JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **2**, December **2023**, pp. 125–173, DOI 10.15421/1423014

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

QUALITATIVE ANALYSIS OF AN OPTIMAL SPARSE CONTROL PROBLEM FOR QUASI-LINEAR PARABOLIC EQUATION WITH VARIABLE ORDER OF NONLINEARITY

Ciro D'Apice[*], Peter Kogut[†], Rosanna Manzo[‡]

Abstract. In this work, we study a sparse optimal control problem involving a quasi-linear parabolic equation with variable order of nonlinearity as a state equation and with a pointwise control constraints. We show that in the case if the cost functional contains the terminal term of the tracking type, the proposed optimal control problem is ill-posed, in general. In view of this, we provide a sufficiently mild relaxation of the proposed problem and establish the existence of optimal solutions for the relaxed version. Using the compensated compactness technique and the consept of variational convergence of minimization problems, we study the attainability of optimal pairs to the relaxed problem by optimal solutions of the special approximating problems. We also discuss the optimality conditions for approximating problems and provide their substantiation.

Key words: Weak solution, parabolic equation, variable order of nonlinearity, noncoercive problem, compensated compactness technique..

2010 Mathematics Subject Classification: 35K20, 49J20, 35D30, 35K92.

Communicated by Prof. O. M. Stanzhytskyi

1. Introduction

1.1. Motivation

Over the past few decades, the role of optical satellite multi-band images in remote sensing of the Earth surface has been increasingly contributing to many agricultural monitoring services. In spite of the fact that optical images have a high resolution and are easily captured by low-cost cameras, the real-life satellite images frequently suffer from different types of noise, blur, and other atmosphere artifacts , which greatly reduce the effective information is such images. Hence, removing noise is a crucial step for image quality improvement in image processing task. In the last decades, models based on partial differential equations (PDEs) have been widely used in the image de-noising problems. Since 1990s, originated from the pioneering work of Perona and Malikl [51], many different

[*]Dipartimento di Scienze Aziendali - Management and Innovation Systems, University of Salerno, 132, Via Giovanni Paolo II, Fisciano, SA, Italy (cdapice@unisa.it)

[†]Department of Mathematical Analysis and Optimization, Oles Honchar Dnipro National University, Gagarin av., 72, 49010 Dnipro, Ukraine, EOS Data Analytics Ukraine, Gagarin av., 103a, Dnipro, Ukraine (p.kogut@i.ua, peter.kogut@eosda.com)

[‡]Dipartimento di Scienze Politiche e della Comunicazione, University of Salerno, Via Giovanni Paolo II, 132, Fisciano (SA), Italy (rmanzo@unisa.it)

models have been proposed to separate noise from the noisy images. Without being too exhaustive, we refer to [1, 14, 16–18, 21, 22, 47, 54] for a wide variety of different variational models related to the image denoising problems.

However, since the noise, edges, and texture are high-frequency components, it is difficult to distinguish them in the process of denoising, and, as a result, the denoised images could inevitably lose some details. This problems becomes much more difficult if the original image is contaminated by an impulse noise. In view of this, we mainly focus on those approaches where the denoising problem can be stated in the form of some optimal control problem with special class of controls simulating the presence of both the white Gaussian additive noise n and the noise v with a strong impulsive nature which the Gaussian model fails to describe (see, for instance, [2, 13, 48]). In this case the observed image can be represented as $f = u + v + n$, and the question is how to separate a true image u eliminating both Gaussian noise n and impulse noise v from f.

1.2. Statement of the problem

Inspired in the work [2], the first goal of this paper is to analyze the consistency and well-posedness of the following optimal control problem (OCP):

$$\text{Minimize } J(v,u) = \|v\|^2_{L^2(0,T;L^1(\Omega))} + \frac{\mu}{2}\int_\Omega |u(T) - f_0|^2\,dx \tag{1.1}$$

subject to the following constraints

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(|R_\eta \nabla u|^{p_u(t,x)-2} R_\eta \nabla u\right) = \kappa\,(f - u - v) \;\text{ in }\; Q_T := (0,T)\times\Omega, \tag{1.2}$$

$$\partial_\nu u = 0 \quad \text{on} \;\; (0,T)\times\partial\Omega, \tag{1.3}$$

$$u(0,\cdot) = f_0(\cdot) \quad \text{in} \;\; \Omega, \tag{1.4}$$

$$v_a(x) \leqslant v(t,x) \leqslant v_b(x), \quad \text{a.e. in } \; Q_T. \tag{1.5}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded simple-connected open set with a sufficiently smooth boundary $\partial\Omega$, $T > 0$ is a positive value, $\kappa \in \mathbb{R}$ is a given positive parameter, $f \in L^2(\Omega)$, $f_0 \in L^2(\Omega)$ and $v_a, v_b \in L^2(\Omega)$, $v_a(x) \leqslant v_b(x)$ a.e. in Ω, are given distributions,

$$\|v\|^2_{L^2(0,T;L^1(\Omega))} = \int_0^T \left(\int_\Omega |v|\,dx\right)^2 dx \tag{1.6}$$

is the so-called directional sparsity term, $R_\eta : L^1(\Omega;\mathbb{R}^2) \to L^1(\Omega;\mathbb{R}^2)$ is a linear bounded operator, and the exponent $p_u : Q_T \to \mathbb{R}$ is defined by the rule

$$p_u(t,x) := 1 + g\left(\frac{1}{h}\int_{t-h}^t \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(x)\right|\,d\tau\right), \quad \forall\,(t,x) \in Q_T, \tag{1.7}$$

where $g : [0,\infty) \to (0,1]$ is a continuous non-increasing function such that $g(0) =$

1 and $g(s) > 0$ for all $s > 0$ with $\lim_{s\to\infty} g(s) = 0$,

$$|g(s) - g(y)| \leqslant C_g|s - y|, \quad \forall s, y \in [0, \infty) \text{ with some constant } C_g > 0, \tag{1.8}$$

$$G_\sigma(x) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^2} \exp\left(-\frac{|x|^2}{2\sigma^2}\right), \quad \sigma > 0, \tag{1.9}$$

$$\left(G_\sigma * \widetilde{u}(t, \cdot)\right)(x) = \int_{\mathbb{R}^2} G_\sigma(x - y)\widetilde{u}(t, y)\, dy, \tag{1.10}$$

$\widetilde{u}$ denotes zero extension of u from Q_T to $\mathbb{R} \times \mathbb{R}^2$, and $h > 0$ and $\sigma > 0$ are given small positive values.

