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Flows on Signed Graphs

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ABSTRACT

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This dissertation focuses on integer flow problems within specific signed graphs. The theory of integer flows, which serves as a dual problem to vertex coloring of planar graphs, was initially introduced by Tutte as a tool related to the Four-Color Theorem. This theory has been extended to signed graphs.

In 1983, Bouchet proposed a conjecture asserting that every flow-admissible signed graph admits a nowhere-zero 6-flow. To narrow dawn the focus, we investigate cubic signed graphs in Chapter 2. We prove that every flow-admissible 3-edge-colorable cubic signed graph admits a nowhere-zero 10-flow. This together with the 4-color theorem implies that every flow-admissible bridgeless planar signed graph admits a nowhere-zero 10-flow. As a byproduct of this research, we also demonstrate that every flow-admissible hamiltonian signed graph can admit a nowhere-zero 8-flow.

In Chapter 3, we delve into triangularly connected signed graphs. Here, A triangle-path in a graph G is defined as a sequence of distinct triangles $T_1, T_2, \ldots, T_m$ in G such that for any i, j with $1 \leq i < j \leq m$, $|E(T_i) \cap E(T_{i+1})| = 1$ and $E(T_i) \cap E(T_j) = \emptyset$ if $j > i + 1$. We categorize a connected graph G as triangularly connected if it can be demonstrated that for any two nonparallel edges e and e', there exists a triangle-path $T_1 T_2 \cdots T_m$ such that $e \in E(T_1)$ and $e' \in E(T_m)$. For ordinary graphs, Fan *et al.* characterized all triangularly connected graphs that admit nowhere-zero 3-flows or 4-flows. Corollaries of this result extended to integer flow in certain families of ordinary graphs, such as locally connected graphs due to Lai and certain types of products of graphs due to Imrich et al. In this dissertation, we extend Fan's result for triangularly connected graphs to signed graphs. We proved that a flow-admissible triangularly connected signed graph (G, σ) admits a nowhere-zero 4-flow if and only if (G, σ) is not the wheel W_5 associated with a specific signature. Moreover, this result is proven to be sharp since we identify infinitely many unbalance triangularl connected signed graphs that can admit a nowhere-zero 4-flow but not 3-flow.

Chapter 4 investigates integer flow problems within K_4-minor free signed graphs. A minor of a graph G refers to any graph that can be derived from G through a series of vertex and edge deletions and edge contractions. A graph is considered K_4-minor free if K_4 is not a minor of G. While Bouchet's conjecture is known to be tight for some signed graphs with a flow number of

6. Kompišová and Máčajová extended those signed graph with a specific signature to a family $\mathcal{M}$, and they also put forward a conjecture that suggests if a flow-admissible signed graph does not admit a nowhere-zero 5-flow, then it belongs to $\mathcal{M}$. In this dissertation, we delve into the members in $\mathcal{M}$ that are K_4-minor free, designating this subfamily as $\mathcal{N}$. We provide proof demonstrating that every flow-admissible, K_4-minor free signed graph admits a nowhere-zero 5-flow if and only if it does not belong to the specified family $\mathcal{N}$.

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DEDICATION

To

my mother Jianxiu Kou, my father Xitong Li

Contents

Chapter 1

Introduction

1.1 Notations and Terminology

We consider finite graphs and may have multiple edges or loops. For terminology and notations not defined here we follow [1, 5, 31]. Throughout this dissertation, Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. For a graph G, if $X \subseteq V(G)$, then $G[X]$ is the subgraph induced by X. For each vertex $v \in V(G)$, the set of vertices adjacent to v and the set of edges incident with v are respectively denoted by $N_G(v)$ and $E_G(v)$, and $d_G(v) = |E_G(v)|$. This is called the number of *neighbours* of v in G. When G is understood from this dissertation, we often use $N(v)$ and $d(v)$ for $N_G(v)$ and $d_G(v)$, respectively.

Let G be a graph. Let U_1 and U_2 be two disjoint vertex sets. Denote by $\delta_G(U_1, U_2)$ the set of edges with one end in U_1 and the other in U_2. For convenience, we write $\delta_G(U_1)$ for $\delta_G(U_1, V(G) \setminus U_1)$. We use $B(G)$ to denote the set of bridges of G. A path in G is said to be a *subdivided edge* of G if every internal vertex of P has degree 2.

1.2 Signed Graphs

A *signed graph* (G, σ) is a graph G together with a *signature* $\sigma : E(G) \to \{-1, 1\}$. An edge $e \in E(G)$ is *positive* if $\sigma(e) = 1$ and *negative* otherwise. Denote the set of all negative edges of (G, σ) by $E_N(G, \sigma)$. For a signed graph (G, σ), *switching* a vertex u means reversing the signs of all edges incident with u such that in the resulting signed graph (G, σ'), $\sigma'(e) = -\sigma(e)$ for each edge $e \in E(v)$ and $\sigma'(e) = \sigma(e)$ for all other edges. See Figure 1.1 for an illustration. Two signed graphs are *equivalent* if one can be obtained from the other by a sequence of switching operations. The *negativeness* of (G, σ) is denoted by $\epsilon(G, \sigma) = min\{|E_N(G, \sigma')| : \sigma' \text{ is equivalent to } \sigma\}$.

For convenience, the signature σ is usually omitted if no confusion arises or is written as σ_G if it needs to emphasize G. If there is no confusion from the context, we simply use $E_N(G)$ for

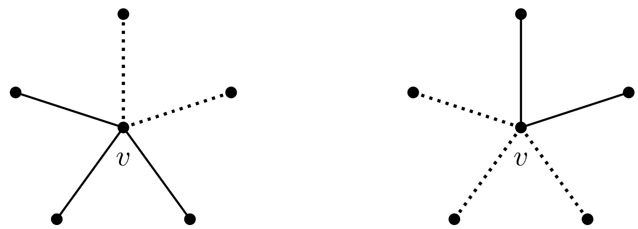


Figure 1.1: A switching at the vertex v

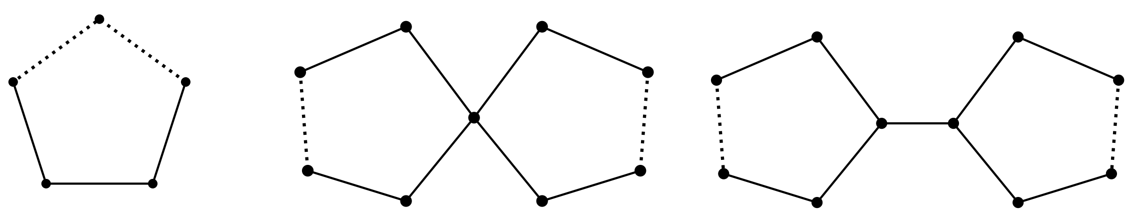
$E_N(G, \sigma)$ and use $\epsilon(G)$ for $\epsilon(G, \sigma)$.

Let $e = uv$ be an edge. By contracting e, we mean to first identify u with v and then to delete the loop if $\sigma(e) = 1$ otherwise to keep the negative loop. An unsigned graph is regarded as a signed graph with all-positive signature. A circuit is *balanced* if it contains an even number of negative edges, and it is *unbalanced* otherwise. A signed graph is called *balanced* if it contains no unbalanced circuit and is called *unbalanced* otherwise. A signed circuit is defined as a signed graph of any of the following three types (see Figure 1.2):

(1) a balanced circuit;

(2) a short barbell, that is, the union of two unbalanced circuits that meet at a single vertex;

(3) a long barbell, that is, the union of two disjoint unbalanced circuits with a path that meets the circuits only at its ends.



(1) A balanced circuit (2) A short barbell (3) A long barbell

Figure 1.2: Three types of signed circuits (dotted edges are negative edges)

We regard an edge $e = uv$ of a signed graph as two half edges h_e^u and h_e^v, where h_e^u is incident with u and h_e^v is incident with v. Let $H_G(v)$ (or simply $H(v)$ if no confusion may cause) be the set of all half edges incident with v, and $H(G)$ be the set of all half edges of (G, σ). An *orientation* of a signed graph (G, σ) is a mapping $\tau : H(G) \to \{-1, 1\}$ such that for each $e = uv \in E(G)$, $\tau(h_e^u)\tau(h_e^v) = -\sigma(e)$. It is convenient to consider τ as an assignment of orientations on $H(G)$. Namely, for $h_e^u \in H(G)$, h_e^u is *oriented away from* u if $\tau(h_e^u) = 1$ and h_e^u is *oriented toward* u if $\tau(h_e^u) = -1$. A signed graph (G, σ) together with an orientation τ is called an *oriented signed graph*, denoted by (G, τ).

1.3 Background on Nowhere-zero Flow Problems

The theory of integer flows which is a dual problem to vertex coloring of planar graphs was introduced by Tutte [28, 27] as a tool related to the Four-Color Theorem. He discovered the following duality relation between these two categories of problems.

Theorem 1.3.1 (Tutte [28])**.** *Let G be a graph strongly embedded on an orientable surface S. If G is k-face colorable, then G admits a nowhere-zero k-flow. Furthermore, if S is a sphere, then they are equivalent.*

It has been extended to signed graphs. The concept of integer flows on signed graphs naturally comes from the study of graphs embedded on non-orientable surfaces, where nowhere-zero flow emerges as the dual notion to local tension.

The collection $\pi = \{\pi_v | v \in V(G)\}$ is called a *rotation system*, which means for each vertex v, π_v is a cyclic permutation of the edges incident with v. Thus the embedding of the graph together with π naturally induces a signature as seen in the following definition.

Definition 1 (Mohar and Thomassen [22])**.** *Let (G, π, S) be an embedding of G on a non-orientable surface S where $\pi = \{\pi_v | v \in V(G)\}$ is the rotation system of the embedding. The signature σ_π induced by the embedding is a mapping $\sigma_\pi : E(G) \to \{\pm 1\}$ where $\sigma_\pi(e) = -1$ if and only if e passes through the cross-caps of S odd times.*

Lu et al. [18] showed the following proposition for the existence of an embedding (G, π, S).

Proposition 1.3.2 (Lu et al. [18])**.** *For any signed graph (G, σ), there exists a non-orientable surface S and an embedding (G, π, S) of G on S such that σ is the induced signature σ_π of the embedding.*

The following result extends Theorem 1.3.1 to all the surfaces (including non-orientable cases).

Theorem 1.3.3 (Bouchet [2])**.** *Let (G, π, S) be a signed graph strongly embedded on a surface S, π be a rotation system of the embedding. If (G, π, S) is k-face colorable on S, then (G, σ_π) admits a nowhere-zero k-flow. Furthermore, if S is a sphere or a projective plane, then they are equivalent.*

Theorem 1.3.3 is a natural extension of the fundamental result by Tutte (Theorem 1.3.1) to graphs embedded on all surfaces. If S is orientable, then the rotation system π can be selected as clockwise on one side of the surface and thus $\sigma_\pi(e) = 1$ for each edge, which is an ordinary graph and has already been well studied in Tutte's flow theory.

For ordinary graphs, Tutte [28] conjectured that every bridgeless graph admits a nowhere-zero 5-flow and Seymour [24] showed that every such graph admits a nowhere-zero 6-flow. there are signed graphs which have no nowhere-zero 5-flow (see [[2],[14],[25]]) and Bouchet proposed the following 6-flow conjecture in 1983.

Conjecture 1.3.1. (Bouchet [2]) *Every flow-admissible signed graph admits a nowhere-zero 6-flow.*

Bouchet [2] himself proved that such signed graphs admit nowhere-zero 216-flows and Zýka [38] further reduced to 30-flows. Recently DeVos et al. [4] further proved the following best known result.

Theorem 1.3.4. (DeVos et al. [4]) *Every flow-admissible signed graph admits a nowhere-zero 11-flow.*

1.4 Definition of Integer Flows on Signed Graphs

Definition 2. *Let (G, σ) be a signed graph and τ be an orientation of (G, σ). Let $k \geq 2$ be an integer and $f : E(G) \to \mathbb{Z}$ be a mapping.*

(1) The boundary *of (τ, f) is the mapping $\partial(\tau, f) : V(G) \to \mathbb{R}$ defined as*

$$\partial(\tau, f)(v) = \sum_{h \in H(v)} \tau(h) f(e_h)$$

for each vertex v, where e_h is the edge of (G, σ_τ) containing h.

(2) The support *of f, denoted by* $\text{supp}(f)$*, is the set of edges e with $|f(e)| > 0$.*

(3) If $\partial(\tau, f)(v) = 0$ for each vertex v, then (τ, f) is called a flow *of (G, σ). A flow (τ, f) is said to be* nowhere-zero *of (G, σ_τ) if* $\text{supp}(f) = E(G)$.

(4) If $1 \leq |f(e)| \leq k - 1$ for each edge $e \in E(G)$, then the flow (τ, f) is called a nowhere-zero k-flow of (G, σ_τ).

(4) If $\partial(\tau, f)(v) \equiv 0 \pmod{k}$ for each vertex v, then (τ, f) is called a $\mathbb{Z}_k$-flow *of (G, σ). A $\mathbb{Z}_k$-flow (τ, f) is said to be* nowhere-zero *if* $\text{supp}(f) = E(G)$.

For a mapping $f : E(G) \to \mathbb{Z}$, denote $E_{f=\pm i} = \{e \in E(G) : |f(e)| = i\}$.

For convenience, we shorten the notation of nowhere-zero k-flow and nowhere-zero Z_k-flow as k-NZF and $\mathbb{Z}_k$-NZF, respectively. If the orientation is understood from the context, we use f instead of (τ, f) to denote a flow. Observe that G admits a k-NZF under an orientation τ if and only if it admits a k-NZF under any orientation τ' which is equivalent to τ.

A signed graph is *flow-admissible* if it admits a nowhere-zero k-flow for some integer k. Note that switching a vertex does not change the parity of the number of negative edges in a circuit and although technically it changes the flows, it only reverses the directions of the half edges incident with the vertex, but the directions of other half edges and the flow values of all edges remain the same. Bouchet [2] gave a characterization for all flow-admissible signed graphs.

Proposition 1.4.1. (Bouchet [2]) *Let (G, σ) be a connected signed graph. The following three statements are equivalent:*

(1) (G, σ) is flow-admissible;

(2) (G, σ) is not equivalent to a signed graph with exactly one negative edge and it has no cut-edge b such that $(G - b, \sigma|_{G-b})$ has a balanced component;

(3) every edge in (G, σ) is contained in a signed circuit.

Given a signed graph (G, σ), let H be a signed subgraph of (G, σ) and C be a balanced circuit. Define the following operation:

$$\Phi_2\text{-operation}: \text{ add a balanced circuit } C \text{ into } H \text{ if } |E(C) - E(H)| \leq 2.$$

We use $\langle H \rangle_2$ to denote the maximal subgraph of G obtained from H via Φ_2-operations. Zýka [38] proved the following result.

Lemma 1.4.2. (Zýka [38]) *Let (G, σ) be a signed graph and H be a subgraph of G. If $\langle H \rangle_2 = G$, then (G, σ) admits a $\mathbb{Z}_3$-flow ϕ such that $E(G) - E(H) \subseteq \text{supp}(\phi)$.*

The next lemma gives a characterization of signed graphs admitting a nowhere-zero 2-flow.

Lemma 1.4.3. (Xu and Zhang [33]) *A signed graph (G, σ) admits a nowhere-zero 2-flow if and only if each component of (G, σ) is eulerian and has an even number of negative edges.*

The next proposition is a characterization of signed circuits admitting a nowhere-zero 3-flow.

Proposition 1.4.4. (Bouchet [2]) *Every balanced circuit or short barbell has a nowhere-zero 2-flow and every long barbell has a nowhere-zero 3-flow where each edge has flow value 2 or -2 if and only if it belongs to the path connecting the two unbalanced circuits.*

We further study the relation between modulo flows and integer flows on signed graphs. The equivalency of modulo flow and integer flow is a fundamental result in the theory of flows on unsigned graphs.

Theorem 1.4.5. (Tutte [26], or Younger [34]) *An unsigned graph admits a nowhere-zero modulo k-flow if and only if it admits a nowhere-zero k-flow.*

However, there is no equivalent result in regard to Theorem 1.4.5 for signed graphs in general. See an example in Figure 1.3. The next lemmas show how to convert a modulo flow to an integer-valued flow.

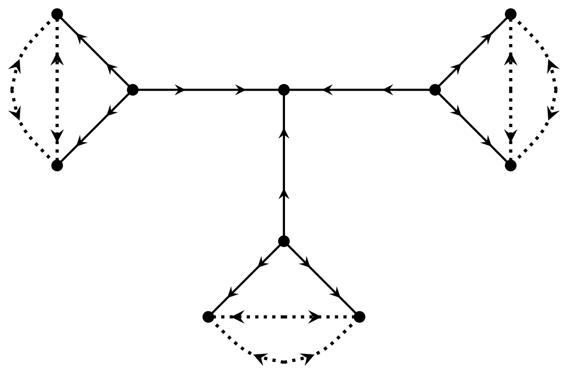


Figure 1.3: (G, σ) admits a $\mathbb{Z}_3$-NZF with all edges assigned with 1 but no 3-NZF. Dotted edges are negative.

Lemma 1.4.6. (Xu and Zhang [33]) *If a signed graph (G, σ) admits a $\mathbb{Z}_3$-flow f_1 such that* $\text{supp}(f_1)$ *has no cut edge, then it also admits an integer-valued* 3*-flow* f_2 *with* $\text{supp}(f_1) = \text{supp}(f_2)$.

The next lemma strengthens Lemma 1.4.6.

Lemma 1.4.7. (DeVos et al. [4]) *Let (G, σ) be a bridgeless signed graph admitting a $\mathbb{Z}_3$- NZF. Then for any edge $e' \in E(G)$ and for any $i \in \{1, 2\}$, (G, σ) admits a 3-NZF f such that $f(e') = i$.*

Let f be a $\mathbb{Z}_2$-flow of (G, σ). Then $\text{supp}(f)$ is a vertex-disjoint union of Eulerian subgraphs. A component of $\text{supp}(f)$ is called *balanced* if it contains an even number of negative edges; otherwise it is called *unbalanced*.

Lemma 1.4.8. (Cheng et al. [3]) *Let (G, σ) be a connected signed graph. If (G, σ) admits a $\mathbb{Z}_2$-flow f_1 such that* $\text{supp}(f_1)$ *contains an even number of unbalanced components, then it admits a* 3*-flow* f_2 *such that* $\text{supp}(f_1) = E_{f_2=\pm 1}$ *and* $\text{supp}(f_2)/\text{supp}(f_1)$ *is acyclic.*

Lemma 1.4.9. (Cheng et al. [3]) *Let (G, σ) be a bridgeless signed graph. If (G, σ) admits a $\mathbb{Z}_3$-flow f_1, then it admits a* 4*-flow* f_2 *with* $\text{supp}(f_1) \subseteq E_{f_2=\pm 1} \cup E_{f_2=\pm 2}$.

The following lemma converts a modulo flow to an integer-valued flow if G does not contain long barbells.

Lemma 1.4.10. (Lu et al. [16]) *Let (G, σ) be a signed graph without long barbells, and let k be an integer with $k = 3$ or $k \geq 5$. Then (G, σ) admits a nowhere-zero $\mathbb{Z}_k$-flow if and only if it admits a nowhere-zero k-flow.*

We use $B(G)$ to denote the set of cut-edges of G.

Lemma 1.4.11. (DeVos et al. [4]) *Let G be a signed graph admitting a $\mathbb{Z}_3$-NZF. Then G admits a 5-NZF g such that $E_{g=\pm 3} = \emptyset$ and $E_{g=\pm 4} \subseteq B(G)$.*

Integer flows on signed graphs also have been studied for many specific families of graphs. We list some following results.

