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Combinatorially Orthogonal Paths

Sean Bailey*[1], LeRoy Beasley†[2], and David Brown‡[2]

[1]Department of Mathematics, Computer Science, & MIS, Texas A&M University Texarkana
[2]Department of Mathematics and Statistics, Utah State University

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Abstract

Vectors $\vec{x} = (x_1, x_2, \ldots, x_n)^T$ and $\vec{y} = (y_1, y_2, \ldots, y_n)^T$ are combinatorially orthogonal if $|\{i : x_i y_i \neq 0\}| \neq 1$. An undirected graph $G = (V, E)$ is a combinatorially orthogonal graph if there exists $f : V \to \mathbb{R}^n$ such that for any $u, v \in V$, $uv \notin E$ iff $f(u)$ and $f(v)$ are combinatorially orthogonal. We will show that every graph has a combinatorially orthogonal representation. We will show the bounds for the combinatorially orthogonal dimension of any path P_n.

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*sbailey@tamut.edu
†lerory.b.beasley@gmail.com
‡david.e.brown@usu.edu

1 Introduction

Combinatorial orthogonality was first introduced by Beasley, Brualdi, and Shader [1]. They defined vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ to be *combinatorially orthogonal* if

$$|\{i : x_i y_i \neq 0\}| \neq 1.$$

This definition means that the combinatorial orthogonality of two vectors is only dependent on the positions of the nonzero coordinates. An alternate definition is vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ with $x_i, y_i \in \{0, 1\}$ for $1 \leq i \leq n$ are combinatorially orthogonal if $x \cdot y \neq 1$.

This definition can be extended to matrices: A matrix A is *combinatorially row-orthogonal* if its rows are pairwise combinatorially orthogonal. Similarly A is *combinatorially column-orthogonal* if its columns are pairwise combinatorially orthogonal. If A is a square matrix that is both combinatorially row-orthogonal and combinatorially column-orthogonal, it is called a *combinatorially orthogonal matrix*. Beasley et al. used this definition to determine the minimum number of nonzero entries possible in an orthogonal matrix of order n which cannot be decomposed into two smaller orthogonal matrices. Some work has also been done on the combinatorial orthogonality of the digraph of orthogonal matrices [1, 2, 3].

2 Combinatorially Orthogonal Graphs

Let $G = (V, E)$ be a simple undirected graph. Then we say G has a *k-combinatorially orthogonal representation of type I* if there exists a function $f : V \to \mathbb{R}^k$ such that for any $u, v \in V$ $uv \notin E$ if and only if $f(u)$ and $f(v)$ are combinatorially orthogonal. We say G has *a k-combinatorially orthogonal representation of type II* if there exists $g : V \to \mathbb{R}^k$ with $g(v)_i \in \{0, 1\}$ for $v \in V$ and $1 \leq i \leq k$ such that for any $u, v \in V$ $uv \in E$ if and only if $g(u) \cdot g(v) \neq 1$. The equivalence of these representations is given in Proposition 2.1.

Proposition 2.1. *Let G be a simple undirected graph that has a k-combinatorially orthogonal representation of type I, then there is a k-combinatorially orthogonal representation of type II of the graph G that is equivalent. Further, if G has a k-combinatorially orthogonal representations of type II, then there is a k-combinatorially orthogonal representation of type I of the graph G that is equivalent. (That is every representation of type I is equivalent to a representation of type II and vice versa.)*

Proof. It is sufficient to consider the function

$$F(\vec{v})_i \;=\; \begin{cases} 1 & \text{if } \vec{v}_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $\vec{v} \in \mathbb{R}^k$. We can then observe that for any $\vec{x}, \vec{y} \in \mathbb{R}^k$, $|\{i : x_i y_i \neq 0\}| = F(\vec{x}) \cdot F(\vec{y})$. From this it follows immediately that Type I and Type II combinatorially orthogonal representations of a graph are equivalent. □

Since both of the representations are equivalent, we will primarily use the representations of type II. In either case, the question arises whether there exists a combinatorially orthogonal representation for every graph. Theorem 2.2 shows that every graph has a combinatorially orthogonal representation.

Theorem 2.2. *For every graph $G = (V, E)$, there exists an integer k such that G has a k-combinatorially orthogonal representation.*

Proof. Let $\bar{G} = (V, \bar{E})$ be the complement of G and $k = |\bar{E}|$. Label the edges of $\bar{G}$ $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k\} = \bar{E}$. Define $f : V \to \mathbb{R}^k$ such that for $v \in V$ $f(v)_i = 1$ if $\bar{e}_i$ is incident to v and 0 otherwise. Note that $uv \notin E$ if and only if $f(u) \cdot f(v) = 1$. Similarly, $uv \in E$ if and only if $f(u) \cdot f(v) = 0$. Thus G is a k-combinatorially orthogonal graph. □

We define the *combinatorially orthogonal dimension* of G, denoted $\rho_{co}(G)$, to be the minimum k such that there exists a combinatorially orthogonal representation of G. If the combinatorially orthogonal dimension of a graph G is at most k, we refer to G as a k-combinatorially orthogonal graph. From Theorem 2.2, we can see that $\rho_{co}(G) \leq |\bar{E}|$. This bound is interesting in that it is based on the non-adjacencies of G, which is opposite of the general bound for dot product graphs which is based on adjacencies.

