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NUMERICAL COMPARISON OF RUNGE-KUTTA (RK5) AND NEW ITERATIVE METHOD (NIM) FOR SOLVING METASTATIC CANCER MODEL

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ABSTRACT

This paper attempts to present and employ Runge-Kutta Method of fifth-order (RK5) and New Iterative Method for the numerical solution of metastatic cancer model which occur in two compartments of cancer environment. These methods have been proved to be powerful mathematical tools for various phenomena in biomathematics and it is extremely effective for linear and non-linear systems of differential equations. Our numerical experiments illustrate the effect of parameters β_1, β_2 and β_3 on cancer models which are responsible for the spread or reduction of cancer cells through the boundary of an organ tissue. The results obtained are compared with analytical solutions and show that (RK5) and NIM are powerful numerical techniques to solve systems of differential equations. Finally, all computations and algorithms are implemented using MAPLE 18 software version.

Keywords: *Analytical solutions, MAPLE 18 Mathematical software, Metastatic cancer model, New Iterative Method (NIM), Runge-Kutta method (RK5).*

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1. Introduction

Mathematical modeling is a prominent practice in almost all fields of pure and applied sciences. Modeling serves as a tool for researchers to formalize and quantitatively understand physical and chemical phenomena observed in reality via the language of mathematics (Rasmuson *et al.*, 2014). It is with this concern that researchers and concerned individuals have actively worked on ways of understanding and proffering solutions over the past decades. In a recent study conducted by the United Nation (UN), it was revealed that by every year there are approximately 8.2 million cancer-related deaths worldwide. Metastasis is the major cause of cancer death which occurs when the disease reaches its lethal stage via the uncontrolled spreading of cancer cells to invade a nearby connective tissue and other key organs in the human body. The metastasizing primary tumor cells are not the only agents that drive the progression of metastasis. Instead, metastasis is a systematic process that involves the interaction of cancer cells among a community of various biochemical and cellular factors localized in the tumor microenvironment at both the primary and secondary tumor compartments (Cox *et al.*, 2016).

Several scientists have worked on understanding the behaviors and development of the cancerous cells including (Hanin & Zaider, 2011; Lorenzo-Herrero *et al.,* 2019; Liu *et al.,* 2016; Thakur & Rao, 2016 and Benzekry *et al.*, 1970) and many have as well worked on

developing Mathematical models for understanding cancer cells with proposed cure parameters through surgery, Radiotherapy, chemotherapy etc. (see Dan *et al.,* 2016; Pinho *et al.,* 2011; Kolev *et al*., 2011). Moreover, the numerical solution of mathematical models by systems of ordinary differential equations has been widely studied in recent years.

Yen *et a*l. (2018) used a remodeling extracellular matrix to obtain numerical solutions of the Metastatic Cancer model. The Quantitative Analysis of the Tumor/Metastasis system and its optimal therapeutic control was carried out by Sebastien *et al*. (2017), modelling and simulation of tumor development, treatment and control was presented by Bellomo & De Angelis (2003), Mathematical population dynamics models to describe the spread of metastatic cancer was considered by Daniel *et al.* (2011), while Nikos & Mastorakis (2016) presented numerical solution of mathematical models of cancer growth and optimal cancer therapy and to mention a few.

This study is concerned with metastatic cancer model tracking cancerous cells in two compartments; the first compartment encompasses proliferating cells which includes active cells that can grow rapidly to another compartment with the target tissue cells. We determine the equilibrium point and analyze the stability. Furthermore, to reduce computational time and length involved in calculating Ks in Runge-Kutta and evaluation involved in New iterative method, we hereby formulate a three step Maple 18 software code for the numerical solutions of equation (1) with suitable initial conditions of parameters.

This paper is arranged as follows: In section 1, a brief introduction on Cancer metastasis was presented. In section 2, Mathematical modeling and metastatic cancer in two compartments was described, section 3 is majorly on equilibrium point and stability analysis of the metastatic model. Section 4 describes two numerical techniques employed and was discussed, in section 5, numerical experiment was demonstrated and finally, section 6 presented numerical results tables, discussion and conclusion.

2. Mathematical Modeling of Metastatic Cancer in Two Compartments

Mathematical models have been used on several occasions as an abstract description of physical, biological and chemical phenomenon. In applied Mathematics, systems of differential equations are the most important tools in modeling metastatic cancer in two compartments. Understanding of the phenomenon, formulations of mathematical models with parameters of the situation and analysis of the metastatic model will further help to understand the behaviors of cancerous cells in the body of humans. Consider two-compartment metastatic cancer model Figure 1, the first compartment (A) houses the proliferating cells and the second compartment (B) encloses quiescent cells. Let be the number X of arrested cells on the boundary of the tissue in compartment A and Y the number of cells that have invaded the target tissue in compartment B. Cancerous cells pass from compartment A to compartment B at the rate of β_2 in X. Also, cells die from compartment A at the rate of β_1 in X and die from compartment B at the rate β_3 in Y.

