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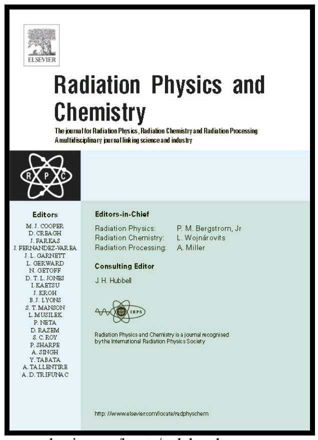

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Bayesian treatment of results from radioanalytical measurements. Effect of prior information modification in the final value of the activity.

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Abstract

We address the problem of evaluating measurements of radionuclide activity concentration from a robust Bayesian perspective. As shown in previous studies, Bayesian incorporation of available prior information on the activity levels, together with the measured values, leads in general to an improvement in the quality of radioanalytical results. Since the specific form of the employed prior is a critical aspect of the Bayesian framework, in the present paper we distort the prior distribution in order to evaluate how it influences the final activity estimate. We applied this procedure to inter-laboratory proficiency test data obtained by the laboratories that perform radiochemical analysis for the Spanish radioactive monitoring network. We found that in the present application the Bayesian methodology is indeed robust, as modifying the specific form of the prior has little effect on the activity estimate. Similar sensitivity analysis could be applied to the Bayesian evaluation of measurements of other quantities for which prior knowledge is available.

Keywords: Robust Bayesian analysis, Radiation measurement, class of priors, radionuclide analysis, proficiency tests, uncertainty of measurement.

1 Introduction

Bayesian statistics constitutes a powerful methodology to evaluate measurement data. One of its advantages is that it allows incorporating prior knowledge (prior information) about the specific quantity under study (in our context, the mensurand). The use of this prior information, together with the experimental value of the measurement performed, can be used to obtain in general a better estimation of the physical quantity under study, see for example [20]. In the scope of radionuclide activity measurements of environmental samples, previous studies [3] have shown the effectiveness of the Bayesian methodology; see also [9], [12], [16], [19], [23], [29], [31] and [32] for related analyses oriented to the calculation characteristic limits such as decision thresholds, detection limits and confidence limits. A thorough review of the Bayesian approach can be found in [6], [7], [11], [17], [18] and [28]; also [10] and [24] provide some guidance on the uncertainty components of the measurement processes. However, paradoxically, the advantage brought about by the introduction of prior information is also to some extent a disadvantage of the Bayesian framework: if using other priors is acceptable, the one selected may appear to be somewhat arbitrary. This is precisely the problem addressed by Robust Bayesian Analysis [14], also called Bayesian Sensitivity Analysis, which quantifies and interprets the uncertainty induced by incomplete knowledge about the precise details of the analysis. Thus, the present work is intended to clarify and to support the Bayesian treatment of prior information from a robust viewpoint in the context of radioanalytical measurements.

As in [3], the case studied below is the Inter-laboratory Proficiency Test periodically organized among the Spanish environmental radioactivity laboratories that perform radiochemical analysis for the Spanish radioactive monitoring network, see [26]. The scheme of these exercises consists in the determination of different radionuclide activities in a test sample distributed to the laboratories of the network. If θ is the parameter that represents the actual unknown activity of a certain radionuclide, its state of knowledge before any measurements are taken is encoded by a probability density function (PDF), denoted by $\pi(\theta)$, which is termed the prior.

Which is the prior in our case? At the beginning of the exercise, when each laboratory receives the sample, the coordinator that assesses the laboratories' performances provides them with an interval on the activity level, $\Theta = [m, M]$ (minimum m and maximum M activity levels for each radionuclide to be determined (the coordinator knows these intervals since it has prepared the sample). Then, uniform distributions with support on the corresponding intervals seem to be appropriate priors. The inclusion of such intervals in the protocol for the Proficiency Test is a general policy of the Spanish laboratory intercomparisons. Its purpose is to assure participants that the sample activity conforms to their routine analysis for environmental activity levels.

