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New optimality conditions for multiobjective fuzzy programming problems

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Abstract

In this paper we study fuzzy multiobjective optimization problems defined for n variables. Based on a new p-dimensional fuzzy stationary-point definition, necessary efficiency conditions are obtained. And we prove that these conditions are also sufficient under new fuzzy generalized convexity notions. Furthermore, the results are obtained under general differentiability hypothesis.

Keywords: Multiobjective fuzzy optimization, generalized differentiable fuzzy functions, fuzzy generalized convexity.

1 Introduction

Fuzzy number and fuzzy mapping concepts have been widely studied by many authors since their introduction by Zadeh [17] and Chang and Zadeh [6]. One of these research lines has been the fuzzy optimization.

In classic optimization methods, it is a priority to establish necessary optimality conditions which enable the identification of possible candidates for optimal solutions. In the crisp optimization problem, the optimizers are also stationary points and all stationary points can be found by setting the gradient to zero. From these stationary points, we can separately ascertain whether each stationary point is an optimal solution or not. Thus, a further major part of Optimization Theory involves establishing sufficient optimality conditions for the elimination of non-optimal candidates. It is therefore necessary to use certain intrinsic properties of problems.

In [8], a scalar G-differentiable fuzzy mapping defined on $\mathbb{R}$ is considered and a necessary optimality condition based on a new fuzzy stationary point definition is obtained. This new stationary-point notion presents major advantages, from practical and theoretical points of view, as compared to the existing notions which are restrictive and difficult to compute. It is proved that this condition is also sufficient under new fuzzy generalized convexity conditions.

Now, in this paper, the results obtained in [8] from different approaches are generalized in order to provide results of a more applicable nature. We first consider fuzzy mappings defined on $\mathbb{R}^n$, with $n \geq 1$. Futhermore, let us suppose that such functions are gH-differentiable, which is a less strict hypothesis than the G-differentiability assumed in [8]. And finally, we consider that our objective is to simultaneously optimize p objectives that are usually in conflict. Our objectives are modelled by p fuzzy functions.

In [15] and [16] the sufficient KKT optimality conditions for multiobjective programming problems with level-wise continuously differentiable fuzzy-valued objective functions are derived. The solution concepts for these kinds of problems follows the concept of nondominated solution adopted in the crips multiobjective programming problems. To the best of our knowledge, for multiobjective fuzzy problems, no necessary optimality conditions for efficiency can be found in the literature that enable us to suitably identify the possible candidates for an efficient solution. This is

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the main contribution of our paper, and these conditions are established for gH-differentiable fuzzy functions. The gH-differentiability is the most general fuzzy differentiability concept defined to date (see [1, 11, 14]).

This paper is organised as follows: in Section 2 we introduce the arithmetic and basic properties for intervals and fuzzy numbers. In Section 3, the generalized Hukuhara difference (in short, gH-difference) is employed to define the difference between any fuzzy numbers. Using gH-difference and limit concepts for fuzzy functions, we consider the gH-derivative for a fuzzy function. In Section 4, optimum solution concepts are defined for a p-dimensional fuzzy function. In Section 5, we define a p-dimensional stationary point for a p-dimensional fuzzy function and we prove necessary efficiency conditions for the solution concepts given in Section 4. In Section 6, a new generalized convexity concept is introduced in order to prove sufficient efficiency conditions.

2 Preliminaries

The family of all bounded closed intervals in $\mathbb{R}$ is denoted by $\mathcal{K}_C$, that is, $\mathcal{K}_C = \{[\underline{a}, \overline{a}] \ / \ \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$. For $A = [\underline{a}, \overline{a}]$, $B = \left[\underline{b}, \overline{b}\right] \in \mathcal{K}_C$, and $\nu \in \mathbb{R}$, we consider the following operations

$$A + B = [\underline{a}, \overline{a}] + \left[\underline{b}, \overline{b}\right] = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right], \tag{1}$$

$$\nu A = \nu\,[\underline{a}, \overline{a}] = \begin{cases} [\nu\underline{a}, \nu\overline{a}] & \text{if} \quad \nu \geq 0, \\ [\nu\overline{a}, \nu\underline{a}] & \text{if} \quad \nu < 0, \end{cases} \tag{2}$$

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a) \ \ A = B + C, & \text{or} \\ (b) \ \ B = A + (-1)C. \end{cases} \tag{3}$$

This difference (3), called the generalized Hukuhara difference ($gH-$difference in short) has highly interesting properties compared to other definitions (Minskowki, Hukuhara differences), for example $A \ominus_{gH} A = \{0\} = [0, 0]$. Futhermore, the gH-difference of two intervals $A = [\underline{a}, \overline{a}]$ and $B = \left[\underline{b}, \overline{b}\right]$ always exists and is equal to ([13])

$$A \ominus_{gH} B = \left[\min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}, \max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}\right].$$

Given two intervals, we define the distance between A and B as $H(A, B) = \max\left\{|\underline{a} - \underline{b}|, \left|\overline{a} - \overline{b}\right|\right\}$. It is well-known that $(\mathcal{K}_C, H)$ is a complete metric space.

We consider the following order relation in $\mathcal{K}_C$

Definition 2.1. *Let $A, B \in \mathcal{K}_C$ be. It is said that*

- $A \underline{\preceq} B \Leftrightarrow \underline{a} \leq \underline{b}$ *and* $\bar{a} \leq \bar{b}$.
- $A \preceq B \Leftrightarrow A \underline{\preceq} B$ *and* $A \neq B$, *i.e.* $\underline{a} \leq \underline{b}$ *and* $\bar{a} \leq \bar{b}$, *with a strict inequality.*
- $A \prec B \Leftrightarrow \underline{a} < \underline{b}$ *and* $\bar{a} < \bar{b}$.

It is clear that $A \prec B \Rightarrow A \preceq B \Rightarrow A \underline{\preceq} B$.

A fuzzy set on $\mathbb{R}^n$ is a mapping defined as $u : \mathbb{R}^n \to [0, 1]$. The α-level set of a fuzzy set, $0 \leq \alpha \leq 1$, is defined as

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R}^n | u(x) \geq \alpha\} & \text{if } \alpha \in (0, 1], \\ cl(supp \ \ u) & \text{if } \alpha = 0, \end{cases}$$

where $cl(supp \ \ u)$ denotes the closure of the support of u, $supp(u) = \{x \in \mathbb{R}^n |\ u(x) > 0\}$.

Definition 2.2. *A fuzzy number is a fuzzy set u on $\mathbb{R}$ with the following properties:*

1. *u is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;*
2. *u is an upper semi-continuous function;*
3. *$u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$, $x, y \in \mathbb{R}$, $\lambda \in [0, 1]$;*
4. *$[u]^0$ is compact.*

Let $\mathcal{F}_C$ be the set of all fuzzy numbers on $\mathbb{R}$.

Definition 2.3. *A p-dimensional fuzzy number u on $\mathbb{R}$ is defined as a mapping, $u : \mathbb{R} \to [0,1]^p$, $u = (u_1, ..., u_p)$ where each u_i is a fuzzy number.*
Let $\mathcal{F}_C^p$ be the family of all p-dimensional fuzzy numbers, that is, $u \in \mathcal{F}_C^p$ if $u = (u_1, ..., u_p)$ where each $u_i \in \mathcal{F}_C$.

There exists a well-known L-U representation of a fuzzy number.

Theorem 2.4. [7] *A fuzzy number is completely determined by any pair $u = (\underline{u}, \overline{u})$ of functions $\underline{u}, \overline{u} : [0,1] \to \mathbb{R}$, that define the endpoints of the α-level sets, and satisfy the following three conditions:*
- $\underline{u}(\alpha) = \underline{u}_\alpha \in \mathbb{R}$ *is a bounded nondecreasing left-continuous function in $(0,1]$ and it is right-continuous at 0;*
- $\overline{u}(\alpha) = \overline{u}_\alpha \in \mathbb{R}$ *is a bounded nonincreasing left-continuous function in $(0,1]$ and it is right-continuous at 0;*
- $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, *for all $\alpha \in [0,1]$.*

Obviously, if $u \in \mathcal{F}_C$, then $[u]^\alpha \in \mathcal{K}_C$ for all $\alpha \in [0,1]$ and thus the α-level sets of a fuzzy number are given by $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, $\underline{u}_\alpha, \overline{u}_\alpha \in \mathbb{R}$ for all $\alpha \in [0,1]$.

