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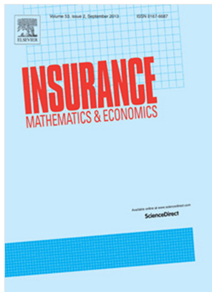

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Stochastic orders and multivariate measures of risk contagion

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Abstract

Co-risk measures and risk contributions measures are used in portfolio risk analysis to assess and quantify the risk of contagion, given that one or more assets in the portfolio are in distress. In this paper, given two random vectors $\mathbf{X}$ and $\mathbf{Y}$ that represent two portfolios of n assets ($n \geq 2$) and exhibit some kind of positive dependence, we give sufficient conditions based on stochastic orders to compare the risk of contagion of the portfolios. The measures of risk contagion that we consider are the conditional value at risk (CoVaR), the conditional expected shortfall (CoES) and the recently introduced marginal mean excess (MME).

JEL: G220

Keywords: risk measure, contagion risk, stochastic orders, CoVaR, CoES.

1 Introduction and background

In the last decades, especially since the 2007-2009 financial crisis, there has been an increasing interest in modelling the contagion risk (also referred to as systemic risk) caused by interconnections among financial institutions, markets or services. In portfolio risk analysis, the risk of contagion appears when the collapse of one or more components of the portfolio can eventually cause the collapse of other components, putting the entire portfolio at risk. From the point of view of risk measurement, the risk of contagion puts the focus on conditional rather than unconditional risk distributions. This makes it necessary to adapt the risk measures typically used in the financial industry, such as the value-at-risk (VaR) and the expected shortfall, to incorporate the effect of interaction.

Recall that, given a risk X with continuous distribution function F, its value at risk (or VaR) at level p (or p-quantile) is defined by

$$\mathrm{VaR}_p\left[X\right] = F^{-1}(p) = \inf\left\{x : F(x) \geq p\right\},\ p \in (0,1)$$

and its corresponding expected shortfall is defined by

$$\mathrm{ES}_p\left[X\right] = E\left[X|X > \mathrm{VaR}_p\left[X\right]\right].$$

The expected shortfall is a coherent risk measure in the sense of Artzner er al. (1999). Due to the continuity of F, the following identity holds:

$$\mathrm{ES}_p\left[X\right] = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_t\left[X\right]dt,\ p \in (0,1).$$

Let $\mathbf{X} = (X_1, ..., X_n)$ be a portfolio of risks (or random vector) with joint distribution function F and respective marginal distribution functions $F_1, F_2, ...F_n$, which we assume to be continuous with finite expectations. The joint distribution function F can be expressed as

$$F\left(x_1, ..., x_n\right) = C\left(F_1\left(x_1\right), ..., F_n\left(x_n\right)\right),$$

where C is the copula of $\mathbf{X}$, that is, the joint distribution function of the vector-copula $\left(F_1\left(X_1\right), ..., F_n\left(X_n\right)\right)$ (see Nelsen, 1999). The copula contains the information about the dependence of the random vector $(X_1, ..., X_n)$ apart from the behavior of the marginal distributions. We denote by $\overline{C}$ the joint tail function for n uniform random variables whose joint distribution function is the copula C, that is,

$$\overline{C}\left(p_1, ..., p_n\right) = P\left[X_1 > F_1^{-1}\left(p_1\right), ..., X_n > F_n^{-1}\left(p_n\right)\right].$$

We also use the notation $\overline{C}\left(\mathbf{p}\right)$, where $\mathbf{p} = (p_1, ...p_n) \in (0,1)^n$.

In this framework, we adopt the following dependence-adjusted version of the VaR, called CoVaR (see Girardi and Ergün, 2013).

Definition 1 *For* $\mathbf{p} = (p_1, ...p_n) \in (0,1)^n$ *we set*

$$\mathrm{CoVaR}_{\mathbf{p}}\left[X_1|X_2,...,X_n\right] = \mathrm{VaR}_{p_1}\left[X_1|\bigcap_{j=2}^{n}\left\{X_j > \mathrm{VaR}_{p_j}(X_j)\right\}\right]. \quad (1)$$

Sometimes, to avoid ambiguity, we shall denote the CoVaR in Definition 1 by $\mathrm{CoVaR}_{p_1,...,p_n}\left[X_1|X_2,...,X_n\right]$. In words, (1) is the VaR at level p_1 of the conditional distribution of X_1 given the joint event $\{X_2 > \mathrm{VaR}_{p_2}\left[X_2\right], ..., X_n > \mathrm{VaR}_{p_n}\left[X_n\right]\}$. Therefore, fixed $\mathbf{p}^* = (p_2, ...p_n) \in (0,1)^{n-1}$, (1) is increasing[1] in p_1. Fixed p_1, however, the monotonicity of (1) with respect to $\mathbf{p}^* = (p_2, ...p_n) \in (0,1)^{n-1}$ reveals a structure of dependence between X_1 and $(X_2, ..., X_n)$ known as right-tail increasing (decreasing). The formal definition of this notion is the following, see Joe (1997, page 22).

Definition 2 *Let* $\mathbf{X} = (X_1, ..., X_n)$ *be a random vector with tail function* $\bar{\mathbf{F}}$. *We say that* $X_i, i \in A^c$, *is right-tail increasing (decreasing) or RTI (RTD) in* $X_j, j \in A$, *if*

$$P\left(X_i > x_i, i \in A^c | X_j > x_j, j \in A\right)$$

increases (decreases) in $x_k, k \in A$, *where* A *is a non-empty subset of* $\{1, ..., n\}$ *and* A^c *is the complementary set of* A.

The risk contribution measure

$$\Delta\,\mathrm{CoVaR}_{\mathbf{p}}[X_1 \mid X_2, ..., X_n] = \mathrm{CoVaR}_{\mathbf{p}}[X_1 \mid X_2, ..., X_n] - \mathrm{VaR}_{p_1}[X_1]$$

quantifies how a joint stress situation for the components $X_2, ..., X_n$ affects X_1 by comparing $\mathrm{CoVaR}_{\mathbf{p}}[X_1 \mid X_2, ..., X_n]$, which represents the risk of X_1 when is affected by the joint behavior of $(X_2, ..., X_n)$, with $\mathrm{VaR}_{p_1}[X_1]$, which is the risk of X_1 when considered in isolation.

The expected shortfall is also generalized to incorporate dependence (see Mainik and Schaanning, 2014).

Definition 3 *For* $\mathbf{p} = (p_1, ...p_n) \in (0,1)^n$ *we set*

$$\begin{aligned}&\mathrm{CoES}_{\mathbf{p}}\left[X_1|X_2,...,X_n\right] \\ &= \frac{1}{1-p_1}\int_{p_1}^{1}\mathrm{CoVaR}_{t,p_2,...,p_n}\left[X_1|X_2,...,X_n\right]dt \quad (2)\end{aligned}$$

[1] Here and throughout the paper, increasing stands for non-decreasing.

Clearly, fixed $\mathbf{p}^* = (p_2, ...p_n) \in (0,1)^{n-1}$, (2) is increasing with respect to p_1 and fixed p_1, (2) inherits the monotonicity of CoVaR. Therefore, fixed p_1, if X_1 is RTI (RTD) in $(X_2, ..., X_n)$, then (2) increases (decreases) with respect to $\mathbf{p}^* = (p_2, ...p_n) \in (0,1)^{n-1}$.

Risk contributions can be measured in terms of CoES by using

$$\Delta\,\mathrm{CoES}_{\mathbf{p}}[X_1 \mid X_2, ..., X_n] = \mathrm{CoES}_{\mathbf{p}}[X_1 \mid X_2, ..., X_n] - \mathrm{ES}_{p_1}[X_1].$$

Other dependence-adjusted extensions of VaR and ES can be found in Adrian and Brunnermeier (2016). Some other measures have been recently introduced to manage and quantify the impact of risk contagion. We consider in this paper the marginal mean excess (MME) measure introduced by Das and Fasen-Hartmann (2018a).

Definition 4 *Assume $E|X_1| < \infty$. For $\mathbf{p}^* = (p_2, ...p_n) \in (0,1)^{n-1}$ we set*

$$\mathrm{MME}_{\mathbf{p}^*}[X_1|X_2, ..., X_n] = E\left[(X_1 - A_{\mathbf{X},\mathbf{p}^*})_+ | \bigcap_{j=2}^{n} \{X_j > \mathrm{VaR}_{p_j}(X_j)\}\right]$$

with $A_{\mathbf{X},\mathbf{p}^*} = a_1 \mathrm{VaR}_{p_2}(X_2) + ... + a_{n-1}\mathrm{VaR}_{p_n}(X_n)$, where $a_1, ..., a_{n-1}$ are given positive constants, such that $\sum_{i=1}^{n-1} a_i = 1$.