Theorem 1.4.12. (Lu et al. [16]) *Let (G, σ) be a flow-admissible signed graph. If (G, σ) contains no long barbells, then it admits a nowhere-zero 6-flow.*

Theorem 1.4.13. (Lu et al. [17]) *Every flow-admissible signed graph without edge-disjoint unbalanced circuits admits a nowhere-zero 6-flow.*

Theorem 1.4.14. (Kaiser and Rollová [13]) *Every flow-admissible signed series-parallel graph has a nowhere-zero 6-flow.*

Theorem 1.4.15. (Máčajová and Rollová [19]) *The flow number of a flow-admissible signed graph whose underlying graph is either complete or complete bipartite is at most 4.*

Theorem 1.4.16. (Máčajová and Škoviera [21]) *Every flow-admissible signed Eulerian graph admits a nowhere-zero 4-flow.*

Theorem 1.4.17. (Wang et al. [29]) *Every flow-admissible signed graph with two negative edges admits a nowhere-zero 6-flow such that each negative edge has flow value 1.*

Theorem 1.4.18. (Wu et al. [32]) *A flow-admissible 8-edge-connected signed graph admits a nowhere-zero 3-flow.*

Theorem 1.4.19. (Raspaud and Zhu [23]) *A flow-admissible 6-edge-connected signed graph admits a nowhere-zero 4-flow.*

Theorem 1.4.20. (Wei et al. [30]) *Every 3-edge-connected flow-admissible signed graph admits a nowhere-zero 15-flow.*

1.5 Main Results

We call a signed graph G *antibalanced* if its signature is equivalent to the all-negative signature. Clearly, G is antibalanced if and only if every circuit contains an even number of positive edges, or equivalently, if and only if all even circuits of G are balanced and all odd circuits of G are unbalanced. Consequently, an antibalanced graph is balanced if and only if its underlying unsigned graph is bipartite. The following is a direct consequence of Harary's balance theorem.

Theorem 1.5.1. *(Harary's balance theorem [9]) A signed graph is antibalanced if and only if its vertex set can be partitioned into two sets (either of which may be empty) in such a way that each edge between the sets is positive and each edge within a set is negative.*

A circuit C with an antibalanced bipartition $\{A_1, A_2\}$ will be called *half-odd* if for some $i \in \{1, 2\}$, each component of $C - A_i$ is either a path of odd length or the entire C.

Schubert and Steffen [25] verified Bouchet's Conjecture for Kotzig graphs. Máčajová and Škoviera [20] characterized cubic signed graphs that admit a nowhere-zero 3-flow and that admit a nowhere-zero 4-flow with the following theorem.

Theorem 1.5.2. (Máčajová and Škoviera [20]) *Let G be a cubic signed graph.*

(1) Then G has a nowhere-zero 3-flow if and only if it is antibalanced and has a perfect matching.

(2) Then G admits a nowhere-zero 4-flow if and only if it is switching equivalent to one that has an antibalanced 2-factor with all components half-odd and with complement an all-negative perfect matching.

We investigated integer flows in 3-edge-colorable cubic signed graphs and prove the following theorem.

Theorem 1.5.3. *Every flow-admissible* 3*-edge-colorable cubic signed graph admits a nowhere-zero* 10*-flow.*

By the 4-color theorem, every bridgeless cubic planar graph is 3-edge-colorable. Therefore we have the following corollary for bridgeless signed planar graphs.

Corollary 1.5.4. *Every flow-admissible bridgeless planar signed graph admits a nowhere-zero* 10*-flow.*

Theorem 1.5.3 follows from the following stronger result which shows that every connected flow-admissible 3-edge-colorable cubic signed graph admits a nowhere-zero 8-flow except one case which has a nowhere-zero 10-flow.

Theorem 1.5.5. *Let (G, σ) be a connected* 3*-edge-colorable cubic signed graph and $E_N(G, \sigma)$ be the set of negative edges in (G, σ). Let R, B, Y be the three color classes such that $|R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| \pmod 2$. If (G, σ) is flow-admissible, then it has a nowhere-zero* 8*-flow unless $R \cup B$ contains no unbalanced circuits and the numbers of unbalanced circuits in $R \cup Y$ and $B \cup Y$ are both odd and at least* 3*, in which case it has a nowhere-zero* 10*-flow.*

As a byproduct, we also prove the following 8-flow theorem for Hamiltonian signed graphs.

Theorem 1.5.6. *If (G, σ) is a flow-admissible Hamiltonian signed graph, then (G, σ) admits a nowhere-zero 8-flow.*

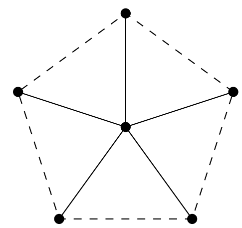


Figure 1.4: (W_5, σ^*) has a 5-NZF but no 4-NZF. Dotted edges are negative.

Secondly, we considered nowhere-zero integer flows in triangularly connected signed graphs. For triangularly connected ordinary graphs, Fan *et al.* [7] show that every triangularly connected ordinary graph admits a nowhere-zero 4-flow and they also characterize all such graphs not admitting a nowhere-zero 3-flow. For its signed counterpart, we prove the following result.

Theorem 1.5.7. *If (G, σ) is a flow-admissible triangularly connected signed graph, then (G, σ) admits a nowhere-zero 4-flow if and only if $(G, \sigma) \neq (W_5, \sigma^*)$ where (W_5, σ^*) is the signed graph in Figure 1.4. Moreover, there are infinitely many triangularly connected unbalanced signed graphs that admit a nowhere-zero 4-flow but no 3-flow.*

A graph G is *locally connected* if the subgraph induced by the neighbor of each vertex is connected. It is known that locally connected graphs, square of graphs, chordal graphs, triangulations on surfaces and some types of products of graphs are triangularly connected (such as [11], [15], for ordinary graphs) and thus we have the following corollary.

Corollary 1.5.8. *Let (G, σ) be a flow-admissible signed graph. If G is locally connected, then (G, σ) admits a nowhere-zero 4-flow if and only if $(G, \sigma) \neq (W_5, \sigma^*)$. In particular, if G is the square of a connected graph or is the strong product of graphs, then (G, σ) admits a nowhere-zero 4-flow.*

Lastly, we worked on the nowhere-zero integer flows in K_4-minor-free signed graphs. There are flow-admissible signed graphs that admit a nowhere-zero 6-flow but no nowhere-zero 5-flow (see Figure 1.5). Note that the third graph, denoted by N^{σ}_{4k+2}, is the smallest signed graph of an infinite family, in which all the members are signed graphs obtained from a positive circuit of length $4k+2$ by replacing every even index edge with an unbalanced 2-circuit, where $k \geq 1$.

We define *contraction* in signed graphs as follows. For an edge $e \in E(G)$, the *contraction* G/e is the signed graph obtained from G by identifying the two ends of e, and then deleting the resulting positive loop if e is a positive edge, but keeping the resulting negative loop if e is a negative edge in $E(G)$. Let (G, σ) be a flow-admissible signed graph. We define the following operations:

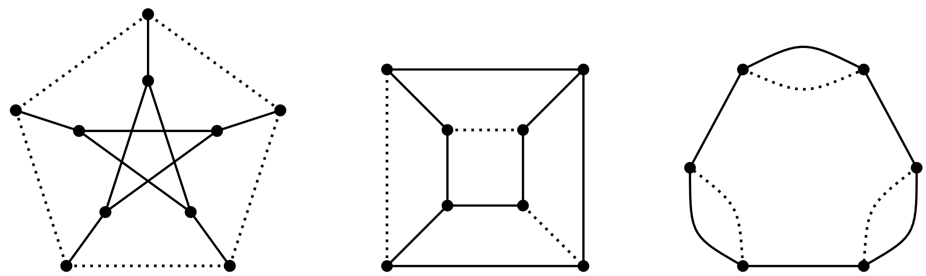


Figure 1.5: Signed graphs with flow number 6. Dotted edges are negative.

(O1) Let $X \subseteq V(G)$ such that $2 \leq \delta_G(x) \leq 3$ and $G[X]$ is balanced. First switch at some vertices so that $G[X]$ contains no negative edges and then contract $G[X]$.

(O2) Let x be a cut-vertex of G and H be a balanced component of G at x. First switch at some vertices such that H contains no negative edges and then delete H.

(O3) Suppress a degree 2-vertex.

Let (G_1, σ_1) and (G_2, σ_2) be two signed graphs. We say that (G_1, σ_1) is reducible to (G_2, σ_2) if (G_2, σ_2) can be obtained from (G_1, σ_1) by a sequence of Operations (O1), (O2), and (O3).

The following lemma is proved in [18].

Lemma 1.5.9. (Lu et al. [18]) *Let (G_1, σ_1) and (G_2, σ_2) be two signed graphs. If (G_1, σ_1) is reducible to (G_2, σ_2), then either both admit a nowhere-zero 6-flow or neither admits a nowhere-zero 6-flow.*

Let $\mathcal{M}$ be the set of signed graphs switching equivalent to signed graphs reducible to one of the three types of signed graphs in Figure **??**. Kompišová and Máčajová [**?**] proposed the following conjecture.

Conjecture 1.5.1. (Kompišová and Máčajová [**?**]) *If a flow-admissible signed graph does not admit a nowhere-zero 5-flow, then it belongs to $\mathcal{M}$.*

They also proved the following lemma.

Lemma 1.5.10. (Kompišová and Máčajová [**?**]) *Let (G, σ) be a flow-admissible signed graph in $\mathcal{M}$. Then (G, σ) admits a nowhere-zero 6-flow.*

Let $\mathcal{N}$ be the family of flow-admissible signed graphs switching equivalent to signed graphs reducible to N^{σ}_{4k+2} $(k \geq 1)$.

It is easily to see that every K_4-minor free ordinary graph admits a nowhere-zero 3-flow. For its signed counterpart, we prove the following result, which implies that Conjecture 1.3.1 is true for flow-admissible K_4-minor free signed graphs. Let H be a bridgeless and balanced graph. Let $\mathcal{N}$ be the family of signed graphs switching equivalent to N^{σ}_{4k+2} $(k \geq 1)$ and all their cut-vertex reducible signed graphs, 2-edge-cut reducible signed graphs, 3-edge-cut reducible signed graphs such that the bridgeless, balanced component H is nontrivial.

Theorem 1.5.11. *Let (G, σ) be a flow-admissible, K_4-minor free signed graph. Then (G, σ) admits a nowhere-zero 5-flow if and only if (G, σ) does not belong to the family $\mathcal{N}$.*

Chapter 2

Flows of 3-edge-colorable cubic signed graphs

The main purpose of this chapter is to consider the first introduce the following basic notation and terminology.

2.1 Notation and Terminology

Let G be a graph. A *leaf vertex* is a vertex of degree 1. A path is *nontrivial* if it contains at least two vertices. Let u, v be two vertices in $V(G)$. A (u, v)*-path* is a path with u and v as its endvertices. Let $C = v_1 \cdots v_r v_1$ be a circuit where $v_1, v_2, \ldots, v_r$ appear in clockwise on C. A *segment* of C is the path $v_i v_{i+1} \cdots v_{j-1} v_j$ contained in C and is denoted by $v_i C v_j$, where the indices are taken modulo r.

2.2 Uesful Lemmas

It is clear that a signed graph admits a $\mathbb{Z}_2$-NZF if and only if each component of (G, σ) is eulerian. We introduced that Lemma 1.4.3 gives a characterization of signed graphs admitting a 2-NZF. We introduced Lemma 1.4.8 and Lemma 1.4.9 to show how to convert a modulo flow to an integer-valued flow. Lemma 1.4.8 can be extended to the case when the support of a $\mathbb{Z}_2$-flow contains an odd number of odd components in the following lemma.

Lemma 2.2.1. *Let (G, σ) be a connected signed graph. If (G, σ) admits a $\mathbb{Z}_2$-flow f_1 such that the number of odd components of* $\text{supp}(f_1)$ *is odd and is at least three, then (G, σ) has a 5-flow f_2 satisfying*

(1) $\text{supp}(f_2)/\text{supp}(f_1)$ *is acyclic;*

(2) $\text{supp}(f_1) \subseteq \{e \in E(G)\colon\ 1 \leq |f_2(e)| \leq 3\}$ *and* $|f_2(e)| \in \{1,2\}$ *for each negative loop* $e \in \text{supp}(f_1)$*.*

Proof. Let (G,σ) together with a $\mathbb{Z}_2$-flow (τ, f_1) be a counterexample to Lemma 2.2.1 such that $|E(G)|$ is minimized. In the following, we always assume the flows are under the orientation τ or its restriction on according subgraphs.

Denote by $\mathcal{B}$ the set of components of $\text{supp}(f_1)$ and let $H = G/\text{supp}(f_1)$. Thus $V(H)$ can be partitioned into three parts: X, Y and W where X and Y are the sets of vertices corresponding to even and odd components in $\mathcal{B}$ respectively and W is corresponding to the vertices which are also the vertices in $V(G)$. For $u \in X \cup Y$, let B_u denote the corresponding component in $\mathcal{B}$.

Claim 2.2.1.1. *G contains no leaf vertices and H is a tree.*

Proof. If G contains a leaf vertex, say x, then $f_1(e) = 0$ where e is the edge incident with x and $G - x$ remains connected. This contradicts to the minimality of G.

Clearly H is connected since G is connected. If H is not a tree, then there is an edge $e \in E(G)$ such that $f_1(e) = 0$ and $G - e$ is connected, a contradiction to the minimality of G again. □

Let u be a leaf vertex of H and v be its neighbor. By Claim 2.2.1.1, $u \in X \cup Y$. Since u is a leaf vertex of H, there is only one edge in G with one endvertex in B_u and the other one in B_v. Let $x_u x_v$ be the only edge in G where $x_u \in V(B_u)$ and $x_v \in V(B_v)$.

Claim 2.2.1.2. *$u \in Y$ and $v \notin Y$.*

Proof. Suppose to the contrary that either $u \in X$ or $u \in Y$ and $v \in Y$. Let $G' = G - V(B_u)$. Since B_u is a leaf block, G' is connected.

If $u \in X$, then B_u is an even component and thus $\mathcal{B} - B_u$ and $\mathcal{B}$ have the same number of odd components. Since G' is connected, by the minimality of G, there is an integer 5-flow g_1 of $(G', \sigma|_{E(G')})$ such that $\text{supp}(f_1) - E(B_u) \subseteq \text{supp}(g_1)$ and g_1 satisfies (1) and (2). Then g_1 can be considered as a flow of (G,σ) under the orientation τ such that $E(B_u) \cap \text{supp}(g_1) = \emptyset$. Since B_u is an even eulerian component, by Lemma 1.4.3, there is a 2-flow g_2 of (G,σ) such that $\text{supp}(g_2) = E(B_u)$. Therefore $g_1 + g_2$ is a 5-flow of (G,σ) satisfying (1) and (2), a contradiction.

Now assume that $u \in Y$ and $v \in Y$. Let $\hat{B_v}$ be the subgraph obtained from B_v by deleting as many negative loops in B_v as possible so that $\hat{B_v}$ remains odd. Then $\hat{B_v}$ contains at most one negative loop and $\mathcal{B} - \{B_u, B_v\} + \{\hat{B_v}\}$ has an even number of odd components. By Lemma 1.4.8, there is a 3-flow g_3 such that $\text{supp}(f_1) - [E(B_u) \cup E(B_v)] + E(\hat{B_v}) \subseteq \text{supp}(g_3)$ and g_3 satisfies (1) and (2). Note that $g_3(x_u x_v) = 0$ and $g_3(e) = 0$ for each $e \in E(B_u) \cup [E(B_v) \setminus E(\hat{B_v})]$. By Lemma 1.4.8 again, there is a 3-flow g_4 such that $\text{supp}(g_4) = E(B_u) \cup E(B_v) + x_u x_v$, $E(B_u) \cup E(B_v) = E_{g_4 = \pm 1}$, and $\{x_u x_v\} = E_{g_4 = \pm 2}$. Since $\hat{B_v}$ contains at most one negative loop, we have that $\text{supp}(g_3)$ contains at most one loop in B_v. Therefore either $g_3 + 2g_4$ or $g_3 - 2g_4$ is a desired 5-flow, a contradiction. This proves the claim. $\square$

Let (G_1, σ_1) be the signed graph obtained from $G - V(B_u)$ by adding a negative loop e_1 at x_v where σ_1 is defined as $\sigma_1(e) = \sigma(e)$ for each $e \in E(G_1) - \{e_1\}$ and $\sigma_1(e_1) = -1$. The orientation τ_1 of (G_1, σ_1) is defined as $\tau_1(h) = \tau(h)$ for each $h \in H(G_1)$ and h is not an half edge of the loop e_1; for each half edge h of e_1, $\tau_1(h) = \tau(h^v_{uv})$.

Let (G_2, σ_2) be the signed graph obtained from B_u by adding a negative loop e_2 at x_u. Its signature σ_2 and orientation τ_2 are defined similarly to σ_1 and τ_1, respectively.

Denote $B'_v = B_v \cup \{e_1\}$ and $\mathcal{B}' = \mathcal{B} - B_u - B_v + B'_v$ if $v \in X \cup Y$; otherwise denote $B'_v = \{e_1\}$ and $\mathcal{B}' = \mathcal{B} - B_u + B'_v$. Note that there is a $\mathbb{Z}_2$-flow of (G_1, σ_1) whose support is $\bigcup_{B \in \mathcal{B}'} E(B)$.

By Claim 2.2.1.2, both B_u and B'_v are odd. Thus $\mathcal{B}'$ and $\mathcal{B}$ have the same number of odd components. By the minimality of G, there is a 5-flow (τ_1, g_5) of (G_1, σ_1) satisfying (1) and (2).

By Claim 2.2.1.2, (G_2, σ_2) is a signed eulerian graph with even number of negative edges. By Lemma 1.4.3, there is a 2-flow (τ_2, g_6) of (G_2, σ_2) such that $\text{supp}(g_6) = E(G_2)$. We may assume $g_5(e_1)g_6(e_2) > 0$ otherwise replacing g_6 with $-g_6$. Let $a = g_5(e_1)$. Then $|a| \in \{1, 2\}$.

Let (τ, g_7) be the integer flow of (G, σ) defined as follows: for each $e \in E(G)$,

$$g_7(e) = \begin{cases} g_5(e) & \text{if } e \in \text{supp}(g_5); \\ ag_6(e) & \text{if } e \in \text{supp}(g_6); \\ 2a & \text{if } e = uv; \\ 0 & \text{otherwise.} \end{cases}$$

Then g_7 is a 5-flow of (G, σ) satisfying (1) and (2), a contradiction. This completes the proof of the lemma. $\square$

The following lemma is due to Zaslavsky [35].

Lemma 2.2.2. (Zaslavsky [35]) *Let T be a spanning tree of a signed graph (G, σ). For every $e \notin E(T)$, let C_e be the unique circuit contained in $T + e$. If the circuit C_e is balanced for every $e \notin E(T)$, then G is balanced.*

The proof of the following lemma is inspired by the proof of Theorem 4.2 in [21] due to Máčajová and Škoviera.

Lemma 2.2.3. *Let C be an unbalanced circuit of a signed graph (G, σ). If (G, σ) is flow-admissible and $G - E(C)$ is balanced, then (G, σ) has a 4-flow f satisfying the following:*

(1) $E(C) \subseteq \text{supp}(f)$;

(2) In $H = G[\text{supp}(f)]$ the subgraph induced by $\text{supp}(f)$, each vertex in $V(H) - V(C)$ has degree at most 3 in H and at most one vertex in $V(H) - V(C)$ has degree 3.

Proof. Denote by $G' = G - E(C)$. Since G' is balanced, with some switching operations, we may assume that all edges in $E(G')$ are positive and thus $E_N(G,\sigma) \subseteq E(C)$. Fix an orientation τ of (G,σ) and in the following we always assume the flows are under the orientation τ or its restriction on according subgraphs.

Let M be a component of G'. The circuit C is divided by the vertices of M into segments whose endvertices lie in M and all inner vertices lie outside M. An endvertex of a segment is called an *attachment* of M. A segment is called positive (negative) if it contains an even (odd) number of negative edges. Let S be a segment. Note that $M \cup S$ is unbalanced (balanced) if and only if the segment S is negative (positive). Since C is unbalanced, the number of negative segments determined by each component M is odd.

We prove the lemma by contradiction. Suppose to the contrary that (G,σ) has no 4-flow satisfying (1) and (2).

Claim 2.2.3.1. *Each component of G' determines exactly one negative segment.*

Proof. Suppose to the contrary that M determines more than one negative segments. Thus M determines at least three negative segments. Let u_1Cu_1', u_2Cu_2', u_3Cu_3' be three consecutive negative segments (in clockwise) where u_i and u_i' are attachments for $i = 1,2,3$. Then $u_1'Cu_2$, $u_2'Cu_3$, $u_3'Cu_1$ all contain even number of negative edges. This implies that C can be partitioned into three negative segments: u_1Cu_2, u_2Cu_3, and u_3Cu_1.

