Theorem 3.2 .18 we have that $\operatorname{Pol}(\mathbb{A}) \mid=\Sigma_{p}$ for some prime $p>|A|$. Let $\mathbb{C}_{p}$ be the directed cycle of length $p$. Consider the structure $\mathbb{B}:=\mathbb{A} \otimes \mathbb{C}_{p}$. Let $q$ be a prime such that $q>p \cdot|A|$. We prove the following:

$$
(\star) \operatorname{Pol}(\mathbb{B}) \not \models \Sigma_{p} \quad \text { and } \quad(\star \star) \operatorname{Pol}(\mathbb{B}) \models \Sigma_{q}
$$

$(\star)$ : From Proposition 3.2 .1 it follows that $\mathbb{B} \leq_{\text {Con }} \mathbb{C}_{p}$; suppose that $\operatorname{Pol}(\mathbb{B}) \models \Sigma_{p}$, from Theorem 3.1.10 we would have that $\operatorname{Pol}\left(\mathbb{C}_{p}\right)$ satisfies $\Sigma_{p}$, too; this is in contradiction with Lemma 3.2.20.
$(\star \star)$ : From Theorem 3.2 .18 we know that $\operatorname{Pol}(\mathbb{A}) \models \Sigma_{q}$ and from Lemma 3.2.20 we have $\operatorname{Pol}\left(\mathbb{C}_{p}\right) \models \Sigma_{q}$. Let $f^{\mathbb{A}} \in \operatorname{Pol}(\mathbb{A})$ and let $f^{\mathbb{C}_{p}} \in \operatorname{Pol}\left(\mathbb{C}_{p}\right)$ be the operation that witnesses $\operatorname{Pol}\left(\mathbb{C}_{p}\right) \models \Sigma_{q}$. Consider the function $f$ defined as follows:

$$
f\left(x_{1}, \ldots, x_{q}\right):= \begin{cases}f^{\mathbb{A}}\left(x_{1}, \ldots, x_{q}\right) & \text { if } x_{1}, \ldots, x_{q} \in A \\ f^{\mathbb{C}_{p}}\left(x_{1}, \ldots, x_{q}\right) & \text { if } x_{1}, \ldots, x_{q} \in E_{p} \\ 0 & \text { otherwise }\end{cases}
$$

By definition, $f \in \operatorname{Pol}(\mathbb{B})$ and $f$ satisfies $\Sigma_{q}$.
From $(\star)$ it follows that $\overline{\mathbb{B}}<_{\text {Con }} \overline{\mathbb{A}}$ in $\mathfrak{P}_{\text {fin }}$, moreover $(\star \star)$ implies that $\overline{\mathbb{B}} \neq \overline{\mathbb{K}_{3}}$.
The reader might ask whether it would be possible to achieve the latter result via the use of the infinite descending chain of minor conditions of the form $\Sigma_{\mathbb{G}}$ introduced in [25]. The answer is negative since, for clones over finite sets, all non-trivial conditions of the form $\Sigma_{\mathbb{G}}$ are equivalent; in particular, they are equivalent to the 6-ary Siggers identity that we will define in Definition 3.3.3.

Theorem 3.2.23. $\mathfrak{P}_{\mathrm{fin}}$ has no atoms.
Proof. The statement follows from Proposition 3.2 .22 .

### 3.2.5 Minimal Taylor clones

We have proved in Section 3.2 .4 that $\mathfrak{P}_{\text {fin }}$ has no atoms. However, the scenario changes significantly when we fix the size of the domain. In fact, the concept of atoms in $\mathfrak{P}_{n}$ is closely related to the concept of minimal Taylor clone on a $n$-element domain: every atom of $\mathfrak{P}_{n}$ is the $\equiv_{\mathrm{m}}$-class of a minimal Taylor clone on $E_{n}$; we will show that the converse is not true even if we consider minimal Taylor clones up to isomorphism. Minimal Taylor clones have been recently introduced and studied by Barto, Brady, Bulatov, Kozik, and Zhuk [9] and play a crucial role in the ongoing attempt to unify and simplify the two existing proofs of the CSP Dichotomy Theorem [37], [106, 107]. In this direction, the notion of minimal Taylor clone (or algebra) seems to be the right framework: the authors studied the core relational structures whose CSP is in P and such that the addition of any supplemental relation to them makes their CSP NP-complete [9]; by the Inv-Pol Galois connection, these maximal relational structures correspond to minimal Taylor clones. Recall the notion of Taylor clone and Taylor algebra from Definition 3.2.15

Definition 3.2.24. A clone $\mathcal{C}$ on a finite domain is called a minimal Taylor clone if $\mathcal{C}$ is Taylor and every proper subclone of $\mathcal{C}$ is not Taylor. A finite algebra $\mathbf{A}$ is a minimal Taylor algebra if $\operatorname{Clo}(\mathbf{A})$ is a minimal Taylor clone.

Proposition 3.2.25 ([9]). Every Taylor clone on a finite domain contains a minimal Taylor clone.

Brady [32] classified all minimal Taylor algebras on a three-element set.
Theorem 3.2.26 ([32], Theorem 4.4.23). There are exactly 24 minimal Taylor algebras over $\{0,1,2\}$, up to term-equivalence and isomorphism.

The next proposition will come in handy in Section 6.2 and provides us all minimal Taylor clones on $\{0,1,2\}$ without any ternary cyclic operation.

Proposition 3.2.27 ([32], Theorem 4.4.12). Let $\mathcal{C}$ be a clone on $E_{3}$ such that $\mathcal{C} \not \vDash \Sigma_{3}$, then $\mathcal{C}$ is isomorphic to one of the following:
(1) $\mathcal{Z}_{3}:=\langle f\rangle$ where $f(x, y, z)=x-{ }_{3} y+{ }_{3} z$ and $+{ }_{3}$ is the sum modulo 3;
(2) $\mathcal{M}_{3}:=\left\langle M_{3}\right\rangle$ where $M_{3}$ is the majority operation on $E_{3}$ that returns the first projection whenever $|\{x, y, z\}|=3$;
(3) $\mathcal{L}:=\langle m\rangle$ where $m$ is the minority operation on $E_{3}$ that returns the first projection whenever $|\{x, y, z\}|=3$;
(4) $\mathcal{W}:=\langle w\rangle$ where $w$ is the binary symmetric idempotent operation on $E_{3}$ such that $w(0,1)=1, w(1,2)=2$, and $w(0,2)=0$.

We now provide an example of a minimal Taylor clone on $E_{3}$ whose $\equiv_{\mathrm{m}}$ is not an atom in $\mathfrak{P}_{3}$. From the classification presented in Theorem 3.2 .26 it turns out that, up to isomorphism, there are exactly three minimal Taylor clones on $E_{3}$ with a majority operation. We denote these three clones by $\mathcal{M}_{1}, \mathcal{M}_{2}$, and $\mathcal{M}_{3}$ where $\mathcal{M}_{i}:=\left\langle M_{i}\right\rangle$, for $i=1,2,3$ and $M_{i}$ is a majority operation defined as follows: $M_{1}$ returns the constant value 0 whenever $|\{x, y, z\}|=3, M_{3}$ is the operation defined in Proposition 3.2.27, and

$$
\begin{aligned}
& M_{2}(0,1,2)=M_{2}(1,2,0)=M_{2}(2,0,1)=0 \\
& M_{2}(0,2,1)=M_{2}(2,1,0)=M_{2}(1,0,2)=1 .
\end{aligned}
$$

Proposition 3.2.28. There is a minor-preserving map from $\mathcal{M}_{2}$ to $\mathcal{M}_{1}$ and from $\mathcal{M}_{3}$ to $\mathcal{M}_{1}$. Moreover, the following inequalities hold:

- $\mathcal{M}_{1} \preceq_{\mathrm{m}} \mathcal{M}_{2}$ and $\mathcal{M}_{1} \preceq_{\mathrm{m}} \mathcal{M}_{3}$;
- $\mathcal{M}_{2} \not \varliminf_{\mathrm{m}} \mathcal{M}_{3}$ and $\mathcal{M}_{3} \preceq_{\mathrm{m}} \mathcal{M}_{2}$.

Proof. The maps $\xi_{1}: M_{2} \mapsto M_{1}$ and $\xi_{2}: M_{3} \mapsto M_{1}$ are minor-preserving maps from $\mathcal{M}_{2}$ to $\mathcal{M}_{1}$ and from $\mathcal{M}_{3}$ to $\mathcal{M}_{1}$, respectively. Clearly, $\mathcal{M}_{1} \models \mathrm{FS}(3)$ while neither $\mathcal{M}_{2}$ nor $\mathcal{M}_{3}$ do have a fully symmetric operation of arity 3 . Moreover, we know from Proposition 3.2 .27 that $\mathcal{M}_{3} \not \vDash \Sigma_{3}$ while $\mathcal{M}_{2}$ clearly does; in addition, we have that $\mathcal{M}_{3}$ satisfies the following minor condition

$$
\begin{aligned}
m(x, x, y) \approx m(x, y, x) & \approx m(y, x, x) \approx m(x, x, x), \\
m(x, y, z) & \approx m(x, z, y)
\end{aligned}
$$

and it is easy to check that $\mathcal{M}_{2}$ does not.
It follows that $\overline{\mathcal{M}}_{1}$ is not an atom in $\mathfrak{P}_{3}$. More recently, a joint work of the author with Barto, Brady, and Zhuk led to the complete classification of the atoms of $\mathfrak{P}_{3}$, a result that will be the object of a future publication. Note that this classification provides a concrete list of the hardest tractable CSPs over $\{0,1,2\}$, refining [34].

### 3.3 Open problems

We conclude this chapter by discussing some open problems that are related to the study of pp-constructability poset. Additional open problems can be found in Chapter 7

### 3.3.1 Loop-conditions and splitting-theorems

We call splitting-theorem every theorem of the following form: let $\Gamma$ be a (possibly infinite) fixed set of finite relational structures and let $\mathbb{A}$ be any finite relational structure, then
either $\mathbb{A}$ pp-constructs some relational structure from $\Gamma$ or $\operatorname{Pol}(\mathbb{A})$ satisfies a particular set of minor conditions $\Sigma_{\Gamma}$; in this case, we say that $\Gamma$ is the set of obstructions for $\Sigma_{\Gamma}$. If $\Gamma=\{\mathbb{A}\}$ we also say that $\mathbb{A}$ is a blocker for $\Sigma_{\Gamma}$. In other words, every splitting-theorem induces a partition of $\mathfrak{P}_{\text {fin }}$ into two classes:

$$
\{\overline{\mathbb{S}} \mid \mathbb{S} \leq \text { Con } \mathbb{B}, \text { where } \mathbb{B} \in \Gamma\} \quad \text { and } \quad\left\{\overline{\mathbb{S}} \mid \operatorname{Pol}(\mathbb{S}) \models \Sigma_{\Gamma}\right\}
$$

The set of minor conditions $\Sigma_{\Gamma}$ and the fixed set of finite structures $\Gamma$ are closely related: in particular, if $\Gamma=\{\mathbb{G}\}$ and $\mathbb{G}$ is a digraphs, then $\Sigma_{\Gamma}$ falls into a special case of minor conditions, known in the literature as loop conditions [82, 83].

Definition 3.3.1. Let $\sigma, \tau: E_{m} \rightarrow E_{n}$ be maps. A loop condition is a minor identity of the form

$$
f_{\sigma} \approx f_{\tau}
$$

An example of loop condition is the condition $\Sigma_{p}$ from Definition 3.2.17. To any loop condition $\Sigma$ we can assign a digraph in a natural way.

Definition 3.3.2. Let $\sigma, \tau: E_{m} \rightarrow E_{n}$ be maps and let $\Sigma$ be the loop condition, given by the identity $f_{\sigma} \approx f_{\tau}$. We define the digraph $\mathbb{G}_{\Sigma}:=\left(E_{n},\left\{(\sigma(i), \tau(i)) \mid i \in E_{m}\right)\right.$ and we refer to it as the digraph associated to the loop condition $\Sigma$.

The name loop-condition comes from the following observation: if a digraph with a polymorphism satisfying $\Sigma$ contains $\mathbb{G}_{\Sigma}$ as a subdigraph, then it also contains a loop. This particular class of minor conditions has been generalized in the literature [10, 25] and also adapted to the case of oligomorphic clones [54].

Definition 3.3.3. We define the following minor conditions:

- We call 6-ary Siggers the identity given by:

$$
s(x, y, z, x, y, z) \approx s(y, x, x, z, z, y)
$$

- We call 4-ary Siggers the identity given by:

$$
s(x, y, y, z) \approx(y, x, z, x)
$$

Example 3.3.4. Here we provide some examples of concrete loop-conditions with their associated digraph:

- The cyclic identity of length $p$ from Definition 3.2.17 is associated to the digraph $\mathbb{C}_{p}$, i.e., the directed cycle of length $p$.
- The 6 -ary Siggers identity is associated to the digraph $\mathbb{K}_{3}$, i.e., the complete graph over $\{0,1,2\}$ which we introduced at the beginning of Section 3.2.3.
- The 4 -ary Siggers is associated to the digraph $\mathbb{S}_{4}$ (Figure 3.2).


Figure 3.2: The digraph $\mathbb{S}_{4}$

As we anticipated in Section 3.2 .3 there are other conditions which are equivalent to the seven items listed in Theorem 3.2.18. In particular, we would like to mention the following results [70, 97] which are in turn our first two examples of splitting-theorems:

Theorem 3.3.5 ( 70,97 ). Let $\mathbb{A}$ be a finite structure. Then either $\mathbb{A}$ pp-constructs $\mathbb{K}_{3}$ or $\operatorname{Pol}(\mathbb{A})$ satisfies the 6 -ary Siggers identity.

Theorem 3.3.6 ([70]). Let $\mathbb{A}$ be a finite structure. Then either $\mathbb{A} p p$-constructs $\mathbb{S}_{4}$ or $\operatorname{Pol}(\mathbb{A})$ satisfies the 4-ary Siggers identity.

Definition 3.3.7. We call quasi Mal'cev the following minor condition:

$$
m(x, y, y) \approx m(y, y, x) \approx m(x, x, x)
$$

Remark 3.3.8. The condition $\Sigma_{\mathrm{M}}^{\prime}$ originally appears in the literature in a slightly different guise and it is known as Mal'cev:

$$
\begin{equation*}
m(x, y, y) \approx m(y, y, x) \approx x . \tag{M}
\end{equation*}
$$

Note that $\Sigma_{\mathrm{M}}$ is not a minor condition. Such a difference is stressed in Definition 3.3.7 by the presence of prefix "quasi" in the name of the condition. Following a general convention, an operation satisfying $\Sigma_{\mathrm{M}}^{\prime}$ is called a quasi Mal'cev operation, while an operation satisfying $\Sigma_{\mathrm{M}}$ is a Mal'cev operation. Note that an idempotent quasi Mal'cev operation is a Mal'cev operation. We use the same convention every time that we present a condition with the prefix "quasi" in its name. For instance, we invite the reader to compare the conditions $\operatorname{QHM}(n)$ from Theorem 3.3.16 and $\operatorname{HM}(n)$ from Theorem 4.2.2 "Q" indeed stands for quasi. Once this clarification has been made, please note that from Corollary 3.1.13 it follows that, in our setting, every strong linear Mal'cev condition can be considered as a minor condition.
Remark 3.3.9. A note on terminology: please be careful not to confuse the notion of Mal'cev condition with the condition " Mal'cev" (also denoted by $\Sigma_{\mathrm{M}}$ ), which is a particular instance of a Mal'cev condition. In fact, $\Sigma_{\mathrm{M}}$ was the very first (strong) Mal'cev condition presented in the literature [77]. See also Section 4.2.1

An example of an operation that satisfies $\Sigma_{\mathrm{M}}^{\prime}$ is the operation $m(x, y, z):=x+y+z$ from Example 2.3.2. Recall that in Example 2.3.2 we showed that $m \notin \operatorname{Pol}\left(\mathbb{B}_{2}\right)$; indeed,
it can be proved that $\mathbb{B}_{2}$ is a blocker for $\Sigma_{\mathrm{M}}^{\prime}$, as stated in the following splitting-theorem which can be deduced from Proposition 7.7 in [84.

Theorem 3.3.10 ([84], Proposition 7.7). Let $\mathbb{A}$ be a finite structure. Then either $\mathbb{A}$ pp-constructs $\mathbb{B}_{2}$ or $\operatorname{Pol}(\mathbb{A})$ satisfies $\Sigma_{M}^{\prime}$.

The next result is a consequence of Lemma 6.8 in [27] due to Bodirsky, Starke, and the author of this dissertation. The same result could be reached using Theorem 4.1.8 in [32] who in turn refers the reader to [12] and [108].

Theorem 3.3.11 ([27, [108]). Let $\mathbb{A}$ be a finite structure. Then for every prime $p$ either $\mathbb{A}$ pp-constructs $\mathbb{C}_{p}$ or $\operatorname{Pol}(\mathbb{A})$ satisfies the cyclic identity $\Sigma_{p}$.

The latter two theorems will come in handy in Section 3.3.2 in the investigation of submaximal elements of $\mathfrak{P}_{\text {fin }}$.

Sometimes splitting-theorems are closely related to complexity classes; for example, the partition induced by Theorem 3.3 .5 marks the boundary between finite between finite structures whose CSP is NP-complete and finite structures whose CSP is in P, as stated in the CSP Dichotomy theorem. Analogously, there are other splitting-theorems which are conjectured to mark a boundary for finer complexity classes. Next we present the most important results in this direction that appear in the literature; we would like to mention that the following results are reworded from the survey [17].

Recall the relation $\leq_{2}$ from Example 2.2.2. First, we define the following relational structures:

$$
\begin{aligned}
\mathbb{B} \leq & :=\left(E_{2} ; \leq_{2},\{0\},\{1\}\right) \\
\mathbb{H O R N} & :=\left(E_{2} ; R_{110}, R_{111},\{0\},\{1\}\right) \\
3 \mathbb{L} \mathbb{N} \mathbb{N}_{p} & :=\left(E_{p} ; \text { all affine subspaces } R_{a b c d} \text { of } \mathbb{Z}_{p}^{3} \text { of dimension } 2\right)
\end{aligned}
$$

where $\mathbb{Z}_{p}$ is the Galois field with $p$ elements, and for all $a, b, c, d \in E_{p}$ :

$$
\begin{aligned}
R_{a b c} & :=\{0,1\}^{3} \backslash\{(a, b, c)\}, \\
R_{a b c d} & :=\left\{(x, y, z) \in \mathbb{Z}_{p}^{3} \mid a x+b y+c z=d\right\} .
\end{aligned}
$$

Note that all the structures above, with the exception of $3 \mathbb{L} \mathbb{N} \mathbb{N}_{p}$ for $p \geq 3$, are Boolean structures and we will encounter them again in Section 5.3.

The first splitting-theorem we are going to present is about the family of relational structures $3 \mathbb{L} \mathbb{N}_{p}$, where $p$ is a prime.

Theorem 3.3.12 ( 11,73 ). Let $\mathbb{A}$ be a finite relational structure, then either
(1) $\mathbb{A} \leq_{\text {Con }} 3 \mathbb{L} \mathbb{N}_{p}$, for some prime $p$, or
(2) $\operatorname{Pol}(\mathbb{A}) \models \mathrm{WNU}(n)$ for every $n \geq 3$.

A rather significant result in the literature is the proof that the class of structures satisfying the second item of Theorem 3.3 .12 coincides with the class of finite relational structures whose CSP has bounded width [11, 13, 35]. We do not define here what does it mean for a structure to have bounded width, however it might be someone's cup of tea to know that this notion is equivalent to solvability by a Datalog program 49]. Other conditions which are equivalent to the second item of Theorem 3.3.12 can be find, e.g, in [31, 58, 66, 73, 79].

There is an analogous splitting-theorem for $\mathbb{H O R} \mathbb{N}$; the minor condition involved in the next theorem comes from [58].

Theorem 3.3.13 ([58). Let $\mathbb{A}$ be a finite relational structure, then either
(1) $\mathbb{A} \leq_{\text {Con }} \mathbb{H O R} \mathbb{N}$, or
(2) $\operatorname{Pol}(\mathbb{A})$ satisfies a Hobby-McKenzie condition, i.e., a minor condition of the form

$$
t\left(\left[\begin{array}{cccc}
x & ? & \cdots & ? \\
x & x & \cdots & ? \\
\vdots & \vdots & \ddots & \vdots \\
x & x & \cdots & x
\end{array}\right]\right) \approx t\left(\left[\begin{array}{cccc}
y & ? & \cdots & ? \\
? & y & \cdots & ? \\
\vdots & \vdots & \ddots & \vdots \\
? & ? & \cdots & y
\end{array}\right]\right)
$$

where each $? \in\{x, y\}$.
The next result comes from [58] and is related to another important complexity class: indeed, the class described by the second item of Theorem 3.3.14 contains the class of finite relational structures whose CSP has bounded linear width [1]. Moreover, it has been proved that CSPs with bounded linear width are in the complexity class NL [44].

Theorem 3.3.14 ([58). Let $\mathbb{A}$ be a finite relational structure, then either
(1) $\mathbb{A} \leq_{\text {Con }} 3 \mathbb{L} \mathbb{N}_{p}$, for some prime $p$, or $\mathbb{A} \leq_{\text {Con }} \mathbb{H O R} \mathbb{N}$, or
(2) for some $n \geq 2, \operatorname{Pol}(\mathbb{A})$ satisfies the following minor condition $\operatorname{QSD}_{\vee}(n)$ :

$$
\begin{aligned}
d_{0}(x, y, z) & =d_{0}(x, x, x), \\
d_{i}(x, y, y) & =d_{i+1}(x, y, y) \quad \text { and } \quad d_{i}(x, y, x)=d_{i+1}(x, y, x) \text { if } i \text { is even, } i<n, \\
d_{i}(x, x, y) & =d_{i+1}(x, x, y) \text { if } i \text { is odd, } i<n, \\
d_{n}(x, y, z) & =d_{n}(z, z, z),
\end{aligned}
$$

It has been conjectured that the two classes of relational structures mentioned above coincide.

Conjecture $3.3 .15([75])$. Let $\mathbb{A}$ be a structure such that $\operatorname{Pol}(\mathbb{A})$ satisfies the minor condition of the second item in Theorem 3.3.14, then $\operatorname{CSP}(\mathbb{A})$ has bounded linear width.

The next minor condition comes from [56].
Theorem 3.3.16 ([56]). Let $\mathbb{A}$ be a finite relational structure, then either
(1) $\mathbb{A} \leq$ Con $\mathbb{B} \leq$, or
(2) for some $n \geq 2, \operatorname{Pol}(\mathbb{A})$ satisfies the minor condition $\operatorname{QHM}(n)$, that is:

$$
\begin{aligned}
p_{0}(x, y, z) & =p_{0}(x, x, x) \\
p_{i}(x, x, y) & =p_{i+1}(x, y, y) \text { for all } i<n . \\
p_{n}(x, y, z) & =p_{n}(z, z, z)
\end{aligned}
$$

The minor condition $\operatorname{QHM}(n)$ is a generalization of the condition quasi Mal'cev from Definition 3.3.7, as will be clarified in Chapter 4

The next minor condition comes again from [58] and it is related to another interesting descriptive complexity class: symmetric Datalog [47].

Theorem 3.3.17 ([58]). Let $\mathbb{A}$ be a finite relational structure, then either
(1) $\mathbb{A} \leq$ Con $3 \mathbb{L} \mathbb{N}_{p}$, for some prime $p$, or $\mathbb{A} \leq_{\text {Con }} \mathbb{B} \leq$, or
(2) for some $n \geq 2, \operatorname{Pol}(\mathbb{A})$ satisfies the following minor condition $4-\operatorname{HMcK}(n)$ :

$$
\begin{aligned}
f_{0}(x, y, y, z) & =f_{0}(x, x, x, x) \\
f_{i}(x, x, y, x) & =f_{i+1}(x, y, y, x) \quad \text { and } \quad f_{i}(x, x, y, y)=f_{i+1}(x, y, y, y) \text { for all } i<n \\
f_{n}(x, x, y, z) & =f_{n}(z, z, z, z)
\end{aligned}
$$

Analogously to Conjecture 3.3.15 it was conjectured that the class of structures satisfying the second item of Theorem 3.3.17 and the class of structures with a CSP having bounded symmetric width coincide. One of the implications was proved in [48].

Conjecture 3.3.18 ([47]). Let $\mathbb{A}$ be a structure such that $\operatorname{Pol}(\mathbb{A})$ satisfies the minor condition of the second item in Theorem 3.3 .17 , then $\operatorname{CSP}(\mathbb{A})$ has bounded symmetric width.

The two conjectures are linked in the following way: Kazda [67] proved that if a structure $\mathbb{A}$ satisfies the minor condition $\operatorname{QHM}(n)$, for some $n$, (see Theorem 3.3.16) and $\operatorname{CSP}(\mathbb{A})$ has bounded linear width, then $\operatorname{CSP}(\mathbb{A})$ has bounded symmetric width. Thus, Conjecture 3.3 .18 reduces to Conjecture 3.3.15. Furthermore, the class of relational structures $\mathbb{A}$ such that $\operatorname{CSP}(\mathbb{A})$ has bounded linear (respectively symmetric) width is conjectured to coincide with the class of relational structures $\mathbb{A}$ such that $\operatorname{CSP}(\mathbb{A})$ is in NL (respectively in L).

In Figure 3.3 we present a figure that aims to overview what is presented in this section. In the figure: black arrows either follow from definition or are folklore. Red


Figure 3.3: An overview of the results presented in Section 3.3.1
arrows are presented in Chapter 3 of the dissertation. Green arrows either follow from definition or from the pp-constructability order. Cyan arrows can be deduced combining green and red arrows and will be discussed again in Chapter 4

### 3.3.2 Submaximal elements

In Section 3.2 .2 we proved that $\mathfrak{P}_{\text {fin }}$ has a unique coatom: $\overline{\mathbb{I}_{2}}$. In analogy to classical clone theory, our investigation continues with the intention of determining whether there are (and what are the) elements that have $\overline{\mathbb{I}_{2}}$ as their only cover-element. We say that an element $\overline{\mathbb{A}}$ is submaximal in $\mathfrak{P}_{\text {fin }}$ if $\overline{\mathbb{A}}$ is covered by $\overline{\mathbb{I}_{2}}$. In Chapter 5 we present a complete description of $\mathfrak{P}_{2}$, from which it comes out that $\overline{\mathbb{C}_{2}}$ and $\overline{\mathbb{B}_{2}}$ are the only submaximal elements in $\mathfrak{P}_{2}$. In Section 6.1 we prove that $\overline{\mathbb{C}_{2}}, \overline{\mathbb{C}_{3}}$, and $\overline{\mathbb{B}_{2}}$ are the only submaximal elements in $\mathfrak{P}_{3}$. In particular, we prove that if $\mathbb{A}$ is a structure on $E_{3}$ such that $\operatorname{Pol}(\mathbb{A})$ satisfies $\Sigma_{M}^{\prime}\left(\right.$ Definition 3.3.7), $\operatorname{Pol}(\mathbb{A}) \models \Sigma_{2}$, and $\operatorname{Pol}(\mathbb{A}) \models \Sigma_{3}$, then there is a minor-preserving map from $\mathcal{I}_{2}=\operatorname{Pol}\left(\mathbb{I}_{2}\right)$ to $\operatorname{Pol}(\mathbb{A})$. Note that, in this case, Theorem 3.2 .18 implies that $\operatorname{Pol}(\mathbb{A}) \models \Sigma_{p}$ for every prime $p$.

In addition, let $\mathfrak{P}_{\mathrm{D}}:=\left(\{\overline{\mathbb{D}} \mid \mathbb{D}\right.$ is finite directed graph $\left.\} ; \leq_{\text {Con }}\right)$ : it has been proved that $\overline{\mathbb{C}_{p}}$, for every prime $p$, and $\overline{\mathbb{B}_{2}}$ are the only submaximal elements in $\mathfrak{P}_{\mathrm{D}}$ [26]. In the light of these developments, the temptation to try to prove that the last result can be generalised to the case of $\mathfrak{P}_{\text {fin }}$ was strong. However, there is a structure in the literature
that provides a counterexample.
Definition 3.3.19. We define the following minor conditions:

- We call n-ary (fully) symmetric condition the set of all identities of the form

$$
f\left(x_{1}, x_{2}, \ldots, x_{n}\right) \approx f\left(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}\right), \quad(\operatorname{FS}(n))
$$

where $\pi$ is a permutation of the set $\{1,2, \ldots, n\}$.

- We call $n$-ary totally symmetric condition the set of all identities of the form

$$
\begin{equation*}
f\left(x_{1}, x_{2}, \ldots, x_{n}\right) \approx f\left(y_{1}, y_{2}, \ldots, y_{n}\right) \tag{n}
\end{equation*}
$$

where $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}=\left\{y_{1}, y_{2}, \ldots, y_{n}\right\}$. If $f$ is a totally symmetric operation we also write $f\left(\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}\right)$ instead of $f\left(x_{1}, x_{2}, \ldots, x_{n}\right)$.

Carvalho and Krokhin [41] presented a structure $\mathbb{K}$ with 21 elements that has cyclic polymorphisms of all arities, a Mal'cev polymorphism, and that does not have any fully symmetric polymorphism of arity 5 . Indeed, from Theorem 3.3.11 and $\operatorname{Pol}(\mathbb{K}) \models \Sigma_{p}$ for every prime $p$, it follows that $\mathbb{K} \not$ Con $^{\mathbb{C}_{p}}$. Carvalho and Krokhin also show that $\operatorname{Pol}(\mathbb{K})$ satisfies $\Sigma_{\mathrm{M}}$, it follows from Theorem 3.3 .10 that $\mathbb{K} \not$ Con $\mathbb{B}_{2}$. Moreover, we have that $\mathbb{I}_{2} \not \backslash_{\text {Con }} \mathbb{K}$, since $\mathcal{I}_{2} \models \mathrm{FS}(5)$ while $\operatorname{Pol}(\mathbb{K}) \notin \mathrm{FS}(5)$.

The structure $\mathbb{K}$ is defined as follows: $\mathbb{K}:=(K ; R, S)$ where

$$
K=\{0,1,2, \ldots, 9,10, a, b, c, d, e, f, g, h, i, j\}
$$

and $R$ and $S$ are binary relations that are graphs of the following permutations $r$ and $s$, respectively (see Figure 3.4,

$$
\begin{aligned}
& r=(012)\left(\begin{array}{ll}
5 & 6
\end{array}\right)(8910)(e b a)(d g i)(f h c), \\
& s=(14)(23)(56)(78)(j e)(b c)(a d)(i f) \text {. }
\end{aligned}
$$

The author would like to thank Zarathustra Brady for notifying us of the existence of the structure $\mathbb{K}$ in the literature and for many other fruitful conversations. In the light of this, we have to weaken our pretensions; we then formulate the following conjecture.

Conjecture 3.3.20. Let $\mathbb{A}$ be a finite relational structure such that $\operatorname{Pol}(\mathbb{A}) \models \operatorname{TS}(n)$, for every $n \geq 2$, and such that $\operatorname{Pol}(\mathbb{A})$ satisfies $\Sigma_{M}$. Then there is a minor-preserving map from $\mathcal{I}_{2}$ to $\operatorname{Pol}(\mathbb{A})$.

### 3.3.3 Cardinality

As already mentioned, Post fully described the lattice $\mathfrak{L}_{2}$ of all clones on a two-element set ordered by inclusion 91 (see Theorem 2.1 .12 for the definition of $\mathfrak{L}_{n}$ ). The result is


Figure 3.4: The structure $\mathbb{K}:=(K ; R, S)$.
the countably infinite lattice displayed in Figure 5.1. The scenario changes drastically already when we consider clones on a set of size three. Indeed, Janov and Mučnik [61] showed that there are continuum many clones over a set with at least three elements.

Theorem 3.3.21 ([61]). Let $A$ be a finite set such that $|A| \geq 3$. Then the number of clones on $A$ is continuum.

With the current state of knowledge, a complete classification of $\mathfrak{L}_{3}$ seems beyond hope. Subsequent research in universal algebra therefore focused on understanding particular aspects of clone lattices on finite domains, for example on the description of maximal clones [92] or minimal clones [43, 42, 93].

However, it is easy to show that all the clones considered in the proof of Janov and Mučnik are minor-equivalent to some Boolean clone. Since there are only countably infinite many Boolean clones, the family of uncountable clones considered in [61] does not imply the existence of uncountably many elements in $\mathfrak{P}_{\mathrm{fin}}$. Similarly, all the families of uncountable clones with at least a constant operation are minor-equivalent to $\langle 0\rangle$ (see Theorem 3.2.3). In fact, it is a consequence of Corollary 3.1.13 the family of uncountable clones that would witness that $\mathfrak{P}_{\text {fin }}$ is uncountable necessarily has to be a family of idempotent clones. Uncountably many idempotent clones can be find in the literature already over a three-element set: Marchenkov [78] proved that there are uncountably many clones of self-dual operations, and Zhuk [104] presented a complete description of their lattice. The proof that there are only countably many $\equiv_{\mathrm{m}}$-classes of clones of self-dual operations on a three-element domain is non-trivial and we prove this in Section 6.2. This encouraging result led us to formulate the following conjecture.

Conjecture 3.3.22. The poset $\mathfrak{P}_{3}$ is at most countably infinite.

In addition, there is hope of achieving a similar result even if one does not place a bound on the cardinality of the domain. It follows from the complete description of the lattice

$$
\mathfrak{P}_{\mathrm{SD}}:=\left(\{\overline{\mathbb{C}} \mid \mathbb{C} \text { is a disjoint union of directed cycles }\} ; \leq_{\text {Con }}\right)
$$

that such lattice is countable. Moreover, we would like to point out that Corollary 3.2.21 does not imply that $\mathfrak{P}_{\text {fin }}$ has $2^{\omega}$ elements.

Proposition 3.3.23. Let $\mathbb{A}$ be a finite relational structure such that $\mathbb{A} \leq$ Con $\mathbb{C}_{p}$ for all but finitely many primes $p$. Then $\overline{\mathbb{A}}=\overline{\mathbb{K}_{3}}$.

Proof. Let $\mathbb{A}$ be a finite structure satisfying the hypothesis of the proposition and suppose $\overline{\mathbb{A}} \neq \overline{\mathbb{K}_{3}}$. Then by Theorem 3.2 .18 we have that $\operatorname{Pol}(\mathbb{A}) \vDash \Sigma_{q}$, for every prime $q>|A|$. Hence, there exists a prime $q_{0}$ big enough such that $\mathbb{A} \leq \operatorname{Con} \mathbb{C}_{q_{0}}$ and $\operatorname{Pol}(\mathbb{A})=\Sigma_{q_{0}}$. This would be a contradiction to Theorem 3.3.11.

The last two mentioned results tempt us to formulate the following provocative conjecture.