For $n = 2$ and $p \in (0,1)$, the MME given in Definition 4 reduces to

$$\mathrm{MME}_p[X_1|X_2] = E\left[(X_1 - \mathrm{VaR}_p(X_2))_+ | X_2 > \mathrm{VaR}_p(X_2)\right], \quad p \in (0,1),$$

as defined originally by Das and Fasen-Hartmann (2018a). In this case, we interpret the MME as the expected excess of the risk X_1 over the VaR of X_2 at level p, given that X_2 is greater than its VaR. The extension given in Definition 4 to the case $n > 2$, which is taken from Hashorva (2019), represents the expected excess of the risk X_1 over a linear combination of the values at risks $\mathrm{VaR}_{p_2}(X_2), ..., \mathrm{VaR}_{p_n}(X_n)$ under a stress scenario for the other risks, given by

$$\{X_2 > \mathrm{VaR}_{p_2}(X_2), ..., X_n > \mathrm{VaR}_{p_n}(X_n)\}.$$

The contribution of $(X_2, ..., X_n)$ to the expected excess of X_1 over the linear combination $A_{\mathbf{X},\mathbf{p}^*}$ can be quantified by:

$$\begin{aligned} &\Delta\mathrm{MME}_{\mathbf{p}^*}[X_1|X_2, ..., X_n] \qquad (3)\\ &= E\left[(X_1 - A_{\mathbf{X},\mathbf{p}^*})_+ | \bigcap_{j=2}^{n} \{X_j > \mathrm{VaR}_{p_j}(X_j)\}\right] - E\left[(X_1 - A_{\mathbf{X},\mathbf{p}^*})_+\right].\end{aligned}$$

The study of CoVaR, CoES and MME has attracted growing attention among researchers, specially in the copula framework. To mention only some of the most recent works: Bernardi et al. (2017) provide a directory of CoVaR values for different families of copulas, Jaworski (2017) deals with CoVaR for copulas with tail dependence, Karimalis and Nomikos (2017) use dynamic copula models to estimate the CoVaR and the CoES, Choi and Shin (2019) propose a new method for estimation of CoVaR, and Bernardi et al. (2018) and Bernardi and Catania (2019) provide applications in different contexts. Regarding the marginal mean excess, Das and Fasen-Hartmann (2018a,2018b) study its asymptotic behavior under hidden regular variation assumptions and Hashorva (2019) provides approximations for Gaussian random vectors. Some generalizations of MME can be found in Ling (2019).

An important reason for measuring the risk of contagion is to make comparisons and express preferences among different portfolios. However, given two portfolios $\mathbf{X}$ and $\mathbf{Y}$, there are several issues that complicate the process of decision-making based on the direct comparisons of CoVaR, CoES and MME. One issue is that, in general, these measures cannot be calculated explicitly since they depend on quantiles which sometimes do not have closed forms. Another issue is the frequent lack of consensus about the vector $\mathbf{p}$ of probabilities that defines the stress scenario. One possibility to overcome these issues is to find conditions (that do not require the explicit expression of the underlying measures) to ensure that

$$\Phi_{\mathbf{p}}\left[X_1|X_2,...,X_n\right] \leq \Phi_{\mathbf{p}}\left[Y_1|Y_2,...,Y_n\right], \quad \text{for all } \mathbf{p} \in \Omega, \tag{4}$$

where Φ stands for CoVaR, CoES and MME and $\Omega = (0,1)^n$ (in the case of CoVaR and CoES) or $\Omega = (0,1)^{n-1}$ (in the case of MME). In this paper, we give sufficient conditions on the vectors $\mathbf{X}$ and $\mathbf{Y}$, in terms of well-known stochastic orders and dependence notions, under which (4) holds. Regarding CoVaR and CoES, our results complement and extend those obtained by Sordo et al. (2018), who derived sufficient conditions for (4) in the bivariate case. The results connecting MME with stochastic orders are novel and offer an alternative to the approach of Hashorva (2019) and Das and Fasen-Hartmann (2018a, 2018b), who derived approximations of MME under different asymptotic conditions. Other recent works that provide sufficient conditions based on stochastic orders for the comparisons of random vectors under different co-risk measures include Fang and Li (2018), Navarro and Sordo (2018), Dhaene et al. (2019) and Longobardi and Pellerey (2019).

We recall the definitions of the univariate stochastic orders considered in this paper.

Definition 5 *Let X and Y be two random variables with respective distribution functions F_X and F_Y and tail functions $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$, respectively. Then, X is said to be smaller than Y :*
(i) in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(t) \leq \bar{G}(t)$, for all t.
(ii) in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\overline{F}_Y(t)/\overline{F}_X(t)$ increases in t,
(iii) in the increasing convex order (denoted by $X \leq_{icx} Y$) if $\int_t^\infty \overline{F}_X(x)dx \leq \int_t^\infty \overline{F}_Y(x)dx, \ \forall t$,
(iv) in the increasing concave order (denoted by $X \leq_{icv} Y$) if $\int_{-\infty}^t F_X(x)dx \geq \int_{-\infty}^t F_Y(x)dx, \ \forall t$,
(v) in the dispersive order (denoted by $X \leq_{disp} Y$) if $F^{-1}(p) - F^{-1}(q) \leq G^{-1}(p) - G^{-1}(q)$ for all $0 \leq q < p \leq 1$.
(vi) in the excess wealth order (denoted by $X \leq_{ew} Y$) if $\int_{F_X^{-1}(p)}^\infty \overline{F}_X(x)dx \leq \int_{F_Y^{-1}(p)}^\infty \overline{F}_Y(x)dx, \ \forall p \in (0,1)$.

It is well-known (see Shaked and Shanthikumar (2007)) that

$$X \leq_{hr} Y \Longrightarrow X \leq_{st} Y \Longrightarrow X \leq_{icx(icv)} Y \tag{5}$$

and

$$X \leq_{disp} Y \Longrightarrow X \leq_{ew} Y$$

but the converses are false. It is also well-known (see Lemma 2.1 in Sordo and Ramos, 2007) that

$$X \leq_{icx} Y \text{ if and only if } \mathrm{ES}_p[X] \leq \mathrm{ES}_p[Y], \forall p \in (0,1), \tag{6}$$

where $\mathrm{ES}_p(X)$ is the expected shortfall. Classical references for these orders are the books by Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

The following multivariate stochastic order introduced by Hu et al. (2003) plays an important role in this paper.

Definition 6 *Let $\mathbf{X} = (X_1, ..., X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ be two random vectors with respective joint survival functions $\bar{F}$ and $\bar{G}$, defined, respectively, by $\bar{F}(\mathbf{x}) = P[\mathbf{X} > \mathbf{x}]$ and $\bar{G}(\mathbf{x}) = P[\mathbf{Y} > \mathbf{x}]$, for $\mathbf{x} \in \mathbb{R}^n$. We say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the weak multivariate hazard rate order (denoted by $\mathbf{X} \leq_{whr} \mathbf{Y}$) if*

$$\frac{\bar{G}(\mathbf{x})}{\bar{F}(\mathbf{x})} \quad \text{is increasing in } \mathbf{x} \in \left\{\mathbf{x} : \bar{\mathbf{F}}(\mathbf{x}) > 0\right\}.$$

In the literature on stochastic orders, given two random vectors $\mathbf{X}$ and $\mathbf{Y}$ ordered stochastically by certain order $\leq_*$, one uses equivalently the notations $\mathbf{X} \leq_* \mathbf{Y}$ and $F \leq_* G$, where F and G are the respective distribution functions of $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we will use the latter notation, for consistency with the literature, when we compare copulas. We recall the definition of concordance order (Definition 2.8.1 in Nelsen, 1999).

Definition 7 *Given two n-dimensional copulas C and C', we say that C is smaller than C' in the concordance order (and write $C \leq_c C'$) if $C(\mathbf{p}) \leq C'(\mathbf{p})$ for all $\mathbf{p} \in (0,1)^n$.*

According to Definition 6, we will write $C \leq_{whr} C'$ if

$$\frac{\bar{C}'(\mathbf{p})}{\bar{C}(\mathbf{p})} \text{ is increasing in } \mathbf{p} \in \left\{\mathbf{p} \in [0,1]^n : \bar{C}(\mathbf{p}) > 0\right\}.$$

It is easy to see, see section 3 in Hu et al. (2003), that

$$C \leq_{whr} C' \text{ implies } C \leq_c C'.$$

These orders are partial orders because not every pair of copulas are comparable (in fact, all the stochastic orders considered in this paper are partial orders).

Besides RTI, we will use other notions that formalize the idea of positive dependence of random vectors, which intuitively means that large values of one component are associated with large values for the others (see Barlow and Proschan (1975), Block et al. (1985) and Joe (1997) for properties and applications).

Definition 8 *Let $\mathbf{X} = (X_1, ..., X_n)$ be a random vector with tail function $\bar{\mathbf{F}}$.*
(i) If $\mathbf{X}$ has a density function $\mathbf{f}$, $\mathbf{X}$ is said to be multivariate totally positive of order 2 (or MTP$_2$) if $\mathbf{f}(\mathbf{x} \vee \mathbf{y})\mathbf{f}(\mathbf{x} \wedge \mathbf{y}) \geq \mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathrm{R}^n$ where $\vee$ represents the maximum and $\wedge$ the minimum.
(ii) $(X_1, ..., X_{i-1}, X_{i+1}, .., X_n)$ is said to be stochastically increasing in X_i (denoted $(X_1, ..., X_{i-1}, X_{i+1}, .., X_n) \uparrow_{SI} X_i$) if the conditional distribution $\{(X_1, ..., X_{i-1}, X_{i+1}, .., X_n | X_i = x_i)\}$ is stochastically increasing as x_i increases.
(iii) $\mathbf{X}$ is said to be positive dependent through the stochastic order (or PDS) if $(X_1, ..., X_{i-1}, X_{i+1}, .., X_n) \uparrow_{SI} X_i$ for $i = 1, .., n$.