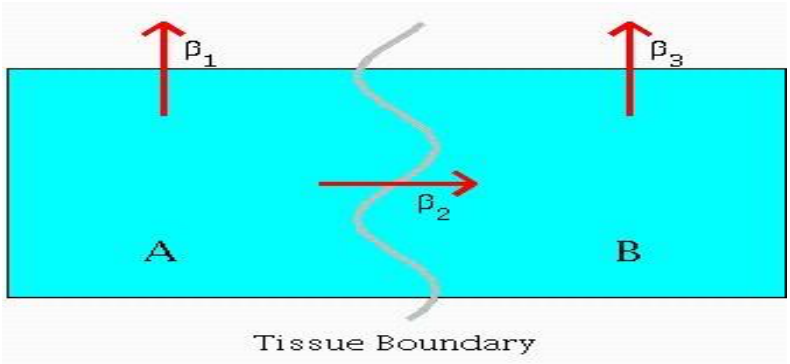

Figure 1. Two Compartments Model for Metastatic in Cancer (CERD)

$$\begin{cases} \frac{dX(t)}{dt} = -(\beta_1 + \beta_2)X(t) \\ \frac{dY(t)}{dt} = \beta_2 X(t) - \beta_3 Y(t) \end{cases} \quad (1)$$

subject to initial conditions

$$\begin{cases} \mathrm{X}(0) = \omega \\ \mathrm{Y}(0) = \varphi \end{cases} \quad (2)$$

where $\beta_1, \beta_2, \beta_3$ are constants, $\beta_1, \beta_2, \beta_3 > 0$ and ω, φ are the initial numbers of cancerous cells in compartments A and B respectively. The size of the tumor, M is defined as

$$M(t) = X(t) + Y(t) \quad (3)$$

3. Equilibrium Point and Stability Analysis

The metastatic model given in equation (1) will have equilibrium point on $\frac{dX(t)}{dt} = \frac{dY(t)}{dt} = 0$, so the equation will become:

$$\begin{cases} -(\beta_1 + \beta_2)\mathrm{X}(t) = 0 \\ \beta_2 X(t) - \beta_3 Y(t) = 0 \end{cases} \quad (4)$$

The point (0, 0) is clearly the equilibrium. We observe that $\big(X(t), Y(t)\big) = (0,0)$
The Jacobian matrix as equation (4), can be written as follows;

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial x} \end{pmatrix}$$

where

$$f_1 = -(\beta_1 + \beta_2)X$$
$$f_2 = \beta_2 X - \beta_3 Y$$

Implies,

$$J = \begin{pmatrix} -(\beta_1 + \beta_2) & 0 \\ \beta_2 & -\beta_3 \end{pmatrix}$$

with characteristic equation,

$$(\lambda + \beta_3)(\lambda + \beta_1 + \beta_2) = 0$$
$$\begin{cases} \lambda_1 = -\beta_3 \\ \lambda_2 = -(\beta_1 + \beta_2) \end{cases}$$

are eigenvalues. Since λ_1 and $\lambda_2 < 0$, it implies the point (0, 0) is a saddle.

4. Description of Numerical Methods

In this section, we present and employ two numerical techniques Runge-Kutta (RK5) and New Iterative Method proposed by Daftardar-Gejji & Jafari (2006) to solve the Metastatic cancer model in two compartments.

4.1 Runge-Kutta fifth order (RK5)

The Runge-Kutta method is an important method for obtaining the approximate solutions of ordinary differential equations. It is one of the most used methods by scientists and engineers. It was named for its creators Carl Runge (1856-1927), which the original idea for such formulas seems to be due and Wilhelm Kutta (Tamer, 2019), which the idea was used more effectively for first-order equations. It has been used extensively to obtain approximate numerical solutions of differential equations of first, second, and higher orders. It transforms second and higher orders into a system of equations of first order. In recent years, several authors have applied Runge-Kutta to solve ordinary differential equations such as direct explicit integrators of RK type for solving special fourth-order ordinary differential equations with an application by Mohammed & Murtaza (2016), Salih *et al.* (2016) presented fifth order Runge-Kutta-nystrom methods for solving linear second order oscillatory problems, Bazuaye (2018) proposed a new 4th order hybrid Runge-Kutta methods for solving initial value problems (IVP), authors (Constantin *et al*., 2019) applied of the Euler and Runge-Kutta generalized methods for FDE and symbolic packages in the analysis of some fractional attractors, Nizam *et al.* (2018) presented diagonally implicit Runge–Kutta type method for directly solving special fourth-order ordinary differential equations with ill-posed problem of a Beam on elastic foundation and Anthony *et al.* (2018) presented an analysis and comparative study of numerical solutions of initial value problems (IVP) in ordinary differential equations (ODE) with Euler and Runge Kutta methods.