Conjecture 3.3.24. The poset $\mathfrak{P}_{\mathrm{fin}}$ is at most countably infinite.
Moreover, Aichinger, Mayr, and McKenzie proved that modulo term-equivalence and a renaming of the elements, there are only countably many finite algebras with a Mal'cev term [2]. Thus, from Theorem 3.3.10 it follows that in order to solve Conjecture 3.3 .24 it is enough to determine the cardinality of the subposet $\left\{\overline{\mathbb{A}} \mid \overline{\mathbb{A}} \leq_{\text {Con }} \overline{\mathbb{B}_{2}}\right\}$. The main result from [2] is a generalization of a previous result of Bulatov who proved that there are only finitely many clones on $\{0,1,2\}$ containing a Mal'cev operation [33].

## Chapter 4

## The geography of linear Mal'cev conditions

This chapter is a survey of results from the literature, thus the reader might make "encyclopedic" use of it. In fact, we opted not to go into depth on several of the topics covered in this chapter, preferring instead to provide references to the literature for the reader interested in more details. For example, we deal only superficially with tame congruence theory, simply referring to the five types described by Hobby and McKenzie [58] and providing the equational conditions to which they correspond.

We decided to include this chapter in the dissertation for the following reasons: first, we are not aware of a survey in the literature that covers such a large number of Mal'cev conditions ordered by strength. Second, we will use most of the Mal'cev conditions we present here in Chapter 5 and Chapter 6 as a witness of $\mathcal{A} \npreceq_{\mathrm{m}} \mathcal{B}$ for some clones $\mathcal{A}$ and $\mathcal{B}$ on a finite set (see Remark 3.1.11 and Remark 3.3.8). Many of the classes dealt with in this chapter are of interest in different communities: the same class therefore appears in the literature in different guises; in this sense, this chapter serves as a dictionary.

### 4.1 Mal'cev filters

We have already (informally) defined strong Mal'cev conditions as finite sets of particular identities satisfied by clones, see Definition 2.1.15 Formally, a strong Mal'cev condition is a primitive positive sentence in the language of clones [69]. However, for our purpose it is more convenient to look at Mal'cev conditions from a slightly different perspective: by identifying a (strong) Mal'cev condition $\Sigma$ with the class of varieties satisfying $\Sigma$, it turns out that $\Sigma$ can be associated to a (principal) filter in the lattice of interpretability types of varieties. Thus, implications among Mal'cev conditions translate into inclusion among the corresponding associated filters.

Definition 4.1.1. Let $\mathcal{V}$ and $\mathcal{W}$ be any two varieties and let $F$ be the set of basic operation symbols of $\mathcal{V}$. An interpretation of $\mathcal{V}$ in $\mathcal{W}$ is a function $\iota$ from $F$ to the set of terms of $\mathcal{W}$, such that the following statements hold:
(1) for every operation $f \in F, \iota(f)$ is a term of $\mathcal{W}$ of the same arity of $f$;
(2) for any algebra $\mathbf{A} \in \mathcal{W}$, the algebra $\mathbf{A}^{\iota}:=\left(A ;\left\{\iota(f)^{\mathbf{A}} \mid f \in F\right\}\right) \in \mathcal{V}$.

Likewise in Chapter 2 and Chapter 3 with similar notions, we write $\mathcal{V} \leq_{\iota} \mathcal{W}$ if there exists an interpretation of $\mathcal{V}$ in $\mathcal{W}$. We write $\mathcal{V} \equiv_{\iota} \mathcal{W}$ if $\mathcal{V} \leq_{\iota} \mathcal{W} \leq_{\iota} \mathcal{V}$. The relation $\equiv_{\iota}$ induces a partition on the class of all varieties in $\equiv_{\iota}$-classes, called interpretability types of varieties. As usual, $\overline{\mathcal{V}}$ denotes the interpretability type of $\mathcal{V}$, and we write $\overline{\mathcal{V}} \leq_{\iota} \overline{\mathcal{W}}$ if and only if $\mathcal{V} \leq_{\iota} \mathcal{W}$. We then define

$$
\mathfrak{L}_{\text {Var }}:=\left(\overline{\mathcal{V}} \mid \mathcal{V} \text { is a variety } ; \leq_{\iota}\right)
$$

which turns out to be a lattice and is known in the literature as the lattice of interpretability types of varieties [51, 80].

Example 4.1.2. In Example 2.1 .9 we introduced Sets, i.e., the variety of all sets. Note that $\mathcal{S}$ ets is trivially interpretable in any variety $\mathcal{V}$, i.e., for every variety $\mathcal{V}$ we have that $\overline{\mathcal{S e t s}} \leq_{\iota} \overline{\mathcal{V}}$, and hence $\overline{\mathcal{S} \text { ets }}$ is the bottom element of $\mathfrak{L}_{\text {Var }}$.

Definition 4.1.3. A Mal'cev condition is a countable set $\Sigma:=\left\{\overline{\mathcal{V}_{i}} \mid i \geq 0\right\}$ of interpretability types of finitely presentable varieties, such that $\overline{\mathcal{V}_{i+1}} \leq_{\iota} \overline{\mathcal{V}_{i}}$ for every $i \geq 0$. If $\overline{\mathcal{V}_{i+1}}=\overline{\mathcal{V}_{i}}$, for every $i \geq 0$, we say that $\Sigma=\left\{\overline{\mathcal{V}_{0}}\right\}$ is a strong Mal'cev condition. A strong Mal'cev condition $\{\overline{\mathcal{V}}\}$ is linear if $\mathcal{V}=\operatorname{Mod}(\Gamma)$ and $\Gamma$ is a finite set of linear identities, i.e., identities where for every $(p \approx q) \in \Gamma$ at most one function symbol appears both in $p$ and in $q$ (the use of nested terms is forbidden).

Consequently, we also adapt the definition of satisfiability of a Mal'cev condition.
Definition 4.1.4. Let $\Sigma:=\left\{\overline{\mathcal{M}_{i}} \mid i \geq 0\right\}$ be a Mal'cev condition, and $\mathcal{V}$ a variety. We say that $\mathcal{V}$ satisfies $\Sigma$ if there exists $i \geq 0$ such that $\overline{\mathcal{M}_{i}} \leq \iota \overline{\mathcal{V}}$.

Let $C_{\Sigma} \subseteq L_{\text {Var }}$, where $L_{\text {Var }}$ is the class of all interpretability types of varieties; $C_{\Sigma}$ is a (strong) Mal'cev class if there exists a (strong) Mal'cev condition $\Sigma:=\left\{\overline{\mathcal{M}_{i}} \mid i \geq 0\right\}$ with

$$
C_{\Sigma}=\left\{\overline{\mathcal{V}} \mid \exists i\left(\overline{\mathcal{M}_{i}} \leq_{\iota} \overline{\mathcal{V}}\right)\right\} .
$$

Note that a (strong) Mal'cev class $C_{\Sigma}$ generates a (principal) filter in $\mathfrak{L}_{\text {Var }}$, to which we refer to as the Mal'cev filter $C_{\Sigma}$ (see also [96]).

Frequently, when $\mathcal{V}_{0}$ is finitely presentable, it is convenient to denote the (strong) Mal'cev condition $\left\{\overline{\mathcal{V}_{0}}\right\}$ by $\Sigma$ where $\Sigma$ is such that $\mathcal{V}_{0}=\operatorname{Mod}(\Sigma)$.

Definition 4.1.5. Let $\Sigma, \Gamma$ be linear Mal'cev conditions. We say that $\Sigma$ implies $\Gamma$ (equivalently, $\Sigma$ is stronger than $\Gamma$ ), denoted $\Sigma \Rightarrow \Gamma$, if $C_{\Sigma} \subseteq C_{\Gamma}$. Two linear Mal'cev conditions $\Sigma$ and $\Gamma$ are equivalent if $\Sigma \Rightarrow \Gamma$ and $\Gamma \Rightarrow \Sigma$.

As already mentioned in the final part of Chapter 2, the lattice $\mathfrak{L}_{\text {Var }}$ is isomorphic to the lattice that arises from the order defined in Theorem 2.5.9 Define $\operatorname{Clo}(\mathcal{V})$ to be equal to $\operatorname{Clo}(\mathbf{A})$ for some generator $\mathbf{A}$ of $\mathcal{V}$, the choice of generator is irrelevant. The
notion of interpretability between varieties translates to clone homomorphism as follows: $\mathcal{V} \leq_{\iota} \mathcal{W}$ if $\operatorname{Clo}(\mathcal{V}) \leq_{\mathrm{h}} \operatorname{Clo}(\mathcal{W})$, i.e., if there exists a clone homomorphism from $\operatorname{Clo}(\mathcal{V})$ to $\operatorname{Clo}(\mathcal{W})$. Since in this dissertation we deal more with clones than with varieties, we consider a lattice which is isomorphic to $\mathfrak{L}_{\text {Var }}$ :

$$
\mathfrak{L}_{\mathrm{Clo}}:=\left(\overline{\operatorname{Clo}(\mathcal{V})} \mid \mathcal{V} \text { is a variety } ; \leq_{\mathrm{h}}\right)
$$

Thus, Definition 4.1.3, Definition 4.1.4 and Definition 4.1.5 can easily be adapted to clones. For instance, a Mal'cev condition can be seen as the set $\left\{\overline{\operatorname{Clo}\left(\mathcal{V}_{i}\right)} \mid i \geq 0\right\}$ for some finitely presentable variety $\mathcal{V}_{i}$. Implication between Mal'cev conditions translates then to inclusion between clones in $\mathfrak{L}_{\text {Clo }}$.

Note that, since in this dissertation we only focus on clones over finite sets and from Lemma 3.1.7 we have that $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{C}^{\text {id }}$, we will state most of the results for varieties that are locally finite and idempotent (a variety $\mathcal{V}$ is idempotent if every algebra contained in $\mathcal{V}$ is idempotent) at the cost of presenting weaker results even when some of the statements are still true in a broader setting. We would also like to point out that in $[58]$ a theory of the local structure of finite algebras is developed, known as tame congruence theory. One of the main achievements of the theory is that there are exactly five types of local behaviour that can be associated with locally finite varieties. For each locally finite variety $\mathcal{V}$ and each type $\boldsymbol{i} \in\{\mathbf{1}, \ldots, \boldsymbol{5}\}, \mathcal{V}$ either admits $\boldsymbol{i}$ or omits $\boldsymbol{i}$. The types are often are often addressed by the following names: type $\mathbf{1}$ (also, the unary type), type $\mathbf{2}$ (the affine type), type $\mathbf{3}$ (the Boolean type), type $\mathbf{4}$ (the lattice type), and type $\mathbf{5}$ (the semilattice type). Avoiding entering into Tame Congruence Theory, we will use this theory as a black-box to easily obtain implications between Mal'cev conditions (in the sense of Definition 4.1.5 and for the sake of exhaustiveness.

### 4.1.1 Filters of varieties omitting lattices

There is another way of generating filters in $\mathfrak{L}_{\text {Var }}$ that has received considerable attention in the literature: filters arising from varieties that omit particular lattices (see, e.g., [69]). We would attempt to keep this topic as succinct as possible so as not to deviate too much from the topics on which this dissertation is focused on.

Recall that a congruence of an algebra $\mathbf{A}$ is an equivalence relation $\alpha \subseteq A^{2}$ of $\mathbf{A}$ that is preserved by every operation of $\mathbf{A}$ in the sense of Definition 2.3.1 As usual, for $X \subseteq A^{2}$ the congruence generated by $X$ is the smallest congruence containing $X$. Let us denote by $\operatorname{Con}(\mathbf{A})$ the set of all congruences of an algebra $\mathbf{A}$; it is well known that $\operatorname{Con}(\mathbf{A})=(\operatorname{Con}(\mathbf{A}) ; \wedge, \vee)$ is a lattice where for every two congruences $\alpha$ and $\beta, \alpha \wedge \beta$ is the intersection of $\alpha$ and $\beta$, whereas $\alpha \vee \beta$ is the congruence generated by $\alpha \cup \beta$. We refer to $\mathbf{C o n}(\mathbf{A})$ as the congruence lattice of $\mathbf{A}$.

Definition 4.1.6. Let $\mathbf{L}$ be a finite lattice and $\mathcal{V}$ be a variety. We say that $\mathcal{V}$ admits $\mathbf{L}$ if $\mathbf{L}$ is a sublattice of $\operatorname{Con}(\mathbf{A})$, for some $\mathbf{A}$ in $\mathcal{V}$. we say that $\mathcal{V}$ omits $\mathbf{L}$ otherwise.

We then define the following class for a fixed variety $\mathcal{V}$

$$
\operatorname{Adm}(\mathcal{V}):=\{\mathbf{L} \mid \mathcal{V} \text { admits } \mathbf{L}\} .
$$

On the other side, for a fixed finite lattice $\mathbf{L}$, we define

$$
\mathfrak{F}(\mathbf{L}):=\{\mathcal{V} \mid \mathbf{L} \notin \operatorname{Adm}(\mathcal{V})\} .
$$

It is well known that $\mathbf{A}$ is subdirectly irreducible if and only if $\operatorname{Con}(\mathbf{A})$ has exactly one atom; given this, it is easy to check that all the lattices from Figure 4.1 are subdirectly irreducible. The next proposition establishes a sufficient condition for $\mathfrak{F}(\mathbf{L})$ to be a filter in $\mathfrak{L}_{\text {Var }}$.

Proposition 4.1.7. If $\mathbf{L}$ is a subdirectly irreducible lattice, then $\mathfrak{F}(\mathbf{L})$ is a filter in $\mathfrak{L}_{\text {Var }}$.
Determining when $\mathfrak{F}(\mathbf{L})$ is a Mal'cev filter is more complicated, although there are some results in this direction in the literature. However, such a question is beyond the scope of this dissertation. For our purposes, it suffices to point out that if $\mathbf{L}$ is one of the lattices whose Hasse diagram is shown in Figure 4.1 then $\mathfrak{F}(\mathbf{L})$ is a Mal'cev filter. For instance, the filter $\mathfrak{F}\left(\mathbf{D}_{1}\right)$ coincides with a Mal'cev class that corresponds to a Mal'cev condition we have already dealt with (c.f. Theorem 3.2.18).
Theorem 4.1.8 ([58, 69, 81, 97, 101]). Let $\mathcal{V}$ be an idempotent and locally finite variety. The following are equivalent:
(1) $\mathcal{V} \not ڭ_{\iota}$ Sets,
(2) $\mathcal{V} \in \mathfrak{F}\left(\mathbf{D}_{1}\right)$,
(3) $\mathcal{V}$ omits type $\mathbf{1}$,
(4) there is no clone homomorphism from $\operatorname{Clo}(\mathcal{V})$ to $\mathcal{P}_{2}$,
(5) $\operatorname{Clo}(\mathcal{V})$ satisfies a Taylor identity,
(6) $\operatorname{Clo}(\mathcal{V})$ satisfies the 6-ary Siggers identity,
(7) $\mathrm{Clo}(\mathcal{V})$ satisfies the Olšák condition, that is

$$
t(x, y, y, y, x, x) \approx t(y, x, y, x, y, x) \approx t(y, y, x, x, x, y)
$$

Proof. The equivalence between (1) and (4) follows from the fact that $\operatorname{Clo}(\mathcal{S e t s})=\mathcal{P}_{2}$ and the observation that $\mathcal{V} \leq_{\iota} \mathcal{W}$ if and only if $\operatorname{Clo}(\mathcal{V}) \leq_{h} \operatorname{Clo}(\mathcal{W})$. From Theorem 3.2.16 it follows (4) if and only if (5). The equivalence between (2) and (5) follows from Theorem 4.16 and Theorem 4.23 in [69]. The equivalence (3) if and only if (5) follows from Lemma 9.4 and Theorem 9.6 of [58]. The equivalence between (5) and (6) follows from Theorem 3.2.18 and Theorem 3.3.5. To conclude, the equivalence between (5) and (7) comes from Theorem 6.1 in [81].

There is a similar characterization of the Mal'cev filter $\mathfrak{F}\left(\mathbf{D}_{2}\right)$. We denote by $\mathcal{S}$ the variety of semilattices. Let $\mathbf{S}_{2}:=(\{0,1\} ; \vee)$, where $\vee$ is a semilattice operation; it is well known that $\mathcal{S}=\boldsymbol{V}\left(\mathbf{S}_{2}\right)$ and clearly $\operatorname{Clo}\left(\mathbf{S}_{2}\right)=\langle\mathrm{V}\rangle$.

Theorem 4.1.9 ([58, 69]). Let $\mathcal{V}$ be an idempotent and locally finite variety. The following are equivalent:
(1) $\mathcal{V} \not \mathbb{Z}_{\iota} \mathcal{S}$,
(2) $\mathcal{V} \in \mathfrak{F}\left(\mathbf{D}_{2}\right)$,
(3) $\mathcal{V}$ omits types $\mathbf{1}$ and $\mathbf{5}$,
(4) there is no clone homomorphism from $\operatorname{Clo}(\mathcal{V})$ to $\langle\vee\rangle$,
(5) $\mathrm{Clo}(\mathcal{V})$ satisfies the Hobby-McKenzie condition from Theorem 3.3.13.

Proof. The theorem can be obtained by combining Lemma 9.5 and Theorem 9.8 in [58]. We also invite the reader to check Theorems $5.25,5.28$, and 8.11 in [69]; they prove that all the statements of this theorem, with the exception of item (3) are equivalent even if we drop the assumption that $\mathcal{V}$ is locally finite.

We deal with the Mal'cev filters $\mathfrak{F}\left(\mathbf{M}_{3}\right)$ and $\mathfrak{F}\left(\mathbf{N}_{5}\right)$ in the next section in more details.



Figure 4.1: The Hasse diagram of the lattices $\mathbf{D}_{1}, \mathbf{D}_{2}, \mathbf{M}_{3}$, and $\mathbf{N}_{5}$.

### 4.2 Congruence identities

Between the 1960s and 1970s, one of the most prosperous research topics in universal algebra was the study of so-called congruence identities satisfied by varieties. If $\mathcal{V}$ be a variety of algebras, then any identity in the language of lattices that holds in the class $\{\operatorname{Con}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{V}\}$ of congruence lattices of algebras in $\mathcal{V}$ is called a congruence
identity of $\mathcal{V}$. A.I. Mal'cev can certainly be considered the founder of this research line: he proved that any variety $\mathcal{V}$ has permutable congruences if and only if $\mathcal{V}$ belongs to the class of varieties satisfying a precise condition, that is the condition $\Sigma_{\mathrm{M}}$ from Remark 3.3.8. Subsequently, Pixley [86], Jónsson [65], and Day [45] presented analogous conditions for congruence arithmetical varieties, congruence distributive varieties, and congruence modular varieties, respectively; the study of Mal'cev conditions was thus initiated. These results are instances of a more general theorem that was obtained independently by Pixley [87] and Wille [102] that can be considered as a groundwork of the theory: they presented an algorithm to generate Mal'cev conditions associated with congruence identities.

In order to keep this chapter as brief as possible and not to deviate too much from the focus of the dissertation, with the exception of the congruence permutability property, we will not give the definitions of the congruence-properties that we are going to mention; rather, we will use the names of these properties to indicate Mal'cev classes in which we are interested. Below we present the Mal'cev conditions we have just mentioned as well as other Mal'cev conditions derived from congruence identities.

### 4.2.1 Congruence $n$-permutability

The first instance of a Mal'cev condition to appear in the literature comes from a result due to A.I. Mal'cev who characterized congruence permutable varieties [77].

Let $\mathbf{A}$ be any algebra and let $\alpha, \beta \in \mathbf{C o n}(\mathbf{A})$, we define the relational product of $\alpha$ and $\beta$ as follows

$$
\alpha \circ \beta:=\left\{(x, y) \in A^{2} \mid \exists z \in A,(x, z) \in \alpha \text { and }(z, y) \in \beta\right\}
$$

Note that in general $\alpha \circ \beta$ is not a congruence on $\mathbf{A}$; it is well known that $\alpha \circ \beta$ is a congruence if and only if $\alpha \circ \beta=\beta \circ \alpha$ and it is known that in this case $\alpha \circ \beta$ is the smallest congruence containing $\alpha$ and $\beta$, i.e., $\alpha \vee \beta$.

Iterating, we define

$$
\alpha \circ \circ_{n} \beta:=\underbrace{\alpha \circ \beta \circ \alpha \circ \ldots}_{n-1 \text { occurences of } \circ} .
$$

We say that two congruences $\alpha$ and $\beta$ n-permute, if $\alpha \circ_{n} \beta=\beta \circ_{n} \alpha$; we say that $\alpha$ and $\beta$ permute if they 2-permute. As usual, an algebra $\mathbf{A}$ is congruence $n$-permutable if all pairs of congruences of $\mathbf{A} n$-permute. A class of algebras (e.g., a variety) is congruence $n$-permutable if all its members are.

Theorem 4.2.1 ([77]). For any variety $\mathcal{V}$ the following are equivalent:
(1) $\mathcal{V}$ is congruence permutable,
(2) $\operatorname{Clo}(\mathcal{V})$ satisfies the following condition

$$
\begin{equation*}
m(x, y, y) \approx m(y, y, x) \approx x \tag{M}
\end{equation*}
$$

Hagemann and Mitschke provided a similar characterization for varieties that are congruence $n$-permutable [56]. Please confront the following theorem with Theorem 3.3.16.
Theorem 4.2.2 ([56]). For any variety $\mathcal{V}$ the following are equivalent:
(1) $\mathcal{V}$ is congruence $n$-permutable,
(2) $\operatorname{Clo}(\mathcal{V})$ satisfies the $\operatorname{HM}(n)$ condition for some $n \geq 2$, i.e.,

$$
\begin{aligned}
& p_{0}(x, y, z) \approx x \\
& p_{i}(x, x, y) \approx p_{i+1}(x, y, y) \text { for every } i \leq n \\
& p_{n}(x, y, z) \approx z
\end{aligned}
$$

If we additionally assume that $\mathcal{V}$ is a locally finite variety, then the condition from Theorem 3.3.16, that is " $\mathcal{V}$ is congruence $n$-permutable, for some $n \geq 2$ " is equivalent to the condition $\mathcal{V}$ omits types $\mathbf{1}, \mathbf{4}$, and $\mathbf{5}$.

### 4.2.2 Congruence meet-semidistributivity

The Mal'cev condition we deal with in this section is closely related to Theorem 3.3.12 As already discussed, we are not going to define when a variety $\mathcal{V}$ is congruence meetsemidistributive, ( $\mathcal{V}$ is $\mathrm{SD}_{\wedge}$, for short) however it is worth mentioning that the class of varieties enjoying the $\mathrm{SD}_{\wedge}$ property play a key role both in tractable CSPs and in commutator theory. Indeed, Barto and Kozik proved that for every finite relational structure $\mathbb{A}$, the polymorphism algebra $\operatorname{Pol}(\mathbb{A})$ generates a $\mathrm{SD}_{\wedge}$ variety if and only if the CSP of $\mathbb{A}$ has bounded width [11, 13]. Moreover, it has been proved that a variety $\mathcal{V}$ is $\mathrm{SD}_{\wedge}$ if and only if the commutator trivializes in $\mathcal{V}$, i.e., $[\alpha, \beta]=\alpha \wedge \beta$, for every $\alpha, \beta \in \operatorname{Con}(\mathbf{A})$ and every $\mathbf{A} \in \mathcal{V}[72$ (see also [58] for locally finite varieties). We refer the reader to [50, 69] for a basic introduction to Commutator Theory.
Theorem 4.2.3 ([58, 69, [73, 79]). Let $\mathcal{V}$ be an idempotent and locally finite variety. The following are equivalent:
(1) $\mathcal{V}$ is $\mathrm{SD}_{\wedge}$,
(2) $\mathcal{V} \in \mathfrak{F}\left(\mathbf{M}_{3}\right)$,
(3) $\mathcal{V}$ omits types $\mathbf{1}$ and $\mathbf{2}$,
(4) there is an integer $m>1$ such that $\operatorname{Clo}(\mathcal{V}) \models \mathrm{WNU}(k)$ for all $k \geq m$.
(5) $\operatorname{Clo}(\mathcal{V}) \models \mathrm{WNU}(n)$ for every $n \geq 3$;
(6) $\operatorname{Clo}(\mathcal{V})$ satisfies the $\mathrm{WNU}(3,4)$ minor condition, that is:

$$
\begin{aligned}
g(y, x, x) \approx & g(x, y, x) \approx g(x, x, y) \approx \\
& h(x, x, x, y) \approx h(x, x, y, x) \approx h(x, y, x, x) \approx h(y, x, x, x) .
\end{aligned}
$$

Proof. The equivalence of items (1), (2), and (3) can be found in [58], Theorem 9.10. Note that the equivalence of (1) and (2) holds even without assuming $\mathcal{V}$ to be idempotent and locally finite, see Theorem 8.1 in [69]. The equivalence between (3) and (4) follows from Theorem 1.2 in [79]. The equivalence of (3), (5) and (6) can be obtained combining Theorem 1.6 and Theorem 2.8 in [73].

### 4.2.3 Congruence join-semidistributivity

The property we deal with in this section implies, in the sense of Definition 4.1.5 both the Mal'cev class from Theorem 4.2.3 and the one from Theorem 4.1.9 indeed, it is the intersection of the two mentioned filters. More recently, this class has also attracted attention in terms of descriptive complexity of the CSP of finite relational structures and is indeed the subject of what can perhaps be considered the most important open problem still unresolved in CSP for finished structures. In fact, it has been conjectured that the polymorphism algebra $\operatorname{Pol}(\mathbb{A})$ of any finite structure $\mathbb{A}$ generates a $S D_{\vee}$ variety if and only if $\operatorname{CSP}(\mathbb{A})$ has bounded linear width, see also Theorem 3.3.14 and Conjecture 3.3.15

Theorem 4.2.4 ([58, 69]). Let $\mathcal{V}$ be an idempotent and locally finite variety. The following are equivalent:
(1) $\mathcal{V}$ is $\mathrm{SD}_{\vee}$,
(2) $\mathcal{V} \in \mathfrak{F}\left(\mathbf{M}_{3}\right)$ and $\mathcal{V} \in \mathfrak{F}\left(\mathbf{D}_{2}\right)$,
(3) $\mathcal{V}$ omits types $\mathbf{1}, \mathbf{2}$, and $\mathbf{5}$,
(4) $\operatorname{Clo}(\mathcal{V})$ satisfies the $\mathrm{SD}_{\vee}(n)$ condition for some $n \geq 2$, that is:

$$
\begin{aligned}
d_{0}(x, y, z) & =x \\
d_{i}(x, y, y) & =d_{i+1}(x, y, y) \quad \text { and } \quad d_{i}(x, y, x)=d_{i+1}(x, y, x) \text { if } i \text { is even, } i<n, \\
d_{i}(x, x, y) & =d_{i+1}(x, x, y) \text { if } i \text { is odd, } i<n \\
d_{n}(x, y, z) & =z
\end{aligned}
$$

Proof. The equivalence of all of the four items was proved in Theorem 9.11 in [58]. The equivalence of (1), (2), and (4), without the assumption of locally finiteness, follows from Theorem 8.14 in [69].

### 4.2.4 Congruence modularity

In this section we present a Mal'cev condition, known as congruence modularity, which was first described through 4-ary terms by Day [45]. Subsequently, characterizations of the same Mal'cev filter were found via the use of terms that are easier to work with; the results of Gumm [55] and more recently of Kazda, Kozik, McKenzie, and Moore [68] point in this direction. Remarkably, one of the most important conjectures about Mal'cev
conditions concerns this Mal'cev filter: the so called Taylor's modularity conjecture [51] regarding the primeness of the filter of congruence modular varieties. This conjecture has remained open for almost 40 years now, although partial results have been achieved in [19], 96], and more recently in [84.
Theorem 4.2.5 ([22, 45, [55, 68]). Let $\mathcal{V}$ be an idempotent and locally finite variety. The following are equivalent:
(1) $\mathcal{V}$ is congruence modular,
(2) $\mathcal{V} \in \mathfrak{F}\left(\mathbf{N}_{5}\right)$,
(3) $\mathrm{Clo}(\mathcal{V})$ satisfies the directed-Gumm condition, that is:

$$
\begin{aligned}
p_{1}(x, y, y) & \approx x \\
p_{i}(x, y, y) & \approx x \text { for all } 1 \leq i \leq m, \\
p_{i}(x, y, y) & \approx p_{i+1}(x, y, y) \text { for all } 1 \leq i \leq m-1, \\
p_{m}(x, y, y) & \approx q(x, y, y) \\
q(x, y, y) & \approx y .
\end{aligned}
$$

Proof. The equivalence between (1) and (2) is a well-known result due to Dedekind and can be find in Theorem 1.7.12 of [22]. The equivalence of (1) and (3) was first proved by Day [45] and Gumm [55] via the so-called Day-terms and Gumm-terms, respectively. However, here we decided to use the characterization provided in [68].

Barto proved that a finitely related algebra A (see Definition 2.3.7) generates a congruence modular variety if and only if for some $k \geq 2$, the variety generated by $\mathbf{A}$ satisfies the so-called $k$-edge condition [8].

Theorem 4.2.6 ([8]). Let A be a finite algebra. Then the following are equivalent:
(1) $\mathbf{A}$ is finitely related and $\boldsymbol{V}(\mathbf{A})$ is congruence modular,
(2) $\boldsymbol{V}(\mathbf{A})$ satisfies the $k$-edge condition, that is:

$$
f\left(\left[\begin{array}{cccccc}
y & y & x & x & \cdots & x \\
y & x & y & x & \cdots & x \\
x & x & x & y & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x & x & x & x & \cdots & y
\end{array}\right]\right) \approx\left[\begin{array}{c}
x \\
x \\
x \\
\vdots \\
x
\end{array}\right]
$$

where the arity of the operation symbol $f$ is $k+1$ and all but the first column have exactly one $y$ and $k-1$ occurences of $x$.
We would like to point out that the 2 -edge condition is equivalent to the Mal'cev condition $\Sigma_{M}$ (up to permuting the inputs).

### 4.2.5 Congruence distributivity

The Mal'cev condition we deal with in this section turned out to be the intersection of the Mal'cev filters from Theorem 4.2.3 and Theorem 4.2.5 and a characterization through terms was found by Jónsson [65].

Theorem 4.2.7 ([65]). For any variety $\mathcal{V}$ the following are equivalent:
(1) $\mathcal{V}$ is congruence distributive,
(2) $\mathcal{V} \in \mathfrak{F}\left(\mathbf{M}_{3}\right)$ and $\mathcal{V} \in \mathfrak{F}\left(\mathbf{N}_{5}\right)$,
(3) $\operatorname{Clo}(\mathcal{V})$ satisfies the Jónsson condition of length $n$ for some $n \geq 2$, i.e.,

$$
\begin{array}{rlr}
d_{0}(x, y, z) & \approx x & \\
d_{i}(x, y, x) & \approx x & 0 \leq i \leq n \\
d_{i}(x, x, z) & \approx d_{i+1}(x, x, z) & \text { for even } i \\
d_{i}(x, z, z) & \approx d_{i+1}(x, z, z) & \text { for odd } i \\
d_{n}(x, y, z) & \approx z . &
\end{array}
$$

Following Remark 3.3 .8 we define the minor condition $\mathrm{QJ}(n)$ by replacing the first, second, and last line in the condition presented in the third item of Theorem 4.2.7 with $d_{0}(x, y, z) \approx d_{0}(x, x, x), d_{i}(x, y, x) \approx d_{i}(x, x, x)$, and $d_{n}(x, y, z) \approx d_{n}(z, z, z)$, respectively.

Barto proved a result of a flavour similar to Theorem 4.2 .6 also for the case of congruence distributive varieties [7].

Theorem 4.2.8 ([7). Let A be a finite algebra. Then the following are equivalent:
(1) $\mathbf{A}$ is finitely related and $\boldsymbol{V}(\mathbf{A})$ is congruence distributive,
(2) $\boldsymbol{V}(\mathbf{A})$ satisfies the $k$-near unanimity condition, that is:

$$
f\left(\left[\begin{array}{ccccc}
y & x & x & \cdots & x  \tag{k}\\
x & y & x & \cdots & x \\
x & x & y & \cdots & x \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x & x & x & \cdots & y
\end{array}\right]\right) \approx\left[\begin{array}{c}
x \\
x \\
x \\
\vdots \\
x
\end{array}\right]
$$

where $k$ is the arity of the operation symbol $f$.

### 4.2.6 Congruence arithmeticity

A variety is called congruence arithmetical if it is both congruence distributive and congruence permutable. A characterization of congruence arithmetical via terms was provided by Pixley in [86].

Theorem 4.2.9 ([86]). For any variety $\mathcal{V}$ the following are equivalent:
(1) $\mathcal{V}$ is congruence arithmetical,
(2) $\operatorname{Clo}(\mathcal{V})$ satisfies the Pixley condition, i.e.,

$$
p(x, y, y) \approx x, \quad p(x, y, x) \approx x, \quad p(y, y, x) \approx x .
$$

Remark 4.2.10. We would like to point out that any operation satisfying the Pixley condition (see Definition 2.1.15) automatically satisfies the Mal'cev condition $\Sigma_{\mathrm{M}}$. Furthermore, if $p$ is any operation satisfying the Pixley condition, then the operation $m$ defined by $m(x, y, z):=p(x, p(x, y, z), z)$ is a majority operation, i.e., $m$ satisfies $\mathrm{NU}(3)$.

### 4.3 Linear Mal'cev conditions ordered by strength

We conclude this chapter by collecting all the Mal'cev conditions that we introduced throughout this chapter and in Chapter 3 and ordering them by strength, as established in Definition 4.1.5. Please note that, in order to remain in line with the main purpose of the dissertation, we are particularly interested in those implications that hold for locally finite varieties. The outcome of this work is displayed in Figure 4.2 which is intended as a summary map.
Remark 4.3.1. For those readers who are more interested in clones rather than in varieties, the implication-order arising from Definition 4.1.5 can be interpreted as follows: given two linear Mal'cev conditions $\Sigma$ and $\Gamma$, we say that $\Sigma$ implies $\Gamma$, denoted $\Sigma \Rightarrow \Gamma$, if $\mathcal{C} \models \Sigma$ implies that $\mathcal{C} \models \Gamma$ for all idempotent clones $\mathcal{C}$ over some finite set.
Remark 4.3.2. We would like to emphasise that, despite being very similar notions, there are some differences between linear Mal'cev conditions and minor conditions. Note that, e.g., the linear Mal'cev condition $\Sigma_{\mathrm{M}}=\{m(x, y, y) \approx m(y, y, x), m(y, y, x) \approx x\}$ is not a minor condition, since there is no function symbol occurring on the right-hand side of the equality in the second identity. However, we have already discussed in Remark 3.3.8 how this difference - for finite idempotent clones - can be adjusted considering the quasi-version of a linear Mal'cev conditions. On the other side, consider the minor condition $\mathrm{WNU}(n)$ from Definition 3.2.17. By solely looking at Definition 4.1.3, the set $\exists n$. $\mathrm{WNU}(n)=\{\mathrm{WNU}(n) \mid n \geq 3\}$ is not a linear Mal'cev condition, since $\mathrm{WNU}(n)$ does not imply $\mathrm{WNU}(n+1)$. However, from Theorem 4.1.8, it follows that $\exists n$. $\mathrm{WNU}(n)$ is equivalent to the Olšák condition. Thus, for finite idempotent clones, the set $\exists n$. WNU ( $n$ ) is even a strong linear Mal'cev condition.