For any $u \in \mathcal{F}_C^p$, therefore

$$[u]^\alpha = \begin{cases} \{x = (x_1, ..., x_p) \in \mathbb{R}^p |\ u_i(x_i) \geq \alpha,\ \forall i = 1, ..., p\} \text{ if } \alpha \in (0,1), \\ \{x = (x_1, ..., x_p) \in \mathbb{R}^p |\ x_i \in cl(supp\, u_i),\ \forall i = 1, ..., p\} \text{ if } \alpha = 0. \end{cases}$$

Geometrically, this is a Cartesian product of closed intervals in $\mathbb{R}^p$

$$[u]^\alpha = \prod_{i=1}^{p} [u_i]^\alpha. \tag{4}$$

Lemma 2.5. *It is verified that $[u]^\alpha \subseteq [u]^0$, for all $\alpha \in [0,1]$, for all $u \in \mathcal{F}_C^p$.*

Proof. If $x \in [u]^\alpha$ then $u_i(x_i) \geq \alpha \geq 0$, for all $i = 1, ...p$ and then $x \in [u]^0$. □

For fuzzy numbers, $u, v \in \mathcal{F}_C$, represented by $[\underline{u}_\alpha, \overline{u}_\alpha]$ and $[\underline{v}_\alpha, \overline{v}_\alpha]$ respectively, and for any real number θ, we define the following operations between fuzzy numbers:

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}, \quad (\theta u)(x) = \begin{cases} u\left(\frac{x}{\theta}\right), & \text{if } \theta \neq 0, \\ 0, & \text{if } \theta = 0, \end{cases}$$

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} & (i)\ u = v + w, \\ \text{or} & (ii)\ v = u + (-1)w. \end{cases}$$

It is known that, for every $\alpha \in [0,1]$,

$$[u+v]^\alpha = [(\underline{u+v})_\alpha, (\overline{u+v})_\alpha] = [\underline{u}_\alpha + \underline{v}_\alpha\ ,\ \overline{u}_\alpha + \overline{v}_\alpha]\,, \tag{5}$$

$$[\theta u]^\alpha = \left[(\underline{\theta u})_\alpha, (\overline{\theta u})_\alpha\right] = \theta [u]^\alpha = \theta\, [\underline{u}_\alpha, \overline{u}_\alpha] = [\min\{\theta \underline{u}_\alpha, \theta \overline{u}_\alpha\}, \max\{\theta \underline{u}_\alpha, \theta \overline{u}_\alpha\}]\,, \tag{6}$$

and if $u \ominus_{gH} v$ exists, then in terms of α-level sets, we can deduce that (see [13, 12]):

$$[u \ominus_{gH} v]^\alpha = [u]^\alpha \ominus_{gH} [v]^\alpha = [\min\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}, \max\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}]\,. \tag{7}$$

Given $u, v \in \mathcal{F}_C$, we define the distance between u and v as

$$D(u,v) = \sup_{\alpha \in [0,1]} H\left([u]^\alpha, [v]^\alpha\right) = \sup_{\alpha \in [0,1]} \max\left\{|\underline{u}_\alpha - \underline{v}_\alpha|\,, |\overline{u}_\alpha - \overline{v}_\alpha|\right\}.$$

Therefore, $(\mathcal{F}_C, D)$ is a complete metric space.

We recall the usual order relations between fuzzy numbers [8]:

Definition 2.6. *For $u, v \in \mathcal{F}_C$, it is said that*

1. *$u \underline{\preceq} v$ if $[u]^\alpha \underline{\preceq} [v]^\alpha$ for every $\alpha \in [0,1]$.*
2. *$u \preceq v$ if $u \underline{\preceq} v$ and $u \neq v$, i. e. $[u]^\alpha \underline{\preceq} [v]^\alpha$ for every $\alpha \in [0,1]$, and $\exists \alpha_0 \in [0,1]$, such that $[u]^{\alpha_0} \preceq [v]^{\alpha_0}$.*
3. *$u \prec v$ if $u \underline{\preceq} v$ and $\exists \alpha_0 \in [0,1]$, such that $[u]^{\alpha_0} \prec [v]^{\alpha_0}$.*

Note that $\underline{\preceq}$ is a partial order relation on $\mathcal{F}_C$. Hence, $v \underline{\succeq} u$ can be written instead of $u \underline{\preceq} v$. We observe that if $u \prec v$ then $u \preceq v$ and therefore $u \underline{\preceq} v$.

Remark 2.1. *We recall that if $a, b \in \mathbb{R}^p$ then $a \geqq b \Leftrightarrow a_j \geq b_j$, $\forall j = 1, ..., p$; $a \geq b \Leftrightarrow a_j \geq b_j$, $\forall j = 1, ..., p$ and $\exists r$ such that $a_r > b_r$ and $a > b \Leftrightarrow a_j > b_j$, $\forall j = 1, ..., p$.*

3 gH-Differentiable fuzzy functions

Henceforth, S denotes an open subset of $\mathbb{R}^n$ and T denotes an open subset of $\mathbb{R}$. Let us consider $\tilde{f} : S \to \mathcal{F}_C$ as a fuzzy function or fuzzy mapping. For each $\alpha \in [0,1]$, we associate with $\tilde{f}$ the family of interval-valued functions $\tilde{f}_\alpha : S \to \mathcal{K}_C$ given by $\tilde{f}_\alpha(x) = [\tilde{f}(x)]^\alpha$. For any $\alpha \in [0,1]$, we denote

$$\tilde{f}_\alpha(x) = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right] = \left[\underline{f}(\alpha, x), \overline{f}(\alpha, x)\right].$$

Here, for each $\alpha \in [0,1]$, the real-valued endpoint functions $\underline{f}_\alpha, \overline{f}_\alpha : S \to \mathbb{R}$ are called lower and upper functions of $\tilde{f}$, respectively.

We can now present the gH-differentiable fuzzy functions concept based on the gH-difference of fuzzy numbers.

Definition 3.1. [2] *The gH-derivative of a fuzzy function $\tilde{f} : T \to \mathcal{F}_C$ at $t_0 \in T$ is defined as*

$$\tilde{f}'(t_0) = \lim_{h \to 0} \frac{1}{h}\left[\tilde{f}(t_0 + h) \ominus_{gH} \tilde{f}(t_0)\right]. \tag{8}$$

If $\tilde{f}'(t_0) \in \mathcal{F}_C$ that satisfies (8) exists, then we can say that $\tilde{f}$ is generalized Hukuhara differentiable (gH-differentiable, in short) at t_0.

The following results establish relationships between the gH-differentiability of $\tilde{f}$ and the gH-differentiability of the associated family of interval-valued functions $\tilde{f}_\alpha$, (see [4]), as well as relationships between the gH-differentiability of $\tilde{f}$ and the differentiability of its real-valued endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$ for each $\alpha \in [0,1]$ (see [5]).

