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Semi-infinite interval equilibrium problems: optimality conditions and existence results

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Abstract
This paper aims to obtain new Karush–Kuhn–Tucker optimality conditions for solutions to semi-infinite interval equilibrium problems with interval-valued objective functions. The Karush–Kuhn–Tucker conditions for the semi-infinite interval programming problem are particular cases of those found in this paper for constrained equilibrium problem. We illustrate this with some examples. In addition, we obtain solutions to the interval equilibrium problem in the unconstrained case. The results presented in this paper extend the corresponding results in the literature.

Keywords Equilibrium problem · Semi-infinite programming · Interval-valued functions

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1 Introduction

Sometimes, more than looking for minima or maxima of functions, searching for equilibrium points is of interest, as occurs in economics, with supply and demand.

Equilibrium theory in Euclidean spaces was firstly introduced by Fan (1961). An *equilibrium problem* consists in finding $x \in S$ such that

$$F(x, y) \geq 0, \quad \forall y \in S$$

where $S \subseteq \mathbb{R}^n$ is a nonempty closed set and $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bifunction.

The motivation for studying these equilibrium problems in this paper is that other mathematical problems can be reformulated as equilibrium problems:

- The optimality problem where $F(x, y) = f(y) - f(x)$, $\forall x, y \in X$ where $f : X \to \mathbb{R}$.
- The variational inequality problem involving

$$F(x, y) = \langle T(x), y - x \rangle, \quad \forall x, y \in X,$$

  where $\mathcal{L}(X, Y)$ is the space of all continuous linear mappings from X to Y and $T : X \to \mathcal{L}(X, Y)$.
- The Nash equilibrium problem when we assume that there are n companies, and each company i may possess I_i generating units. Let x denote the vector whose entry x_j stands for the power generation by unit j. Suppose that the price $p_i(s)$ is a decreasing affine function of s with $s = \sum_{i=1}^{N} x_i$, where N is the number of all generating units. Then the profit made by company i can be proposed as:

$$f_i(x) = -\sum_{j \in I_i} c_j(x_j) + p_i(s) \sum_{j \in I_i} x_j,$$

  where $c_j(x_j)$ is the cost for generating x_j by generating unit j. Let us suppose that K_i is the strategy set of company i, which means that $\sum_{j \in I_i} x_i \in K_i$ must be fulfilled for every i. Then the strategy set of the model is the following form: $K = K_1 \times K_2 \times \cdots \times K_n$. We recall that $\bar{x} \in K$ is said to be an equilibrium point of the model if

$$f_i(\bar{x}) \geq f_i(\bar{x}[x_i]), \quad \forall x_i \in K_i, \ \forall i = 1, 2, \ldots, n,$$

  where $\bar{x}[x_i]$ signifies the vector obtained from $\bar{x}$ by replacing $\bar{x}_i$ with x_i. By pickering,

$$F(x, y) = \phi(x, y) - \phi(x, x)$$

  with $\phi(x, y) = -\sum_{i=1}^{n} f_i(\bar{x}[x_i])$.

In this paper, we study the equilibrium problems with infinite constraints. *Semi-infinite programming* is the class of mathematical programming problems in which infinite constraints define the feasible set. The origin of the mathematical theory of semi-infinite programming was conceived by Haar (1924). The term 'semi-infinite programming' was proposed by Charnes et al. (1962)

Semi-infinite programming has a vast range of applications in physics, portfolio problems, engineering design, etc. (see Goberna 2005; Goberna and López 2002; López and Still 2007) and the references cited therein. As an example, in Vaz et al. (2004), the authors describe that robot trajectory planning can be formulated as a semi-infinite programming problem. Similarly, Vaz and Ferreira (2009) used semi-infinite optimization techniques for stationary pollution source planning and control.

Another concern linked to the taking of measurements is the treatment of the errors associated with them. The complexity of the mechanisms and structures we work with daily makes the appearance of inaccuracies in the data inevitable. In recent years, along with the fuzzy theory, one of the most interesting is *Interval Analysis*, introduced by Moore (1966) as a technique to control them and prevent an accumulation of errors from causing disastrous final results. Therefore, imprecision on an equilibrium problem's objective function can be interesting.

Springback is one of the main failures in sheet metal forming and is difficult to control. Springback depends on several factors such as sheet thickness, temperature or coefficient of friction. This coefficient can be considered uncertain, and it cannot be given precise values. For its calculation, we could use fuzzy techniques, but sometimes it is difficult to specify the membership function. Jiang (2007) treat the coefficients of friction as intervals and use a nonlinear interval number programming method.

For a given molecular protein, the distances between its atoms are measured by nuclear magnetic resonance (NMR), but the machines that perform this task have measurement errors. We must take into account the inaccuracies in the distance data. Costa (2017) assume that the NMR machine distance data are represented by intervals and propose a new methodology to compute possible conformations using a constraint interval analysis approach.

In the world of the stock market, the portfolio model proposed by Markowitz is well known and consists of solving the following minimization problem (see Osuna-Gómez et al. 2017).

$$
\begin{aligned}
&\min f(x) = \frac{1}{2}\sum_{i,j=1}^{n}[a_{ij}^L, a_{ij}^U]x_i x_j\\
&\text{subject to :}\\
&\qquad\qquad \sum_{i=1}^{n} x_i = m_p\\
&\qquad\qquad x_i \geq 0,\ i \in \{1, \ldots, n\}
\end{aligned}
$$

The aim is to minimize the variance of the portfolio, while achieving an average rate of return, where the intervals of the objective function represent the uncertainty through the covariance matrix between the return R_i and R_j, x_i is the investment fund devoted to the investment type $i \in \{1, \ldots, n\}$, m_p represents the mean return fixed by investor and $f(x)$ is the risk of investment of $x = \{x_1, \ldots, x_n\}$.