So, after a measurement x is taken, the posterior belief is obtained from Bayes' theorem and is denoted by $\pi_x(\theta)$. This PDF represents a combination between the experimental observation and the prior knowledge. From it, a Bayesian estimation of the activity value, x_{B}, is obtained, normally as the expectation of $\pi_x(\theta)$. Depending on the amount of information encoded by the prior, the estimate x_{B} is usually more accurate than the experimental observation x. In what follows we will distort the flat prior in order to analyze how such a modification affects the estimate x_{B} for the activity of a specific radionuclide. In other words, we will assess the relative invariance or independence of the activity estimate when the prior is modified. Roughly speaking, we will compute the range of the quantity of interest x_{B} when the prior belief π varies in a class of priors Γ recently published in [2].

The rest of the paper is structured as follows. In Section 2, we recall the definition of distortion band from [2] and we adapt it to our context. In Section 3, we discuss the model from a Bayesian point of view by considering a uniform prior. In Section 4, we study the robustness of the model by using some particular functions for distorting the prior. In Section 5, we apply our method to the results of an Inter-laboratory Proficiency Test. Finally, in Section 6 we present some concluding remarks.

2 The distorted class

In order to introduce the concept of distortion band, let us first recall the definition of likelihood ratio order. Let π_1 and π_2 be two PDFs for the same parameter θ. The former is said to be smaller than the latter in the likelihood ratio order, if the ratio $\pi_2(\theta)/\pi_1(\theta)$ increases over the union of the supports of the two PDFs. We denote this occurrence by $\pi_1 \leq_{\mathrm{lr}} \pi_2$. Roughly speaking, this means that π_2 takes on larger values and more variability than π_1. It is well known that

$$\pi_1 \leq_{lr} \pi_2 \Rightarrow E^{\pi_1}[\phi(\theta)] \leq E^{\pi_2}[\phi(\theta)], \tag{1}$$

for all non-decreasing functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the two expectations E^{π_1} and E^{π_2} exist. For more details on the concept of likelihood ratio order see [22] and [27].

The distorted band class of priors was recently defined in [2] and fulfills all the requirements that [5] discussed about the choice of a class of priors. First, its elicitation and interpretation is easy. Second, the prior uncertainty can be reflected by using different metrics. Finally, the range of quantities of interest can be computed by just looking for the extremal distributions generating the class.

The distortion band is based on the concept of distortion functions, see Section 2.6 in [8]. These are non-decreasing continuous functions $h : [0,1] \to [0,1]$ such that $h(0) = 0$ and $h(1) = 1$. If π is a prior with cumulative distribution function (CDF) F_π, the distorted CDF is given by

$$F_{\pi_h}(\theta) = h \circ F_\pi(\theta) = h\left[F_\pi(\theta)\right], \forall\theta \in \Theta. \tag{2}$$

By just taking derivatives in (2), we can easily compute the density of the distorted prior, π_h, provided the derivatives exist.

Clearly, priors can be distorted according to various distortion functions. The authors of [2] argue for the use of convex and concave distortion functions based on two arguments. First, they represent satisfactorily a change in the weighting of the underlying prior. Thus, a convex [concave] distortion function gives more weight to higher [smaller] events. Second, they have desirable properties when we compare the original prior with the distorted one.

If h_1 is concave and h_2 is convex, it follows from Lemma 1 in [2], that:

$$\pi_{h_1} \leq_{lr} \pi \leq_{lr} \pi_{h_2}. \tag{3}$$

Clearly based on property (3), the distortion band associated with a specific prior π can now be defined as the class of all priors π' larger and smaller than π_{h_1} and π_{h_2}, respectively, in the likelihood ratio order, i.e.

$$\Gamma_{h_1,h_2,\pi} = \{\pi' : \pi_{h_1} \leq_{lr} \pi' \leq_{lr} \pi_{h_2}\}. \tag{4}$$

In other words, the distortion band may be regarded as a "neighborhood" of the prior π.