Theorem 3.2. *If $\tilde{f} : T \to \mathcal{F}_C$ is gH-differentiable at $t_0 \in T$, then $\tilde{f}_\alpha$ is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$ and $\tilde{f}'_\alpha(t_0) = [\tilde{f}'(t_0)]^\alpha$, for all $\alpha \in [0,1]$.*

Theorem 3.3. *Let $\tilde{f} : T \to \mathcal{F}_C$ be a fuzzy function. If $\tilde{f}$ is gH-differentiable at $t_0 \in T$, then the lateral derivatives of real-valued endpoint functions $(\underline{f}_\alpha)'_-(t_0)$, $(\underline{f}_\alpha)'_+(t_0)$, $(\overline{f}_\alpha)'_-(t_0)$ and $(\overline{f}_\alpha)'_+(t_0)$ exist uniformly in $\alpha \in [0,1]$ and satisfy one of the following cases:*

(a) $(\underline{f}_\alpha)'_-(t_0) = (\underline{f}_\alpha)'_+(t_0)$ *and* $(\overline{f}_\alpha)'_+(t_0) = (\overline{f}_\alpha)'_-(t_0)$*, and therefore $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at t_0, uniformly in $\alpha \in [0,1]$. Moreover*

$$[\tilde{f}'(t_0)]^\alpha = \left[\min\left\{(\underline{f}_\alpha)'(t_0), (\overline{f}_\alpha)'(t_0)\right\}, \max\left\{(\underline{f}_\alpha)'(t_0), (\overline{f}_\alpha)'(t_0)\right\}\right] \quad \forall \alpha \in [0,1];$$

(b) $(\underline{f}_\alpha)'_-(t_0) = (\overline{f}_\alpha)'_+(t_0)$ *and* $(\underline{f}_\alpha)'_+(t_0) = (\overline{f}_\alpha)'_-(t_0)$*. Moreover*

$$\begin{aligned} [\tilde{f}'(t_0)]^\alpha &= \left[\min\left\{(\underline{f}_\alpha)'_-(t_0), (\overline{f}_\alpha)'_-(t_0)\right\}, \max\left\{(\underline{f}_\alpha)'_-(t_0), (\overline{f}_\alpha)'_-(t_0)\right\}\right] \\ &= \left[\min\left\{(\underline{f}_\alpha)'_+(t_0), (\overline{f}_\alpha)'_+(t_0)\right\}, \max\left\{(\underline{f}_\alpha)'_+(t_0), (\overline{f}_\alpha)'_+(t_0)\right\}\right] \\ &\qquad \forall \alpha \in [0,1]. \end{aligned}$$

Remark 3.1. *For convenience, we denote $[\tilde{f}'(t_0)]^\alpha = \left[(\tilde{f}'_\alpha(t_0))^L, (\tilde{f}'_\alpha(t_0))^U\right]$, where*

$$(\tilde{f}'_\alpha(t_0))^L = \min\left\{(\underline{f}_\alpha)'_+(t_0), (\overline{f}_\alpha)'_+(t_0)\right\} = \min\left\{(\underline{f}_\alpha)'_-(t_0), (\overline{f}_\alpha)'_-(t_0)\right\},$$

$$(\tilde{f}'_\alpha(t_0))^U = \max\left\{(\underline{f}_\alpha)'_+(t_0), (\overline{f}_\alpha)'_+(t_0)\right\} = \max\left\{(\underline{f}_\alpha)'_-(t_0), (\overline{f}_\alpha)'_-(t_0)\right\}.$$

Let us define the partial gH-derivative for a fuzzy function $\tilde{f}$ defined on $S \subset \mathbb{R}^n$.

Definition 3.4. ([3], [15]) *Let $\tilde{f}$ be a fuzzy function defined on $S \subset \mathbb{R}^n$ and let $x_0 = \left(x_1^{(0)}, ..., x_n^{(0)}\right)$ be a fixed element of S. We consider the fuzzy function $\tilde{h}_i(x_i) = \tilde{f}(x_1^{(0)}, ..., x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, ..., x_n^{(0)})$. If $\tilde{h}_i$ is gH-differentiable at $x_i^{(0)}$, then we say that $\tilde{f}$ has the ith partial gH-derivative at x_0 (denoted by $\left(\partial\tilde{f}/\partial x_i\right)(x_0)$) and $(\partial\tilde{f}/\partial x_i)(x_0) = (\tilde{h}_i)'(x_i^{(0)})$.*

We say that $\tilde{f}$ is gH-differentiable at x_0 if all the partial gH-derivatives $\left(\partial\tilde{f}/\partial x_1\right)(x_0)$,..., $\left(\partial\tilde{f}/\partial x_n\right)(x_0)$ exist in a neighborhood of x_0.

From the definition above we can give the definition of gradient for a fuzzy function as follows:

Definition 3.5. *Let $\tilde{f} : S \to \mathcal{F}_C$ be a gH-differentiable fuzzy function at $x_0 \in S$. The gradient of $\tilde{f}$ at x_0, denoted by $\tilde{\nabla}\tilde{f}(x_0)$, becames a p-dimensional fuzzy number, defined by*

$$\tilde{\nabla}\tilde{f}(x_0) = \left(\left(\frac{\partial \tilde{f}}{\partial x_1}\right)(x_0), ..., \left(\frac{\partial \tilde{f}}{\partial x_n}\right)(x_0)\right) \in \mathcal{F}_C^n, \tag{9}$$

where each fuzzy number $(\partial\tilde{f}/\partial x_j)(x_0)$ is the jth partial gH-derivative of $\tilde{f}$ at x_0.

Notice that if $\tilde{f}$ is gH-differentiable at x_0, then $\left(\partial\tilde{f}/\partial x_i\right)(x_0)$ is a fuzzy number whose α-level sets, $\left[\dfrac{\partial \tilde{f}}{\partial x_i}(x_0)\right]^\alpha$, are given by Theorem 3.3:

$$\left[\tilde{\nabla}\tilde{f}(x_0)\right]^\alpha = \left(\left[\frac{\partial \tilde{f}}{\partial x_1}(x_0)\right]^\alpha, ..., \left[\frac{\partial \tilde{f}}{\partial x_n}(x_0)\right]^\alpha\right),$$

where $\left[\frac{\partial \tilde{f}}{\partial x_i}(x_0)\right]^\alpha$ is a closed interval of $\mathbb{R}$ for all $\alpha \in [0,1]$. In other words, $\left[\tilde{\nabla}\tilde{f}(x_0)\right]^\alpha$ is a n-upla of which each component is a real closed interval.

Remark 3.2. *There are other derivative concepts for fuzzy functions which are used in fuzzy optimization. Notice that the gH-differentiability coincides with the H-differentiability [11], only when $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable and $(\underline{f}_\alpha)'(x) \leq (\overline{f}_\alpha)'(x)$ for all $\alpha \in [0,1]$. The gH-differentiability coincides with level-wise differentiability ([14]) if Theorem 3.3 (a) is verified. Futhermore, G-differentiability implies gH-differentiability (see [1]).*

Definition 3.6. *Given a p-dimensional fuzzy function $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_p) \in \mathcal{F}_C^p$, we say that $\tilde{f}$ is a p-dimensional gH-differentiable fuzzy function at $x_0 \in S$, if and only if, $\tilde{f}_j$ is gH-differentiable at x_0, for all $j = 1, ..., p$.*

From definition above, if $\tilde{f} : S \subset \mathbb{R}^n \to \mathcal{F}_C^p$, then for each $j = 1, ..., p$

$$[\tilde{\nabla}\tilde{f}_j(x_0)]^\alpha = \left(\left[\frac{\partial \tilde{f}_j}{\partial x_1}(x_0)\right]^\alpha, ..., \left[\frac{\partial \tilde{f}_j}{\partial x_n}(x_0)\right]^\alpha\right).$$

Therefore, for each $i = 1, ..., n$:

$$\left[\frac{\partial \tilde{f}}{\partial x_i}(x_0)\right]^\alpha = \left(\left[\frac{\partial \tilde{f}_1}{\partial x_i}(x_0)\right]^\alpha, ..., \left[\frac{\partial \tilde{f}_p}{\partial x_i}(x_0)\right]^\alpha\right) \in \mathcal{K}_C^p$$

From Remark 3.1, for each $j = 1, ..., p$ and for each $i = 1, ...n$

$$\left[\frac{\partial \tilde{f}_j}{\partial x_i}(x_0)\right]^\alpha = \left[\left(\frac{\partial \tilde{f}_{j_\alpha}}{\partial x_i}(x_0)\right)^L, \left(\frac{\partial \tilde{f}_{j_\alpha}}{\partial x_i}(x_0)\right)^U\right].$$

For convenience, if we denote $\tilde{f}_\alpha = (\tilde{f}_{1_\alpha}, ..., \tilde{f}_{p_\alpha})$, then

$$\left[\frac{\partial \tilde{f}}{\partial x_i}(x_0)\right]^\alpha = \left[\left(\frac{\partial \tilde{f}_\alpha}{\partial x_i}(x_0)\right)^L, \left(\frac{\partial \tilde{f}_\alpha}{\partial x_i}(x_0)\right)^U\right]. \tag{10}$$

4 Multiobjective fuzzy optimization

We need to interpret the meaning of "minimize a p-dimensional fuzzy function". We are going to follow a similar solution concept to that of the nondominated solution introduced by Pareto, which is usually considered in real-valued multiobjective optimization.