It is well-known that a continuous random vector $\mathbf{X}$ has the property MTP2 (resp. SI, RTI) if and only if its copula is MTP2 (resp. SI, RTI) (see Theorem 3.10.19 in Müller and Stoyan (2002) and the discussion on this issue in Cai and Wei (2012)).

The rest of the paper is organized as follows. In sections 2, we give sufficient conditions for comparisons of CoVaR, CoES and their associated risk contribution measures. The results in this section extend to the multivariate setting those obtained by Sordo et al. (2018) for the bivariate case. Given the random vectors $\mathbf{X}$ and $\mathbf{Y}$, under the hypothesis that at least one of the vectors satisfies a particular property of positive dependence, the conditions are either given in terms of the order $\leq_{whr}$ of the vectors or in terms of the order $\leq_{whr}$ of their copulas and the order $\leq_{icx}$ or $\leq_{st}$ of their components. In Section 3, we give sufficient conditions for comparisons of the form (4), where Φ is MME or its associated risk contribution measure (4). In this case, the conditions are given in terms of the order $\leq_{whr}$ of their copulas and a particular variability order of their components. Sections 2 and 3 contain several examples based on multivariate parametric families that are relevant in actuarial science. In Section 4 we provide an empirical illustration in the context of Spanish banking sector. Section 5 contains conclusions.

2 Sufficient conditions for the comparisons of CoVaRs and CoESs

In the remainder, we will make repeatedly use of the following assumption, which, to improve the readability of the paper, will be referred as (A).

Assumption (A): $\mathbf{X} = (X_1, ..., X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ are two random vectors with absolutely continuous joint distribution functions F and G, marginal distribution functions $F_1, F_2, ..., F_n$ and $G_1, G_2, ..., G_n$ and copulas C and C' respectively.

Given two random vectors, our first result shows the relationships between the orderings of their respective CoVaRs, CoESs and MMEs.

Theorem 9 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying (A) and let $\mathbf{p} = (p_1, p_2, .., p_n) \in [0,1]^n$ and $\mathbf{p}^* = (p_2, p_3, .., p_n) \in [0,1]^{n-1}$.*
(i) If

$$\mathrm{CoVaR}_{\mathbf{p}}\left[X_1|X_2, ..., X_n\right] \leq \mathrm{CoVaR}_{\mathbf{p}}\left[Y_1|Y_2, ..., Y_n\right] \tag{7}$$

then

$$\mathrm{CoES}_{\mathbf{p}}\left[X_1|X_2, ..., X_n\right] \leq \mathrm{CoES}_{\mathbf{p}}\left[Y_1|Y_2, ..., Y_n\right]. \tag{8}$$

(ii) If (8) holds and $X_i \geq_{st} Y_i$ for all $i = 2, .., n$ (in particular, if $\mathbf{X}$ and $\mathbf{Y}$ have the same marginals) then

$$\mathrm{MME}_{\mathbf{p}^*}\left[X_1|X_2, ..., X_n\right] \leq \mathrm{MME}_{\mathbf{p}^*}\left[Y_1|Y_2, ..., Y_n\right]. \tag{9}$$

Proof. Part (i) is obvious from Definition 3. In order two prove (ii), note that $\mathrm{CoES}_{\mathbf{p}}[X_1|X_2,...,X_n]$ is the expected shortfall of the conditional random variable $\left\{X_1|\bigcap_{j\neq 1}\left\{X_j > F_j^{-1}(p_j)\right\}\right\}$. Therefore, it follows from (6) that (8) is the same as

$$\left\{X_1|\bigcap_{i=2}^{n}\{X_j > F_j^{-1}(p_j)\}\right\} \leq_{icx} \left\{Y_1|\bigcap_{i=2}^{n}\{Y_j > G_j^{-1}(p_j)\}\right\}. \tag{10}$$

Now observe that the assumption $X_i \geq_{st} Y_i$ for all $i = 2,..,n$ implies $A_{\mathbf{X},\mathbf{p}^*} \geq A_{\mathbf{Y},\mathbf{p}^*}$ (see Definition 4). Therefore, using that $Z \leq_{icx} Z + k$ for any random variable Z and for all $k > 0$, we have

$$\left\{Y_1|\bigcap_{i=2}^{n}\{Y_i > G_i^{-1}(p_i)\}\right\} \leq_{icx} \left\{Y_1 + A_{X,\mathbf{p}^*} - A_{Y,\mathbf{p}^*}|\bigcap_{i=2}^{n}\{Y_i > G_i^{-1}(p_i)\}\right\} \tag{11}$$

From (10), (11) and the transitivity of the increasing convex order we get

$$\left\{X_1|\bigcap_{i=2}^{n}\{X_i > F_i^{-1}(p_i)\}\right\} \leq_{icx} \left\{Y_1 + A_{X,\mathbf{p}^*} - A_{Y,\mathbf{p}^*}|\bigcap_{i=2}^{n}\{Y_i > G_i^{-1}(p_i)\}\right\} \tag{12}$$

Since $Z_1 \leq_{icx} Z_2$ implies $E[\phi(Z_1)] \leq E[\phi(Z_2)]$ for all increasing and convex function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ (see Chapter 4 in Shaked and Shanthikumar, 2007) (9) follows from (12) by taking $\phi(x) = (x - A_{\mathbf{X},\mathbf{p}^*})^+$. ■

Given two bivariate random vectors with the same marginals, Mainik and Schaaning (2014) prove that the concordance order is a sufficient condition for the ordering of the respective CoVaRs. The following theorem is an extension of this result to the case of multivariate random vectors with possibly different marginals.

Theorem 10 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A). If* $\mathbf{X} \leq_{whr} \mathbf{Y}$ *and* X_1 *is RTI in* X_j *(or* Y_1 *is RTI in* Y_j*)* $j = 2,...,n$, *then (7) holds for all* $\mathbf{p} \in (0,1)^n$.

Proof. Let $\mathbf{X} \leq_{whr} \mathbf{Y}$ be and assume that X_1 is RTI in $X_j, j = 2,...,n$. On the one hand, it follows from Theorem 2.5 of Hu et al. (2001) and from the relationship (5) between the hazard rate order and the usual stochastic order, that

$$\left\{X_1|\bigcap_{j=2}^{n}\{X_j > x_j\}\right\} \leq_{st} \left\{Y_1|\bigcap_{j=2}^{n}\{Y_j > x_j\}\right\}, \ x_j \in \mathbb{R}, \quad j = 2,...,n,$$

or, equivalently,

$$\left\{X_1 | \bigcap_{j=2}^{n} \{X_j > G_j^{-1}(p_j)\}\right\} \le_{st} \left\{Y_1 | \bigcap_{j=2}^{n} \{Y_j > G_j^{-1}(p_j)\}\right\}, 0 \le p_j \le 1. \tag{13}$$

On the other hand, $\mathbf{X} \le_{whr} \mathbf{Y}$ implies $X_i \le_{st} Y_i$ for $i = 1, ..., n$ (Hu et al., 2001) or, equivalently, $F_j^{-1}(p_j) \le G_j^{-1}(p_j)$ for $0 \le p_j \le 1$. Using that X_1 is RTI in X_j, $j = 2, ..., n$, it follows

$$\left\{X_1 | \bigcap_{j=2}^{n} \{X_j > F_j^{-1}(p_j)\}\right\} \le_{st} \left\{X_1 | \bigcap_{j=2}^{n} \{X_j > G_j^{-1}(p_j)\}\right\}, 0 \le p_j \le 1. \tag{14}$$

Combining (13) and (14) we get

$$\left\{X_1 | \bigcap_{j=2}^{n} \{X_j > F_j^{-1}(p_j)\}\right\} \le_{st} \left\{Y_1 | \bigcap_{j=2}^{n} \{Y_j > G_j^{-1}(p_j)\}\right\}, \ 0 \le p_j \le 1,$$

which is equivalent to (7) for all $\mathbf{p} \in (0,1)^n$. The proof is similar when Y_1 is RTI in Y_j, $j = 2, ..., n,$. ■

Regarding marginal risk contributions, we have the following result, which extends Theorem 15 in Sordo et al. (2018).

Theorem 11 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) such that* $C \le_{whr} C'$, $(X_2, ..., X_n) \uparrow_{SI} X_1$, *(or* $(Y_2, ..., Y_n) \uparrow_{SI} Y_1$*) and* $X_1 \le_{disp} Y_1$ *then*

$$\Delta\, \mathrm{CoVaR}_{\mathbf{p}}[X_1 \mid X_2, ..., X_n] \le \Delta\, \mathrm{CoVaR}_{\mathbf{p}}[Y_1 \mid Y_2, ..., Y_n]$$

holds for all $\mathbf{p} = (p_1, ..., p_n) \in (0,1)^n$.