In order to apply Runge-Kutta (RK5), we consider a weighted average of the slopes at v points very close to the current point i.e., it considers the Metastatic equation (1) at the grid point t_i with v as slopes:

$$\begin{cases} \mathrm{X_{i+1}} = \mathrm{X_i} + \left(-(\beta_1 + \beta_2)\sum_{r=1}^{v} \mathrm{w_r K_r}\right)\mathrm{X_i} \\ \mathrm{Y_{i+1}} = \mathrm{Y_i} + \left(\left(\beta_2 \sum_{r=1}^{v} \mathrm{w_r K_r}\right)\mathrm{X_i} - \left(\beta_3 \sum_{r=1}^{v} \mathrm{w_r K_r}\right)\mathrm{Y_i}\right) \end{cases} \tag{5}$$

with initial conditions

$$\begin{cases} \mathrm{X_i} = \omega \\ \mathrm{Y_i} = \varphi \end{cases} \tag{6}$$

where

$$K_r = hf\left(t_i + \epsilon_i h, X_i + \sum_{j=1}^{r-1} a_{rj} K_j\right), K_r = hf\left(t_i + \epsilon_i h, Y_i + \sum_{j=1}^{r-1} a_{rj} K_j\right)$$

and $\epsilon_1 = 0,\ r = 1,2,..v,\quad \epsilon_2, \epsilon_3, \epsilon_4 \ldots\ldots, \epsilon_v \ and \quad a_{2j},\ldots,a_{vv-1}$ are parameters to be determined.

After further simplification, one can construct a fifth order Runge-Kutta formula as follows:

$$\begin{cases} X_{i+1} = X_i + \frac{1}{192}(23K_1 + 125K_3 - 81K_5 + 125K_6) \\ K_1 = hf(t_i, X_i) \\ K_2 = hf\left(t_i + \frac{h}{3}, X_i + \frac{K_1}{3}\right) \\ K_3 = hf\left(t_i + \frac{2h}{5}, X_i + \frac{1}{25}(4K_1 + 6K_2)\right) \\ K_4 = hf\left(t_i + h, X_i + \frac{1}{4}(K_1 - 12K_2 + 15K_3)\right) \\ K_5 = hf\left(t_i + \frac{2h}{3}, X_i + \frac{1}{81}(6K_1 + 90K_2 - 50K_3 + 8K_4)\right) \\ K_6 = hf\left(t_i + \frac{4h}{5}, X_i + \frac{1}{75}(6K_1 + 36K_2 + 10K_3 + 8K_4)\right) \end{cases} \quad (7)$$

Similarly,

$$\begin{cases} Y_{i+1} = Y_i + \frac{1}{192}(23K_1 + 125K_3 - 81K_5 + 125K_6) \\ K_1 = hf(t_i, Y_i) \\ K_2 = hf\left(t_i + \frac{h}{3}, Y_i + \frac{K_1}{3}\right) \\ K_3 = hf\left(t_i + \frac{2h}{5}, Y_i + \frac{1}{25}(4K_1 + 6K_2)\right) \\ K_4 = hf\left(t_i + h, Y_i + \frac{1}{4}(K_1 - 12K_2 + 15K_3)\right) \\ K_5 = hf\left(t_i + \frac{2h}{3}, Y_i + \frac{1}{81}(6K_1 + 90K_2 - 50K_3 + 8K_4)\right) \\ K_6 = hf\left(t_i + \frac{4h}{5}, Y_i + \frac{1}{75}(6K_1 + 36K_2 + 10K_3 + 8K_4)\right) \end{cases} \quad (8)$$

The setback of the Runge-Kutta method is that it involves considerably more computational effort per step length.

4.2 New Iterative Method (NIM)

New Iterative Method (NIM) was proposed by Daftardar-Gejji & Jafari (2006). NIM is simple in its principles and easy to implement on computers using symbolic computation packages such as Maple. This method is better than numerical methods as it is free from rounding off errors and does not require large computer power. It has proven successful over other methods in many cases (see Bhalekar & Gejji, 2008; Zead & Ali, 2018; Aisha *et al.,* 2018; Mohammad *et al.*, 2018) and just to mention a few.

Consider the following general functional equation;

$$x(\bar{t}) = f(\bar{t}) + N\big(x(\bar{t})\big) \quad (9)$$

where N is a nonlinear operator from a Banach space $B \rightarrow B$ and f a known function $\underline{x} = (x_1, x_2, \dots, x_n)$.

We are looking for a solution $X(t)$ and $Y(t)$ of equation (9) having the series form

$$\begin{cases} X(\bar{t}) = \sum_{i=0}^{\infty} X_i(\bar{t}) \\ Y(\bar{t}) = \sum_{i=0}^{\infty} Y_i(\bar{t}) \end{cases} \tag{10}$$

The nonlinear operator N can be decomposed as;

$$\begin{cases} N\left(\sum_{i=0}^{\infty} X_i(\bar{t})\right) = N(\mathrm{X}_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} X_j\right) - N\left(\sum_{j=0}^{i-1} X_j\right)\right\} \\ N\left(\sum_{i=0}^{\infty} Y_i(\bar{t})\right) = N(\mathrm{Y}_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} Y_j\right) - N\left(\sum_{j=0}^{i-1} Y_j\right)\right\} \end{cases} \tag{11}$$