One interesting property of the distortion band is that the posteriors corresponding to its lower and upper bounds are also lower and upper bounds for the class of all posteriors π'_x, in the likelihood ratio order sense. Thus, for all $\pi' \in \Gamma_{h_1,h_2,\pi}$ then

$$\pi_{h_1,x} \leq_{lr} \pi'_x \leq_{lr} \pi_{h_2,x}. \tag{5}$$

A similar property is that

$$E^{\pi_{h_1}}[\theta] \leq E^{\pi'}[\theta] \leq E^{\pi_{h_2}}[\theta] \quad \text{and} \quad E^{\pi_{h_1,x}}[\theta] \leq E^{\pi'_x}[\theta] \leq E^{\pi_{h_2,x}}[\theta], \tag{6}$$

for all priors $\pi' \in \Gamma_{h_1,h_2,\pi}$.

Expression (6), which follows by just taking the identity function in (1) and using (4) and (5), means that the expectations and posterior expectations of the lower and upper bounds of the distortion band are also lower and upper bounds for the expectations and posterior expectations, respectively, of all priors in $\Gamma_{h_1,h_2,\pi}$. This fact allows us to evaluate both the amount of uncertainty in the distorted class and the influence of a modification of the specific prior, by computing the difference between expectations or posterior expectations of the distorted priors, respectively. Incidentally, in [2], those differences were interpreted in terms of the Kantorevich probability metric. For more details about probability metrics see [4].

3 The Model: the posterior distribution activity

Every statistical study requires a model for the probabilistic relation between an observation and the study's object. In our case, the model is quite simple. The laboratory makes an experimental observation x, which is assumed to be drawn from a Gaussian distribution centered on the unknown true value of the activity θ. The standard deviation of this distribution, u (the standard uncertainty), is assumed to have been established from the laboratory's experience. The likelihood function is then

$$f_{N(x,u)}(\theta) \propto \exp\left[-\frac{1}{2}\left(\frac{\theta - x}{u}\right)^2\right]. \tag{7}$$

By considering the Principle of Maximum Information Entropy (PME), the Gaussian distribution described in (7) is neither an approximation nor a probability distribution from repeated or counting measurements. If no other information is available, the observation x is taken as the best estimate of the true activity, see [15],[10], [21] and [33] for further information. However, as explained in the Introduction, the organizer of the Proficiency Test provides the laboratory with a reference activity interval, $\Theta = [m, M]$, within which the true activity of the radionuclide is assumed to be contained. Therefore, the prior becomes the uniform PDF

$$\pi(\theta) \propto \mathbf{1}_{[m,M]}(\theta), \tag{8}$$

where $\mathbf{1}_{[m,M]}$ represents the well-known indicator function:

$$\mathbf{1}_{[m,M]}(x) = \begin{cases} 0 & \text{if } x \notin [m, M], \\ 1 & \text{if } x \in [m, M]. \end{cases}$$

From Bayes' theorem and using expressions (7) and (8), the posterior PDF is a truncated normal distribution that vanishes outside the activity interval with density

$$\pi_x(\theta) \propto f_{N(x,u)}(\theta)\mathbf{1}_{[m,M]}(\theta).$$

Its expectation, which we shall denote by x_B is given by

$$x_\mathrm{B} = E^{\pi_x}[\theta] = \int_m^M \theta\, \pi_x(\theta)\, \mathrm{d}\theta. \tag{9}$$

We will take this expectation as the Bayesian estimate of the activity. It combines the experimental data and the prior knowledge, see [3] for further details. Additionally, the standard uncertainty is summarized by the standard deviation of the posterior distribution, u_B, given by

$$u_\mathrm{B} = \sqrt{E^{\pi_x}[(\theta - x_\mathrm{B})^2]} = \sqrt{\int_m^M (\theta - x_\mathrm{B})^2\, \pi_x(\theta)\, \mathrm{d}\theta}. \tag{10}$$