In particular, the function g in (1.7) can be defined in the form of the Cauchy law

$$\begin{gathered} g(s) = \delta + \tfrac{a^2(1-\delta)}{a^2+s^2}, \quad \forall\, s \in [0, +\infty) \\ \text{with an appropriate } a > 0 \text{ and } 0 < \delta \ll 1. \end{gathered} \tag{1.11}$$

Moreover, it will be shown further that, for each function u with properties $u \in L^1(Q_T) \cap L^\infty(0, T; L^2(\Omega))$, there exists a positive value $\delta > 0$ such that $p_u(t, x) \in [p^-, p^+] \subset (1, 2]$ almost everywhere in Q_T with $p^- = 1 + \delta$ and $p^+ = 2$.

We can indicate here a few main characteristic features of the addressed OCP (1.1)–(1.5). The first one is a special character of the linear operator R_η. In fact, this operators plays the role of the so-called Directional Total Variation along a given vector field. In practice, having some vector field $\theta \in L^\infty(\Omega; \mathbb{R}^2)$, we determine this operator as follows:

$$R_\eta \nabla u = \left[I - \eta^2\, \theta \otimes \theta\right] \nabla u, \quad \forall\, u \in W^{1,1}(\Omega),$$

where $\eta \in (0, 1)$ is a given threshold. So, $R_\eta \nabla u$ can be reduced to $(1 - \eta^2)\nabla v$ if the gradient $\nabla u(t, x)$ at this point is co-linear to θ, and to $\nabla u(t, x)$ provided $\nabla u(t, x)$ is orthogonal to θ. In other words, this operator impose some anisotropy effect in the standard diffusivity of u.

The second characteristic point of OCP (1.1)–(1.5) is related to the variable character of the exponent $p = p(t, x)$. As follows from representation (1.7) this characteristic depends not only on (t, x) but also on $u(t, x)$. So, in contrast to the recent paper [19], where the authors study the solvability issues for the nonlinear parabolic equation having nonstandard growth condition with respect to the gradient and with well predefined variable exponent, the function $p_u(t, x)$ in (1.2) is unknown a priori and strictly depends on the current solution of the initial-boundary value problem (IBVP) (1.2)–(1.4). It is worth also to emphasize that we do not assume here that the dependency $u \mapsto p_u$ is local whereas it is the crucial assumption in the most of existing publications (see for instance [9, 12]). The next difficulty in the analysis of this IBVP relies that its weak formulation cannot be written as equality in terms of duality in a fixed Banach space (for the details we refer to [23]). In fact, we show that each weak solution to the IBVP (1.2)–(1.4) lives in the corresponding 'personal' functional space, and, in view of our assumptions on the structure of exponent $p_u(t, x)$, the problem (1.2)–(1.4) can admit the weak solutions that may not possess the usual properties of solutions to parabolic equations. In particular, it would be rather questionable assertion that a weak solution to the above is unique, belong to the space $C([0,T];L^2(\Omega))$, and satisfies the standard energy equality.

It is well-known that the variable character of exponent p causes a gap between the monotonicity and coercivity conditions. Because of this gap, the problem (1.1)–(1.4) can be termed an optimal control problem for the quasi-linear parabolic equations with nonstandard growth conditions, and it can be viewed as a generalization of the evolutional version of $p(t,x)$-Laplacian equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p(t,x)-2}\nabla u\right) \tag{1.12}$$

with an exponent that depends only on t and x. During the last decades equation (1.12) was intensively studied by many authors. There is extensive literature devoted to equation (1.12). We limit ourselves by referring here to the following ones [9, 10, 15, 50, 52, 58] which provide an excellent insight to the theory of evolutional $p(t,x)$-Laplacian equations.

Albeit PDEs with variable nonlinearity are rather interesting from the purely mathematical point of view as was mentioned before, their study is often motivated by various applications where the problem (1.2)–(1.4), or some special cases of it, appear in the most natural way [2, 3, 14, 21]. It was recently shown that the model (1.2)–(1.4) naturally appears as the Euler-Lagrange equation in the problem of restoration of cloud contaminated satellite optical images [27]. Moreover, the above mentioned problem can be considered as a model for the deblurring and denoising of multi-spectral satellite images. In particular, this model has been proposed in [28, 43] in order to avoid the blurring of edges and other localization problems presented by linear diffusion models in images processing. We also refer to [40], where the authors study some optimal control problems associated with a special case of the model (1.2)–(1.4) and show that, in contrast to the case of the problem (1.2)–(1.4), the proposed in [40] class of optimal control problems is well posed.

It is also worth to notice that the model (1.2)–(1.4) can be considered as a natural generalization of the well-know Perona-Malik model [51]. In spite of the fact that Perona-Malik model reduces the diffusivity of color in places having higher likelihood of being edges, its major defect is that this model is ill-posed and there are no results of existence and its consistency (see [40]). To overcome this problem it has been proposed to modify this model by applying a Gaussian filter on the gradient (we can refer to the pioneering works [7, 20]).

The next characteristic feature of OCP (1.1)–(1.5) is that this control problem is formulated with $L^1(\Omega;L^2(0,T))$ control cost functional (together with some additional pointwise control constraints). Because of this the resulting optimal control may have directional sparsity, i.e., its support is a constant in time and the control v is identically zero on some parts of the domain Ω.

All of this leads us to the followings conclusion: OCP (1.1)–(1.5) is sufficiently challenging and its consistency is an open question. In fact, it will be shown in the next sections that because of the variable character of exponent p and its dependence on t and x, we can lose the continuity of the mapping $t \mapsto \|u(t,\cdot)\|_{L^2(\Omega)}$. Hence, the cost functional (1.1) is not well-defined and, as a result, we can assert that the OCP (1.1)–(1.5) is ill-posed, in general. Because of this, the original OCP requires some relaxation and approximations.