We first show that no (u_1,u_2)-path in M passes through u_3. Otherwise let P be a (u_1,u_2)-path in M that passes through u_3. Then $C_1 = u_1Cu_3 + u_1Pu_3$ and $C_2 = u_3Cu_2 + u_3Pu_2$ both are balanced circuits. By Lemma 1.4.3, there is a 2-flow f_i of (G,σ) such that $\text{supp}(f_i) = E(C_i)$ for each $i = 1,2$. Therefore $2f_1 + f_2$ is a 4-flow of (G,σ) and $\text{supp}(2f_1 + f_2) = E(C) \cup E(P)$, which is a desired 4-flow, a contradiction.

By symmetry, no (u_i,u_j)-path passes through u_k where $\{i,j,k\} = \{1,2,3\}$. This implies that u_1 and u_2 are not adjacent. Otherwise, a (u_1,u_3)-path together with u_1u_2 gives a (u_2,u_3)-path containing u_1.

Let P_1 be a (u_1,u_2)-path. Since M is connected, there is a path P_2 from u_3 to P_1 such that $|V(P_2) \cap V(P_1)| = 1$. Let v be the only common vertex in P_1 and P_2. Then C, P_1, and P_2 form a signed graph as illustrated in Figure 2.1 which has a desired 4-NZF, a contradiction again. This completes the proof of the claim. □

Let $\mathcal{M}$ denote the set of all components of G'. For each component M, denote by $S_M = uCv$ the negative segment determined by M where u and v are two attachments of M on C. Denote by $S'_M = vCu$ the cosegment of S_M. Then $E(S_M) \neq \emptyset$ and $E(S'_M) = E(C) - E(S_M)$.

Claim 2.2.3.2. $\bigcap_{M\in\mathcal{M}} E(S_M) = \emptyset$. *Therefore* $\bigcup_{M\in\mathcal{M}} E(S'_M) = C$ *and* $|\mathcal{M}| \geq 2$.

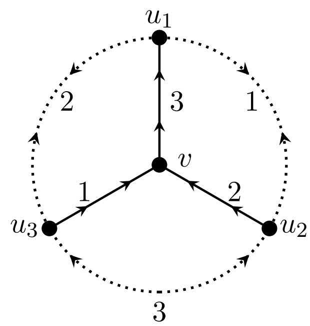


Figure 2.1: a 4-flow covers C

Proof. Suppose to the contrary $\bigcap_{M\in\mathcal{M}} E(S_M) \neq \emptyset$. Let $e^* \in \bigcap_{M\in\mathcal{M}} E(S_M)$. Then there is a spanning tree T of $G - e^*$ containing the path $P^* = C - e^*$. Let $e = uv \in E(G) - e^* - E(T)$. Denote the unique circuit contained in $T + e$ by C_e.

If $E(C_e) \cap E(P^*) = \emptyset$, then C_e contains no negative edges and thus is balanced.

Assume that C_e and P^* have common edges. Since T contains all the edges in $C - e^*$, $E(C_e) \cap E(C)$ is a path P on C. Let u' and v' be the two endvertices of P in clockwise order on C. Then $C_e[(V(C_e) - V(P)) \cup \{u', v'\}]$ is a also a path and thus it is contained in some component $M \in \mathcal{M}$. This implies that u' and v' are two attachments of M on C. Since e^* belongs to the only negative segment of C determined by M, $u'Cv'$ is the union of some positive segments of C determined by M. Therefore C_e has an even number of negative edges and thus is balanced. By Lemma 2.2.2, $G - e^*$ is balanced, contradicting Lemma 1.4.1. This proves $\bigcap_{M\in\mathcal{M}} E(S_M) = \emptyset$.

Since $E(S'_M) = E(C) - E(S_M)$ and $\bigcap_{M\in\mathcal{M}} E(S_M) = \emptyset$, we have $\bigcup_{M\in\mathcal{M}} E(S'_M) = C$.

Since $E(S_M) \neq \emptyset$ and $\bigcap_{M\in\mathcal{M}} E(S_M) = \emptyset$, we have $|\mathcal{M}| \geq 2$. □

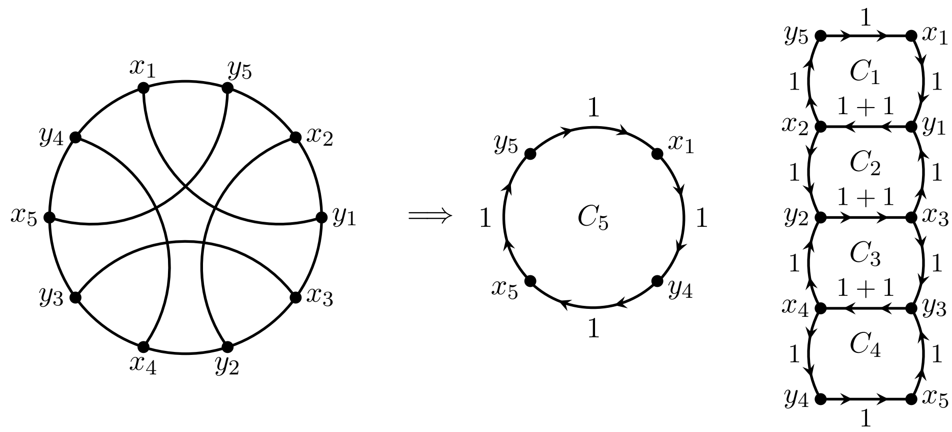
Let $\mathcal{S} = \{S'_1, S'_2, \ldots, S'_t\}$ be a minimal cosegment cover of C. Then $S'_i \not\subseteq S'_j$ for any i, j.

Claim 2.2.3.3. *(i) For each pair $i, j \in \{1, 2, \ldots, t\}$, either $S'_i \cap S'_j$ consists of some nontrivial paths or S'_i and S'_j are vertex-disjoint;*

(ii) Each edge $e \in E(C)$ is contained in at most two cosegments.

Proof. (i) Note that for any two segments S_i and S_j, their endvertices belong to two vertex-disjoint components M_i and M_j. Thus no component of $S'_i \cap S'_j$ is an isolated vertex. This proves (i).

(ii) Suppose to the contrary that there is an edge $e = uv$ that belongs to three cosegments, say S'_1, S'_2, S'_3. Let $S'_i = u_iCv_i$ for each $i \in \{1, 2, 3\}$. Without loss of generality, we may assume that u_1, u_2, u_3, u, v appear in this clockwise cyclic order. Then there exists a pair i, j such that u_iCv_j contains all the $u_l, v'_l s$ $(l \in \{1, 2, 3\})$ and u, v. Hence, there is a $k \in \{1, 2, 3\} - \{i, j\}$ such



(a) a minimal cosegment cover with $t = 5$ (b) 2-flow f_5 of C_5 (c) 2-flows f_1, f_2, f_3, f_4

Figure 2.2: Minimum cosegment cover and 4-flow

that u_k and v_k are properly included in u_iCv_j. In this case, either $\mathcal{S} \setminus \{S'_k\}$ is still a cover of C or $S'_k \cup S'_i = C$, both in contradiction to the minimality of $\mathcal{S}$. This completes the proof of the claim. □

The final step. For each $i = 1, \ldots, t$, denote by $S'_i = x_iCy_i$ and let P_i be a path in M_i connecting x_i and y_i. Then $C_i = S'_i \cup P_i$ is a balanced eulerian subgraph. By Claim 2.2.3.3, we may assume that the vertices $x_1, y_t, x_2, y_1, \ldots, x_t, y_{t-1}, x_1$ appear on C in the acyclic order. Then $C_i \cap C_j \neq \emptyset$ if and only if $|j - i| \equiv 1 \pmod{t}$. Moreover $C_i \cap C_{i+1} = x_{i+1}Cy_i$ where the subindices are taken modulo t. See Figure 2.2 for an illustration with $t = 5$.

For each $i \in \{1, 2, \ldots, t\}$, let (τ, f_i) be a 2-flow of (G, σ) such that $\text{supp}(f_i) = E(C_i)$. We may assume that for each $i = 1, \ldots, t-1$, $f_i(e) = f_{i+1}(e)$ for each $e \in E(C_i) \cap E(C_{i+1})$. Then $\phi = \sum_{i=1}^{t-1} f_i + 2f_t$ is a 4-flow of (G, σ) satisfying $E(C) \subseteq \text{supp}(\phi) = E(C) \cup [\cup_{i=1}^{t} E(P_i)]$. Since $P_1, \ldots, P_t$ belong to different components of G', they are pairwise vertex-disjoint. Thus for each vertex $v \in V(\text{supp}(\phi)) - V(C)$, the degree of v in $\text{supp}(\phi)$ is two. Therefore ϕ is a 4-flow satisfying (1) and (2), a contradiction to the assumption that (G, σ) is a counterexample. This contradiction completes the proof of the lemma. □

The proof of the following lemma is straightforward and thus is omitted.

Lemma 2.2.4. *Let (G, σ) be a signed graph and C be a chordless circuit whose edges are all positive. Suppose that $2 \leq |\delta(V(C))| \leq 3$ and $k \geq 4$ is an integer. If $(G/C, \sigma)$ has a k-NZF f, then f can be extended to be a k-NZF of (G, σ).*

2.3 Proofs of Theorem 1.5.5 and Corollary 1.5.4

Let's first recall Theorem 1.5.5.

Theorem 1.5.5 *Let (G, σ) be a connected 3-edge-colorable cubic signed graph and $E_N(G, \sigma)$ be the set of negative edges in (G, σ). Let R, B, Y be the three color classes such that $|R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| \pmod 2$. If (G, σ) is flow-admissible, then it has a nowhere-zero 8-flow unless $R \cup B$ contains no unbalanced circuits and the numbers of unbalanced circuits in $R \cup Y$ and $B \cup Y$ are both odd and at least 3, in which case it has a nowhere-zero 10-flow.*

Proof. Let τ be an orientation of (G, σ). In the following we always assume the flows are under the orientation τ or its restriction on according subgraphs. Denote by M_1M_2 the 2-factor induced by $M_1 \cup M_2$ for each pair $M_1, M_2 \in \{R, B, Y\}$. Since $|R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| \pmod 2$, RB has an even number of odd components.

Case 1. RB contains an unbalanced circuit.

Then by Lemma 1.4.8, (G, σ) has a 3-flow (τ, f_1) such that $RB = E_{f_1=\pm 1}$ and $|f_1(e)| = 2$ only if $e \in Y$.

Subcase 1.1. $|Y \cap E_N(G, \sigma)| \equiv |R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| \pmod 2$.

Then RY has an even number of unbalanced circuits. By Lemma 1.4.8 again, (G, σ) has a 3-flow (τf_2) such that $RY = E_{f_2=\pm 1}$ and $|f_2(e)| = 2$ only if $e \in B$.

Then $f = f_1 + 3f_2$ is a 9-NZF of (G, σ). Since $E_{f_2=\pm 2} \cap E_{f_1=\pm 2} = \emptyset$, $|f(e)| \neq 8$. Thus f is indeed an 8-NZF of (G, σ).

Subcase 1.2. RY or BY has an odd number of unbalanced circuits.

In this case, both RY and BY have an odd number of unbalanced circuits.

Let C be an unbalanced circuit in RB. Let $R' = R \triangle C$ and $B' = B \triangle C$ with R and B, respectively (this is equivalent to swap colors R and B on C). This implies that $|Y \cap E_N(G, \sigma)| \equiv |R' \cap E_N(G, \sigma)| \equiv |B' \cap E_N(G, \sigma)| \pmod 2$. We are back to Subcase 1.1.

Case 2. RB contains no unbalanced circuit.

Then by Lemma 1.4.3, (G, σ) has a 2-flow f_3 such that $\text{supp}(f_3) = RB$.

Subcase 2.1. The number of unbalanced circuits in RY or BY is even.

Let f_2 be the 3-flow in Subcase 1.1. Then $\text{supp}(f_2) \cup \text{supp}(f_3) = E(G)$. Thus $3f_3 + f_2$ is a 6-NZF of (G, σ).

Subcase 2.2. The number of unbalanced circuits in RY or BY is equal to one.

By symmetry, assume that RY has exactly one odd component, say C_1. Let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be the set of components of RY, where each C_i $(i \geq 2)$ is balanced, and, with some switching operations, we may assume that the edges of each C_i $(i \geq 2)$ are all positive. Let H be the signed graph obtained from (G,σ) by contracting $\mathcal{C} - C_1$. Then $V(H)$ can be partitioned into K and $\overline{K}$, where $K = V(C_1)$ and $\overline{K}$ is the set of vertices corresponding to the balanced circuits in $\mathcal{C}$. For $u \in \overline{K}$, denote the corresponding circuit in $\mathcal{C}$ by C_u. Since G is flow-admissible, H remains flow-admissible. Note that C_1 is an unbalanced circuit in H.

We consider the following two cases.

Subcase 2.2.1. H contains an unbalanced circuit C' that is edge-disjoint from C_1.

Since G is cubic, C' is vertex-disjoint from C_1. Thus there is a long barbell Q in H with P as the path connecting C_1 and C'. Let τ_1 be the orientation of Q which is a restriction of τ on $H(Q)$. By Lemma 1.4.8, let (τ_1, f'') be a 3-NZF in Q. Since $d_Q(u) = 2$ or 3 for any $u \in V(Q) - V(C_1)$, u is corresponding to an all-positive circuit C_u in (G,σ) with $|\delta_Q(V(C_u))| = 2$ or 3. Hence by Lemma 2.2.4 we can extend f'' to a 4-flow f' (G,σ) with $\bigcup_{u \in V(Q)} E(C_u) \cup E(C_1) \subseteq \text{supp}(f')$. Since for each $v \in V(H) - V(Q)$, C_v is a balanced circuit in (G,σ) , (G,σ) admits a 2-flow ϕ_v with $E(C_v) = \text{supp}(\phi_v)$. Thus $f_4 = f' + \sum_{u \in V(H)-V(Q)} \phi_u$ is a 4-flow of (G,σ) with $RY \subseteq \text{supp}(f_4)$. Therefore, $f_3 + 2f_4$ is an 8-NZF of (G,σ).

Subcase 2.2.2. H contains no unbalanced circuit that is edge-disjoint from C_1.

In this case, $H - E(C_1)$ is balanced and thus $G - E(C_1)$ is balanced. With some switching operations we may assume $E_N(G,\sigma) \subseteq E_G(C_1)$. By Lemma 2.2.3, (G,σ) has a 4-flow f'' such that $C_1 \subseteq \text{supp}(f'')$ and every vertex in $\text{supp}(f'') - E(C_1)$ has degree at most 3 in H. By Lemma 2.2.4, we can extend f'' to a 4-flow f_5 of (G,σ) with $RY \subseteq \text{supp}(f_5)$ in (G,σ). Therefore, $f_3 + 2f_5$ is an 8-NZF of (G,σ).

Subcase 2.3. The number of unbalanced circuits in RY or BY is odd and is at least 3.

By symmetry, assume that the number of unbalanced circuits in RY is odd and is at least 3. By Lemma 2.2.1, (G,σ) has a 5-flow f_6 such that $RY \subseteq \text{supp}(f_6)$ and $E_{f_6=\pm 4} \subseteq B$. Then $\text{supp}(f_3) \cup \text{supp}(f_6) = E(G)$. Thus $5f_3 + f_6$ is a 10-NZF of (G,σ). □

Next we will prove Corollary 1.5.4.

Corollary 1.5.4 Every flow-admissible bridgeless planar signed graph admits a nowhere-zero 10-flow.

Proof. Let (G,σ) be a flow-admissible bridgeless planar signed graph. Let τ be an orientation of (G,σ). In the following we always assume the flows are under the orientation τ or its restriction on according subgraphs.

We may assume that the minimum degree of G is at least 3 otherwise we can suppress all degree 2 vertices. We may also assume that G contains no positive loops.

If G is cubic, then by Theorem 1.5.3 and the 4-color theorem, G admits a nowhere-zero 10-flow.

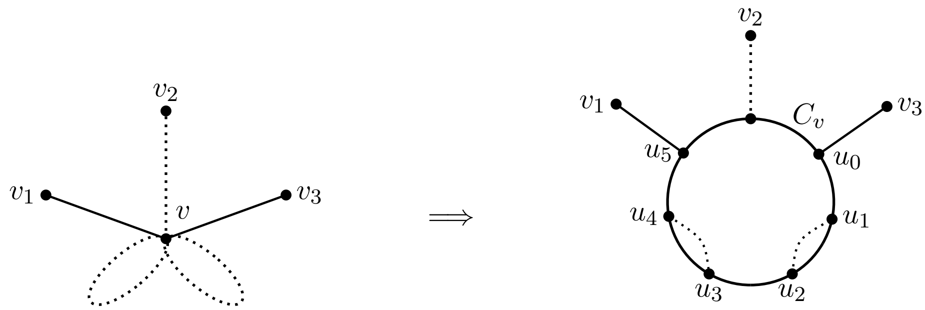


Figure 2.3: blowing up of a vertex v with $d(v) = 7$ and $t = 2$. Dotted lines are negative edges.

Suppose that G is not cubic and that G is already embedded in a sphere. Let v be a vertex with $d_G(v) \geq 4$ and t be the number of negative loops adjacent to v. First delete the t negative loops and then blow up v into a circuit C_v of length $d_G(v) - 2t$ where each edge of C_v is positive. Let xy be an edge in C_v. Replace it with a subdivided edge $u_0u_1u_2\cdots u_{2t+1}$ where $x = u_0$ and $y = u_{2t+1}$ and then replace each u_iu_{i+1} with an unbalanced digon for each $i = 1, 3, \ldots, 2t-1$ (see Figure 2.3). Let (G', σ') be the resulting signed graph obtained from (G, σ) by applying the above operations on each vertex in G of degree at least 4. Then (G', σ') is cubic, planar, and flow-admissible. By Theorem 1.5.3 and the 4-color theorem, (G', σ') admits a nowhere-zero 10-flow. Note that (G, σ) can be obtained from (G', σ') by contracting an all-positive subgraph of (G', σ'). Thus (G, σ) admits a nowhere-zero 10-flow. □

2.4 Proof of Theorem 1.5.6

Let's first recall Theorem 1.5.6.

Theorem 1.5.6 *If (G, σ) is a flow-admissible hamiltonian signed graph, then (G, σ) admits a nowhere-zero 8-flow.*

Proof. Let τ be an orientation of (G, σ). In the following we always assume the flows are under the orientation τ or its restriction on according subgraphs. Let C_0 be a hamiltonian circuit of G. We consider two cases according to whether C_0 is balanced or unbalanced.

Case 1. C_0 is balanced.

We may assume that C_0 is all-positive with some switching operations. It is known that every ordinary graph with a hamiltonian circuit admits a 4-NZF (See Corollary 3.3.7 [36]). Thus we may further assume that (G, σ) is unbalanced. Hence, by Lemma 1.4.1, G contains at least two negative edges. Clearly, $\langle C_0 \rangle_2 = (G, \sigma)$. By Lemma 1.4.2, (G, σ) admits a $\mathbb{Z}_3$-flow

ϕ such that $E(G) - E(C_0) \subseteq \text{supp}(\phi)$. By Lemma 1.4.9, (G, σ) admits a 4-flow f_1 such that $E(G) - E(C_0) \subseteq \text{supp}(\phi) \subseteq \text{supp}(f_1)$.

Since C_0 is balanced, (G, σ) has a 2-flow f_2 such that $E(C_0) = \text{supp}(f_2)$. Note that $\text{supp}(f_2) \cup \text{supp}(f_1) = E(G)$. Therefore $f = 2f_1 + f_2$ is an 8-NZF of (G, σ).

Case 2. C_0 is unbalanced.

Since C_0 is unbalanced, for each edge $e \notin E(C_0)$, there is a balanced circuit in $C_0 + e$ containing e, denoted by C_e. Let $H = \triangle_{e \notin E(C_0)} C_e$. Then H admits a $\mathbb{Z}_2$-NZF and has an even number of negative edges.