Theorem 2.3. *Let H be an induced subgraph of G, then $\rho_{co}(H) \leq \rho_{co}(G)$.*

Proof. Let $f : V(G) \to \mathbb{R}^k$ be a function such that G is a k-combinatorially orthogonal graph. It can easily be noted that f restricted to $V(H)$ is a k-combinatorially orthogonal representation. □

Theorem 2.3 allows the characterization of k-combinatorially orthogonal graphs, for any fixed k, by forbidden induced subgraphs or substructures.

Since we may use representations of the form $f : V \to \{0, 1\}^k$, there are 2^k different vectors to choose from for each vertex of the graph being represented. Therefore, a k-combinatorially orthogonal representations of a graph $G = (V, E)$ is tantamount to a partition of V into 2^k classes, each class characterized by a behavior. For example, if $f(v) = \vec{0}$, then v is a universal vertex. We record this observation, in a form useful to us, as the following lemma.

Lemma 2.4. *Let G be a graph and v be a universal vertex of G. Then $\rho_{co}(G) = \rho_{co}(G - v)$.*

Proof. Let $\rho_{co}(G) = k$. By Theorem 2.3, $\rho_{co}(G - v) \leq \rho_{co}(G)$.

We will now use a proof by contradiction. Suppose that $\rho_{co}(G-v) < \rho_{co}(G)$. Then there exists a $k-1$-combinatorially orthogonal representation of $G - v$, namely $f : V(G - v) \to \{0, 1\}^{k-1}$. Now consider the combinatorially orthogonal representation of G given as follows for any $u \in V(G)$

$$F(u) = \begin{cases} \vec{f(u)} & \text{if } u \neq v \\ \vec{0} & \text{if } u = v \end{cases}$$

A brief examination shows that this representation holds for G. But this is a contradiction that $\rho_{co}(G) = k$ since this is a $k - 1$-combinatorially orthogonal representation of G.

Therefore, $\rho_{co}(G - v) = \rho_{co}(G)$. □

3 Combinatorially Orthogonal Dimension of Paths

After considering general bounds on combinatorially orthogonal graphs and some of the characterized behavior, we turn to a specific class of graphs - paths. To do this we can first note that for $n > 3$, no pair of vertices has the same neighborhood and there is no universal vertices. Thus each vertex will map to a distinct vector and no $\vec{0}$ will be used in any k-combinatorially orthogonal representation. We now consider what other vectors are not possible or restricted in any k-combinatorially orthogonal representation of P_n. The use of $\vec{1}$ is not possible in any k-combinatorially orthogonal representation of P_n, as shown in the following lemma.

Lemma 3.1. *If $n > 5$, then any k-combinatorially orthogonal representation of P_n can have no all 1 vector from $\{0, 1\}^k$.*

Proof. Suppose that $v \in V(P_n)$ is represented by $\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$. Then the only vertices non-adjacent to v are represented by vectors with exactly 1 nonzero element. There can be at most 2 such vectors or they will form a complete subgraph isomorphic to K_3. Thus P_n has at most 2 vertices non-adjacent to v. Thus $n \leq 5$. □

We can next examine the use of unit vectors in any k-combinatorially orthogonal representation of P_n.

Lemma 3.2. *If $n > 6$, then any k-combinatorially orthogonal representation of P_n can have at most one unit vector from $\{0, 1\}^k$.*

Proof. Suppose that P_n has k-combinatorially orthogonal representation with at least 2 unit vectors from $\{0, 1\}^k$. Without loss of generality, suppose those vectors are $\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix}$ and they represent vertices u and v in P_n, respectively. Because these two vectors are orthogonal, they are also combinatorially orthogonal. So $uv \in E(P_n)$.

Now consider all $w \in V(P_n)$ such that $uw \notin E(P_n)$ and $vw \notin E(P_n)$. For all such w, $f_1(w) = f_2(w) = 1$. This however means for any two such vertices, namely w_1 and w_2, $w_1 w_2 \in E(P_n)$ since $f(w_1) \cdot f(w_2) \geq 2$. Thus at most two such w can exist. In the case where two such vertices exist, then $n \leq 6$ as there are at most 2 other vertices adjacent to either u or v. □

These lemmas limit the vector options when building k- combinatorially orthogonal representations of P_n when $n > 6$. These limitations lead to following upper bound of combinatorially orthogonal dimension of P_n and associated proof.