4 Bayesian robustness: the distorted class of priors

Since the particular form of the prior information given in (8) is a critical point of the Bayesian treatment, it should be interesting to analyze its influence on the estimate x_B. For this, consider the class of priors $\Gamma_{h_1,h_2,\pi}$ defined in (4) where we shall use the classical power distortion functions given by

$$h_1(x) = 1 - (1 - x)^\alpha \text{ and } h_2(x) = x^\alpha,\ \alpha > 1, \tag{11}$$

where α is a distortion parameter related to the strength of the perturbation. The concave distortion function, h_1, gives more probability weight to smaller values of the activity, i.e. to values closer to the left endpoint m of the activity interval. Evidently, the opposite effect holds for the convex distortion function h_2. A straightforward computation shows that the densities of the distorted priors are

$$\pi_{h_1}(\theta) \propto \left(\frac{M-\theta}{M-m}\right)^{\alpha-1} \mathbf{1}_{[m,M]}(\theta) \quad \text{and} \quad \pi_{h_2}(\theta) \propto \left(\frac{x-m}{M-\theta}\right)^{\alpha-1} \mathbf{1}_{[m,M]}(\theta).$$

Note that for $\alpha = 1$ both distortion functions reduce to the uniform density and that, when α increases, the effect of the distortion increases too. This fact can be directly observed by computing the differences between the expectations of the uniform prior and the distorted ones. These differences are

$$\begin{aligned} E^{\pi}(\theta) - E^{\pi_{h_1}}(\theta) &= \int_m^M \theta\ \pi(\theta)\ \mathrm{d}\theta - \int_m^M \theta\ \pi_{h_1}(\theta)\ \mathrm{d}\theta, \\ &= \frac{M+m}{2} - \frac{M\alpha+m}{\alpha+1} = \frac{(M-m)(\alpha-1)}{2(\alpha+1)}. \end{aligned} \tag{12}$$

$$\begin{aligned} E^{\pi_{h_2}}(\theta) - E^{\pi}(\theta) &= \int_m^M \theta\ \pi_{h_2}(\theta)\ \mathrm{d}\theta - \int_m^M \theta\ \pi(\theta)\ \mathrm{d}\theta, \\ &= \frac{M\alpha+m}{\alpha+1} - \frac{M+m}{2} = \frac{(M-m)(\alpha-1)}{2(\alpha+1)}. \end{aligned} \tag{13}$$

Both expressions (12) and (13) are equal and strictly increasing in α. Therefore, the convex and concave distortion functions play a symmetrical role in the perturbation.

The choice of the parameter α depends on how much we want to distort the uniform prior. We shall determine a value of α such that (12) and (13) represent a given percentage of deviation from the expectation of the uniform prior, i.e., from the midpoint of the activity interval. In other words, if σ represents a certain fraction of deviation, then α is given by the solution of

$$\frac{(M-m)(\alpha-1)}{2(\alpha+1)} = \sigma(\frac{M+m}{2}) \tag{14}$$

which is

$$\alpha = \frac{2(M-m)}{(M-m)+(M+m)\sigma} - 1. \tag{15}$$

Because the original and distorted priors share the same support -the interval provided- it is apparent that the right-hand side of equation (14) is bounded by the half length of the activity interval, $(M-m)/2$. Therefore, σ cannot be larger than $(M-m)/(M+m)$.

Given the experimental value x of the laboratory and a value for α, the posterior expectations corresponding to the distorted priors, based on h_1 and h_2, are given by

$$x_{\mathrm{B}_i} = E^{\pi_{h_i,x}}[\theta] = \int_m^M \theta\ \pi_{h_i,x}(\theta)\ \mathrm{d}\theta,\ i=1,2, \tag{16}$$

where from Bayes's theorem:

$$\pi_{h_i,x}(\theta) \propto f_{N(x,u)}(\theta)\pi_{h_i}(\theta),\ i=1,2.$$

According to the expression (6) we conclude that x_{B_1} and x_{B_2} are lower and upper bounds for all posterior expectations based on any prior in $\Gamma_{h_1,h_2,\pi}$, respectively. In particular, we see that

$$x_{\mathrm{B}_1} \leq x_{\mathrm{B}} \leq x_{\mathrm{B}_2}. \tag{17}$$

The inequalities (17) will be the key to study the robustness of the technique. We will use them to determine the relative invariance or independence of the result x_{B} when a modification of prior information is performed, and also to determine if we are able to improve the relative performance of the Bayesian final result from a more accurate prior information. Of course, associated with π_{h_i} we can also compute the uncertainties

$$u_{\mathrm{B}_i} = \sqrt{E^{\pi_{h_i,x}}[(\theta - x_{\mathrm{B}_i})^2]} = \sqrt{\int_m^M (\theta - x_{\mathrm{B}_i})^2\ \pi_{h_i,x}(\theta)\ \mathrm{d}\theta},\ i=1,2. \tag{18}$$