Proof. The proof is similar to the proof of Theorem 15 in Sordo et al. (2008) with the difference that we must use the distortion function

$$h_C(u) = \frac{\bar{C}(1-u, p_2, ..., p_n)}{\bar{C}(0, p_2, ..., p_n)}, \ u \in [0,1]. \tag{15}$$

■

Looking again at the proof of Theorem 10, we easily see that, under equal marginals, the RTI condition is no longer necessary since $F_i^{-1}(p_i) = G_i^{-1}(p_i)$ for $i = 1, ..., n$. Moreover, in this case, (13) is equivalent to

$$\frac{\bar{C}(p_1, ..., p_n)}{\bar{C}(0, p_2, ..., p_n)} \le \frac{\bar{C}'(p_1, ..., p_n)}{\bar{C}'(0, p_2, ..., p_n)}, \ p_i \in [0,1],$$

which is implied by the stronger condition $C \le_{whr} C'$. This observation together with Theorem 9 proves the following result.

Corollary 12 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying (A) such that $X_i =_{st} Y_i$ for all $i = 1, .., n$. If $\mathbf{X} \leq_{whr} \mathbf{Y}$ or $C \leq_{whr} C'$, then (7) and (8) hold for all $\mathbf{p} \in (0,1)^n$ and (9) holds for all $\mathbf{p}^* \in (0,1)^{n-1}$.*

In the case of bivariate random vectors, the condition $C \leq_{whr} C'$ in Corollary 12 can be weakened to the concordance order of the corresponding copulas since, under the the assumption of equally distributed components, $C(p_1, p_2) \leq C'(p_1, p_2)$ is equivalent to

$$\mathrm{CoVaR}_{\mathbf{p}}\left[X_1|X_2\right] \leq \mathrm{CoVaR}_{\mathbf{p}}\left[Y_1|Y_2\right], \ \mathbf{p} = (p_1, p_2).$$

We illustrate the above results by providing several examples based on parametric families of distributions.

Example 13 (Multivariate Pareto Distribution) *A random vector $\mathbf{X} = (X_1, ..., X_n)$ follows a multivariate Pareto distribution $\mathbb{P}_n(\alpha, a)$, with $\alpha > 0, a > 0$, if its joint tail function is given by*

$$\bar{F}(x_1, .., x_n) = \left(\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)\right)^{-a}, \ x_i \geq \alpha > 0, \ a > 0.$$

It can be checked that $\mathbf{X}$ has identically distributed Pareto marginals (that is, $\bar{F}_1(x_i) = (\alpha/x_i)^a, i = 1, ..., n$) linked by a Clayton-Oakes survival copula (see Arias-Nicolas et al., 2009). Since, given $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, the conditional tail function

$$\begin{aligned}\bar{F}_{\left\{X_1|\bigcap_{j=2}^{n}\{X_j > x_j\}\right\}}(x_1) &= \left(\frac{\sum_{i=2}^{n} \frac{x_i}{\alpha} - (n-1)}{\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)}\right)^{a} \\ &= \left(1 - \frac{\frac{x_1}{\alpha}}{\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)}\right)^{a}\end{aligned}$$

is increasing in x_i for $i = 2, 3.., n$, X_1 is RTI in $\{X_j, j = 2, .., n\}$. Let $\mathbf{Y} = (Y_1, ..., Y_n)$ be another random vector that follows a multivariate Pareto distribution $\mathbb{P}_n(\gamma, b)$ with $\gamma > 0, b > 0$. Now it will be shown that if $\alpha \leq \gamma$ and $a \geq b$, then $\mathbf{X} \leq_{whr} \mathbf{Y}$. In order to prove it, we consider a third random vector $\mathbf{Z} = (Z_1, ..., Z_n) \sim \mathbb{P}_n(\alpha, b)$, with joint tail function $\bar{H}$. Then,

$$\frac{\bar{H}(x_1, .., x_n)}{\bar{F}(x_1, .., x_n)} = \frac{\left(\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)\right)^{-b}}{\left(\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)\right)^{-a}} = \left(\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)\right)^{a-b}$$

which is increasing in x_i, for $i = 2, 3.., n$, that is, $\mathbf{X} \leq_{whr} \mathbf{Z}$. Similarly,

$$\frac{\bar{G}(x_1, .., x_n)}{\bar{H}(x_1, .., x_n)} = \frac{\left(\sum_{i=1}^{n} \frac{x_i}{\gamma} - (n-1)\right)^{-b}}{\left(\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)\right)^{-b}} = \left(\frac{\sum_{i=1}^{n} \frac{x_i}{\gamma} - (n-1)}{\sum_{i=1}^{n} \frac{x_i}{\alpha} - (n-1)}\right)^{-b}. \tag{16}$$

To study the monotony of (16) we write, for $i = 2, ..., n$,

$$\frac{d}{dx_i}\left(\frac{\bar{G}(x_1,..,x_n)}{\bar{H}(x_1,..,x_n)}\right) = -b \cdot f(x_1,..,x_n)^{-b-1} \cdot \frac{df(x_1,..,x_n)}{dx_i} \tag{17}$$

where

$$f(x_1,..,x_n) = \left(\frac{\sum_{i=1}^n \frac{x_i}{\gamma} - (n-1)}{\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1)}\right).$$

It can be checked that

$$\frac{df(x_1,..,x_n)}{dx_i} = \frac{(\alpha-\gamma)(n-1)}{\alpha\,\gamma\,\left(\sum_{i=1}^n \frac{x_i}{\alpha} - (n-1)\right)^2}$$

which is negative if and only if $\alpha \leq \gamma$. From this fact and (17) it follows that (16) is increasing, that is, $\mathbf{Z} \leq_{whr} \mathbf{Y}$. By the transitivity of the order $\leq_{whr}$, we have that if $\alpha \leq \gamma$ and $b \leq a$, then $\mathbf{X} \leq_{whr} \mathbf{Y}$. It follows from Theorem 10 that (7) holds for all $\mathbf{p} \in (0,1)^n$. This example is illustrated in the bivariate case in Figure 1.

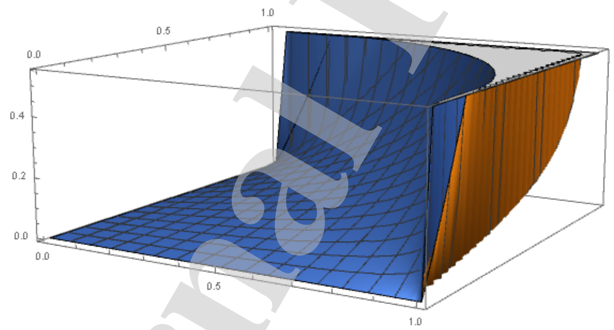

Figure 1: $\mathrm{CoVaR}_{p_1,p_2}[X_1|X_2]$ as function of p_1 and p_2. The orange surface corresponds to the CoVaR of $(X_1, X_2) \sim \mathbb{P}_2(1,6)$. The blue one corresponds to the CoVaR of $(Y_1, Y_2) \sim \mathbb{P}_2(3,4)$. Here $\mathbf{X} \leq_{whr} \mathbf{Y}$.

Example 14 (Multivariate Gumbel Exponential Distributions) *Given $\lambda = \{\lambda_I : I \subseteq \{1,..,n\}, \lambda_I \geq 0, I \neq \emptyset\}$, a random vector $\mathbf{X}_\lambda = (X_1, ..., X_n)$ follows a multivariate Gumbel Exponential distribution (see Kotz et al., 2000) if its joint tail function is given by*

$$\bar{F}_\lambda(x_1, x_2, ..., x_n) = exp\left\{-\sum_I \lambda_I \prod_{i\in I} x_i\right\},\ x_i \geq 0, \quad i = 1, ..., n.$$

Let $\mathbf{X}_\lambda = (X_1, ..., X_n)$ and $\mathbf{Y}_{\lambda^*} = (Y_1, ..., Y_n)$ be two multivariate Gumbel Exponential vectors such that $X_i =_{st} Y_i$ for all $i = 1, .., n$ (that is, $\lambda_i =$

λ_i^*). Under this assumption, Khaledi and Kochar (2005) show that $\lambda \geq \lambda^*$ implies $\mathbf{X} \leq_{whr} \mathbf{Y}$. It follows from Corollary 12 that (7) and (8) hold for all $\mathbf{p} \in (0,1)^n$ and (9) holds for all $\mathbf{p}^* \in (0,1)^{n-1}$. This example is illustrated in Figure 2.