Theorem 3.3. *For any n,*

$$\rho_{co}(P_n) \leq \begin{cases} n - 2 & \text{if } n \text{ is odd} \\ n - 1 & \text{if } n \text{ is even} \end{cases}$$

Proof. We will break our proof into two cases, namely when n is even and when n is odd.
Case 1: Suppose that n is even. Consider the following function $f : V(P_n) \to \{0,1\}^{n-1}$.

$$\begin{aligned}
f(v_1) &= \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}^T \\
f_i(v_2) &= \begin{cases}1 & \text{if } 1 < i \leq \lceil\frac{n-3}{2}\rceil + 1 \\ 0 & \text{otherwise}\end{cases} \\
f_i(v_3) &= \begin{cases}1 & \text{if } i = 1 \text{ or } i = \lceil\frac{n-3}{2}\rceil + 2 \\ 0 & \text{otherwise}\end{cases} \\
\text{For } j > 3 \ f_i(v_j) &= \begin{cases}1 & \text{if } i = 1 \text{ or } i = \lceil\frac{j-3}{2}\rceil + 1 \text{ or } i = \lceil\frac{j-2}{2}\rceil + \lceil\frac{n-3}{2}\rceil + 1 \\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

For example, when $n = 8$, the vectors $f(v_1), ..., f(v_8)$ are the columns of this matrix:

$$\begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}$$

To show that this representation satisfies the adjacencies of P_n, we will first consider $f(v_2) \cdot f(v_j)$. Trivial examinations shows that $f(v_2) \cdot f(v_j) = 0$ when $j = 1$ or 3. Similarly, since $2 \leq \lceil\frac{j-3}{2}\rceil + 1 \leq \lceil\frac{n-3}{2}\rceil + 1$, $f(v_2) \cdot f(v_j) = 1$ when $3 \leq j \leq n$.

When we examine $f(v_2) \cdot f(v_j)$ with $j \geq 3$, it is trivial to note that $f(v_2) \cdot f(v_j) = 1$.

Now consider $f(v_3) \cdot f(v_j)$ for $j \geq 4$. When $j = 4$, $f_1(v_3) = f_1(v_4) = 1$ and $\lceil\frac{4-2}{2}\rceil + \lceil\frac{n-3}{2}\rceil + 1 = \frac{n-3}{2}\rceil + 2$, so $v_3v_4 \in E(P_n)$. Similarly for $j \geq 5$, $f(v_3) \cdot f(v_j) = 1$ since $\lceil\frac{j-2}{2}\rceil + \lceil\frac{n-3}{2}\rceil + 1 > \frac{n-3}{2}\rceil + 2$. So $v_2v_j \notin E(P_n)$.

Finally we can consider $f(v_j) \cdot f(v_k)$, where $j, k \geq 4$. Since $f_1(v_j) = f_1(v_k) = 1$, $f(v_j) \cdot f(v_k) \geq 1$. Thus any two such vertices are adjacent if and only if

$$\lceil\frac{j-3}{2}\rceil + 1 = \lceil\tfrac{k-3}{2}\rceil + 1 \text{ or} \tag{1}$$

$$\lceil\frac{j-2}{2}\rceil + \lceil\frac{n-3}{2}\rceil + 1 = \lceil\tfrac{k-2}{2}\rceil + \lceil\tfrac{n-3}{2}\rceil + 1. \tag{2}$$

First, (1) is true if and only if $\lceil\frac{j-3}{2}\rceil = \lceil\frac{k-3}{2}\rceil$. To examine this equality, suppose that $k = j + m$ for some non-zero integer m. Then $\lceil\frac{k-3}{2}\rceil = \lceil\frac{j+m-3}{2}\rceil = \lceil\frac{j-3}{2} + \frac{m}{2}\rceil$. If $|m| \geq 2$, then $\lceil\frac{j-3}{2}\rceil \neq \lceil\frac{k-3}{2}\rceil$. This leaves $m = 1$ or $m = -1$. This also implies either j or k is even and the other is odd. Without losss of generality, we will suppose that j is even and k is odd. Namely $j = 2t$ and $k = 2t \pm 1$ for some integer t. So $\lceil\frac{j-3}{2}\rceil = \lceil\frac{2t-3}{2}\rceil = \lceil t - 1 - \frac{1}{2}\rceil = t - 1$. If $k = 2t + 1$, then $\lceil\frac{k-3}{2}\rceil = \lceil\frac{2t-2}{2}\rceil = \lceil t - 1\rceil = t - 1$. However if $k = 2t - 1$, then $\lceil\frac{k-3}{2}\rceil = \lceil\frac{2t-4}{2}\rceil = \lceil t - 2\rceil = t - 2$. Thus $v_jv_k \in E(G)$ if $k = j + 1$.