We would like to emphasise that a full classification of cyclic loop conditions ordered by strength has been presented in a joint work by Bodirsky, Starke, and Vucaj [27].

Most of the results we are going to present below either follow by definition or are well-known in the literature. Nevertheless, we explicitly disclose some of these results with the goal of being self-contained.

Definition 4.3.3. We call minority condition the following set of identities

$$
m(x, y, y) \approx m(y, x, y) \approx m(y, y, x) \approx x
$$

(Minority)
It follows by definition that the Minority condition implies the Mal'cev condition $\Sigma_{\mathrm{M}}$.
Proposition 4.3.4. It holds that $\operatorname{HM}(2) \Leftrightarrow \Sigma_{\mathrm{M}}$. Moreover, for every $n \geq 2$, it holds that $\mathrm{HM}(n) \Rightarrow \mathrm{HM}(n+1)$.

Proof. Note that for any clone $\mathcal{C}, \mathcal{C} \models \Sigma_{\mathrm{M}}$ if and only if $\mathcal{C} \models \mathrm{HM}(2)$. Indeed, if $\mathcal{C}$ has a Mal'cev operation $m$, then $\mathcal{C}$ satisfies $\mathrm{HM}(2)$ by assigning $p_{0}=\operatorname{pr}_{1}^{3}, p_{1}=m$, and $p_{2}=\operatorname{pr}_{3}^{3}$. If $\mathcal{C}$ satisfies $\operatorname{HM}(2)$, then $p_{1}^{\mathcal{C}}$ is a Mal'cev operation. The second claim simply follows by definition. Indeed, for every $n \geq 2$, if $\alpha$ and $\beta$ are $n$-permutable congruences, then $\alpha$ and $\beta$ are also ( $n+1$ )-permutable.

Theorem 4.3.5 ([64]). For any variety $\mathcal{V}$, if $\mathcal{V}$ is congruence 3-permutable, then it is congruence modular.

The latter result is optimal, i.e., it cannot be pushed beyond 3-permutability: the so-called Polin's variety [88] is an example of a variety which is congruence 4 -permutable but not congruence modular.

In the next proposition recall the condition $4-\operatorname{HMcK}(n)$ from Theorem 3.3.17.
Proposition 4.3.6 ([58, 86]). The Pixley condition implies the following conditions: $\mathrm{NU}(3)$, $4-\operatorname{HMcK}(n)$, and $\Sigma_{\mathrm{M}}$.

Proof. From Remark 4.2 .10 it follows that the Pixley condition implies both $\Sigma_{\mathrm{M}}$ and $\mathrm{NU}(3)$. From Theorem 9.15 in [58] it follows that $\operatorname{Clo}(\mathcal{V})$ satisfies the $4-\operatorname{HMcK}(n)$ condition if and only if $\mathcal{V}$ is $\mathrm{SD}_{\wedge}$ and it satisfies the condition $\mathcal{V}$ is congruence $n$ permutable, for some $n \geq 2$. The implication follows then by definition, since the Mal'cev filter generated by the Pixley condition is the intersection of congruence distributivity and congruence permutability.

Proposition 4.3.7. The following implications hold:
(1) for every $n \geq 3$, it holds that $\mathrm{NU}(n) \Rightarrow \mathrm{NU}(n+1)$;
(2) if it exists some $k \geq 3$ such that $\operatorname{Clo}(\mathcal{V}) \models \mathrm{NU}(k)$, then $\operatorname{Clo}(\mathcal{V})$ satisfies the Jónsson condition;
(3) if it exists some $k \geq 3$ such that $\operatorname{Clo}(\mathcal{V}) \models \mathrm{NU}(k)$, then $\operatorname{Clo}(\mathcal{V})$ satisfies the $k$-edge condition;
(4) if $\operatorname{Clo}(\mathcal{V})$ satisfies the $k$-edge condition for some $k$, then $\operatorname{Clo}(\mathcal{V})$ satisfies the directedGumm condition.

Proof. Let $\mathcal{C}$ be an idempotent clone on a finite set and let $f_{k} \in \mathcal{C}$ be an operation satisfy$\operatorname{ing} \mathrm{NU}(k)$, for some $k \geq 3$. Define $f_{k+1}\left(x_{1}, \ldots, x_{k+1}\right):=f_{k}\left(f_{k}\left(x_{1}, \ldots, x_{k}\right), x_{2}, \ldots, x_{k+1}\right)$; clearly $f_{k+1} \in \mathcal{C}$ and $f_{k+1}$ satisfies $\mathrm{NU}(k+1)$; this proves the implication in (1). The implication in (2) follows from Theorem 4.2.8. The implication in (3) follows from the observation that from a $k$-edge term one obtains a $k$-ary near-unanimity term by ignoring its first input. Finally, the implication in (4) follows from Theorem 4.2.6

Recall the conditions $\operatorname{FS}(n)$ and $\operatorname{TS}(n)$ from Definition 3.3.19.
Proposition 4.3.8. The following implications hold:

$$
\forall n \geq 2, \operatorname{Clo}(\mathcal{V}) \models \mathrm{TS}(n) \quad \Rightarrow \quad \forall n \geq 2, \operatorname{Clo}(\mathcal{V}) \models \mathrm{FS}(n) \quad \Rightarrow \quad \mathcal{V} \text { is } \mathrm{SD}_{\wedge}
$$

Proof. The first implication follow immediately by the definition of the conditions $\operatorname{TS}(n)$ and $\operatorname{FS}(n)$. By Theorem 4.2.3 we have that $\mathcal{V}$ is $\mathrm{SD}_{\wedge}$ if and only if $\operatorname{Clo}(\mathcal{V}) \models \mathrm{WNU}(n)$ for every $n \geq 3$. The second implication follows then by definition.

All the remaining implications, which in Figure 4.2 are witnessed by the existence of directed edges, follow simply by definition. For instance, one can deduce that the directed-Gumm condition (see Theorem 4.2.5) implies the Hobby-McKenzie condition (from Theorem 3.3.13) as follows:

Proposition 4.3.9. If $\operatorname{Clo}(\mathcal{V})$ satisfies the directed-Gumm condition, then it satisfies the Hobby-McKenzie condition.

Proof. From Theorem 4.2.5 we know that $\operatorname{Clo}(\mathcal{V})$ satisfies the directed-Gumm condition if and only if $\mathcal{V} \in \mathfrak{F}\left(\mathbf{N}_{5}\right)$. Analogously, from Theorem 4.1.9 we know that $\operatorname{Clo}(\mathcal{V})$ satisfies the Hobby-McKenzie condition if and only if $\mathcal{V} \in \mathfrak{F}\left(\mathbf{D}_{2}\right)$. Since $\mathbf{N}_{5}$ is a sublattice of $\mathbf{D}_{2}$, it follows that for every variety $\mathcal{V}$, if $\mathcal{V} \in \mathfrak{F}\left(\mathbf{N}_{5}\right)$ then $\mathcal{V} \in \mathfrak{F}\left(\mathbf{D}_{2}\right)$.

Note that in Figure 4.2 by "All idempotent" we denote the class of all linear Mal'cev conditions that are satisfied by the clone $\mathcal{I}_{2}$ (see Section 3.2.2).


Figure 4.2: The geography of linear Mal'cev conditions.

## Chapter 5

## Description of $\mathfrak{P}_{2}$

In this chapter we describe systematically the poset $\mathfrak{P}_{2}$; we refer the reader to Chapter 3 for the formal definition of $\mathfrak{P}_{n}$. Recall that $\mathfrak{P}_{2}:=\left(\{\overline{\mathcal{C}} \mid \mathcal{C}\right.$ is a clone on $\left.\{0,1\}\} ; \preceq_{\mathrm{m}}\right)$. The results presented in this chapter come from a joint work of the author of the dissertation and Manuel Bodirsky [28]. We first present a succinct description of Post's lattice [91]: a complete classification of all clones of operations over the Boolean domain $\{0,1\}$.

### 5.1 Post's lattice

We label the clones of Post's lattice by generators: if $f_{1}, \ldots, f_{n}$ are operations on $\{0,1\}$, then $\left\langle f_{1}, \ldots, f_{n}\right\rangle$ denotes the clone generated by $f_{1}, \ldots, f_{n}$. As usual, we may apply functions componentwise, i.e., if $f$ is a $k$-ary map, and $\mathbf{t}_{1}, \ldots, \mathbf{t}_{k} \in\{0,1\}^{m}$, then $f\left(\mathbf{t}_{1}, \ldots, \mathbf{t}_{k}\right)$ denotes the $m$-tuple

$$
\left(f\left(t_{1,1}, \ldots, t_{1, k}\right), \ldots, f\left(t_{m, 1}, \ldots, t_{m, k}\right)\right) .
$$

In the description of Post's lattice, we use the following operations.

- 0 and 1 denote the two unary constant operations.
- $\neg(x)$ is the usual Boolean negation, i.e., the non-identity permutation on $\{0,1\}$.
- If $f\left(x_{1}, \ldots, x_{n}\right)$ is an $n$-ary operation, then $f^{\Delta}\left(x_{1}, \ldots, x_{n}\right)$ denotes its dual operation, given by $\left.f^{\Delta}\left(x_{1}, \ldots, x_{n}\right):=\neg\left(f\left(\neg\left(x_{1}\right), \ldots, \neg\left(x_{n}\right)\right)\right)\right)$.
- $x \vee y$ is the operation defined in Example 2.3.2 and $x \wedge y:=x \vee^{\Delta} y$.
- $x \oplus y:=(x+y) \bmod 2$ and $x \oplus^{\prime} y:=\neg(x \oplus y)$.
- $x \rightarrow y:=\neg(x) \vee y$ and $x * y:=\neg(x) \wedge y$.
- $d_{n}\left(x_{1}, \ldots, x_{n}\right):=\bigvee_{i=1}^{n} \wedge_{j=1, j \neq i} x_{j}$. For $n=3$ we obtain the majority operation $d_{3}(x, y, z)=(x \wedge y) \vee(x \wedge z) \vee(y \wedge z)$.


Figure 5.1: Post's Lattice.

- The minority operation $m(x, y, z):=x \oplus y \oplus z$.
- $p(x, y, z):=x \wedge(y \vee z)$.
- $q(x, y, z):=x \wedge\left(y \oplus^{\prime} z\right)=x \wedge((y \wedge z) \vee(\neg(y) \wedge \neg(z)))$.

Post's lattice has 7 atoms, 5 coatoms and it is countably infinite because of the presence of some infinite descending chains; see Figure 5.1

### 5.2 Collapses

First, we prove that certain Boolean clones are in the same $\equiv_{\mathrm{m}}$-class. Later, in Section 5.3 we prove that certain Boolean clones lie in different $\equiv_{\mathrm{m}}$-classes and, for each separation, we provide a concrete minor condition as a witness.

Recall that if $\mathcal{A} \equiv_{\mathrm{m}} \mathcal{B}$, then we say that $\mathcal{A}$ and $\mathcal{B}$ collapse. The next result is a corollary of Theorem 3.2.3.

Proposition 5.2.1. All clones with a constant operation collapse.

Proposition 5.2.2. Let $\mathcal{C}$ and $\mathcal{C}^{\Delta}$ be Boolean clones such that $\mathcal{C}:=\left\langle f_{1}, \ldots, f_{m}\right\rangle$ and $\mathcal{C}^{\Delta}:=\left\langle f_{1}^{\Delta}, \ldots, f_{m}^{\Delta}\right\rangle$. Then $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{C}^{\Delta}$.

Proof. To prove that $\mathcal{C} \preceq_{\mathrm{m}} \mathcal{C}^{\Delta}$, define $\xi(f):=f^{\Delta}$ for any $f \in \mathcal{C}$. Then

$$
\begin{aligned}
\xi\left(f_{\pi}\right)\left(x_{1}, \ldots, x_{n}\right) & =f_{\pi}^{\Delta}\left(x_{1}, \ldots, x_{n}\right)=\neg\left(f_{\pi}\left(\neg\left(x_{1}, \ldots, x_{n}\right)\right)\right. \\
& =\neg\left(f\left(\neg\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)\right)\right)=\neg\left(f\left(\neg\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)\right)\right) \\
& =f^{\Delta}\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)=\xi(f)\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right) \\
& =\xi(f)_{\pi}\left(x_{1}, \ldots, x_{n}\right) .
\end{aligned}
$$

The same argument can be used to prove that $\mathcal{C}^{\Delta} \preceq_{\mathrm{m}} \mathcal{C}$.
Recall that the idempotent reduct of a clone $\mathcal{C}$ is the clone $\mathcal{C}^{\text {id }}$ that consists of all idempotent operations in $\mathcal{C}$ (see Section 2.1.3).

Lemma 5.2.3. Let $\mathcal{C}$ be a Boolean clone with no constant operations. Let $\mathcal{D}:=\langle\mathcal{C} \cup\{\neg\}\rangle$ be the clone generated by $\mathcal{C}$ and the Boolean negation $c$. Then we have $\mathcal{D} \equiv_{\mathrm{m}} \mathcal{D}^{\text {id }}$.

Proof. Since $\mathcal{C}$ contains no constant operations, for every $f$ in $\mathcal{C}$ either $f(x, \ldots, x) \approx x$ holds or $f(x, \ldots, x) \approx \neg(x)$ holds. We claim that there exists a minor-preserving map $\xi: \mathcal{D} \rightarrow \mathcal{D}^{\text {id }}$. We define $\xi: \mathcal{D} \rightarrow \mathcal{D}^{\text {id }}$ as follows: for an $n$-ary operation $f \in \mathcal{D}$

$$
\xi(f)\left(x_{1}, \ldots, x_{n}\right):= \begin{cases}f\left(x_{1}, \ldots, x_{n}\right) & \text { if } f \text { is idempotent } \\ \neg\left(f\left(x_{1}, \ldots, x_{n}\right)\right) & \text { otherwise. }\end{cases}
$$

By definition $\xi(f) \in \mathcal{D}^{\text {id }}$. We claim that $\xi$ is minor preserving: if $f$ is idempotent, then $\xi$ is the identity, and the claim trivially holds; in the other case, the claim follows by the definition of negation:

$$
\begin{aligned}
\xi\left(f_{\pi}\right)\left(x_{1}, \ldots, x_{n}\right) & \approx\left(\neg \circ f_{\pi}\right)\left(x_{1}, \ldots, x_{n}\right) \approx(\neg \circ f)\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right) \\
& \approx \xi(f)\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right) \approx \xi(f)_{\pi}\left(x_{1}, \ldots, x_{n}\right) .
\end{aligned}
$$

Proposition 5.2.4. The following equivalences hold: $\langle\emptyset\rangle \equiv_{\mathrm{m}}\langle\neg\rangle ;\left\langle d_{3}, \neg\right\rangle \equiv_{\mathrm{m}}\left\langle d_{3}, m\right\rangle$, and $\langle m, \neg\rangle \equiv_{\mathrm{m}}\langle m\rangle$.

Proof. Checking in Post's lattice we have that that $\langle\neg\rangle^{\text {id }}=\langle\emptyset\rangle,\left\langle d_{3}, \neg\right\rangle^{\text {id }}=\left\langle d_{3}, m\right\rangle$, and $\langle m, \neg\rangle^{\text {id }}=\langle m\rangle$. The statement follows from Lemma 5.2.3

Note that with the collapses we have reported so far we can make some observations on the number of atoms in $\mathfrak{P}_{2}$. We already pointed out that $\overline{\langle 0\rangle}$ and $\overline{\langle 1\rangle}$ are not atoms in $\mathfrak{P}_{2}$, since $\overline{\langle 0\rangle}=\overline{\langle 1\rangle}$ is the top-element in $\mathfrak{P}_{2}$. Furthermore, we have that $\langle\mathrm{V}\rangle \equiv_{\mathrm{m}}\langle\Lambda\rangle$ because of Proposition 5.2.2. Altogether, we get that $\mathfrak{P}_{2}$ has at most three atoms: $\overline{\langle\Lambda\rangle}$, $\overline{\langle m\rangle}$, and $\overline{\left\langle d_{3}\right\rangle}$. We prove in Section 5.3 that these are distinct elements in $\mathfrak{P}_{2}$.

Another case of collapse is the following: $\langle\vee, \wedge\rangle \equiv_{\mathrm{m}}\left\langle d_{3}, p\right\rangle$. This time we prove this via a concrete pp-construction. We consider the binary relations

$$
\leq_{2}:=\{(0,0),(0,1),(1,1)\} \quad \text { and } \quad B_{2}:=\{(0,1),(1,0),(1,1)\}
$$

and define:

$$
\begin{aligned}
& \mathbb{B}_{2}:=\left(\{0,1\} ; B_{2},\{0\},\{1\}\right) \\
& \mathbb{B}^{\leq}:=\left(\{0,1\} ; \leq_{2},\{0\},\{1\}\right) \\
& \mathbb{B}_{2} \leq=\left(\{0,1\} ; B_{2}, \leq_{2},\{0\},\{1\}\right)
\end{aligned}
$$

where $\{0\}$ and $\{1\}$ are unary relations.
Proposition 5.2.5. $\langle\vee, \wedge\rangle \equiv_{\mathrm{m}}\left\langle d_{3}, p\right\rangle$.
Proof. It is known that $\langle\vee, \wedge\rangle=\operatorname{Pol}(\mathbb{B} \leq)$ and $\left\langle d_{3}, p\right\rangle=\operatorname{Pol}(\mathbb{B} \leq)$ (see, e.g., [91, 103]). Since $\left\langle d_{3}, p\right\rangle \subseteq\langle\vee, \wedge\rangle$ it follows that $\left\langle d_{3}, p\right\rangle \preceq_{\mathrm{m}}\langle\vee, \wedge\rangle$. For the other inequality it suffices to prove that $\mathbb{B}_{2} \leq$ is homomorphically equivalent to a pp-power of $\mathbb{B} \leq$. We consider the relational structure $\mathbb{S}$ with domain $\{0,1\}^{2}$ and relations defined by

$$
\begin{aligned}
\Phi_{0}\left(x_{1}, x_{2}\right) & :=\left(x_{1}=0\right) \wedge\left(x_{2}=1\right) \\
\Phi_{1}\left(x_{1}, x_{2}\right) & :=\left(x_{1}=1\right) \wedge\left(x_{2}=0\right) \\
\Phi_{\leq}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) & :=\left(x_{1} \leq_{2} y_{1}\right) \wedge\left(y_{2} \leq_{2} x_{2}\right) \\
\Phi_{B_{2}}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) & :=x_{2} \leq_{2} y_{1} .
\end{aligned}
$$

Note that $\mathbb{S}$ is indeed a pp-power of $\mathbb{B} \leq$. We define the map

$$
f: \mathbb{S} \rightarrow \mathbb{B}_{2}^{\leq}
$$

as follows:

$$
f((0,1)):=0 ; \quad f((0,0)):=f((1,0)):=f((1,1)):=1 .
$$

Let $g: \mathbb{B}_{2}^{\leq} \rightarrow \mathbb{S}$ be a map such that $g(0):=(0,1)$ and $g(1):=(1,0)$. It is easy to check that both $f$ and $g$ are homomorphisms. This proves that $\mathbb{S}$ is homomorphically equivalent to $\mathbb{B} \frac{\leq}{2}$.

In Figure 5.2 we propose a visual representation of the pp-construction presented in the proof that we hope will be helpful to the reader.

Remark 5.2.6. Alternatively, one could prove Proposition 5.2 .5 by explicitly providing a minor-preserving map from $\langle\vee, \wedge\rangle$ to $\left\langle d_{3}, p\right\rangle$. For the sake of completeness, we would like to point out that the minor-preserving map that yields the result is $\xi: f \mapsto f \vee f^{\Delta}$.


Figure 5.2: The structure $\mathbb{B} \leq \mathrm{pp}$-constructs $\mathbb{B}_{2}^{\leq}$. The red edges connect elements in $B_{2}$; the blue edges connect elements in $\leq_{2}$.

### 5.3 Separations

Recall that if $\mathcal{A} \npreceq_{\mathrm{m}} \mathcal{B}$ then there is a minor condition $\Sigma$ which is satisfied by some operations in $\mathcal{A}$ but by none of the operations in $\mathcal{B}$. In this case we say that $\Sigma$ is a witness of $\mathcal{A} \npreceq_{\mathrm{m}} \mathcal{B}$ (see Remark 3.1 .11 . We write $\mathcal{A} \mid \mathcal{B}$ if $\mathcal{A} \not \varliminf_{\mathrm{m}} \mathcal{B}$ and $\mathcal{A} \not \varliminf_{\mathrm{m}} \mathcal{B}$.

Proposition 5.3.1. The quasi Jónsson condition $\mathrm{QJ}(4)$ is a witness of $\langle p\rangle \preceq_{\mathrm{m}}\langle\wedge\rangle$.
Proof. Define the operations:

$$
\begin{array}{rlrl}
t_{0}^{\langle p\rangle}(x, y, z) & :=p(x, x, x) & & t_{1}^{\langle p\rangle}(x, y, z):=p(x, y, z) \\
t_{2}^{\langle p\rangle}(x, y, z):=p(x, z, z) & & t_{3}^{\langle p\rangle}(x, y, z):=p(z, x, y) \\
t_{4}^{\langle p\rangle}(x, y, z) & :=p(z, z, z) ; & &
\end{array}
$$

then $t_{0}^{\langle p\rangle}, \ldots, t_{4}^{\langle p\rangle}$ are witnesses for $\langle p\rangle \models \mathrm{QJ}(4)$. On the other hand, let us suppose that there are witnesses $t_{0}^{\langle\Lambda\rangle}, \ldots, t_{4}^{\langle\Lambda\rangle}$ for $\langle\Lambda\rangle \vDash$ QJ(4). Since any operation in $\langle\Lambda\rangle$ is idempotent, we have $t_{0}^{\langle\Lambda\rangle}(x, y, z)=x$. Moreover, from the identity

$$
t_{i}(x, y, x) \approx t_{i}(x, x, x) \quad 0 \leq i \leq n
$$

we have that $t_{1}^{\langle\wedge\rangle}(x, y, z)$ does not depend on the second argument. Moreover, $t_{0}(x, x, z) \approx$ $t_{1}(x, x, z)$ implies that $t_{1}^{(\wedge\rangle}(x, y, z)$ does not depend on the third argument. Hence, we conclude that $t_{1}^{(\Lambda\rangle}(x, y, z)=x$. From the identity $t_{1}(x, z, z) \approx t_{2}(x, z, z)$ and using $t_{1}^{\langle\Lambda\rangle}(x, y, z)=x$, we get that $t_{2}^{\langle\Lambda\rangle}(x, y, z)=x$. Similarly, we obtain also that $t_{3}^{\langle\Lambda\rangle}(x, y, z)=$ $x$ and $t_{4}^{\{\wedge\rangle}(x, y, z)=x$. This is in contradiction with the identity $t_{4}(x, y, z) \approx t_{4}(z, z, z)$. Hence, we conclude that $\langle\wedge\rangle$ does not satisfy QJ(4).

The following structures have already been defined in Section 3.3.1 and play an
essential role in the next proposition:

$$
\begin{aligned}
\mathbb{C}_{2}^{\leq} & :=\left(\{0,1\} ; C_{2}, \leq 2,\{0\},\{1\}\right) \\
\mathbb{H} \mathbb{R} \mathbb{N} & :=\left(\{0,1\} ; R_{110}, R_{111},\{0\},\{1\}\right) \\
3 \mathbb{L} \mathbb{N}_{2} & :=\left(\{0,1\} ; R_{1110}, R_{1111}\right)
\end{aligned}
$$

where for all $a, b, c, d \in\{0,1\}$ :

$$
\begin{aligned}
R_{a b c} & :=\{0,1\}^{3} \backslash\{(a, b, c)\}, \\
R_{a b c d} & :=\left\{(x, y, z) \in \mathbb{Z}_{2}^{3} \mid a x+b y+c z=d\right\} .
\end{aligned}
$$

These structures are the relational counterparts of the atoms of $\mathfrak{P}_{2}$ in the sense that $\left\langle d_{3}\right\rangle=\operatorname{Pol}\left(\mathbb{C}_{2}^{\leq}\right),\langle m\rangle=\operatorname{Pol}\left(3 \mathbb{L} \mathbb{N}_{2}\right)$, and $\langle\wedge\rangle=\operatorname{Pol}(\mathbb{H O R} \mathbb{N})$ (see, for instance, [91, [103]).

Proposition 5.3.2. The following holds:
(1) $\langle\wedge\rangle \mid\left\langle d_{3}\right\rangle$.
(2) $\left\langle d_{3}\right\rangle \mid\langle m\rangle$.
(3) $\langle m\rangle \mid\langle\wedge\rangle$.

Proof. (1) By definition, $d_{3}$ is a quasi majority operation. Let $f$ be any Boolean quasi majority operation. Then it is easy to check that $f$ does not preserve $R_{110}$ and thus $f \notin\langle\Lambda\rangle=\operatorname{Pol}(\mathbb{H} \mathbb{O} \mathbb{R})$. Hence, the quasi majority condition is a witness of $\left\langle d_{3}\right\rangle \preceq_{\mathrm{m}}\langle\Lambda\rangle$.

We claim that the minor identity $f(x, y) \approx f(y, x)$ is a witness of $\langle\wedge\rangle \npreceq_{\mathrm{m}}\left\langle d_{3}\right\rangle$. This identity is clearly satisfied by $\wedge$. Let $f$ be any Boolean binary commutative operation. Then $f$ does not preserve $C_{2}$. Hence, $f \notin\left\langle d_{3}\right\rangle=\operatorname{Pol}\left(\mathbb{C}_{2}^{\leq}\right)$and thus the claim is proved.
(2) Let $f$ be a Boolean quasi majority operation. Then $f$ does not preserve the relation $R_{1111}=\{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$ and thus $f \notin\langle m\rangle=\operatorname{Pol}\left(3 \mathbb{L} \mathbb{N}_{2}\right)$. Hence, the quasi majority condition is a witness of $\left\langle d_{3}\right\rangle \not \varliminf_{\mathrm{m}}\langle m\rangle$. Let $g$ be any Boolean quasi minority operation, then $g$ does not preserve $R_{110}$ and therefore the quasi minority condition is a witness of $\langle m\rangle \not \varliminf_{\mathrm{m}}\left\langle d_{3}\right\rangle$.
(3) Similar to case (1).

Corollary 5.3.3. $\left\langle d_{3}, p\right\rangle \preceq_{\mathrm{m}}\left\langle d_{3}\right\rangle$.
Proof. If $\langle\wedge\rangle \subseteq\left\langle d_{3}, p\right\rangle$, then $\left\langle d_{3}, p\right\rangle$ satisfies the minor identity $f(x, y) \approx f(y, x)$. In the proof of Proposition 5.3.2 it was shown that $\left\langle d_{3}\right\rangle$ does not satisfy $f(x, y) \approx f(y, x)$. Hence, $\left\langle d_{3}, p\right\rangle \npreceq_{\mathrm{m}}\left\langle d_{3}\right\rangle$.

It is easy to check that $\left\langle d_{3}, m\right\rangle=\operatorname{Pol}\left(\mathbb{C}_{2}\right)$, where $\mathbb{C}_{2}$ is the relational structure $\mathbb{C}_{2}:=\left(\{0,1\} ; C_{2},\{0\},\{1\}\right)$.

Proposition 5.3.4. The following holds:
(1) $\left\langle d_{3}, m\right\rangle \not \varliminf_{\mathrm{m}}\left\langle d_{3}\right\rangle$.
(2) $\left\langle d_{3}, m\right\rangle \preceq_{\mathrm{m}}\langle m\rangle$.
(3) $\left\langle d_{3}, m\right\rangle \mid\langle\wedge\rangle$.

Proof. (1) and (2) follow immediately from Proposition 5.3.2, the quasi minority condition is a witness of $\left\langle d_{3}, m\right\rangle \not \varliminf_{\mathrm{m}}\left\langle d_{3}\right\rangle$ and the quasi majority condition is a witness of $\left\langle d_{3}, m\right\rangle \not \varliminf_{\mathrm{m}}\langle m\rangle$. Concerning (3), it follows from Proposition 5.3 .2 that the quasi majority condition is a witness of $\left\langle d_{3}, m\right\rangle \not \varliminf_{\mathrm{m}}\langle\wedge\rangle$. Conversely, suppose that $g$ is a Boolean binary commutative operation. Then $g$ does not preserve $\mathbb{C}_{2}$. Therefore, the minor identity $f(x, y) \approx f(y, x)$ is a witness of $\langle\wedge\rangle \not \varliminf_{\mathrm{m}}\left\langle d_{3}, m\right\rangle$.

We now prove that $\mathfrak{P}_{2}$ contains an infinite descending chain.

$$
\begin{equation*}
\overline{\left\langle d_{3}, q\right\rangle} \succ_{\mathrm{m}} \overline{\left\langle d_{4}, q\right\rangle} \succ_{\mathrm{m}} \overline{\left\langle d_{5}, q\right\rangle} \succ_{\mathrm{m}} \cdots \succ_{\mathrm{m}} \overline{\langle q\rangle} \tag{C1}
\end{equation*}
$$

In order to prove this fact, we introduce the following relational structures, also known as cube term blockers [84]:

$$
\begin{gathered}
\mathbb{B}_{k}:=\left(\{0,1\} ; B_{k},\{0\},\{1\}\right) \text { where } B_{k}:=\{0,1\}^{k} \backslash\{(\underbrace{0, \ldots, 0}_{k})\} \\
\mathbb{B}_{\infty}:=\bigcup_{n \in \mathbb{N}} \mathbb{B}_{n} .
\end{gathered}
$$

Cube blockers are the relational counterparts of the clones considered in the chain (C1), because the same chain can be rewritten as:

$$
\overline{\operatorname{Pol}\left(\mathbb{B}_{2}\right)} \succ_{\mathrm{m}} \overline{\operatorname{Pol}\left(\mathbb{B}_{3}\right)} \succ_{\mathrm{m}} \overline{\operatorname{Pol}\left(\mathbb{B}_{4}\right)} \succ_{\mathrm{m}} \cdots \succ_{\mathrm{m}} \overline{\operatorname{Pol}\left(\mathbb{B}_{\infty}\right)}
$$

We use the QNU identities to prove that the order is strict: in fact, $\operatorname{Pol}\left(\mathbb{B}_{n-1}\right)$ satisfies $\mathrm{QNU}(n)$ but $\operatorname{Pol}\left(\mathbb{B}_{n}\right)$ does not.

Proposition 5.3.5. For any natural number $n>2$, the quasi near-unanimity condition $\mathrm{QNU}(n)$ is a witness of $\operatorname{Pol}\left(\mathbb{B}_{n-1}\right) \not \varliminf_{\mathrm{m}} \operatorname{Pol}\left(\mathbb{B}_{n}\right)$.

Proof. Let $f$ be an $n$-ary quasi near-unanimity operation. Suppose for contradiction that $f$ is in $\operatorname{Pol}\left(\mathbb{B}_{n}\right)$. Note that $f$ is idempotent since it has to preserve the unary relations $\{0\}$ and $\{1\}$. In the following $(n \times n)$-matrix every column is an element of $B_{n}$. Then we get a contradiction since, applying $f$ row-wise, we obtain the missing $n$-tuple $(0, \ldots, 0)$.

$$
f\left(\begin{array}{cccc}
0 & \ldots & 0 & 1 \\
\vdots & . \cdot & 1 & 0 \\
0 & . \cdot & . \cdot & \vdots \\
1 & 0 & \ldots & 0
\end{array}\right)=\left(\begin{array}{c}
f(0, \ldots, 0,1) \\
\vdots \\
f(1,0, \ldots, 0)
\end{array}\right)=\left(\begin{array}{c}
0 \\
\vdots \\
0
\end{array}\right) .
$$

Let $g$ be an $n$-ary operation defined as:

$$
g\left(x_{1}, \ldots, x_{n}\right):= \begin{cases}1 & \text { if at least two variables are evaluated to } 1 . \\ 0 & \text { otherwise }\end{cases}
$$

Then, by definition, $g$ is a QNU operation. We claim that $g \in \operatorname{Pol}\left(\mathbb{B}_{n-1}\right)$. Indeed, from an analysis of any $(n-1) \times n$-matrix $\mathbf{M}$ such that applying $g$ to the rows of $\mathbf{M}$ one gets the tuple $(0, \ldots, 0)$, we conclude that one of the columns of $\mathbf{M}$ must be equal to a 0 -vector. Thus the claim follows.

With the same argument we can prove that there is another infinite descending chain, namely

$$
\begin{equation*}
\overline{\left\langle d_{3}, p\right\rangle} \succ_{\mathrm{m}} \overline{\left\langle d_{4}\right\rangle} \succ_{\mathrm{m}} \overline{\left\langle d_{5}\right\rangle} \succ_{\mathrm{m}} \cdots \succ_{\mathrm{m}} \overline{\langle p\rangle} . \tag{C2}
\end{equation*}
$$

For every $k$, we define

$$
\begin{aligned}
& \mathbb{B}_{k}^{\leq}:=\left(\{0,1\} ; B_{k}, \leq,\{0\},\{1\}\right), \\
& \mathbb{B} \leq \infty \\
& \leq \bigcup_{n \in \mathbb{N}} \mathbb{B}_{n}^{\leq}
\end{aligned}
$$

Again we consider the relational counterparts of the clones involved in (C2) and rewrite the chain as follows:

$$
\overline{\operatorname{Pol}\left(\mathbb{B}_{2}^{\leq}\right)} \succ_{\mathrm{m}} \overline{\operatorname{Pol}\left(\mathbb{B}_{3}^{\leq}\right)} \succ_{\mathrm{m}} \overline{\operatorname{Pol}\left(\mathbb{B}_{4}^{\leq}\right)} \succ_{\mathrm{m}} \cdots \succ_{\mathrm{m}} \overline{\operatorname{Pol}\left(\mathbb{B}_{\bar{\infty})}^{\leq}\right)}
$$

Proposition 5.3.6. For any natural number $n>2$, the quasi near-unanimity condition $\operatorname{QNU}(n)$ is a witness of $\operatorname{Pol}\left(\mathbb{B}_{n-1}^{\leq}\right) \npreceq_{\mathrm{m}} \operatorname{Pol}\left(\mathbb{B}_{n}^{\leq}\right)$.

The next step is to show that (C1) and (C2) are two distinct chains in $\mathfrak{P}_{2}$. In particular, we prove that there is no minor-preserving map from $\langle q\rangle$ to $\left\langle d_{3}, p\right\rangle$. The minor condition which we use as a witness of this fact is the quasi-version of a celebrated set of identities from universal algebra [56.