Historical background

In Euclidean spaces, some of the early works related to equilibrium problems are given by Ansari and Flores-Bazán (2003). Gong (2008) studied necessary and sufficient conditions for weakly efficient solutions of vector equilibrium problems with constraints under convexity and by Feng and Qiu (2014) under generalized convexity. Wei and Gong (2010) studied optimality conditions for weakly constrained vector equilibrium solutions in real normed spaces. Tran et al. (2008) obtained an extragradient method for a solution of a pseudomonotone equilibrium problem and Nguyen et al. (2014) provided an algorithm for finding a common solution to an equilibrium problem an a fixed point problem. Noor and Noor (2012) suggested an iterative method for solving the equilibrium problems on Hadamard manifolds.

In 2019, necessary and sufficient optimality conditions for vector equilibrium problems on Hadamard manifolds were provided in Ruiz-Garzón et al. (2019), but not with interval-valued functions.

Recently, the most interesting works were those produced by Tung and Chen. Tung (2020) dealt with convex semi-infinite programming problem with multiple interval-valued objective functions, but not with equilibrium problems. Chen (2020) studied the KKT optimality conditions in an optimization problem with interval-valued objective function on Hadamard manifolds with a finite number of constraints.

Finally, Antczak and Farajzadeh (2023) consider the necessary and sufficient optimality conditions for nondifferentiable semi-infinite interval-valued vector optimization problem.

To the best of our knowledge, no paper has studied the optimality conditions for semi-infinite interval equilibrium problem. This paper makes the following contributions in this direction:

- Working with equilibrium problems, infinite constraints and interval-valued objective functions to handle imprecision.
- Obtaining a new definition of an optimal solution for the semi-infinite equilibrium problem with interval-valued functions.
- Extending recent studies given by other authors such as Wei and Gong (2010) and Tung (2020).
- Obtaining the classical KKT conditions for the semi-infinite interval programming problem are particular cases of those found here for constrained equilibrium problems.
- We generalize the KKM theorem for the existence of solutions of the unrestricted case of the equilibrium problem to interval-valued functions.

Organization. The paper's organization is as follows. In Sect. 2, we discuss the notations we use for intervals and semi-infinite programming. Section 3 is devoted to proving necessary and sufficient optimality conditions for semi-infinite interval equilibrium problems with constraints. We illustrate the results with an example. In Sect. 4, we study the solvability of the Interval Equilibrium Problem under pseudo-monotonicity conditions. In Sect. 5, we see that the optimality conditions of the semi-infinite interval programming problem given by several authors can be considered as particular cases of those given here. We end with some conclusions and references.

2 Notations

In this section, we will recall those indispensable concepts and techniques about intervals and semi-infinite programming that we use in this article.

2.1 Notations for intervals

Let us now recall the basic arithmetic operations with intervals. We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals. By definition, we have:

(a) $A + B = [a^L + b^L, a^U + b^U]$ and $\lambda A = \begin{cases} [\lambda a^L, \lambda a^U], & \lambda \geq 0 \\ [\lambda a^U, \lambda a^L], & \lambda < 0 \end{cases}$, where λ is a real number.

(b) By Chalco-Cano et al. (2011), we have that gH-difference between the two intervals is

$$A \ominus_{gH} B = \left[\min\{a^L - b^L, a^U - b^U\}, \max\{a^L - b^L, a^U - b^U\}\right].$$

We have to define an order between two intervals to decide when one interval is greater than another.

Definition 1 Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in $\mathbb{R}$. We write

- $A \underline{\preceq} B \Leftrightarrow a^L \leq b^L$ and $a^U \leq b^U$.
- $A \preceq B \Leftrightarrow A \underline{\preceq} B$ and $A \neq B$, i.e., $a^L \leq b^L$ and $a^U \leq b^U$, with a strict inequality.
- $A \prec B \Leftrightarrow a^L < b^L$ and $a^U < b^U$.

Let D be an open and non-empty subset of $M = \mathbb{R}$. The function $f : D \to \mathcal{K}_C$ is called an *interval-valued function*, i.e., $f(x)$ is a closed interval in $\mathbb{R}$ for each $x \in M$. We will denote $f(x) = [f^L(x), f^U(x)]$, where f^L and f^U are real-valued functions and satisfy $f^L(x) \leq f^U(x)$ for every $x \in M$.

We can define:

Definition 2 Let D be a nonempty open convex subset of M and let $f : D \to \mathcal{K}_C$ be called gH-directionable differentiable function at $\bar{x} \in D$ in the direction of v if there exists a closed interval $f'_{\bar{x}}(v)$ such that the limit

$$f'_{\bar{x}}(v) = \lim_{t \to 0} \frac{1}{t}(f(\bar{x} + tv) \ominus_{gH} f(\bar{x}))$$

exists, then $f'_{\bar{x}}(v)$ is the derivative of f at $\bar{x}$ in the direction v. Let $f_{\bar{x}}$ be a differentiable function if f is differentiable in any direction; in this case, $f'_{\bar{x}}(v) = \langle \nabla f_{\bar{x}}, v \rangle$.