5 Application to Proficiency Test data CSN/CIEMAT-04

The preceding theory will now be applied to data from the Inter-laboratory Proficiency Test of environmental radioactivity laboratories CSN/CIEMAT-2004. This test consisted in the measurement of a synthetically prepared water sample containing plutonium 239 and 240. The measurements were carried out by ten participating laboratories and were reported in the form of pairs (x_i, u_i), where x_i is the experimental observation obtained by the ith laboratory and u_i is the standard deviation related to its own measurement experience. These data are shown in Table 1, see [26] for additional details.

Lab No	1	2	3	4	5	6	7	8	9	10
x (Bq m^{-3})	47.60	34.90	41.20	40.70	53.40	43.05	43.50	42.00	53.60	62.00
u (Bq m^{-3})	1.10	1.00	4.25	1.62	1.10	1.49	1.75	2.50	4.50	1.50

Table 1: Pairs (x, u) of the activity, $^{239+240}$Pu, for each laboratory.

The metrologically certified reference activity, denoted by x_{Ref}, was in the characteristic range of environmental samples. The laboratories did not know that x_{Ref} took the value 49.8 Bq m^{-3} with relative precision $\sigma_p = 14\%$ (measured as a percentage of x_{Ref}), see [25]. The precision σ_p was set by the coordinator of the test ("fitness for purpose" according to common evaluation protocols, see [30]), in order to determine the z-score of the laboratories. Based on clause 6.4.7 of Supplement 1 to the GUM, see [15], a Gaussian distribution is appropriate for characterizing the state of knowledge about the reference activity.

From expert judgment, the bounds for the activity interval of $^{239+240}$Pu informed to the laboratories were taken as $m = 40$ Bq m^{-3} and $M = 100$ Bq m^{-3}. At first glance, it would appear that these bounds are unreasonable, since the lower one is too close to the reference activity. It should be mentioned, however, that plutonium is an artificial radioelement introduced into the environment by the bombs of Hiroshima and Nagasaki, the atmospheric nuclear tests conducted in the 50s and 60s and, more recently, by the nuclear accidents of Chernobyl and Fukushima, as well as by the satellites carrying plutonium as fuel that have disintegrated in the atmosphere. Its 239 and 240 isotopes are determined jointly through the measurement of their alpha emissions. This yields a very asymmetric frequency distribution (see [1] and [13]), so it is natural that the activity interval supplied to the laboratories also be asymmetric with respect to x_{Ref}. The pairs $(x_{\text{B}}, u_{\text{B}})$ shown in Table 2 resulted from applying the models in Sections 3 and 4 with a uniform prior supported on this interval.

Lab No	1	2	3	4	5	6	7	8	9	10
x_{B} (Bq m^{-3})	47.60	40.18	43.87	41.58	53.40	43.12	43.60	42.92	53.62	62.00
u_{B} (Bq m^{-3})	1.10	0.18	2.79	1.11	1.10	1.41	1.65	1.89	4.47	1.50

Table 2: Pairs $(x_{\text{B}}, u_{\text{B}})$ of the activity, $^{239+240}$Pu, for each laboratory.