Proposition 5.3.7. The minor condition $\mathrm{QHM}(3)$ is a witness of $\langle q\rangle \npreceq_{\mathrm{m}}\left\langle d_{3}, p\right\rangle$.
Proof. Note that $\langle q\rangle \models \operatorname{QHM}((3)$; defining:

$$
\begin{aligned}
p_{0}^{\langle q\rangle}(x, y, z) & :=q(x, x, x) \\
p_{1}^{\langle q\rangle}(x, y, z) & :=q(x, y, z) \\
p_{2}^{\langle q\rangle}(x, y, z) & :=q(z, x, y) \\
p_{3}^{\langle q\rangle}(x, y, z) & :=q(z, z, z) .
\end{aligned}
$$

Suppose for contradiction that $\mathcal{C}:=\operatorname{Pol}\left(\mathbb{B}_{2}^{\leq}\right)$satisfies the minor condition $\mathrm{QHM}(3)$ via $p_{0}^{\mathcal{C}}, \ldots, p_{3}^{\mathcal{C}}$. Then we have

$$
1=p_{0}^{\mathcal{C}}(1,1,0)=p_{1}^{\mathcal{C}}(1,0,0) \leq_{2} p_{1}^{\mathcal{C}}(1,1,0)=\cdots=p_{3}^{\mathcal{C}}(1,0,0)=0
$$

which is a contradiction.

Proposition 5.3.8. $\langle m\rangle \varliminf_{\mathrm{m}}\left\langle d_{3}, p\right\rangle$.
Proof. We claim that the quasi minority condition is a witness of $\langle m\rangle \not \varliminf_{\mathrm{m}}\left\langle d_{3}, p\right\rangle$. In fact, let $f$ be any Boolean quasi minority operation. Then $f$ does not preserve $B_{2}$ since the missing tuple $(0,0)$ can be obtained by applying $f$ to tuples in $B_{2}$. Hence, $f \notin\left\langle d_{3}, p\right\rangle=\operatorname{Pol}\left(\mathbb{B}_{2}^{\leq}\right)$and thus the claim follows.

Corollary 5.3.9. Let $\mathcal{C}$ be a Boolean clone such that $\langle p\rangle \subseteq \mathcal{C} \subseteq\left\langle d_{3}, p\right\rangle$. Then $\mathcal{C} \mid\langle m\rangle$.
Proof. Let $\mathcal{C}$ be as in the hypothesis. Let us suppose that $\langle m\rangle \preceq_{\mathrm{m}} \mathcal{C}$. Then we get $\langle m\rangle \preceq_{\mathrm{m}} \mathcal{C} \preceq_{\mathrm{m}}\left\langle d_{3}, p\right\rangle$, contradicting Proposition5.3.8. Let us suppose now that $\mathcal{C} \preceq_{\mathrm{m}}\langle m\rangle$. Then we get $\langle\wedge\rangle \prec_{\mathrm{m}}\langle p\rangle \preceq_{\mathrm{m}} \mathcal{C} \preceq_{\mathrm{m}}\langle m\rangle$, contradicting Proposition 5.3.2.

Corollary 5.3.10. $\langle m\rangle \not \varliminf_{\mathrm{m}}\left\langle d_{3}, q\right\rangle$.

Proof. Since $\left\langle d_{3}, q\right\rangle=\operatorname{Pol}\left(\mathbb{B}_{2}\right)$, the argument is essentially the same as the one of Proposition 5.3.8.

Corollary 5.3.11. Let $\mathcal{C}$ be a Boolean clone such that $\langle q\rangle \subseteq \mathcal{C} \subseteq\left\langle d_{3}, q\right\rangle$. Then $\mathcal{C} \mid\langle m\rangle$.
Proof. The proof is essentially the same as the one of Corollary 5.3.9.
Proposition 5.3.12. Let $\mathcal{C}$ be a Boolean clone such that $\langle\wedge\rangle \subseteq \mathcal{C} \subseteq\left\langle d_{4}, q\right\rangle$. Then $\mathcal{C} \mid\left\langle d_{3}\right\rangle$.

Proof. Since $\langle\wedge\rangle \subseteq \mathcal{C}$ and since, by Proposition 5.3 .2 , we have $\langle\wedge\rangle \not \varliminf_{\mathrm{m}}\left\langle d_{3}\right\rangle$, it follows that $\mathcal{C} \not \varliminf_{\mathrm{m}}\left\langle d_{3}\right\rangle$. Since $d_{3}$ is the Boolean majority operation, we have that $\left\langle d_{3}\right\rangle$ satisfies the quasi majority condition $\mathrm{QNU}(3)$. By Proposition 5.3.5, we have that $\left\langle d_{4}, q\right\rangle$ does not satisfy $\mathrm{QNU}(3)$ and hence no subclone of $\left\langle d_{4}, q\right\rangle$ satisfies $\mathrm{QNU}(3)$. Since $\mathcal{C}$ is a subclone of $\left\langle d_{4}, q\right\rangle$ it follows that the quasi majority condition is a witness of $\mathcal{C} \not \varliminf_{\mathrm{m}}\left\langle d_{3}\right\rangle$.

Proposition 5.3.13. $\langle 0\rangle \not \varliminf_{\mathrm{m}}\langle m, q\rangle$.
Proof. Note that $\langle m, q\rangle=\operatorname{Pol}(\{0,1\} ;\{0\},\{1\})$ is the clone of all the idempotent operations on $\{0,1\}$. Hence, $\langle m, q\rangle$ contains no constants and thus $\langle m, q\rangle \not \vDash f(x) \approx f(y)$ while $\langle 0\rangle$ does.

### 5.4 The lattice $\mathfrak{P}_{2}$

Putting all the results of the sections 5.2 and 5.3 together, we display a Hasse diagram of $\mathfrak{P}_{2}$ which turns out to be a lattice (Figure 5.3). We then use this diagram to revisit the complexity of Boolean CSPs. In Figure 5.4 we indicate for each element of $\mathfrak{P}_{2}$ the corresponding complexity class.


Figure 5.3: The lattice $\mathfrak{P}_{2}$.
Theorem 5.4.1. The pp-constructability poset restricted to the case of Boolean clones is the lattice $\mathfrak{P}_{2}$ in Figure 5.3.

Proof. Recall that every element in $\mathfrak{P}_{2}$ is a $\equiv_{\mathrm{m}}$-class; for every $\equiv_{\mathrm{m}}$-class we list explicitly the clones on $\{0,1\}$ that are in the considered class. The list is justified by the results
proved in Section 5.2.

$$
\begin{aligned}
\overline{\langle\emptyset\rangle} & =\{\langle\emptyset\rangle,\langle\neg\rangle\} & \overline{\langle\wedge\rangle} & =\{\langle\wedge\rangle,\langle\vee\rangle\} \\
\overline{\left\langle d_{3}\right\rangle} & =\left\{\left\langle d_{3}\right\rangle\right\} & \overline{\langle m\rangle} & =\{\langle m\rangle,\langle m, \neg\rangle\} \\
\overline{\left\langle d_{3}, m\right\rangle} & =\left\{\left\langle d_{3}, m\right\rangle,\left\langle d_{3}, \neg\right\rangle\right\} & \overline{\langle p\rangle} & =\left\{\langle p\rangle,\left\langle p^{\Delta}\right\rangle\right\} \\
\overline{\langle q\rangle} & =\left\{\langle q\rangle,\left\langle q^{\Delta}\right\rangle\right\} & \overline{\left\langle d_{3}, p\right\rangle} & =\left\{\left\langle d_{3}, p\right\rangle,\left\langle d_{3}, p^{\Delta}\right\rangle,\langle\wedge, \vee\rangle\right\} \\
\overline{\langle m, q\rangle} & =\{\langle\vee, q\rangle\} & \overline{\langle 0\rangle} & =\{\mathcal{C} \mid 0 \in \mathcal{C} \text { or } 1 \in \mathcal{C}\}
\end{aligned}
$$

Moreover, for every $i>3$,

$$
\overline{\left\langle d_{i}, p\right\rangle}=\left\{\left\langle d_{i}, p\right\rangle,\left\langle d_{i}, p^{\Delta}\right\rangle\right\} \quad \overline{\left\langle d_{i}, q\right\rangle}=\left\{\left\langle d_{i}, q\right\rangle,\left\langle d_{i}, q^{\Delta}\right\rangle\right\}
$$

Note that all the clones of Post's lattice appear in this list. We have to show that there are no further collapses. Recall that if $\mathcal{C}$ and $\mathcal{D}$ are elements of Post's lattice such that $\mathcal{C} \subseteq \mathcal{D}$, then $\mathcal{C} \preceq_{\mathrm{m}} \mathcal{D}$. Using this remark together with the results proved in Section 5.2 and Section 5.3 we get the following inequalities.

$$
\begin{aligned}
& \overline{\langle\emptyset\rangle} \preceq_{\mathrm{m}} \overline{\mathcal{C}} \preceq_{\mathrm{m}} \overline{\langle m, q\rangle} \prec_{\mathrm{m}} \overline{\langle 0\rangle} \text {, for every } \overline{\mathcal{C}} \neq \overline{\langle 0\rangle} \quad \text { (Propositions 5.2.4, 5.3.13, 5.2.1) } \\
& \overline{\left\langle d_{3}\right\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{3}, m\right\rangle} \text { and } \overline{\langle m\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{3}, m\right\rangle} \quad \text { (Proposition 5.3.4) } \\
& \left\langle\overline{\wedge\rangle} \prec_{\mathrm{m}} \overline{\langle p\rangle} \prec_{\mathrm{m}} \overline{\langle q\rangle} \quad\right. \text { (Propositions 5.3.1, 5.3.7) } \\
& \overline{\left\langle d_{3}\right\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{3}, p\right\rangle} \\
& \overline{\langle p\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{i+1}, p\right\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{i}, p\right\rangle} \text {, for every } i \geq 3 \\
& \overline{\langle q\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{i+1}, q\right\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{i}, q\right\rangle} \text {, for every } i \geq 3 \\
& \overline{\left\langle d_{i}, p\right\rangle} \prec_{\mathrm{m}} \overline{\left\langle d_{i}, q\right\rangle} \text {, for every } i \geq 3 \\
& \text { (Corollary 5.3.3) } \\
& \text { (Proposition 5.3.6) } \\
& \text { (Proposition 5.3.5) } \\
& \text { (Proposition 5.3.7) }
\end{aligned}
$$

It remains to prove that there are no other comparable elements in $\mathfrak{P}_{2}$. Propositions 5.3.2, 5.3.4, 5.3.12, Corollary 5.3.9, and Corollary 5.3.11ensure that this is indeed the case.

Now we move to the relational side via the Inv-Pol Galois connection. We already pointed out that the bottom element of $\mathfrak{P}_{2}$ represents the class of all the Boolean relational structures $\mathbb{B}$ such that $\operatorname{CSP}(\mathbb{B})$ is NP-complete, and Schaefer's theorem 95 ] implies that the CSP of every other Boolean structure is in P. It is well known that $\operatorname{Pol}(1 \mathbb{N} 3)=\langle\emptyset\rangle$ and $\operatorname{Pol}(\mathbb{N A E})=\langle\neg\rangle[91]$ (see also [24] for a more modern approach).

Following [3], we describe the complexity of Boolean CSPs within P. Combining Theorem 5.4.1 with the main result in [3] we obtain the following.

Theorem 5.4.2. Let $\mathbb{A}$ be a Boolean relational structure and finite relational signature.

- If $\operatorname{Pol}(\mathbb{A}) \equiv_{\mathrm{m}}\langle\emptyset\rangle$, then $\operatorname{CSP}(\mathbb{A})$ is NP-complete.
- If $\operatorname{Pol}(\mathbb{A}) \equiv_{\mathrm{m}}\langle\wedge\rangle$, then $\operatorname{CSP}(\mathbb{A})$ is $\underline{P \text {-complete. }}$.
- If $\operatorname{Pol}(\mathbb{A}) \equiv_{\mathrm{m}}\langle m\rangle$, then $\operatorname{CSP}(\mathbb{A})$ is $\oplus$ L-complete.
- If $\operatorname{Pol}(\mathbb{A}) \equiv_{\mathrm{m}}\left\langle d_{3}\right\rangle$ or $\langle p\rangle \preceq_{\mathrm{m}} \operatorname{Pol}(\mathbb{A}) \preceq_{\mathrm{m}}\left\langle d_{3}, p\right\rangle$, then $\operatorname{CSP}(\mathbb{A})$ is NL-complete.
- If $\left\langle d_{3}, m\right\rangle \preceq_{\mathrm{m}} \operatorname{Pol}(\mathbb{A})$ or $\langle q\rangle \preceq_{\mathrm{m}} \operatorname{Pol}(\mathbb{A})$, then $\operatorname{CSP}(\mathbb{A})$ is in $L$.


Figure 5.4: The lattice $\mathfrak{P}_{2}$ split according to Theorem 5.4.2
In Figure 5.4 each element of $\mathfrak{P}_{2}$ is marked with the colour associated with the corresponding complexity class, as is illustrated in the legenda. We invite the reader to confront Figure 5.4 with the splitting-theorems from Section 3.3.1 in particular with Theorem 3.3.14, Theorem 3.3.16, and Theorem 3.3.17 (see also Figure 3.3).

## Chapter 6

## Clones over a three-element set

As we already anticipated in Section 3.3.3, almost twenty years after Post [91] succeeded in completely describing the lattice $\mathfrak{L}_{2}$ of all clones of operations on $\{0,1\}$, Janov and Mučnik 61] proved that there exists a continuum of clones on a $n$-element set, for $n \geq 3$. This result constituted a significant setback for clone theory: the goal of continuing Post's work and thus providing a complete description of $\mathfrak{L}_{n}$, seemed to be hopeless even in the case of $n=3$. Nevertheless, Jablonskij [60] described all maximal elements of $\mathfrak{L}_{3}$. Later, it was proved that all maximal clones in $\mathfrak{L}_{3}$, except the clone of all linear functions, contain a continuum of subclones 46, 78; in particular, Marchenkov proved that there are $2^{\omega}$ clones of self-dual operations over $E_{3}=\{0,1,2\}$ and Zhuk [104] presented a complete description of their lattice. This lattice has a remarkably rich structure (see Figure 6.1) and is large in the sense that its top element $\mathcal{C}_{3}:=\operatorname{Pol}(\{0,1,2\} ;\{(0,1),(1,2),(2,0)\})$ is one of the 18 maximal clones found by Jablonskij [60].

In this chapter we show that, in contrast to $\mathfrak{L}_{3}$, the poset $\mathfrak{P}_{3}$ is rather tame. First, we prove that it has only 3 submaximal elements, i.e., $\overline{\mathcal{C}_{2}}, \overline{\mathcal{C}_{3}}$, and $\overline{\mathcal{B}_{2}}$. Here $\mathcal{C}_{n}=\operatorname{Pol}\left(\mathbb{C}_{n}\right)$ and $\mathcal{B}_{2}=\operatorname{Pol}\left(\mathbb{B}_{2}\right)$, where $\mathbb{C}_{n}$ and $\mathbb{B}_{2}$ are the relational structures defined in Section 3.2.4 and in Example 2.3.2, respectively. Although we are still do not know the cardinality of $\mathfrak{P}_{3}$, we know that a possible family of $2^{\omega}$ elements would necessarily have to be located below $\overline{\mathcal{B}_{2}}$ (see the discussion at the end of Section 3.3.3.

### 6.1 Submaximal elements of $\mathfrak{P}_{3}$

In this section we prove that $\overline{\mathcal{C}_{2}}, \overline{\mathcal{C}_{3}}$, and $\overline{\mathcal{B}_{2}}$ are the only submaximal elements of $\mathfrak{P}_{3}$. In particular, we show that if $\mathcal{S}$ is a clone on $\{0,1,2\}$ such that

$$
\mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{2}, \mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{3}, \text { and } \mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{B}_{2}
$$

then there exists a minor-preserving map from $\mathcal{I}_{2}$ to $\mathcal{S}$, i.e., $\mathcal{I}_{2} \preceq_{\mathrm{m}} \mathcal{S}$. Our proof is of syntactic nature: we prove several statements that entail the existence of operations satisfying suitable identities in some clone $\mathcal{C}$, provided that $\mathcal{C}$ has certain operations. Results
in this direction are Lemma 6.1.5, Corollary 6.1.6, Lemma 6.1.7, and theorems 6.1.10 and 6.1.11. Statements of this form constitute results of independent interest in universal algebra as they lay in the foundations of the well-known Galois connection between identities and varieties. We are going to prove that any clone on a three-element set satisfying (\$) has an oddition of arity $k$ for every odd $k \geq 3$ and a totally symmetric operation of arity $n$ for every $n \geq 2$. Recall that we say that an $n$-ary operation $f$ is totally symmetric if it satisfies the condition $\operatorname{TS}(n)$ from Definition 3.3.19.

As a first step, we want to prove that every clone $\mathcal{S}$ on $E_{3}$ satisfying ( $\dagger$ ) has a majority operation. For this purpose, we need to introduce some more notions and terminology concerning relations. In particular, we will consider essential and key relations.

The author would like to express his sincere gratitude to Dmitriy Zhuk: the formulae used in Theorem 6.1.10 and Theorem 6.1.11 are due to him.

Definition 6.1.1. Let $R$ be an $n$-ary relation on a finite set $A$. We say that

- $R$ is an essential relation if there do not exist relations $R_{1}, \ldots, R_{m}$ of arity smaller than $n$ such that $R$ is obtained as a conjunction of relations from $\left\{R_{1}, \ldots, R_{m}\right\}$. A tuple $\left(a_{1}, \ldots, a_{n}\right) \in A^{n} \backslash R$ is essential for $R$ if if for every $i \in\{1, \ldots, n\}$ there exists $b$ such that $\left(a_{1}, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{n}\right) \in R$. We denote by $\operatorname{Ess}(R)$ the set of all essential tuples for $R$.
- $R$ is a key relation if there exists a tuple $\boldsymbol{b} \in A^{n} \backslash R$ such that for every $\boldsymbol{a} \in A^{n} \backslash R$ there exists a tuple $\Psi:=\left(\Psi_{1}, \ldots, \Psi_{n}\right)$, where $\Psi_{i}: A \rightarrow A$, which preserves $R$ and gives $\boldsymbol{\Psi}(\boldsymbol{a})=\boldsymbol{b}$.

Key relations are a generalization of critical relations introduced by Kearnes and Szendrei in [71]. We only need one more ingredient in order to prove Lemma 6.1.7 the notion of block of a relation $R$ over a finite domain $A$.

We denote by $\tilde{R}$ the relation $R \cup \operatorname{Ess}(R)$. Again following [105], we define a graph $\mathbb{G}_{\tilde{R}}:=(\tilde{R} ; E)$ as follows: if $\boldsymbol{a}, \boldsymbol{b} \in \tilde{R} \subseteq A^{n}$, then we have $(\boldsymbol{a}, \boldsymbol{b}) \in E$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ differ just in one element, i.e., there exists a unique $i \in\{1, \ldots, n\}$ such that $a_{i} \neq b_{i}$. A block of $R$ is a connected component of $\mathbb{G}_{\tilde{R}}$. A block is called trivial if it only contains tuples from $R$; it is nontrivial otherwise.

Another notation that that will come in handy throughout the current chapter is the following: we write $+_{n}$ to denote the addition modulo $n$, i.e., the operation of the group of integers modulo $n$; please note that sometimes, e.g., in Chapter 5 we denote +2 by $\oplus$.

Theorem 6.1.2 (c.f. [105], Theorem 3.11). Let $R$ be a key essential relation of arity $n \geq 3$, preserved by a Mal'cev operation. Then

- Every block of $R$ equals $B_{1} \times \cdots \times B_{n}$, for some $B_{1}, \ldots, B_{n} \subseteq A$.
- For every nontrivial block $\boldsymbol{B}:=B_{1} \times \cdots \times B_{n}$ of $R$, the intersection $R \cap \boldsymbol{B}$ can be defined as follows: there exists an abelian group $(G ;+,-, 0)$ whose order is a power
of a prime, and surjective mappings $\phi_{i}: B_{i} \rightarrow G$, for $i=1,2, \ldots, n$ such that

$$
R \cap \boldsymbol{B}=\left\{\left(x_{1}, \ldots, x_{n}\right) \mid \phi_{1}\left(x_{1}\right)+\phi_{2}\left(x_{2}\right)+\ldots+\phi_{n}\left(x_{n}\right)=0\right\} .
$$

Theorem 6.1.3 (4). Let $\mathcal{C}$ be an idempotent clone on a finite set. Then, for every $k \geq 2$, the following are equivalent:

- $\mathcal{C} \models \mathrm{NU}(k+1)$;
- every $(k+1)$-ary relation in $\operatorname{Inv}(\mathcal{C})$ can be obtained as a conjunction of relations of arity $k i n \operatorname{Inv}(\mathcal{C})$.

We want to remark that if an idempotent clone $\mathcal{C}$ does not have a near unanimity operation of arity $k$ then $\operatorname{Inv}(\mathcal{C})$ has an essential relation $R$ of arity $k$. Furthermore, a result of Zhuk [105] yields that $R$ can be chosen to be a key relation. In fact, it almost follows from definition that one can choose $R$ to be critical (see [71) and Zhuk proved the following.

Lemma 6.1.4 ([105], Lemma 2.4). Let $R$ be a critical relation, then $R$ is a key relation.
Lemma 6.1.5. Let $\mathcal{C}$ be a clone over $E_{n}$, for some natural $n \geq 2$, such that
(1) $\mathcal{C} \models \Sigma_{p}$, for every prime $p \leq n$, and
(2) $\mathcal{C} \models \Sigma_{\mathrm{M}}$, i.e., $\mathcal{C}$ has a Mal'cev operation.

Then $\mathcal{C}$ has a majority operation.
Proof. Let $\mathcal{C}$ be a clone satisfying all the hypotheses and suppose that $\mathcal{C}$ does not have a ternary near unanimity operation, i.e., a majority operation. Then by Theorem 6.1 .3 we have that $\operatorname{Inv}(\mathcal{C})$ has a critical relation $R$ of arity $k \geq 3$. Therefore, by Theorem 6.1.2 for every nontrivial block $\mathbf{B}$ of $R$, there exists an abelian group $\mathbf{G}=(G,+,-, 0)$ whose order $\ell \leq n$ is the power of some prime and surjective mappings $\phi_{i}: B_{i} \rightarrow G$, for $i=1,2, \ldots, k$ such that $R \cap \mathbf{B}=\left\{\left(x_{1}, \ldots, x_{k}\right) \mid \phi_{1}\left(x_{1}\right)+\phi_{2}\left(x_{2}\right)+\ldots+\phi_{k}\left(x_{k}\right)=0\right\}$.

Let us show that the relation $R$ cannot be preserved by a cyclic operation $c_{p}$ of arity $p$, where $p$ divides $\ell$. Choose a mapping $\psi_{i}: G \rightarrow B_{i}$, for every $i$, such that $\phi_{i}\left(\psi_{i}(x)\right)=x$, for every $x \in G$. Let $a$ be an element in $G$ of order $p$. Notice, that

$$
B_{1}\left(x_{1}\right)=\exists x_{2} \ldots \exists x_{k} \bigwedge_{i=2}^{k} R\left(x_{1}, \psi_{2}(0), \ldots, \psi_{i-1}(0), x_{i}, \psi_{i+1}(0), \ldots, \psi_{k}(0)\right),
$$

which means that $B_{1}$ is pp-definable from $R$ and constants. Combining this with the idempotency of $\mathcal{C}$ we derive that the cyclic operation $c_{p}$ preserves $B_{1}$. Similarly, we show
that $c_{p}$ preserves $B_{i}$, for every $i$. Applying $c_{p}$ to the rows of the matrices

$$
\begin{gathered}
\left(\begin{array}{ccccc}
\psi_{1}(0) & \psi_{1}(a) & \psi_{1}(2 a) & \ldots & \psi_{1}((p-1) a) \\
\psi_{2}(0) & \psi_{2}(-a) & \psi_{2}(-2 a) & \ldots & \psi_{2}(-(p-1) a) \\
\psi_{3}(0) & \psi_{3}(0) & \psi_{3}(0) & \ldots & \psi_{3}(0) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\psi_{k}(0) & \psi_{k}(0) & \psi_{k}(0) & \ldots & \psi_{k}(0)
\end{array}\right) \\
\left(\begin{array}{ccccc}
\psi_{1}(0) & \psi_{1}(a) & \ldots & \psi_{1}((p-2) a) & \psi_{1}((p-1) a) \\
\psi_{2}(-a) & \psi_{2}(-2 a) & \ldots & \psi_{2}(-(p-1) a) & \psi_{2}(0) \\
\psi_{3}(a) & \psi_{3}(a) & \ldots & \psi_{3}(a) & \psi_{3}(a) \\
\psi_{4}(0) & \psi_{4}(0) & \ldots & \psi_{4}(0) & \psi_{4}(0) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\psi_{k}(0) & \psi_{k}(0) & \ldots & \psi_{k}(0) & \psi_{k}(0)
\end{array}\right)
\end{gathered}
$$

we get respectively the tuples

$$
\begin{aligned}
& \left(c, d, \psi_{3}(0), \psi_{4}(0), \ldots, \psi_{k}(0)\right), \text { and } \\
& \left(c, d, \psi_{3}(a), \psi_{4}(0), \ldots, \psi_{k}(0)\right)
\end{aligned}
$$

from $R \cap \mathbf{B}$, which contradicts the definition of $R \cap \mathbf{B}$. This contradiction proves that such a relation $R$ cannot exist in $\operatorname{Inv}(\mathcal{C})$, thus $\mathcal{C}$ has a majority operation.

Corollary 6.1.6. Let $\mathcal{S}$ be a clone on $E_{3}$ such that $\mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{2}, \mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{3}$, and $\mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{B}_{2}$. Then $\mathcal{S}$ has a fully symmetric majority operation.

Proof. From Theorem 3.3 .10 it follows that $\mathcal{S}$ has a Mal'cev operation, and from Theorem 3.3 .11 it follows that there exist $c_{2}, c_{3} \in \mathcal{C}$ such that $c_{2} \models \Sigma_{2}$ and $c_{3} \models \Sigma_{3}$. Thus, it follows from Lemma 6.1.5 that $\mathcal{S}$ has a majority operation $M^{\prime}$. We define the operation $M$ as follows:

$$
\begin{array}{r}
M(x, y, z):=c_{2}\left(c_{3}\left(M^{\prime}(x, y, z), M^{\prime}(y, z, x), M^{\prime}(z, x, y)\right)\right. \\
\left.c_{3}\left(M^{\prime}(x, z, y), M^{\prime}(z, y, x), M^{\prime}(y, x, z)\right)\right) \tag{৩}
\end{array}
$$

It is easy to check that $M$ is a fully symmetric majority.
Lemma 6.1.7. Let $\mathcal{S}$ be a clone on $E_{3}$ such that $\mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{2}, \mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{3}$, and $\mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{B}_{2}$. Then $\mathcal{S}$ has a fully symmetric minority operation.

Proof. Let $\mathcal{S}$ be as in the hypothesis. It follows from Theorem 3.3.10 that $\mathcal{S}$ has a Mal'cev operation $d$. Also, from Corollary 6.1.6 we know that $\mathcal{S}$ has a majority operation $M$. We define $m_{3}^{\prime}$ as follows:

$$
m_{3}^{\prime}(x, y, z):=M(d(x, y, z), d(y, z, x), d(z, x, y))
$$

It is easy to check that $m_{3}^{\prime}$ is indeed a minority operation: note that, since $d$ is a Mal'cev operation, whenever we identify two variables in $m_{3}^{\prime}$ at least two of the values among $d(x, y, z), d(y, z, x)$, and $d(z, x, y)$ are equal to the variable that occurs only once. Hence, applying $M$ we obtain this variable, again. Furthermore, it follows from Theorem 3.3.11 that $\mathcal{S}$ has a binary cyclic operation $c_{2}$ and a ternary cyclic operation $c_{3}$. We then define a fully symmetric minority $m_{3}$ in the same way we obtained a fully symmetric majority in Corollary 6.1.6 we simply replace every occurrence of $M$ in (Q) by $m_{3}$.

Remark 6.1.8. Note that the value of a fully symmetric minority $m_{3}(x, y, z)$ has to be a constant $c \in\{0,1,2\}$ whenever the three values in the scope of $m_{3}$ are all distinct, i.e.,

$$
m_{3}(0,1,2)=m_{3}(0,2,1)=m_{3}(1,0,2)=m_{3}(1,2,0)=m_{3}(2,0,1)=m_{3}(2,1,0)=c .
$$

In this case we also denote the fully symmetric minority operation by $m_{3}^{c}$. We follow the same convention for fully symmetric majority operations.

## The "oddition"

An oddition ${ }^{\text {W }}$ of arity $n$, for some odd number $n$, is an idempotent operation $m_{n}^{c}: A \rightarrow A$ defined as follows: $m_{n}^{c}\left(x_{1}, \ldots, x_{n}\right)$ returns the constant $c \in A$ if there are at least three distinct values occurring an odd number of times in the tuple ( $x_{1}, \ldots, x_{n}$ ), otherwise $m_{n}^{c}\left(x_{1}, \ldots, x_{n}\right)$ returns the only value occurring an odd number of times.

Remark 6.1.9. Let $m_{2 n+1}^{c}$ be an oddition over a finite set $A$, for some $n \in \mathbb{N}$. If $\left(a_{1}, \ldots, a_{2 n+1}\right) \in A^{2 n+1}$ is such that $a_{i}=a_{j}$, for some $i, j \in\{1, \ldots, 2 n+1\}$ with $i<j$, then

$$
\begin{aligned}
m_{2 n+1}^{c}\left(a_{1}, \ldots, a_{2 n+1}\right) & \stackrel{\star}{=} m_{2 n+1}^{c}\left(a_{1}, \ldots, a_{2 n+1}, a_{i}, a_{j}\right) \\
& \stackrel{\star \star}{=} m_{2 n-1}^{c}\left(a_{1}, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{2 n+1}\right) .
\end{aligned}
$$

The equality $\stackrel{\star}{=}$ holds since $m_{2 n+1}^{c}$ is fully symmetric by definition, and $\stackrel{\star \star}{=}$ holds since the output of $m_{2 n+1}^{c}\left(a_{1}, \ldots, a_{2 n+1}\right)$ only depends on the parity of the values occurring as arguments in $m_{2 n+1}^{c}$.

Note that the minority operation $m_{3}(x, y, z)=x+{ }_{2} y+{ }_{2} z$ on the Boolean set $\{0,1\}$ is indeed an oddition of arity 3 . Also note that if a clone $\mathcal{C}$ over $\{0,1\}$ contains the minority operation $m_{3}(x, y, z)$ then, for every $n \geq 2$, the oddition

$$
\begin{aligned}
m_{2 n+1}\left(x_{1}, \ldots, x_{2 n+1}\right) & :=m_{3}\left(m_{2 n-1}\left(x_{1}, \ldots, x_{2 n-1}\right), x_{2 n}, x_{2 n+1}\right) \\
& =x_{1}+{ }_{2} x_{2}+2 \ldots+2 x_{2 n+1}
\end{aligned}
$$

[^3]is also in $\mathcal{C}$. We prove an analogous result for the three-element case: we show that every clone $\mathcal{S}$ on $E_{3}$ satisfying condition ( has an oddition of every odd arity. Recall that by Theorem 6.1.7 we know that $\mathcal{S}$ has a fully symmetric minority operation $m_{3}^{c}$ where $c \in\{0,1,2\}$ is the constant value that $m_{3}^{c}(x, y, z)$ returns whenever $|\{x, y, z\}|=3$, see Remark 6.1.8. We define the following auxiliary operation
\[

D^{c}(x, y, z):= $$
\begin{cases}c+{ }_{3} 1 & \text { if }(x, y, z) \in\left\{\left(c+{ }_{3} 2, c, c+{ }_{3} 1\right),\left(c+{ }_{3} 2, c+{ }_{3} 1, c\right)\right\} \\ c+{ }_{3} 2 & \text { if }(x, y, z) \in\left\{\left(c+{ }_{3} 1, c, c+{ }_{3} 2\right),\left(c+{ }_{3} 1, c+{ }_{3} 2, c\right)\right\} \\ x & \text { otherwise }\end{cases}
$$
\]

Note that $D^{c}(x, y, z)=m_{3}^{c}\left(m_{3}^{c}(x, y, z), y, z\right)$, hence $D^{c}(x, y, z) \in \mathcal{S}$.

Theorem 6.1.10. Let $\mathcal{S}$ be a clone on $E_{3}$ such that $\mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{2}, \mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{3}$, and $\mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{B}_{2}$. Then $\mathcal{S}$ has an oddition of every odd arity.

Proof. From Lemma 6.1.7 we know that $\mathcal{S}$ has a symmetric minority operation $m_{3}^{c}$; let $D^{c}$ the operation defined as in $\left.\Delta\right\rangle$. For every $n \geq 2$, we define the operation

$$
\begin{array}{r}
m_{2 n+1}^{c}\left(x_{1}, \ldots, x_{2 n+1}\right):=m_{3}^{c}\left(t_{2 n+1}\left(x_{1}, x_{2}, x_{3}, x_{4}, \ldots, x_{2 n+1}\right)\right. \\
t_{2 n+1}\left(x_{2}, x_{1}, x_{3}, x_{4}, \ldots, x_{2 n+1}\right) \\
\left.t_{2 n+1}\left(x_{3}, x_{1}, x_{2}, x_{4}, \ldots, x_{2 n+1}\right)\right)
\end{array}
$$

where

$$
\begin{aligned}
t_{2 n+1}\left(x_{1}, \ldots, x_{2 n+1}\right):=m_{3}^{c}( & D^{c}\left(m_{2 n-1}^{c}\left(x_{1}, x_{4}, x_{5}, \ldots, x_{2 n+1}\right), x_{1}, x_{1}\right) \\
& D^{c}\left(m_{2 n-1}^{c}\left(x_{1}, x_{4}, x_{5}, \ldots, x_{2 n+1}\right), x_{1}, x_{2}\right) \\
& \left.D^{c}\left(m_{2 n-1}^{c}\left(x_{1}, x_{4}, x_{5}, \ldots, x_{2 n+1}\right), x_{1}, x_{3}\right)\right)
\end{aligned}
$$

The first argument of $m_{3}^{c}$ in the latter formula equals to $m_{2 n-1}^{c}\left(x_{1}, x_{4}, x_{5}, \ldots, x_{2 n+1}\right)$ however, we instead prefer to write $D^{c}\left(m_{2 n-1}^{c}\left(x_{1}, x_{4}, x_{5}, \ldots, x_{2 n+1}\right), x_{1}, x_{1}\right)$ in the definition, for the sake of symmetry.