From equations (10) and (11), equation (9) is equivalent to

$$\begin{cases} \sum_{i=0}^{\infty} X_i = f(t) + N(X_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} X_j\right) - N\left(\sum_{j=0}^{i-1} X_j\right)\right\} \\ \sum_{i=0}^{\infty} Y_i = f(t) + N(Y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} Y_j\right) - N\left(\sum_{j=0}^{i-1} Y_j\right)\right\} \end{cases} \tag{12}$$

We define the recurrence relation:

$$\begin{cases} X_0 = f, \\ X_1 = N(X_0), \\ X_{m+1} = N(X_0 + X_1 + \cdots + X_m) - N(X_0 + X_1 + \cdots + X_{m-1}) \\ Y_0 = f, \\ Y_1 = N(Y_0), \\ Y_{m+1} = N(Y_0 + Y_1 + \cdots + Y_m) - N(Y_0 + Y_1 + \cdots + Y_{m-1}) \\ m = 1,2,3, \ldots \end{cases} \tag{13}$$

then,

$$\begin{cases} (X_0 + X_1 + \cdots + X_{m+1}) = N(X_0 + X_1 + \cdots + X_m), \\ (Y_0 + Y_1 + \cdots + Y_{m+1}) = N(Y_0 + Y_1 + \cdots + Y_m), \\ m = 1,2, \ldots \end{cases}$$

and

$$\begin{cases} \sum_{i=0}^{\infty} X_i = f(t) + N\left(\sum_{i=0}^{\infty} X_i\right) \\ \sum_{i=0}^{\infty} Y_i = f(t) + N\left(\sum_{i=0}^{\infty} Y_i\right) \end{cases} \tag{14}$$

The k −term approximate solution of (9) and (10) is given by;

$$\begin{cases} x(t) = x_0 + x_1 + x_2 \ldots + x_{k-1} \\ y(t) = y_0 + y_1 + y_2 \ldots + y_{k-1} \end{cases} \tag{15}$$

5. Computational Experiments

In this section, we present and employ an algorithm scheme based on two numerical techniques discussed in section four. The behaviors and effects of three parameters β_1, β_2 and β_3 are investigated at three different cases beginning with an initial condition guess of $X(0) = 1$ and $Y(0) = 1$. All the numerical computations are executed in MAPLE 18 software and the test parameter values used for the numerical simulation are stated in Table 1, Table 2 and Table 3.

Table 1. Description of Parameters in the Model (Case1)

Parameter	Symbol	Value	Unit
Per Capital Death Rate Cells from A in X	β_1	0.001	Daily
Per Capital Movement of Cancerous Cells from A to B	β_2	0.005	Daily
Per Capital Death Rate Cells from B in Y	β_3	0.008	Daily

Table 2. Description of Parameters in the Model (Case 2)

Parameter	Symbol	Value	Unit
Per Capital Death Rate Cells from A in X	β_1	0.007	Weekly
Per Capital Movement of Cancerous Cells from A to B	β_2	0.035	Weekly
Per Capital Death Rate Cells from B in Y	β_3	0.056	Weekly

Table 3. Description of Parameters in the Model (Case 3)

Parameter	Symbol	Value	Unit
Per Capital Death Rate Cells from A in X	β_1	0.030	Monthly
Per Capital Movement of Cancerous Cells from A to B	β_2	0.150	Monthly
Per Capital Death Rate Cells from B in Y	β_3	0.240	Monthly

In order to examine and obtain numerical solutions for equation (1) using the parameters given in Tables 1 to 3. We develop an algorithm using Runge-Kutta (RK5) discussed in 4.1 as follow:

Restart:
$Digits := 26;$

Step 1:
$f := (t, x) \to -(\beta_1 + \beta_2) * x$;
$g := (t, y, x) \to (\beta_2) * x - (\beta_3) * y;$
$t[0] := 0;$
$x[0] := 1.0;\ y[0] := 1.0;$
$h[0] := 0.1;$
$\beta_1 := [0.001, 0.007, 0.0030];$
$\beta_2 := [0.005, 0.0035, 0.0150];$
$\beta_3 := [0.008, 0.0056, 0.0240];$

Step 2:
$for\ n\ from\ 1\ to\ 10\ do$
$t[n] := n * h:$
$k1 := f(t[n-1], x[n-1]);$
$k2 := f\left(t[n-1] + \frac{h}{3}, x[n-1] + \frac{h}{3} * k1\right);$
$k3 := f\left(t[n-1] + \frac{2*h}{5}, x[n-1] + \frac{1}{25} * (4 * k1 + 6 * k2\right);$

$k4 := f\left(t[n-1]+h, x[n-1]+\frac{1}{4}*(k1-12*k2+15*k3)\right);$
$k5 := f\left(t[n-1]+\frac{2*h}{3}, x[n-1]+\frac{1}{81}*(6*k1+90*k2-50*k3+8*k4)\right);$
$k6 := f\left(t[n-1]+\frac{4*h}{5}, x[n-1]+\frac{1}{75}*(6*k1+36*k2+10*k3+8*k4)\right);$
$x[n] := x[n-1]+\frac{h}{192}*(23*k1+125*k3-81*k5+125*k6);$
od;