Definition 4.1. *Let $\tilde{f} : S \subseteq \mathbb{R}^n \to \mathcal{F}_C^p$ be a p-dimensional fuzzy function. It is said that $x^* \in S$ is:*

1. *a strongly efficient solution if there exists no* $x \in S$ *such that* $\tilde{f}(x) \underline{\preceq} \tilde{f}(x^*)$ *and* $\tilde{f}(x) \neq \tilde{f}(x^*)$*;*

2. *an efficient solution if there exists no* $x \in S$ *such that* $\tilde{f}_j(x) \underline{\preceq} \tilde{f}_j(x^*)$*,* $\forall j = 1, ..., p$ *and* $\exists k$ *such that* $\tilde{f}_k(x) \prec \tilde{f}_k(x^*)$*;*

3. *a mildly weakly efficient solution if there exists no* $x \in S$ *such that* $\tilde{f}_j(x) \preceq \tilde{f}_j(x^*)$*,* $\forall j = 1, ..., p$*;*

4. *a weakly efficient solution if there exists no* $x \in S$ *such that* $\tilde{f}(x) \prec \tilde{f}(x^*)$*.*

The following relations are immediate:

$$\begin{array}{ccc} \text{efficient} & \Leftarrow & \text{strongly efficient} \\ \Downarrow & & \Downarrow \\ \text{weakly efficient} & \Leftarrow & \text{mildly weakly efficient} \end{array}$$

We focus our study, therefore, on the weakly efficient solutions because these represent the most general class of solutions and we prove necessary and sufficient conditions for this solution class.

Remark 4.1. *If* $\tilde{f} : \mathbb{R}^n \to \mathcal{F}_C^p$*, such that* $\tilde{f} = \chi_{\{f\}}$*, where* $f : \mathbb{R}^n \to \mathbb{R}^p$*, that is,* $\tilde{f}$ *is a real-valued or crisp vector function:*

- *fuzzy strongly efficient* $\equiv$ *fuzzy efficient* $\equiv$ *crisp efficient Pareto solution, and*
- *fuzzy mildly weakly efficient* $\equiv$ *fuzzy weakly efficient* $\equiv$ *crisp weakly efficient Pareto solution.*

Remark 4.2. *From* [8]*, that is, if* $\tilde{f} : T \subseteq \mathbb{R} \to \mathcal{F}_C$*, then relationships between the minimum and efficient solution definitions would be:*

- *efficient solution* $\equiv$ *weakly efficient solution* $\equiv$ *weak minimum, and*
- *strongly efficient solution* $\equiv$ *mildly weakly efficient solution* $\equiv$ *minimum.*

Example 4.2. *Trapezoidal fuzzy numbers are a special type of fuzzy numbers which are well determined by four real numbers* $a \leq b \leq c \leq d$*. We write* (a, b, c, d) *to denote trapezoidal fuzzy numbers whose core or 1-level is given by* $[b, c]$*. If* $b = c$*, then we denote* (a, b, c) *which is a triangular fuzzy number, whose* α*-levels are*

$$[(a,b,c)]^\alpha = [a + (b-a)\alpha,\ (b-c)\alpha + c] \quad \textit{and} \quad [(a,b,c,d)]^\alpha = [(b-a)\alpha + a,\ (c-d)\alpha + d].$$

Consider the 2*-dimensional fuzzy function* $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) : \mathbb{R}^2 \to \mathcal{F}_C^2$ *defined by*

$$\tilde{f}_1(x_1, x_2) = (1,1,1) \cdot x_1^2 + (0,0,1) \cdot x_2^2$$

$$\tilde{f}_2(x_1, x_2) = (1,1,1,1) \cdot x_1^2 + (1,2,4,6) \cdot x_2$$

Therefore,

$$\left[\tilde{f}_1(x_1,x_2)\right]^\alpha = \left[x_1^2\ ,\ x_1^2 + (1-\alpha)x_2^2\right],$$

$$\left[\tilde{f}_2(x_1,x_2)\right]^\alpha = \begin{cases} \left[x_1^2 + (1+\alpha)x_2,\ x_1^2 + 2(3-\alpha)x_2\right] & \textit{if} \quad x_1 \in \mathbb{R} \textit{ and } x_2 \geq 0; \\ \left[x_1^2 + 2(3-\alpha)x_2,\ x_1^2 + (1+\alpha)x_2\right] & \textit{if} \quad x_1 \in \mathbb{R} \textit{ and } x_2 < 0. \end{cases}$$

Hence,

- *Let* $x^* = (0, x_2)$ *with* $x_2 > 0$*.* x^* *is not a strongly efficient or efficient or mildly weakly efficient solution because, for* $x = (0, -x_2)$*,* $\tilde{f}_1(x) = \tilde{f}_1(x^*)$ *and*

$$[\tilde{f}_2(x)]^\alpha = [-2(3-\alpha)x_2,\ -(1+\alpha)x_2] \prec [0,\ 0] \prec [(1+\alpha)x_2,\ 2(3-\alpha)x_2] = [\tilde{f}_2(x^*)]^\alpha.$$

  However, x^* *is a weakly efficient solution because another* x *such that* $[\tilde{f}_1(x)]^{\alpha_0} \prec [\tilde{f}_1(x^*)]^{\alpha_0}$ *for some* α_0 *cannot exist.*

- *Let* $x^* = (0, -1)$*.* x^* *is a strongly efficient solution for* $\tilde{f}$*, and therefore it is a weakly efficient solution, because if* $\tilde{f}_1(x) \preceq \tilde{f}_1(x^*)$ *for* $x = (x_1, x_2)$ *then* $x_1 = 0$ *and* $|x_2| < 1$*. Moreover, if* $\tilde{f}_2(x) \preceq \tilde{f}_2(x^*)$*, then* $x_2 < -1$*, which is a contradiction.*

- *Let* $x^* = (0, x_2)$*.* x^* *is a weakly efficient solution* $\forall x_2 \in \mathbb{R}$*, because* $\tilde{f}_1(y) \prec \tilde{f}_1(x^*)$ *cannot verified.*

Example 4.3. *Consider the 2-dimensional fuzzy function $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) : \mathbb{R}^2 \to \mathcal{F}_C^2$ defined by*

$$\tilde{f}_1(x_1, x_2) = (0, 0, 1, 1) \cdot x_1^2 + (1, 2, 3, 4) \cdot x_2$$

$$\tilde{f}_2(x_1, x_2) = (1, 1, 1, 1) \cdot x_1^2 + (1, 2, 4, 6) \cdot x_2$$

Therefore,

$$\left[\tilde{f}_1(x_1, x_2)\right]^\alpha = \begin{cases} \left[(1+\alpha)x_2,\, x_1^2 + (\alpha+4)x_2\right] & \text{if} \quad x_1 \in \mathbb{R} \text{ and } x_2 \geq 0; \\ \left[(\alpha+4)x_2,\, x_1^2 + (1+\alpha)x_2\right] & \text{if} \quad x_1 \in \mathbb{R} \text{ and } x_2 < 0. \end{cases}$$

$$\left[\tilde{f}_2(x_1, x_2)\right]^\alpha = \begin{cases} \left[x_1^2 + (1+\alpha)x_2,\, x_1^2 + 2(3-\alpha)x_2\right] & \text{if} \quad x_1 \in \mathbb{R} \text{ and } x_2 \geq 0; \\ \left[x_1^2 + 2(3-\alpha)x_2,\, x_1^2 + (1+\alpha)x_2\right] & \text{if} \quad x_1 \in \mathbb{R} \text{ and } x_2 < 0. \end{cases}$$