Lemma 1 (Chen 2020) *Let $D \subseteq M$ be a nonempty open set and consider $f : D \to \mathcal{K}_C$. Then f is a gH-directional derivative at $\bar{x} \in D$ in the direction of v if and only if f^L and f^U have directional derivative at $\bar{x}$ in the direction v. Furthermore, we have*

$$f'_{\bar{x}}(v) = \left[min\{f'^L_{\bar{x}}(v), f'^U_{\bar{x}}(v)\}, max\{f'^L_{\bar{x}}(v), f'^U_{\bar{x}}(v)\}\right],$$

where $f'^L_{\bar{x}}(v)$ and $f'^U_{\bar{x}}(v)$ are the directional derivatives of f^L and f^U at $\bar{x}$ in the direction v, respectively.

Example 1 Suppose that $S = \{x \in \mathbb{R} \mid x \geq 1\}$ and $f : D \subseteq S \to \mathcal{K}_C$ is defined by

$$f(x) = [x, x^2], \quad \forall x \in S.$$

We have that

$$\begin{aligned} f'_{\bar{x}}(v) &= \lim_{t \to 0} \frac{1}{t}(f(\bar{x} + tv) \ominus_{gH} f(\bar{x})) \\ &= \lim_{t \to 0} \frac{1}{t}\left[\min\{(x + tv - x), (x + tv)^2 - x^2\}, \max\{(x + tv - x), (x + tv)^2 - x^2\}\right] \\ &= [\min\{v, 2xv\}, \min\{v, 2xv\}] = v[1, 2x]. \end{aligned}$$

2.2 Tools for semi-infinite programming

The following lemmas are employed commonly in the proof of the optimality conditions of semi-infinite programming (see Tung 2020):

Lemma 2 *If K is a nonempty compact subset of $\mathbb{R}^n$, then,*

(i) *the convex hull of K, $co(K)$, is a compact set;*
(ii) *if $0 \notin co(K)$, then the convex cone containing the origin generated by K, $pos(K)$ is a closed cone.*

Lemma 3 *Suppose that S, P are arbitrary (possibly infinite) index sets, and $a_s = (a_1(s), \ldots, a_n(s))$ maps S onto $\mathbb{R}^n$ and so does a_p. Suppose that the set $co(a_s,\ S \in S) + pos(a_p,\ p \in P)$ is closed. Then the following statements are equivalent:*

I: $\begin{cases} \langle a_s, x\rangle < 0,\ s \in S,\ S \neq \emptyset, \\ \langle a_p, x\rangle \leq 0,\ \ p \in P \end{cases}$ *has no solution $x \in \mathbb{R}^n$;*
II: $0 \in co(a_s,\ S \in S) + pos(a_p, p \in P)$.

Lemma 4 *Let $\{C_t \mid t \in v\}$ be an arbitrary collection convex sets in $\mathbb{R}^n$ and $K = pos(\cup_{t\in v} C_t)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

3 Optimality conditions

This section aims to give necessary and sufficient optimality conditions for the semi-infinite interval equilibrium problem.

Let $D \subset M = \mathbb{R}$ be a nonempty closed convex subset and let $F : D \times D \to \mathcal{K}_C$ be an interval-valued function, and let $g_j : M \to \mathbb{R},\ j \in J$ be functions.

Definition 3 We define the semi-infinite interval equilibrium problem with constraints (SIEPC): find $x \in S$ such that

$$F(x, y) = [F^L(x, y),\ F^U(x, y)] \succeq [0, 0],\ \forall y \in S,$$

where we denote the feasible solution set of (SIEPC):

$$S = \{x \in M \mid\ g_t(x) \leq 0,\ t \in T\}.$$

The index set T is an arbitrary nonempty set, not necessarily finite and $T(\bar{x}) = \{t \in T \mid g_t(\bar{x}) = 0\}$. The set of active constraint multipliers at $\bar{x} \in S$ is

$$\Lambda(\bar{x}) = \{\mu \in \mathbb{R}_+^{|T|} \mid\ \mu_t g_t(\bar{x}) = 0,\ \forall t \in T\}.$$

Remark 1 The Definition 3 is equivalent to finding $x \in S$ such that $F^L(x, y) \geq 0,\ \forall y \in S$ because $F^U(x, y) \geq F^L(x, y)$.

Definition 4 A point $\bar{x} \in S$ is said to be an optimal solution to (SIEPC) if there exists no $\bar{x} \in S$ such that $F(\bar{x}, y) \prec [0, 0],\ \forall y \in S$.

Notation 5 *Let $x \in S$ be given. Denote the mapping $H_{\bar{x}} : S \to \mathcal{K}_C$ by*

$$H_{\bar{x}}(y) = F(\bar{x}, y) = [H_{\bar{x}}^L(y), H_{\bar{x}}^U(y)], \quad \forall y \in S,$$

where $H_{\bar{x}}^L(y) = H_{\bar{x}}^U(y) : S \to \mathbb{R}$.

Remark 2 The Definition 4 is equivalent in that there exists no $\bar{x} \in S$ such that $F_{\bar{x}}^{L}(y) < 0,\ \forall y \in S$. In this case, the (SIEPC) is a problem of real functions, not interval-valued functions.

We recall the classical concepts:

Definition 5 Let S be a given nonempty subset of M and $\bar{x} \in clS$.

- The contingent cone of S at $\bar{x}$ is:

$$\mathcal{T}(S, \bar{x}) = \{v \in \mathbb{R}^n \mid \exists t_k \to 0,\ \exists v_k \in \mathbb{R}^n,\ v_k \to v,\ \forall k \in \mathbb{N},\ \bar{x} + t_k v_k \in S\}.$$

- The negative polar cone of S in M is:

$$S^{-} = \{x^* \in M \mid \langle x^*, x \rangle \leq 0, \quad \forall x \in S\}.$$

- The strictly negative polar cone of S in M is:

$$S^{s} = \{x^* \in M \mid \langle x^*, x \rangle < 0, \quad \forall x \in S \backslash \{0\}\}.$$

Constraint qualification plays an important role in obtaining the necessary optimality conditions.