Step 3:
$for\ n\ from\ 1\ to\ 10\ do$
$t[n] := n*h$:
$k1 := f(t[n-1], y[n-1])$:
$k2 := f\left(t[n-1]+\frac{h}{3}, y[n-1]+\frac{h}{3}*k1\right)$:
$k3 := f\left(t[n-1]+\frac{2*h}{5}, y[n-1]+\frac{1}{25}*(4*k1+6*k2\right)$:
$k4 := f\left(t[n-1]+h, y[n-1]+\frac{1}{4}*(k1-12*k2+15*k3)\right)$:
$k5 := f\left(t[n-1]+\frac{2*h}{3}, y[n-1]+\frac{1}{81}*(6*k1+90*k2-50*k3+8*k4)\right)$:
$k6 := f\left(t[n-1]+\frac{4.h}{5}, y[n-1]+\frac{1}{75}*(6*k1+36*k2+10*k3+8*k4)\right)$:
$y[n] := y[n-1]+\frac{h}{192}*(23.k1+125*k3-81*k5+125*k6)$:
od;

Output (see Table 4, Table 5 and Table 6 for Case 1, Case 2 and Case 3 respectively)

We also obtain approximate solutions using New Iterative Method discussed in section 4.2 as follows:

Case 1: $\beta_1 = 0.001, \beta_2 = 0.005, \quad \beta_3 = 0.008$

$$X(t) = \begin{cases} 1 - 0.006t + 0.000018t^2 - 3.6 \times 10^{-8}t^3 + \\ 5.4 \times 10^{-11}t^4 + -6.48 \times 10^{-14}t^5 + \\ 6.48 \times 10^{-17}t^6 - 5.5543 \times 10^{-20}t^7 \end{cases} \tag{16}$$

$$Y(t) = \begin{cases} 1 - 0.003t - 0.000003t^2 + 3.8 \times 10^{-8}t^3 - \\ 1.21 \times 10^{-10}t^4 + 2.476 \times 10^{-13}t^5 \\ -3.8413 \times 10^{-16}t^6 + 4.853 \times 10^{-19}t^7 \end{cases} \tag{17}$$

Case 2: $\beta_1 = 0.007, \beta_2 = 0.035, \beta_3 = 0.056$

$$X(t) = \begin{cases} 1 - 0.042t + 0.000882t^2 - 1.2348 \times 10^{-5}t^3 + \\ 1.29654 \times 10^{-7}t^4 - 1.0891 \times 10^{-9}t^5 + \\ 7.62366 \times 10^{-12}t^6 - 4.574193 \times 10^{-14}t^7 \end{cases} \tag{18}$$

$$Y(t) = \begin{cases} 1 - 0.021t - 0.000147t^2 + 1.3034 \times 10^{-5}t^3 \\ -2.90521 \times 10^{-7}t^4 + 4.16141 \times 10^{-9}t^5 \\ -4.5192903 \times 10^{-11}t^6 + 3.996615 \times 10^{-13}t^7 \end{cases} \tag{19}$$

Case 3: $\beta_1 = 0.030, \beta_2 = 0.150, \beta_3 = 0.240$

$$X(t) = \begin{cases} 1 - 0.180t + 0.0162t^2 - 2.916 \times 10^{-3}t^3 + \\ 4.374 \times 10^{-5}t^4 - 1.57464 \times 10^{-6}t^5 \\ 4.72392 \times 10^{-8}t^6 - 1.214722286 \times 10^{-9}t^7 \end{cases} \tag{20}$$

$$Y(t) = \begin{cases} 1 - 0.090t - 0.0027t^2 + 1.026 \times 10^{-3}t^3 - \\ 9.8010 \times 10^{-5}t^4 + 6.061668 \times 10^{-6}t^5 \\ -2.800332 \times 10^{-7}t^6 + 1.061340686 \times 10^{-8}t^7 \end{cases} \quad (21)$$

5.1 Numerical results

Table 4, Table 5 and Table 6 show the decrease in cancer compartments A and B (X(t) and Y(t)) as the rate of β_1, β_2 and β_3 increases on a daily, weekly and monthly basis. The role and significant of relationship in increase the parameters values β_1, β_2 and β_3 are great reduction in cancer tumor within the compartment A and B. Thus, increases in values of β_1, β_2 and β_3 (daily, weekly and monthly) are responsible for the reductions in cancer tumor in the two compartments under study. Furthermore Figure 2, Figure 3 and Figure 4 depict the absolute errors performance which indicate that new iterative method (NIM) solutions are better and closer to analytical solutions compared to Runge-Kutta method (RK5).