Consider now a distorted band based on the power functions given in (11) with three different values for the dispersion parameter, namely $\alpha = 1.61, 1.97$ and 2.45, which were computed from (15) using $\sigma = 0.10, 0.14$ and 0.18, respectively. Tables 3, 4 and 5 show the corresponding pairs $(x_{\mathrm{B}_1}, u_{\mathrm{B}_1})$ and $(x_{\mathrm{B}_2}, u_{\mathrm{B}_2})$. From the last columns of these tables, which show the differences $x_{\mathrm{B}_2} - x_{\mathrm{B}_1}$, we conclude that the Bayesian approach is rather robust, as these differences are small in comparison with the uncertainties u shown in Table 1 . This is true even for the broadest range observed, 2.91 Bq m^{-3} at laboratory 3 for $\alpha = 2.45$, for which $u_3 = 4.25$ Bq m^{-3}. Such robustness indicates that the Bayesian final results are relatively insensitive to the specific prior employed. Expectations (9) and (16) and uncertainties (10) and (18) shown in Table 1 and 2 have been solved numerically by using Mathematica software.

With respect to the improvement produced by the Bayesian Technique, we will compute the z-score index in order to evaluate the performance based on the metrologically certified values of the activity. For any measurement v of the activity, the z-score index is defined as

$$z = \frac{v - x_{\mathrm{Ref}}}{\sqrt{u_v^2 + (\sigma_p x_{\mathrm{Ref}})^2}} = \frac{v - 49.8}{\sqrt{u_v^2 + 6.972^2}}, \tag{19}$$

where u_v is the uncertainty of the corresponding estimate (i.e., u for the laboratory and u_{B_1}, u_{B} and u_{B_2} by incorporating the uniform prior and the distorted ones), $x_{\mathrm{Ref}} = 49.8$ Bq m^{-3} and $u_{\mathrm{Ref}} = \sigma_p x_{\mathrm{Ref}} = 6.972$ Bq m^{-3}. The z-score is a direct indicator of the deviation of the measurement v with respect to the reference value x_{Ref} expressed in terms of the established deviation for the difference of independent random variables, see [34] for a very interesting study on how to compare measurement results using the Bayesian Decision theory.

Lab No	x_{B_1}(Bq m^{-3})	u_{B_1}(Bq m^{-3})	x_{B_2}(Bq m^{-3})	u_{B_2}(Bq m^{-3})	$x_{\mathrm{B}_2} - x_{\mathrm{B}_1}$(Bq m^{-3})
1	47.59	1.10	47.70	1.09	**0.11**
2	40.18	0.18	40.29	0.22	**0.11**
3	43.78	2.75	45.18	2.91	**1.40**
4	41.57	1.10	42.09	1.15	**0.52**
5	53.38	1.10	53.46	1.10	**0.07**
6	43.10	1.41	43.54	1.35	**0.44**
7	43.57	1.64	44.09	1.58	**0.52**
8	42.88	1.88	43.72	1.91	**0.84**
9	53.36	4.45	54.56	4.32	**1.20**
10	61.96	1.50	62.06	1.50	**0.10**

Table 3: Estimates of the activity with distortion parameter $\alpha = 1.61$.

For each of the three values of α we show in Table 6 the ratios $|z_{\mathrm{B}}|/|z|$, $|z_{\mathrm{B}_1}|/|z|$ and $|z_{\mathrm{B}_2}|/|z|$ of the absolute values of the z-scores, where the denominator $|z|$ is based on the raw experimental observations and the numerators $|z_{\mathrm{B}_1}|$, $|z_{\mathrm{B}}|$ and $|z_{\mathrm{B}_2}|$ are based on the Bayesian estimates. Since a small value of a z-score represents a more accurate measurement, ratios smaller than one indicate that the Bayesian activity results are more

Lab No	x_{B_1}(Bq m^{-3})	u_{B_1}(Bq m^{-3})	x_{B_2}(Bq m^{-3})	u_{B_2}(Bq m^{-3})	$x_{B_2} - x_{B_1}$(Bq m^{-3})
1	47.58	1.10	47.75	1.09	**0.18**
2	40.18	0.18	40.35	0.24	**0.17**
3	43.73	2.72	45.82	2.95	**2.09**
4	41.56	1.10	42.34	1.16	**0.78**
5	53.37	1.10	53.49	1.10	**0.11**
6	43.09	1.41	43.74	1.33	**0.65**
7	43.55	1.64	44.33	1.56	**0.78**
8	42.86	1.87	44.11	1.91	**1.25**
9	53.20	4.44	55.05	4.26	**1.85**
10	61.94	1.50	62.10	1.50	**0.16**

Table 4: Estimates of the activity with distortion parameter $\alpha = 1.97$.