1.3. Organization of the paper

The paper is organized as follows. In Section 2 we give some preliminaries and introduce the main assumptions on the structure of the operator R_η and the variable exponent $p_u(t,x)$. We also give here the main auxiliary results concerning the Orlicz spaces, Sobolev-Orlicz spaces with variable exponent, weighted energy space, and convergence of fluxes to flux. In Section 3 we focus on the solvability issues for IBVP (1.2)–(1.4). With that in mind we follows the indirect approach using the technique of passing to the limit in some special approximation scheme. In this section we show that the IBVP (1.2)–(1.4) admits at least one weak solutions that can be attained by the solutions of more regular Caushy-Neumann problem for quasi-linear parabolic equations. In Section 4 we propose rather mild scheme of relaxation for the original OCP, and show that at each level of relaxation the corresponding OCP is well-posed and admits at least one solution. The questions attainability of the solutions to the relaxed problems are the subject of Section 5. In fact, in this section we introduce the family of OCPs for the special class of parabolic equations

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u - \operatorname{div} A_u^\varepsilon(t,x,\nabla u) + \kappa u = \kappa(f - v) \quad \text{in} \quad Q_T := (0,T) \times \Omega,$$

where the flux $A_u^\varepsilon(t,x,\nabla u)$ we define as follows

$$A_u^\varepsilon(t,x,\nabla u) := \left(|R_\eta \nabla u| + \varepsilon\right)^{p_u(t,x)-2} R_\eta \nabla u.$$

We show that due to this approximation, some optimal solutions to the relaxed OCP can be attained in an appropriate topology by the solutions of the proposed family of OCPs.

The last Section 6 is devoted to the deriving of some optimality conditions for approximating OCPs and their substantiation.

2. Main Assumptions and Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with a sufficiently smooth boundary $\partial\Omega$, and let $T > 0$ be a given value. We suppose that the unit outward normal $\nu = \nu(x)$ is well-defined for a.e. $x \in \partial\Omega$, where a.e. means here with respect to the 1-dimensional Hausdorff measure $\mathcal{H}^1$. We set $Q_T = (0,T) \times \Omega$. For any measurable subset $D \subset \Omega$ we denote by $|D|$ its 2-dimensional Lebesgue measure $\mathcal{L}^2(D)$. We denote its closure by $\overline{D}$ and its boundary by ∂D.

For vectors $\xi \in \mathbb{R}^2$ and $\eta \in \mathbb{R}^2$, $(\xi, \eta) = \xi^t \eta$ denotes the standard vector inner product in $\mathbb{R}^2$, where ^{t} stands for the transpose operator. The norm $|\xi|$ is the Euclidean norm given by $|\xi| = \sqrt{(\xi, \xi)}$. We also make use of the following notation $\operatorname{diam} \Omega = \sup_{x,y \in \Omega} |x - y|$.

2.1. Functional Spaces

Let X denote a real Banach space with norm $\|\cdot\|_X$, and let X' be its dual. Let $\langle \cdot, \cdot \rangle_{X';X}$ be the duality form on $X' \times X$. By $\rightharpoonup$ and $\stackrel{*}{\rightharpoonup}$ we denote the weak and weak* convergence in normed spaces X and X', respectively.

For given $1 \leqslant p \leqslant +\infty$, the space $L^p(\Omega; \mathbb{R}^2)$ is defined by

$$L^p(\Omega; \mathbb{R}^2) = \left\{ f : \Omega \to \mathbb{R}^2 \ : \ \|f\|_{L^p(\Omega;\mathbb{R}^2)} < +\infty \right\},$$

where $\|f\|_{L^p(\Omega;\mathbb{R}^2)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}$ for $1 \leqslant p < +\infty$. The inner product of two functions f and g in $L^p(\Omega; \mathbb{R}^2)$ with $p \in [1, \infty)$ is given by

$$(f, g)_{L^p(\Omega;\mathbb{R}^2)} = \int_\Omega (f(x), g(x)) \, dx = \int_\Omega \sum_{k=1}^{2} f_k(x) g_k(x) \, dx.$$

We denote by $C_c^\infty(\mathbb{R}^2)$ the locally convex space of all infinitely differentiable functions with compact support in $\mathbb{R}^2$. We recall here some functional spaces that will be used throughout this paper. We define the Banach space $W^{1,p^-}(\Omega)$ with $p^- > 1$ as the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|y\|_{W^{1,p^-}(\Omega)} = \left(\int_\Omega \left(|y|^{p^-} + |\nabla y|^{p^-} \right) dx \right)^{1/p^-}.$$

We denote by $\left(W^{1,p^-}(\Omega) \right)'$ the dual space of $W^{1,p^-}(\Omega)$. Let us remark that in this case the embedding $L^2(\Omega) \hookrightarrow \left(W^{1,p^-}(\Omega) \right)'$ is continuous.

Given a real separable Banach space X, we will denote by $C([0,T]; X)$ the space of all continuous functions from $[0,T]$ into X. We recall that a function $u : [0,T] \to X$ is said to be Lebesgue measurable if there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ of step functions (i.e., $u_k = \sum_{j=1}^{n_k} a_j^k \chi_{A_j^k}$ for a finite number n_k of Borel subsets $A_j^k \subset [0,T]$ and with $a_j^k \in X$) converging to u almost everywhere with respect to the Lebesgue measure in $[0,T]$.

Then for $1 \leqslant p < \infty$, $L^p(0,T; X)$ is the space of all measurable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty,$$

while $L^\infty(0,T; X)$ is the space of measurable functions such that

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t \in [0,T]} \|u(t)\|_X < \infty.$$

This choice makes $L^p(0,T;X)$ a Banach space and guarantees that its dual can be identified with $L^{p'}(0,T;X')$, where $p' = p/(p-1)$ and X' is the dual space to X. In particular, for functions $f \in L^2(0,T;L^1(\Omega))$ the continuous Minkowski inequality (see [55, p.499]) yields $f \in L^1(0,T;L^2(\Omega))$ and moreover

$$\begin{aligned}\|f\|_{L^2(0,T;L^1(\Omega))} &:= \left(\int_0^T \left(\int_\Omega |f|\,dx\right)^2 dx\right)^{1/2}\\ &\leqslant \int_\Omega \left(\int_0^T |f|^2\,dt\right)^{1/2} dx =: \|f\|_{L^1(0,T;L^2(\Omega))}.\end{aligned}$$

Hence, we have $L^2(0,T;L^1(\Omega)) \hookrightarrow L^1(0,T;L^2(\Omega))$. The full presentation of this topic can be found in [29].