We are going to prove the claim of the theorem by induction over $n$. Let us first make a few remarks on the symmetries of $m_{2 n+1}^{c}$ in order to make the formula more digestible for the reader. As an inductive hypothesis we assume that $m_{2 n-1}^{c}$ is an oddition. It follows from the symmetry of $m_{2 n-1}^{c}$ that $t_{2 n+1}$ is invariant under any permutation of the variables $x_{4}, \ldots, x_{2 n+1}$. Hence $m_{2 n+1}^{c}$ is also invariant under any permutation of the variables $x_{4}, \ldots, x_{2 n+1}$. Since $m_{3}^{c}$ is symmetric, $t_{2 n+1}$ is invariant under permutation of $x_{2}$ and $x_{3}$ and therefore $m_{2 n+1}^{c}$ is invariant under any permutation of the variables $x_{1}$, $x_{2}$, and $x_{3}$. Notice that $m_{1}^{c}(x):=x$. Moreover, since $m_{2 n-1}^{c}$ is an oddition, it holds

$$
m_{2 n-1}^{c}\left(x_{1}, x_{2}, \ldots, x_{2 n-3}, x, x\right)=m_{2 n-3}^{c}\left(x_{1}, x_{2}, \ldots, x_{2 n-3}\right)
$$

thus, we obtain that

$$
t_{2 n+1}\left(x_{1}, x_{2}, \ldots, x_{2 n-1}, x, x\right)=t_{2 n-1}\left(x_{1}, x_{2}, \ldots, x_{2 n-1}\right)
$$

and therefore $m_{2 n+1}^{c}\left(x_{1}, x_{2}, \ldots, x_{2 n-1}, x, x\right)=m_{2 n-1}^{c}\left(x_{1}, x_{2}, \ldots, x_{2 n-1}\right)$.
Combining this with the symmetry of $m_{2 n+1}^{c}$ over permutation of the last $2 n-2$ coordinates, we obtain that $m_{2 n+1}^{c}$ behaves as an oddition for all the tuples having repetitive elements in $x_{4}, \ldots, x_{2 n+1}$. Thus, if $n \geq 3$ then $2 n+1-3>3$ and $m_{2 n+1}^{c}$ is an oddition. It only remains to consider the case when there are no repeated elements in the first three components and no repeated elements in the last two components of $m_{5}^{c}$. By making use of the known symmetries, it suffices to verify that $m_{5}^{c}\left(x_{1}, x_{2}, x_{3}, x_{1}, x_{2}\right)=x_{3}$.

Let us check how the identification of variables transforms $m_{5}^{c}$.

$$
\begin{align*}
m_{5}^{c}\left(x_{1}, x, x, x_{4}, x_{5}\right)= & m_{3}^{c}\left(t_{5}\left(x_{1}, x, x, x_{4}, x_{5}\right)\right.  \tag{1}\\
& \quad t_{5}\left(x, x_{1}, x, x_{4}, x_{5}\right) \\
& \left.t_{5}\left(x, x_{1}, x, x_{4}, x_{5}\right)\right) \\
= & t_{5}\left(x_{1}, x, x, x_{4}, x_{5}\right) \\
= & m_{3}^{c}\left(D^{c}\left(m_{3}^{c}\left(x_{1}, x_{4}, x_{5}\right), x_{1}, x_{1}\right),\right. \\
& D^{c}\left(m_{3}^{c}\left(x_{1}, x_{4}, x_{5}\right), x_{1}, x\right) \\
& \left.D^{c}\left(m_{3}^{c}\left(x_{1}, x_{4}, x_{5}\right), x_{1}, x\right)\right) \\
= & D^{c}\left(m_{3}^{c}\left(x_{1}, x_{4}, x_{5}\right), x_{1}, x_{1}\right) \\
= & m_{3}^{c}\left(x_{1}, x_{4}, x_{5}\right)
\end{align*}
$$

This proves that $m_{5}^{c}$ behaves well on all the tuples having repetitive elements in the first 3 coordinates. To prove for the case when the first 3 coordinates are different we will need the following identities:

$$
\begin{aligned}
& t_{5}\left(x_{1}, x_{2}, x_{3}, x_{1}, x_{2}\right)= m_{3}^{c}\left(D^{c}\left(m_{3}^{c}\left(x_{1}, x_{1}, x_{2}\right), x_{1}, x_{1}\right)\right. \\
& D^{c}\left(m_{3}^{c}\left(x_{1}, x_{1}, x_{2}\right), x_{1}, x_{2}\right) \\
&\left.D^{c}\left(m_{3}^{c}\left(x_{1}, x_{1}, x_{2}\right), x_{1}, x_{3}\right)\right) \\
&= m_{3}^{c}\left(x_{2}, x_{2}, D^{c}\left(x_{2}, x_{1}, x_{3}\right)\right)=D^{c}\left(x_{2}, x_{1}, x_{3}\right) \\
& t_{5}\left(x_{1}, x_{2}, x_{3}, x_{2}, x_{3}\right)=m_{3}^{c}\left(D^{c}\left(m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right), x_{1}, x_{1}\right)\right. \\
& D^{c}\left(m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right), x_{1}, x_{2}\right) \\
&\left.D^{c}\left(m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right), x_{1}, x_{3}\right)\right)=m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right)
\end{aligned}
$$

We check the last equality as follows. If $\left|\left\{x_{1}, x_{2}, x_{3}\right\}\right|<3$ then $m_{3}^{c}$ is the minority and $D^{c}$ is the first projection, which implies the equality. Otherwise, if $\left|\left\{x_{1}, x_{2}, x_{3}\right\}\right|=3$, then $m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right)=c$ and, since $D^{c}$ returns $c$ whenever the first coordinate is $c$, we get the equality.

Finally, we obtain the following equation

$$
\begin{array}{r}
m_{5}^{c}\left(x_{1}, x_{2}, x_{3}, x_{1}, x_{2}\right)=m_{3}^{c}\left(t_{5}\left(x_{1}, x_{2}, x_{3}, x_{1}, x_{2}\right),\right.  \tag{2}\\
t_{5}\left(x_{2}, x_{1}, x_{3}, x_{1}, x_{2}\right), \\
\left.t_{5}\left(x_{3}, x_{1}, x_{2}, x_{1}, x_{2}\right)\right) \\
=m_{3}^{c}\left(D^{c}\left(x_{2}, x_{1}, x_{3}\right),\right. \\
D^{c}\left(x_{1}, x_{2}, x_{3}\right), \\
\left.m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right)\right)=x_{3} .
\end{array}
$$

The equation ( $\left(\star_{2}\right)$ above can be checked manually. If $\left\{x_{1}, x_{2}, x_{3}\right\} \neq\{0,1,2\}$, then it again follows from the fact that $m_{3}^{c}$ is the minority and $D^{c}$ is the first projection on every 2 -element subset.

If $\left\{x_{1}, x_{2}, x_{3}\right\}=\{0,1,2\}$ and $x_{3}=c$, then

$$
m_{3}^{c}\left(D^{c}\left(x_{2}, x_{1}, x_{3}\right), D^{c}\left(x_{1}, x_{2}, x_{3}\right), m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right)\right)=m_{3}^{c}\left(x_{1}, x_{2}, c\right)=c .
$$

If $\left\{x_{1}, x_{2}, x_{3}\right\}=\{0,1,2\}$ and $x_{1}=c$, then

$$
m_{3}^{c}\left(D^{c}\left(x_{2}, x_{1}, x_{3}\right), D^{c}\left(x_{1}, x_{2}, x_{3}\right), m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right)\right)=m_{3}^{c}\left(x_{3}, c, c\right)=x_{3}
$$

Similarly, if $\left\{x_{1}, x_{2}, x_{3}\right\}=\{0,1,2\}$ and $x_{2}=c$, then

$$
m_{3}^{c}\left(D^{c}\left(x_{2}, x_{1}, x_{3}\right), D^{c}\left(x_{1}, x_{2}, x_{3}\right), m_{3}^{c}\left(x_{1}, x_{2}, x_{3}\right)\right)=m_{3}^{c}\left(c, x_{3}, c\right)=x_{3} .
$$

The equations ( $\star_{1}$ and $\left(\star_{2}\right.$ ) imply that $m_{5}^{c}$ is an oddition.

## Totally symmetric operations of every arity

Now we prove that every clone $\mathcal{S}$ on $E_{3}$ satisfying condition (\$) has totally symmetric operations of every arity $n \geq 2$ (see Definition 3.3.19).
Theorem 6.1.11. Let $\mathcal{S}$ be a clone on $E_{3}$ such that $\mathcal{S} \npreceq_{\mathrm{m}} \mathcal{C}_{2}, \mathcal{S} \npreceq_{\mathrm{m}} \mathcal{C}_{3}$, and $\mathcal{S} \npreceq_{\mathrm{m}} \mathcal{B}_{2}$. Then $\mathcal{S}$ has totally symmetric operations of every arity $n \geq 2$, that is, $\mathcal{S} \models \mathrm{TS}(n)$, for every $n \geq 2$.

Proof. From Corollary 6.1.6 and Lemma 6.1.7 it follows that $\mathcal{S}$ has a symmetric majority operation $M^{c}$ and a symmetric minority operation $m$, respectively. Also, from Theorem 3.3 .11 it follows that there exists a binary cyclic operation $s_{2} \in \mathcal{S}$, thus $\mathcal{S} \models \operatorname{TS}(2)$. For every $n \geq 3$ we define:

$$
\begin{aligned}
s_{n}\left(x_{1}, \ldots, x_{n}\right):= & m\left(s_{n-1}\left(x_{1}, M^{c}\left(x_{1}, x_{2}, x_{3}\right), x_{4}, \ldots, x_{n}\right),\right. \\
& s_{n-1}\left(x_{2}, M^{c}\left(x_{1}, x_{2}, x_{3}\right), x_{4}, \ldots, x_{n}\right), \\
& \left.s_{n-1}\left(x_{3}, M^{c}\left(x_{1}, x_{2}, x_{3}\right), x_{4}, \ldots, x_{n}\right)\right) .
\end{aligned}
$$

We will prove by induction on $n \geq 2$ that
(i) if $\left\{x_{1}, \ldots, x_{n}\right\}=\{a, b\} \subset\{0,1,2\}$, then $s_{n}\left(x_{1}, \ldots, x_{n}\right)=s_{2}(a, b)$;
(ii) if $\left\{x_{1}, \ldots, x_{n}\right\}=\{0,1,2\}$, then

$$
s_{n}\left(x_{1}, \ldots, x_{n}\right)=m\left(s_{2}(0, c), s_{2}(1, c), s_{2}(2, c)\right)
$$

For $n=2$ this is obvious. Notice that, for every $n \geq 3$,

$$
\begin{gathered}
s_{n}\left(x, x, x_{3}, x_{4}, \ldots, x_{n}\right):=m\left(s_{n-1}\left(x, M^{c}\left(x, x, x_{3}\right), x_{4}, \ldots, x_{n}\right)\right. \\
s_{n-1}\left(x, M^{c}\left(x, x, x_{3}\right), x_{4}, \ldots, x_{n}\right) \\
\left.s_{n-1}\left(x_{3}, M^{c}\left(x, x, x_{3}\right), x_{4}, \ldots, x_{n}\right)\right) \\
=s_{n-1}\left(x_{3}, x, x_{4}, \ldots, x_{n}\right)
\end{gathered}
$$

Hence, by the inductive assumption we have the required properties (i) and (ii) on all tuples whose first two elements are equal. Since the operations $M^{c}$ and $m$ are symmetric, $s_{n}$ is symmetric under any permutation of the first 3 variables. Therefore, the property (i) always holds and the property (ii) holds on all tuples such that the first three elements are not different.

Let us prove the property (ii) on all tuples $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ such that $\left\{x_{1}, x_{2}, x_{3}\right\}=$ $\{0,1,2\}$. For $s_{3}$ it immediately follows from the definition. To prove this for $n>3$ consider 3 cases.

Case 1. If $\left\{x_{4}, \ldots, x_{n}\right\}=\{a\} \subset\{0,1,2\}$ then

$$
\begin{array}{r}
s_{n}\left(x_{1}, \ldots, x_{n}\right)=m\left(s_{n-1}\left(x_{1}, c, a, \ldots, a\right),\right. \\
s_{n-1}\left(x_{2}, c, a, \ldots, a\right), \\
\left.s_{n-1}\left(x_{3}, c, a, \ldots, a\right)\right) \\
\stackrel{\star}{=} m\left(s_{n-1}(0, c, a, \ldots, a),\right. \\
s_{n-1}(1, c, a, \ldots, a), \\
\left.s_{n-1}(2, c, a, \ldots, a)\right),
\end{array}
$$

The equality $\stackrel{\star}{=}$ holds because $m$ is symmetric. In case $a=c$, we obtain, by the induction hypothesis, that

$$
s_{n}\left(x_{1}, \ldots, x_{n}\right)=m\left(s_{2}(0, c), s_{2}(1, c), s_{2}(2, c)\right)
$$

If $a \neq c$, then in $\left(\bullet_{1}\right)$ we have an argument of the form $s_{n-1}(a, c, a, \ldots, a)$, one of the form $s_{n-1}(c, c, a, \ldots, a)$, and one where 0,1 , and 2 occur. Therefore, by property (i) of the induction hypothesis we get $s_{n-1}(a, c, a, \ldots, a)=s_{2}(a, c)$ and $s_{n-1}(c, c, a, \ldots, a)=$ $s_{2}(a, c)$. Moreover, by properties of $m$, we get

$$
s_{n}\left(x_{1}, \ldots, x_{n}\right)=m\left(s_{2}(a, c), s_{2}(a, c), s_{3}(0,1,2)\right)=s_{3}(0,1,2)
$$

Case 2. If $\left\{x_{4}, \ldots, x_{n}\right\}=\{a, b\} \subset\{0,1,2\}$ then, by using the fact that $s_{n-1}$ and $m$ are symmetric, we get

$$
\begin{array}{r}
s_{n}\left(x_{1}, \ldots, x_{n}\right)=m\left(s_{n-1}(0, c, a, \ldots, a, b, \ldots, b)\right. \\
\\
s_{n-1}(1, c, a, \ldots, a, b, \ldots, b) \\
\\
\left.s_{n-1}(2, c, a, \ldots, a, b, \ldots, b)\right)
\end{array}
$$

If $c \notin\{a, b\}$ then each argument of $m$ in the latter formula is equal to $s_{3}(0,1,2)$, by the induction hypothesis. Otherwise, if $c \in\{a, b\}$ then in $\left(\bullet_{2}\right)$ we have an argument of the form $s_{n-1}(a, \ldots, a, b, \ldots, b)$, one of the form $s_{n-1}(b, a, \ldots, a, b \ldots, b)$, and one where 0 , 1 , and 2 occur. By the induction hypothesis, we get

$$
s_{n}\left(x_{1}, \ldots, x_{n}\right)=m\left(s_{2}(a, b), s_{2}(a, b), s_{3}(0,1,2)\right)
$$

Case 3. If $\left\{x_{4}, \ldots, x_{n}\right\}=\{0,1,2\}$, then, by the symmetry of $m$ and $s_{n-1}$, we have

$$
\begin{array}{r}
s_{n}\left(\sigma\left(x_{1}\right), \ldots, \sigma\left(x_{n}\right)\right)=m\left(s_{n-1}(0, c, 0, \ldots, 0,1, \ldots, 1,2 \ldots, 2)\right.  \tag{3}\\
\quad s_{n-1}(1, c, 0, \ldots, 0,1, \ldots, 1,2 \ldots, 2) \\
\left.\quad s_{n-1}(2, c, 0, \ldots, 0,1, \ldots, 1,2 \ldots, 2)\right)
\end{array}
$$

It follows, by the the induction hypothesis, that each argument of $m$ in $\bullet_{3}$ is equal to $s_{3}(0,1,2)$; hence, we obtain $s_{n}\left(x_{1}, \ldots, x_{n}\right)=s_{3}(0,1,2)$. This concludes the proof.

## The collapse of $\mathcal{S}$ and $\mathcal{I}_{2}$

In Section 3.2 .2 we proved that $\overline{\mathcal{I}_{2}}$ is the unique coatom in $\mathfrak{P}_{\mathrm{fin}}$, where $\mathcal{I}_{2}$ is the clone over $\{0,1\}$ generated by the operations $\wedge$ and $m(x, y, z):=x \oplus y \oplus z$, defined in Section 5.1. Here we prove that, whenever a clone has totally symmetric operations and odditions of an arbitrary large arity, there exists a minor homomorphism from $\mathcal{I}_{2}$ to this clone.

It is well-known that every operation over $\{0,1\}$ has a unique polynomial representation if we forbid repetitive monomials and disrespect the order of monomials. Applying this fact to idempotent operations from $\mathcal{I}_{2}$ we obtain the following lemma, in which operations $\oplus$ and $\wedge$ denote the usual sum and multiplication modulo 2 , respectively.

Lemma 6.1.12. For every operation $f \in \mathcal{I}_{2}$ there exists an up to the order of monomials unique representation of the form $f\left(x_{1}, \ldots, x_{n}\right):=\bigoplus_{i=1}^{\ell} \wedge W_{i}$, where $\ell$ is odd and the sets $W_{1}, \ldots, W_{l} \subseteq\left\{x_{1}, \ldots, x_{n}\right\}$ are different and nonempty.

Proof. It is sufficient to check that every polynomial preserving $\{0\}$ does not have the constant 1 as a monomial, and that every polynomial preserving $\{1\}$ has an odd number of monomials different from constants.

Theorem 6.1.13. Let $\mathcal{S}$ be a clone over $E_{k}$, for some $k \geq 2$, such that

- $\mathcal{S}$ has a totally symmetric operation of arity $n$, for every $n \geq 2$, and
- $\mathcal{S}$ has an oddition of arity $n$, for every odd $n \geq 3$.

Then there exists a minor homomorphism from $\mathcal{I}_{2}$ to $\mathcal{S}$.
Proof. Let $f$ be any operation in $\mathcal{I}_{2}$. Notice that the identification of two variables of a totally symmetric operation of arity $n$ gives a totally symmetric operation of a smaller arity. Similarly, the identification of three variables of an oddition gives an oddition of a smaller arity. Then by König's lemma there exist an infinite sequence of symmetric operations $s_{2}, s_{3}, s_{4}, \ldots$, and an infinite sequence of odditions $m_{3}, m_{5}, m_{7}, \ldots$, such that $s_{n}$ and $m_{n}$ are of arity $n$ for every $n$, and they are compatible in the following sense. The identification of two variables of $s_{n}$ gives $s_{n-1}$ and the identification of three variables of $m_{2 k+1}$ gives $m_{2 k-1}$.

By Lemma 6.1.12 there exists an up to permutation of monomials unique representation $f\left(x_{1}, \ldots, x_{k}\right)=\bigoplus_{i=1}^{\ell} \wedge W_{i}$, where $\ell$ is odd and the sets $W_{1}, \ldots, W_{l} \subseteq\left\{x_{1}, \ldots, x_{n}\right\}$ are different. Notice that, for every $i \geq 2$, the operation $s_{i}$ only depends on the set of variables occurring in it, i.e., the order of the variables and their multiplicity can be ignored. Thus, we write $s_{\left|W_{i}\right|}\left(W_{i}\right)$ to stress this fact; moreover, we set $s_{1}(\{x\}):=x$, for every $x \in\left\{x_{1}, \ldots, x_{k}\right\}$. We define the map $\xi: \mathcal{I}_{2} \rightarrow \mathcal{S}$ as follows

$$
\xi:\left(\bigoplus_{i=1}^{\ell} \bigwedge W_{i}\right) \mapsto m_{\ell}\left(s_{\left|W_{1}\right|}\left(W_{1}\right), \ldots, s_{\left|W_{\ell}\right|}\left(W_{\ell}\right)\right) .
$$

Since $m_{\ell}$ is symmetric, the map $\xi$ is well defined.
Note that, both the operation $\oplus$ and $m_{\ell}$ only depend on the parity of the elements occurring among their arguments.

Let $\pi:\{1, \ldots, k\} \rightarrow\{1, \ldots, r\}$ be a map. By first applying $\xi$ we obtain

$$
m_{\ell}\left(s_{\left|W_{1}\right|}\left(W_{1}\right), \ldots, s_{\left|W_{\ell}\right|}\left(W_{\ell}\right)\right)
$$

and then, via $\pi$, we obtain

$$
m_{\ell}\left(s_{\left|W_{1}^{\pi}\right|}\left(W_{1}^{\pi}\right), \ldots, s_{\left|W_{\ell}^{\pi}\right|}\left(W_{\ell}^{\pi}\right)\right) \text {, where } W_{i}^{\pi}:=\left\{x_{\pi(j)} \mid x_{j} \in W_{i}\right\} .
$$

Let $\left\{U_{1}, \ldots, U_{t}\right\}$ be the set of all different subsets in $\left\{W_{1}^{\pi}, \ldots, W_{\ell}^{\pi}\right\}$. Without loss of generality we assume that $U_{i}$ appears an odd number of times in $W_{1}^{\pi}, \ldots, W_{\ell}^{\pi}$ for $i \in\{1,2 \ldots, d\}$ and $U_{i}$ appears an even number of times in $W_{1}^{\pi}, \ldots, W_{\ell}^{\pi}$ for $i \in\{d+$ $1, d+2 \ldots, t\}$. Then using properties of $m_{\ell}$ we have

$$
m_{\ell}\left(s_{\left|W_{1}^{\pi}\right|}\left(W_{1}^{\pi}\right), \ldots, s_{\left|W_{\ell}^{\pi}\right|}\left(W_{\ell}^{\pi}\right)\right)=m_{d}\left(s_{\left|U_{1}\right|}\left(U_{1}\right), \ldots, s_{\left|U_{d}\right|}\left(U_{d}\right)\right) .
$$

On the other side, if we first apply $\pi$, we get $\oplus_{i=1}^{\ell} \wedge W_{i}^{\pi}=\bigoplus_{i=1}^{d} \wedge U_{i}$. Since all the monomials in $\bigoplus_{i=1}^{d} \wedge U_{i}$ are different, $\xi$ applied to it gives us $m_{d}\left(s_{\left|U_{1}\right|}\left(U_{1}\right), \ldots, s_{\left|U_{d}\right|}\left(U_{d}\right)\right)$,
which is exactly what we need.

Corollary 6.1.14. Let $\mathcal{S}$ be an idempotent clone over $E_{3}$ such that $\mathcal{S} \npreceq_{\mathrm{m}} \mathcal{C}_{2}, \mathcal{S} \not \varliminf_{\mathrm{m}} \mathcal{C}_{3}$, and $\mathcal{S} \npreceq_{\mathrm{m}} \mathcal{B}_{2}$. There is a minor homomorphism from $\mathcal{I}_{2}$ to $\mathcal{S}$.

Proof. From Theorem 6.1.10 we know that $\mathcal{S}$ has an oddition $m_{\ell}$, for every odd number $\ell \geq 3$. Moreover, from Theorem 6.1.11 it follows that $\mathcal{S}$ has a totally symmetric operation $s_{n}$ of arity $n$, for every $n \geq 2$. Thus, the claim follow from Theorem 6.1.13.

### 6.2 Clones of self-dual operations

In this section we provide a complete description of the subposet $\downarrow \overline{\mathcal{C}_{3}}$ of $\mathfrak{P}_{3}$ that consists of all subclones of $\mathcal{C}_{3}:=\operatorname{Pol}(\{(0,1),(1,2),(2,0)\})$ factored by minor-equivalence and ordered by the existence of minor-preserving maps, i.e., we describe the subposet

$$
\downarrow \overline{\mathcal{C}_{3}}:=\left\{\overline{\mathcal{A}} \mid \overline{\mathcal{A}} \preceq_{\mathrm{m}} \overline{\mathcal{C}_{3}}\right\} .
$$

Moreover, it follows from our description that $\downarrow \overline{\mathcal{C}_{3}}$ is a lattice and that it is countably infinite (Corollary 6.2.12). Note that every element of $\downarrow \overline{\mathcal{C}_{3}}$ is the $\equiv_{\mathrm{m}}$-class of a suitable clone of operations that preserve the relation $\{(0,1),(1,2),(2,0)\}$. We call such operations self-dual. For this reason, we will often refer to $\downarrow \overline{\mathcal{C}_{3}}$ as the lattice of clones of self-dual operations up to minor-equivalence. The results presented in this section come from a joint work of Bodirsky, Zhuk, and the author of the dissertation [29], an article with which this section has therefore strong analogies.

### 6.2.1 A continuum of clones up to homomorphic equivalence

The following relations are defined on the set $\{0,1,2\}$.

$$
\begin{align*}
C_{3} & :=\{(0,1),(1,2),(2,0)\}  \tag{6.1}\\
R_{3}^{=} & :=\{(x, y, z) \mid x \in\{0,1\} \wedge((x=0) \Rightarrow(y=z))\}  \tag{6.2}\\
B_{2} & =\{(0,1),(1,0),(1,1)\} \tag{6.3}
\end{align*}
$$

Marchenkov [78] proved that there are $2^{\omega}$ many distinct operation clones between

$$
\mathcal{W}:=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, R_{3}^{\overline{=}}\right) \quad \text { and } \quad \mathcal{B}_{2}^{3}:=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, B_{2}\right) ;
$$

(also see [104] and Figure 6.1). Our terminology is a simplified version of the terminology used in [104]. We prove that there are $2^{\omega}$ of these clones even when considered up to $\equiv_{\mathrm{h}}$-equivalence (Corollary 6.2.4).

Remark 6.2.1. Note that

- We freely use the terms $x+31$, and $x+32$ in primitive positive definitions over structures that contain the relation $C_{3}$, since we can express $y=x+31$ as $C_{3}(x, y)$ and $y=x+32$ as $C_{3}(y, x)$.
- $\mathcal{B}_{2}^{3} \subseteq \mathcal{W}$, because $B_{2}$ is pp-definable in the structure $\left(\{0,1,2\} ; C_{3}, R_{3}^{=}\right)$by the formula

$$
\exists u, v\left(R_{3}^{=}(x, y, u) \wedge R_{3}^{=}(y, x, v) \wedge C_{3}(u, v)\right)
$$



Figure 6.1: The lattice of clones of self-dual operations ordered by inclusion.

We obtain Corollary 6.2.4 as a consequence of Theorem 6.2.3 in the proof of which we use the fact that $\mathcal{W}$ contains the operation $\vee_{3}$ which is defined as follow:

$$
\vee_{3}(x, y):= \begin{cases}x & \text { if } x=y  \tag{6.4}\\ 1 & \text { if }\{x, y\}=\{0,1\} \\ 2 & \text { if }\{x, y\}=\{1,2\} \\ 0 & \text { if }\{x, y\}=\{0,2\}\end{cases}
$$

The operation $\vee_{3}$ is also known as the rock-paper-scissors operation; indeed by identifying 0 with rock, 1 with paper, and 2 with scissors, we get that $\vee_{3}(x, y)$ is simply the winner between $x$ and $y$ in the world-famous game (assuming that both players win in the event of a draw). We also need the following lemma which states that if a subdirect relation has a pp-definition in $\left(\{0,1,2\} ; C_{3}, R_{3}^{=}\right)$, then we can also find a pp-definition without existential quantifiers and without using $R_{3}^{=}$.

Lemma 6.2.2 ([104], Lemma 17). Suppose that $R \subseteq\{0,1,2\}^{n}$ is a subdirect relation preserved by $\vee_{3}$. Then $R$ can be defined by a conjunction of atomic formulae over $\mathbb{C}_{3}$.

Theorem 6.2.3. Let $\mathbb{A}$ and $\mathbb{B}$ be structures such that $\mathcal{W} \subseteq \operatorname{Pol}(\mathbb{A}), \operatorname{Pol}(\mathbb{B}) \subseteq \mathcal{B}_{2}^{3}$. If $\operatorname{Pol}(\mathbb{B}) \preceq_{\mathrm{h}} \operatorname{Pol}(\mathbb{A})$, then $\operatorname{Pol}(\mathbb{B}) \subseteq \operatorname{Pol}(\mathbb{A})$.

Proof. By Theorem 2.5.9, the structure $\mathbb{A}$ has a $d$-dimensional pp-interpretation $\Gamma$ in $\mathbb{B}$, for some $d \geq 1$; choose $\Gamma$ such that $d$ is smallest possible. Let $S:=\Gamma^{-1}\left(C_{3}\right) \subseteq B^{2 d}$. Let $I \subseteq\{1, \ldots, 2 d\}$ be the set of all $i$ such that $\operatorname{pr}_{i}(S)=\{0,1,2\}$. Note that $S^{\prime}:=\operatorname{pr}_{I}(S)$ is subdirect. By Lemma 6.2 .2 the relation $S^{\prime}$ can be defined by a conjunction of atomic formulae over $\left(\{0,1,2\} ; C_{3}\right)$. We distinguish two cases.

Case 1. $S^{\prime}=\{0,1,2\}^{|I|}$ (Note that this includes the case that $I=\emptyset$ ). We will prove that this case is impossible. Choose $t=\left(t_{1}, \ldots, t_{2 d}\right) \in S$ such that $\operatorname{pr}_{I}(t)=(0, \ldots, 0)$. We know that $\Gamma(t) \in C_{3}$. Then $\Gamma(t)=\left(u, u+{ }_{3} 1\right)$ for some $u \in\{0,1,2\}$ such that $\Gamma\left(\left(t_{1}, \ldots, t_{d}\right)\right)=u$ and $\Gamma\left(\left(t_{d+1}, \ldots, t_{2 d}\right)\right)=u+_{1} 1$. Note that

$$
\begin{aligned}
t^{\prime} & :=\left(t_{d+1}, \ldots, t_{2 d}, s_{1}, \ldots, s_{d}\right) \in \Gamma^{-1}\left(\left(u+{ }_{3} 1, u+{ }_{3} 2\right)\right) \subseteq S \\
\text { and } t^{\prime \prime} & :=\left(s_{1}, \ldots, s_{d}, t_{1}, \ldots, t_{d}\right) \in \Gamma^{-1}\left(\left(u+{ }_{3} 2, u\right)\right) \subseteq S
\end{aligned}
$$

Then for every $i \in\{1, \ldots, 2 d\}$ we have $\left|\left\{\operatorname{pr}_{i}(t), \operatorname{pr}_{i}\left(t^{\prime}\right), \operatorname{pr}_{i}\left(t^{\prime \prime}\right)\right\}\right|<3$ :

- if $i \notin I$, then this holds by the definition of $I$;
- if $i \in I$, then the choice of $t$ implies that $\operatorname{pr}_{i}(t)=t_{i}=0$. Also note that $i+d \in I$, too, since $\operatorname{pr}_{i}(S)=\operatorname{pr}_{i}\left(\Gamma^{-1}(\{0,1,2\})\right)=\operatorname{pr}_{i+d}(S)$, as $C_{3}$ is a subdirect relation on $\{0,1,2\}$. Hence, $\operatorname{pr}_{i}\left(t^{\prime}\right)=t_{i+d}=0$.

Let $r:=\vee_{3}\left(\vee_{3}\left(t, t^{\prime}\right), t^{\prime \prime}\right)$. We claim that $\Gamma(r)$ is of the form $(a, a)$ for some $a \in\{0,1,2\}$, which is a contradiction since $r \in S$ and therefore $\Gamma(r) \in C_{3}$. To see this, note that the operation $\vee_{3}$ is associative and commutative when restricted to sets of size two, hence

$$
\vee_{3}\left(\vee_{3}\left(\left(t_{1}, \ldots, t_{d}\right),\left(t_{d+1}, \ldots, t_{2 d}\right)\right), s\right)=\vee_{3}\left(\vee_{3}\left(\left(t_{d+1}, \ldots, t_{2 d}\right), s\right),\left(t_{1}, \ldots, t_{d}\right)\right)
$$

Here the left hand side is the projection of $r$ to its first $d$ coordinates, while the right hand side is the projection of $r$ to its last $d$ coordinates. Therefore, $\Gamma(r)$ has the form ( $a, a$ ) for some $a \in\{0,1,2\}$, as claimed.

Case 2. $S^{\prime} \neq\{0,1,2\}^{|I|}$. Then there are distinct $i, j \in I$ such that $\operatorname{pr}_{\{i, j\}}(S)$ is the relation $C_{3}$ or $=$ on $\{0,1,2\}$. If $i, j \in\{1, \ldots, d\}$ or if $i, j \in\{d+1, \ldots, 2 d\}$, then we obtain a contradiction to the assumption that the interpretation $\Gamma$ has a smallest possible dimension. If for example $1,2 \in I$ are such that $\operatorname{pr}_{\{1,2\}}(S)$ equals $=$, then $\Gamma^{\prime}\left(a_{1}, \ldots, a_{d-1}\right):=\Gamma\left(a_{1}, a_{1}, a_{2}, \ldots, a_{d-1}\right)$ is a $(d-1)$-dimensional interpretation of $\mathbb{A}$ in $\mathbb{B}$. If $\operatorname{pr}_{\{1,2\}}(S)$ equals $C_{3}$, then $\Gamma^{\prime}\left(a_{1}, \ldots, a_{d-1}\right):=\Gamma\left(a_{1}, a_{1}+{ }_{3} 1, a_{2}, \ldots, a_{d-1}\right)$ is a $(d-1)$-dimensional interpretation of $\mathbb{A}$ in $\mathbb{B}$.

First consider the case that $i \in\{1, \ldots, d\}$ and that $j \in\{d+1, \ldots, 2 d\}$; the case that $j \in\{1, \ldots, d\}$ and that $i \in\{d+1, \ldots, 2 d\}$ can be treated similarly. We claim that for every $a \in\{0,1,2\}$ we have $\left|\operatorname{pr}_{i}\left(\Gamma^{-1}(a)\right)\right|=1$. To see this, let $c=\left(c_{1}, \ldots, c_{d}\right), c^{\prime}=$ $\left(c_{1}^{\prime}, \ldots, c_{d}^{\prime}\right)$ be elements of $B^{d}$ be such that $\Gamma(c)=\Gamma\left(c^{\prime}\right)=a$. Choose $e \in B^{d}$ such that $\Gamma(e)=a+3$ 1. Since $\left(a, a+{ }_{3} 1\right) \in C_{3}$, the tuples $(c, e)=\left(c_{1}, \ldots, c_{d}, e_{1}, \ldots, e_{d}\right)$ and $\left(c^{\prime}, e\right)=\left(c_{1}^{\prime}, \ldots, c_{d}^{\prime}, e_{1}, \ldots, e_{d}\right)$ both belong to $\Gamma^{-1}\left(C_{3}\right)=S$. Hence, $\left(c_{i}, e_{j}\right),\left(c_{i}^{\prime}, e_{j}\right) \in$ $\operatorname{pr}_{\{i, j\}}(S)$, which implies that $c_{i}=c_{i}^{\prime}$. We write $f(a)$ for the element of $\operatorname{pr}_{i}\left(\Gamma^{-1}(a)\right.$; note that $f$ is a permutation of $\{0,1,2\}$.