Lab No	x_{B_1}(Bq m^{-3})	u_{B_1}(Bq m^{-3})	x_{B_2}(Bq m^{-3})	u_{B_2}(Bq m^{-3})	$x_{B_2} - x_{B_1}$(Bq m^{-3})
1	47.57	1.10	47.83	1.08	**0.26**
2	40.18	0.18	40.43	0.26	**0.25**
3	43.67	2.69	46.58	2.97	**2.91**
4	41.55	1.10	42.63	1.16	**1.08**
5	53.36	1.10	53.53	1.09	**0.17**
6	43.07	1.40	43.98	1.31	**0.91**
7	43.53	1.64	44.62	1.53	**1.09**
8	42.83	1.86	44.56	1.90	**1.73**
9	53.00	4.43	55.65	4.19	**2.65**
10	61.91	1.50	62.15	1.49	**0.23**

Table 5: Estimates of the activity with distortion parameter $\alpha = 2.45$.

accurate than the values x initially determined. Ratios shown in bold highlight the smaller values in each line.

Figure 1 summarizes graphically the information contained in Table 6. It can be seen that the Bayesian approach based on the uniform prior leads in general to an improvement with respect to results produced ignoring prior knowledge. The only exception is laboratory 9, for which the ratio $|z_B|/|z|$ is larger than one for all three values of alpha (for all other laboratories, this ratio is at most equal to one). It is interesting to note that a concave distortion of the uniform prior will provide a more accurate result than the experimental one when the laboratory overestimates the activity (all laboratories except 5, 9 and 10). On the contrary, when the laboratory underestimates the activity (all laboratories except 5, 9 and 10) a convex distortion will lead to better results.

6 Conclusions

In this paper we have used a robust Bayesian approach to evaluate the modification induced in the results of radionuclide activity measurements when available prior infor-

Lab No	$\lvert z_{B_1}\rvert/\lvert z\rvert$	$\lvert z_B\rvert/\lvert z\rvert$	$\lvert z_{B_2}\rvert/\lvert z\rvert$	$\lvert z_{B_1}\rvert/\lvert z\rvert$	$\lvert z_B\rvert/\lvert z\rvert$	$\lvert z_{B_2}\rvert/\lvert z\rvert$	$\lvert z_{B_1}\rvert/\lvert z\rvert$	$\lvert z_B\rvert/\lvert z\rvert$	$\lvert z_{B_2}\rvert/\lvert z\rvert$
1	1.006	1.000	**0.955**	1.013	1.000	**0.929**	1.013	1.000	**0.894**
2	0.652	0.652	**0.644**	0.652	0.652	**0.640**	0.652	0.652	**0.635**
3	0.763	0.750	**0.581**	0.780	0.750	**0.500**	0.780	0.750	**0.404**
4	0.917	0.916	**0.858**	0.920	0.916	**0.831**	0.920	0.916	**0.798**
5	**0.996**	1.000	1.016	**0.990**	1.000	1.024	**0.990**	1.000	1.037
6	0.995	0.990	**0.930**	0.999	0.990	**0.901**	0.999	0.990	**0.866**
7	0.993	0.989	**0.911**	1.000	0.989	**0.873**	1.000	0.989	**0.829**
8	0.910	0.905	**0.799**	0.917	0.905	**0.747**	0.917	0.905	**0.689**
9	**0.939**	1.007	1.266	**0.845**	1.007	1.404	**0.845**	1.007	1.570
10	**0.997**	1.000	1.005	**0.993**	1.000	1.008	**0.993**	1.000	1.012
α		1.61			1.97			2.45	

Table 6: Quotient $|z_{B_1}|/|z|$, $|z_B|/|z|$ and $|z_{B_2}|/|z|$ for the different values of α.