2.2. Variable Exponent

Let $u \in L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given function. We associate with $u : Q_T \mapsto \mathbb{R}$ the exponent $p_u : Q_T \to \mathbb{R}$ defined by the rule (1.7).

Since $G_\sigma \in C^\infty(\mathbb{R}^2)$, it follows from (1.7) and from absolute continuity of the Lebesgue integral that $1 < p_u(t,x) \leqslant 2$ in Q_T and $p_u \in C^1([0,T];C^\infty(\mathbb{R}^2))$ even if u is just an absolutely integrable function in Q_T. Moreover, for each $t \in [0,T]$, $p_u(t,x) \approx 1$ in those places of Ω where some discontinuities are present in $u(t,\cdot)$, and $p_u(t,x) \approx 2$ in places where $u(t,x)$ is smooth or contains homogeneous features. In view of this, $p_u(t,x)$ can be interpreted as a characteristic of the sparse texture of the function u.

The following result plays a crucial role in the sequel (for comparison, we refer to [41, Lemma 2.1]).

Lemma 2.1. *Let $\{u_k\}_{k\in\mathbb{N}} \subset L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a sequence of measurable functions such that each element of this sequence is extended by zero outside of Q_T and*

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(0,T;L^2(\Omega))} < +\infty,\\ &u_k \to u \quad \text{weakly in } L^1(0,T;L^1(\Omega)) \text{ for some } \ u \in L^1(0,T;L^1(\Omega)).\end{aligned} \tag{2.1}$$

Let

$$\left\{p_{u_k} = 1 + g\left(\frac{1}{h}\int_{t-h}^t |(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot))|\,d\tau\right)\right\}_{k\in\mathbb{N}}$$

be the corresponding sequence of variable exponents. Then there exist constants $C > 0$ and $\delta \in (0,1)$ depending on Ω, G, g, $\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(0,T;L^2(\Omega))}$, and

$\sup_{k\in\mathbb{N}} \|u_k\|_{L^1(0,T;L^1(\Omega))}$ *such that*

$$p^- := 1+\delta \leqslant p_{u_k}(t,x) \leqslant p^+ := 2, \quad \forall\,(t,x)\in Q_T,\ \forall\,k\in\mathbb{N}, \tag{2.2}$$

$$\{p_{u_k}(\cdot)\} \subset \mathfrak{S} = \left\{ q \in C^{0,1}(Q_T) \;\middle|\; \begin{array}{c} |q(t,x)-q(s,y)| \leqslant C\left(|x-y|+|t-s|\right), \\ \forall\,(t,x),(s,y)\in \overline{Q_T}, \\ 1<p^- \leqslant q(\cdot,\cdot) \leqslant p^+ \text{ in } \overline{Q_T}. \end{array} \right\} \tag{2.3}$$

$$p_{u_k} \to p_u = 1 + g\left(\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(\cdot)\right|\, d\tau\right) \quad \text{uniformly in } \overline{Q_T} \text{ as } k\to\infty. \tag{2.4}$$

Proof. Since the sequence $\{u_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T;L^1(\Omega))$ and the Gaussian filter kernel G_σ is smooth, it follows that

$$\begin{aligned}
\frac{1}{h}\int_{t-h}^{t} \Big|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(x)\Big|\, d\tau &\leqslant \frac{1}{h}\int_{t-h}^{t}\left(\int_\Omega |\nabla G_\sigma(x-y)|\,|\widetilde{u}_k(\tau,y)|\,dy\right)d\tau \\
&\leqslant \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\frac{1}{h}\|u_k\|_{L^1(0,T;L^1(\Omega))}, \\
2 \geqslant p_{u_k}(t,x) = 1 + g&\left(\int_{t-h}^{t}\left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(x)\right|\,d\tau\right) \\
&\geqslant 1 + g\left(\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\tfrac{1}{h}\sup_{k\in\mathbb{N}}\|u_k\|_{L^1(0,T;L^1(\Omega))}\right), \\
&\qquad \forall\,(t,x)\in Q_T,
\end{aligned}$$

where

$$\begin{aligned}
\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} &= \max_{\substack{z=x-y\\ x\in\overline{\Omega},y\in\overline{\Omega}}}\Big[|G_\sigma(z)| + |\nabla G_\sigma(z)|\Big] \\
&= \frac{e^{-1}}{\left(\sqrt{2\pi}\sigma\right)^2}\left[1+\frac{1}{\sigma^2}\operatorname{diam}\Omega\right].
\end{aligned} \tag{2.5}$$

Then L^1-boundedness of $\{u_k\}_{k\in\mathbb{N}}$ guarantees the existence of a positive value $\delta\in(0,1)$ such that $p_{u_k}(t,x)\geqslant 1+\delta$. Hence, the estimate (2.2) holds true for all $k\in\mathbb{N}$.