Definition 6 The Abadie constraint qualification (ACQ) holds at $\bar{x} \in S$ if the negative polar cone $\left(\bigcup_{l \in J(\bar{x})} \nabla g_j(\bar{x})\right)^{-} \subseteq \mathcal{T}(S, \bar{x})$ and the set

$$pos \left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x}) \right)$$

is closed.

We will begin by proving the necessary condition of optimality:

Theorem 6 (Necessary optimality condition) *Let S be a nonempty convex subset of M and let $F : S \times S \to \mathcal{K}_C$, $g_j : S \to \mathbb{R},\ j \in J$ be differential mappings at $\bar{x} \in S$ a feasible point. Let $F(\bar{x}, \bar{x}) = F_{\bar{x}}(\bar{x}) = [F_{\bar{x}}^{L}(\bar{x}), F_{\bar{x}}^{U}(\bar{x})] = [0, 0]$.*

Suppose that $\bar{x}$ is an optimal solution of (SIEPC) and (ACQ) holds at $\bar{x}$. Then, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{x})$ such that

$$0 \in \lambda^L \nabla F_{\bar{x}}^{L}(y) + \sum_{j \in J} \mu_j \nabla g_j(\bar{x}), \ \ \forall y \in S \tag{1}$$

$$\mu_j g_j(\bar{x}) = 0. \tag{2}$$

Proof Our idea is to apply the Lemma 4. We first verify that

$$\left(\nabla F_{\bar{x}}^{L}(y)\right)^{s} \cap \mathcal{T}(S, \bar{x}) = \emptyset. \tag{3}$$

Case 1. If $[0, 0] = \nabla F_{\bar{x}}(y) = (\nabla F_{\bar{x}}^{L}(y), \nabla F_{\bar{x}}^{L}(y))$, then

$$\left(\nabla F_{\bar{x}}^{L}(y)\right)^{s} = \emptyset.$$

Then, expression (3) holds.

Case 2. Let $[0,0] \neq \nabla F_{\bar{x}}(y)$. By reductio ad absurdum, suppose, on the contrary, that there exists $v \in \left(\nabla F_{\bar{x}}^L(y)\right)^s \cap \mathcal{T}(S, \bar{x})$. Then, we have

$$\langle \nabla F_{\bar{x}}^L(y), v\rangle < 0.$$

Since $v \in \mathcal{T}(S, \bar{x})$, there exist $t_k \to 0$ and $v_k \to v$ such that $\bar{x} + t_k v_k \in S$ for all k. It follows that

$$\lim_{t_k \to 0} \frac{1}{t_k}[F_{\bar{x}}^L(\bar{x} + t_k v_k) - F_{\bar{x}}^L(\bar{x})] < 0.$$

Then,

$$F_{\bar{x}}^L(\bar{x} + t_k v_k) - F_{\bar{x}}^L(\bar{x}) < 0.$$

By hypothesis $F(\bar{x}, \bar{x}) = F_{\bar{x}}(\bar{x}) = [0, 0]$, then we have

$$F_{\bar{x}}^L(\bar{x} + t_k v_k) < 0,$$

which contradicts the fact that $\bar{x}$ is an optimal solution of (SIEPC) and therefore the expression (3) holds.

From claims (3) and (ACQ),

$$\left(\nabla F_{\bar{x}}^L(y)\right)^s \cap \left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x})\right)^- \subseteq \left(\nabla F_{\bar{x}}^L(y)\right)^s \cap \mathcal{T}(S, \bar{x}) = \emptyset.$$

Therefore, there is no $v \in \mathcal{R}^n$ fulfilling

$$\begin{cases} \langle \nabla F_{\bar{x}}^L(y), v\rangle < 0, \\ \langle \nabla g_j(\bar{x}), v\rangle < 0, \quad j \in J(\bar{x}) \end{cases}.$$

In addition, we have from Lemma 2 that $co\left(\nabla F_{\bar{x}}^L(y)\right)$ is a compact set, and thus,

$$co\left(\nabla F_{\bar{x}}^L(y)\right)) + pos\left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x})\right)$$

is closed.

By the Lemma 3, we obtain

$$0 \in co\left(\nabla F_{\bar{x}}^L(y)\right) + pos\left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x})\right).$$

According to Lemma 4, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{x})$ such that

$$0 \in \lambda^L \nabla F_{\bar{x}}^L(y) + \sum_{j \in J} \mu_j \nabla g_j(\bar{x}), \quad \forall y \in S.$$

The first expression of KKT (1) is fulfilled.

We will prove the second KKT condition (2). Let W be the set that $(y, z) \in S \times S$, $\exists x \in S$ such that

$$y - \langle \nabla F_{\bar{x}}^L(x),\ x - \bar{x}\rangle > 0$$

$$z - [g(\bar{x}) + \langle \nabla g_j(\bar{x}),\ x - \bar{x}\rangle] > 0,$$

and we can see that W is a nonempty open convex set. It is clear that

$$([\langle \nabla F_{\bar{x}}^L(\bar{x}),\ \bar{x} - \bar{x})\rangle + t'c], \langle \nabla g_j(\bar{x}),\ \bar{x} - \bar{x})\rangle + t'k) \in W$$

for all c, k and $t' > 0$. By separation theorem, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{x})$ such that

$$\begin{aligned}&\lambda^L[\langle \nabla F_{\bar{x}}^L(\bar{x}),\ \bar{x} - \bar{x})\rangle + t'c] + \mu[g(\bar{x}) + \langle \nabla g_j(\bar{x}),\ \bar{x} - \bar{x})\rangle + t'k]\\ &= t'\lambda^L c + \mu g(\bar{x}) + t'\mu k > 0.\end{aligned}$$

Letting $t' \to 0$, we obtain $\mu g(\bar{x}) \geq 0$. Noting that $g(\bar{x}) \leq 0$ and $\mu \geq 0$, we have that $\mu g(\bar{x}) \leq 0$, and thus

$$\mu g(\bar{x}) = 0.$$

Therefore, the KKT conditions hold. □

Convexity plays a very crucial role in obtaining the following sufficient optimality conditions.