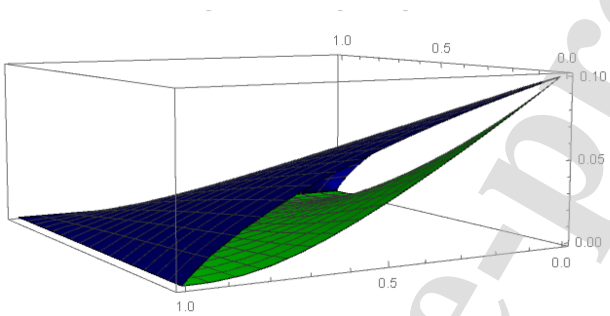

Figure 2: $\mathrm{MME}_{(p_2,p_3)}[X_1|X_2,X_3]$ as function of p_2 and p_3. The green surface corresponds to the MME of (X_1,X_2,X_3) a multivariate Gumbel Exponential Distribution with parameters $\lambda = \{\lambda_I = 10, I \subseteq \{1,..,n\}, I \neq \emptyset\}$. The blue one corresponds to the $\mathrm{MME}_{(p_2,p_3)}[Y_1|Y_2,Y_3]$, where $\mathbf{Y} = (Y_1,Y_2,Y_3)$ is a multivariate Gumbel Exponential Distribution with parameters $\lambda^* = \{\lambda_i^* = 10 \text{ for } i = 1,2,3, \text{ and } \lambda_{12}^* = \lambda_{13}^* = \lambda_{23}^* = \lambda_{123}^* = 100\}$.

Intuitively, a random vector $\mathbf{X} = (X_1,...,X_n)$ with a positive dependence structure among the components exhibits more risk of contagion when compared to a random vector with the same marginals but independent components. The following example formalizes this intuition.

Example 15 (MTP$_2$ Distributions) *Hu et al. (2003) proved that if $\mathbf{X} = (X_1,...,X_n)$ is MTP$_2$ and $\mathbf{Y} = (Y_1,...,Y_n)$ is a random vector with independent components such that $X_i =_{st} Y_i$ for all $i = 1,..,n$, then $\mathbf{Y} \leq_{whr} \mathbf{X}$. It follows from Corollary 12 that (7) and (8) hold for all $\mathbf{p} \in (0,1)^n$ and (9) holds for all $\mathbf{p}^* \in (0,1)^{n-1}$.*

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors satisfying (A). For $n = 2$, Sordo et al. (2018) showed that under SI dependence, the conditions $X_1 \leq_{icx} Y_1$ and $C \leq_c C'$ imply $\mathrm{CoES}_{p_1,p_2}[X_1|X_2] \leq \mathrm{CoES}_{p_1,p_2}[Y_1|Y_2]$ for $0 \leq p_i \leq 1, i = 1,2$. The next result is an extension (under the stronger assumption $C \leq_{whr} C'$) to the general multivariate case. In order to prove it, we recall the notion of distortion function. A distortion function h is a non-decreasing function from $[0,1]$ to $[0,1]$ such that $h(0) = 0$ and $h(1) = 1$. Given a random variable Z with tail function $\overline{F}_Z$ and a continuous distortion function h, the transformation $h\left(\overline{F}_Z(x)\right) = h \circ \overline{F}_Z(x)$ defines a new tail function associated to certain random variable Z_h, which is called distorted random variable.

Theorem 16 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A). If* $X_1 \leq_{icx} Y_1, C \leq_{whr} C'$ *and* $(X_2, ..., X_n) \uparrow_{SI} X_1$, *(or* $(Y_2, ..., Y_n) \uparrow_{SI} Y_1$*) then (8) holds for all* $\mathbf{p} \in (0,1)^n$.

Proof. Suppose $(X_2, ..., X_n) \uparrow_{SI} X_1$ (the proof of the case $(Y_2, ..., Y_n) \uparrow_{SI} Y_1$ is similar). It follows from theorems 2 and 4 in Sordo et al. (2015) that

$$X_{1,h_C} = \left\{ X_1 | \bigcap_{i=2}^{n} \{X_i > F_i^{-1}(p_i)\} \right\}, \ 0 \leq p_i \leq 1,$$

is a distorted random variable induced from X_1 by the concave distortion function

$$h_C(u) = \frac{\bar{C}(1-u, p_2, ..., p_n)}{\bar{C}(0, p_2, ..., p_n)}, \ u \in [0,1]. \tag{18}$$

Similarly, given a random vector $\hat{\mathbf{Y}} = (\hat{Y}_1, ..., \hat{Y}_n)$ with copula C and such that $\hat{Y}_i =_{st} Y_i, i = 1, ..., n$, we have that

$$\hat{Y}_{1,h_C} = \left\{ \hat{Y}_1 | \bigcap_{i=2}^{n} \left\{ \hat{Y}_i > G_i^{-1}(p_i) \right\} \right\}, \ 0 \leq p_i \leq 1,$$

is a distorted random variable induced from $\hat{Y}_1$ (or by Y_1, because $\hat{Y}_1$ and Y_1 have the same distribution function) by the distortion function (18). Therefore, since $X_1 \leq_{icx} \hat{Y}_1$ and h_C is concave, it follows from Theorem 13 in Sordo et al. (2015) that

$$X_{1,h_C} \leq_{icx} \hat{Y}_{1,h_C}. \tag{19}$$

Now consider the conditional random variable

$$Y_{1,h_{C'}} = \left\{ Y_1 | \bigcap_{i=2}^{n} \{Y_i > G_i^{-1}(p_i)\} \right\}, \ 0 \leq p_i \leq 1,$$

which is a distorted random variable induced from Y_1 by the (non-necessarily concave) distortion function

$$h_{C'}(u) = \frac{\bar{C}'(1-u, p_2, ..., p_n)}{\bar{C}'(0, p_2, ..., p_n)}, \ u \in [0,1].$$

Since $C \leq_{whr} C'$, we have $h_C(u) \leq h_{C'}(u)$ for all $u \in (0,1)$. In particular, $h_C(\overline{F}_1(x)) \leq h_{C'}(\overline{F}_1(x))$ for all x, which means that

$$\hat{Y}_{1,h_C} \leq_{st} Y_{1,h_{C'}}. \tag{20}$$

Since the stochastic order is stronger than the increasing convex order, it follows from (19) and (20) that $X_{1,h_C} \leq_{icx} Y_{1,h_{C'}}$, which is equivalent to say that (8) holds for all $\mathbf{p} \in (0,1)^n$. ■

Remark 17 *Observe, using the same argument as in Example 15, that if $\hat{C}$ is any MTP$_2$ copula and $\hat{C}^I$ is the independence copula, then $\hat{C} \leq_{whr} \hat{C}^I$. Examples of MTP$_2$ copulas can be found in Müller and Scarsini (2005). For instance, if $\hat{C}_\psi$ is a survival Archimedean copula with a generator function ψ such that $(-1)^n\psi^{(n)}$ is log-convex, then $\hat{C}_\psi$ is MTP$_2$.*

As mentioned before, for $n = 2$, the condition $C \leq_{whr} C'$ in Theorem 16 can be weakened to the condition $C \leq_c C'$. This is shown in Theorem 12 of Sordo et al. (2018). Now, making use of Theorem 9(ii), this result can also be used to compare the respective MMEs. This is illustrated in the following example.

Example 18 (Gumbel copulas) *Let (X_1, X_2) and (Y_1, Y_2) have Gumbel copulas with dependence parameters $\theta < \theta'$, which implies $C \leq C'$. Let $X_1 \sim G(\alpha_{X1}, \beta_{X1})$, $X_2 \sim G(\alpha_{X2}, \beta_{X2})$, $Y_1 \sim G(\alpha_{Y1}, \beta_{Y1})$ and $Y_2 \sim G(\alpha_{Y2}, \beta_{Y2})$, with $\alpha_{X1} > \alpha_{Y1}$, $\alpha_{X1}\beta_{X1} \leq \alpha_{Y1}\beta_{Y1}$ (which implies $X_1 \leq_{icx} Y_1$), and $\alpha_{X2} \geq \alpha_{Y2}$, $\beta_{X2} \geq \beta_{Y2}$ (which implies $X_2 \geq_{st} Y_2$). It follows from Theorem 12 of Sordo et al. (2018) and Theorem 9(ii) that $\mathrm{MME}_p[X_1|X_2] \leq \mathrm{MME}_p[Y_1|Y_2]$ for all $0 < p < 1$ (see Figure 3).*

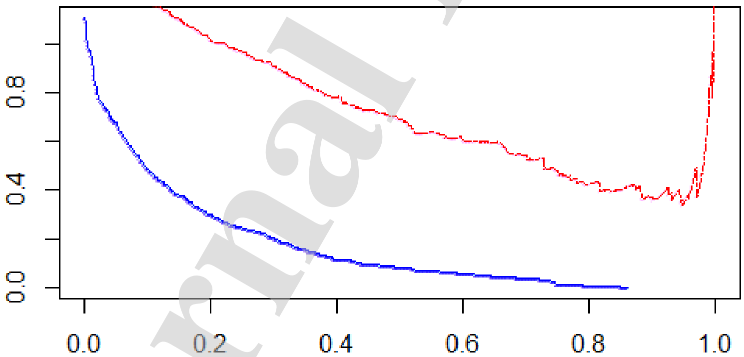

Figure 3: $\mathrm{MME}_p[X_1|X_2]$ as a function of p. The blue line corresponds to a Gumbel copula C with dependence parameter $\rho = 1.05$ and marginals $X_1 \sim G(1.1, 1)$ and $X_2 \sim G(2, 2)$. The red one corresponds to a Gumbel copula C' with dependence parameter $\rho = 1.1$ and marginals $Y_1 \sim G(1, 1.5)$ and $Y_2 \sim G(2, 1)$. Here $X_1 \leq_{icx} Y_1$, $X_2 \geq_{st} Y_2$ and $C \leq C'$.