If H doesn't contain an unbalanced circuit, then we may assume that $E(H)$ are all positive with some switching operations. Thus $E_N(G) \subseteq E(C_0)$ and (G, σ) has a 2-flow f_3 such that $\text{supp}(f_3) = E(H)$. By Lemma 2.2.3, there exists a 4-flow f_4 such that $E(C_0) \subseteq \text{supp}(f_4)$. Since $E(C_0) \cup E(H) = E(G)$, $f_3 + 2f_4$ is an 8-NZF of (G, σ).

Now assume that H contains an unbalanced circuit, say C_0'. Since H admits a $\mathbb{Z}_2$-NZF and has an even number of negative edges, by Lemma 1.4.8, (G, σ) has a 3-flow f_5 such that $E_{f_5 = \pm 1} = E(H)$ and $E_{f_5 = \pm 2} \subseteq E(C_0) - E(H)$.

Let $H' = C_0 \triangle C_0'$. Then H' admits a $\mathbb{Z}_2$-NZF and has an even number of negative edges. Since $C_0 \cup C_0'$ is connected, by Lemma 1.4.8 again, (G, σ) has a 3-flow f_6 such that $\text{supp}(f_6) \subseteq E(C_0) \cup E(C_0')$, $E_{f_6 = \pm 1} = E(H')$, and $E_{f_6 = \pm 2} \subseteq E(C_0) \cap E(C_0')$.

Therefore, $3f_5 + f_6$ is a 9-NZF of (G, σ). Since $E_{f_5 = \pm 2} \cap E_{f_6 = \pm 2} = \emptyset$, $|(3f_5 + f_6)(e)| \neq 8$ for each edge $e \in E(G)$. Thus, $3f_5 + f_6$ is indeed an 8-NZF of (G, σ). $\square$

Chapter 3

Integer flows on triangularly connected signed graphs

3.1 Notations and Terminology

A *triangle-path of length* m, denoted by $T_1T_2\cdots T_m$ in G is a sequence of distinct triangles $T_1, T_2, \ldots, T_m$ in G such that for any $1 \le i < j \le m$,

$$|E(T_i) \cap E(T_{i+1})| = 1 \text{ and } E(T_i) \cap E(T_j) = \emptyset \text{ if } j > i+1.$$

A connected graph G is *triangularly connected* if for any two nonparallel distinct edges e and e', there is a triangle-path $T_1T_2\cdots T_m$ such that $e \in E(T_1)$ and $e' \in E(T_m)$. Trivially, the graph with a single edge is triangularly connected. Let $H_1, H_2, \ldots, H_t$ be subgraphs of G. Denote by $H_1 \triangle H_2 \triangle \cdots \triangle H_t$ the symmetric difference of those subgraphs.

3.2 Useful Lemmas

In this section, we will present some lemmas that will be used in the proof of our main result.

Lemma 3.2.1. *Let (G,σ) be a triangularly connected signed graph. Let T be an unbalanced triangle if there is one otherwise let T be any balanced triangle. Then $\langle T\rangle_2 = (G,\sigma)$ and (G,σ) has a $\mathbb{Z}_3$-flow ϕ such that $E_{\phi=0} \subseteq E(T)$ and for any triangle T', if there are two edges $e_1, e_2 \in E(T')$ such that T' is the only triangle containing them, then $\phi(e_1) = \phi(e_2)$.*

Proof. If there is a triangle T' containing two edges uv, uw such that each is contained in exactly one triangle which is T', then split u into two vertices u_1 and u_2 such that u_1 is adjacent to v and w, and u_2 is adjacent to each vertex in $N_G(u) - \{v, w\}$. Then the degree of u_1 is 2. Repeating this operation until every pair of such edges share a degree 2-vertex. Denote the resulting graph by (G', σ).

It is clear that $\langle T \rangle_2 = (G', \sigma)$. Thus by Lemma 1.4.2, (G', σ) has a $\mathbb{Z}_3$-flow ϕ such that $E_{\phi=0} \subseteq E(T)$. Then ϕ is a desired $\mathbb{Z}_3$-flow of (G, σ). □

The next lemma is proved in [18]. For the purpose of self-containment, we include their proof here.

Lemma 3.2.2. *Let (G, σ) be a signed graph with a path containing all the bridges. Then (G, σ) admits a 3-NZF if (G, σ) admits a $\mathbb{Z}_3$-NZF.*

Proof. Let (τ, ϕ) be a nowhere-zero $\mathbb{Z}_3$-flow of (G, σ). We may assume $\phi(e) = 1$ for each edge e. By Lemma 1.4.3, we may further assume that G has bridges. Since there is a path containing all the bridges, G has exactly two leaf blocks, say G_1 and G_2. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be the two bridges such that $u_i \in V(G_i)$ for each $i = 1, 2$.

For each $i = 1, 2$, denote by G'_i the signed graph obtained from G_i by adding a negative loop e'_i at u_i such that the two half edges of e'_i are oriented the same as the half edge of e_i incident with u_i. Then both G_1 and G_2 are bridgeless and each admits a $\mathbb{Z}_3$-NZF. By Lemma 1.4.7, G'_i admits a nowhere-zero 3-flow g_i such that $g(e_i) = 1$ for each $i = 1, 2$

If e_1 and e_2 are distinct, denote by G'_3 the signed graph obtained by deleting G_1, G_2, e_1 and e_2, and then adding a new edge $e_3 = v_1v_2$ where v_1v_2 consists of the half-edge of e_1 incident with v_1 with the same orientation and the half-edge of e_2 incident with v_2 with the same orientation. Then G_3 is bridgeless and admits a $\mathbb{Z}_3$-NZF. By Lemma 1.4.7, G_3 admits a nowhere-zero 3-flow g_3 such that $g(e_3) = 2$.

If $e_1 = e_2$, then e_1 is the only bridge of G and thus $G = G_1 \cup G_2 \cup \{e_1\}$. It is easy to see that one can obtain a 3-NZF of (G, σ) from g_1 and g_2 by deleting the negative loops e'_1, e'_2 and assigning e_1 with the flow value 2, a contradiction

If $e_1 \neq e_2$, one can merge g_1, g_2 and g_3 to obtain a nowhere-zero 3-flow of (G, σ), a contradiction. This completes the proof of the lemma. □

The following lemma directly follows from the definition of triangularly connected graphs.

Lemma 3.2.3. *Let G be a triangularly connected graph and U, W be two disjoint vertex set with $|\delta_G(U, W)| = 3$. Then either the three edges in $\delta_G(U, W)$ share a common end vertex or the three edges induce a path on four vertices. Moreover in the latter case, the four vertices of the path induce a K_4 minus one edge.*

Lemma 3.2.4. *Let G be a triangularly connected graph with $\delta(G) \geq 3$ and E_0 be a set of edges of G. If $|E_0| \leq 4$ and each component of $G - E_0$ is either an isolated vertex or has minimum degree at least 2, then in each nontrivial component, there is a path containing all the bridges of the component.*

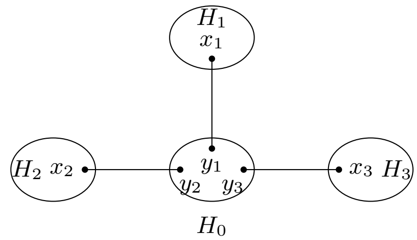


Figure 3.1: The structure of a graph with three bridges not contained in a path

Proof. Suppose to the contrary that $G' = G - E_0$ has a component say H that contains three bridges, say x_1y_1, x_2y_2, x_3y_3, which don't belong to a path (see Figure 3.1). Deleting these three edges, we will get four components and denote the component containing x_i by H_i for $i = 1, 2, 3$ and denote the component containing y_1, y_2, y_3 by H_0.

Since G is triangularly connected and $\delta(G) \geq 3$, G has no cut-vertex and has no 2-edge-cut. Thus G is 3-edge connected. Since the minimum degree of each nontrivial component of $G - E_0$ is at least 2, $|V(H_i)| \geq 2$ for each $i = 1, 2, 3$.

Claim 3.2.4.1. *G' is connected.*

Proof. Suppose to the contrary that G_1 and G_2 are two components of G', where $H_i \subseteq G_1$ for each $i = 1, 2, 3$. This implies that $2|E_0| = |\delta_G(G_2)| + \sum_{i=0}^{3} |\delta_G(V(H_i))| - 3 \times 2 \geq 9$. It contradicts the hypothesis $|E_0| \leq 4$. □

Claim 3.2.4.2. *There exists an integer $i \in \{1, 2, 3\}$ such that $\delta_G(H_i) = 3$ and for any H_j with $|\delta_G(H_j)| = 3$, $\delta_G(H_j) \neq \delta_G(H_j, H_0)$.*

Proof. We first prove that there exists an $i \in \{1, 2, 3\}$ such that $\delta_G(H_i) = 3$. Suppose to the contrary that $\delta(H_j) \geq 4$ for each $j \in \{1, 2, 3\}$. It follows that $2|E_0| = \sum_{j=1}^{3} |\delta_G(H_j)| + |\delta_G(H_0)| - 3 \times 2 \geq 3 \times 4 + 3 - 6 = 9$, a contradiction.

Without loss of generality, assume that $|\delta_G(H_1)| = 3$. Suppose to the contrary that $\delta_G(H_1) = \delta_G(H_1, H_0)$. It follows that $|\delta_G(H_2)| = |\delta_G(H_3)| = 3$, otherwise $2|E_0| = \sum_{j=1}^{3} |\delta_G(H_j)| + |\delta_G(H_0)| - 3 \times 2 \geq 9$, a contradiction. If $|\delta_G(H_2) \cap \delta_G(H_3)| \leq 1$, then $|E_0| \geq \sum_{j=1}^{3}(|\delta_G(H_j)| - 1) - 1 \geq 5$, a contradiction. Thus $|\delta_G(H_2) \cap \delta_G(H_3)| = 2$. This implies that $\{x_2y_2, x_3y_3\}$ is a 2-edge-cut of G. It contradicts that G is 3-edge-connected. Therefore $\delta_G(H_1) \neq \delta_G(H_1, H_0)$. This completes the proof of this claim. □

By Claim 3.2.4.2, in the following without loss of generality we assume that $|\delta_G(H_1)| = 3$ and $\delta_G(H_1, H_2) \neq \emptyset$.

Claim 3.2.4.3. $\delta_G(H_1, H_3) = \emptyset$, $|\delta_G(H_3)| = 3$, *and* $\delta_G(H_2, H_0) = \{x_2y_2\}$.

Proof. Suppose to the contrary that $\delta_G(H_1, H_3) \neq \emptyset$. Since $\delta_G(H_1, H_2) \neq \emptyset$, by Claim 3.2.4.2, we have $|\delta_G(H_1, H_i)| = 1$ for each $i = 0, 2, 3$. Since $\delta_G(H_1)$ is an edge cut with $|\delta_G(H_1)| = 3$ and clearly the three edges in $\delta_G(H_1)$ don't induce a path, by Lemma 3.2.3, the three edges share a common end vertex which is x_1. Since $|V(H_1)| \geq 2$, we have that x_1 is a cut-vertex, a contradiction. This proves $\delta_G(H_1, H_3) = \emptyset$.

Since $\delta_G(H_1, H_3) = \emptyset$ and $|E_0| \leq 4$, we have $3 \leq |\delta_G(H_3)| \leq 4-2+1 = 3$. Thus $|\delta_G(H_3)| = 3$.

Since $(\delta_G(H_1) \cup \delta_G(H_3)) \setminus \{x_1y_1, x_3y_3\} \subseteq E_0$ and $|(\delta_G(H_1) \cup \delta_G(H_3)) \setminus \{x_1y_1, x_3y_3\}| = 4$, we have $(\delta_G(H_1) \cup \delta_G(H_3)) \setminus \{x_1y_1, x_3y_3\} = E_0$. Therefore $\delta_G(H_2, H_0) = \{x_2y_2\}$. □

The final step. By Claims 3.2.4.2 and 3.2.4.3, there is an edge $u_1u_2 \in \delta_G(H_1, H_2)$ where $u_1 \in V(H_1)$ and $u_1 \neq x_1$. By Lemma 3.2.3, u_2 and y_1 are adjacent. Since $\delta_G(H_2, H_0) = \{x_2y_2\}$ by Claim 3.2.4.3, we have $u_2 = x_2$ and $y_1 = y_2$. Similarly there is an edge $v_3v_2 \in \delta_G(H_3, H_2)$ where $v_3 \in V(H_3)$ and $v_3 \neq x_3$ and $v_2 = x_2$. By Lemma 3.2.3, all the edges in $\delta_G(H_2)$ share a common end vertex x_2. Since $|V(H_2)| \geq 2$, x_2 is a cut-vertex, a contradiction to the fact that G has no cut-vertex. This contradiction completes the proof of the lemma. □

The following is a corollary of Lemmas 3.2.2 and 3.2.4.

Lemma 3.2.5. *Let (G, σ) be a triangularly connected signed graph and ϕ be a $\mathbb{Z}_3$-flow of (G, σ) with $|E_{\phi=0}| \leq 4$, then (G, σ) admits a 3-flow f with* $\text{supp}(f) = \text{supp}(\phi)$.

Lemma 3.2.6. *Let $k \geq 3$ be an integer and C be a balanced circuit of (G, σ). Let g be a 2-flow of (G, σ) with* $\text{supp}(g) = E(C)$ *and f_1 be an integer k-flow of (G, σ) such that* $|\text{supp}(f_1) \cap E(C)| \leq k-2$ *and $|f_1(e)| \leq \frac{k}{2}$ for each $e \in E(C)$. Then there is an $\alpha \in \{\pm 1, \pm 2, \cdots, \pm\lfloor \frac{k}{2} \rfloor\}$ such that $f_2 = f_1 - \alpha g$ is an integer k-flow with* $\text{supp}(f_2) = \text{supp}(f_1) \cup E(C)$.

Proof. Since $|\text{supp}(f_1) \cap E(C)| \leq k-2$, we have $|f_1(C)| \leq k-1$.

If k is odd, then there exists an integer $\alpha \in \{\pm 1, \ldots, \pm\lfloor \frac{k}{2} \rfloor\} \setminus f_1(C)$.

If k is even, then there exists at least two integers in $\{\pm 1, \ldots, \pm\frac{k}{2}\} \setminus f_1(C)$. If $\{\pm\frac{k}{2}\} \cap f_1(C) = \emptyset$, let $\alpha = \frac{k}{2}$; otherwise pick one $\alpha \in \{\pm 1, \cdots, \pm(\frac{k}{2} - 1)\} \setminus f_1(C)$. Let $f_2 = f_1 - \alpha g$.

Clearly, when $|\alpha| < \frac{k}{2}$, f_2 is an integer k-flow with $\text{supp}(f_2) = \text{supp}(f_1) \cup E(C)$.

If $\alpha = \frac{k}{2}$, then $\{\pm\frac{k}{2}\} \cap f_1(C) = \emptyset$. Thus for each $e \in E(C)$, $|f_1(e)| \leq \frac{k}{2} - 1$, so $-(k-1) \leq f_2(e) = f_1(e) - \alpha g(e) \leq k-1$ and $f_2(e) \neq 0$. Therefore, f_2 is an integer k-flow with $\text{supp}(f_2) = \text{supp}(f_1) \cup E(C)$. This completes the proof of the lemma. □

Lemma 3.2.7. *Let C be a balanced circuit of (G, σ) with length at most 4 and g be a 2-flow of (G, σ) with* $\text{supp}(g_1) = E(C)$. *Then for any $\mathbb{Z}_3$-flow ϕ of (G, σ), there is an $\alpha \in \mathbb{Z}_3$ such that $\phi_1 = \phi - \alpha g$ is a $\mathbb{Z}_3$-flow satisfying $|E_{\phi_1=0} \cap E(C)| \in \{0, |E(C)| - 2\}$.*

Proof. Let ϕ be a $\mathbb{Z}_3$-flow of (G,σ). If $|E_{\phi=0}\cap E(C)| \in \{0, |E(C)|-2\}$, take $\alpha = 0$.

If $|E_{\phi=0}\cap E(C)| \geq |E(C)|-1$, we can easily find some $\alpha \in \mathbb{Z}_3$ such that $\phi_1 = \phi - \alpha g_1$ is a $\mathbb{Z}_3$-flow satisfying $|E_{\phi_1=0}\cap E(C)| = 0$.

Now we assume $|E_{\phi=0}\cap E(C)| \leq |E(C)|-3$ and $|E_{\phi=0}\cap E(C)| \notin \{0, |E(C)|-2\}$. Then $|E(C)| = 4$ and $|E_{\phi=0}\cap E(C)| = |E(C)|-3 = 1$. Thus $|\phi(C)| \in \{2,3\}$. If $|\phi(C)| = 2$, then choose an α in $\mathbb{Z}_3 \setminus \phi(C)$. If $|\phi(C)| = 3$, then there is an $\alpha \in \phi(C)\setminus\{0\}$ such that there are exactly two edges e in $E(C)$ with $\phi(e) = \alpha$. Then $\phi_1 = \phi - \alpha g$ is a $\mathbb{Z}_3$-flow satisfying $\phi(e) = \phi_1(e)$ for each $e \in E(G) - E(C)$ and $|E_{\phi_1=0}\cap E(C)| \in \{0, |E(C)|-2\}$. □

Lemma 3.2.8. *Let (G,σ) be a triangularly connected signed graph and $C_1,\ldots,C_t$ $(1 \leq t \leq 2)$ be pairwise edge-disjoint balanced circuits of length at most 4. If ϕ is a $\mathbb{Z}_3$-flow of (G,σ) such that $E_{\phi=0} \subseteq \cup_{i=1}^{t} E(C_i)$ and $|E_{\phi=0}| \leq 4$, then (G,σ) admits a 4-NZF.*

Proof. By Lemma 3.2.7, we may assume that $|E_{\phi=0}\cap E(C_i)| \in \{0, |E(C_i)|-2\}$ for each $i = 1,\ldots,t$. Then $|E_{\phi=0}| \leq 4$. By Lemma 3.2.5, there is a 3-flow f such that $\mathrm{supp}(f) = \mathrm{supp}(\phi)$ and of course f is a 4-flow. Taking $k = 4$, we have $|f(e)| \leq \frac{k}{2}$ and $|E_{f\neq 0}\cap E(C_i)| = 2 = k-2$ for each C_i with $E_{f=0}\cap E(C_i) \neq \emptyset$. Applying Lemma 3.2.6 on every C_i with $E_{f=0}\cap E(C_i) \neq \emptyset$, one can obtain a desired 4-NZF. □

By Lemma 2.2 of [7], the proof of the following lemma is straightforward.

Lemma 3.2.9. *Let f be a $\mathbb{Z}_3$-flow of (G,σ) and $H = T_1T_2\cdots T_m$ be a triangle-path in G such that each T_i is balanced for $1 \leq i \leq m$. Given an edge $e_0 \in E(H)$, then there is another $\mathbb{Z}_3$-flow g of (G,σ) satisfying:*

(1) $f(e) = g(e)$ for each $e \notin E(H)$;

(2) $g(e) \neq 0$ for each edge $e \in E(H) - \{e_0\}$.

Lemma 3.2.10. *Let (G,σ) be a triangularly connected signed graph, C_1 be a balanced triangle and C_2 be a balanced circuit of length at most 4 such that $|E(C_1)\cap E(C_2)| \leq 1$. If ϕ is a $\mathbb{Z}_3$-flow of G such that $E_{\phi=0} \subseteq E(C_1)\cup E(C_2)$, then (G,σ) admits a 4-NZF.*

Proof. If C_1 and C_2 are edge-disjoint, then by Lemma 3.2.8, (G,σ) admits a 4-NZF.

If C_1 and C_2 are not edge-disjoint, then $|E(C_1)\cap E(C_2)| = 1$. Let e_0 be the common edge of C_1 and C_2. Applying Lemma 3.2.9 on $H = C_1$ and e_0, we may assume $E_{\phi=0} \subseteq E(C_2)$. By Lemma 3.2.8, (G,σ) admits a 4-NZF. □

3.3 Sharpness of Theorem 1.5.7

Fan et al. [7] give a complete characterization of triangularly connected ordinary graphs that admit a 4-NZF but no 3-NZF. In this subsection we present a family of unbalanced triangularly

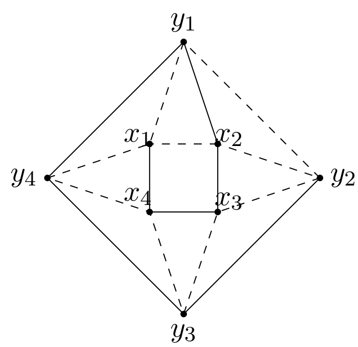


Figure 3.2: an unbalanced signed graph (G_8, σ)

connected signed graphs that admit a 4-NZF but no 3-NZF. Interestingly all those graphs do not contain an unbalanced triangle. This indicates that there are unbalanced triangularly connected signed graphs without unbalanced triangles.