Theorem 7 (Sufficient optimality condition) *Let S be a nonempty convex subset of M and let $F : S \times S \to \mathcal{K}_C$, $g_j : S \to \mathbb{R}$, $j \in J$ be differential mappings at $\bar{x} \in S$ a feasible point. Let $F(\bar{x}, \bar{x}) = F_{\bar{x}}(\bar{x}) = [F_{\bar{x}}^L(\bar{x}), F_{\bar{x}}^U(\bar{x})] = [0, 0]$.*

Assume that the convexity of $F_{\bar{x}}^L(x)$ and g_j and there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{x})$ such that KKT optimality conditions (1) and (2) hold, then $\bar{x}$ is an optimal solution for (SIEPC).

Proof As $\bar{x} \in S$ satisfying (1), there exist $\nabla F_{\bar{x}}^L(y)$ and $\nabla g_j(\bar{x})$, $j \in J$, where J is a finite subset of $J(\bar{x})$, such that

$$\sum_{t \in J} \mu_j \nabla g_j(\bar{x}) = -\lambda^L \nabla F_{\bar{x}}^L(\bar{x}). \tag{4}$$

Since $x \in S$ and $g_j(x) \leq 0$, $\forall j \in J$, we get that $g_j(x) \leq 0 = g_j(\bar{x})$, $\forall j \in J$. Due to convexity g_j, $j \in J$ at $\bar{x}$, we have that

$$\sum_{t \in J} \mu_j \langle \nabla g_j(\bar{x}), x - \bar{x}\rangle \leq \sum_{t \in J} \mu_j (g_j(x) - g_j(\bar{x})) \leq 0. \tag{5}$$

Then by (4) and (5),

$$\lambda^L \langle \nabla F_{\bar{x}}^L(\bar{x}), x - \bar{x}\rangle \geq 0. \tag{6}$$

Let us now assume by reductio ad absurdum that $\bar{x}$ is not an optimal solution of (SIEPC). Then there exists $x \in S$ satisfying

$$F(\bar{x}, x) = F_{\bar{x}}(x) \prec [0, 0].$$

The above inequalities, together with $\lambda^L \in \mathbb{R}_+$, imply that

$$\lambda^L (F_{\bar{x}}^L(x) - F_{\bar{x}}^L(\bar{x})) < 0.$$

Using the convexity of $F_{\bar{x}}^L(x)$ at $\bar{x}$,

$$0 > F_{\bar{x}}^L(x) - F_{\bar{x}}^L(\bar{x}) \geq \langle \nabla F_{\bar{x}}^L(\bar{x}), x - \bar{x}\rangle. \tag{7}$$

From inequality (7). we obtain that

$$\lambda^L \langle \nabla F_{\bar{x}}^L(\bar{x}), x - \bar{x}\rangle < 0,$$

which contradicts (6). □

Remark 3 The results obtained here extend the theorems given by Ruiz-Garzón et al. (2019) and Wei and Gong (2010) to interval-valued function and semi-infinite programming.

In summary, we have obtained necessary and sufficient KKT optimality conditions for solutions to the semi-infinite interval equilibrium problems with constraints. We will illustrate the above Theorem with an example:

Example 2 Consider the problem (SIEPC): find $\bar{x}$ such that

$$F(x, y) = [x(y - x),\ 2x(y - x)] \underline{\succ} 0,\ \forall y \in S,$$
$$\text{subject to:}$$
$$g_j(x) = tx - t - 1 \leq 0, \quad j \in J = [-1, 1],$$

then $g_j(x) \leq 0,\ \forall j \in J \Leftrightarrow x \in [0, 2],\ S = [0, 2]$,

Where $F = [F^L, F^U] : S \times S \to \mathcal{K}_C$, with $F^L, F^U, g_j : S \to \mathbb{R}$ are differentiable functions. For $\bar{x} = 0 \in S$,

$$\nabla F^L_{\bar{x}}(y) = y, \quad \nabla g_j(\bar{x}) = \{t\},\ \forall j \in J,\ J(\bar{x}) = \{-1\}$$
$$\left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x}) \right)^{-} = \mathbb{R}_+$$
$$pos \left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x}) \right) = -\mathbb{R}_+$$

is closed, i.e., (ACQ) holds at $\bar{x}$. Now, there exist $\lambda^L = 1$ and

$$\mu(t) = \begin{cases} y, & \text{if t=-1;} \\ 0, & \text{otherwise,} \end{cases}$$

such that

$$0 = \lambda^L \nabla F^L_{\bar{x}}(y) + \sum_{j \in J} \mu_j \nabla g_j(\bar{x})$$
$$0 = y + y(-1)$$

and that F^L and $g_t(x)$ are convex at $\bar{x} = 0$. Hence, all assumptions in Theorem 7 hold, then $\bar{x}$ is a solution of (SIEPC).