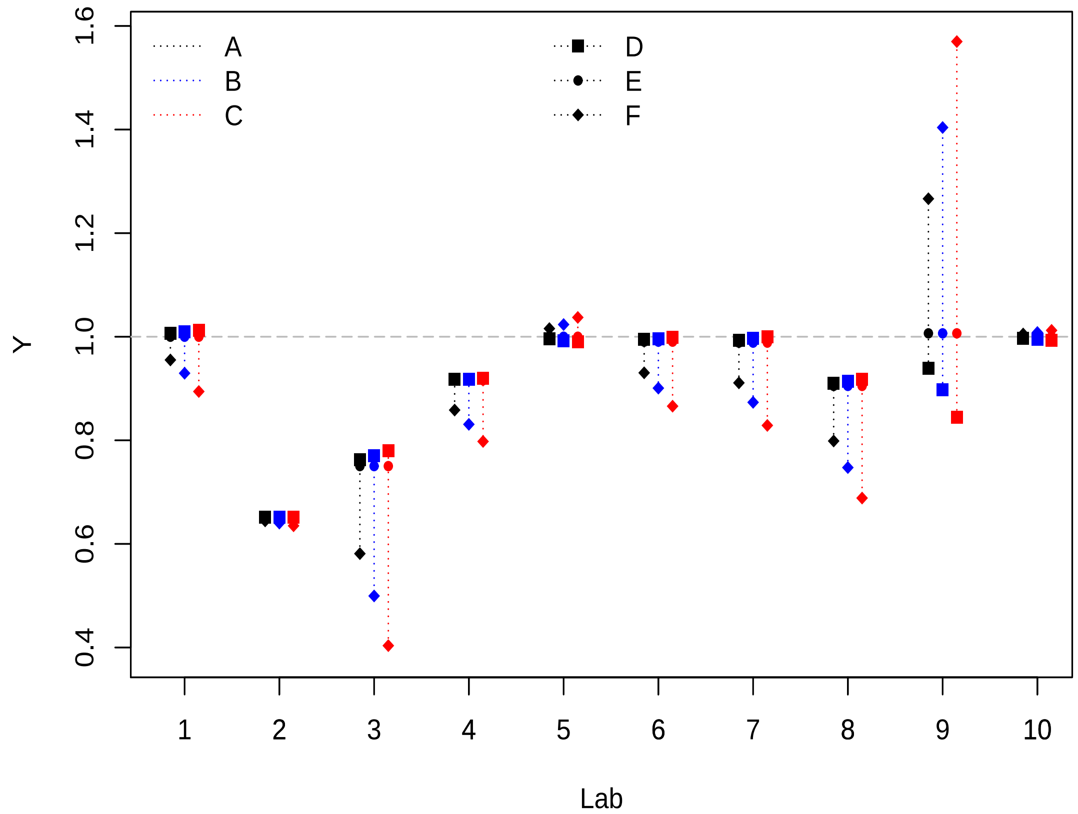

Figure 1: Quotient $|z_{B_1}|/|z|$, $|z_B|/|z|$ and $|z_{B_2}|/|z|$ (Bayesian z-score with respect to the classic one) for the $^{239+240}Pu$ analysis in the INTER/CSN-2004 Proficiency Test.

mation is taken into account. For this purpose, we have developed a methodology based on a class of priors instead of considering only a uniform specific prior. This class of priors, recently published in the literature, is intended to overcome the common criticism about the arbitrariness and bias implied by choosing a single prior to represent the available information. In particular, it allows us to assess both the uncertainty contained in such information and the influence produced by distorting the chosen prior in the final value of the activity.

We applied this methodology to an example involving inter-laboratory proficiency test data obtained by the laboratories that perform radiochemical analysis for the Spanish radioactive monitoring network. We have shown that, in our example, perturbing the prior information does not cause a significant change in the final Bayesian result of the activity, even for the largest distortions. Therefore, the model can be considered robust from a Bayesian point of view. Moreover, with respect to the performance of the Bayesian approach itself, we have shown that in general the results of each laboratory improve when the available prior information is incorporated. In particular, the use of a flat (uniform) prior distribution based on knowing the minimum and maximum activity levels is then justified. The choice of a different and more sophisticated prior distribution would lead basically to the same (improved) final results.

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