For each integer $t \geq 4$, construct the signed graph (G_{2t}, σ) as follows (see Figure 3.2 for an illustration with $t = 4$):

(1) The graph G_{2t} is constructed from the two circuits $C_1 = x_1x_2 \cdots x_tx_1$ and $C_2 = y_1y_2 \cdots y_ty_1$ by adding the edges y_ix_i and y_ix_{i+1} for each $i \in Z_t$;

(2) $E_N(G_{2t}, \sigma)$ consists of the edges x_1x_2, y_1y_2 and all edges y_ix_i, y_ix_{i+1} except y_1x_2.

Theorem 3.3.1. *For each $t \geq 4$, (G_{2t}, σ) is flow-admissible and admits a 4-NZF but no 3-NZF.*

Since (G_{2t}, σ) is bridgeless and every edge is contained in a balanced triangle, by Proposition 1.4.1, it is flow-admissible. Since G_{2t} is Eulerian, the second part of Theorem 3.3.1 follows from the following result due to Mačajova and Škoviera.

Theorem 3.3.2. *(Mačajova and Škoviera[21]) Let (G, σ) be an Eulerian signed graph with an odd number of negative edges. Then (G, σ) admits a 4-NZF if it it flow-admissible. Moreover (G, σ) admits a 3-NZF if and only if (G, σ) can be decomposed into three signed Eulerian subgraphs that have a vertex in common and that each has an odd number of negative edges.*

3.4 Proof of Theorem 1.5.7

Theorem 1.5.7. If (G, σ) is a flow-admissible triangularly connected signed graph, then (G, σ) admits a nowhere-zero 4-flow if and only if $(G, \sigma) \neq (W_5, \sigma^*)$ where (W_5, σ^*) is the signed graph in Figure 1.4. Moreover there are infinitely many triangularly connected unbalanced signed graphs that admit a nowhere-zero 4-flow but no 3-flow.

Proof. We prove Theorem 1.5.7 by contradiction. Let (G, σ) be a counterexample such that $\beta(G) = \sum_{v \in V(G)}(d(v) - 2)$ is as small as possible. Let τ be a fixed orientation of (G, σ) in the proof.

Hu and Li [10] show that (W_5, σ^*) in Figure 1.4 admits a 5-NZF but no 4-NZF. Then (G, σ) does not admit a 4-NZF. By Lemma 3.2.8 we have the following fact which will be applied frequently in the proof.

Fact A (G, σ) does not admit a $\mathbb{Z}_3$-flow ϕ such that $E_{\phi=0} \subseteq E(C_1) \cup \cdots \cup E(C_t)$ where $1 \leq t \leq 2$ and $C_1, \ldots, C_t$ are edge-disjoint balanced circuits of length at most four.

If G contains two parallel edges e_1 and e_2, then after inserting a degree 2-vertex into e_1, the resulting graph G' remains triangularly connected, flow-admissible, and $\beta(G') = \beta(G)$. Thus in the following proof, we assume that G is simple.

If G contains no unbalanced triangle, let T be a triangle. By Lemma 3.2.1, let ϕ be a $\mathbb{Z}_3$-flow ϕ with $E_{\phi=0} \subseteq E(T)$, a contradiction to Fact A. Thus G contains an unbalanced triangle.

(I) (G, σ) *contains two edge-disjoint unbalanced triangles.*

Proof of (I). Suppose to the contrary that (G, σ) contains no edge-disjoint unbalanced triangles. Let T be an unbalanced triangle and ϕ be a $\mathbb{Z}_3$-flow ϕ with $E_{\phi=0} \subseteq E(T)$.

We consider two cases in the following.

Case I.1. (G, σ) contains at least two unbalanced triangles.

Let $T_1, T_2, \ldots, T_t$ be all the unbalanced triangles where $T = T_1$. Then $t \geq 2$. Since (G, σ) contains no edge-disjoint unbalanced triangles, all unbalanced triangles share a common edge, denoted by uv. For each i denote by w_i the third vertex of T_i. Then for any $1 \leq i < j \leq t$, $T_i \triangle T_j$ is a balanced circuit of length 4.

Since $T_1 \triangle T_2$ is a balanced 4-circuit, by Fact A, $\phi(uv) = 0$ and uv is not contained in a balanced triangle. This implies that no other triangle than $T_1, T_2, \ldots, T_t$ contains uv.

Since (G, σ) is flow-admissible, there is a signed circuit C containing uv. By Proposition 1.4.4, let f be a 2-flow (if C is a balanced circuit or a short barbell) or a 3-flow (if C is a long barbell) such that $\text{supp}(f) = E(C)$. Let $\phi_1 = \phi + f$ be the $\mathbb{Z}_3$-flow. Then $\phi_1(uv) \neq 0$.

Let $e \in E_{\phi_1=0} - \bigcup_{i=1}^{t} E(T_i)$. Then there is a triangle-path $S_1 S_2 \cdots S_k$ where $e \in S_k$, $uv \in S_1 \in \{T_1, T_2, \ldots, T_t\}$, and $S_2, S_3, \ldots, S_k$ are balanced. Let $H = S_2 S_3 \cdots S_k$ and $e' = E(S_1) \cap E(S_2)$. By Lemma 3.2.9, there is a $\mathbb{Z}_3$-flow g of (G, σ) satisfying:

(1) $\phi_1(e) = g(e)$ for each $e \notin E(H)$;

(2) $g(e) \neq 0$ for each edge $e \in E(H) - \{e'\}$.

By applying the above operation on each edge in $E_{\phi_1=0} - \bigcup_{i=1}^{t} E(T_i)$, one can obtain a $\mathbb{Z}_3$-flow ϕ_2 such that $E_{\phi_2=0} \subseteq \bigcup_{i=1}^{t} E(T_i) - \{uv\}$.

Denote $C_i = T_1 \triangle T_i$ for each $i = 2, \ldots, t$. Then each C_i is a balanced 4-circuit. For each $i = 2, \ldots, t$, let f_i be a 2-flow of (G, σ) with $\text{supp}(f_i) = E(C_i)$ and let $\alpha_i \in \mathbb{Z}_3 - \{\phi_2(uw_i)f_i(uw_i), \phi_2(vw_i)f_i(vw_i)\}$. Let $\phi_3 = \phi_2 - \sum_{i=2}^{t} \alpha_i f_i$. Then ϕ_3 is a $\mathbb{Z}_3$-flow such that $E_{\phi_3=0} \subseteq \{uw_1, vw_1\} \subseteq E(C_2)$, a contradiction to Fact A.

Case I.2. (G, σ) contains only one unbalanced triangle.

Denote $E(T) = \{e_1, e_2, e_3\}$. If every edge in $E_{\phi=0}$ is contained in a triangle other than T, then every edge in $E_{\phi=0}$ is contained in a balanced triangle since T is the only unbalanced triangle in (G, σ). By Lemma 3.2.8, $|E_{\phi=0}| \geq 2$ and those balanced triangles are not edge-disjoint. This implies that there is a K_4 containing T where T is the only unbalanced triangle in the K_4. However, T is the symmetric difference of the other three balanced triangles in the K_4. Thus T is balanced, a contradiction. Therefore there is one edge in $E_{\phi=0}$ that is contained in only one triangle which is T.

Since (G, σ) is flow-admissible, there is another edge in $E(T)$ which is contained in a balanced triangle. Without loss generality, assume that e_1 is contained in only one triangle, $\phi(e_1) = 0$ and e_3 is contained a balanced triangle. Note that by Lemma 3.2.1, if e_2 is not contained in a balanced triangle, then $\phi(e_1) = \phi(e_2) = 0$.

Since (G, σ) is flow-admissible, by Proposition 1.4.1, there is a signed circuit C_1 containing e_1 and there is a signed circuit C_2 containing e_2. We choose $C_2 = C_1$ if there is a signed circuit containing both e_1 and e_2; otherwise choose any signed circuit C_2 containing e_2.

By Lemma 1.4.4, let f_i be a 2-flow or 3-flow of (G, σ) with $\text{supp}(f_i) = E(C_i)$ for each $i = 1, 2$.

We construct another $\mathbb{Z}_3$-flow ϕ_1 of (G, σ) as follows:

Let $\alpha \in \mathbb{Z}_3 - \{0, \phi(e_2)f_2(e_2)\}$. If $C_1 = C_2$, then $f_1 = f_2$ and let $\phi_1 = \phi - \alpha f_1$; if $C_1 \neq C_2$, then $f_1(e_2) = f_2(e_1) = 0$ and let $\phi_1 = \phi - \alpha(f_1 + f_2)$.

Then $E_{\phi_1=0} \cap \{e_1, e_2\} = \emptyset$ and every edge in $E_{\phi_1=0}$ is contained in a balanced triangle. Similar to the argument in Case I.1, there is a $\mathbb{Z}_3$-flow ϕ_2 such that $E_{\phi_2=0} \subseteq \{e_3\}$ if e_2 is not contained in a balanced triangle or $E_{\phi_2=0} \subseteq \{e_2, e_3\}$ otherwise, a contradiction to Fact A.

We obtain a contradiction in either case and thus completes the proof of (I). □

(II) *G is locally connected.*

Proof of (II). Suppose to the contrary that G is not locally connected. Then there is a vertex $v \in V(G)$ such that $G[N_G(v)]$ is not connected. Since G is triangularly connected, each component of $G[N_G(v)]$ is nontrivial. Let H be a component of $G[N_G(v)]$. Split v into two nonadjacent vertices v' and v'' where v' is adjacent to all vertices in H and v'' is adjacent to all vertices in $N_G(v) - V(H)$. The signs of all edges remain the same. Denote the resulting signed graph by (G', σ). By (I), (G', σ) contains two edge-disjoint unbalanced triangles. Since G' is connected and bridgeless, by Proposition 1.4.1, (G', σ) is flow-admissible. Obviously $\beta(G') < \beta(G)$ and G' remains triangularly connected. By the minimality of $\beta(G)$, (G', σ) admits a 4-NZF

f. Identifying v' and v'', one can easily obtain a 4-NZF of (G, σ), a contradiction. Therefore G is locally connected. □

(III) (G, σ) *does not contain any of the* 11 *configurations in Figure 3.3.*

Proof of (III). For a balanced circuit or a short barbell C, denote by $\chi(C)$ a 2-flow of (G, σ) with $\mathrm{supp}(\chi(C)) = E(C)$ guaranteed by Lemma 1.4.4. In the following argument, all cases only involve one $\chi(C)$ except one which involves three balanced circuits with one common edge. Thus without loss of generality, we assume that $\chi(C)$ is a nonnegative 2-flow.

Take $T = T_1$ if (G, σ) contains FC_i if $i \in \{1, 2, 3, 9, 10\}$, $T = T_2$ if (G, σ) contains FC_4 or FC_{11}, and $T = T_3$ if (G, σ) contains FC_i if $i \in \{5, 6, 7, 8\}$.

Since in FC_1 or FC_2, $E(T_1)$ is contained in two edge-disjoint balanced circuits of length at most 4, a contradiction to Fact A. This proves that (G, σ) does not contain FC_1 or FC_2.

In FC_3, any two edges in T_1 are contained in a balanced 4-circuit, thus by Fact A, $E_{\phi=0} = E(T_1)$. Let $C = T_2 \triangle T_3$. Then C is a balanced 4-circuit and contains the two edges uv_1 and uv_2. Let $\phi_1 = \phi + \phi(v_2v_3)\chi(C)$. Then ϕ_1 is a $\mathbb{Z}_3$-flow such that $E_{\phi_1=0} \subseteq E(T_1 \triangle T_3)$. This contradicts Fact A since $T_1 \triangle T_3$ is a balanced 4-circuit. This proves that (G, σ) does not contain FC_3.

Similarly, in FC_4, by Fact A, $E_{\phi=0} = E(T_2)$. Let $C = T_2 \triangle T_3$ which is a balanced 4-circuit and let $\phi_1 = \phi + \phi(v_4v_5)\chi(C)$. Then ϕ_1 is a $\mathbb{Z}_3$-flow such that $E_{\phi_1=0} \subseteq \{v_3v_4, v_3v_5\} \subseteq E(T_3 \triangle T_4)$. This contradicts Fact A since $T_3 \triangle T_4$ is a balanced 4-circuit. This proves that (G, σ) does not contain FC_4.

Suppose that G contains FC_i for some $i = 5, 6, 7, 8$. By Fact A, $\phi(v_4v_5) = 0$ in FC_5 and in FC_i where $i = 6, 7, 8$, $\phi(v_3v_5) = 0$. Let $C = T_2 \triangle T_3$, which is a balanced 4-circuit. Let $\phi_1 = \phi + \phi(v_2v_3)\chi(C)$. Then ϕ_1 is a $\mathbb{Z}_3$-flow such that $E_{\phi_1=0} \subseteq E(T_1 \triangle T_2) \cup E(T_4)$ when $i = 5, 6$ and $E_{\phi_1=0} \subseteq E(T_1 \triangle T_2) \cup E(T_4 \triangle T_5)$ when $i = 7, 8$. In the former case, $T_1 \triangle T_2$ is a balanced 4-circuit and T_4 is a balanced 3-circuit and they are edg-disjoint. In the latter case, $T_1 \triangle T_2$ and $T_4 \triangle T_5$ are edge-disjoint balanced 4-circuits. This contradicts Fact A and thus proves that (G, σ) does not contain FC_i for each $i = 5, 6, 7, 8$.

Now we consider the case when (G, σ) contains FC_9. Similar to the above argument, we have $\phi(v_1v_3) = \phi(v_2v_3) = 0$.

If $\phi(v_1v_2) = 0$, let $\phi_1 = \phi + \phi(v_3v_5)\chi(C)$ where $C = T_1 \triangle T_4$ is a balanced 4-circuit. Then ϕ_1 is a $\mathbb{Z}_3$-flow such that $E_{\phi_1=0} \subseteq E(T_3 \triangle T_4)$. This contradicts Fact A since $T_3 \triangle T_4$ is a balanced 4-circuit.

Now we further assume $\phi(v_1v_2) = \alpha \neq 0$. Note that $C = T_1 \cup T_3$ is a short barbell. If one of $\phi(v_3v_4)$ and $\phi(v_3v_5)$ is not equal to $-\alpha$, without loss of generality, assume $\phi(v_3v_4) \neq -\alpha$. Let $\phi_1 = \phi + \alpha\chi(C)$. Then ϕ_1 is a $\mathbb{Z}_3$-flow such that $E_{\phi_1=0} \subseteq \{v_3v_5, v_5v_4\} \subseteq E(T_2 \triangle T_3)$. This contradicts Fact A since $T_2 \triangle T_3$ is a balanced 4-circuit. If $\phi(v_3v_4) = \phi(v_3v_5) = -\alpha$, let $\phi_1 = \phi - \alpha\chi(C)$. Then ϕ_1 is a $\mathbb{Z}_3$-flow such that $E_{\phi_1=0} \subseteq \{v_1v_2, v_5v_4\}$, a contradiction to Fact A again since v_1v_2 and v_5v_4 are contained in the balanced 4-circuit $v_1v_2v_4v_5v_1$. This proves that (G, σ) does not contain FC_9.

Suppose that (G, σ) contains FC_{10}. Similarly as before we have that $\phi(v_1v_3) = 0$ and at least one of $\phi(v_2v_1)$ and $\phi(v_2v_3)$ is 0. Let $\phi_1 = \phi + \phi(v_4v_5)\chi(C)$ where $C = T_1 \cup T_3$ is a short barbell. Then ϕ_1 is a $\mathbb{Z}_3$-flow such that $E_{\phi_1=0} \subseteq E(T_1 \triangle T_4) \cup E(T_2)$. Since $T_1 \triangle T_4$ is a balanced 4-circuit, T_2 is a balanced triangle, and they share one common edge, by Lemma 3.2.10, (G, σ) admits a 4-NZF, a contradiction. Thus (G, σ) does not contain FC_{10}.

Finally suppose that (G, σ) contains FC_{11}. Denote $C_1 = T_1 \triangle T_2$, $C_2 = T_2 \triangle T_3$, and $C_3 = T_4$. Note that C_1, C_2, C_3 are all balanced circuits sharing a common edge v_2v_4.

Claim 3.4.0.1. *There is a* 3*-flow* f *such that* $v_2v_4 \in E_{f=0} \subseteq E(C_1) \cup E(C_2)$ *and* $|E_{f=0} \cap E(C_i)| \geq 2$ *for each* $i = 1, 2$.

Proof. With a similar argument as before, we have $\phi(v_2v_3) = \phi(v_3v_4) = 0$. If $\phi(v_2v_4) = 0$, then by Lemma 3.2.5, let f be a 3-flow with $\text{supp}(f) = \text{supp}(\phi)$ which is a desired 3-flow.

Since ϕ is a $\mathbb{Z}_3$-flow, we may assume that $\phi(e) \in \{a, b, 0\}$ for each $e \in E(G)$ with $a + b = 0$. Assume $\phi(v_2v_4) = a$. If $\phi(v_1v_2) = \phi(v_1v_3) = b$, let $\phi_1 = \phi + b\chi(C_1)$, where C_1 corresponds to the orientation of the $\mathbb{Z}_3$-flow ϕ of $E(G)$. Then $E_{\phi_1=0} = \{v_2v_3, v_2v_4\} \subseteq E(C_1)$, a contradiction to Fact A. Thus $\phi(v_1v_2) \neq \phi(v_1v_3)$. Then $a \in \{\phi(v_1v_2), \phi(v_1v_3)\}$. Let $\phi_2 = \phi - a\chi(C_1)$. Then $v_2v_4 \in E_{\phi_2=0}$ and $|E_{\phi_2=0} \cap E(C_i)| = 2$ for each $i = 1, 2$. By Lemma 3.2.5, let g be the corresponding 3-flow of ϕ_2 with $\text{supp}(g) = \text{supp}(\phi_2)$ which is a desired 3-flow. This prove the claim. □

Let f be a 3-flow described in Claim 3.4.0.1. Note $|\{\pm 1, \pm 2\} \setminus f(C_i)| \geq 2$ for each $i = 1, 2$.

If $\{1, -1\} \setminus f(C_i) \neq \emptyset$, take $\alpha_i \in \{1, -1\} \setminus f(C_i)$. Otherwise $f(C_i) = \{0, 1, -1\}$ and take $\alpha_i \in \{2, -2\}$. In the case when both $|\alpha_1| = |\alpha_2| = 2$, we choose $\alpha_1 = 2$ and $\alpha_2 = -2$. Then $g = f + \alpha_1\chi(C_1) + \alpha_2\chi(C_2)$ is a 4-flow such that $E_{g=\pm 3} \subseteq E(C_1) \cup E(C_2)$ and $E_{g=0} \subseteq \{v_2v_4\}$. Since (G, σ) does not admit a 4-NZF, $g(v_2v_4) = 0$. Since T_4 is a balanced triangle and $|g(e)| \leq 2$ for each $e \in E(T_4)$, one can extend g to be a 4-NZF of (G, σ), a contradiction. This proves that (G, σ) does not contain FC_{11} and thus completes the proof of (III). □

(IV) *There is no triangle-path* $T_1T_2 \cdots T_m$ *in* (G, σ) *such that* $m \geq 3$*,* T_1 *and* T_m *are unbalanced, and* T_i *is balanced for each* $i \in \{2, \ldots, m-1\}$.

Proof of (IV). Suppose to the contrary that there is a triangle-path $H = T_1T_2 \cdots T_m$ such that $m \geq 3$, T_1 and T_m are unbalanced and T_i is balanced for each $i = \{2, \ldots, m-1\}$. Denote by $H' = T_2 \cdots T_{m-1}$. Denote $E(T_1) = \{e_1, e_2, e_3\}$ and $E(T_m) = \{e_4, e_5, e_6\}$ where $e_3 \in E(T_1) \cap E(T_2)$ and $e_6 \in E(T_m) \cap E(T_{m-1})$. Let x be the common endvertex of e_1 and e_2 and y be the common endvertex of e_4 and e_5. Let $C = T_1 \triangle T_2 \triangle \cdots \triangle T_m$. Then C is a balanced circuit containing e_i for each $i = 1, 2, 4, 5$.