It can therefore be deduced that $\tilde{f}$ does not have weakly efficient solutions because, for all x, it is possible to find another y such that $\tilde{f}(y) \prec \tilde{f}(x)$. For example, for $x = (0, 0)$, there exists $y = (0, -1)$ and

$$[\tilde{f}(x)]^\alpha = ([0, 0], [0, 0]),$$

$$[\tilde{f}(y)]^\alpha = ([-(4+\alpha),\, -(1+\alpha)],\, [-2(3-\alpha),\, -(1+\alpha)]),$$

such that

$$[\tilde{f}(y)]^\alpha \prec [\tilde{f}(x)]^\alpha, \quad \forall \alpha \in [0, 1].$$

Remark 4.3. *For convenience, we introduce the following notations.*

1. *Let $A = [\underline{a}, \overline{a}] \in \mathcal{K}_C$ and let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. We denote as $\lambda \times A$ the lineal combination:*

   $$\lambda \times A = \lambda_1 \underline{a} + \lambda_2 \overline{a}.$$

   $\lambda \times A = 0$ with $\lambda \geq 0$ is equivalent to $0 \in [\underline{a}, \overline{a}]$, because $\lambda_1 \underline{a} + \lambda_2 \overline{a} = 0$ with $\lambda_1, \lambda_2 \geq 0$, but the two λ_1, λ_2 are not simultaneously zero, if and only, if $\underline{a} \leq 0 \leq \overline{a}$, that is, $0 \in [\underline{a}, \overline{a}]$.

2. *Let $A = (A_1, ..., A_p)$, where $A_j \in \mathcal{K}_C$. Let $\Lambda \in \mathcal{M}^{p \times 2}$, we denote as $\Lambda \times A$ the lineal combination:*

   $$\Lambda \times A = \sum_{j=1}^{p} \lambda_j \times A_j = \sum_{j=1}^{p} \lambda_{j1} \underline{a}_j + \lambda_{j2} \overline{a}_j.$$

   If $0 \in A_j$, for some $j = 1, ..., p$, then there exists $\Lambda \in \mathcal{M}^{p \times 2}$, with $\lambda_{ij} \geq 0$, but not all zero, such that $\Lambda \times A = 0$. If $\Lambda \times A = 0$ with $\lambda_{ij} \geq 0$, nor all λ_{ij} simultaneosuly zero, then there exist $\underline{\lambda}, \overline{\lambda}$, $(\underline{\lambda}, \overline{\lambda}) \geq 0$ such that $0 \in \left[\underline{\lambda}^T \underline{a},\, \overline{\lambda}^T \overline{a}\right]$, where $\underline{a} = (\underline{a}_1, ..., \underline{a}_p)$ and $\overline{a} = (\overline{a}_1, ..., \overline{a}_p)$.

3. *If $a \in \mathbb{R}^m$ and $u \in \mathcal{F}_C^m$, then*

   - *$au = (a_1 u_1, ..., a_m u_m) \in \mathcal{F}_C^m$ such that $[au]^\alpha = \prod_{i=1}^m [a_i u_i]^\alpha = \prod_{i=1}^m a_i [u_i]^\alpha$.*
   - *$a^T u = \displaystyle\sum_{i=1}^{m} a_i u_i \in \mathcal{F}_C$ and, with respect to α-level sets,*

   $$[a^T u]^\alpha = \sum_{i=1}^{m} a_i [u_i]^\alpha = \left[\sum_{a_i \geq 0} a_i \underline{u_i}_\alpha + \sum_{a_i < 0} a_i \overline{u}_{i\alpha},\, \sum_{a_i \geq 0} a_i \overline{u}_{i\alpha} + \sum_{a_i < 0} a_i \underline{u_i}_\alpha\right].$$

Lemma 4.4. *If $a \in \mathbb{R}^m$ and $u \in \mathcal{F}_C^m$, then*

$$a^T u \prec \tilde{0} \Rightarrow a^T \underline{u}_\alpha < 0 \quad \text{and} \quad a^T \overline{u}_\alpha < 0, \quad \forall \alpha \in [0, 1].$$

Proof.

$$a^T u \prec \tilde{0} \Leftrightarrow [a^T u]^\alpha \prec [0,0], \quad \forall\alpha \Leftrightarrow \sum_{i=1}^{m} a_i [u_i]^\alpha \prec [0,0], \quad \forall \alpha \Leftrightarrow$$

$$\sum_{a_i \geq 0} a_i \underline{u_i}_\alpha + \sum_{a_i<0} a_i \overline{u_i}_\alpha < 0, \quad \forall \alpha \in [0,1], \tag{11}$$

$$\sum_{a_i \geq 0} a_i \overline{u_i}_\alpha + \sum_{a_i<0} a_i \underline{u_i}_\alpha < 0, \quad \alpha \in [0,1]. \tag{12}$$

If $a_i \geq 0$, then $a_i \overline{u_i}_\alpha \geq a_i \underline{u_i}_\alpha$, from (12)

$$0 > \sum_{a_i \geq 0} a_i \overline{u_i}_\alpha + \sum_{a_i<0} a_i \underline{u_i}_\alpha \geq \sum_{a_i \geq 0} a_i \underline{u_i}_\alpha + \sum_{a_i<0} a_i \underline{u_i}_\alpha = a^T \underline{u}_\alpha, \quad \forall \alpha \in [0,1].$$

If $a_i < 0$, then $a_i \overline{u_i}_\alpha \leq a_i \underline{u_i}_\alpha$ and

$$0 > \sum_{a_i \geq 0} a_i \overline{u_i}_\alpha + \sum_{a_i<0} a_i \underline{u_i}_\alpha \geq \sum_{a_i \geq 0} a_i \overline{u_i}_\alpha + \sum_{a_i<0} a_i \overline{u_i}_\alpha = a^T \overline{u}_\alpha, \quad \forall \alpha \in [0,1].$$

□

The reciprocal result of Lemma 4.4 does not hold in general. For example, $[u_1]^\alpha = [1,2], \forall\alpha$; $[u_2]^\alpha = [\frac{3}{2}, \frac{5}{2}], \forall \alpha$, and $a = (1,-1)$, then

$$a^T \underline{u}_\alpha = 1 \cdot 1 + (-1) \cdot \frac{3}{2} < 0 \quad \text{and} \quad a^T \overline{u}_\alpha = 1 \cdot 2 + (-1) \cdot \frac{5}{2} < 0, \quad \forall \alpha.$$

However, $[a^T u]^\alpha = [1,2] + [-\frac{5}{2}, -\frac{3}{2}] = [-\frac{3}{2}, \frac{1}{2}] \not\prec [0,0], \forall \alpha$, and thus $a^T u \not\prec \tilde{0}$.

5 Necessary efficiency condition for a multiobjective fuzzy problem

In the crisp multiobjetive optimization problem, the efficient solutions are also stationary points, and the stationary points can be found by setting the gradient to zero (see [9]). From these stationary points, we can separately ascertain whether each stationary point is an efficient solution or not. However, for the fuzzy multiobjective programming problems, we lack adequate definitions of stationary points. Futhermore, the statement that all efficient solutions for a multiobjective fuzzy optimization problem are stationary points remain to be proved.

We now establish a necessary efficiency condition for the solutions of p-dimensional fuzzy optimization problems. It is important to emphasize that in previous work, no result of this type exits.