4 Existence results

The following concepts will play an important role in obtaining solutions to the Stampacchia Interval Equilibrium Problem (IEP) without constraints:

Definition 7 (*KKM mapping*) Let K be a nonempty closed convex subset of M and $\mathcal{F} : K \to 2^K$ be a multi-valued mapping. $\mathcal{F}$ is called a KKM mapping if $co\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^n \mathcal{F}(x_i)$ for any finite set $\{x_1, x_2, \ldots, x_n\}$ of K, where $co\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of the set $\{x_1, x_2, \ldots, x_n\}$.

Lemma 8 (Fan) *Let K be a nonempty closed convex subset of M and $\mathcal{F} : K \to 2^K$ be a multi-valued mapping. Suppose that $\mathcal{F}$ is a KKM mapping. If $\mathcal{F}(x)$ is closed for each $x \in K$ and compact for some $x \in K$, then $\bigcap_{x\in K} \mathcal{F}(x) \neq \emptyset$.*

We need to extend the hemicontinuous concept or continuous over linear segments to geodesics:

Definition 8 (Jana and Nahak 2016) A function $f : K \to \mathcal{K}_C$ is said to be hemicontinuous if for every direction v, whenever $t \to 0$, $f(v(t)) \to f(v(0))$.

We recall the concept of pseudomonotonicity:

Definition 9 A bifunction $F : S \times S \to \mathcal{K}_C$ is said to be pseudomonotone (PM) if for every $x, y \in S$, such that

$$F(x, y) \not\prec [0, 0] \Rightarrow F(y, x) \not\succ [0, 0].$$

Let us look at some examples of pseudomonotone functions.

Example 3 Let $S = \{x \mid 1 \leq x \leq 2\}$ be a subset of $M = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$. We define the bifunction $F : S \times S \to \mathcal{K}_C$ by

$$F(x) = \left[f^L(x), f^U(x)\right] = [x(x - y), 2x(x - y)]\,.$$

The mapping F is pseudo monotone (PM) because for every $x, y \in S$, we have

$$F(x, y) \not\prec [0, 0] \Rightarrow F(y, x) \not\succ [0, 0].$$

Next, we will relate the solutions of the two pairs of Minty and Stampacchia type problems through pseudomonotonicity.

Lemma 9 (Minty) *Let K be a nonempty, compact, and convex subset of a M. Suppose that:*

(a) *Let $F : S \times S \to \mathcal{K}_C$ be a pseudomonotone (PM) mapping and let $F^L, F^U : K \to \mathcal{K}_C$ be a differentiable functions along the direction v.*
(b) *Let F be a hemicontinuous mapping in the first argument.*
(c) *For fixed $x \in K$, let the mapping $z \to F(x, z)$ be a convex (CX) function and all $F(x, x) = 0,\ \forall x \in K$.*

There exists $x \in K$, such that for all $y \in K$ the Interval Equilibrium Problem

$$F(x, y) \not\prec [0, 0] \quad (IEP), \tag{8}$$

if and only if for all $y \in K$ the Minty Interval Equilibrium Problem

$$F(y, x) \not\succ [0, 0] \quad (MIEP). \tag{9}$$

Proof The sufficient condition is fulfilled by the pseudomonotonicity of F.

To prove the necessary condition, let $v(t) = x+t(y-x)$ for each $t \in [0, 1]$ with $v(0) = x$. Since K is convex, then $v(t) \in K$. Suppose $x \in K$ satisfies Minty type inequality (9), and we will prove that (8) holds; thus, x is a solution of IEP.

Since K is convex, then we have

$$F(v(t)), x) \not\succ [0, 0],\ \ \forall t \in [0, 1]. \tag{10}$$

By convexity in the second argument of F, then

$$[0,0] = F(v(t)), v(t)) \preceq tF(v(t)), y) + (1-t)F(v(t)), x).$$

Therefore,

$$t(F(v(t)), y) \ominus_{gH} F(v(t)), x) \succeq -F(v(t)), x) \succeq [0,0].$$

Thus,

$$F(v(t)), y) \ominus_{gH} F(v(t)), x) \nprec [0,0]. \tag{11}$$

Thanks to the hemicontinuity of F and taking $t \to 0$ in (11), one has

$$F(x, y) \nprec [0,0], \; \forall y \in K.$$

Then, x is a solution of IEP. This proof is completed. □

Next, we will prove our main theorem, the Existence Theorem of Solutions of the IEP:

Theorem 10 *Let K be a nonempty, compact, and convex subset of a M. Suppose that:*

(a) *Let $F : S \times S \to \mathcal{K}_C$ be a pseudomonotone (PM) mapping and let $F^L, F^U : K \to \mathcal{K}_C$ be a differentiable function along the direction v.*
(b) *Let F be a hemicontinuous mapping in the first argument.*
(c) *For fixed $x \in K$, let the mapping $z \to F(x,z)$ be a convex (CX) function and all $F(x,x) = 0, \; \forall x \in K$.*

Then IEP is solvable.