Moreover, as follows from (1.8) and the relations

$$\begin{aligned}
&\big|p_{u_k}(t,x) - p_{u_k}(t,y)\big| \\
&\leqslant C_g\left|\int_{t-h}^{t}\left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(x)\right|\,d\tau - \int_{t-h}^{t}\left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(y)\right|\,d\tau\right| \\
&\leqslant C_g\int_0^T \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(x) - \left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(y)\right|\,d\tau \\
&\leqslant C_g\int_0^T\int_\Omega |u(\tau,z)|\,dz\,d\tau \max_{z\in\Omega}|\nabla G_\sigma(x-z) - \nabla G_\sigma(y-z)| \\
&= C_g\gamma_1\max_{z\in\Omega}|\nabla G_\sigma(x-z)-\nabla G_\sigma(y-z)|, \quad \forall\,x,y\in\overline{\Omega}
\end{aligned} \tag{2.6}$$

with $\gamma_1 = \sup_{k\in\mathbb{N}} \|u_k\|_{L^1(0,T;L^1(\Omega))}$, and from smoothness of the function $\nabla G_\sigma(\cdot)$, there exists a positive constant $C_G > 0$ independent of k such that, for each $t \in [0,T]$, we have the following estimate

$$|p_{u_k}(t,x) - p_{u_k}(t,y)| \leqslant \gamma_1 C_g C_G |x-y|, \quad \forall\, x,y \in \overline{\Omega}.$$

Arguing in a similar manner, we see that

$$\begin{aligned}
&\left|p_{u_k}(t,y) - p_{u_k}(s,y)\right| \\
&\leqslant C_g \left| \int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(y)\right| \, d\tau - \int_{s-h}^{s} \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(y)\right| \, d\tau \right| \\
&\leqslant C_g \left| \int_{s}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(y)\right| \, d\tau - \int_{s-h}^{t-h} \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(y)\right| \, d\tau \right| \\
&\leqslant 2\gamma_1\gamma_2 C_g \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} |t-s|, \quad \forall\, t,s \in [0,T],
\end{aligned} \tag{2.7}$$

where $\gamma_2 = \sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(0,T;L^2(\Omega))}$.

As a result, utilizing the estimates (2.6)–(2.7), and setting

$$C := C_g \gamma_1 \left(1 + 2\gamma_2 \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\right), \tag{2.8}$$

we see that

$$\begin{aligned}
|p_{u_k}(t,x) - p_{u_k}(s,y)| &\leqslant |p_{u_k}(t,x) - p_{u_k}(t,y)| + |p_{u_k}(t,y) - p_{u_k}(s,y)| \\
&\leqslant C\left[|x-y| + |t-s|\right], \\
&\forall\, (t,x),(s,y) \in \overline{Q_T} := [0,T] \times \overline{\Omega}.
\end{aligned} \tag{2.9}$$

Thus, $\{p_{u_k}\} \subset \mathfrak{S}$. Since $\max_{(t,x)\in\overline{Q_T}} |p_{u_k}(t,x)| \leqslant p^+$ and each element of the sequence $\{p_{u_k}\}_{k\in\mathbb{N}}$ has the same modulus of continuity, it follows that this sequence is uniformly bounded and equi-continuous. Hence, by Arzelà–Ascoli Theorem the sequence $\{p_{u_k}\}_{k\in\mathbb{N}}$ is relatively compact with respect to the strong topology of $C(\overline{Q_T})$. Taking into account the estimate (2.9) and the fact that the set $\mathfrak{S}$ is closed with respect to the uniform convergence and

$$\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(x)\right| \, d\tau \to \frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(x)\right| \, d\tau$$
$$\text{as } k\to\infty,\ \forall\,(t,x)\in Q_T$$

by definition of the weak convergence in $L^1(0,T;L^1(\Omega))$, we deduce: $p_{u_k} \to p_u$ uniformly in $\overline{Q_T}$ as $k\to\infty$, where

$$p_u(t,x) = 1 + g\left(\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(\cdot)\right| \, d\tau\right)$$

in Q_T. The proof is complete. □

2.3. Anisotropic Diffusion Tensor

Let $E_I \in W^{1,1}(\Omega)$ be a given function. Then for each $\lambda \in \mathbb{R}$ the upper level set of E_I can be defined as follows

$$Z_\lambda(E_I) = \{E_I \geqslant \lambda\} := \{x \in \Omega \ : \ E_I(x) \geqslant \lambda\}.$$

It was proven in [8] that for each function $E_I \in W^{1,1}(\Omega)$ its upper level sets $Z_\lambda(E_I)$ are sets of finite perimeter. So, the boundaries of level sets can be described by a countable family of Jordan curves with finite length, i.e., by continuous maps from the circle into the plane $\mathbb{R}^2$ without crossing points. As a result, at almost all points of almost all level sets of $E_I \in W^{1,1}(\Omega)$ we can define a unit normal vector $\theta(x)$. This vector field formally satisfies the following relations

$$(\theta, \nabla E_I) = |\nabla E_I| \quad \text{and} \quad |\theta| \leqslant 1 \text{ a.e. in } \Omega.$$

In the sequel, we will refer to θ as the vector field of unit normals to the topographic map of a function E_I. In fact, this vector field can be defined by the rule $\theta(x) = \frac{\nabla U(t,x)}{|\nabla U(t,x)|}$ with $t > 0$ small enough, where $U(t,x)$ is a solution the following initial-boundary value problem

$$\frac{\partial U}{\partial t} = \operatorname{div}\left(\frac{\nabla U}{|\nabla U| + \delta}\right), \quad t \in (0, +\infty), \ x \in \Omega, \tag{2.10}$$

$$U(0,x) = E_I(x), \quad x \in \Omega, \tag{2.11}$$

$$\frac{\partial U(0,x)}{\partial \nu} = 0, \quad t \in (0, +\infty), \ x \in \partial\Omega \tag{2.12}$$

with a relaxed version of the $1D$-Laplace operator in the principle part of (2.10). Here, $\delta > 0$ is a sufficiently small positive value and it can be chosen as in (1.8).

Let $\eta \in (0,1)$ be a given threshold. For the simplicity, we set $\eta = 1 - \delta$. Then, we associate with the vector field $\theta : \Omega \to \mathbb{R}^2$ the following linear operator $R_\eta : \mathbb{R}^2 \to \mathbb{R}^2$:

$$R_\eta \nabla v := \nabla v - \eta^2 (\theta, \nabla v)\, \theta = \left[I - \eta^2 \theta \otimes \theta\right] \nabla v, \quad \forall\, v \in W^{1,1}(\Omega). \tag{2.13}$$

In fact, this operator can be interpreted as the Directional Total Variation of v along the vector field θ (see [16] for the details).