Since $\Gamma$ is an interpretation of $\mathbb{A}$ in $\mathbb{B}$, we know that for every relation $R$ of $\mathbb{A}$ the relation $\Gamma^{-1}(R)$ over $B$ is pp-definable in $\mathbb{B}$. The same holds for every relation $R$ that is pp-definable in $\mathbb{A}$ : if $\phi$ is the pp-definition of $R$ in $\mathbb{A}$, then we may obtain a pp-definition of $\Gamma^{-1}(R)$ by replacing each atomic formula in $\phi$ by its defining formula in $\mathbb{B}$; the resulting formula can be rewritten into a pp-formula over the signature of $\mathbb{B}$ by moving the existential quantifiers to the front. In particular,
$(*)$ for every relation $R$ that is pp-definable in $\mathbb{A}$ the relation $f(R)=\operatorname{pr}_{i}\left(\Gamma^{-1}(R)\right)$ is pp-definable in $\mathbb{B}$.

The relation $B_{2}$ is pp-definable in $\mathbb{A}$, because we assumed that $\operatorname{Pol}(\mathbb{A}) \subseteq \mathcal{B}_{2}$. Therefore, $f\left(B_{2}\right)$ is pp-definable in $\mathbb{B}$, by $(*)$. Our assumption that $\mathcal{W} \subseteq \operatorname{Pol}(\mathbb{B})$ together with $\vee_{3} \in \mathcal{W}$ implies that $\vee_{3}$ preserves every relation is pp-definable in $\mathbb{B}$. In particular, $\vee_{3}$ preserves $f\left(B_{2}\right)$. Since $\vee_{3}$ fails to preserve $f\left(B_{2}\right)$ if the permutation $f$ is a transposition, we conclude that $f$ is either a cyclic permutation or the identity permutation. In both cases, the graph of $f$ is pp-definable in $\mathbb{B}$. It follows that for every relation $R$ of $\mathbb{A}$, the relation $R=f^{-1}(f(R))$ is pp-definable in $\mathbb{B}$, because $f(R)$ is pp-definable in $\mathbb{B}$, by $(*)$. This proves that $\mathbb{A}$ is pp-definable in $\mathbb{B}$, and hence $\operatorname{Pol}(\mathbb{B}) \subseteq \operatorname{Pol}(\mathbb{A})$.

Corollary 6.2.4. There are $2^{\omega}$ many clones of self-dual operations with respect to homomorphic equivalence.

### 6.2.2 The decrease of cardinality up to minor-equivalence

We now describe clones of self-dual operations with respect to $\preceq_{\mathrm{m}}$. The lattice of such clones with respect to inclusion is drawn in Figure 6.1 all the clones that appear in this picture will be defined progressively in the text when we present results concerning them. Clones flagged with the same colour in Figure 6.1 are minor-equivalent. In

Proposition 6.2.5 we prove that in order to get a complete description of the subposet $\downarrow \overline{\mathcal{C}_{3}}:=\left\{\overline{\mathcal{A}} \mid \overline{\mathcal{A}} \preceq_{\mathrm{m}} \overline{\mathcal{C}_{3}}\right\}$ is indeed enough to provide a complete description of the poset of clones of self-dual operations factored by $\equiv_{\mathrm{m}}$.

Note that, a priori, it could happen that there is a clone $\mathcal{C}$ on $E_{3}$ which is not a subclone of $\mathcal{C}_{3}$ and however there exists a minor-preserving map from $\mathcal{C}$ to $\mathcal{C}_{3}$; Proposition 6.2.5 shows that this is not the case. Note that we are only interested in Taylor clones, since we already know from Theorem 3.2 .18 and Proposition 3.2 .11 that if $\mathcal{A}$ is a not a Taylor clone, then $\mathcal{A} \equiv{ }_{\mathrm{m}} \mathcal{P}_{3}$.

Proposition 6.2.5. Let $\mathcal{A}$ be a Taylor clone on $\{0,1,2\}$. If $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{C}_{3}$, then $\mathcal{A} \subseteq \mathcal{C}_{3}$.
Proof. Suppose $\mathcal{A}$ is a Taylor clone on $\{0,1,2\}$ such that $\mathcal{A} \nsubseteq \mathcal{C}_{3}$ and $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{C}_{3}$. By Proposition 3.2 .25 we can assume without loss of generality that $\mathcal{A}$ is a minimal Taylor clone. From $\mathcal{A} \preceq_{\mathrm{m}} \mathcal{C}_{3}$ and Theorem 3.3 .11 we have that $\mathcal{A} \not \vDash \Sigma_{3}$. If follows that $\mathcal{A}$ must be isomorphic to one of the clones listed in Proposition 3.2 .27 and therefore $\mathcal{A} \subseteq \mathcal{C}_{3}$, a contradiction.

Let $\mathbb{Q}:=\left(\{0,1,2\} ; C_{3}, R_{2}^{=}\right)$and $\mathbb{R}:=\left(\{0,1,2\} ; C_{3}, R_{2}^{\overrightarrow{2}}\right)$, where

$$
\begin{aligned}
R_{2}^{=} & :=\{(x, y, z) \mid x \in\{0,1\} \wedge x=0 \Rightarrow y=z \in\{0,1\}\} \\
R_{2}^{\Rightarrow} & :=\{(x, y, z) \mid x, y \in\{0,1\} \wedge x=y=0 \Rightarrow z=0\}
\end{aligned}
$$

Zhuk proved that the interval between the clones $\mathcal{Q}=\operatorname{Pol}(\mathbb{Q})$ and $\mathcal{R}:=\operatorname{Pol}(\mathbb{R})$ is a countably infinite chain of clones [104]. Theorem 6.2.7 below implies that the entire chain from $\mathcal{Q}$ to $\mathcal{R}$ collapses in our poset: they are all minor-equivalent (Corollary 6.2.8).
Remark 6.2.6. Note that for every $n \geq 2$ the relation $B_{n}:=\{0,1\}^{n} \backslash\{(0, \ldots, 0)\}$ has the following pp-definition in $\mathbb{R}$ :

$$
\begin{aligned}
B_{n}\left(x_{1}, \ldots, x_{n}\right) \Leftrightarrow \exists u_{1}, \ldots, u_{n-1} & \left(R_{2}^{\Rightarrow}\left(x_{1}, x_{2}, u_{1}\right) \wedge C_{3}\left(u_{n-1}, x_{1}\right)\right. \\
& \left.\wedge \bigwedge_{i \in\{2, \ldots, n-1\}} R_{2}^{\Rightarrow}\left(u_{i-1}, x_{i+1}, u_{i}\right)\right)
\end{aligned}
$$

Let $\leq 2$ be the relation defined in Example 2.2.2. Note that $x \leq_{2} y$ if and only if $R_{2}^{\Rightarrow}(y, y, x) \wedge R_{2}^{\vec{~}}(x, x, x)$ and hence $\leq_{2}$ is pp-definable in $\mathbb{R}$.
Theorem 6.2.7. The structure $\mathbb{R} p p$-constructs the structure $\mathbb{Q}$.
Proof. We first define a fourth pp-power $\mathbb{R}^{\prime}$ of $\mathbb{R}$ with domain $R^{\prime}:=\{0,1,2\}^{4}$, and then show that there exists a homomorphism $h: \mathbb{R}^{\prime} \rightarrow \mathbb{Q}$ and a homomorphism $g: \mathbb{Q} \rightarrow \mathbb{R}^{\prime}$. Our intuition for defining $\mathbb{R}^{\prime}$ will be guided by the choice of $g$ :

$$
\begin{aligned}
g(0) & :=(0,1,0,0) \\
g(1) & :=(1,0,1,0) \\
g(2) & :=(2,0,0,1)
\end{aligned}
$$

The following relations are primitively positively definable over $\mathbb{R}$ :

$$
\begin{aligned}
& C_{3}^{\mathbb{R}^{\prime}}:=\left\{(x, y) \in\left(R^{\prime}\right)^{2} \mid C_{3}\left(x_{0}, y_{0}\right) \wedge x_{1}=y_{2} \wedge x_{2}=y_{3} \wedge x_{3}=y_{1}\right\} \\
&\left(R_{2}^{=}\right)^{\mathbb{R}^{\prime}}:=\left\{(x, y, z) \in\left(R^{\prime}\right)^{3} \mid B_{3}\left(x_{1}, x_{2}, x_{3}\right) \wedge B_{3}\left(y_{1}, y_{2}, y_{3}\right) \wedge B_{3}\left(z_{1}, z_{2}, z_{3}\right)\right. \\
& \wedge x_{3}=0 \wedge x_{0}=x_{2} \wedge y_{3} \leq_{2} x_{0} \wedge z_{3} \leq_{2} x_{0} \\
&\left.\wedge R_{2}^{\vec{F}}\left(x_{0}, y_{2}, z_{2}\right) \wedge R_{2}^{\overrightarrow{2}}\left(x_{0}, z_{2}, y_{2}\right)\right\}
\end{aligned}
$$

Claim. $g$ is a homomorphism from $\mathbb{Q}$ to $\mathbb{R}^{\prime}$.

- Let that $(a, b) \in C_{3}$. Then $\left(g(a)_{0}, g(b)_{0}\right)=(a, b) \in C_{3}$. Moreover, $g(a)_{1}=g(b)_{2}$, $g(a)_{2}=g(b)_{3}$, and $g(a)_{3}=g(b)_{1}$. Hence $(g(a), g(b)) \in C_{3}^{\mathbb{R}^{\prime}}$.
- Let that $(a, b, c) \in R_{2}^{=}$. The definition of $g$ implies that the first three conjuncts of the definition of $\left(R_{2}^{=}\right)^{\mathbb{R}^{\prime}}$ are satisfied by $g(a), g(b), g(c)$. Moreover, $a \in\{0,1\}$ and hence $g(a)_{3}=0$.
Suppose that $a=0$. We have either $b=c=0$ or $b=c=1$ by the definition of $R_{2}^{=}$. Then $g(a)_{0}=g(a)_{2}=0$ and $0=g(b)_{3}=g(c)_{3} \leq_{2} g(a)_{0}=0$. Moreover, the last two conjuncts in the definition of $\left(R_{2}^{\overline{=}}\right)^{\mathbb{R}^{\prime}}$ hold: if $b=c=0$ then the conclusion in the implication $x=y=0 \Rightarrow z=0$ from the definition of $R_{2}^{\Rightarrow}$ is satisfied in each of the two conjuncts, and if $b=c=1$ then the premise in the implication $x=y=0 \Rightarrow z=0$ is not satisfied in each of the two conjuncts; moreover, for each conjunct the first two arguments of $R_{2}^{\overrightarrow{2}}$ are from $\{0,1\}$.
Finally, suppose that $a=1$. In this case $b$ and $c$ may take any value in $\{0,1,2\}$. Note that $g(a)_{0}=g(a)_{2}=1$ and $g(b)_{3}, g(c)_{3} \leq_{2} 1$. Since $g(a)_{0}=1$ the last two conjuncts in the definition of $\left(R_{2}\right)^{\mathbb{R}^{\prime}}$ hold again, because the premise in the implication of the definition of $R_{2}^{=}$is not fulfilled and because the first argument $x_{0}$ of these conjuncts equals 1 . This shows that $(g(a), g(b), g(c)) \in\left(R_{2}^{=}\right)^{\mathbb{R}^{\prime}}$.
Define $h: R^{\prime} \rightarrow\{0,1,2\}$ as follows.

$$
h\left(x_{0}, x_{1}, x_{2}, x_{3}\right):= \begin{cases}0 & \text { if }\left(x_{1}, x_{2}, x_{3}\right) \in\{(1,0,0),(1,0,1)\} \\ 1 & \text { if }\left(x_{1}, x_{2}, x_{3}\right) \in\{(0,1,0),(1,1,0)\} \\ 2 & \text { if }\left(x_{1}, x_{2}, x_{3}\right) \in\{(0,0,1),(0,1,1)\} \\ x_{0} & \text { otherwise }\end{cases}
$$

Claim. $h$ is a homomorphism from $\mathbb{R}^{\prime}$ to $\mathbb{Q}$.

- Let $(a, b) \in C_{3}^{\mathbb{R}^{\prime}}$. From the definition of $C_{3}^{\mathbb{R}^{\prime}}$ it follows that for a fixed $a$ there is a unique $b$ such that $(a, b) \in C_{3}^{\mathbb{R}^{\prime}}$. It is easy to check that if $a$ is such that $\left(a_{1}, a_{2}, a_{3}\right) \in$ NAE $:=\{0,1\}^{3} \backslash\{(0,0,0),(1,1,1)\}$, then $b \in$ NAE and $h$ is defined such that $(h(a), h(b)) \in C_{3}$. Otherwise, we have $(h(a), h(b))=\left(a_{0}, b_{0}\right)$ and the first conjunct in the definition of $C_{3}^{\mathbb{R}^{\prime}}$ implies that $\left(a_{0}, b_{0}\right) \in C_{3}$.
- Let $(a, b, c) \in\left(R_{2}^{=}\right)^{\mathbb{R}^{\prime}}$. Then $a_{3}=0$ and $B_{3}\left(a_{1}, a_{2}, a_{3}\right)$ implies that $a_{1}=1$ or $a_{2}=1$. Moreover, $a_{0}=a_{2}$ implies that $a \in\{(0,1,0,0),(1,1,1,0),(1,0,1,0)\}$ and thus $h(a) \in\{0,1\}$.
If $h(a)=1$ then $R_{2}^{=}(h(a), h(b), h(c))$ holds trivially. If $h(a)=0$ then $\left(a_{0}, a_{1}, a_{2}, a_{3}\right)=$ $(0,1,0,0)$ by the definition of $h$ and the observations above. Hence, $b_{3}=c_{3}=0$ since $b_{3}, c_{3} \leq_{2} a_{0}=0$. This implies that $h(b), h(c) \in\{0,1\}$. From $R_{2}^{\overrightarrow{2}}\left(0, b_{2}, c_{2}\right)$ and $R_{2}^{\overrightarrow{2}}\left(0, c_{2}, b_{2}\right)$ it follows that $u:=b_{2}=c_{2}$. Now we distinguish two cases: if $u=0$, we have $B_{3}\left(b_{1}, 0,0\right)$ and $B_{3}\left(c_{1}, 0,0\right)$, thus $b_{1}=c_{1}=1$. Therefore, $h(b)=h(c)=0$. If $u=1$, then it follows from the definition of $h$ that $h\left(b_{0}, b_{1}, 1,0\right)=h\left(c_{0}, c_{1}, 1,0\right)=1$. This concludes the proof.

Corollary 6.2.8. $\mathcal{R} \equiv_{\mathrm{m}} \mathcal{Q}$.
Proof. It is an immediate consequence of Theorem 6.2.7 and Theorem 3.1.10 that $\mathcal{P} \preceq_{\mathrm{m}} \mathcal{Q}$. Conversely, $\mathcal{Q} \preceq_{\mathrm{m}} \mathcal{P}$ follows from the fact that $\mathcal{Q} \subseteq \mathcal{P}$.

Below we define the clones $\mathcal{B}_{n} \pi_{\infty}$ for every $n \in\{3, \ldots, \infty\}$ and $\mathcal{M}_{n}$ for every $n \in\{2,3,4, \ldots, \infty\}$. There are $2^{\omega}$ many clones between $\mathcal{B}_{\infty} \pi_{\infty}$ and $\mathcal{M}_{\infty}$. In this section we prove that the clones $\mathcal{B}_{\infty} \pi_{\infty}$ and $\mathcal{M}_{\infty}$ (and therefore all the $2^{\omega}$ many clones between them) are minor-equivalent. We obtain this result as a direct consequence of Theorem 6.2.11. The proof is similar to the proof of the minor-equivalence of $\mathcal{Q}$ and $\mathcal{R}$ presented in Theorem 6.2.7

Definition 6.2.9. Define, for every $k \geq 2$,

$$
\begin{aligned}
\mathbb{B}_{k}^{3} & :=\left(\{0,1,2\} ; C_{3}, B_{k}\right) & \mathcal{B}_{k}^{3} & :=\operatorname{Pol}\left(\mathbb{B}_{k}^{3}\right) \\
\mathbb{B}_{\infty}^{3} & :=\left(\{0,1,2\} ; C_{3}, B_{2}, B_{3}, \ldots\right) & \mathcal{B}_{\infty}^{3} & :=\operatorname{Pol}\left(\mathbb{B}_{\infty}^{3}\right) \\
\mathbb{M}_{k} & :=\left(\{0,1,2\} ; C_{3}, \leq_{2}, B_{k}\right) & \mathcal{M}_{k} & :=\operatorname{Pol}\left(\mathbb{M}_{k}\right) \\
\mathbb{M}_{\infty} & :=\left(\{0,1,2\} ; C_{3}, \leq_{2}, B_{2}, B_{3}, \ldots\right) & \mathcal{M}_{\infty} & :=\operatorname{Pol}\left(\mathbb{M}_{\infty}\right) .
\end{aligned}
$$

Note that this definition is compatible with the definition of $\mathcal{B}_{2}^{3}$ that was already defined earlier.

To define $\mathcal{B}_{n} \pi_{\infty}$ we need to introduce new relations on $\{0,1,2\}$. Let

$$
W:=\left(\begin{array}{lllll}
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 2
\end{array}\right) \subseteq\{0,1,2\}^{2} .
$$

Note that $W(x, y)$ holds if $x \in\{0,1\}$ and $x=0$ implies $y \in\{0,1\}$.
For $m, k \in \mathbb{N}$ and $S_{1} \cup \cdots \cup S_{m}=\{1, \ldots, k\}$, the relation $R_{S_{1}, \ldots, S_{m}}$ consists of all tuples $\left(x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{k}\right) \in\{0,1,2\}^{m+k}$ such that
(1) $x_{1}, \ldots, x_{m} \in\{0,1\}$,
(2) for every $i \in\{1, \ldots, m\}$, if $x_{i}=0$ then $y_{j} \in\{0,1\}$ for every $j \in A_{i}$, and
(3) not $x_{1}=\cdots=x_{m}=y_{1}=\cdots=y_{n}=0$.

Definition 6.2.10. We define $\mathbb{B}_{\infty} \pi_{\infty}$ to be the structure on $\{0,1,2\}$ with the relations $C_{3}, W$, and the relation $R_{S_{1}, \ldots, S_{m}}$ for every $m, k \in \mathbb{N}$ and all $S_{1}, \ldots, S_{m} \subseteq\{1, \ldots, k\}$ such that $S_{1} \cup \cdots \cup S_{m}=\{1, \ldots, k\}$. For $n \in\{3, \ldots, \infty\}$, let $\mathbb{B}_{n} \pi_{\infty}$ be the reduct of $\mathbb{B}_{\infty} \pi_{\infty}$ that contains all relations of $\mathbb{B}_{\infty} \pi_{\infty}$ of arity at most $n$. As usual, $\mathcal{B}_{n} \pi_{\infty}:=\operatorname{Pol}\left(\mathbb{B}_{n} \pi_{\infty}\right)$.

It is known that $\mathcal{B}_{n} \pi_{\infty} \subseteq \mathcal{M}_{n}$ for every $n \in\{3,4, \ldots, \infty\}$; in particular $\mathcal{M}_{n}$ contains the generator operation of $\mathcal{B}_{n} \pi_{\infty}$ (see [104, Theorem 29 and Theorem 30). It immediately follows that, for every $n \in\{3,4, \ldots, \infty\}$, there exists a minor-preserving map from $\mathcal{B}_{n} \pi_{\infty}$ to $\mathcal{M}_{n}$, therefore $\mathbb{B}_{n} \pi_{\infty}$ pp-constructs $\mathbb{M}_{n}$. Now we prove the other direction.
Theorem 6.2.11. For every $n \in\{3,4, \ldots, \infty\}$, the structure $\mathbb{M}_{n}$ pp-constructs $\mathbb{B}_{n} \pi_{\infty}$.
Proof. As in the proof of Theorem 6.2.7, we use a fourth pp-power $\mathbb{M}_{n}^{\prime}$ of $\mathbb{M}_{n}$, this time with the signature of $\mathbb{B}_{n} \pi_{\infty}$. The relation $C_{3}^{\mathbb{M}_{n}^{\prime}}$ is defined as in the proof of Theorem 6.2.7.

Let $k \in \mathbb{N}$ and $S_{1}, \ldots, S_{m} \subseteq\{1, \ldots, k\}$ be such that $S_{1} \cup \cdots \cup S_{m}=\{1, \ldots, k\}$ and $m+k \leq n$. Then the following relations are primitively positively definable over $\mathbb{M}_{n}$.

$$
\begin{aligned}
W^{\mathbb{M}_{n}^{\prime}}:= & \left\{(x, y) \in\left(\{0,1,2\}^{4}\right)^{2} \mid B_{3}\left(x_{1}, x_{2}, x_{3}\right) \wedge B_{3}\left(y_{1}, y_{2}, y_{3}\right)\right. \\
& \left.\wedge x_{3}=0 \wedge x_{0}=x_{2} \wedge y_{3} \leq_{2} x_{0}\right\} \\
R_{S_{1}, \ldots, S_{m}}^{\mathbb{M}_{n}^{\prime}}:= & \left\{\left(x^{1}, \ldots, x^{m}, y^{1}, \ldots, y^{k}\right) \mid B_{m+k}\left(x_{2}^{1}, \ldots, x_{2}^{m}, y_{2}^{1}, \ldots, y_{2}^{k}\right)\right. \\
\wedge & \bigwedge_{i \in\{1, \ldots, m\}}\left(B_{3}\left(x_{1}^{i}, x_{2}^{i}, x_{3}^{i}\right) \wedge x_{3}^{i}=0 \wedge x_{0}^{i}=x_{2}^{i}\right. \\
& \left.\left.\wedge \bigwedge_{j \in S_{i}}\left(y_{3}^{j} \leq_{2} x_{0}^{i}\right) \wedge B_{3}\left(y_{1}^{j}, y_{2}^{j}, y_{3}^{3}\right)\right)\right\}
\end{aligned}
$$

Let $g:\{0,1,2\} \rightarrow\{0,1,2\}^{4}$ and $h:\{0,1,2\}^{4} \rightarrow\{0,1,2\}$ be defined as in the proof of Theorem 6.2.7
Claim. $g$ is a homomorphism from $\mathbb{B}_{n} \pi_{\infty}$ to $\mathbb{M}_{n}^{\prime}$. We have already verified in the proof of Theorem 6.2.7 that $g$ preserves $C_{3}$.

- Let $(a, b) \in W$. By the definition of $g$ it holds that $B_{3}\left(g(a)_{1}, g(a)_{2}, g(a)_{3}\right)$ and $B_{3}\left(g(b)_{1}, g(b)_{2}, g(b)_{3}\right)$. If $a \neq 0$, then $a=1$ and

$$
g(b)_{3} \leq_{2} g(a)_{0}=g(a)_{2}=1
$$

since $g(b)_{3} \in\{0,1\}$ by the definition of $g$. If $a=0$ then $b \in\{0,1\}$. Also, $g(a)_{3}=0$ and $g(b)_{3}=0 \leq_{2} g(a)_{0}=g(a)_{2}=0$.

- Let $\left(a^{1}, \ldots, a^{m}, b^{1}, \ldots, b^{k}\right) \in R_{S_{1}, \ldots, S_{m}}$. Then there exists $i \in\{1, \ldots, m\}$ such that $a^{i}=1$ or $j \in\{1, \ldots, k\}$ such that $b^{j}=1$. If $a^{i}=1$ then $g\left(a^{i}\right)=(1,0,1,0)$. If $b^{j}=1$ then $g\left(b^{j}\right)=(1,0,1,0)$. In both cases we have

$$
B_{m+k}\left(g\left(a^{1}\right)_{2}, \ldots, g\left(a^{m}\right)_{2}, g\left(b^{1}\right)_{2}, \ldots, g\left(b^{k}\right)_{2}\right)
$$

To verify the other conjuncts in the definition of $R_{S_{1}, \ldots, S_{m}}^{\mathbb{M}_{n}^{\prime}}$, let $i \in\{1, \ldots, m\}$. Clearly, $B_{3}\left(g\left(a^{i}\right)_{1}, g\left(a^{i}\right)_{2}, g\left(a^{i}\right)_{3}\right)$. Since $a^{i} \in\{0,1\}$ it follows that $g\left(a^{i}\right)_{3}=0$ and $g\left(a^{i}\right)_{0}=g\left(a^{i}\right)_{2}$. Let $j \in A_{i}$. Clearly, $B_{3}\left(g\left(b^{j}\right)_{1}, g\left(b^{j}\right)_{2}, g\left(b^{j}\right)_{3}\right)$. If $a^{i}=1$ then $g\left(b^{j}\right)_{3} \leq_{2} g\left(a^{i}\right)_{0}=1$ because $g\left(b^{j}\right)_{3} \in\{0,1\}$. If $a^{i}=0$, then $g\left(a^{i}\right)=$ $(0,1,0,0)$. We have $g\left(b^{j}\right)_{3}=0 \leq_{2} 0=g\left(a^{i}\right)_{0}$. We can therefore conclude that $\left(g\left(a^{1}\right), \ldots, g\left(a^{m}\right), g\left(b^{1}\right), \ldots, g\left(b^{k}\right)\right) \in R_{S_{1}, \ldots, S_{m}}^{\mathbb{N}_{n}^{\prime}}$.

Claim. $h$ is a homomorphism from $\mathbb{M}_{n}^{\prime}$ to $\mathbb{B}_{n} \pi_{\infty}$. We have already verified in the proof of Theorem 6.2.7 that $h$ preserves $C_{3}$.

- Let $(a, b) \in W^{\mathbb{M}_{n}^{\prime}}$. Since $a_{3}=0$ and $B_{3}\left(a_{1}, a_{2}, a_{3}\right)$ we have that $a_{0}=a_{2} \in\{0,1\}$. If $a_{0}=1$ then $h(a)=1$, and $(h(a), h(b)) \in W$. If $a_{0}=0$, then $b_{3}=0$ since $b_{3} \leq_{2} a_{0}$. Then $B_{3}\left(b_{1}, b_{2}, b_{3}\right)$ implies that $\left(b_{1}, b_{2}, b_{3}\right) \in\{(1,0,0),(0,1,0),(1,1,0)\}$ and therefore $h(b) \in\{0,1\}$ and $(h(a), h(b)) \in W$.
- Let $\left(a^{1}, \ldots, a^{m}, b^{1}, \ldots, b^{k}\right) \in R_{S_{1}, \ldots, S_{m}}^{\mathbb{M}_{n}^{\prime}}$. We have to show that

$$
\left(h\left(a^{1}\right), \ldots, h\left(a^{m}\right), h\left(b^{1}\right), \ldots, h\left(b^{k}\right)\right)
$$

satisfies the items (1), (2), and (3) in the definition of $R_{S_{1}, \ldots, S_{m}}$.
For every $i \in\{1, \ldots, m\}$ we have that $B_{3}\left(a_{1}^{i}, a_{2}^{i}, a_{3}^{i}\right)$ and $a_{3}^{i}=0$ which implies that $\left(a_{1}^{i}, a_{2}^{i}, a_{3}^{i}\right) \in\{(1,0,0),(0,1,0),(1,1,0)\}$ and thus $h\left(a^{i}\right) \in\{0,1\}$, showing (1).
Let $j \in S_{i}$. If $h\left(a^{i}\right)=0$ then $b_{3}^{j} \leq_{2} a_{0}^{i}=a_{2}^{i}=0$ implies that $b_{0}^{j} \in\{0,1\}$. Since $B_{3}\left(b_{1}^{j}, b_{2}^{j}, b_{3}^{j}\right)$ we have $h\left(b^{j}\right) \in\{0,1\}$, showing (2).
To prove (3), suppose for contradiction that

$$
\begin{equation*}
h\left(a^{1}\right)=\cdots=h\left(a^{m}\right)=h\left(b^{1}\right)=\cdots=h\left(b^{k}\right)=0 . \tag{6.5}
\end{equation*}
$$

For every $i \in\{1, \ldots, m\}$ we have $a_{3}^{i}=0, B_{3}\left(a_{1}^{i}, a_{2}^{i}, a_{3}^{i}\right)$, and $a_{0}^{i}=a_{2}^{i}$, and hence $a^{i}=(0,1,0,0)$. Since $b_{3}^{j} \leq_{2} a_{0}^{i}$ we obtain $b_{3}^{j}=0$ for all $j \in\{1, \ldots, k\}$. Since we assumed (6.5), it follows that $b^{j} \in\{(0,0,0,0),(0,1,0,0)\}$ for every $j \in\{1, \ldots, k\}$; in both cases we obtain a contradiction since $B_{m+k}\left(a_{2}^{1}, \ldots, a_{2}^{m}, b_{2}^{1}, \ldots, b_{2}^{k}\right)$ must hold. This shows (3), and thus $\left(a^{1}, \ldots, a^{n}, b^{1}, \ldots, b^{m}\right) \in R_{S_{1}, \ldots, S_{m}}$.

Corollary 6.2.12. The minor-equivalence relation has only countably many equivalence classes of clones of self-dual operations.

### 6.2.3 Other collapses

We now show other collapses with the goal of completing the classification of the lattice of clones of self-dual operations. We start from two rather simple examples of collapse: one is an immediate consequence of Lemma 3.1.7 the other follows from a precise notion of duality.

## The idempotent reduct

Recall that every finite core structure $\mathbb{A} p p$-constructs the expansion of $\mathbb{A}$ by all constant unary relations, see Lemma 3.1.4. Note that every expansion of $\left(\{0,1,2\} ; C_{3}\right)$ is a core. Therefore, when working with a structure $\mathbb{A}$, we will often tacitly work instead with the expansion of $\mathbb{A}$ by the unary relations, and allow the constants 0,1 , and 2 in primitive positive formulae. We define the following clones:

$$
\begin{array}{ll}
\mathcal{L}_{3}:=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, L_{3}\right) & \mathcal{L}_{3}^{0}:=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, L_{3},\{0\}\right) \\
\mathcal{P}_{3}:=\operatorname{Pol}\left(\{0,1,2\} ; R_{\neq},\{0\},\{1\},\{2\}\right) & \mathcal{K}_{3}^{3}:=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, R_{\neq}\right) \\
\mathcal{C}_{3}^{0}:=\operatorname{Pol}\left(\{0,1,2\} ; C_{3},\{0\}\right) . &
\end{array}
$$

where

$$
\begin{align*}
R_{\neq} & :=\left\{(x, y) \in\{0,1,2\}^{2} \mid x \neq y\right\}, \text { and }  \tag{6.6}\\
L_{3} & :=\left\{\left(x_{1}, x_{2}, x_{3}\right) \mid x_{1}+3 x_{2}+3 x_{3}=0\right\} . \tag{6.7}
\end{align*}
$$

Proposition 6.2.13. The following clones are minor-equivalent: $\mathcal{C}_{3} \equiv{ }_{\mathrm{m}} \mathcal{C}_{3}^{0} ; \mathcal{L}_{3} \equiv{ }_{\mathrm{m}} \mathcal{L}_{3}^{0}$; and $\mathcal{K}_{3}^{3} \equiv{ }_{\mathrm{m}} \mathcal{P}_{3}$.

Proof. The claims follow immediately from Lemma 3.1.7.

## The conservative reduct

Here we prove that clones $\mathcal{C}_{3}, \mathcal{C}_{3}^{0}$, which we already defined, are minor-equivalent to $\mathcal{C}_{3}^{0,1}:=\operatorname{Pol}\left(\mathbb{C}_{3}^{0,1}\right)$, where $\mathbb{C}_{3}^{0,1}:=\left(\{0,1,2\} ; C_{3},\{0,1\}\right)$. Note that $\mathcal{C}_{3}^{0,1}$ is the clone of all conservative (see Section 2.1.3) self-dual operations on $\{0,1,2\}$.

Proposition 6.2.14. The structure $\mathbb{C}_{3}$ pp-constructs $\mathbb{C}_{3}^{0,1}$.
Proof. Note that by Lemma 3.1.4 we can use the constants 0,1 and 2 in our pp-formulae. Analogously to the proof of Theorem 6.2.7, we define a fourth pp-power $\mathbb{S}$ of $\mathbb{C}_{3}$. More precisely: we consider $\mathbb{S}:=\left(\{0,1,2\}^{4} ; C_{3}^{\mathbb{S}}, R_{\{0,1\}}^{\mathbb{S}}\right)$ where $C_{3}^{\mathbb{S}}$ is defined by the same primitive positive formula that defines $C_{3}^{\mathbb{R}^{\prime}}$ in Theorem 6.2 .7 and $R_{\{0,1\}}^{\mathbb{S}}$ is defined as follows:

$$
R_{\{0,1\}}^{\mathbb{S}}:=\left\{\left(x_{0}, x_{1}, x_{2}, x_{3}\right) \in\{0,1,2\}^{4} \mid\left(x_{0}=x_{1}\right) \wedge\left(x_{3}=1\right)\right\} .
$$

We define $g: \mathbb{C}_{3}^{0,1} \rightarrow \mathbb{S}$ as follows:

$$
g(0):=(0,0,1,1) ; \quad g(1):=(1,1,0,1) ; \quad g(2):=(2,1,1,0)
$$

It is immediate to check that $g$ is a homomorphism.

Define $h:\{0,1,2\}^{4} \rightarrow\{0,1,2\}$ as follows:

$$
h\left(x_{0}, x_{1}, x_{2}, x_{3}\right):= \begin{cases}x_{0}-1 & \text { if }\left(x_{1}, x_{2}, x_{3}\right) \in A_{1}:=\{(1,1,2),(1,2,1),(2,1,1)\} \\ x_{0} & \text { if }\left(x_{1}, x_{2}, x_{3}\right) \in A_{2}:=\{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\} \\ x_{0}+1 & \text { if }\left(x_{1}, x_{2}, x_{3}\right) \in A_{3}:=\{0,1,2\}^{3} \backslash\left(A_{1} \cup A_{2}\right)\end{cases}
$$

Claim. $h$ is a homomorphism from $\mathbb{S}$ to $\mathbb{C}_{3}^{0,1}$.

- Let $(a, b) \in C_{3}^{\mathbb{S}}$. From the definition of $h$ it follows that if $\left(a_{1}, a_{2}, a_{3}\right) \in A_{i}$, for some $i \in\{1,2,3\}$, then $\left(b_{1}, b_{2}, b_{3}\right) \in A_{i}$. Therefore,

$$
\left(h\left(a_{0}, a_{1}, a_{2}, a_{3}\right), h\left(b_{0}, b_{1}, b_{2}, b_{3}\right)\right) \in\left\{\left(a_{0}-1, b_{0}-1\right),\left(a_{0}, b_{0}\right),\left(a_{0}+1, b_{0}+1\right)\right\} .
$$

Since $\left(a_{0}, b_{0}\right) \in C_{3}$, it follows that $\left(h\left(a_{0}, a_{1}, a_{2}, a_{3}\right), h\left(b_{0}, b_{1}, b_{2}, b_{3}\right)\right) \in C_{3}$.