First, we introduce the fuzzy p-dimensional stationary point definition:

Definition 5.1. *Let $\tilde{f}$ be a p-dimensional gH-differentiable function on S, $x^* \in S$ is said to be a fuzzy p-dimensional stationary point for $\tilde{f}$, if for every $i = 1, ..., n$, there exists a nonnegative matrix $\Lambda^i \in \mathcal{M}^{p\times 2}$ such that*

$$\Lambda^i \times \left[\frac{\partial \tilde{f}}{\partial x_i}(x^*) \right]^0 = 0. \tag{13}$$

If $\tilde{f} : T \to \mathcal{F}_C$, then Definition 5.1 coincides with Definition 6 and Corollary 1 of [8]. If $\tilde{f}$ is a crisp function, then Definition 5.1 coincides with Definition 2.3 of [9].

Remark 5.1. *From Remark 4.3 (2), condition (13) is equivalent to the following statement: for each $i = 1, ..., n$, there exist $\underline{\lambda}_i, \overline{\lambda}_i \in \mathbb{R}^p$, $(\underline{\lambda}_i, \overline{\lambda}_i) \geq 0$ such that $0 \in \left[\underline{\lambda}_i^T \left(\frac{\partial \tilde{f}_0}{\partial x_i}(x^*) \right)^L, \overline{\lambda}_i^T \left(\frac{\partial \tilde{f}_0}{\partial x_i}(x^*) \right)^U \right]$. Futhermore, from Remark 4.3 (2), if for each $i = 1, ..., n$, there exists $j = 1, ..., p$, such that $0 \in \left[\frac{\partial \tilde{f}_j}{\partial x_i}(x^*) \right]^0$, then x^* is a fuzzy p-dimensional stationary-point for $\tilde{f}$.*

In crisp optimization, the stationary points are those whose gradient is zero. From Definition 5.1, in fuzzy environment, p-dimensional stationary points are those such that zero is in the 0-level set of the gradient.

Proposition 5.2. *Let $\tilde{f}$ be a p-dimensional fuzzy gH-differentiable function on S. If $x^* \in S$ is a weakly efficient solution for $\tilde{f}$, then the following system has no solution at $y \in \mathbb{R}$, for any $i = 1, ..., n$.*

$$y\left(\frac{\partial \tilde{f}}{\partial x_i}(x^*)\right) \prec \tilde{0}^p. \tag{14}$$

Proof. Arguing by contradiction, let us suppose that for some $i = 1, ..., n$, $\exists y \in \mathbb{R}$ such that $y\frac{\partial \tilde{f}_j}{\partial x_i}(x^*) \prec \tilde{0} \quad \forall j = 1, ..., p$. Therefore, from Definition 3.4, for some i

$$y[h_j'(x_i^*)]^\alpha \prec [0,0], \quad \forall \alpha \in [0,1], \quad \forall j = 1, ..., p.$$

From Theorem 3.3, for each j, one of the following cases holds:

(a) There exist $\underline{h_j'}(\alpha, x^*)$ and $\overline{h_j'}(\alpha, x^*)$, uniformly in $\alpha \in [0,1]$ and

$$y[h_j'(x_i^*)]^\alpha \prec [0,0] \Rightarrow y\underline{h_j'}(\alpha, x_i^*) < 0 \text{ and } y\overline{h_j'}(\alpha, x_i^*) < 0, \forall \alpha \in [0,1].$$

Since $\underline{h_j}(\alpha, x)$ and $\overline{h_j}(\alpha, x)$ are real-valued functions,

$$y\underline{h_j'}(\alpha, x_i^*) = \lim_{t\to 0} \frac{1}{t}(\underline{h_j}(\alpha, x_i^* + ty) - \underline{h_j}(\alpha, x_i^*)) < 0, \quad \forall \alpha \in [0,1].$$

Hence, $\forall \nu > 0$, $\exists \underline{\epsilon_j} > 0$ such that if $|t| < \underline{\epsilon_j}$, then

$$\left|\frac{1}{t}(\underline{h_j}(\alpha, x_i^* + yt) - \underline{h_j}(\alpha, x_i^*)) - y\underline{h_j'}(\alpha, x_i^*)\right| < \nu, \quad \forall \alpha \in [0,1].$$

If we consider $\nu = -\frac{1}{2}y\underline{h_j'}(\alpha, x_i^*) > 0$, then, for $|t| < \underline{\epsilon_j}$

$$-\nu < \left(\tfrac{1}{t}(\underline{h_j}(\alpha, x_i^* + yh) - \underline{h_j}(\alpha, x^*)) - y\underline{h_j}'(\alpha, x_i^*)\right) < \nu$$

and

$$\tfrac{1}{t}(\underline{h_j}(\alpha, x_i^* + yt) - \underline{h_j}(\alpha, x_i^*)) < y\underline{h_j'}(\alpha, x_i^*) + \nu = \tfrac{1}{2}y\underline{h_j'}(\alpha, x_i^*) < 0, \quad \forall \alpha \in [0,1].$$

If $h \in (0, \underline{\epsilon_j})$ holds, then

$$\underline{h_j}(\alpha, x_i^* + yt) - \underline{h_j}(\alpha, x_i^*) < 0, \quad \forall \alpha \in [0,1].$$

$$\underline{f_j}(\alpha, (x_1^*, ..., x_i^* + yt, ..., x_n^*)) - \underline{f_j}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1]. \tag{15}$$

Analogously, $\exists \overline{\epsilon_j} > 0$ such that, $\forall t \in (0, \overline{\epsilon_j})$,

$$\overline{h_j}(\alpha, x_i^* + yt) - \overline{h_j}(\alpha, x_i^*) < 0, \quad \forall \alpha \in [0,1],$$

$$\overline{f_j}(\alpha, (x_1^*, ..., x_i^* + yt, ..., x_n^*)) - \overline{f_j}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1]. \tag{16}$$

(b) There exist $(\underline{h_j}(\alpha, (x_i^*))'_+$ and $(\overline{h_j}(\alpha, x_i^*))'_+$, uniformly in $\alpha \in [0,1]$, which satisfy

$$h_j'(x_i^*) = \left[\min\left\{(\underline{h_j}(\alpha, x_i^*))'_+, (\overline{h_j}(\alpha, x_i^*))'_+\right\}, \max\left\{(\underline{h_j}(\alpha, x_i^*))'_+, (\overline{h_j}(\alpha, x_i^*))'_+\right\}\right].$$

Since

$$y\left[h_j'(x_i^*)\right]^\alpha \prec [0,0] \Leftrightarrow \begin{cases} y(\underline{h_j}(\alpha, x_i^*))'_+ < 0 \\ y(\overline{\overline{h_j}}(\alpha, x_i^*))'_+ < 0 \end{cases}, \quad \forall \alpha \in [0,1].$$

Given that

$$y(\underline{h_j}(\alpha, x_i^*))'_+ = \lim_{t\to 0^+} \frac{1}{t}(\underline{h_j}(\alpha, x_i^* + yt) - \underline{h_j}(\alpha, x_i^*)) < 0,$$

it can therefore be deduced that, there exist $\underline{\epsilon_j^+} > 0$, such that, for all t, with $0 < t < \underline{\epsilon_j^+}$,

$$\underline{h_j}(\alpha, x_i^* + yt) - \underline{h_j}(\alpha, x_i^*) < 0, \quad \forall \alpha \in [0,1].$$

$$\underline{f_j}(\alpha, (x_1^*, ..., x_i^* + yt, ..., x_n^*) - \underline{f_j}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1]. \tag{17}$$

Analogously, there exists $\overline{\epsilon_j^+} > 0$ such that, for all t, with $0 < t < \overline{\epsilon_j^+}$,

$$\overline{h_j}(\alpha, x_i^* + yt) - \overline{h_j}(\alpha, x_i^*) < 0, \quad \forall \alpha \in [0,1],$$

$$\overline{f_j}(\alpha, (x_1^*, ..., x_i^* + yt, ..., x_n^*) - \overline{f_j}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1]. \tag{18}$$

By taking $\epsilon = \min\{\underline{\epsilon_j}, \overline{\epsilon_j}, \underline{\epsilon_j^+}, \overline{\epsilon_j^+} : \quad j = 1, ..., p\}$, $t \in (0, \epsilon)$ and from (15), (16), (17), (18), we deduce that

$$\underline{f}(\alpha, x) - \underline{f}(\alpha, x^*) < 0 \quad \text{and} \quad \overline{f}(\alpha, x) - \overline{f}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1],$$

where $x = (x_1^*, ..., x_i^* + ty, ..., x_n^*)$, and we suppose that $x \in S$. Hence, $\exists x \in S$ and $\tilde{f}(x) \prec \tilde{f}(x^*)$, and this is a contradiction to the affirmation that x^* is a weakly efficient solution for $\tilde{f}$. □

We can now, we prove the main result of this section.