Table 4. Metastatic cancer model case one (Daily)

t	Solution	$X(t)$	$Y(t)$
0	Analytical	1.000000000000000000000000000	1.000000000000000000000000000
	RK5	1.000000000000000000000000000	1.000000000000000000000000000
	NIM	1.000000000000000000000000000	1.000000000000000000000000000
0.1	Analytical	0.999400179964005399352064 79	0.999699970037987902475615 94
	RK5	0.999400179963962937186519 49	0.999699970038151415668501 36
	NIM	0.999400179964005397643267 88	0.999699970037987901345984 32
0.2	Analytical	0.998800719712086379268146 49	0.999399880303806479207421 76
	RK5	0.998800719711928823616801 50	0.999399880304413445510138 92
	NIM	0.998800719712086378764523 18	0.999399880303806477905432 78
0.3	Analytical	0.998201619028437242583227 06	0.999099731025020501388072 97
	RK5	0.998201619028109030845321 46	0.999099731026285465346640 13
	NIM	0.998201619028437241238795 46	0.999099731025020501184598 76
0.4	Analytical	0.997602877697381736713329 83	0.998799522428904933851384 76
	RK5	0.997602877696842609424407 16	0.998799522430983786423101 22
	NIM	0.997602877697381735498543 27	0.998799522428904932298756 34
0.5	Analytical	0.997004495503372976012066 23	0.998499254742445231501706 06
	RK5	0.997004495502596288904240 01	0.998499254745441655407988 18
	NIM	0.997004495503372963498187 43	0.998499254742445229742567 82
0.6	Analytical	0.996406472230993364173754 65	0.998198928192337635467451 65
	RK5	0.996406472229964477832853 38	0.998198928196309254393135 17
	NIM	0.996406472230993363425781 68	0.998198928192337635401567 43
0.7	Analytical	0.995808807664954516683083 41	0.997898543004989468979033 67
	RK5	0.995808807663669263756134 06	0.997898543009953700896746 58
	NIM	0.995808807664954516683217 89	0.997898543004989468978754 22
0.8	Analytical	0.995211501590097183311290 00	0.997598099406519432971438 18
	RK5	0.995211501588560413217821 21	0.997598099412459047848394 59
	NIM	0.995211501590097183311197 64	0.997598099406519432971397 64
0.9	Analytical	0.994614553791391170658828 74	0.997297597622757901411688 78
	RK5	0.994614553789615371759507 19	0.997297597629626283619021 84
	NIM	0.994614553791391170658754 26	0.997297597622757901411679 93
1.0	Analytical	0.994017964053935264744498 77	0.996997037879247216351442 37
	RK5	0.994017964051939263920636 88	0.996997037886973331990939 18
	NIM	0.994017964053935264744493 24	0.996997037879247216351441 99

Table 5. Metastatic cancer model case two (Weekly)

t	Solution	$X(t)$	$Y(t)$
0	Analytical	1.000000000000000000000000	1.000000000000000000000000
	RK5	1.000000000000000000000000	1.000000000000000000000000
	NIM	1.000000000000000000000000	1.000000000000000000000000
0.1	Analytical	0.995808807664954516683083 41	0.997898543004989468979033 60
	RK5	0.995808807663519804674550 44	0.997898543010532040009543 23
	NIM	0.995808807664954515643276 87	0.997898543004989467654398 73
0.2	Analytical	0.991635181423098377377050 28	0.995794223808495164986240 80
	RK5	0.991635181420111704706790 95	0.995794223820102021958736 30
	NIM	0.991635181423098376965873 21	0.995794223808495163765432 98
0.3	Analytical	0.987479047651556450208614 34	0.993687119574859275659724 81
	RK5	0.987479047648635720144088 75	0.993687119586349248787738 44
	NIM	0.987479047651556450167513 87	0.993687119574859274362542 87
0.4	Analytical	0.983340333036021233241207 35	0.991577306781090813885135 21
	RK5	0.983340333034366925746457 40	0.991577306787817703515794 70
	NIM	0.983340333036021233115638 97	0.991577306781090813043876 51
0.5	Analytical	0.979218964569459588189545 21	0.989464861221778634081534 13
	RK5	0.979218964568995450986555 33	0.989464861223956049241235 58
	NIM	0.979218964569459588178643 93	0.989464861221778634076645 32
0.6	Analytical	0.975114869550824894461936 68	0.987349858013972508181456 91
	RK5	0.975114869524698043403895 89	0.987349858117199286950576 07
	NIM	0.975114869550824894460654 37	0.987349858013972508181348 76
0.7	Analytical	0.971027975583774600812698 77	0.985232371602032458549405 34
	RK5	0.971027974876442989309507 30	0.985232374393744619104551 63
	NIM	0.971027975583774600812689 74	0.985232371602032458549405 16
0.8	Analytical	0.966958210575393151982256 55	0.983112475762446543902410 15
	RK5	0.966958208429883535413237 28	0.983112484250930196428985 31
	NIM	0.966958210575393151982256 12	0.983112475762446543902410 00
0.9	Analytical	0.962905502734920267797320 32	0.980990243608617293122618 00
	RK5	0.962905498598890976673863 82	0.980990260016270973498472 67
	NIM	0.962905502734920267797320 11	0.980990243608617293122617 23
1.0	Analytical	0.958869780572484552297950 42	0.978865747595616980685067 12
	RK5	0.958869774073906771038713 51	0.978865773450366681386591 61
	NIM	0.958869780572484552297950 12	0.978865747595616980685066 99