- Let $\left(a_{0}, a_{1}, a_{2}, a_{3}\right) \in R_{\{0,1\}}$. It follows that $a_{0}=a_{1}$ and $a_{3}=1$. We distinguish three cases. If $a_{0}=0$, we have:

$$
h\left(0,0, a_{2}, 1\right):= \begin{cases}1 & \text { if } a_{2}=0 \\ 0 & \text { if } a_{2}=1 \\ 1 & \text { if } a_{2}=2\end{cases}
$$

If $a_{0}=1$ :

$$
h\left(1,1, a_{2}, 1\right):= \begin{cases}1 & \text { if } a_{2}=0 \\ 1 & \text { if } a_{2}=1 \\ 0 & \text { if } a_{2}=2\end{cases}
$$

If $a_{0}=2$ :

$$
h\left(2,2, a_{2}, 1\right):= \begin{cases}0 & \text { if } a_{2}=0 \\ 1 & \text { if } a_{2}=1 \\ 0 & \text { if } a_{2}=2\end{cases}
$$

Therefore, we can conclude that $h\left(a_{0}, a_{1}, a_{2}, a_{3}\right) \in\{0,1\}$.
Corollary 6.2.15. The clones $\mathcal{C}_{3}$ and $\mathcal{C}_{3}^{0,1}$ are minor-equivalent.
Proof. Since $\mathcal{C}_{3}^{0,1} \subset \mathcal{C}_{3}$, there exists a minor-preserving map from $\mathcal{C}_{3}^{0,1}$ to $\mathcal{C}_{3}$. From Proposition 6.2.14 and Theorem 3.1.10 it follows that there exists a minor-preserving map from $\mathcal{C}_{3}$ to $\mathcal{C}_{3}^{0,1}$. Therefore, $\mathcal{C}_{3} \equiv_{\mathrm{m}} \mathcal{C}_{3}^{0,1}$.

## Duals

In this chapter we consider two different dualities: with respect to a cyclic permutation and with respect to a transposition.

Formally, for a permutation $\pi$ of $\{0,1,2\}$ we define $f^{\pi}$ by

$$
f^{(\pi)}\left(x_{1}, \ldots, x_{n}\right):=\pi\left(f\left(\pi^{-1}\left(x_{1}\right), \ldots, \pi^{-1}\left(x_{n}\right)\right)\right),
$$

and say that $f^{(\pi)}$ is dual to $f$ with respect to $\pi$. As it follows from the definition, if $\pi$ is a cyclic permutation, then $f=f^{\pi}$ if and only if $f \in \operatorname{Pol}\left(\{0,1,2\} ; C_{3}\right)$, that is why we call such operations self-dual.

Sometimes we will need to consider duality with respect to the transposition

$$
\sigma:\{0,1,2\} \rightarrow\{0,1,2\} \text { defined by } \sigma(0,1,2)=(1,0,2) .
$$

In the following we write $f^{*}$ instead of $f^{(\sigma)}$ and denote $\mathcal{C}^{*}=\left\{f^{*} \mid f \in \mathcal{C}\right\}$. We call $f^{*}$ dual to $f$ with respect to the transposition. For example, $\wedge_{3}$ is dual with respect to the transposition to the rock-paper-scissors operation $\vee_{3}$ whose composition table can be found below, and we write $\left(\wedge_{3}\right)^{*}=\vee_{3}$.

| $\vee_{3}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 2 |
| 2 | 0 | 2 | 2 |


| $\wedge_{3}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 2 |
| 1 | 0 | 1 | 1 |
| 2 | 2 | 1 | 2 |

Note that $f$ preserves $R$ if and only if $f^{*}$ preserves

$$
R^{*}:=\left\{\left(\sigma\left(x_{1}\right), \ldots, \sigma\left(x_{n}\right)\right) \mid\left(x_{1}, \ldots, x_{n}\right) \in R\right\} .
$$

Proposition 6.2.16. Let $\mathcal{C}$ be a clone of self-dual operations. Then $\mathcal{C} \equiv_{\mathrm{m}} \mathcal{C}^{*}$.

Proof. The map $\xi: \mathcal{C} \rightarrow \mathcal{C}^{*}$, given by $\xi: f \mapsto f^{*}$ is a minor-preserving map.

The clones such that $\mathcal{C}^{*}=\mathcal{C}$ are drawn in the middle of the diagram. We refer to the middle part as the spine; all clones that are not in the spine are either in the left wing or in the mirror symmetric right wing; the symmetry is given by taking the dual with respect to the transposition and clearly visible in Figure 6.1 o avoid confusion we either write "self-dual", meaning the duality with respect to a cyclic permutation, or "dual with respect to the transposition".

## A hinge between the spine and the wings

Here we study the minor-equivalence class of $\mathcal{M}_{2}=\operatorname{Pol}\left(\mathbb{M}_{2}\right)$. Let $N$ be the relation

$$
N:=\left(\begin{array}{llll}
0 & 1 & 1 & 1  \tag{6.8}\\
0 & 0 & 1 & 2
\end{array}\right) \subseteq\{0,1,2\}^{2}
$$

Note that $N(x, y)$ holds if $x \in\{0,1\}$ and $x=0$ implies $y=0$. Define

$$
\begin{aligned}
\mathbb{N}_{2} & :=\left(\{0,1,2\} ; C_{3}, N, B_{2}\right), \text { and } \\
\mathbb{M} & :=\left(\{0,1,2\} ; C_{3}, \leq_{2}\right) .
\end{aligned}
$$

We will show that there is a pp-construction of $\mathbb{N}_{2}$ in $\mathbb{M}$. Note that $N(y, x) \wedge N(x, x)$ is equivalent to $x \leq_{2} y$, so $\mathbb{M}$ also has a pp-construction in $\mathbb{N}_{2}$ (even a pp-definition). It will follow that $\mathcal{M}:=\operatorname{Pol}(\mathbb{M})$ and $\mathcal{N}_{2}:=\operatorname{Pol}\left(\mathbb{N}_{2}\right)$ are minor-equivalent. Note that $\mathbb{M}^{*}=\mathbb{M}$, so $\mathbb{M}$ is already part of the spine.

Instead of directly specifying the pp-construction of $\mathbb{M}$ in $\mathbb{N}_{2}$, we found it more convenient to prove this in two steps: first we show that $\mathbb{N}_{2}$ has a pp-construction in $\mathbb{M}_{2}$, and then we show that $\mathbb{M}_{2}$ has a pp-construction in $\mathbb{M}$.

Lemma 6.2.17. There is a pp-construction of $\mathbb{N}_{2}$ in $\mathbb{M}_{2}$.
Proof. We again use a four-dimensional pp-power $\mathbb{A}$ of $\mathbb{M}_{2}$ and the maps $h$ and $g$ from the proof of Theorem6.2.7. The structure $\mathbb{A}$ has domain $A:=\{0,1,2\}^{4}$ and signature $\left\{C_{3}, N, B_{2}\right\}$. The relation $C_{3}^{\mathbb{A}}$ is defined as in the proof of Theorem 6.2.7. and

$$
\begin{aligned}
& N^{\mathbb{A}}:=\left\{(x, y) \in A^{2} \mid x_{3}=0 \wedge x_{0}=x_{2} \wedge B_{2}\left(x_{1}, x_{2}\right)\right. \\
&\left.\wedge x_{1} \leq_{2} y_{1} \wedge y_{2} \leq_{2} x_{2} \wedge y_{3} \leq_{2} x_{0}\right\} \\
& B_{2}^{\mathbb{A}}:=\left\{(x, y) \in A^{2} \mid B_{2}\left(x_{0}, y_{0}\right) \wedge x_{0}=x_{2} \wedge y_{0}=y_{2} \wedge x_{3}=y_{3}=0\right\}
\end{aligned}
$$

Claim. The map $g$ is a homomorphism from $\mathbb{N}_{2}$ to $\mathbb{A}$.
We have already verified in the proof of Theorem 6.2.7 that $g$ preserves $C_{3}$.

- Let $(a, b) \in N$. If $a=0$, then $b=0$, and hence $\left(x_{0}, x_{1}, x_{2}, x_{3}\right):=g(a)=(0,1,0,0)$ and $\left(y_{0}, y_{1}, y_{2}, y_{3}\right):=g(b)=(0,1,0,0)$ satisfies the formula from the definition of $N^{\mathbb{A}}$. If $a=1$, then putting $\left(x_{0}, x_{1}, x_{2}, x_{3}\right):=g(a)=(1,0,1,0)$ satisfies $x_{3}=0, x_{0}=x_{2}$, $B_{2}\left(x_{1}, x_{2}\right)$. Moreover, putting $\left(x_{0}, x_{1}, x_{2}, x_{3}\right):=g(b)$ all the remaining conjuncts in the definition of $N^{\mathbb{A}}$ are satisfied as well: for $x_{1} \leq_{2} y_{1}$ since $x_{1}=0$, and for $y_{2} \leq_{2} x_{2}$ and $y_{3} \leq_{2} x_{0}$ since $y_{2}, y_{3} \in\{0,1\}$ and $x_{0}=x_{2}=1$.
- Let $(a, b) \in B_{2}$. Then $a, b \in\{0,1\}$, and hence $(g(a), g(b))$ satisfies $x_{0}=x_{2}, y_{0}=y_{2}$, and $x_{3}=y_{3}=0$. Moreover, $B_{2}\left(x_{0}, y_{0}\right)$ holds because otherwise $a=b=0$, contrary to the assumption that $B_{2}(a, b)$.

Claim. The map $h$ is a homomorphism from $\mathbb{A}$ to $\mathbb{N}_{2}$.
We have already verified in the proof of Theorem 6.2.7 that $h$ preserves $C_{3}$. Let us assume that that $(x, y) \in N^{\mathbb{A}}$. We must have $x_{3}=0$ and $x_{0}=x_{2}$ therefore $h(x) \in$ $\{0,1\}$. Since $x_{3}=0$ and $B_{2}\left(x_{1}, x_{2}\right)$ it is impossible that $x_{1}=x_{2}=x_{3}$ and therefore $x \in\{(0,1,0,0),(1,0,1,0),(1,1,1,0)\}$. If $x \in\{(1,0,1,0),(1,1,1,0)\}$ then $h(x)=1$ and $(h(x), h(y)) \in N$. If $x=(0,1,0,0)$ then $1=x_{1} \leq_{2} y_{1}, y_{2} \leq_{2} x_{2}=0$, and $y_{3} \leq_{2} x_{0}=0$, and hence $h(y)=0$. Thus, $(h(x), h(y))=(0,0) \in N$.

Let $(x, y) \in B_{2}^{\mathbb{A}}$. Again the conjuncts $x_{3}=0$ and $x_{0}=x_{2}$ imply that $h(x) \in\{0,1\}$, and similarly $h(y) \in\{0,1\}$. If $h(x)=0$ then $x=(0,0,0,0)$. In this case the conjunct $B_{2}\left(x_{0}, y_{0}\right)$ implies that $y_{0} \neq 0$, and hence $h(y) \neq 0$. So $(h(x), h(y)) \in B_{2}$.

Lemma 6.2.18. The structure $\mathbb{M} p$ p-constructs $\mathbb{M}_{2}$.
Proof. Let $\mathbb{A}$ be the structure with domain $A:=\{0,1,2\}^{2}$ and signature $\left\{C_{3}, \leq_{2}, B_{2}\right\}$ such that

$$
\begin{aligned}
& C_{3}^{\mathbb{A}}:=\left\{(x, y) \in A^{2} \mid C_{3}\left(x_{1}, y_{1}\right) \wedge C_{3}\left(y_{2}, x_{2}\right)\right\}, \\
& \leq_{2}^{\mathbb{A}}:=\left\{(x, y) \in A^{2} \mid x_{1} \leq_{2} y_{1} \wedge y_{2} \leq_{2} x_{2}\right\}, \\
& B_{2}^{\mathbb{A}}:=\left\{(x, y) \in A^{2} \mid x_{2} \leq_{2} y_{1} \wedge x_{1} \leq_{2} x_{1} \wedge y_{2} \leq_{2} y_{2}\right\} .
\end{aligned}
$$

Let $g:\{0,1,2\} \rightarrow\{0,1,2\}^{2}$ be defined as follows: $g(0):=(0,1), g(1):=(1,0)$, and $g(2):=(2,2)$.

Claim 1: $g$ is a homomorphism from $\mathbb{M}_{2}$ to $\mathbb{A}$. Let $(a, b) \in C_{3}$. If $(a, b)=(0,1)$ then $(g(a), g(b))=((0,1),(1,0)) \in C_{3}^{\mathbb{A}}$. If $(a, b)=(1,2)$ then $(g(a), g(b))=((1,0),(2,2)) \in C_{3}^{\mathbb{A}}$. If $(a, b)=(2,0)$ then $(g(a), g(b))=((2,2),(0,1)) \in C_{3}^{\mathbb{A}}$.

Now suppose that $a \leq_{2} b$. If $a=b=0$ then

$$
(g(a), g(b))=((0,1),(0,1)) \in \leq_{2}^{\mathbb{A}}
$$

and similarly for $a=b=1$. If $a=0$ and $b=1$ then $(g(a), g(b))=((0,1),(1,0)) \in \leq_{2}^{\mathbb{A}}$.
Finally, suppose that $(a, b) \in B_{2}$. Then $a, b \in\{0,1\}$ and hence the entries of $h(a)$ and $h(b)$ are from $\{0,1\}$ as well. Thus, the final two conjuncts in the definition of $B_{2}^{\mathbb{A}}$ are always satisfied. If $a=b=1$ then $(g(a), g(b))=((1,0),(1,0)) \in B_{2}^{\mathbb{A}}$ since $0 \leq_{2} 1$. If $a=1$ and $b=0$ then $(g(a), g(b))=((1,0),(0,1)) \in B_{2}^{\mathbb{A}}$ since $0 \leq_{2} 0$. And if $a=0$ and $b=1$ then $(g(a), g(b))=((0,1),(1,0)) \in B_{2}^{\mathbb{A}}$ since $1 \leq_{2} 1$.

Let $h:\{0,1,2\}^{2} \rightarrow\{0,1,2\}$ be defined by

$$
\begin{aligned}
& h(0,1)=h(0,2)=h(2,1)=0, \\
& h(0,0)=h(1,0)=h(1,1)=1, \\
& h(2,0)=h(1,2)=h(2,2)=2 .
\end{aligned}
$$

Note that $h(a, b)=a$ for every $a, b \in\{0,1,2\}$ with three exceptions: $h(2,1)=0$, $h(0,0)=1$, and $h(1,2)=2$; we call $(2,1),(1,2),(0,0)$ the exceptional points and all other points regular.

Claim 2: $h$ is a homomorphism from $\mathbb{A}$ to $\mathbb{M}_{2}$. Let $(a, b) \in C_{3}^{\mathbb{A}}$. Then $C_{3}\left(a_{1}, b_{1}\right)$ and $C_{3}\left(b_{2}, a_{2}\right)$. In particular, note that $a$ equals $(2,1)$ if and only if $b$ equals $(0,0)$. In this case, $(h(a), h(b))=(0,1) \in C_{3}$ and we are done. Similarly, we verify the statement if $a$ equals $(1,2)$ or $(0,0)$. Also note that $a$ is an exceptional point for $h$ if and only if $b$ is. If $a$ and $b$ are regular then $(h(a), h(b))=\left(a_{1}, b_{1}\right) \in C_{3}$ and we are done.

Now let $(a, b) \in \leq_{2}^{\mathbb{A}}$. Then by definition we have $a_{1} \leq_{2} b_{1}$ and $b_{2} \leq_{2} a_{2}$. Note that in particular, neither $a$ nor $b$ can be the exceptional points $(1,2)$ or $(2,1)$ for $g$. If $a=(0,0)$ then $b_{1}, b_{2} \in\{0,1\}$, hence $h(b) \in\{0,1\}$, and thus $h(b) \in\{0,1\}$ and $h(b) \leq_{2} h(a)=1$. For the regular points the verification is again immediate.

Finally, let $(a, b) \in B_{2}^{\mathbb{A}}$. Note that then $a_{1}, a_{2}, b_{1}, b_{2} \in\{0,1\}$, hence $h(a), h(b) \in\{0,1\}$. Suppose that $h(b)=0$. By the definition of $B_{2}^{\mathbb{A}}$, we have $a_{2} \leq_{2} b_{1}=0$, so we have $h(a)=h(0,0)=1$. This shows that $(h(a), h(b)) \in B_{2}$.

## The Spine

We now discuss collapses in the spine; again, some of them can be proved by exhibiting pp-constructions, while in one case it was more convenient to directly exhibit a minorpreserving map (in Proposition 6.2.21). The following structures are at the bottom of the spine in Figure $6.1, \mathbb{T}_{\leq}, \mathbb{L}_{\leq}$, and $\mathbb{K}_{3}^{c}$; we show that they are equivalent with respect to pp-constructability. Recall that by $\mathbb{K}_{3}^{c}$ we denote the rigid core of the complete graph over $\{0,1,2\}$, i.e., $\mathbb{K}_{3}^{c}:=\left(\{0,1,2\} ; R_{\neq},\{0\},\{1\},\{2\}\right)$ (see Section 3.2.3). Following [104] we define

$$
\begin{aligned}
\mathbb{K}_{3}^{3} & :=\left(\{0,1,2\} ; C_{3}, R_{\neq}\right) \\
\mathbb{T L}_{\leq} & :=\left(\{0,1,2\} ; C_{3}, T, L_{2}, \leq_{2}\right) \\
\mathbb{L}_{\leq} & :=\left(\{0,1,2\} ; C_{3}, L_{2}, \leq_{2}\right),
\end{aligned}
$$

where

$$
\begin{align*}
L_{2} & :=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in\{0,1\}^{3} \mid x_{1}+{ }_{2} x_{2}+{ }_{2} x_{3}=0\right\}  \tag{6.9}\\
\text { and } \quad T & :=\{(0,1),(1,0),(2,2)\} . \tag{6.10}
\end{align*}
$$

Lemma 6.2.19. The structures $\mathbb{K}_{3}^{c}, \mathbb{K}_{3}^{3}, \mathbb{T}_{\leq}$, and $\mathbb{L}_{\leq}$pairwise pp-construct each other.
Proof. Every finite structure has a pp-interpretation in $\left(\{0,1,2\} ; R_{\neq}\right)$(see Theorem 3.2.7). The structure $\mathbb{T} \mathbb{L}_{\leq}$is an expansion of $\mathbb{L}_{\leq}$. Finally, observe that $\mathbb{L}_{\leq}$interprets primitively positively the structure

$$
\left(\{0,1\} ; L_{2}, \leq_{2},\{0\},\{1\}\right)
$$

and that all polymorphisms of this structure are projections [91]. Hence, $\mathbb{L}_{\leq}$interprets
all finite structures (see, e.g., [24], Theorem 6.3.10). Finally, note that $\mathcal{P}_{3}=\operatorname{Pol}\left(\mathbb{K}_{3}^{c}\right)$ and $\mathcal{K}_{3}^{3}=\operatorname{Pol}\left(\mathbb{K}_{3}^{3}\right) ;$ from Proposition 6.2 .13 if follows that $\mathcal{K}_{3}^{3} \equiv_{\mathrm{m}} \mathcal{P}_{3}$.

In the lattice of clones of self-dual operations (Figure 6.1) the clone $\mathcal{L}_{\leq}:=\operatorname{Pol}\left(\mathbb{L}_{\leq}\right)$is covered by

$$
\mathcal{L}_{2}:=\operatorname{Pol}\left(\mathbb{L}_{2}\right):=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, L_{2}\right)
$$

and the clone $\mathcal{T} \mathcal{L}_{\leq}:=\operatorname{Pol}\left(\mathbb{T L}_{\leq}\right)$is covered by

$$
\mathcal{T} \mathcal{L}_{2}:=\operatorname{Pol}\left(\mathbb{T L}_{2}\right):=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, T, L_{2}\right)
$$

Proposition 6.2.20. The structures $\mathbb{T L}_{2}$ and $\mathbb{L}_{2}$ pp-construct each other.
Proof. Since $\mathbb{T L}_{2}$ is an expansion of $\mathbb{L}_{2}$ it suffices to prove that $\mathbb{L}_{2}$ pp-constructs $\mathbb{T}_{2}$. We consider the pp-power $\mathbb{A}$ of $\mathbb{L}_{2}$ with domain $\{0,1,2\}^{2}$ and the same signature as $\mathbb{T}_{2}$ whose relations are defined as follows.

$$
\begin{aligned}
& C_{3}^{\mathbb{A}}:=\left\{(x, y) \in A^{2} \mid C_{3}\left(x_{1}, y_{1}\right) \wedge C_{3}\left(y_{2}, x_{2}\right)\right\} \\
& T^{\mathbb{A}}:=\left\{(x, y) \in A^{2} \mid x_{1}=y_{2} \wedge x_{2}=y_{1}\right\} \\
& L_{2}^{\mathbb{A}}:=\left\{(x, y, z) \in A^{3} \mid L_{2}\left(x_{1}, y_{1}, z_{1}\right) \wedge L_{2}\left(x_{1}, x_{2}, 1\right) \wedge L_{2}\left(y_{1}, y_{2}, 1\right) \wedge L_{2}\left(z_{1}, z_{2}, 1\right)\right\}
\end{aligned}
$$

We prove that $\mathbb{T L}_{2}$ and $\mathbb{A}$ are homomorphically equivalent. Let $g:\{0,1,2\} \rightarrow\{0,1,2\}^{2}$ be defined by $g(0):=(0,1), g(1):=(1,0)$, and $g(2):=(2,2)$. Then $g$ is a homomorphism from $\mathbb{T L}_{2}$ to $\mathbb{A}$ : the proof that $g$ preserves $C_{3}$ we have already seen this in the proof of Lemma 6.2.18. Now suppose that $(x, y) \in T$. If $(x, y)=(0,1)$ then $(g(0), g(1))=((0,1),(1,0)) \in T^{\mathbb{A}}$. For $(x, y) \in\{(1,0),(2,2)\}$ the argument is similarly straightforward. Finally, suppose that $(x, y, z) \in L$. Then $x, y, z \in\{0,1\}^{2}$, and hence $g(x), g(y), g(z) \in\{(0,1),(1,0)\}$. Hence, the conjuncts $L_{2}\left(x_{1}, x_{2}, 1\right), L_{2}\left(y_{1}, y_{2}, 1\right)$, $L_{2}\left(z_{1}, z_{2}, 1\right)$ in the definition of $L_{2}^{\mathbb{A}}$ are satisfied. Moreover, $x+2 y+2 z=0$ implies that $g(x)_{1}+{ }_{2} g(y)_{1}+{ }_{2} g(z)_{1}=0$, and hence $(g(x), g(y), g(z)) \in L_{2}^{\mathbb{A}}$.

Let $h:\{0,1,2\}^{2} \rightarrow\{0,1,2\}$ be defined as

$$
h(x, y):= \begin{cases}0 & \text { if } C_{3}(x, y) \\ 1 & \text { if } C_{3}(y, x) \\ 2 & \text { if } x=y\end{cases}
$$

We prove that $h$ is a homomorphism from $\mathbb{A}$ to $\mathbb{T}_{2}$. Let $\left(\left(x_{1}, x_{2}\right),\left(y_{1}, y_{2}\right)\right) \in C_{3}^{\mathbb{A}}$. Then $C_{3}\left(x_{1}, y_{1}\right)$ and $C_{3}\left(y_{2}, x_{2}\right)$. If $h\left(x_{1}, x_{2}\right)=0$, then $C_{3}\left(x_{1}, x_{2}\right)$, and hence $C_{3}\left(y_{2}, y_{1}\right)$, and therefore $h\left(y_{1}, y_{2}\right)=1$. Hence, $\left(h\left(x_{1}, x_{2}\right), h\left(y_{1}, y_{2}\right)\right) \in C_{3}$. The verification if $h\left(x_{1}, x_{2}\right) \in\{1,2\}$ is similarly straightforward. The proof that $h$ preserves $T$ is similar as well. Finally, suppose that

$$
\left(\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right),\left(a_{3}, b_{3}\right)\right) \in L_{2}^{\mathbb{A}}
$$

Then $L\left(a_{i}, b_{i}, 1\right)$, for $i \in\{1,2,3\}$, implies that $\left(a_{i}, b_{i}\right) \in\{(0,1),(1,0)\}$, and hence $h\left(a_{i}, b_{i}\right) \in\{0,1\}$. Note that in this case $h\left(a_{i}, b_{i}\right)=a_{i}$, and hence $L_{2}\left(a_{1}, a_{2}, a_{3}\right)$ implies that $\left(h\left(a_{1}, b_{1}\right), h\left(a_{2}, b_{2}\right), h\left(a_{3}, b_{3}\right)\right) \in L$.

Define the relational structures

$$
\begin{aligned}
\mathbb{T N} & :=\left(\{0,1,2\} ; C_{3}, T, N, N^{*}\right), \\
\mathbb{D M} & :=\left(\{0,1,2\} ; C_{3}, C_{2}, \leq_{2}\right)
\end{aligned}
$$

where $C_{2}:=\{(0,1),(1,0)\}$ and $N^{*}=\{(0,0),(0,1),(1,1),(0,2)\}$ is the dual of the relation $N$ defined in (6.8). Zhuk (104, Theorem 28) proved that

$$
\mathcal{T \mathcal { N }}:=\operatorname{Pol}(\mathbb{T N})=\langle m\rangle \quad \text { and } \quad \mathcal{D M}:=\operatorname{Pol}(\mathbb{D M})=\langle m, p\rangle
$$

where the operations $m$ and $p$ are defined as follows:

$$
\begin{align*}
m(x, y, z) & := \begin{cases}\left(x \wedge_{3} y\right) \vee_{3}\left(x \wedge_{3} z\right) \vee_{3}\left(y \wedge_{3} z\right) & \text { if }|\{x, y, z\}| \leq 2 \\
x & \text { otherwise }\end{cases}  \tag{6.11}\\
p(x, y, z) & := \begin{cases}x & \text { if }|\{x, y, z\}| \leq 2 \\
x+31 & \text { otherwise } .\end{cases} \tag{6.12}
\end{align*}
$$

Rather than proving that $\mathbb{T N}$ and $\mathbb{D} \mathbb{M}$ pp-construct each other, we follow a different strategy and directly work with the respective clones and minor preserving maps.
Proposition 6.2.21. The clones $\mathcal{T N}$ and $\mathcal{D M}$ are minor-equivalent.
Proof. Since $\mathcal{D M}$ contains $\mathcal{T} \mathcal{N}$, it suffices to find a minor-preserving map $\Xi$ from $\mathcal{D M}$ to $\mathcal{T} \mathcal{N}$. We first define a minor-preserving map $\xi$ over $\mathcal{D} \mathcal{M}^{(3)}$, i.e., the set of all operations of arity at most three in $\mathcal{D M}$ (see Claim 1). Then we extend $\xi$ to a minor-preserving map $\Xi$ from $\mathcal{D M}$ to the clone of all operations on $\{0,1,2\}$, and finally we show that the image of $\Xi$ lies in $\mathcal{T N}$ (see Claim 2 and Claim 3).

Note that every binary operation of $\mathcal{D M}$ must be a projection: this follows by induction over the generation process from the fact that $\mathcal{D M}=[m, p]$ is generated by $m$ and $p$, since any operation obtained from the ternary operations $m$ and $p$ by identifying arguments is a projection. Moreover, every ternary operation of $\mathcal{D M}$ restricted to $\{0,1\}$ must be either a projection or the ternary majority operation; again, this is easy to show by induction over the generation process.

For $i \in\{1,2\}$, we define $\xi\left(\operatorname{pr}_{i}^{2}\right):=\operatorname{pr}_{i}^{2}$. Let $f \in \mathcal{D} \mathcal{M}^{(3)}$. Note that every operation in $\mathcal{D M}$ preserves $\{0,1\}$; if the restriction of $f$ to $\{0,1\}$ is a projection to the $i$-th argument, then we define $\xi(f):=\operatorname{pr}_{i}^{3}$. Suppose now that the restriction of $f$ to $\{0,1\}$ is a majority operation. Note that for $b_{1}, b_{2}, b_{3} \in\{0,1,2\}$ and $i \in\{1,2\}$ we have $f\left(b_{1}+{ }_{3} i, b_{2}+{ }_{3} i, b_{3}+{ }_{3} i\right)=f\left(b_{1}, b_{2}, b_{3}\right)+{ }_{3} i$ since $f$ preserves $C_{3}$. Hence, $f$ is fully determined by its values for $(0,1,2)$ and on $(0,2,1)$. Moreover, for every $i \in\{1,2,3\}$, let
$\sigma_{i}:\{1,2,3\} \rightarrow\{1,2,3\}$ be the map that fixes $i$ and permutes the remaining two elements. For every $f \in \mathcal{D M}^{(3)}$, there exists an $i \in\{1,2,3\}$ such that $f=f_{\sigma_{i}}$ :

- If $f(0,1,2)=f(0,2,1)$, then we have $f=f_{\sigma_{1}}$ and we define $\xi(f):=m(x, y, z)$. Note that $m(x, y, z)=m_{\sigma_{1}}$.
- If $f(0,1,2)=f(0,2,1)+{ }_{3} 1=f(1,0,2)$, then we have $f=f_{\sigma_{3}}$ and we define $\xi(f):=m(z, y, x)=m_{\sigma_{2}}$.
- If $f(0,1,2)=f(0,2,1)+{ }_{3} 2=f(2,1,0)$, then we have $f=f_{\sigma_{2}}$ and we define $\xi(f):=m(y, x, z)=m_{\sigma_{3}}$.

For the sake of notation, let us define a map $\nu:\{1,2,3\} \rightarrow\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}\right\}$ such that $\nu(1)=\sigma_{1}$, $\nu(2)=\sigma_{3}$, and $\nu(3)=\sigma_{2}$. Note that, if $f=f_{\sigma_{i}}$, then $\xi(f)=m_{\nu(i)}$.

Claim 1: The map $\xi$ is minor-preserving. If $f \in \mathcal{D} \mathcal{M}^{(3)}$ is such that its restriction to $\{0,1\}$ is a projection, then it can be easily checked that $\xi\left(f_{\pi}\right)=\xi(f)_{\pi}$ for every $\pi:\{1,2,3\} \rightarrow\{1,2,3\}$. Now let us consider the case where the restriction of $f$ to $\{0,1\}$ is the majority operation. If $\pi:\{1,2,3\} \rightarrow\{1,2,3\}$ in not injective, then $\pi(j)=\pi(k)=i$ for some $i, j, k \in\{1,2,3\}$; in this case, $\xi\left(f_{\pi}\right)=\operatorname{pr}_{i}^{3}=\xi(f)_{\pi}$. Let us finally consider the case where $\pi$ is injective. If $f \in \mathcal{D M}^{(3)}$ is such that $f=f_{\sigma_{i}}$ for some $i \in\{1,2,3\}$, then first applying $\pi$ we get $f_{\pi}=f_{\pi \circ \sigma_{i}}$ and then via $\xi$ we get $m_{\pi \circ \nu(i)}$. On the other side, if we first apply $\xi$ we get $m_{\nu(i)}$ which is mapped to $m_{\pi \circ \nu(i)}$ by $\pi$.

Since $\mathcal{D M}$ is defined on a set of cardinality three, the map $\xi$ naturally extends to a map $\Xi$ from $\mathcal{D M}$ to the set of all operations on $\{0,1,2\}$ as follows: for every $f \in \mathcal{D M}$ of arity $n$ and $a_{1}, \ldots, a_{n} \in\{0,1,2\}$, let $f^{\prime}$ be the ternary operation in $\mathcal{D} \mathcal{M}$ defined by $f^{\prime}\left(x_{0}, x_{1}, x_{2}\right):=f\left(x_{a_{1}}, \ldots, x_{a_{n}}\right)$. Then let $\Xi(f)$ be the $n$-ary operation on $\{0,1,2\}$ that maps $\left(a_{1}, \ldots, a_{n}\right)$ to $\xi\left(f^{\prime}\right)(0,1,2)$.

The map $\Xi$ is minor-preserving by definition, so we are left with showing that $\Xi(f) \in \mathcal{T N}$ for every $f \in \mathcal{D} \mathcal{M}$.

Claim 2: The operation $\Xi(f)$ preserves $C_{3}$ and $T$. Observe that, since $\left|C_{3}\right|=|T|=3$, it is sufficient to prove the claim for ternary operations in $\mathcal{D M}$; indeed, if $f$ does not preserve $C_{3}$ or $T$, then there is a ternary minor of $f$ that does not preserve $C_{3}$ or $T$. If $g$ is a ternary operation in $\mathcal{D} \mathcal{M}$, then $\Xi(g)=\xi(g)$ and therefore the claim holds since $\xi(g) \in \mathcal{T} \mathcal{N}$ by the definition of $\xi$.

Claim 3: $\Xi(f)$ preserves $N$. It suffices to show that every four-variable minor of $f$ preserves $N$, because $|N|=4$. Let $g \in \mathcal{D} \mathcal{M}^{(4)}$ and let $\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right),\left(a_{3}, b_{3}\right),\left(a_{4}, b_{4}\right) \in$ $N$. We may suppose that if $i, j \in\{1,2,3,4\}$ are distinct, then $\left(a_{i}, b_{i}\right) \neq\left(a_{j}, b_{j}\right)$. Indeed, suppose that for $i=1$ and $j=2$ we have $\left(a_{i}, b_{i}\right)=\left(a_{j}, b_{j}\right)$. Then let $g^{\prime}$ be the ternary minor of $g$ defined by $g^{\prime}(x, y, z):=g(x, x, y, z)$, and note that

$$
\left(\Xi(g)\left(a_{1}, a_{2}, a_{3}, a_{4}\right), \Xi(g)\left(b_{1}, b_{2}, b_{3}, b_{4}\right)\right)=\left(\xi\left(g^{\prime}\right)\left(a_{1}, a_{3}, a_{4}\right), \xi\left(g^{\prime}\right)\left(b_{1}, b_{3}, b_{4}\right)\right)
$$

and we conclude that $\Xi(g)$ preserves $N$ since $\xi\left(g^{\prime}\right) \in \mathcal{T} \mathcal{N}$. The argument for different pairs of distinct elements $i, j \in\{1,2,3,4\}$ is similar. By permuting arguments of $g$, we may therefore assume without loss of generality that

$$
\left(\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right),\left(a_{3}, b_{3}\right),\left(a_{4}, b_{4}\right)\right)=((0,0),(1,0),(1,1),(1,2)) .
$$

Let $g_{1}, g_{2} \in \mathcal{D M}$ be the ternary operations given by

$$
\begin{aligned}
g_{1}\left(x_{0}, x_{1}, x_{2}\right) & :=g\left(x_{a_{1}}, x_{a_{2}}, x_{a_{3}}, x_{a_{4}}\right)=g\left(x_{0}, x_{1}, x_{1}, x_{1}\right) \\
\text { and } g_{2}\left(x_{0}, x_{1}, x_{2}\right) & :=g\left(x_{b_{1}}, x_{b_{2}}, x_{b_{3}}, x_{b_{4}}\right)=g\left(x_{0}, x_{0}, x_{1}, x_{2}\right) .
\end{aligned}
$$

Since $g_{1}$ must be a projection to the first or second argument we obtain

$$
\Xi(g)(0,1,1,1)=\xi\left(g_{1}\right)(0,1,2) \in\{0,1\} .
$$

To show that $(\Xi(g)(0,1,1,1), \Xi(g)(0,0,1,2)) \in N$ it therefore suffices to show that if $\Xi(g)(0,1,1,1)=0$, then $\Xi(g)(0,0,1,2)=0$. Suppose that $\Xi(g)(0,1,1,1)=0$. Since $\Xi(g)(0,1,1,1)=\xi\left(g_{1}\right)(0,1,2)$ and $\xi\left(g_{1}\right)$ is a projection, we then must have $\xi\left(g_{1}\right)=\operatorname{pr}_{1}^{3}$. By the definition of $\xi$, this implies that $g_{1}=\operatorname{pr}_{1}^{3}$, hence $g(0,1,1,1)=0$.