Theorem 5.3. *Let $\tilde{f} : S \to \mathcal{F}_C^p$ be a p-dimensional gH-differentiable fuzzy function at $x^* \in S$. If x^* is a weakly efficient solution for $\tilde{f}$, then x^* is a fuzzy p-dimensional stationary-point for $\tilde{f}$.*

Proof. If x^* is a weakly efficient solution for $\tilde{f}$, then (14) has no solution for any $i = 1, ..., n$. From Lemma 2.5,

$$y\left(\frac{\partial \tilde{f}}{\partial x_i}(x^*)\right) \prec \tilde{0}^p \Leftrightarrow y\left[\frac{\partial \tilde{f}_j}{\partial x_i}(x^*)\right]^\alpha \prec [0,0], \quad \forall \alpha \in [0,1], \quad \forall j = 1, ..., p \Leftrightarrow y\left[\frac{\partial \tilde{f}_j}{\partial x_i}(x^*)\right]^0 \prec [0,0], \quad \forall j = 1, ..., p.$$

Now, for every $i = 1, ..., n$, let us consider the following lineal system

$$\left.\begin{array}{l} yA_i < 0 \\ yB_i < 0 \end{array}\right\} \tag{19}$$

where A_i, B_i are

$$A_i = \begin{pmatrix} \left(\frac{\partial \tilde{f}_{1_0}}{\partial x_i}(x^*)\right)^L \\ \vdots \\ \left(\frac{\partial \tilde{f}_{p_0}}{\partial x_i}(x^*)\right)^L \end{pmatrix}, \quad B_i = \begin{pmatrix} \left(\frac{\partial \tilde{f}_{1_0}}{\partial x_i}(x^*)\right)^U \\ \vdots \\ \left(\frac{\partial \tilde{f}_{p_0}}{\partial x_i}(x^*)\right)^U \end{pmatrix}.$$

If (19) has a solution for some $i = 1, ..., n$, then the system (14) also has a solution for some i. This is impossible from Proposition 5.2.

Since (19) is a system of linear inequalities and it has no solution for any i, from Gordan's alternative theorem, for each i, then there exist $\alpha_i, \beta_i \in \mathbb{R}^p$ with $\alpha_i \geqq 0$, $\beta_i \geqq 0$, but not all zero, such that

$$A_i^T \alpha_i + B_i^T \beta_i = 0 \Leftrightarrow \sum_{j=1}^{p} \left[\ \alpha_{ij} \left(\frac{\partial \tilde{f_{j_0}}}{\partial x_i}(x^*)\right)^L + \beta_{ij} \left(\frac{\partial \tilde{f_{j_0}}}{\partial x_i}(x^*)\right)^U \ \right] = 0.$$

By redefining $\Lambda^i = (\alpha_{ij}, \beta_{ij})$, it can be stated that, for every i, there exits $\Lambda^i \in \mathcal{M}^{p\times 2}$ such that $\Lambda^i \times \left[\frac{\partial \tilde{f}}{\partial x_i}(x^*)\right]^0 = 0$, and the proof is complete. □

Hence, the identification of possible candidates for weakly efficient solutions is reduced to identification of those points whose 0-level set of the gradient contain the zero element.

Remark 5.2. *In deterministic vectorial optimization, x^* is a Vectorial Critical Point, (see [9]) if $\lambda \in \mathbb{R}^p, \lambda \geq 0$ exists such that*

$$\lambda^T \nabla f(x^*) = 0. \tag{20}$$

Since $\nabla f(x^*) \in \mathcal{M}^{p\times n}$, (20) implies that, for each $i = 1, ..., n$, $\displaystyle\sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial x_i}(x^*) = 0$. Thus (20) coincides with Definition 5.1 except in that λ is unique for all i. And, in (13), there are two multipliers for each i, and they are also different.

Example 5.4. *Let $\tilde{f}$ be from Example 4.2. Therefore, $\tilde{f}_1$ and $\tilde{f}_2$ are gH-differentiable fuzzy functions on $\mathbb{R}^2$ and*

$$\left[\tilde{\nabla} \tilde{f}_1(x)\right]^0 = \left([2x_1, 2x_1], \begin{cases} [0, 2x_2] & \text{if} \quad x_1 \in \mathbb{R}, x_2 \geq 0 \\ [2x_2, 0] & \text{if} \quad x_1 \in \mathbb{R}, x_2 < 0 \end{cases} \right),$$

$$\left[\tilde{\nabla} \tilde{f}_2(x)\right]^0 = \left(\; [2x_1, 2x_1], [1, 6] \; \right).$$

If $x_1 \neq 0$, then Λ^1 nonnegative such that $\Lambda^1 \times ([2x_1, 2x_1], [2x_1, 2x_1]) = 0$ cannot exist. Therefore, only fuzzy p-dimensional stationary points for $\tilde{f}$ are those points such that $x^* = (0, x_2)$. We can therefore conclude, that if $x_1^* = 0$ then x^* is a weakly efficient solution for $\tilde{f}$.

Example 5.5. *For $\tilde{f}$, from Example 4.3, $\left[\frac{\partial \tilde{f}}{\partial x_2}(x)\right]^0 = ([1, 3], [1, 6])$. Hence, neither do fuzzy p-dimensional stationary points exist, nor do weakly efficient solutions exits.*

6 Sufficient efficiency condition for a multiobjective fuzzy problem

In order to ensure that the stationary points characterize weakly efficient solutions it is necessary to demand additional hypotheses on the function. In [10], [15] and [16] sufficient optimality conditions are established for fuzzy optimization problems by demanding the convexity hypothesis (the endpoint functions are convex functions), pseudoconvexity hypothesis (the endpoint functions are pseudoconvex functions), and invexity hypothesis (the positive sum of endpoint functions is an invex function). We propose a more general convexity hypothesis.

Definition 6.1. *Let $\tilde{f}$ be a p-dimensional gH-differentiable fuzzy mapping on S. It is said that $\tilde{f}$ is a weak pseudoinvex fuzzy function on S if, $\forall x, y \in S$, there exists $\eta(x, y) \in \mathbb{R}$ such that if $\tilde{f}(x) \prec \tilde{f}(y)$ then $\eta(x, y)\frac{\partial \tilde{f}}{\partial x_i}(y) \prec \tilde{0}^p$, for some $i = 1, ..., n$.*

Theorem 6.2. *If x^* is a fuzzy p-dimensional stationary point for $\tilde{f}$ and $\tilde{f}$ is a weak pseudoinvex function, then x^* is a weakly efficient solution for $\tilde{f}$.*

Proof. Arguing by contradiction, let us suppose that there exists $x, x^* \in S$ $x \neq x^*$ such that $\tilde{f}(x) \prec \tilde{f}(x^*)$, then

$$\eta(x, x^*)\frac{\partial \tilde{f}}{\partial x_i}(x^*) \prec \tilde{0}^p \Leftrightarrow \eta(x, x^*)\left(\frac{\partial \tilde{f}_j}{\partial x_i}(x^*)\right) \prec \tilde{0} \quad \forall j = 1, ..., p \Leftrightarrow$$

$$\Leftrightarrow \eta(x, x^*)\left[\frac{\partial \tilde{f}_j}{\partial x_i}(x^*)\right]^\alpha \prec [0, 0], \quad \forall j = 1, ..., p, \quad \forall \alpha \in [0, 1] \Leftrightarrow$$