Table 6. Metastatic cancer model case three (Monthly)

t	Solution	$X(t)$	$Y(t)$
0	Analytical	1.000000000000000000000000	1.000000000000000000000000
	RK5	1.000000000000000000000000	1.000000000000000000000000
	NIM	1.000000000000000000000000	1.000000000000000000000000
0.1	Analytical	0.982161032358300718000538 30	0.990974016258887824738433 21
	RK5	0.982161032171135058400298 71	0.990974016999501833155910 36
	NIM	0.982161032358300718004873 21	0.990974016258887824729941 72
0.2	Analytical	0.964640293483123030043878 75	0.981900053091550458258692 02
	RK5	0.964640293340648556208897 17	0.981900053692596714412958 10
	NIM	0.964640293483123030032785 43	0.981900053091550458257948 72
0.3	Analytical	0.947432106501798299538115 93	0.972783922537687151225453 17
	RK5	0.947432106474819746863367 98	0.972783922695630285377446 45
	NIM	0.947432106501798299537653 87	0.972783922537687148952873 12
0.4	Analytical	0.930530895811205731746557 87	0.963631215424955103757584 67
	RK5	0.930530890246731686847228 68	0.963631237218055075890727 02
	NIM	0.930530895811205731745784 51	0.963631215424955103697265 71
0.5	Analytical	0.913931185271228186747353 55	0.954447308102334193577036 13
	RK5	0.913931169088936083355077 46	0.954447372344815068696129 57
	NIM	0.913931185271228186746947 24	0.954447308102334193569376 52

0.6	Analytical	0.897627596430434875689033 54	0.945237368987279663961738 59
	RK5	0.897627572890901106166208 38	0.945237464463294005853704 79
	NIM	0.897627596430434875688572 31	0.945237368987279663891267 32
0.7	Analytical	0.881614846783416046013655 13	0.936006364931552016209717 61
	RK5	0.881614822058114483846526 69	0.936006468536024670779752 01
	NIM	0.881614846783416046013594 72	0.936006364931552016208568 23
0.8	Analytical	0.865887748059205016840563 55	0.926759067410488962129468 78
	RK5	0.865887727444795589664984 43	0.926759158360619942556887 48
	NIM	0.865887748059205016840559 82	0.926759067410488962129467 35
0.9	Analytical	0.850441204540232998277187 64	0.917500058540363003317139 85
	RK5	0.850441190353895441593579 99	0.917500126569772795934044 20
	NIM	0.850441204540232998277175 63	0.917500058540363003317138 65
1.0	Analytical	0.835270211411272021312384 97	0.908233736928349939452335 32
	RK5	0.835270202537096702307358 55	0.908233784631256301326472 20
	NIM	0.835270211411272021312379 98	0.908233736928349939452334 56

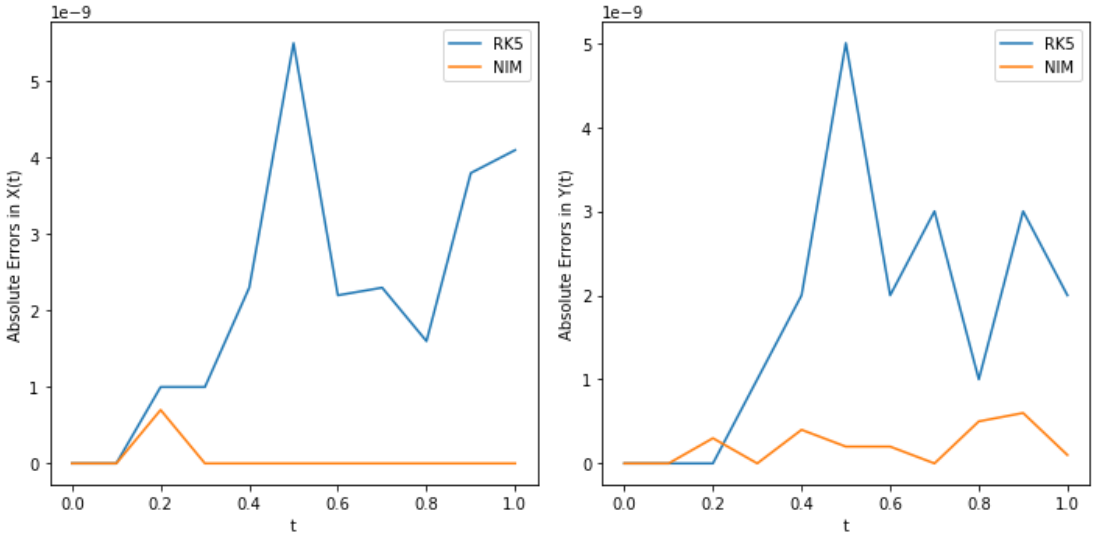

Figure 2. Depict the absolute error for Table 4 $X(t)$ and $Y(t)$)