Note that the restriction $g_{2}^{\prime}$ of $g_{2}$ to $\{0,1\}$ is a projection or a majority operation. Suppose for contradiction that $g_{2}^{\prime}$ is the second projection. Then $g(0,0,1,0)=g_{2}^{\prime}(0,1,0)=$ 1 , but $g(0,1,1,1)=0$, a contradiction to the assumption that $g$ preserves $\leq$. If $g_{2}^{\prime}$ to $\{0,1\}$ is the third projection then $g(0,0,0,1)=g_{2}^{\prime}(0,0,1)=1$, which similarly leads to a contradiction. If $g_{2}^{\prime}$ is the majority operation then $g(0,0,1,1)=g_{2}^{\prime}(0,1,1)=1$, again leading to a contradiction. Hence, $g_{2}^{\prime}$ is must be the first projection. It follows that $\Xi(g)(0,0,1,2)=\xi\left(g_{2}\right)(0,1,2)=\operatorname{pr}_{1}^{3}(0,1,2)=0$.

The proof for $N^{*}$ is analogous.
Define the following relational structures

$$
\begin{aligned}
\mathbb{T D D} & :=\left(\{0,1,2\} ; C_{3}, C_{2}, T\right), \\
\mathbb{D} & :=\left(\{0,1,2\} ; C_{3}, C_{2}\right) .
\end{aligned}
$$

The following can be shown similarly as Proposition 6.2.20
Proposition 6.2.22. The structures $\mathbb{T D}$ and $\mathbb{D} p$ p-construct each other.
Proof. Since $\mathbb{T D}$ is an expansion of $\mathbb{D}$, it suffices to prove that $\mathbb{D}$ pp-constructs $\mathbb{T}$. Let $\mathbb{A}$ be the pp-power of $\mathbb{D}$ with domain $\{0,1,2\}^{2}$ and the same signature as $\mathbb{T D}$; the relations $C_{3}^{\mathbb{A}}$ and $T^{\mathbb{A}}$ are defined as in the proof of Proposition 6.2.20, and

$$
\begin{aligned}
C_{2}^{\mathbb{A}} & :=\left\{(x, y) \in A^{2} \mid C_{2}\left(x_{1}, x_{2}\right) \wedge C_{2}\left(x_{1}, y_{1}\right) \wedge C_{2}\left(x_{2}, y_{2}\right)\right\} \\
& =\{((0,1),(1,0)),((1,0),(0,1))\} .
\end{aligned}
$$

Let $g:\{0,1,2\} \rightarrow\{0,1,2\}^{2}$ and $h:\{0,1,2\}^{2} \rightarrow\{0,1,2\}$ be as in the proof of Proposition 6.2.20. To verify that $h$ is a homomorphism from $\mathbb{A}$ to $\mathbb{T D}$, it suffices to prove that $h$ maps tuples in $C_{2}^{\mathbb{A}}$ to tuples in $C_{2}$, which is straightforward since $(0,1,1,0) \in C_{2}^{\mathbb{A}}$ is mapped to $(0,1) \in C_{2}$ and $(1,0,0,1) \in C_{2}^{\mathbb{A}}$ is mapped to $(1,0) \in C_{2}$. Conversely, $g$ is a homomorphism from $\mathbb{T D}$ to $\mathbb{A}$. We have already shown that $g$ preserves $C_{3}$ and $T$ (see the proof of Proposition 6.2.20). Finally, $(g(0), g(1))=((0,1),(1,0)) \in C_{2}^{\mathbb{A}}$ and $(g(1), g(0))=((1,0),(0,1)) \in C_{2}^{\mathrm{A}}$.

We conclude this section with a description of the clones of self-dual operations that collapse with the clone of all self-dual operations $\mathbb{C}_{3}$. Define

$$
\begin{aligned}
\mathbb{C}_{3}^{0} & :=\left(\{0,1,2\} ; C_{3},\{0\}\right) \\
\text { and } \quad \mathbb{T} & :=\left(\{0,1,2\} ; C_{3}, T\right)
\end{aligned}
$$

Proposition 6.2.23. The structures $\mathbb{C}_{3}^{0}, \mathbb{C}_{3}$, and $\mathbb{T}$ pp-construct each other.
Proof. It follows from Theorem 3.1 .10 and Proposition 6.2 .13 that $\mathbb{C}_{3}^{0}$ and $\mathbb{C}_{3}$ pp-construct each other. Since every relation of $\mathbb{C}_{3}^{0}$ has a pp-definition in $\mathbb{T}$, it suffices to show that $\mathbb{C}_{3}^{0}$ pp-constructs $\mathbb{T}$. This can be shown as in Proposition 6.2.20.

### 6.2.4 Separations and final picture

We conclude this chapter proving that any two clones of self-dual operations whose minor-equivalence was not proved in the previous section are in fact not minor-equivalent; for every pair of self-dual clones $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{C} \not \varliminf_{\mathrm{m}} \mathcal{D}$ we present a minor condition that holds in $\mathcal{C}$ but not in $\mathcal{D}$. Interestingly, all but one of the conditions (i.e., the condition $g \Sigma_{3}$ from Theorem 6.2.25 that we use in this section were already presented in Chapter 4

## The Atoms

We show that there are four smallest clones of self-dual operations that, again with respect to $\preceq_{\mathrm{m}}$, are larger than $\mathcal{P}_{3}$ (see Figure 6.3). Let us define the following clones:

$$
\begin{aligned}
\mathcal{L}_{3} & :=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, L_{3}\right) & & \text { (see 6.7) } \\
\mathcal{T} \mathcal{L}_{2} & :=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, T, L_{2}\right) & & \text { (see 6.9) } \\
\mathcal{T} \mathcal{N} & :=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, T, N, N^{*}\right) & & \text { (see 6.8) } \\
\mathcal{W} & :=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, R_{3}^{=}\right) & & \text {(see } 6.2)
\end{aligned}
$$

The minority operation that returns $x$ whenever $|\{x, y, z\}|=3$ is denoted by 'plus'. The binary operation $\oplus$ is defined to be $(x, y) \mapsto 2(x+y) \bmod 3$. It is known that $\mathcal{T} \mathcal{L}_{2}=\langle$ plus $\rangle$ and that $\mathcal{L}_{3}=\langle\oplus\rangle$ (see [104], Theorem 28). Recall that $\mathcal{W}=\left\langle\vee_{3}\right\rangle$ and $\mathcal{T} \mathcal{N}=\langle m\rangle$, see (6.4) and 6.11 respectively.

|  | $\mathcal{L}_{3} \not \models$ | $\mathcal{T} \mathcal{L}_{2} \not \models$ | $\mathcal{T N} \not \models$ | $\mathcal{W} \not \models$ |
| ---: | :--- | :--- | :--- | :--- |
| $\mathcal{L}_{3} \models$ |  | $\Sigma_{2}$ | $\Sigma_{2}$ | $\Sigma_{\mathrm{M}}$ |
| $\mathcal{T} \mathcal{L}_{2} \models$ | Minority |  | Minority | Minority |
| $\mathcal{T \mathcal { N }} \models$ | $\operatorname{QNU}(3)$ | $\operatorname{QNU}(3)$ |  | $\operatorname{QNU}(3)$ |
| $\mathcal{W} \vDash$ | $\operatorname{WNU}(3)$ | $\Sigma_{2}$ | $\Sigma_{2}$ |  |

Figure 6.2: The minor conditions that show that the clones $\mathcal{L}_{3}, \mathcal{T} \mathcal{L}_{3}, \mathcal{T} \mathcal{N}$, and $\mathcal{W}$ are pairwise incomparable.

Proposition 6.2.24. The clones $\mathcal{L}_{3}, \mathcal{T} \mathcal{L}_{2}, \mathcal{T N}$, and $\mathcal{W}$ are pairwise incomparable with respect to $\preceq_{\mathrm{m}}$.

Proof. We use the minor conditions as specified in Figure 6.2. We claim that $\Sigma_{2}$ holds in $\mathcal{L}_{3}$ and in $\mathcal{W}$ but not in $\mathcal{T} \mathcal{L}_{2}$ and $\mathcal{T \mathcal { N }}$. Clearly, the operation $\oplus$ in $\mathcal{L}_{3}$ and the operation $\vee_{3}$ in $\mathcal{W}$ are binary symmetric. Every binary symmetric operation $f$ that preserves $\{0,1\}$ does not preserve the relation $T$, because $f(0,1)=f(1,0)=a \in\{0,1\}$ and $(a, a) \notin T$. Note that all operations in $\mathcal{T} \mathcal{L}_{2}$ or in $\mathcal{T} \mathcal{N}$ preserve $\{0,1\}$.

It is easy to check that the operation $(x, y, z) \mapsto x \oplus(y \oplus z) \in \mathcal{L}_{3}$ is a quasi Mal'cev operation. But any Mal'cev operation $f$ does not preserve the relation $R_{3}^{=}$because

$$
\left(\begin{array}{l}
f(0,1,1) \\
f(1,1,1) \\
f(1,1,0)
\end{array}\right)=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right) \notin R_{3}^{\overline{\overline{3}}} .
$$

which shows that $\mathcal{W}$ does not satisfy the Mal'cev condition $\Sigma_{\mathrm{M}}$.
Note that $\mathcal{T} \mathcal{L}_{2}=\langle$ plus $\rangle$ satisfies the quasi minority condition but $\mathcal{L}_{3}, \mathcal{T N}$, and $\mathcal{W}$ do not. To see this, let $f$ be a quasi minority operation $f$. Then $f$ does not preserve $L_{3}$ because

$$
\left(\begin{array}{l}
f(0,0,1) \\
f(0,1,0) \\
f(0,2,2)
\end{array}\right)=\left(\begin{array}{l}
1 \\
1 \\
0
\end{array}\right) \notin L_{3} .
$$

Moreover, $f$ does not preserve $N$ because

$$
\binom{f(0,1,1)}{f(0,0,2)}=\binom{0}{2} \notin N .
$$

Finally, we have already seen that $\mathcal{W}$ does not have a quasi minority operation because every quasi minority operation is a particular quasi Mal'cev operation.

The clone $\mathcal{T N}=\langle m\rangle$ (see (6.11) ) satisfies $\mathrm{QNU}(3)$ since $m$ is a quasi majority
operation. However, any quasi majority operation $f$ does not preserve $L_{3}$ as

$$
\left(\begin{array}{l}
f(0,0,1) \\
f(0,1,0) \\
f(0,2,2)
\end{array}\right)=\left(\begin{array}{l}
0 \\
0 \\
2
\end{array}\right) \notin L_{3},
$$

does not preserve $L_{2}$ as

$$
\left(\begin{array}{l}
f(0,0,1) \\
f(0,1,1) \\
f(0,1,0)
\end{array}\right)=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right) \notin L_{2},
$$

and does not preserve $R_{3}^{=}$as

$$
\left(\begin{array}{l}
f(0,0,1) \\
f(0,1,1) \\
f(0,1,0)
\end{array}\right)=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right) \notin R_{3}^{=},
$$

which shows that $\mathcal{L}_{3}, \mathcal{T} \mathcal{L}_{2}$, and $\mathcal{W}$ do not satisfy $\operatorname{QNU}(3)$. Note that $\operatorname{WNU}(3)$ holds in $\mathcal{W}$ since $\mathcal{W}$ contains $(x, y, z) \mapsto x \vee_{3}\left(y \vee_{3} z\right)$. Suppose that there exists a ternary weak near unanimity operation $w \in \mathcal{L}_{3}$. Then

$$
\left(\begin{array}{l}
w(0,1,1) \\
w(1,0,1) \\
w(2,2,1)
\end{array}\right)=\left(\begin{array}{l}
a \\
a \\
b
\end{array}\right) .
$$

Note that $(a, a, b) \in L_{3}$ implies that $a=b$. It follows that

$$
\binom{w(0,1,1)}{w(1,2,2)}=\binom{a}{a} \notin C_{3}
$$

which is a contradiction.

## Separations in the Wings

In the lattice of clones of self-dual operations the clone $\mathcal{W}$ is the smallest clone in the left wing. The clone $\mathcal{Q}:=\operatorname{Pol}(\mathbb{Q})$, where $\mathbb{Q}$ is the structure introduced in Section 6.2 .2 is the unique smallest clone that properly contains $\mathcal{W}$ and lies in the left wing (see Figure 6.1). We now show that the aforementioned clones are not minor-equivalent. In order to prove the separation in Theorem 6.2.25, we will use a minor condition which we call guarded 3 -cyclic, and denote by $g \Sigma_{3}$. Note that, among the minor conditions that we will use to prove separations in the lattice $\downarrow \overline{\mathcal{C}_{3}}$, the minor condition $g \Sigma_{3}$ is the only one that did not appear in Chapter 4 (see Figure 4.2).

Theorem 6.2.25. There is no minor-preserving map from $\mathcal{Q}$ to $\mathcal{W}$.

Proof. It is known ([104], Theorem 29) that $\mathcal{Q}$ contains the operation $r_{4}$ defined as follows.

$$
r_{4}(x, y, z, t):= \begin{cases}x \vee_{3} y \vee_{3} z & \text { if }|\{x, y, z\}| \leq 2 \\ t & \text { otherwise }\end{cases}
$$

Note that $r_{4}$ satisfies the following minor condition, which we call guarded 3-cyclic ( $g \Sigma_{3}$ )

$$
\begin{align*}
f\left(x_{1}, x_{2}, x_{3}, y\right) & \approx f\left(x_{2}, x_{3}, x_{1}, y\right), \text { and }  \tag{6.13}\\
f(x, x, x, y) & \approx f(x, x, x, x) \tag{6.14}
\end{align*}
$$

Suppose for contradiction that these identities can be satisfied by an operation $f \in$ $\operatorname{Pol}(\mathbb{W})$. Let $a \in\{0,1,2\}$ be such that $f(0,1,2,0)=a$. Since $f$ preserves $C_{3}$ we have $f(1,2,0,1)=a+3$. By 6.13 we have $f(0,1,2,1)=a+31$. But then $f$ does not preserve $R_{3}^{=}$, because

$$
f\left(\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 1 & 2 & 0 \\
0 & 1 & 2 & 1
\end{array}\right)=\left(\begin{array}{c}
0 \\
a \\
a+{ }_{3} 1
\end{array}\right) \notin R_{3}^{=}
$$

Proposition 6.2.26. The minor condition $\mathrm{QJ}(4)$ holds in $\mathcal{M}_{\infty}$ but not in $\mathcal{Q}$.

Proof. Zhuk ([104], Theorem 30) proved that $\mathcal{M}_{\infty}$ contains the following operation

$$
f_{\pi}^{\infty}(x, y, z):= \begin{cases}x \vee_{3}\left(y \wedge_{3} z\right) & \text { if }|\{x, y, z\}| \leq 2 \\ x & \text { otherwise }\end{cases}
$$

Note that the operations $t_{0}, t_{1}, t_{2}, t_{3}, t_{4}$ given by

$$
\begin{aligned}
t_{0}(x, y, z):=f_{\pi}^{\infty}(x, x, x) & t_{2}(x, y, z):=f_{\pi}^{\infty}(x, z, z) \quad t_{4}(x, y, z):=f_{\pi}^{\infty}(z, z, z) \\
t_{1}(x, y, z):=f_{\pi}^{\infty}(x, y, z) & t_{3}(x, y, z):=f_{\pi}^{\infty}(z, x, y)
\end{aligned}
$$

witness that $\mathcal{M}_{\infty}$ satisfies $\mathrm{QJ}(4)$; in particular, we have

$$
\begin{aligned}
& t_{1}(x, y, x)=f_{\pi}^{\infty}(x, y, x)=x \vee_{3}\left(y \wedge_{3} x\right)=x \\
& t_{2}(x, y, x)=f_{\pi}^{\infty}(x, x, x)=x \\
& t_{3}(x, y, x)=f_{\pi}^{\infty}(x, x, y)=x \vee_{3}\left(x \wedge_{3} y\right)=x \\
& t_{1}(x, z, z)=f_{\pi}^{\infty}(x, z, z)=f_{\pi}^{\infty}(x, z, z)=t_{2}(x, z, z) \\
& t_{2}(x, x, z)=f_{\pi}^{\infty}(x, z, z)=f_{\pi}^{\infty}(z, x, x)=t_{3}(x, x, z)
\end{aligned}
$$

We claim that $\mathcal{Q}$ does not satisfy $\mathrm{QJ}(4)$. Since $\mathcal{Q}$ has a minor-preserving map to $\mathcal{R}$, it suffices to prove that the clone $\mathcal{R}$ does not satisfy QJ(4). From Theorem 4.2.8 it
follows that every finite structure with a finite relational signature whose polymorphism clone satisfies $\mathrm{QJ}(n)$ for some $n$ also satisfies $\mathrm{QNU}(n)$ for some $n \geq 3$. Suppose for contradiction that $\mathcal{R}$ contains a quasi near unanimity operation $t$ of arity $n \geq 3$. If $n=3$ we have

$$
\left(\begin{array}{l}
t(0,1,0) \\
t(1,0,0) \\
t(1,1,0)
\end{array}\right)=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right) \notin R_{2}^{\Rightarrow}
$$

For $k>3$, note that

$$
\left(\begin{array}{l}
t(0, \ldots, 0,1,0) \\
t(1, \ldots, 1,0,0) \\
t(1, \ldots, 1,1,0)
\end{array}\right)=\left(\begin{array}{c}
0 \\
a_{1} \\
1
\end{array}\right)
$$

Since we assumed that $t \in \operatorname{Pol}(\mathbb{R})$, we obtain that $a_{1}=1$. We repeat the same reasoning $k-4$ times: each time we consider the matrix $M_{i}$, for $2 \leq i \leq k-3$, where the first $k-i-1$ columns are equal to $(0,1,1)$, the $(k-i)$-th column is equal to $(1,0,1)$, and the last $i$ columns are equal to $(0,0,0)$. Note that every column of $M_{i}$ is an element of $R_{2}^{\vec{~}}$. By letting $t$ act row-wise in $M_{i}$, we get the chain of equalities $a_{2}=\cdots=a_{k-3}=1$. Finally we get

$$
\left(\begin{array}{c}
t(0,1,0, \ldots, 0) \\
t(1,0,0, \ldots, 0) \\
t(1,1,0, \ldots, 0)
\end{array}\right)=\left(\begin{array}{c}
0 \\
0 \\
a_{k-3}
\end{array}\right) \notin R_{2}^{\overrightarrow{2}}
$$

a contradiction.

Recall that from Theorem 6.2 .7 it follows that $\mathcal{R}$ and $\mathcal{Q}$ are minor-equivalent and from Theorem 6.2.11 that $\mathcal{B}_{\infty} \pi_{\infty}$ collapses with $\mathcal{M}_{\infty}$. It follows that $\overline{\mathcal{M}_{\infty}}$ is the unique smallest element that properly contains $\overline{\mathcal{Q}}$ in Figure 6.3 .

We will now show all the clones $\mathcal{B}_{k}^{3}$ and $\mathcal{M}_{k}$, for $k \geq 2$, are pairwise distinct. If $t \in A^{n}$, we denote by $\operatorname{Two}(t)$ the set of all entries of $t$ that appear at least twice. For every $n \geq 3$, let $f_{\pi}^{n}:\{0,1,2\}^{n+1} \rightarrow\{0,1,2\}$ be the operation defined as follows

$$
f_{\pi}^{n}\left(x_{1}, \ldots, x_{n+1}\right):= \begin{cases}x_{1} & \text { if } \operatorname{Two}\left(x_{1}, \ldots, x_{n+1}\right)=\{0,1,2\} \\ a \vee_{3} b & \text { if } \operatorname{Two}\left(x_{1}, \ldots, x_{n+1}\right)=\{a, b\} \\ a & \text { if } \operatorname{Two}\left(x_{1}, \ldots, x_{n+1}\right)=\{a\}\end{cases}
$$

It is known that $f_{\pi}^{n} \in \mathcal{B}_{n}^{3}$ and $f_{\pi}^{n} \in \mathcal{M}_{n}$ ([104], Theorem 29).
Proposition 6.2.27. Let $n \geq 2$. The minor condition $\mathrm{QNU}(n+1)$ holds in $\mathcal{B}_{n}^{3}$ and $\mathcal{M}_{n}$, but does not hold in $\mathcal{B}_{n+1}^{3}$ and $\mathcal{M}_{n+1}$.

Proof. If $n \geq 3$, then $\mathcal{B}_{n}^{3}$ and $\mathcal{M}_{n}$ contain the quasi near unanimity operation $f_{\pi}^{n}$, which is a witness for $\mathcal{B}_{n}^{3} \models \mathrm{QNU}(n+1)$ and $\mathcal{M}_{n} \models \mathrm{QNU}(n+1)$. If $n=2$, then note that $m \in \mathcal{B}_{n}^{3}$ and $m \in \mathcal{M}_{n}$ (where $m$ is defined in (6.11) and that $m$ is a majority operation. It is easy to see that every quasi near unanimity operation of arity $n \geq 3$ does not preserve $B_{n}$, so $\mathrm{QNU}(n)$ does not hold in $\mathcal{B}_{n}^{3}$ and in $\mathcal{M}_{n}$.

Let $f_{0}^{\infty}:\{0,1,2\}^{3} \rightarrow\{0,1,2\}$ be defined by $f_{0}^{\infty}(x, y, x)=x \vee_{3} y$, and by $f_{0}^{\infty}(x, y, z)=$ $x$ otherwise. It is known that $f_{0}^{\infty} \in \mathcal{B}_{n}^{3}$ for every $n \geq 2$ ([104], Theorem 29).

Proposition 6.2.28. The minor condition $\mathrm{QHM}(3)$ holds in $\mathcal{B}_{\infty}^{3}$, but not in $\mathcal{M}_{2}$.
Proof. For $x, y, z \in\{0,1,2\}$, define

$$
\begin{aligned}
& p_{0}(x, y, z):=f_{0}^{\infty}(x, x, x) \\
& p_{1}(x, y, z):=f_{0}^{\infty}(x, z, y) \\
& p_{2}(x, y, z):=f_{0}^{\infty}(z, x, y) \\
& p_{3}(x, y, z):=f_{0}^{\infty}(z, z, z) .
\end{aligned}
$$

Then $p_{0}, p_{1}, p_{2}, p_{3}$ witness that $\mathcal{B}_{\infty}^{3}$ satisfies $\operatorname{QHM}(3)$ : in particular, we have

$$
p_{1}(x, x, y)=f_{0}^{\infty}(x, y, x)=x \vee_{3} y=f_{0}^{\infty}(y, x, y)=p_{2}(x, y, y)
$$

Note that $\leq_{2} \in \operatorname{Inv}\left(\mathcal{M}_{2}\right)$ and therefore the same argument as in Proposition 5.3.7 can be used to show that $\mathrm{QHM}(3)$ does not hold in $\mathcal{M}_{2}$.

This implies that for all $k, l \geq 2$, there is no minor-preserving map from $\mathcal{B}_{k}^{3}$ to $\mathcal{M}_{l}$, because $\mathcal{B}_{\infty}^{3} \subseteq \mathcal{B}_{k}^{3}$ and $\mathcal{M}_{l} \subseteq \mathcal{M}_{2}$.

## The Final Picture

We conclude this chapter by collecting all the results presented in this section into a single theorem that accomplishes the goal we set ourselves at the beginning of this section: a complete description of $\downarrow \overline{\mathcal{C}_{3}}$.

Theorem 6.2.29. The lattice $\downarrow \overline{\mathcal{C}_{3}}$ of clones of self-dual operations factored by minorequivalence and ordered by the existence of minor-preserving maps, is a countably infinite lattice, and is exactly of the form as described in Figure 6.3.

Proof. We use the minor conditions as indicated in the table of Figure 6.4. In Proposition 6.2.27 and Proposition 6.2.28 we proved that the clones $\mathcal{M}_{n}$ and $\mathcal{B}_{n}^{3}$, for $n \in$ $\{2,3, \ldots, \infty\}$, form two descending chains as displayed in Figure 6.4 The restriction of this table to the clones $\mathcal{L}_{3}, \mathcal{T} \mathcal{L}_{2}, \mathcal{T}$, and $\mathcal{T}$ has already been described in Figure 6.2 We now describe how to extend this table to the remaining clones $\mathcal{W}, \mathcal{D}:=\operatorname{Pol}(\mathbb{D}), \mathcal{Q}, \mathcal{M}_{n}$, $\mathcal{B}_{n}^{3}$, and $\mathcal{C}_{3}$.


Figure 6.3: The lattice of clones of self-dual operations up to minor-equivalence.

The clone $\mathcal{D}=\operatorname{Pol}\left(\{0,1,2\} ; C_{3}, C_{2}\right)$ contains $\mathcal{T} \mathcal{N}$ and $\mathcal{T} \mathcal{L}_{2}$, and therefore contains the minority operation plus and the majority operation $m$. It does not satisfy $\Sigma_{2}$ because idempotent operations on $\{0,1,2\}$ that satisfy $\Sigma_{2}$ cannot preserve $C_{2}$.

The clone $\mathcal{Q}$ contains $\mathcal{W}$ and therefore contains a binary symmetric operation and a ternary weak near unanimity operation. Moreover, the proof of Theorem 6.2.25 shows that $\mathcal{Q}$ satisfies $g \Sigma_{3}$, which is not satisfied by $\mathcal{W}$. Moreover, $\mathcal{Q}$ does not satisfy $\operatorname{QJ}(4)$, which is satisfied by $\mathcal{M}_{n}$ and $\mathcal{B}_{n}^{3}$ for all $n \in\{2,3, \ldots, \infty\}$ (Proposition 6.2.26).

The clone $\mathcal{M}_{n}$, for each $n \in\{2,3, \ldots, \infty\}$, contains $\mathcal{Q}$ and therefore satisfies $\operatorname{QNU}(3)$, $\Sigma_{2}$, and $g \Sigma_{3}$. It does not satisfy $\mathrm{QHM}(3)$ as we have seen in Proposition 6.2.28 For each $n \in\{2,3, \ldots, \infty\}$, the clone $\mathcal{B}_{n}^{3}$ contains $\mathcal{M}_{n}$, but satisfies the additional minor condition $\mathrm{QHM}(3)$. It is straightforward to verify that any minority operation on $\{0,1,2\}$ does not preserve the relation $B_{2}$, so $\mathcal{B}_{n}^{3}$ does not satisfy the minority condition. The clone $\mathcal{C}_{3}$ contains all the clones discussed so far; since each of these clones does not satisfy some minor condition discussed so far, it follows that $\mathcal{C}_{3}$ does not have a minor-preserving map to any of these clones.

|  | $\mathcal{L}_{3} \nmid=$ | $\mathcal{T} \mathcal{L}_{2} \not \models$ | $\mathcal{T N} \neq$ | $\mathcal{W} \nLeftarrow$ | $\mathcal{D} \nmid=$ | $\mathcal{Q} \not \models$ | $\mathcal{M}_{n} \nmid=$ | $\mathcal{B}_{n}^{3} \neq$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{L}_{3} \models$ |  | $\Sigma_{2}$ | $\Sigma_{2}$ | $\Sigma_{\mathrm{M}}$ | $\Sigma_{2}$ | $\Sigma_{\text {M }}$ | $\Sigma_{\text {M }}$ | $\Sigma_{\text {M }}$ |
| $\mathcal{T} \mathcal{L}_{2} \models$ | Minority |  | Minority | Minority |  | Minority | Minority | Minority |
| $\mathcal{T N} \equiv$ | QNU(3) | QNU(3) |  | QNU(3) |  | QNU(3) | QNU(3) | QNU(3) |
| $\mathcal{W} \models$ | WNU(3) | $\Sigma_{2}$ | $\Sigma_{2}$ |  | $\Sigma_{2}$ |  |  |  |
| $\mathcal{D} \models$ | QNU(3) | QNU(3) | Minority | Minority |  | Minority | Minority | Minority |
| $\mathcal{Q} \models$ | WNU(3) | $\Sigma_{2}$ | $\Sigma_{2}$ | $g \Sigma_{3}$ | $\Sigma_{2}$ |  |  |  |
| $\mathcal{M}_{n} \models$ | WNU(3) | $\Sigma_{2}$ | $\Sigma_{2}$ | $g \Sigma_{3}$ | $\Sigma_{2}$ | QJ(4) |  |  |
| $\mathcal{B}_{n}^{3} \models$ | WNU(3) | $\Sigma_{2}$ | $\Sigma_{2}$ | $g \Sigma_{3}$ | $\Sigma_{2}$ | QJ(4) | QHM(3) |  |
| $\mathcal{C}_{3} \models$ | Minority | $\Sigma_{2}$ | Minority | Minority | $\Sigma_{2}$ | Minority | Minority | Minority |

Figure 6.4: The minor conditions that justify that the existence of a minor-preserving map orders the clones of self-dual operations as depicted in Figure 6.3

## Chapter 7

## Conclusion

As we have already pointed out in Chapter 1 the study of the pp-constructability poset can be approached from different perspectives and different research communities may ask questions that differ according to their focus. For this reason, there are several directions that this line of research could take. In Chapter 3, we already proposed some open problems concerning mainly the posets $\mathfrak{P}_{3}$ and $\mathfrak{P}_{\mathrm{fin}}$. We therefore conclude this dissertation by recalling some problems we consider most relevant and by proposing some new ones. The aim is to suggest a possible direction of research that may hopefully be fruitful and lead to a better understanding of clones over finite sets and, as a byproduct, of the pp-constructability poset.

In Section 3.2 we proved that $\mathfrak{P}_{\text {fin }}$ is a semilattice. We ask the following:
Question 7.0.1. Is $\mathfrak{P}_{\mathrm{fin}}$ a lattice?
In Section 3.3.2 we gave an overview of what is the current status on la recherche of the submaximal elements of $\mathfrak{P}_{\text {fin }}$. In this scenario, we ask the following question which is clearly in symbiosis with Conjecture 3.3.20.

Problem 7.0.2. Find a set $\Gamma_{\mathrm{TS}}$ of finite relational structures such that, for every finite relational structure $\mathbb{A}$, either

- $\mathbb{A}$ pp-constructs some structure in $\Gamma_{\mathrm{TS}}$, or
- $\operatorname{Pol}(\mathbb{A}) \models \mathrm{TS}(n)$, for every $n \geq 2$.

The set $\left\{\overline{\mathbb{A}} \mid \mathbb{A} \in \Gamma_{\mathrm{TS}}\right\} \cup\left\{\overline{\mathbb{B}_{2}}\right\}$ is therefore a candidate for being the set of all submaximal elements of $\mathfrak{P}_{\mathrm{fin}}$.

Problem 7.0.3. Find all the submaximal elements of $\mathfrak{P}_{\mathrm{fin}}$.
In Section 3.3.3 we pointed out that determining the cardinality of $\mathfrak{P}_{3}$ and $\mathfrak{P}_{\mathrm{fin}}$ is still an open problem and conjectured that both the mentioned posets are countably infinite. It follows from the observation we made at the end of Section 3.3.3. based on the results in [2] and [33], that we can focus on the following problem.

Problem 7.0.4. Describe the following posets:

- $\downarrow \overline{\mathcal{B}}_{2}{ }^{*}:=\left\{\overline{\mathcal{A}} \mid \mathcal{A}\right.$ is a clone over $\{0,1,2\}$ and $\left.\overline{\mathcal{A}} \preceq_{\mathrm{m}} \overline{\mathcal{B}_{2}}\right\}$,
- $\downarrow \overline{\mathcal{B}_{2}}:=\left\{\overline{\mathcal{A}} \mid \mathcal{A}\right.$ is a clone over a finite set and $\left.\overline{\mathcal{A}} \preceq_{\mathrm{m}} \overline{\mathcal{B}_{2}}\right\}$.

In Section 3.2.5, we discussed the connection between minimal Taylor clones over an $n$-element set and the atoms of $\mathfrak{P}_{n}$. As already mentioned in the conclusion of Section 3.2.5, the author of the dissertation together with Barto and Zhuk found a complete classification (yet unpublished) of the atoms of $\mathfrak{P}_{3}$, based on Theorem 3.2 .26 We would like to stress that such a classification refines [34] by providing a concrete list of the hardest tractable CSPs over a three-element set. However, we do not yet have an answer to the following:

Question 7.0.5. Is every atom of $\mathfrak{P}_{3}$ of the form $\overline{\mathcal{A}}$ where $\mathcal{A}=\operatorname{Clo}(\mathbf{A})$ and $\mathbf{A}$ is a finitely related algebra? Is it true also for every atom of $\mathfrak{P}_{n}$, for every $n>3$ ?

Turning our attention back to Mal'cev conditions, we conclude this chapter asking a question with a flavour of classic universal algebra; the following open problem already appeared in [27]. Given two strong linear Mal'cev conditions $\Gamma$ and $\Sigma$, we say that $\Gamma$ implies $\Sigma$, denoted $\Gamma \Rightarrow \Sigma$, if $\mathcal{C} \models \Gamma$ implies that $\mathcal{C} \models \Sigma$ for all idempotent clones $\mathcal{C}$ over some finite set (see Chapter 4).

Question 7.0.6. Is the following problem decidable?
Input: two strong linear Mal'cev conditions $\Gamma$ and $\Sigma$.
Output: Does $\Gamma \Rightarrow \Sigma$ hold?
The same question could be asked for general clones, i.e., without assuming that we only consider clones over finite sets.

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I herewith declare that I have produced this thesis without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such. This thesis has not previously been presented in identical or similar form to any other German or foreign examination board.

Albert Vucaj
Dresden, 16 März 2023


[^0]:    ${ }^{1} \mathrm{~A}$ note for the serious reader: you will meet this operation in Chapter 6

[^1]:    ${ }^{1}$ This word will reappear in this dissertation under mathematical guises. Here by core we refer to the everyday definition of the world: a central and often foundational part (Merriam-Webster Dictionary).

[^2]:    ${ }^{2}$ A. A. Bulatov: "Almost everything is a $C S P$ ".

[^3]:    ${ }^{1}$ The name oddition is due to Péter Pál Pálfy. We decided to opt for this name since it is a portmanteau of the words "odd" and "addition".