From Lemma 2.5,

$$\Leftrightarrow \eta(x, x^*)\left[\frac{\partial \tilde{f}_j}{\partial x_i}(x^*)\right]^0 \prec [0, 0], \quad \forall j = 1, ..., p.$$

From Lemma 4.4, the following system has a solution

$$\left.\begin{array}{l} yA_i < 0 \\ yB_i < 0 \end{array}\right\}$$

where

$$A_i = \begin{pmatrix} \left(\frac{\partial \tilde{f}_{1_0}}{\partial x_i}(x^*)\right)^L \\ \vdots \\ \left(\frac{\partial \tilde{f}_{p0}}{\partial x_i}(x^*)\right)^L \end{pmatrix}, \quad B_i = \begin{pmatrix} \left(\frac{\partial \tilde{f}_{1_0}}{\partial x_i}(x^*)\right)^U \\ \vdots \\ \left(\frac{\partial \tilde{f}_{p0}}{\partial x_i}(x^*)\right)^U \end{pmatrix}.$$

Consequently, there can be no positive linear combinations $\alpha_i, \beta_i \in \mathbb{R}^p$ such that

$$0 \in \left[\alpha_i^T \left(\frac{\partial \tilde{f}_0}{\partial x_i}(x^*)\right)^L, \beta_i^T \left(\frac{\partial \tilde{f}_0}{\partial x_i}(x^*)\right)^U\right]$$

And hence, x^* cannot be a fuzzy p-dimensional stationary point which is a contradiction. □

It should be borne in mind that the necessary optimality condition that we have proved and the stationary point definition on which is based are less restrictive than the corresponding condition and definition for real-valued functions (see [9]). This is due to the fact that the multiplier λ can be different for each variable x_i. On the other hand, the notion that guarantees the sufficient optimality condition, that of weak pseudoinvexity, is more restrictive than the notion required by the authors in [9] since if f is a real-valued vector function and there exists $\eta(x,y) \in \mathbb{R}$ such that for some i, $\eta(x,y)\frac{\partial f}{\partial x_i}(x^*) < 0$ then also there exists $\eta(x,y) \in \mathbb{R}^n$ such that $\eta(x,y)^T \nabla f(x) < 0$ (it is sufficient to take $\eta(x,y)_j = 0$ when $j \neq i$). However, if there exists $\eta(x,y) \in \mathbb{R}^n$ such that $\eta(x,y)^T \nabla f(x) < 0$, then it cannot be certain that there exists i such that $\eta(x,y)_i \frac{\partial f_j}{\partial i}(y) < 0$ for all j. These differences are due to the nonlinearity of the fuzzy differentiability notion ($(a+b)^T \tilde{\nabla} \tilde{f}(x) \neq a^T \tilde{\nabla} \tilde{f}(x) + b^T \tilde{\nabla} \tilde{f}(x)$), and therefore the decrease directions of function cannot be characterized as in the real-valued case.

Example 6.3. *Given $\tilde{f}$ from Example 4.2, $\forall x, y$, there exists $\eta(x,y) \in \mathbb{R}$ such that $\eta(x,y)[2x_1, 2x_1] \prec [0,0]$, and hence for $i = 1$, $\eta(x,y)\frac{\partial \tilde{f}}{\partial x_1} \prec \tilde{0}^2$. Thus, $\tilde{f}$ is a weak pseudoinvex fuzzy function on $\mathbb{R}^2$, and it was proved that all the stationary points were weakly efficient solutions.*

Example 6.4. *Given $\tilde{f}$ from Example 4.3, $\forall x, y$, there exists $\eta(x,y) < 0$ such that*

$$\eta(x,y)[1+\alpha, \alpha+4] \prec [0,0] \quad \textit{and} \quad \eta(x,y)[1+\alpha, 2(3-\alpha)] \prec [0,0].$$

Therefore for $i = 2$, $\eta(x,y)\frac{\partial \tilde{f}}{\partial x_2} \prec \tilde{0}^2$. Thus, $\tilde{f}$ is a weak pseudoinvex fuzzy function on $\mathbb{R}^2$.

7 Conclusions

In this paper, we study fuzzy multiobjective optimization problems and provide necessary and sufficient efficiency conditions based on a suitable differentiability notion. First, we prove that the candidates for efficient solutions can be identified by calculating the gradients and by determining the points whose 0-level sets of gradient contain the zero element. Secondly, using the sufficient efficiency conditions proved in Theorem 6.2, the nonefficient solutions can be identified. The results obtained generalize the results that exist in the literature in several aspects. This generalization is reached primarily since our results consider p-dimensional functions instead of scalar functions and they are defined for n variables instead of only one variable. Futhermore, it has been proved that the differentiability notion used is less restrictive than those used in the literature. Consequently, this paper contains new contributions to Fuzzy Optimization Theory.

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References

[1] B. Bede, S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equation*, Fuzzy Sets and Systems, **151**(3) (2005), 581-599.

[2] B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and Systems, **230** (2013), 119-141.

[3] Y. Chalco-Cano, W. A. Lodwick, A. Rufián-Lizana, *Optimality conditions of type KKT for optimization problems with interval-valued objective function via generalized derivative*, Fuzzy Optimization and Decision Making, **12**(3) (2013), 305-322.

[4] Y. Chalco-Cano, R. Rodríguez-López, M. D. Jiménez-Gamero, *Characterizations of generalized differentiable fuzzy functions*, Fuzzy Sets and Systems, **295** (2016), 37-56.

[5] Y. Chalco-Cano, H. Román-Flores, M. D. Jiménez-Gamero, *Generalized derivative and π-derivative for set-valued functions*, Information Sciences, **181** (2011), 2177-2188.

[6] S. S. L. Chang, L. A. Zadeh. *On fuzzy mappings and control*, IEEE Transactions on Systems, Man and Cybernetics, **2**(1) (1972), 30-34.

[7] R. Goestschel, W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Systems, **18**(1) (1986), 31-43.

[8] R. Osuna-Gómez, Y. Chalco-Cano, A. Rufián-Lizana, B. Hernández-Jiménez, *Necessary and sufficient conditions for fuzzy optimality problems*, Fuzzy Sets and Systems, **296** (2016), 112-123.

[9] R. Osuna-Gómez, A. Rufián-Lizana, P. Ruíz-Canales, *Invex functions and generalized convexity in multiobjective programming*, Journal of Optimization Theory and Applications, **98**(3) (1998), 651-661.

[10] M. Panigrahi, G. Panda, S. Nanda, *Convex fuzzy mapping with differentiability and its applicaction in fuzzy optimization*, European Journal of Operational Research, **185**(1) (2008), 47-62.

[11] M. L. Puri, D. A. Ralescu, *Differentials for fuzzy functions*, Journal of Mathematical Analysis and Applications, **91**(2) (1983), 552-558.

[12] L. Stefanini, *A generalization of Hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems, **161**(11) (2010), 1564-1584.

[13] L. Stefanini, B. Bede, *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations*, Nonlinear Analysis, **71**(3-4) (2009), 1311-1328.

[14] H. C. Wu, *The Karush-Kuhn-Tucker optimality conditions for the optimization problem with fuzzy-valued objective function*, Mathematical Methods of Operations Research, **66**(2) (2007), 203-224.

[15] H. C. Wu, *The Karush-Kuhn-Tucker optimality conditions for multi-objective programming problems with fuzzy-valued objective functions*, Fuzzy Optimization and Decision Making, **8**(1) (2009), 1-28.

[16] H. C. Wu, *The optimality conditions for optimization problems with convex constraints and multiple fuzzy-valued objective functions*, Fuzzy Optimization and Decision Making, **8**(3) (2009), 295-321.

[17] L. A. Zadeh, *Fuzzy sets*, Information Control, **8** (1965), 338-353.