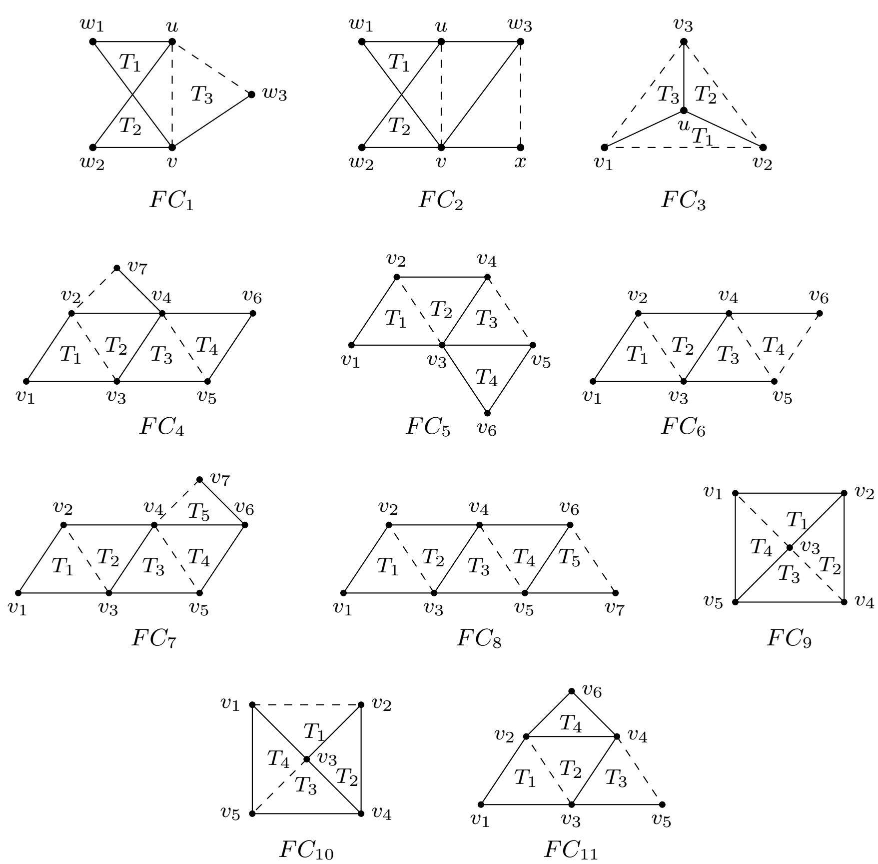


Figure 3.3: Forbidden configurations: the dotted lines are negative edges.

Take $T = T_1$. Then $E_{\phi=0} \subseteq E(T_1)$. Since e_3 belongs to the balanced triangle T_2, by Lemma 3.2.8, either $\phi(e_1) = 0$ or $\phi(e_2) = 0$.

If $d(x) \geq 3$, there is a triangle T_0 such that T_0 and T_1 share exactly one of e_1 and e_2 since by (II), G is locally connected. Let $C_1 = T_0$ if T_0 is balanced otherwise let $C_1 = T_0 \triangle T_1$ which is a balanced 4-circuit. Without loss of generality assume $e_1 \in E(C_1)$.

Similarly if $d(y) \geq 3$, there is a triangle T_{m+1} such that T_{m+1} and T_m share exactly one of e_4 and e_5. Let $C_2 = T_{m+1}$ if T_{m+1} is balanced otherwise let $C_2 = T_{m+1} \triangle T_m$ which is a balanced 4-circuit. Without loss of generality assume $e_4 \in E(C_2)$.

Let $\alpha = \phi(e_5)$ and $\phi_1 = \phi + \alpha\chi(C)$.

We first show $\phi(e_1) \neq \phi(e_2)$. Suppose the contradiction that $\phi(e_1) = \phi(e_2)$. Then $\phi(e_1) = \phi(e_2) = 0$ and thus $E_{\phi_1=0} \subseteq E(H') \cup \{e_4\}$.

If $\phi_1(e_4) \neq 0$, then $E_{\phi_1=0} \subseteq E(H')$. By Lemma 3.2.9, there is a Z_3-flow ϕ_2 such that $E_{\phi_2=0} \subseteq \{e_6\}$, a contradiction to Fact A.

If $\phi_1(e_4) = 0$, then $\phi(e_4) \neq \phi(e_5)$. This implies $d(y) \geq 3$ and thus C_2 exists. If $E(C_2) \cap E(H') \neq \emptyset$, let $e_0 \in E(C_2) \cap E(H')$. Otherwise, let $e_0 = e_6$. By Lemma 3.2.9, there is a Z_3-flow ϕ_3 such that $E_{\phi_3=0} \subseteq \{e_0, e_4\} \subseteq E(C_2)$, a contradiction to Fact A since C_2 is a balanced circuit of length at most 4. This shows that $\phi(e_1) \neq \phi(e_2)$, which implies $d(x) \geq 3$. By symmetry, we also have $d(y) \geq 3$. Therefore both C_1 and C_2 exist.

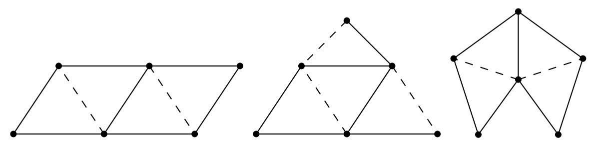
Since $e_1 \in E(C_1)$ and $e_3 \in E(T_2)$, we have $\phi(e_2) = 0$. Then $E_{\phi_1=0} \subseteq E(H') \cup \{e_1, e_4\}$. If $(E(C_1) \cup E(C_2)) \cap E(H') \neq \emptyset$, let e_7 be an edge in $(E(C_1) \cup E(C_2)) \cap E(H')$. Otherwise let $e_7 = e_3$. By Lemma 3.2.9, one can obtained a Z_3-flow ϕ_4 from ϕ_1 such that $E_{\phi_4=0} \subseteq \{e_1, e_4, e_7\}$. Note that if C_i is a circuit of length 4 for some $i = 1, 2$, then $e_7 \in E(C_i) \cap E(H')$.

If C_1 and C_2 are edge-disjoint, then we have either $\{e_1, e_4, e_7\} \subseteq E(C_1) \cup E(C_2)$ or $\{e_1, e_4, e_7\} \subseteq E(C_1) \cup E(C_2) \cup E(T_2)$ where C_1, C_2, T_2 are edge-disjoint balanced triangles. The former case contradicts Fact A. In the latter case, by Lemma 3.2.5, there is an integer 3-flow flow f such that $\text{supp}(f) = \text{supp}(\phi_4)$. By Lemma 3.2.6 (considering f as an integer 4-flow), f can be extended to a 4-NZF of G, a contradiction. Therefore C_1 and C_2 are not edge-disjoint.

If C_1 is a triangle, then by Lemma 3.2.9, one can obtain a Z_3-flow ϕ_5 from ϕ_4 such that $|E_{\phi_5=0}| \leq 4$ and $E_{\phi_5=0} \subseteq E(C_2) \cup \{e_7\}$ since C_1 and C_2 are not edge-disjoint. Since e_7 is contained in a balanced triangle and C_2 is a balanced 4-circuit, by Lemma 3.2.8 or Lemma 3.2.10, (G, σ) has a 4-NZF, a contradiction. Thus C_1 is a 4-circuit. By symmetry, C_2 is also a 4-circuit. This implies $e_3 \in E(C_1)$ and $e_6 \in E(C_2)$ and $\{e_1, e_4\} \subseteq E_{\phi_4=0} \subseteq \{e_1, e_4, e_7\} \subseteq E(C_1) \cup E(C_2)$.

Since C_1 and C_2 are not edge-disjoint, there is a $\beta \in \mathbb{Z}_3$ such that $\phi_6 = \phi_4 + \beta\chi(C_1)$ satisfying $E_{\phi_6=0} \subseteq E(C_2) \cup \{e_3\}$. Since $\{e_3, e_7\} \subseteq E(H')$, by Lemma 3.2.9, one can obtain a Z_3-flow ϕ_7 from ϕ_6 such that $E_{\phi_7=0} \subseteq E(C_2)$, a contradiction to Fact A. This completes the proof of (IV).

(V) For any triangle-path $H = T_1T_2T_3$ with each T_i unbalanced, H is an induced subgraph of



Γ_1 Γ_2 Γ_3

Figure 3.4: Three graphs formed by four unbalanced triangles

(G, σ).

Proof of (V). Suppose to the contrary that H is not induced. Denote $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ where $V(T_i) = \{v_i, v_{i+1}, v_{i+2}\}$ for each $i = 1, 2, 3$.

Since by (III), (G, σ) does not contain FC_9 or FC_{10}, v_1 and v_5 are not adjacent. Then either v_1 and v_4 are adjacent or v_2 and v_5 are adjacent. Without loss of generality, assume v_1 and v_4 are adjacent. Denote $T_4 = v_1v_3v_4$. Since by (III) (G, σ) does not contain FC_3, T_4 is balanced. Then T_2, T_3 and T_4 form a FC_1, a contradiction to (III) again. This completes the proof of (V).

The final step. By (III), (G, σ) does not contain any graph of Figure 3.3 as a subgraph. We can further assume that (G, σ) contains two edge-disjoint unbalanced triangles by (I).

By (IV), let $H = T_1T_2 \ldots T_m$ be a triangle-path such that each triangle T_i is unbalanced and $E(T_1) \cap E(T_m) = \emptyset$. We choose H such that m is as large as possible. Since (G, σ) contains two edge-disjoint unbalanced triangles by (I) and does not contain FC_8 by (III), we have $3 \leq m \leq 4$. One can easily see that H admits a 4-NZF. Since (G, σ) does not admit a 4-NZF, $H \neq G$. Since G is triangularly connected, there must be a triangle $T_5 \neq T_i$ for each $i = 1, 2, 3$ such that $E(T_5) \cap E(H) \neq \emptyset$.

If $m = 4$, then $H = \Gamma_1$ or Γ_3 in Figure 3.4. If $m = 3$, by (V), H is an induced subgraph and hence $|E(T_4) \cap E(H)| = 1$. Since by (III), G does not contain FC_i for each $i = 1, 2, 5, 6, 11$, H must be one of Γ_i in Figure 3.4. It is easy to see that each Γ_i admits a 4-NZF and thus $(G, \sigma) \neq \Gamma_i$ for each i. Since G is triangularly connected, there is a triangle T_6 such that $T_6 \neq T_i$ for each $i = 1, 2, 3, 4$ and $E(T_6) \cap E(H) \neq \emptyset$. By the maximality of m and since (G, σ) does not contain FC_i for each $i = 1, 2, 4, 5, 6, 11$, we have $|E(T_6) \cap E(H)| \geq 2$. By (V), $H = \Gamma_3$ and thus by (IV) $G = (W_5, \sigma^*)$, a contradiction. This completes the proof of the theorem. □

Chapter 4

Flows on K_4-minor free signed graphs

4.1 Notations and Terminology

Let G be a graph. A *block* of a graph is a subgraph which is 2-connected and is maximal with respect to this property. If G is a graph with at least one cut-vertex, then at least two of the blocks of G contains exactly one cut-vertex, each such block is called a *leaf block*. We also call a 2-circuit as a *digon*. If a block H is not a digon, we call it a *nontrivial block*.

A *two-terminal series-parallel signed graph* $(G, \sigma; x, y)$ is defined recursively as follows:

- Let $V(K_2) = \{x, y\}$. For any signature σ, $(K_2, \sigma; x, y)$ is a two-terminal series-parallel signed graph.
- (The parallel construction) Let $(G_1, \sigma_1; x_1, y_1)$ and $(G_2, \sigma_2; x_2, y_2)$ be two disjoint two-terminal series-parallel signed graphs. Define G to be the graph obtained from the union of G_1 and G_2 by identifying x_1 and x_2 into a single vertex x, and identifying y_1 and y_2 into a single vertex y. For any edge $e \in E(G)$, $\sigma(e) = \sigma_1(e)$ if $e \in E(G_1)$, $\sigma(e) = \sigma_2(e)$ if $e \in E(G_2)$. Then $(G, \sigma; x, y)$ is a two-terminal series-parallel signed graph, and is called the *parallel join* of $(G_1, \sigma_1; x_1, y_1)$ and $(G_2, \sigma_2; x_2, y_2)$.
- (The series construction) Let $(G_1, \sigma_1; x_1, y_1)$ and $(G_2, \sigma_2; x_2, y_2)$ be two disjoint two-terminal series-parallel signed graphs. Define G to be the graph obtained from the union of G_1 and G_2 by identifying y_1 and x_2 into a single vertex. For any edge $e \in E(G_1)$, $\sigma(e) = \sigma_1(e)$ if $e \in E(G)$, $\sigma(e) = \sigma_2(e)$ if $e \in E(G_2)$. Then $(G, \sigma; x_1, y_2)$ is a two-terminal series-parallel signed graph, and is called the *series join* of $(G_1, \sigma_1; x_1, y_1)$ and $(G_2, \sigma_2; x_2, y_2)$.
- There are no other two-terminal series-parallel signed graphs.

A *series-parallel graph* is a two-terminal graph obtained by a sequence of series and parallel joins, starting with the copies of K_2 (with some choice of the terminals). Note that the terminals of a series-parallel graph are fixed by the definition.

Proposition 4.1.1. (Dirac [6]) *Let G be a 2-connected graph. Then G is a series-parallel (with fixed terminals) if and only if G is a K_4-minor free graph.*

If $G_1, \cdots, G_n (n \geq 2)$ are series-parallel graphs such that G is either a series join or a parallel join of $G_1, \cdots, G_n$ and n is maximum with this property, then we refer to the G_i as *parts* of G. In the case of a series join, G_1 and G_n are the *endparts* of G.

We introduce the following notation for some specific series-parallel signed graphs: K_2^+ denotes the positive K_2, K_2^- stands for the negative K_2, D is the balanced digon and D_0 is the unbalanced digon.

A *string* is a series join of copies of K_2^+ and D_0 where each nonterminal vertex is contained in a digon. A string is *nontrivial* if it contains more than two vertices. A *necklace* N is a series-parallel signed graph obtained by the parallel join of two strings, at least one of which is nontrivial.

A *tadpole* L_Q consists of two vertices v_0, v_1 and a negative loop L_{v_1} at v_1 and a positive edge v_0v_1. The vertex v_0 is called the *end* of tadpole while L_{v_1} is the *head* of the tadpole L_Q, the edge v_0v_1 is the *tail* of the tadpole L_Q.

A *subdivided digon* is formed by combining two subdivided edges, denoted as e_1^+ and e_2^+, in such a way that every internal vertex, if any exist, in e_1^+ and e_2^+ is incident with either a negative loop or a tadpole. If the product of the signs of all subdivided edges within e_1^+ and e_2^+ equals 1, then we refer to it as a *balanced subdivided digon*, represented as D^+; conversely, if the product is -1, it is termed an *unbalanced subdivided digon*, denoted as D_0^+. Note that if one end of a digon is incident with a negative loop or a tadpole, then it is also considered an internal vertex of this subdivided digon. Additionally, a balanced digon D can be obtained from a subdivided balanced digon D^+ by deleting all incident negative loops and tadpoles and then suppressing any resulting 2-vertices. Similarly, D_0 can be obtained from D_0^+ using the same process.

A *generalized string* is a series join of copies of signed K_2, unbalanced subdivided digons, negative loops and tadpoles. Likewise, a *generalized necklace* N^+ is a parallel join of two generalized strings.

In the following context, we use $\gamma(G)$ to denote the number of subdivided unbalanced digons contained in G, and we use $\gamma'(G)$ to denote the number of subdivided unbalanced digons, negative loops and tadpoles contained in G. For each pair of tadpoles, if their incident vertices are two adjacent internal vertices in G, then we refer to these two tadpoles as *adjacent*.

4.2 Uesful Lemmas

Lemma 4.2.1. (Lu, Luo and Zhang [17]) *Let k be a positive integer, and let G be a graph with an orientation τ and admitting a k-NZF. If a vertex v of G is of degree at most 3 and $g : E_G(v) \mapsto \{\pm 1, \cdots, \pm(k-1)\}$ satisfies $\partial g(v) = 0$, then there exists a k-NZF (τ, f) on G such that $f|_{E_G(v)} = g$.*

By Lemma 4.2.1, we conclude the following lemma.

Lemma 4.2.2. *Let (G, σ) be a bridgeless, K_4-minor free signed graphs with $E_N(G) = \{e_1, e_2\}$. If e_2 is a negative loop, then (G, σ) admits a 4-NZF f such that $f(e_1) = f(e_2) = 1$. Moreover, for each $a \in \{1, 2\}$, (G, σ) admits a 5-NZF f such that $f(e_1) = f(e_2) = a$.*

Proof. Let the negative loop e_2 be incident to the vertex x. We construct an ordinary graph G_1 as follows:

If e_1 is not a negative loop, let $e_1 = uv$ and G_1 be the graph obtained from G by deleting e_2 and replacing the edge e_1 with a path uwv where both uw and wv are positive edges.

If e_1 is a negative loop, let G_1 be the graph obtained from G by deleting e_1 and e_2. For convenience we denote the vertex incident with e_1 by w.

Since G_1 is a bridgeless K_4-minor free ordinary graph, G_1 admits a 3-NZF. Let G_2 be the ordinary graph obtained from G_1 by adding a new edge xw. Then G_2 admits a nowhere-zero 4-flow (see Exercise 2.5 in [37]). Let D_2 be an orientation of G_2 such a way that at the vertex w, xw is oriented away from w and the edges wu and wv are oriented into w. Since the degree of w in G_2 is 3, by Lemma 4.2.1, G_2 admits a 4-NZF (D_2, f_2) such that $f_2(wx) = 2$ and $f_2(wv) = f_2(wu) = 1$.

Let τ be the orientation of (G, σ) obtained from D_2 by orienting the two half edges of the negative loop e_2 into x and the two half edges of the negative edge uv out u and v respectively, and keeping the orientation of the other edges. Then one can obtain a desired 4-NZF (τ, f) of (G, σ) where $f(e_1) = f(e_2) = 1$ and $f(e) = f_2(e)$ for each edge $e \in E(G) \setminus \{e_1, e_2\}$.

The argument for the moreover part is similar to the above by simply choosing f_2 such that $f_2(wx) = 2a$ and $f_2(wv) = f_2(wu) = a$. □

We define f as a *pseudoflow* in G if f has one source u and one sink v under the orientation τ, and $\partial f(v) = 0$ for each $v \in V(G) - \{u, v\}$. Specifically, a pseudoflow f in G is referred to as an *(a,b)-pseudoflow*, where $a, b \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$, if $\partial f(u) = a$ and $\partial f(v) = -b$. Note that a nontrivial string G admits a $(0, 0)$-pseudoflow with the sequence $(0, 2, 4, 2, 4, \cdots, 4, 2, 0)$ if $\gamma(G)$ is even; conversely, if $\gamma(G) \geq 2$ and is odd, the sequence becomes $(0, 2, 4, 2, \cdots, 4, 0)$, where each subsequence (a_i, a_{i+1}) represents an (a_i, a_{i+1})-pseudoflow in a subgraph of G that contains precisely one unbalanced digon.

Lemma 4.2.3. *Let G be a K_4-minor free 2-connected graph. If G contains at most one vertex of degree 2, then G contains a 2-circuit.*

Proof. Let G be a counterexample with $|V(G)|$ minimum. Clearly $\delta(G) = 2$. Thus G has exactly one degree 2 vertex, say v. Let x_1 and x_2 be the two neighbors of v.

Let $\overline{G}$ be the graph obtained from G after suppressing the degree 2 vertex. Then $\overline{G}$ is K_4-minor free and $\delta(\overline{G}) \geq 3$, and thus $\overline{G}$ contains a 2-circuit, which is the triangle $C = x_1x_2vx_1$ in G.