Proof Let us define two set-value mappings $\mathcal{F}, \mathcal{G} : K \to 2^K$ by

$$\mathcal{F}(y) = \{x \in K : F(x,y) \nprec [0,0], \quad \forall y \in K\}$$
$$\mathcal{G}(y) = \{x \in K : F(y,x) \nsucc [0,0], \quad \forall y \in K\}.$$

We wish to prove that

$$\bigcap_{y \in K} \mathcal{F}(y) \neq \emptyset.$$

We will begin by proving mathematically that:

Step 1: Firstly, we prove that $\mathcal{G}$ is a KKM mapping because $\mathcal{F}$ is a KKM mapping. By contradiction, we assume that $\mathcal{F}$ is not a KKM mapping, then there exist $x_1, x_2, \ldots, x_n \in K$, $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$ and $x = \sum_{i=1}^n t_i x_i$ such that $x \notin \mathcal{F}(x_i), \; i = 1, 2, \ldots, n$. That is,

$$F(x, x_i) \prec [0,0], \quad i = 1, 2, \ldots, n.$$

Then by convexity assumption,

$$0 = F(x,x) = \sum_{i=1}^n t_i F(x, x_i) \prec [0,0].$$

Contradiction. Therefore, $\mathcal{F}$ is KKM mapping and so is $\mathcal{G}$.

Step 2: Secondly, we can show that $\mathcal{F}(y) \subset \mathcal{G}(y)$. Since F is hemicontinuous and pseudomonotone, it follows from Lemma 9 that $\mathcal{F}(y) \subset \mathcal{G}(y)$ for all $y \in K$. $\mathcal{F}(y)$ is a KKM function, then $\mathcal{G}(y)$ is a KKM function and $\bigcap_{y \in K} \mathcal{F}(y) = \bigcap_{y \in K} \mathcal{G}(y)$.

Step 3: Furthermore, since K is bounded, then $\mathcal{G}(y)$ is bounded. Moreover, it is obvious that $\mathcal{G}(y)$ is closed in K, and therefore $\mathcal{G}(y)$ is compact. It follows from Lemma 8 that $\bigcap_{y\in K}\mathcal{F}(y)=\bigcap_{y\in K}\mathcal{G}(y)\neq\emptyset$, which implies that there exists $x\in K$ such that $\forall y\in K$,

$$F(x,y)\not\prec[0,0].$$

Therefore, IEP is solvable. □

Just as pseudomonotonicity assures us the existence of the solution, to achieve uniqueness we need a stronger concept:

Definition 10 A bifunction $F: S\times S\to\mathcal{K}_C$ is said to be strictly pseudomonotone (SPM) if for every $x, y\in S$, such that

$$F(x,y)\not\prec[0,0]\Rightarrow F(y,x)\not\succeq[0,0].$$

Theorem 11 *Let K be a nonempty, compact, and convex subset of a M. Suppose that:*

(a) *Let $F: S\times S\to\mathcal{K}_C$ be a strictly pseudomonotone (SPM) mapping and let F^L, F^U : $K\to\mathcal{K}_C$ be a differentiable functions along the direction v.*
(b) *Let F be a hemicontinuous mapping in the first argument.*
(c) *For fixed $x\in K$, let the mapping $z\to F(x,z)$ be a convex (CX) function and all $F(x,x)=0,\ \forall x\in K$.*

Then, IEP has a unique solution.

Proof *Step 1:* By the Theorem 10, we are guaranteed the existence of a solution because the SPM implies the PM.

Step 1: We need to prove the uniqueness. By contradiction, suppose that IEP has two distinct solutions, say x_1 and x_2, then for every $x_1, x_2\in K,\ x_1\neq x_2$, we have

$$F(x_1,x_2)\not\prec[0,0] \tag{12}$$

and

$$F(x_2,x_1)\not\prec[0,0]. \tag{13}$$

From F is SPM, it follows from (12) that

$$F(x_2,x_1)\not\succeq[0,0]. \tag{14}$$

Contradiction with (13). □

The above theorem generalizes results given by Ruiz-Garzón et al. (2003) from mathematical programming problems with real-valued functions to equilibrium problems interval-valued functions.

In the next section, we will show the relationships of the existence results already achieved with mathematical optimization problems.

Example 4 Let $S=\{x\mid 1\leq x\leq 2\}$ be a subset of $M=\mathbb{R}_{++}=\{x\in\mathbb{R}: x>0\}$. We define the bifunction $F: S\times S\to\mathcal{K}_C$ by

$$F(x)=\left[f^L(x), f^U(x)\right]=\left[\left((1-y)^2-(1-x)^2\right), 2\left((1-y)^2-(1-x)^2\right)\right].$$

It is easy to see that the assumptions of the Theorem 10 are satisfied and that the solution is x=1. The extragradient method type algorithm designed by Iusem and Mohebbi (2021) for vector equilibrium problems can be used to generate a sequence that converges to the solution.

5 Application: a particular case

In this section, we will consider a real-life case, such as the Walras model of economic equilibrium, which can be formulated as a Stampacchia Interval Equilibrium problem IEP.

We consider a market structure with perfect competition. The model deals in n commodities. Then, given a price vector $p \in \mathbb{R}^n_+$, we can define the excess demand mapping in a more general way than usual as an interval-valued function, i.e., $E : \mathbb{R}^n_+ \to \mathcal{K}_C$.

We could define:

Definition 11 A price $p^* \in \mathbb{R}^n$ is said to be an equilibrium price vector if it solves

$$p^* \geq 0,\ \exists q^* = [q^{*L}, q^{*U}] \in E(p^*) : \quad q^* = [q^{*L}, q^{*U}] \preceq [0, 0], \quad \langle p^*, q^* \rangle = [0, 0].$$

In a similar way to Dafermos (1990), we can prove the following theorem:

Theorem 12 *A price $p^* \in \mathbb{R}^n$ is said to be a price vector if it solves the Interval Equilibrium Problem (IEP), which consists of finding $p^* \geq 0$ such that $\exists q^* = [q^{*L}, q^{*U}] \in E(p^*)$ so that*

$$F(p^*, p) = \langle -q^*, p - p^* \rangle \succeq [0, 0], \quad \forall p \geq 0.$$

The Theorem 10 of the previous section would give us the conditions under which there exists a solution to the IEP and hence the existence of an equilibrium point.