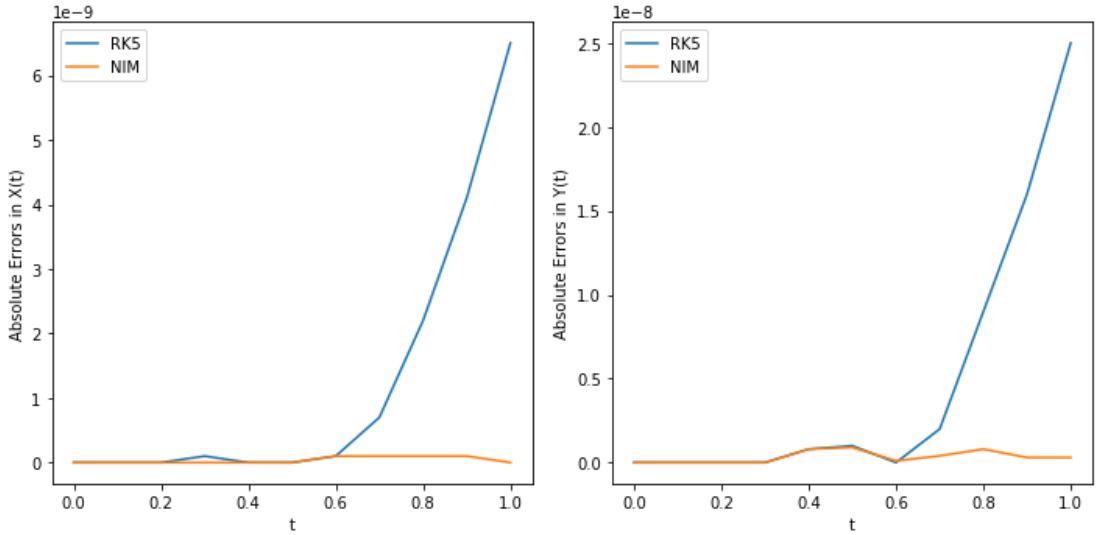

Figure 3. Depict the absolute error for Table 5 $X(t)$ and $Y(t)$)

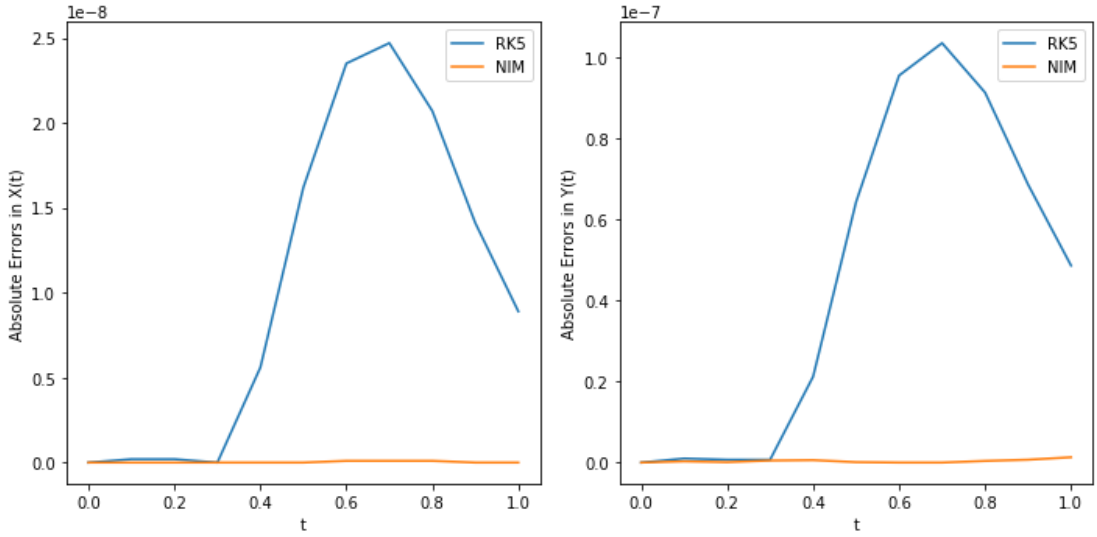

Figure 4. Depict the absolute error for Table 6 $X(t)$ and $Y(t)$)

6. Conclusion

This study compared the Runge-Kutta method of fifth-order (RK5) with New Iterative Method (NIM) for the numerical solutions of metastatic cancer models which occur in two compartments of cancer cells. A numerical solution obtained shows that increase in parameters β_1, β_2 and β_3 leads to decrease in spreading of cancer tumor in the two apartments. Finally, the results are more realistic compare with analytical solutions and from the computational viewpoint, the New Iterative Method (NIM) is easy to utilize and less error obtained (Figure 2, Figure 3 and Figure 4) compare to Runge-Kutta Method of fifth-order (RK5) method.

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