If $d_G(x_1) = d_G(x_2) = 3$, then G/C has exactly one degree 2-vertex v_C which is corresponding to C. Thus G/C has a 2-circuit not containing v_C. Thus the 2-circuit is also a 2-circuit in G, a contradiction.

Thus we may assume that $d_G(x_2) \geq 4$. Then $G - v$ has at most one vertex of degree 2 and thus contains a 2-circuit which is also a 2-circuit in G, a contradiction. □

Lemma 4.2.4. *Let G be a smallest counterexample to Theorem 1.5.11 with respect to $|E(G)|$. Then G does not contain two adjacent vertices both incident with a tadpole or a negative loop.*

Proof. By our assumption, $\delta(G) \geq 3$ and G does not contain any balanced digon. Suppose to the contrary that G contains two adjacent vertices, say u_1 and u_2, incident with either a tadpole or a negative loop. Without loss of generality, we may assume that u_i is incident with a tadpole L_{Q_i} for $i = 1, 2$ since the proof for the other cases is similar. Let L_{v_1} and L_{v_2} be the heads of L_{Q_1} and L_{Q_2} and let v_1u_1 and v_2u_2 be the tails of L_{Q_1} and L_{Q_2} , respectively. Since $\{u_1u_2\} \cup L_{Q_1} \cup L_{Q_2}$ forms a long barbell in G, denoted by B, admits a 3-flow f_1 with $\text{supp}(f_1) = E(B)$ and $f_1(u_1u_2) = 2$ by Proposition 1.4.4.

We first show that G_1 the graph obtained from G by deleting L_{Q_1} and L_{Q_2} is flow-admissible.

Suppose to the contrary, that G_1 is not flow-admissible. Then G_1 is switch equivalent to a signed graph with exactly one negative edge, or G_1 contains a bridge $b = x_1x_2$ such that $G_1 - b$ has a balanced component by Proposition 1.4.1.

We first consider the case when G_1 contains a bridge $b = x_1x_2$ such that $G_1 - b$ has a balanced component. Since G is flow-admissible, then $b = x_1x_2$ connects two unbalanced component in G, denoted by H_1 and H_2, respectively. Without loss of generality, we assume that H_2 contains the balanced component in $G_1 - b$. We may assume that the balanced component of $G_1 - b$ contains no negative edge.

For each $i = 1, 2$, we add a negative loop e'_i at the vertex x_{3-i} in H_i. The resulting graph is denoted by H'_i. Thus H'_2 contains at most three negative edges and all are negative loops. By the minimality of G, H'_1 has a 5-NZF f_1. Let $a = f_1(e'_2)$. If H'_2 has only two negative loop, then by Lemma 4.2.2, H'_2 has a 5-NZF f_2 such that $f_2(e'_1) = a$. Thus one can merge f_1 and f_2 into a 5-NZF of G, a contradiction.

Now we assume that H_2' contains exactly three negative edges, meaning H_2' contains B. Let H_2'' be the graph obtained from H_2' by deleting L_{Q_1} and L_{Q_2} and adding a negative edge e_B connecting u_1 and u_2. Then H_2'' contains an unbalanced digon between u_1 and u_2 and a negative loop e_1'. By Lemma 4.2.2 again H_2'' has a 5-NZF f_3 such that $f_3(e_1') = a = f_3(e_B)$. If a is even, then one can obtain a 5-NZF f_4 of H_2' by letting $f_4(u_iv_i) = a$ or $-a$ and $f_4(L_{v_i}) = \frac{a}{2}$ or $-\frac{a}{2}$ for each $i = 1, 2$. Then we can further obtain a 5-NZF of G by merging f_4 with f_1, a contradiction again.

If a is odd, then $a = 1$, let H_2''' be the ordinary graph obtained from H_2'' by deleting e_1', replacing the edge e_B with a positive path u_1wu_2, and adding a new positive edge x_2w. Similar to the argument in the proof of Lemma 4.2.2, H_2''' has a 5-NZF f_5 such that $f_5(x_2w) = 2$, $f_5(u_1w) = 2$ and $f_5(wu_2) = 4$. Thus H_2' has a 5-NZF f_6 such that $f_6(e_1) = 1 = a$. Therefore one can merge f_6 and f_1 into a 5-NZF of G, a contradiction.

Next we consider the case when G_1 is switch equivalent to a signed graph with exactly one negative edge. WLOG assume that G_1 has only one negative edge e_1. We may further assume that G_1 is bridgeless otherwise we go back to the previous case.

If e_1 is a negative loop, let G_2 be the graph obtained from $G - L(Q_1) - L(Q_2)$ by adding a negative edge e_3 connecting u_1 and u_2. Then by Lemma 4.2.2, G_2 admits a 5-NZF f_7 such that $f_7(e_2) = 2$. Thus one can obtain a 5-NZF of G.

Now assume that e_1 is not a negative loop. If there exists $i \in \{1, 2\}$ such that $d_G(u_i) = 3$, without loss of generality, we may assume that $d_G(u_1) = 3$. Let G_3 be the graph obtained from $G - L(Q_1) - L(Q_2)$ by adding a negative loop e_4 at u_1. By Lemma 4.2.2, G_3 has a 4-NZF f_8 such that $f_8(e_4) = 1$. Since $d_G(u_1) = 3$ and f_8 is a 4-flow, $f_8(u_1v_1) \neq \pm 2$. Let f_9 be a flow of G such that $supp(f_9) = E(B)$ and $f_9(v_1u_1) = f_9(u_1u_2) = f_9(u_2v_2) = 4$. Then $f_8 - f_9$ is a 5-NZF of G, a contradiction.

In the following we further assume that $d_{G_1}(u_1) \geq 4$ and $d_{G_1}(u_2) \geq 4$. Since $\delta(G) \geq 3$, we have $\delta(G_1) \geq 3$, so G_1 (and G) contains a digon C. By the minimality of G, G contains no balanced digons. Thus C is unbalanced which contains the only negative edge e_1 in G_1. Let $V(C) = \{y_1, y_2\}$. Then $G_1 - e_1$ is a bridgeless ordinary graph and $\delta(\overline{G_1 - e_1}) \geq 3$, by Lemma 4.2.3, $\overline{G_1 - e_1}$ contains a balanced digon, say C_1. Assume that $V(C_1) = \{z_1, z_2\}$. Then z_1z_2 is an edge in G and the other edge in D is a subdided path say, P in $G_1 - e_1$ contains the edge in $C - e_1$. By Proposition 1.4.4, C_1 admits a 2-flow f_1'. Also, since $G_1 - e_1$ is a K_4-minor free ordinary graph, $G_1 - e_1$ admits 3-NZF, by Lemma 1.4.7, $G_1 - e_1$ admits a positive 3-NZF (D_1, f_{10}) such that $f_{10}(u_1u_2) = 1$. Under the orientation D_1, we assume that the positive edge in C is oriented away from x_2, u_1u_2 is oriented away from u_1, and there exists a directed path from x_1 to u_1. Let $\beta = \{0, 1\}$. Then $f_{11} = f_{10} + \beta f_1'$ is a 4-NZF of $G_1 - e_1$ such that $f_{11}(x_2x_1) = 1$ or 3 and $f_{11}(u_1u_2) = 1$. After adding a negative edge e_B between u_1 and u_2 in G_1, $f_{12} = f_{11} - 2f_1$ is a 5-NZF of $G_1 \cup \{e_B\}$. Therefore G admits a 5-NZF such that $f_{12}(u_iv_i) = -2$ and $f_{12}(L_{v_i}) = -1$

for each $i = 1, 2$, a contradiction. This contradiction implies that G_1 is flow-admissible.

Next we will show that $G_1 \notin \mathcal{N}$. Suppose to the contrary that $G_1 \in \mathcal{N}$

If u_1u_2 is not contained in an unbalanced digon, then $G_1 - u_1u_2$ admits a $(1, 3)$-pseudoflow g_1, and thus G admits a 5-NZF by letting $g_1(u_1u_2) = 1, g_1(u_1v_1) = 2$, $g_1(u_2v_2) = 4, g_1(L_{v_1}) = 1, g_1(L_{v_2}) = 2$, a contradiction.

If u_1u_2 is not contained in an unbalanced digon, then $G_1 - u_1u_2$ admits a $(1, -1)$-pseudoflow g_2, and $g_2 + f_1$ is a 5-NZF of G, a contradiction again.

By the minimality of G, G_1 admits a 5-NZF g_3 with $g_3(u_1u_2) = a$, for some $a \in \{\pm 1, \cdots, \pm 4\}$. Hence either $g_3 + f_1$ or $g_3 - f_1$is a 5-NZF on G, a contradiction. This completes the proof of the lemma. □

Lemma 4.2.5. *Let $\alpha, \beta \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ with $\alpha \equiv \beta \pmod 2$ and $\alpha \neq \pm\beta$. Let D_0 be the unbalanced digon obtained from D_0^+ by deleting all tadpoles and negative loops and then suppressing all 2-vertices. If D_0 admits an (α, β)-pseudoflow, then it can be extended to D_0^+.*

Proof. Let D_0^+ be a counterexample with minimum $|E(D_0^+)|$. By Lemma 4.2.4, every subdivided edge in D_0^+ contains at most one internal vertex. With a similar proof, we may assume that every internal vertex is incident with a tadpole.

Let $E(D_0^+) = \{e_1^+, e_2^+\}$ and x, y be the two-terminal of D_0^+ and let e_i be the corresponding edge of e_i^+ in D_0. Let g be an (α, β)-pseudoflow on $E(D_0)$ under the orientation τ such that $\partial g(x) = \alpha$ and $\partial g(y) = \beta$. Without loss of generality, we may assume that e_1 is positive, oriented from x to y, while e_2 is negative and both half edges have orientation towards their respective ends, where $\alpha \equiv \beta \pmod 2$ and $\alpha \neq \pm\beta$. We have $g(e_1) = \frac{\alpha+\beta}{2}$ and $g(e_2) = \frac{\beta-\alpha}{2}$ and thus $g(e_1) \neq \pm g(e_2)$, and both e_1,$e_2 \notin E_{g=\pm 4}$. We consider the following two cases according to the number of internal vertices in D_0^+.

We first consider the case where each subdivided edge in D_0^+ contains only one internal vertex, that is, both e_1 and e_2 are incident with one tadpole, denoted by $L_{Q'}$ and $L_{Q''}$, respectively. Let P_1 denote the path connecting $L_{Q'}$ and $L_{Q''}$ that passes through the terminal y. Then $g(e) \in \{\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}\}$ if $e \in E(P_1)$, and there exists a 3-flow f_2 such that $\text{supp}(f_2) = E(P_1) \cup L_{Q'} \cup L_{Q''}$ by Lemma 1.4.4. Therefore we can always find some $\alpha_2 \in \{\pm 1, \pm 2\}$ such that $f = g + \alpha_2 f_2$ is an (α, β)-pseudoflow such that $\text{supp}(f) = E(D_0^+)$ for all possible $g(e_1)$ and $g(e_2)$, a contradiction to the assumption of D_0^+.

It remains to consider the case when D_0^+ contains exactly one internal vertex, either e_1^+ or e_2^+ contains exactly one internal vertex, by some switching operation, we can further assume that e_1^+ is the edge containing the single internal vertex, denoted by w. Suppose that the tadpole incident with w is denoted by $L_Q = L_v \cup \{wv\}$, where L_v is the head and wv is the tail, with w as the internal vertex. According to Lemma 1.4.4, $D_0 \cup L_Q$ admits a 3-flow f_3 with $\text{supp}(f_3) = E(D_0) \cup L_Q$. As a result, there exists an $\alpha_3 \in \{\pm 1, \pm 2\}$ such that $f_3 = g + \alpha_3 f_3$ is an

(α,β)	e_2(oriented into x,y)	arc (x,w)	arc (w,y)	tail (w,v)	L_v
$(1,3)$	2	3	1	2	1
$(1,-3)$	-4	-3	1	-4	-2
$(2,4)$	2	4	2	2	1
$(2,-4)$	-1	1	-3	4	2

Table 4.1: An extension in D_0^+ to a tadpole

(α,β)-pseudoflow such that $\mathrm{supp}(f_3) = E(D_0) \cup L_Q$. The various possible choices are examined in Table 4.1. □

Lemma 4.2.6. *Let $G = H_1 \cup H_2 \cup H_3$ be a flow-admissible parallel join of three generalized strings such that $H_1 \cup H_2$ is a generalized necklace with $\gamma(H_1 \cup H_2) \geq 1$ and H_3 is a string containing an even number of unbalanced digons. If G contains no subdivided unbalanced digons that have at least one internal vertex, then G admits a 5-NZF.*

Proof. Without loss of generality, we can assume that $\gamma(H_1) \geq 1$ since $\gamma(H_1 \cup H_2) \geq 1$. Then $\gamma(H_2) \geq 0$. Therefore we need to consider the following three cases according to the parity of $\gamma'(H_1)$ and $\gamma'(H_2)$ since $H_1 \cup H_2$ is a generalized necklace.

Case I. $\gamma'(H_1)$ is even and $\gamma'(H_2)$ is even.

Note that H_1 admits a $(-3, 1, \cdots, -3, 1, -3)$-pseudoflow, H_2 admits an $(1, 3, \cdots, 1, 3, 1)$-pseudoflow (note that if $\gamma'(H_2) = 0$, then H_2 admits a $(1,1)$-pseudoflow), and H_3 admits a $(2, 4, \cdots, 2)$-pseudoflow. Then the sum of the pseudoflows in all three generalized strings corresponds to a 5-NZF of G.

Case II. $\gamma'(H_1)$ is odd and $\gamma'(H_2)$ is even.

Note that H_1 contains at least one unbalanced digon, which admits a $(2,-4)$-pseudoflow. Depending on the place of the first unbalanced digon counted in $\gamma'(H_1)$, H_1 admits a $(2, 4, 2, 4, \cdots, 2, -4, \cdots, -2, -4)$-pseudoflow. Similarly, H_2 admits a $(1,1)$-pseudoflow, and H_3 admits a $(-3, 1, 3, \cdots, 1, 3)$-flow. Then their sum corresponds to a 5-NZF of G.

Case III. Both $\gamma'(H_1)$ and $\gamma'(H_2)$ are odd.

Note that H_1 admits a $(-3, 1, \cdots, -3, 1)$-pseudoflow, H_2 admits an $(1, -3, \cdots, 1, -3)$- pseudoflow, and H_3 admits a $(2, 4, \cdots, 2)$-pseudoflow. Then their sum corresponds to a 5-NZF of G. □

Lemma 4.2.7. *Let G be a flow-admissible generalized necklace with $\gamma(G) \geq 1$. If G contains no subdivided unbalanced digons that have at least one internal vertex, then there exists an (a,b)-pseudoflow in G such that $a, b \in \{\pm1, \pm2, \pm3, \pm4\}$ such that $a \equiv b \pmod 2$ and $a \neq \pm b$.*

Proof. By the assumption, G is a parallel join of two generalized strings, say H_1 and H_2.

Note that $\gamma(G) \geq 1$, we can assume that $\gamma'(H_1) \geq \gamma'(H_2)$ and $\gamma(H_1) \geq 1$. Without loss of generality, we may assume that $|a| \leq |b|$ by switchings if necessary. For each possible combination of (a, b), it's possible to find suitable choices for (a_1, b_1) and (a_2, b_2) such that $a_1 + a_2 = a$, $b_1 + b_2 = b$ and $a_1 \equiv b_1 \pmod 2$, $a_2 \equiv b_2 \pmod 2$, as shown in Table 4.2.

Note that H_1 contains at least one unbalanced digon. In particular, suppose that H_1 is a generalized odd string. According to the table 4.2, for the case when $(a, b) = (1, 3)$ and $(a_1, b_1) = (-2, 4)$, we can assign a $(-2, 4)$-pseudoflow to the first unbalanced digon counted in $\gamma'(H_1)$, This allows H_1 to admit a $(-2, -4, \cdots, -2, 4, 2, 4, \cdots, 2, 4)$-pseudoflow. For the case when $(a, b) = (2, -4)$ and and $(a_1, b_1) = (4, -2)$, we assign a $(4, -2)$-pseudoflow to the first unbalanced digon counted in $\gamma'(H_1)$, then it enables H_1 to admit a $(4, 2, \cdots, 4, -2, -4, -2, \cdots, -4, -2)$-pseudoflow. Therefore for all possible choices listed in Table 4.2, we can assign an (a_1, b_1)-pseudoflow in H_1 and an (a_2, b_2)-pseudoflow in H_2, and their sum corresponds to an (a, b)-pseudoflow of G. □

a b	a_1 b_1	a_2 b_2	a_1 b_1	a_2 b_2	a_1 b_1	a_2 b_2
	odd string	odd string	odd string	0 or even	even	0 or even
2 4	−1 3	3 1	1 3	1 1	−1 1	3 3
2 −4	−1 −3	3 −1	4 −2	−2 −2	3 −3	−1 −1
1 3	−2 4	3 −1	2 4	−1 −1	−1 1	2 2
1 −3	−2 −4	3 1	3 −1	−2 −2	2 −2	−1 −1

Table 4.2: The choices of (a_1, b_1) and (a_2, b_2)

Lemma 4.2.8. *Let $b \in \{\pm 2, \pm 4\}$ and G is a flow-admissible generalized necklace with $\gamma(G) \geq 1$. Let $x \in V(G)$ be one end of an unbalanced digon. If G contains no subdivided unbalanced digons that have at least one internal vertex, then G can be extended to have a $(0, b)$-pseudoflow f, where $\partial f(x) = b$.*

Proof. Since G is a generalized necklace, we may assume that G consists of two generalized strings, say H_1 and H_2, with x as a terminal.

Since $\gamma(G) \geq 1$ and G contains no subdivided unbalanced digons that have at least one internal vertex, we may assume that H_1 contains at least one unbalanced digon. Referring to Table 4.3, we can get a $(0, b)$-pseudoflow f in G such that $\text{supp}(f) = E(G)$ and $\partial f(x) = b$,

a	a_1	a_2	a_1	a_2	a_1	a_2
b	b_1	b_2	b_1	b_2	b_1	b_2
	odd string	odd string	odd string	0 or even	even	0 or even
0	3	-3	1	-1	-1	1
2	1	1	3	-1	1	1
0	-4	4	-1	1	-2	2
4	2	2	3	1	2	2

Table 4.3: A $(0, b)$-pseudoflow in G

where H_1 admits an (a_1, b_1)-pseudoflow and H_2 admits an (a_2, b_2)-pseudoflow, $a_1 + a_2 = 0$, and $b_1 + b_2 = b$.

Particularly, as described in Table 4.3, when H_1^+ is a generalized odd string , for the case when $(a, b) = (0, 4)$ and $(a_1, b_1) = (-4, 2)$, we can assign a $(-4, 2)$-pseudoflow to the first unbalanced digon counted in $\gamma'(H_1)$, then H_1 admits a $(-4, -2, \cdots, -4, 2, 4, 2 \cdots, 4, 2)$-pseudoflow. □

4.3 Proof of Theorem 1.5.11

Let's first recall Theorem 1.5.11.

Theorem 1.5.11. Let (G, σ) be a flow-admissible, K_4-minor free signed graph. Then (G, σ) admits a nowhere-zero 5-flow if and only if (G, σ) does not belong to the family $\mathcal{N}$.

Proof. If (G, σ) is a flow-admissible, K_4-minor free signed graph which admits a 5-NZF, then (G, σ) does not belong to $\mathcal{M}$ by Lemma 1.5.10, so (G, σ) is also not in $\mathcal{N}$.

If (G, σ) is a flow-admissible, K_4-minor free signed graph and does not belong to $\mathcal{M}$, we shall show that G admits a 5-NZF. Let (G, σ) be a counterexample with $|E(G)|$ minimum. Then (G, σ) is a flow-admissible, K_4-minor free signed graph that does not belong to $\mathcal{N}$ and admits no 5-NZF. By Lemma 1.5.9, (G, σ) can't be reduced via Operations (O1-O3). Thus we have the following claim. (G, σ) does not contain any edge-cut T with $|T| \leq 3$ such that one nontrivial component is balanced, and (G, σ) also contains no balanced leaf block, otherwise G either is not flow-admissible, or contains a contractible configuration, say H. Then G/H admits a 5-NZF, which implies that G admits a 5-NZF, a contradiction with our assumption.