Remark 2.1. In practice, the function E_I is usually associated with the spectral energy for a smoothed version $I = [I_1, I_2, I_3]^t \in L^2(\Omega; \mathbb{R}^3)$ of the original color image which is presumably has been corrupted by some noise. The standard rule for that is the following one

$$E_I(x) := \alpha_1 I_1(x) + \alpha_2 I_2(x) + \alpha_3 I_3(x), \quad \forall\, x \in \Omega,$$

with $\alpha_1 = 0.114$, $\alpha_2 = 0.587$, and $\alpha_3 = 0.299$.

As for the operator $R_\eta : \mathbb{R}^2 \to \mathbb{R}^2$, in this case it accumulates the structural prior information about the spectral energy E_I. Indeed, let us assume that $x \in \Omega$

is a point in which E_I is not expected to change drastically in any direction, i.e. x is not close to a discontinuity or rapid change in the known structure of E_I. In this case, R_η can be represented as a unit matrix. So, at this point we obviously have $R_\eta \nabla v \approx \nabla v$.

On the other hand, if we consider a point that is close to a discontinuity of E_I, then $R_\eta \nabla v$ reduces to $(1-\eta^2)\nabla v$ if the gradient $\nabla v(t,x)$ at this point is co-linear to θ, and to $\nabla v(t,x)$ provided $\nabla v(t,x)$ is orthogonal to θ. So, this operator does not enforce gradients of v in the direction θ. Moreover, the following two-side estimate

$$(1-\eta^2)|\nabla v|^2 \leqslant |(\nabla v, R_\eta \nabla v)| \leqslant |\nabla v|^2, \quad \text{a.e. in } Q_T \tag{2.14}$$

holds for each $v \in L^\infty(0,T;W^{1,1}(\Omega))$. We also make use of the following observation: since $|\xi|^2 \leqslant \left(\xi, \left[I-\eta^2\theta\otimes\theta\right]^{-1}\xi\right) \leqslant (1-\eta^2)^{-1}|\xi|^2$ and

$$(1-\eta^2)|\nabla v|^2 \leqslant \left(R_\eta\nabla v, \left[I-\eta^2\theta\otimes\theta\right]^{-1} R_\eta \nabla v\right) \leqslant (1-\eta^2)^{-1}|R_\eta \nabla v|^2,$$
$$|R_\eta\nabla v|^2 \leqslant \left(R_\eta\nabla v, \left[I-\eta^2\theta\otimes\theta\right]^{-1} R_\eta \nabla v\right) \leqslant |\nabla v|^2$$

it follows that

$$(1-\eta^2)|\nabla v| \leqslant |R_\eta \nabla v| \leqslant |\nabla v|, \quad \text{a.e. in } Q_T. \tag{2.15}$$

2.4. On Orlicz Spaces

Let $w \in L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given function. Let $p_w : Q_T \to \mathbb{R}$ be the corresponding variable exponent which is defined by the rule (1.7). Then

$$1 < p^- \leqslant p_w(t,x) \leqslant p^+ < \infty \ \text{ a.e. in } \ Q_T \tag{2.16}$$

(see Lemma 2.1), where the constants p^- and p^+ are given by (2.2). Let $p_w'(t,x) = \frac{p_w(t,x)}{p_w(t,x)-1}$ be the corresponding conjugate exponent. It is clear that

$$2 = \underbrace{\frac{p^+}{p^+-1}}_{(p^+)'} \leqslant p_w'(t,x) \leqslant \underbrace{\frac{p^-}{p^--1}}_{(p^-)'} = \frac{p^-}{\delta} \ \ \textit{a.e. in } Q_T, \tag{2.17}$$

where $(p^+)'$ and $(p^-)'$ stand for the conjugates of constant exponents. Denote by $L^{p_w(\cdot)}(Q_T)$ the set of all measurable functions $f : Q_T \to \mathbb{R}$ such that the modular is finite, i.e.

$$\rho_{p_w(t,x)}(f) := \int_{Q_T} |f(t,x)|^{p_w(t,x)}\, dxdt < \infty. \tag{2.18}$$

Equipped with the Luxembourg norm

$$\|f\|_{L^{p_w(\cdot)}(Q_T)} = \inf\left\{\lambda > 0 \ : \ \int_{Q_T} |\lambda^{-1} f(t,x)|^{p_w(t,x)}\, dxdt \leqslant 1\right\}. \tag{2.19}$$

$L^{p_w(\cdot)}(Q_T)$ becomes a Banach space (see [24, 30] for the details). The space $L^{p_w(\cdot)}(Q_T)$ is a sort of Musielak-Orlicz space that can be denoted by generalised Lebesgue space, because many of its properties are inherited from the classical Lebesgue spaces. In particular, the two-sides inequality (2.16) implies that $L^{p_w(\cdot)}(Q_T)$ is reflexive, separable, and the set $C_0^\infty(Q_T)$ is dense in $L^{p_w(\cdot)}(Q_T)$. Moreover, under condition (2.16), $L^\infty(Q_T) \cap L^{p_w(\cdot)}(Q_T)$ is also dense in $L^{p_w(\cdot)}(Q_T)$.

Its dual can be identified with $L^{p'_w(\cdot)}(Q_T)$ and, therefore, any continuous functional $F = F(f)$ on $L^{p_w(\cdot)}(Q_T)$ has the form (see [58, Lemma 13.2])

$$F(f) = \int_{Q_T} fg\,dxdt, \quad \text{with} \;\; g \in L^{p'_w(\cdot)}(Q_T).$$

Since the relation between the modular (2.18) and the norm (2.19) that is not so direct as in the classical Lebesgue spaces, it can be proved, from its definitions in (2.18) and (2.19), that

$$\begin{aligned} \min\left\{\|f\|^{p^-}_{L^{p_w(\cdot)}(Q_T)}, \|f\|^{p^+}_{L^{p_w(\cdot)}(Q_T)}\right\} &\leqslant \rho_{p_w(t,x)}(f) \\ &\leqslant \max\left\{\|f\|^{p^-}_{L^{p_w(\cdot)}(Q_T)}, \|f\|^{p^+}_{L^{p_w(\cdot)}(Q_T)}\right\}, \\ \min\left\{\rho^{\frac{1}{p^-}}_{p_w(t,x)}(f), \rho^{\frac{1}{p^+}}_{p_w(t,x)}(f)\right\} &\leqslant \|f\|_{L^{p_w(\cdot)}(Q_T)} \\ &\leqslant \max\left\{\rho^{\frac{1}{p^-}}_{p_w(t,x)}(f), \rho^{\frac{1}{p^+}}_{p_w(t,x)}(f)\right\}. \end{aligned} \tag{2.20}$$