Next,(2) is true if and only if $\lceil\frac{j-2}{2}\rceil = \lceil\frac{k-2}{2}\rceil$. Similar to the prior case, we can suppose that $k = j + m$ for some non-zero integer m. We can see that $\lceil \frac{j-2}{2} \rceil \neq \lceil \frac{j+m-2}{2} \rceil$ if $|m| \geq 2$. This leaves us with the two cases of $m = \pm 1$. This also implies that either j or k is even and the other is odd. Without loss of generality, we will again assumes that j is even and k is odd. So $j = 2t$ and $k = 2t \pm 1$ for some integer t. So $\lceil \frac{j-2}{2} \rceil = \lceil \frac{2k-2}{2} \rceil = t - 1$. Similarly if $k = 2t - 1$ then $\lceil \frac{k-2}{2} \rceil = \lceil \frac{2k-1-2}{2} \rceil = \lceil t - 2 + \frac{1}{2} \rceil = t - 1$. However, if $k = 2t + 1$ then $\lceil \frac{k-2}{2} \rceil = \lceil \frac{2k+1-2}{2} \rceil = \lceil t - 1 + \frac{1}{2} \rceil = t$. Thus $v_j v_k \in E(G)$ if $k = j - 1$.

Thus (1) and (2) establish that $v_j v_k \in E(G)$ if and only if $k = j \pm 1$, which satisfies the desired adjacencies. Further we have shown that the function is an $n - 1$ combinatorially orthogonal representation of P_n when n is even.

Case 2: Suppose that n is odd. Consider the following function $\hat{f} : V(P_n) \to \{0, 1\}^{n-2}$.

$$
\begin{aligned}
\hat{f}(v_1) &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \\
\hat{f}_i(v_2) &= \begin{cases} 1 & \text{if } 1 < i \leq \lceil \frac{n-4}{2} \rceil + 1 \\ 0 & \text{otherwise} \end{cases} \\
\hat{f}_i(v_3) &= \begin{cases} 1 & \text{if } i = 1 \text{ or } i = \lceil \frac{n-4}{2} \rceil + 2 \\ 0 & \text{otherwise} \end{cases} \\
\text{For } 3 < j < n \ \hat{f}_i(v_j) &= \begin{cases} 1 & \text{if } i = 1 \text{ or } i = \lceil \frac{j-3}{2} \rceil + 1 \text{ or } i = \lceil \frac{j-2}{2} \rceil + \lceil \frac{n-4}{2} \rceil + 1 \\ 0 & \text{otherwise} \end{cases} \\
\hat{f}_i(v_n) &= \begin{cases} 1 & \text{if } i = 1 \text{ or } i = \lceil \frac{n-3}{2} \rceil + 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

A brief examination shows that $\hat{f}(v_i) = f(v_i)$ for $i = 1, 2, \cdots, n-1$, where f is the function from Case 1, where the n used to define f is smaller than the corresponding n for $\hat{f}$ by 1. Thus the validity of this representation has been proven for $v_1, \cdots, v_{n-1}$. We will now show that it holds $f(v_k) \cdot f(v_n)$.

First we can proved previously that $\lceil \frac{n-3}{2} \rceil = \lceil \frac{j-3}{2} \rceil$ when $j = n - 1$. Thus $\lceil \frac{n-3}{2} \rceil = \lceil \frac{n-4}{2} \rceil$ so $f(v_k) \cdot f(v_n) = 1$ for $k = 1, 2, \cdots, n-2$. Similarly $f(v_{n-1}) \cdot f(v_n) = 2$.
Thus we have shown that this function $\hat{f}$ is an $(n-2)$ combinatorially orthogonal representation of P_n when n is odd. □

4 Further Work

Theorem 3.3 provides an upper bound for the combinatorially orthogonal dimension of P_n. We conjecture that this bound is the combinatoraily orthogonal dimension of P_n for $n > 6$.

Similarly, we make the following conjecture about k-combinatorial orthogonal representations.

Conjecture 4.1. *Let X be a k-combinatorial orthogonal representation of a graph G, and let U be a $k \times k$ combinatorial orthogonal matrix. Then UX is also a k-combinatorial orthogonal representation of G.*

The proof of this conjecture will likely focus on the linear algebra properties related to combinatorial orthogonal matrices.

References

[1] L. Beasley, R. Brualdi, and B. Shader, *Combinatorial and Graph-Theoretical Problems in Linear Algebra* (1993), Springer New York, 207–218.

[2] J. Lundgren, K. Reid, S. Severini, and D. Stewart, Quadrangularity and Strong Quadrangularity In Tournaments, *Australian Journal of Combinatorics* **34** (2006), 247–260.

[3] S. Severini, On the digraph of a unitary matrix, *SIMAX, SIAM Journal of Matrix Analysis Applications* **25** (2003), 295–300.