Claim 4.3.0.1. *$\delta(G) \geq 3$, G has not cut-vertex having an unbalanced component at it, and there is no $X \subseteq V(G)$ such that $\delta(X) \leq 3$ and $G[X]$ is nontrivial and balanced.*

Claim 4.3.0.2. *The number of negative loops and tadpoles incident with a vertex is at most one.*

Proof. Suppose not, we can assume that there exists a vertex v in G that is incident with two tadpoles (similar proof for other cases), say L_{Q_1} and L_{Q_2}. Let $G' = G \setminus L_{Q_2}$. Then G' is not in $\mathcal{N}$, and G' is also K_4-minor free and flow admissible, otherwise (G, σ) contains 2-edge-cut T such that one nontrivial component is balanced since $\delta(G) \geq 3$, so G' admits a 5-NZF f_1 such that $f_1(L_v) \in \{\pm 1, \pm 2\}$ if $L_v \in L_{Q_1}$. By Lemma 1.4.4, there exists a 3-flow f_2 such that $\text{supp}(f_2) = L_{Q_1} \cup L_{Q_2}$, thus we can pick some $\alpha_1 = 1$ or -1 such that $g_1 = f_1 + \alpha_1 f_2$ is a 5-NZF in G. By Lemma 1.4.4, we can assume that v is incident with at most two negative loops. If v is incident with two negative loops, then there exists a 2-flow f_3 such that $\text{supp}(f_3) = L_{Q_1} \cup L_{Q_2}$ by Lemma 1.4.4, thus we can pick some $\alpha_2 = 1$ or -1 such that $g_1 + \alpha_2 f_3$ is a 5-NZF in G. $\square$

Claim 4.3.0.3. *Any internal vertex in every subdivided balanced digon of* (G, σ) *is not incident with any negative loops.*

Proof. Suppose that there exists a subdivided balanced digon D^+ that contains an internal vertex, say v, incident with a negative loop L_v. Denote by $E(D) = \{e_1, e_2\}$. Without loss of generality, we can assume that v is contained in e_2^+. By Claim 4.3.0.2, every internal vertex of D^+ is incident with at most one negative loop or one tadpole. By Lemma 4.2.4, v is the only internal vertex in e_2^+, as depicted in Figure 4.1-(a).

We replace $e_2 \cup L_v$ in G with one negative edge e_2', as in Figure 4.1-(b) to get a new graph, denoted by G'. Then G' is still K_4-minor free. If G' is not flow-admissible, G' does not contain an unbalanced circuit that is edge-disjoint from the circuit $C_1 = \{e_1^+, e_2'\}$. Since G is flow-admissible, there exists exactly one unbalanced circuit that is not edge-disjoint with e_1^+ in C_1. By some switching operations, $G - L_v$ only contains one negative edge. Then G is switching equivalent to a signed graph containing two negative edges such that one of them is a negative loop. Then G admits a 4-NZF by Lemma 4.2.2, a contradiction. So G' is flow-admissible.

Next we claim that $G' \notin \mathcal{N}$. Otherwise, then $G' = N^{\sigma}_{4k+2}$ by the choice of G, Consider the graphs D^+ and $G - E(D^+)$. Since $G - E(D^+)$ is a string, we can assign a $(-1, 1)$-pseudoflow g_1 in D^+, and $G - E(D^+)$ is a string that admits an $(1, -1)$-pseudoflow g_2. Then $g_1 + g_2$ is a 4-NZF in G, a contradiction with the choice of G. So G' does not belong to $\mathcal{N}$. By the minimality of G, G' admits a 5-NZF f_1.

Then we insert two internal vertices in e_2' and the signs of the edges in the subdivided path are shown in Figure 4.1-(c), denoting this graph by G'', so the flow values on $E(G')$ still preserve in G'' under a certain orientation. After identifying v_1 and v_2 as a single vertex, we can obtain a 5-NZF f in G such that $f(xv) = f(yv) = f(L_v) = f_1(e_2')$, a contradiction again. $\square$

Let G_0 be the graph obtained from deleting all tadpoles and negative loops and then suppressing all degree 2 vertices. In the following context, we use $\{x, y\}$ as the two-terminal of an balanced/unbalanced digon.

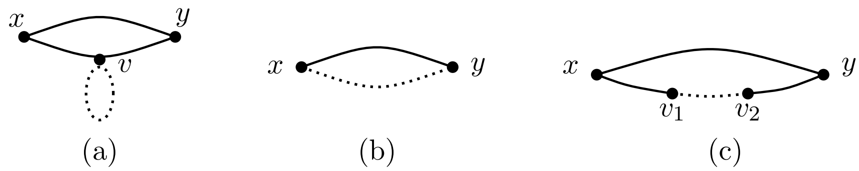


Figure 4.1: An operation in a subdivided digon

Claim 4.3.0.4. *G_0 contains no balanced digon D.*

Proof. Suppose that G_0 contains a balanced digon D, which implies that G contains a subdivided balanced digon D^+. By the choice of G and by Lemma 4.2.4, each subdivided edge in D^+ contains at most one internal vertex. If D is a leaf block of G_0 that share a common vertex x with $G_0 - D$, by the choice of G and Claim 4.3.0.3, D^+ contains exactly one tadpole L_Q, we can replace D^+ in G by the tadpole L_Q to get a smaller flow-admissible, K_4-minor free signed graph that does not belong to $\mathcal{N}$, so G' admits 5-NZF f' by the minimality of G. By Claim 4.3.0.1, the other terminal y of D is incident with the tadpole L_Q. Since D admits a positive 2-flow g such that $\text{supp}(g) = E(D)$, then there exists $\alpha = 1$ or -1 such that $f' + \alpha g$ is a 5-NZF of G, a contradiction with our assumption. therefore it remains to consider the following two cases.
Case I. If at least one edge of $E(D)$ is also an edge in $E(G)$.

Without loss of generality, suppose that e_1 is the edge in $E(D) \cap E(G)$. Let e_2^+ be the other subdivided edge in D^+. By Claim 4.3.0.3, if e_2^+ contains one internal vertex, say v, then v is incident with exactly one tadpole, denoted by $L_Q = L_w \cup \{vw\}$, where L_w is the head of this tadpole. Let $G_1 = G - e_1$. Then G_1 is K_4-minor free. Also G_1 is flow-admissible. This is because, even if one of the edges in $e_2^+ \cup \{vw\}$ is a bridge of G_1, G_1 remains flow-admissible since G does not contain balanced leaf blocks. Moreover, G_1 does not belong to $\mathcal{N}$, hence G_1 admits a 5-NZF f_1 by the minimality of G. If v and L_Q exist in e_2^+, then it is not possible for $|f(xv)| = |f(vy)| = 4$. Since D^+ admits a nonnegative 2-flow g such that $\text{supp}(g) = \{e_1 \cup e_2^+\}$, we can find some $\alpha \in \{\pm 1, \pm 2\}$ such that $f_1 + \alpha g$ is a 5-NZF in G.
Case II. Both e_1 and e_2 contain exactly one internal vertex.

In this case, each internal vertex is incident with exactly one tadpole by Claim 4.3.0.3. Let L_Q be the tadpole that is incident with the internal vertex contained in e_2^+. Then $G - \{e_2^+ \cup L_Q\}$ is a smaller K_4-minor free signed graph that is not part of $\mathcal{N}$. Since G contains no balanced component and no 2-edge-cut reducible configuration in which one component is balanced, $G - \{e_2^+ \cup L_Q\}$ is flow-admissible, and thus it admits a 5-NZF g with an (a, b)-pseudoflow on e_1^+, and it can be extended to a 5-NZF of G by reassigning an (a, b)-pseudoflow on $E(D^+)$, as outlined in Table 4.4. □

(a,b)	e_1^+	e_2^+
$(1,3)$	$(3,1)$	$(-2,2)$
$(1,-3)$	$(-2,-4)$	$(3,1)$
$(2,4)$	$(-1,3)$	$(3,1)$
$(1,-1)$	$(2,-2)$	$(-1,1)$
$(2,-2)$	$(1,-1)$	$(1,-1)$

Table 4.4: An (a,b)-pseudoflow in D^+

Claim 4.3.0.5. *G_0 does not contain a balanced leaf block.*

Proof. If G_0 has a balanced leaf block, say G_1, then G_1 contains at most one cut vertex v in G_0 such that $d_{G_1}(v) = 2$, which is also a cut-vertex of G_0, so G_1 contains a 2-circuit by Lemma 4.2.3, which is a balanced digon of G_0, a contradiction with Claim 4.3.0.4. So there is no balanced circuit in G_1, contradicts that G_1 is balanced. □

Claim 4.3.0.6. *G_0 does not contain any tadpoles or negative loops.*

Proof. Suppose not, then we need to consider the following two cases.

The first case is that G_0 is a negative loop, so G is a generalized unbalanced digon, denoted by D_0^+. Let $V(D_0) = \{x, y\}$. Then x and y are incident with a tadpole or a negative loop respectively by Claim 4.3.0.1, denoted by L_1, L_2. Moreover, $E(G) - \{L_1, L_2\} = E(D_0)$ by Lemma 4.2.4, so G admits a 5-NZF.

The second case is that G_0 contains a tadpole or a negative loop, so G contains a subdivided unbalanced digon D_0^+ as a leaf block. Let $\{x, y\}$ be the terminals of D_0. Then we can use a tadpole $L_Q = L_v \cup \{vy\}$ to replace D_0^+, then the resulting graph is smaller by Claim 4.3.0.1, thus it admits a 5-NZF f' since it is K_4-minor free, flow-admissible and does not belong to $\mathcal{N}$. Assume that $f'(vy) = b$, then $b \in \{\pm 2, \pm 4\}$. Since $\delta(G) \geq 3$, x is incident with a negative loop or tadpole in G, then we can extend the flow f' on the edges of this unbalanced digon D_0 such that $\partial f'(x) = a$ and $a \neq \pm b, a \equiv b \pmod 2$. By Lemma 4.2.5, f' can be extended to a 5-NZF f in G such that $\text{supp}(f) = E(G)$. □

Claim 4.3.0.7. *G contains a generalized necklace H. More specifically, every nontrivial leaf block contains a generalized necklace.*

Proof. Let G_{01} be a nontrivial leaf block in G_0 (Note that $G_{01} = G_0$ is possible). Then we can delete all parallel edges in G_{01} and suppressing all vertices of degree 2 to get a new graph G'_{01}, then $\delta(G'_{01}) \geq 3$, by Lemma 4.2.3, G'_{01} contains a 2-circuit, which corresponds to a necklace of G_0 by Claim 4.3.0.6, as the parallel edges are from the unbalanced digons of G_0 by Claim 4.3.0.4, therefore G contains a generalized necklace. □

Claim 4.3.0.8. *G does not contain any subdivided unbalanced digons with at least one internal vertex.*

Proof. Suppose that G contains a subdivided unbalanced digon, which is an unbalanced digon $D_0 = \{e_1, e_2\}$ in G_0, that contains at least one internal vertex. Suppose that e_1 is the negative edge in D_0, by Lemma 4.2.4, each subdivided edge in the subdivided digon D_0^+ does not contain adjacent internal vertices. So $\gamma'(D_0^+) = 1$ or 2 and each internal vertex is incident with precisely one tadpole or one negative loop by Claim 4.3.0.2.

Suppose that e_2^+ contains an internal vertex that is incident with one tadpole or negative loop, say L_2, then we can delete $e_2^+ \cup L_2$ in G to obtain a smaller K_4-minor free signed graph, denoted by G'.

If G' belongs to $\mathcal{N}$, then we observe that $\gamma'(D_0^+) = 1$, $G - D_0^+$ admits a $(-1, -3)$-pseudoflow, D_0^+ admits an $(1, 3)$-pseudoflow, so the sum of those flows forms a 5-NZF on $E(G)$, a contradiction. Therefore, G' does not belong to $\mathcal{N}$.

Suppose G' is flow-admissible, then G' admits a 5-NZF f_1. According to Lemma 1.4.4, G admits a 3-flow f_2 such that $\text{supp}(f_2) = e_1^+ \cup e_2^+ \cup L$. Then there exists an $\alpha \in \{\pm 1, \pm 2\}$ such that $f_1 + \alpha f_2$ is a 5-NZF of G, leading to a contradiction. So G' is not flow-admissible.

By Lemma 4.2.2, $\gamma'(D_0^+) = 2$. Thus e_1^+ contains an internal vertex that is incident with one tadpole or negative loop, say L_1. Moreover, G' does not contain a bridge such that there is a balanced component, as G does not contain balanced leaf blocks, so G' has only one negative edge, which belongs to L_1. Also D_0 is the only unbalanced digon in G_0 and G contains no other negative loops and tadpoles, so we can use a positive directed subdivided path xvy to replace e_1^+ in G'. The resulting graph, denoted by G_1, is a signed graph with all positive edges. So G_1 admits a 3-NZF g. Define $g_0(xv) = g_0(vy) = 1$ such that $\partial g_0(v) = 0$. By Lemma 4.2.1, there exists a 3-NZF g' in G_1 such that $g'|_{E_{G_1}}(v) = g_0$. Then we can easily extend it to a 5-NZF on $E(G)$ such that D_0^+ admits an $(1, 1)$-pseudoflow. □

The final step of the proof: We denote a series join of G_1 and G_2 by $[G_1, G_2]$.

By Claim 4.3.0.7, G_0 contains a necklace H. Specifically, if G_0 is not 2-connected, then we can select a necklace H in a leaf block G_{02} of G_0. Note that if the cut vertex that separates G_{02} and $G_0 - G_{02}$ is contained in H as a nonterminal of H, denoted by x, we shall choose another necklace in G_{02}. Let $H = H_1 \cup H_2$ be formed of two strings H_1 and H_2 and H_1^+ and H_2^+ be their generalized strings respectively.

Suppose that $x \in V(H_1^+)$. If G_{02}^+ is a generalized necklace, we can use a tadpole L_Q to replace G_{02}^+, then the resulting graph, denoted by G', admits a 5-NZF f since it is K_4-minor free, flow-admissible and is not in $\mathcal{N}$. Therefore, $\partial f_{G-G_{02}^+}(x) \in \{\pm 2, \pm 4\}$, hence G admits a 5-NZF by Lemma 4.2.8, a contradiction. Therefore G_{02}^+ is not a generalized necklace.

Since G_{02} is 2-connected, by Lemma 4.1.1, G is equivalent to a series-parallel graph, and G_{02} is a parallel join of at least two parts. So either H is a proper subgraph of one part of G_{02}, or H is composed of two parts of G_{02}. In either case, $G_{02} - H_1$ is also a nontrivial block, otherwise $G_{02} - H_1$ would be an unbalanced digon by Claim 4.3.0.4, contradicts that G_{02} is not a necklace. By Claim 4.3.0.7, another necklace H' can be found in the nontrivial block $G_{02} - H_1$, which is vertex disjoint from x in G.

Therefore we can always find a generalized necklace H^+ in G such that x is either one terminal of H or not contained in $V(H^+)$. Subject to this condition, we choose H such that $\gamma(H)$ is as small as possible.

Since $\gamma(H) \geq 1$, we can replace the generalized necklace H^+ in G with the graph $[e_0, D_0, e_1]$, where e_0, e_1 are positive edges.The resulting graph, denoted by G', is a K_4-minor free signed graph.

If $G' \in \mathcal{N}$, then G_0 is 2-connected and $G' = N_{4k+2}^{\sigma}$, thus G admits a 5-NZF by Lemma 4.2.6, a contradiction. Therefore G' is not part of the family $\mathcal{N}$.

If G' is not flow-admissible, then D_0 contains the only one negative edge and G' does not contain any negative loops or tadpoles. Therefore G' contains a 2-edge-cut reducible configuration which is also a 2-edge-cut reducible configuration in G such that one component is balanced, a contradiction with the choice of G. Hence G' is flow-admissible.

If G' contains smaller number of edges than G, by the choice of G, G' admits a 5-NZF f' such that $\text{supp}(f') = E(G')$. Under the restriction of f', D_0 has an (α, β)-pseudoflow such that $\alpha \equiv \beta \pmod 2$ and $\alpha \neq \pm\beta$, where $\alpha, \beta \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$. Since H^+ is a generalized necklace in G, by Lemma 4.2.7, H^+ can be extended to have an (α, β)-pseudoflow, so it is a 5-NZF in G, a contradiction with our assumption. It implies that G' has the same number of edges with G, then the necklace H in G is a parallel join of e_0 and $[e_1, D_0]$, where e_0, e_1 are positive edges by some switching operations. and we can use the graph $[e_1, D_0]$ to replace H to get a smaller K_4-minor free flow-admissible signed graph, denoted by G''. Then G'' is not in $\mathcal{N}$, otherwise $G - H$ admits a $(1, -1)$-pseudoflow, and H admits a $(-1, 1)$-pseudoflow, their sum is a 4-NZF in G, a contradiction. So G'' admits a 5-NZF f'' such that D_0 has an (α, β)-pseudoflow, where $\alpha \equiv \beta \pmod 2$ and $\alpha \neq \pm\beta$, where $\alpha, \beta \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7\}$. Let $E(D_0) = \{e_2, e_3\}$ with e_3 negative. Then $|f''(e_1)| \neq |f''(e_2)|$, by Lemma 1.4.4, there exits a 2-flow g_1 such that $\text{supp}(g_1) = \{e_0, e_2, e_1\}$, and there exists $\alpha \in \{\pm 1, \pm 2\}$ such that $f'' + \alpha g_1$ is a 5-NZF of G. □

Chapter 5

Final Remarks

5.1 Flows of 3-edge colorable cubic signed graph

Bouchet's conjecture [2] is equivalent to the restriction to cubic signed graphs: *every flow-admissible, cubic signed graph admits a nowhere-zero* 6*-flow.* So we want to consider the following problem to improve our previous result.

Problem 1. *Let (G, σ) be a connected* 3*-edge colorable cubic signed graph. If (G, σ) is flow admissible, then (G, σ) has a nowhere-zero* 6*-flow.*

The following problem is a weaker problem.

Problem 2. *Let (G, σ) be a connected* 3*-edge colorable cubic signed graph. If (G, σ) is flow admissible and hamiltonian, then (G, σ) has a nowhere-zero* 6*-flow.*

5.2 Flows of signed Kotzig graphs

A cubic graph is a *Kotzig graph* if there is a 3-edge coloring such that every two color classes induce a hamiltonian circuit.

Schubert and Steffen [25] proved the following lemma.

Lemma 5.2.1. *Let (G, σ) be a flow admissible signed graph. If G is a Kotzig graph, then (G, σ) admits a nowhere-zero* 6*-flow.*

So we want to consider if we can further reduce to 4-flow.

Problem 3. *Let (G, σ) be a flow admissible signed graph. If G is a Kotzig graph, then (G, σ) admits a nowhere-zero* 4*-flow.*

We also want to consider a flow for a $(2k+1)$-regular Kotzig graph. Since a $(2k+1)$-regular Kotzig graph is 4-edge-connected and Raspaud and Zhu [23] proved that every flow-admissible 4-edge-connected graph has a nowhere-zero 4-flow. So we want to consider the following problem.

Problem 4. *Let (G, σ) be a flow admissible signed graph and $k \geq 2$. If G is a $(2k+1)$-regular Kotzig graph, then (G, σ) admits a nowhere-zero 3-flow.*

5.3 Converting a modulo flow to an integer flow

We have some results on modulo k-flows and integer flows where k is odd. However we don't have any one related to $\mathbb{Z}_4$-flow and integer flows, so we want to consider the following theorem.

Problem 5. *Let (G, σ) be a flow admissible signed graph. If (G, σ) admits a nowhere-zero $\mathbb{Z}_4$-flow, then it has a nowhere-zero 8-flow.*

5.4 Modulo orientations

Proposition 5.4.1. (Goddyn et al. [8], Jaeger [12]) *Let G be an ordinary graph. If G has a modulo $(2p+1)$-orientation for some $p \geq 1$, then it has a modulo $(2p'+1)$-orientation for each integer p' with $1 \leq p' \leq p$.*

It is unknown whether Proposition 5.4.1 remains true for signed graphs, so we want to consider the following problem.

Problem 6. *Let $p \geq 2$ be an integer. Is it true that for any integer p' with $1 \leq p' < p$, if (G, σ) is modulo-$(2p+1)$-orientable, then it is also modulo-$(2p'+1)$-orientable?*

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