When proving some estimates the following consequence of (2.20) is very useful,

$$\begin{gathered} \|f\|^{p^-}_{L^{p_w(\cdot)}(Q_T)} - 1 \leqslant \int_{Q_T} |f(t,x)|^{p_w(t,x)}\,dxdt \leqslant \|f\|^{p^+}_{L^{p_w(\cdot)}(Q_T)} + 1, \\ \forall\, f \in L^{p_w(\cdot)}(Q_T), \end{gathered} \tag{2.21}$$

$$\begin{gathered} \|f_k - f\|_{L^{p_w(\cdot)}(Q_T)} \to 0 \quad \Longleftrightarrow \quad \int_{Q_T} |f_k(t,x) - f(t,x)|^{p_w(t,x)}\,dxdt \to 0 \\ \text{as } k \to \infty. \end{gathered} \tag{2.22}$$

Moreover, if $f \in L^{p_w(\cdot)}(Q_T)$ then

$$\|f\|_{L^{p^-}(Q_T)} \leqslant (1 + T|\Omega|)^{1/p^-} \|f\|_{L^{p_w(\cdot)}(Q_T)}, \tag{2.23}$$

$$\|f\|_{L^{p_w(\cdot)}(Q_T)} \leqslant (1 + T|\Omega|)^{1/(p^+)'} \|f\|_{L^{p^+}(Q_T)}, \tag{2.24}$$

$$(p^+)' = \frac{p^+}{p^+ - 1}, \;\forall\, f \in L^{p^+}(Q_T),$$

(see, for instance, [24, 30, 57]).

In generalised Lebesgue spaces, there holds a version of Young's inequality,

$$|fg| \leqslant \varepsilon \frac{|f|^{p_w(\cdot)}}{p_w(\cdot)} + C(\varepsilon) \frac{|g|^{p'_w(\cdot)}}{p'_w(\cdot)},$$

valid for some positive constant $C(\varepsilon)$ and any $\varepsilon > 0$.

The following result can be viewed as an analogous of the Hölder inequality in Lebesgue spaces with variable exponents (for the details we refer to [24, 30]).

Proposition 2.1. If $f \in L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)$ and $g \in L^{p'_w(\cdot)}(Q_T;\mathbb{R}^2)$, then $(f,g) \in L^1(Q_T)$ and

$$\int_{Q_T} (f,g)\,dxdt \leqslant 2\|f\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)}\|g\|_{L^{p'_w(\cdot)}(Q_T;\mathbb{R}^2)}. \tag{2.25}$$

As a consequence of (2.25), we have, for a bounded domain $Q_T = (0,T)\times\Omega$ and $p_w(\cdot)$ satisfying to (2.16), the following continuous imbedding

$$L^{p_w(\cdot)}(Q_T) \hookrightarrow L^{r(\cdot)}(Q_T) \quad \text{whenever} \quad p_w(t,x) \geqslant r(t,x) \quad \text{for a.e. } (t,x)\in Q_T. \tag{2.26}$$

Let $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$, with some $\widehat{\delta}\in(0,1]$, be a given sequence of exponents. Hereinafter in this subsection we assume that

$$\begin{gathered} p, p_k \in C^{0,\widehat{\delta}}(\overline{Q_T}) \quad \text{for } k = 1,2,\dots, \text{ and} \\ p_k(\cdot) \to p(\cdot) \quad \text{uniformly in } \overline{Q_T} \text{ as } k\to\infty. \end{gathered} \tag{2.27}$$

We associate with this sequence the another one $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. The characteristic feature of this set of functions is that each element f_k lives in the corresponding Orlicz space $L^{p_k(\cdot)}(Q_T)$. So, we have a sequence in the scale of spaces $\left\{L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. We say that the sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty}\int_{Q_T} |f_k(t,x)|^{p_k(t,x)}\,dxdt < +\infty. \tag{2.28}$$

Definition 2.1. A bounded sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ is weakly convergent in the variable Orlicz space $L^{p_k(\cdot)}(Q_T)$ to a function $f \in L^{p(\cdot)}(Q_T)$, where $p \in C^{0,\delta}(\overline{Q_T})$ is the limit of $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$ in the uniform topology of $C(\overline{Q_T})$, if

$$\lim_{k\to\infty}\int_{Q_T} f_k\varphi\,dxdt = \int_{Q_T} f\varphi\,dxdt, \quad \forall\,\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^2). \tag{2.29}$$

For our further analysis, we make use of the following result concerning the lower semicontinuity property of the variable $L^{p_k(\cdot)}$-norm with respect to the weak convergence in $L^{p_k(\cdot)}(Q_T)$ (for the proof, we refer to [23, Lemma 3.1], see also [58, Lemma 13.3] and [41, Lemma 2.1] for comparison).

Proposition 2.2. If the sequence of exponents $\{p_k\}_{k\in\mathbb{N}}$ satisfies condition (2.16), $p_k \to p$ as $k\to\infty$ a.e.in Q_T, and a bounded sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ converges weakly in $L^{p^-}(Q_T)$ to f, then $f\in L^{p(\cdot)}(Q_T)$, $f_k \rightharpoonup f$ in variable $L^{p_k(\cdot)}(Q_T)$, and

$$\liminf_{k\to\infty}\int_{Q_T} |f_k(t,x)|^{p_k(t,x)}\,dxdt \geqslant \int_{Q_T} |f(t,x)|^{p(t,x)}\,dxdt. \tag{2.30}$$

We recall also the inequality which is classical in the theory of p-Laplace equations: if $1 < p \leqslant 2$ then, for all $\xi, \eta \in \mathbb{R}^N$, the following estimate holds true

$$(p-1)|\xi - \eta|^2 \leqslant \left(\left[|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right], \xi - \eta\right) \left(|\xi|^p + |\eta|^p\right)^{\frac{2-p}{p}}.$$