As a particular case of the results obtained in the previous section, we will obtain the optimality conditions of KKT for a semi-infinite interval programming problem.

Let us consider the semi-infinite interval programming problem (SIP) defined as:

$$\begin{aligned} \text{(SIP)} \quad & \min f(x) = [f^L(x), f^U(x)] \\ & \text{subject to:} \\ & \quad g_j(x) \leq 0, \quad \forall j \in J \\ & \quad x \in S \subseteq M, \end{aligned}$$

where let $f : S \to \mathcal{K}_C$, $g_j : S \to \mathbb{R},\ j \in J$ be differential mappings at $\bar{x} \in S \subset M = \mathbb{R}$.

Definition 12 The point $\bar{x}$ is an optimal solution for (SIP) if there is no $x \in S$ satisfying $f(x) \prec f(\bar{x})$.

As a consequence of the previous theorems and considering SIP as a particular case of SIEPC, we have the KKT classical conditions.

Corollary 13 *Let S be a nonempty convex subset of M and let $F : S \times S \to \mathcal{K}_C$, $g_j : S \to \mathbb{R},\ j \in J$ be differential mappings at $\bar{x} \in S$ a feasible point. Let $F(\bar{x}, \bar{x}) = F_{\bar{x}}(\bar{x}) = [F^L_{\bar{x}}(\bar{x}), F^U_{\bar{x}}(\bar{x})] = [0, 0]$.*

(a) *Suppose that $\bar{x}$ is an optimal solution of (SIEPC) and (ACQ) holds at $\bar{x}$. Then, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{x})$ such that*

$$0 \in \lambda^L \nabla F^L_{\bar{x}}(y) + \sum_{j \in J} \mu_j \nabla g_j(\bar{x}), \ \ \forall y \in S \tag{15}$$

$$\mu_j g_j(\bar{x}) = 0. \tag{16}$$

(b) *Assume the convexity of $F^L_{\bar{x}}(x)$ and g_j and there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{x})$ such that KKT optimality conditions (15) and (16) hold, then $\bar{x}$ is an optimal solution for (SIP).*

Proof The proof is similar to that of Theorem 10 with no more than considering SIP as a particular case of SIEPC, simply taking $F(x, y) = f(y) - f(x), \forall x, y \in M$. □

Example 5 Let us consider the semi-infinite interval programming problem (SIP) defined as:

$$
\begin{aligned}
\text{(SIP)} \quad & \min f(x) = [f^L(x), f^U(x)] = [x^2 + x, x^2 + x + 1] \\
& \text{subject to:} \\
& \quad g_j(x) = tx - t - 1 \leq 0, \quad j \in J = [-1, 1] \\
& \quad x \in S \subseteq M,
\end{aligned}
$$

then $g_j(x) \leq 0,\ \forall j \in J \Leftrightarrow x \in [0, 2],\ S = [0, 2]$.

Here, $f = [f^L, f^U] : S \to \mathcal{K}_C$, with $f^L, f^U, g_j : S \to \mathbb{R}$ are differentiable functions. For $\bar{x} = 0 \in S$,

$$
\begin{aligned}
& \nabla f^L_{\bar{x}} = 1, \quad \nabla g_j(\bar{x}) = \{t\},\ \forall j \in J,\ J(\bar{x}) = \{-1\} \\
& \left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x}) \right)^{-} = \mathbb{R}_+ \\
& pos \left(\bigcup_{j \in J(\bar{x})} \nabla g_j(\bar{x}) \right) = -\mathbb{R}_+
\end{aligned}
$$

is closed, i.e., (ACQ) holds at $\bar{x}$. Now, there exist $\lambda^L = 1$ and

$$
\mu(t) = \begin{cases} 1, & \text{if t=-1;} \\ 0, & \text{otherwise,} \end{cases}
$$

such that

$$
\begin{aligned}
0 &= \lambda^L \nabla f^L_{\bar{x}} + \sum_{j \in J} \mu_j \nabla g_j(\bar{x}) \\
0 &= 1(1) + 1(-1)
\end{aligned}
$$

and that f^L and f^U and $g_t(x)$ are geodesic convex at $\bar{x} = 0$. Hence, all assumptions in Theorem 7 hold, then $\bar{x}$ is a solution of (SIP).

Remark 4 This result agrees with Proposition 1 and 3 given by Tung (2020) in the non-differentiable case with subdifferentiables. These results can be considered particular cases of those obtained in this article for equilibrium problems.

6 Conclusions

This paper's principal contribution is obtaining KKT optimality conditions for semi-infinite interval equilibrium problems, an unexplored field up to this date. Our results have more advantages than others in the literature as it has allowed us to:

- Extend results given by Ruiz-Garzón et al. (2019) and Wei and Gong (2010), for equilibrium problems from real-valued to interval-valued functions and semi-infinite programming.

- Consider recent studies given by other authors such as Tung (2020) on optimality conditions to the semi-infinite interval programming problem (SIP) as particular case of equilibrium problems.
- Generalising solution existence results of classical euclidean optimization problems given by Ruiz-Garzón et al. (2003) to equilibrium problems with interval-valued functions.

Finally, it would be interesting to continue the studies of algorithms in this order to go to find a computational solution to semi-infinite interval equilibrium problems with constraints following recent interesting works done by Iusem and Mohebbi (2021) or Khammahawong et al. (2022) for vector equilibrium problems, for example.

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Declarations

Conflict of interest The authors declare that they have no known competing financial or financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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