Theorem 16 admits a variant for the case of stochastically decreasing dependence between X_1 and $(X_2, ..., X_n)$. Recall that, given a random vector $(X_1, ..., X_n)$, we say that $(X_1, ..., X_{i-1}, X_{i+1}, .., X_n)$ is stochastically decreasing in X_i (denoted $(X_1, ..., X_{i-1}, X_{i+1}, .., X_n) \downarrow_{SD} X_i$) if the conditional distribution $\{(X_1, ..., X_{i-1}, X_{i+1}, .., X_n | X_i = x_i)\}$ is stochastically decreasing as x_i increases. We have the following result (observe that the order $\leq_{icx}$ is replaced by the order $\leq_{icv}$ and the CoES is replaced by a kind of dual conditional measure).

Theorem 19 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A). If* $X_1 \leq_{icv} Y_1, C \leq_{whr} C'$ *and* $(X_2, ..., X_n) \downarrow_{SD} X_1$*, (or* $(Y_2, ..., Y_n) \downarrow_{SD} Y_1$*) then*

$$\int_0^{p_1} \mathrm{VaR}_{p_1}\left[X_1 | \bigcap_{j=2}^{n} \{X_j > \mathrm{VaR}_{p_j}(X_j)\}\right] dt \tag{21}$$
$$\leq \int_0^{p_1} \mathrm{VaR}_{p_1}\left[Y_1 | \bigcap_{j=2}^{n} \{Y_j > \mathrm{VaR}_{p_j}(Y_j)\}\right] dt$$

for all $(p_1, ...p_n) \in (0,1)^n$.

Proof. The proof is similar to the proof of Theorem 16, with the following differences. Under the assumption $(X_2, ..., X_n) \downarrow_{SD} X_1$, the distortion function $h_C(u)$ in (18) is convex. Recall that $\hat{\mathbf{Y}} = (\hat{Y}_1, ..., \hat{Y}_n)$ is a random vector with copula C, such that $\hat{Y}_i =_{st} Y_i, i = 1, ..., n$. Then, since the increasing concave order is preserved by convex distortion functions, the assumption $X_1 \leq_{icv} \hat{Y}_1$ implies $X_{1,h_C} \leq_{icv} \hat{Y}_{1,h_C}$. This, together with (20) implies $X_{1,h_C} \leq_{icv} Y_{1,h_{C'}}$, which is equivalent to (21). ■

To interpret the result, note that the risk measure in the left hand of inequality (21) represents (up to the scale factor $1/p_1$), the expected value of the conditional random variable

$$\left\{X_1 | \bigcap_{j=2}^{n} \{X_j > \mathrm{VaR}_{p_j}(X_j)\}, X_1 \leq \mathrm{VaR}_{p_1}\left[X_1 | \bigcap_{j=2}^{n} \{X_j > \mathrm{VaR}_{p_j}(X_j)\}\right]\right\}$$

which intuitively describes the behavior of small values of X_1 given that $X_2, ..., X_n$ jointly take on large values (taking into account interactions between X_1 and $(X_2, ..., X_n)$).

The following result compares marginal risk contributions based on CoES and extends Theorem 23 in Sordo et al. (2018).

Theorem 20 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) such that* $C \leq_{whr} C'$, $(X_2, ..., X_n) \uparrow_{SI} X_1$*, (or* $(Y_2, ..., Y_n) \uparrow_{SI} Y_1$*) and* $X_1 \leq_{ew} Y_1$*. Then,*

$$\Delta\,\mathrm{CoES}_{\mathbf{p}}[X_1 \mid X_2, ..., X_n] \leq \Delta\,\mathrm{CoES}_{\mathbf{p}}[Y_1 \mid Y_2, ..., Y_n]$$

holds for all $\mathbf{p} = (p_1, ..., p_n) \in (0,1)^n$.

Proof. The proof is similar to the proof of Theorem 23 in Sordo et al. (2018) by using the convex distortion function

$$A_{\mathbf{p}}(t) = \max\left\{1 - \frac{\bar{C}(1-u, p_2, ..., p_n)}{(1-p_1)\bar{C}(0, p_2, ..., p_n)}, 0\right\}.$$

■

3 Sufficient conditions for the comparisons of MMEs

In this section, we provide sufficient conditions for the comparisons of MMEs in the general multivariate case. As in Theorem 16, the copulas C and C' are required to verify $C \leq_{whr} C'$. However, while in Theorem 16 the first components are required to be ordered in the increasing convex order ($X_1 \leq_{icx} Y_1$), in the following results they are required to be ordered according to how much they exceed a threshold given by a linear combination of the VaRs of the other $n-1$ components. We recall from Definition 4 that, given a random vector X satisfying (A) and $\mathbf{p}^* = (p_2, p_3, .., p_n) \in [0,1]^{n-1}$, this threshold is given by

$$A_{\mathbf{X},\mathbf{p}^*} = a_1 \mathrm{VaR}_{p_2}(X_2) + ... + a_{n-1} \mathrm{VaR}_{p_n}(X_n),$$

where $a_1, ..., a_{n-1}$ are given positive constants, such that $\sum_{i=1}^{n-1} a_i = 1$. We start by providing an integral representation for the marginal mean excess.

Lemma 21 *Let* $\mathbf{X}$ *be a random vector satisfying (A) and let* $(U_1, ..., U_n)$ *be the corresponding vector-copula, where* $U_i = F_i(X_i), i = 1,,,,.n$. *Given* $\mathbf{p}^* = (p_2, p_3, .., p_n) \in [0,1]^{n-1}$, *we have*

$$\mathrm{MME}_{\mathbf{p}^*}[X_1 | X_2, ..., X_n] = \int_{F_1(A_{\mathbf{X},\mathbf{p}^*})}^{1} (F_1^{-1}(u) - A_{\mathbf{X},\mathbf{p}^*}) dF_{U_1 | \bigcap_{j=2}^n \{U_j > p_j\}}(u). \tag{22}$$

Proof. Let $Y = \left\{ (X_1 - A_{\mathbf{X},\mathbf{p}^*})_+ \,|\, \bigcap_{j=2}^n \{X_j > F_j^{-1}(p_j)\} \right\}$ be and denote by F_Y the distribution function of Y. Clearly, $F_Y(x) = 0$ for $x < 0$. For $x > 0$, we have

$$F_Y(x) = \frac{\bar{C}(0, p_2, .., p_n) - \bar{C}(F_1(A_{\mathbf{X},\mathbf{p}^*} + x), p_2, .., p_n)}{\bar{C}(0, p_2, .., p_n)}.$$

Since $E[Y] = \mathrm{MME}_{\mathbf{p}^*}[X_1 | X_2, ..., X_n]$ and

$$dF_Y(x) = -\frac{\partial_1 \bar{C}(F_1(A_{\mathbf{p}^*} + x), p_2, .., p_n) \cdot f_1(A_{\mathbf{p}^*} + x)}{\bar{C}(0, p_2, .., p_n)},$$

we see that $\mathrm{MME}_{\mathbf{p}^*}[X_1 | X_2, ..., X_n]$ can be written as

$$\int_0^\infty x \left(\frac{-\partial_1 \bar{C}(F_1(A_{\mathbf{p}^*} + x), p_2, .., p_n) \cdot f_1(A_{\mathbf{p}^*} + x)}{\bar{C}(0, p_2, .., p_n)} \right) dx.$$

The change of variable $u = F_1(x + A_{\mathbf{p}^*})$ yields

$$\mathrm{MME}_{\mathbf{p}^*}[X_1|X_2, ..., X_n] = \int_{F_1(A_{\mathbf{p}^*})}^{1} (F^{-1}(u) - A_{\mathbf{p}^*}) \frac{-\partial_1 \bar{C}(u, p_2, .., p_n)}{\bar{C}(0, p_2, .., p_n)} du. \tag{23}$$

By taking into account that

$$F_{U_1|\bigcap_{j=2}^n \{U_j > p_j\}}(u) = \frac{\bar{C}(0, p_2, .., p_n) - \bar{C}(u, p_2, .., p_n)}{\bar{C}(0, p_2, .., p_n)},$$

we see that (23) is the same as (22).

■

Now we can prove the announced results.

Theorem 22 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) and let* $\mathbf{p}^* = (p_2, .., p_n) \in [0, 1]^{n-1}$. *If* $C \leq_{whr} C'$ *and* $(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y},\mathbf{p}^*})^+$ *then (9) holds.*

Proof. Let $\mathbf{Z} = (Z_1, ..., Z_n)$ be a random vector with the same copula C as $\mathbf{X}$ such that $Z_i =_{st} Y_i, i = 1, ..., n$. Since $(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y},\mathbf{p}^*})^+$ and $A_{\mathbf{Y},\mathbf{p}^*} = A_{\mathbf{Z},\mathbf{p}^*}$, we have $F^{-1}_{(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+}(u) \leq F^{-1}_{(Z_1 - A_{\mathbf{Z},\mathbf{p}^*})^+}(u)$ for all $u \in (0, 1)$. It follows, by using (22), that $\mathrm{MME}_{\mathbf{p}^*}[X_1|X_2, ..., X_n] \leq \mathrm{MME}_{\mathbf{p}^*}[Z_1|Z_2, ..., Z_n]$. Since $\mathbf{Z}$ and $\mathbf{Y}$ have the same marginals and $C \leq_{whr} C'$, it follows from Corollary 12 that $\mathrm{MME}_{\mathbf{p}^*}[Z_1|Z_2, ..., Z_n] \leq \mathrm{MME}_{\mathbf{p}^*}[Y_1|Y_2, ..., Y_n]$. The result follows by transitivity.

■

A particularly interesting corollary is the following statement concerning the dispersive order.

Corollary 23 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) such that* $X_i =_{st} X_1$ *and* $Y_i =_{st} Y_1$ *for* $i = 2, ..., n$. *If* $C \leq_{whr} C'$ *and* $X_1 \leq_{disp} Y_1$ *then (9) holds for all* $\mathbf{p}^* = (p, ..., p) \in (0, 1)^{n-1}$.

Proof. If $\mathbf{X}$ has identically distributed components and $\mathbf{p}^* = (p, ..., p)$, then $A_{\mathbf{X},\mathbf{p}^*} = F_1^{-1}(p)$, where F_1 is the common distribution function of $X_1, ..., X_n$.. If, in addition, $\mathbf{Y}$ has identically distributed components, then $A_{\mathbf{Y},\mathbf{p}^*} = G_1^{-1}(p)$, where G_1 is the common distribution function of $Y_1, ..., Y_n$. Then, the condition $(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y},\mathbf{p}^*})^+$ can be rewritten as $(X_1 - F_1^{-1}(p))^+ \leq_{st} (Y_1 - G_1^{-1}(p))^+$. Taking into account that $X_1 \leq_{disp} Y_1$ if and only if $(X_1 - F_1^{-1}(p))^+ \leq_{st} (Y_1 - G_1^{-1}(p))^+$ for all $p \in (0, 1)$ (Muñoz-Pérez, 1990), the result follows by applying Theorem 22.

■

This result can be interpreted as follows: given a portfolio of identically distributed risks, the greater the dispersion of the individual risks (in the sense of the order $\leq_{disp}$) and the stronger the dependence structure of the vector (in the sense of the order $\leq_{whr}$), the greater the risk of contagion (as measured by MME).

Example 24 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) with multivariate Pareto distributions* $\mathbb{P}_{\mathbf{X},n}(\alpha, a)$ *and* $\mathbb{P}_{\mathbf{Y},n}(\beta, a)$ *respectively, with* $0 < \alpha \leq \beta$ *and* $a > 0$*. Then* $\mathbf{X}$ *has identically distributed Pareto marginals* $\mathbb{P}(\alpha, a)$ *and* $\mathbf{Y}$ *has identically distributed Pareto marginals* $\mathbb{P}(\alpha, b)$*. Moreover,* $C = C'$ *and* $X_i \leq_{disp} Y_i$ *for* $i = 1, .., n$ *(see Arias-Nicolas et al., 2009). It follows from Corollary 23 that (9) holds for all* $\mathbf{p}^* = (p, ..., p) \in (0, 1)^{n-1}$*. This example is illustrated in Figure 4.*

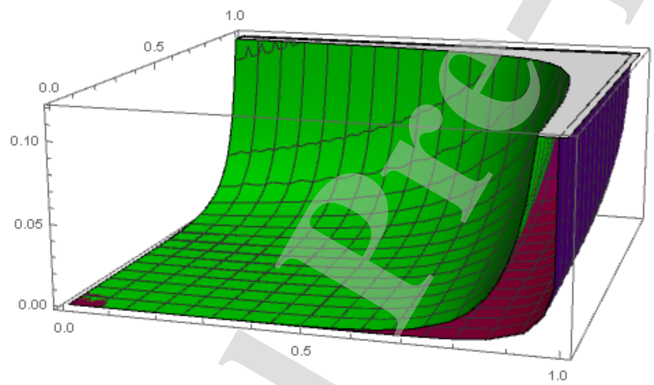

Figure 4: $\text{MME}_{(p_2,p_3)}\left[X_1|X_2, X_3\right]$ as function of p_2 and p_3. The purple surface corresponds to the MME of $(X_1, X_2, X_3) \sim \mathbb{P}(1, 1, 1, 5)$. The green one corresponds to the MME of $(Y_1, Y_2, Y_3) \sim \mathbb{P}(10, 10, 10, 5)$. Here $X_i \leq_{disp} Y_i$ for $i = 1, 2, 3$.

Given two random vectors with the same marginals, comparisons of marginal risk contributions of the form (4) are equivalent to comparisons of MME . The following result follows directly from Corollary 23.

Theorem 25 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) such that* $X_i =_{st} Y_i$ *for all* $i = 1, .., n$*. If* $\mathbf{X} \leq_{whr} \mathbf{Y}$ *or* $C \leq_{whr} C'$*, then*

$$\Delta\text{MME}_{\mathbf{p}^*}\left[X_1|X_2, ..., X_n\right] \leq \Delta\text{MME}_{\mathbf{p}^*}\left[Y_1|Y_2, ..., Y_n\right]$$

holds for all $\mathbf{p}^* \in (0, 1)^{n-1}$*.*

By using the same argument as in Corollary 12, for $n = 2$ the condition $C \leq_{whr} C'$ can be weakened to the concordance order of the corresponding copulas.

The following result says that if $(X_2, ..., X_n) \uparrow_{SI} X_1$, then the condition $(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+ \leq_{st} (Y_1 - A_{\mathbf{Y},\mathbf{p}^*})^+$ in Theorem 22 can be weakened by replacing $\leq_{st}$ by $\leq_{icx}$.

Theorem 26 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) and let* $\mathbf{p}^* = (p_2, .., p_n) \in [0,1]^{n-1}$. *If* $(X_2, ..., X_n) \uparrow_{SI} X_1$ *(or* $(Y_2, ..., Y_n) \uparrow_{SI} Y_1$*),* $C \leq_{whr} C'$ *and* $(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+ \leq_{icx} (Y_1 - A_{\mathbf{Y},\mathbf{p}^*})^+$ *then (9) holds.*

Proof. As in the proof of Theorem 22, let $\mathbf{Z} = (Z_1, ..., Z_n)$ be a random vector with the same copula C as $\mathbf{X}$ such that $Z_i =_{st} Y_i, i = 1, ..., n$. Since $(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+ \leq_{icx} (Y_1 - A_{\mathbf{Y},\mathbf{p}^*})^+$ and $A_{\mathbf{Y},\mathbf{p}^*} = A_{\mathbf{Z},\mathbf{p}^*}$, it follows from Theorem 2.1 in Sordo and Ramos (2007) that

$$\int_0^1 F^{-1}_{(X_1 - A_{\mathbf{X},\mathbf{p}^*})^+}(u) dg(u) \leq \int_0^1 F^{-1}_{(Z_1 - A_{\mathbf{Z},\mathbf{p}^*})^+}(u) dg(u) \tag{24}$$

for all increasing and convex function g. Now observe that (22) can be rewritten as

$$\mathrm{MME}_{\mathbf{p}^*}\left[X_1 | X_2, ..., X_n\right] = \int_0^1 F^{-1}_{(X_1 - A_{\mathbf{p}^*})^+}(u) dF_{U_1 | \bigcap_{j=2}^n \{U_j > p_j\}}(u), \tag{25}$$

where $(U_1, ..., U_n)$ is the corresponding vector-copula of $\mathbf{X}$. From the assumption $(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n) \uparrow_{SI} X_i$ and Theorem 4(i) of Sordo et al. (2015) it follows that $F_{U_1 | \bigcap_{j=2}^n \{U_j > p_j\}}(u)$ is an increasing and convex function of u. Then, by taking $g(u) = F_{U_1 | \bigcap_{j=2}^n \{U_j > p_j\}}(u)$ in (24) and using (25), we have $\mathrm{MME}_{\mathbf{p}^*}\left[X_1 | X_2, ..., X_n\right] \leq \mathrm{MME}_{\mathbf{p}^*}\left[Z_1 | Z_2, ..., Z_n\right]$. Reasoning again as in the proof of Theorem 22, since $\mathbf{Z}$ and $\mathbf{Y}$ have the same marginals and $C \leq_{whr} C'$, it follows from Corollary 12 that $\mathrm{MME}_{\mathbf{p}^*}\left[Z_1 | Z_2, ..., Z_n\right] \leq \mathrm{MME}_{\mathbf{p}^*}\left[Y_1 | Y_2, ..., Y_n\right]$. The result follows by transitivity. The proof is similar if we assume $(Y_1, ..., Y_{i-1}, Y_{i+1}, ..., Y_n) \uparrow_{SI} Y_i$. ■

A corollary similar to Corollary 23 can be stated in terms of the excess wealth order.

Corollary 27 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors satisfying (A) such that* $X_i =_{st} X_1$ *and* $Y_i =_{st} Y_1$ *for* $i = 2, ..., n$. *If* $(X_2, ..., X_n) \uparrow_{SI} X_1$ *(or* $(Y_2, ..., Y_n) \uparrow_{SI} Y_1$*),* $C \leq_{whr} C'$ *and* $X_1 \leq_{ew} Y_1$ *then (9) holds for all* $\mathbf{p}^* = (p, ..., p) \in (0,1)^{n-1}$

Proof. The proof follows from Theorem 26 in the same manner as the proof of Corollary 23 follows from Theorem 22. The only difference is that now, instead of using the characterization of the dispersive order, we use $X_1 \leq_{ew} Y_1$ if and only if $(X_1 - F_1^{-1}(p))^+ \leq_{icx} (Y_1 - G_1^{-1}(p))^+$ for all $p \in (0,1)$ (Belzunce, 1999). ■

The interpretation of this result is similar to that of Corollary 23 with the difference that when $(X_2, ..., X_n)$ is positive dependent on X_1 in the sense SI (or when $(Y_2, ..., Y_n)$ is positive dependent on Y_1), we can weaken the condition $X_1 \leq_{disp} Y_1$ to the condition $X_1 \leq_{ew} Y_1$.

Remark 28 *It follows from the comment below Corollary 12 that, in the case of bivariate random vectors, the condition $C \leq_{whr} C'$ in corollaries 23 and 27 can be weakened to the condition $C \leq_c C'$.*

4 A real numerical example

As an empirical illustration, we apply the methodology based on stochastic orders to compare the risk of contagion in the context of Spanish banking sector. We consider three of the largest banks operating in Spain, namely, the Santander bank, the BBVA and Bankinter. At the end of March, 2019, the Santander bank is the largest one in terms of assets[2] , deposits, loans, number of branches and employers. It is also one of the seven banks in the euro area labelled by the Financial Stability Board as global systemically important banks. This means that distress of this bank could have a significant impact on economic activity and financial system. The BBVA is the second largest bank in Spain in terms of assets and Bankinter is the sixth one. Our aim is to compare how larger positions of Santander affect the behavior of BBVA and Bankinter.

We follow the common approach of testing for contagion based on the analysis of interaction among asset log returns, which are weekly logarithm price differentials. If we denote as p_t the price of an asset at week t, the log return at week t is defined by $r_t = \log(p_t/p_{t-1})$. Our model considers two vectors of log returns, $\mathbf{X} = (X_{BV}, X_S)$, where X_{BV} and X_S represent the log returns of BBVA and Santander, respectively and $\mathbf{Y} = (Y_{BK}, Y_S)$ where X_{BK} and Y_S represent the log returns of Bankinter and Santander, with the assumption $X_S = Y_S$. We have used samples of size $n = 209$ for each financial institution, measuring the share value from June 2015 until June 2019. Data were gathered from the public website *http://es.finance.yahoo.com* and they are also available from authors upon request. In order to eliminate the time dependent effect, data are related to the weekly close of trading.

Figure 5 plots the sample marginal mean excess of BBVA on Santander (blue curve) and the sample marginal mean excess of Bankinter on Santander (green curve) as a function of the p-level. The empirical measures generated from the samples suggest that $\mathrm{MME}_p\,[X_{BK}|X_S] \leq \mathrm{MME}_p\,[X_{BV}|X_S]$ for $p \geq p_0$, where p_0 is close to 0.3, whereas for p-levels lower than p_0 the curves cross each other slightly, experiencing very small differences that could be due to sampling variability. Consequently, we

[2]Data drawn from the website https://www.advratings.com/europe/top-banks-in-spain.

use statistical inference to check the hypothesis

$$\mathrm{MME}_p\left[X_{BK}|X_S\right] \leq \mathrm{MME}_p\left[X_{BV}|X_S\right], \text{ for all } 0 < p < 1. \tag{26}$$

We first perform some tests to study the marginal distributions of X_S, X_{BV} and Y_{BK}. For the classical runs test for randomness, the p-values are 0.6276, 0.6272 and 0.945, respectively. For the Kolmogorov-Smirnof test for normality, the p-values are 0.7431, 0.9804 and 0.6482, respectively. We conclude that there is not significant evidence to reject the hypothesis that the three log return distributions are random, symmetric and normal.

Next, we compare means and standard deviations of X_{BV} and Y_{BK}. An unilateral F-test for paired data performed for testing the hypothesis of equality of variances against $\sigma_{BK} < \sigma_{BV}$ gives a p-value of 0.000014, showing significant evidence that $\sigma_{BK} < \sigma_{BV}$. The p-value of the t-test for testing $\mu_{BV} = \mu_{BK}$ against $\mu_{BV} \neq \mu_{BK}$ is 0.8593, so we can not reject the equality of means. From the assumptions of normality, $\mu_{BV} = \mu_{BK}$ and $\sigma_{BK} < \sigma_{BV}$, it follows $Y_{BK} \leq_{icx} X_{BV}$ (see Table 2.2 in Belzunce et al., 2015).

Finally, we adjust the copulas C and C' of the vectors $\mathbf{X}$ and $\mathbf{Y}$, respectively, by using the goodness of fit test based on the Rosenblatt transformation (see Genest et al., 2009). This test compares the empirical copula with a parametric estimation of a given copula. The elliptical family of copulas is one of the most extensively applied models in finance (see, for example, Owen and Rabinovitch, 1983) and the normal distribution is the archetype of this family. By considering a bivariate normal (BN) copula we obtain p-values 0.6079 and 0.9196, respectively, so there is not statistical evidence to reject the null hypothesis. For a bivariate random vector $\mathbf{X}$ the BN copula parameter is the Pearson correlation coefficient $\rho_{\mathbf{X}}$. We perform the Williams's Test (unilateral) (Steigner (1980) and Williams (1959)) for testing the hypothesis of equality of these coefficients against $\rho_{\mathbf{X}} < \rho_{\mathbf{Y}}$ when the vectors share one component ($X_S = Y_S$). A p-value of 0.010 was obtained, so there is significant evidence that $\rho_{\mathbf{X}} < \rho_{\mathbf{Y}}$. This implies that $C \leq_c C'$ (see Example 3.8.6 in Müller and Stoyan, 2002). Reasoning as in Example 18, it follows from Theorem 12 of Sordo et al. (2018) and Theorem 9(ii) that (26) is supported by statistical significance, which indicates that higger positions of Santander affect more to BBVA than to Bankinter.

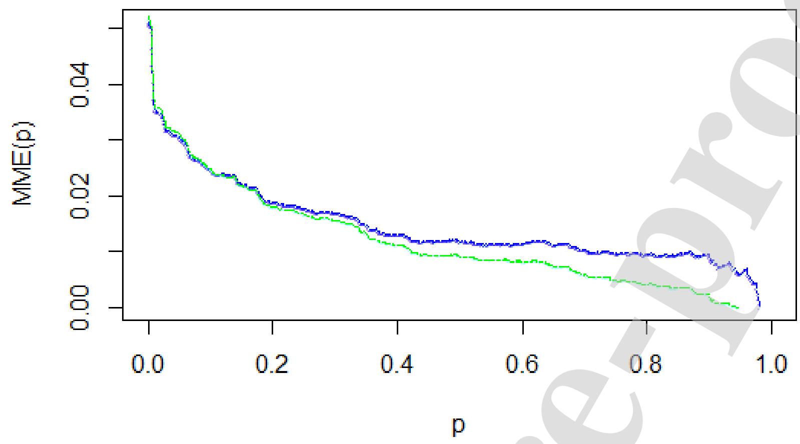

Figure 5: Marginal mean excess in function of the p-level. The blue curve corresponds to $\mathbf{X} = (X_{BV}, X_S)$ and the green curve to $\mathbf{Y} = (Y_{Bk}, Y_S)$.

4.1 CONCLUSIONS

Conditional risk measures (or co-risk measures) and risk contributions measures are used in portfolio risk analysis to quantify the risk of contagion given that one or more assets in the portfolio are in distress. Although these measures can be used directly to compare the systemic risk among different portfolios, there are several issues that hinder the process. One issue is that, sometimes, these measures cannot be calculated explicitly. Another is that different stress scenarios may produce different conclusions, something that happens, for example, when $\text{CoVaR}_{\mathbf{p}}[X_1|X_2]$ and $\text{CoVaR}_{\mathbf{p}}[Y_1|Y_2]$ produce different orderings for different choices of $\mathbf{p} \in [0,1]^2$.

To overcome these issues, given the vectors $\mathbf{X}$ and $\mathbf{Y}$, we have provided sufficient conditions, in terms of well-known stochastic orders and dependence notions, under which the multivariate CoVaRs, CoESs and MMEs are ordered whatever the probability level used to define the stress scenario. While some results concerning CoVaR and CoES have extended to a multivariate setting the bivariate results obtained by Sordo et al. (2018), all results concerning MMEs are new even for the bivariate case. The results have been illustrated with examples based on parametric families of multivariate distributions and with real data.

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