2.5. On Weighted Energy Space with Variable Exponent

Let $R_\eta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by the rule (2.13) and associated with some vector field $\theta \in L^\infty(\Omega; \mathbb{R}^2)$. Let $w \in C([0,T]; L^2(\Omega))$ be a given function. We define the weighted energy space $W_w(Q_T)$ as the set of all functions $u(t,x)$ such that

$$\begin{gathered} u \in L^2(Q_T), \quad u(t,\cdot) \in W^{1,1}(\Omega) \text{ for a.e.} t \in [0,T], \\ \int_{Q_T} |R_\eta \nabla u|^{p_w(t,x)}\, dxdt < +\infty. \end{gathered} \tag{2.31}$$

We equip $W_w(Q_T)$ with the norm

$$\|u\|_{W_w(Q_T)} = \|u\|_{L^2(Q_T)} + \|R_\eta \nabla u\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)}, \tag{2.32}$$

where the second term on the right-hand side is the norm of the vector-valued function $R_\eta \nabla u(t,x)$ in the Orlicz space $L^{p_w(\cdot)}(Q_T; \mathbb{R}^2)$. Due to the estimate (2.14), we see that $W_w(Q_T)$, equipped with the norm (2.32), is a reflexive Banach space. Moreover, due to the fact that the exponent $p_w : Q_T \to \mathbb{R}$ is Lipschitz continuous, the smooth functions are dense in the weighted Sobolev-Orlicz space $W_w(Q_T)$ (see [4]). So, $W_w(Q_T)$ can be considered as the closure of the set $\left\{\varphi \in C^\infty(\overline{Q}_T)\right\}$ with respect to the norm $\|\cdot\|_{W_w(Q_T)}$.

2.6. On the weak convergence of fluxes to flux

Let us consider the following collection of parabolic equations of monotone type

$$\frac{\partial u_k}{\partial t} - \operatorname{div} A_k(t,x,\nabla u_k) = f, \quad (t,x) \in Q_T, \tag{2.33}$$

where $f \in L^2(\Omega)$ and $k = 1, 2, \ldots$. Let u_k be a solution of (2.33) for a given $k \in \mathbb{N}$ and this solution is understood in the sense of distributions. Assume that $A_k(\cdot,\cdot,\xi) \to A(\cdot,\cdot,\xi)$ as $k \to \infty$ pointwise a.e. with respect to the first two arguments and for all $\xi \in \mathbb{R}^N$.

A typical situation arising in the study of most optimization problems and which is of fundamental importance in many others areas of nonlinear analysis, can be stated as follows: suppose it is known that a solution $u_k \in L^2(0,T; W^{1,p^-}(\Omega))$ of (2.33) and the corresponding flow $w_k = A_k(\cdot,\cdot,\nabla u_k) \in L^{(p^+)'}(Q_T;\mathbb{R}^N)$ converge weakly, namely,

$$\begin{gathered} u_k \rightharpoonup u \text{ in } L^2(0,T; W^{1,p^-}(\Omega)), \quad w_k \rightharpoonup w \text{ in } L^{(p^+)'}(Q_T;\mathbb{R}^N), \\ 1 < p^- < p^+, \ (p^+)' = \frac{p^+}{p^+ - 1}. \end{gathered}$$

The main question is whether a flux converges to a flux, i.e., whether the equality for the limit elements $A(t,x,\nabla u) = w$ holds. The situation is not trivial because the function $A(\cdot,\cdot,v)$ is nonlinear in v and the weak convergence $v_k \rightharpoonup v$ is far from sufficient to derive the limit relation $A_k(\cdot,\cdot,v_k) \rightharpoonup A(\cdot,\cdot,v)$. So, the important problem is to show that $w = A(\cdot,\cdot,\nabla u)$, although the validity of this equality is by no means obvious at this stage. The conditions (first of all, on the exponents p^- and p^+) under which the answer to the above question is affirmative, have been obtained by Zhikov and Pastukhova in their celebrated paper [60].

Theorem 2.1. *Assume that the following conditions are satisfied:*

(C1) $A_k(t,x,\xi)$ *and* $A(t,x,\xi)$ *are* $\mathbb{R}^N$*-valued Carathéodory functions, that is, these functions are continuous in* $\xi \in \mathbb{R}^N$ *for a.e.* $(t,x) \in Q_T$ *and measurable with respect to* $(t,x) \in Q_T$ *for each* $\xi \in \mathbb{R}^N$*;*

(C2) $\Big(A_k(t,x,\xi) - A_k(t,x,\zeta), \xi - \zeta\Big) \geqslant 0,\ A_k(t,x,0) = 0\ \ \forall \xi,\zeta \in \mathbb{R}^N$ *and for a.e.* $(t,x) \in Q_T$*;*

(C3) $|A_k(t,x,\xi)| \leqslant c(|\xi|) < \infty$ *and* $\lim_{k\to\infty} A_k(t,x,\xi) = A(t,x,\xi)$ *for all* $\xi \in \mathbb{R}^N$ *and for a.e.* $(t,x) \in Q_T$*;*

(C4) $u_k \rightharpoonup u$ *in* $L^{p^-}(0,T;W^{1,p^-}(\Omega))$, $p^- > 1$, *and* $\{u_k\}_{k\in\mathbb{N}}$ *are bounded in* $L^\infty(0,T;L^2(\Omega))$*;*

(C5) $w_k = A_k(t,x,\nabla u_k) \rightharpoonup w$ *in* $L^{(p^+)'}(Q_T;\mathbb{R}^N)$, $p^+ > 1$*;*

(C6) $u_k \in L^{p^+}(0,T;W^{1,p^+}(\Omega))$ *for all* $k \in \mathbb{N}$, *and* $\sup_{k\in\mathbb{N}} \|\,(w_k,\nabla u_k)\,\|_{L^1(Q_T)} < \infty$*;*

(C7) $1 < p^- < p^+ < 2p^-$.

Then the flux $A_k(t,x,\nabla u_k)$ *weakly converges in the Lebesgue space* $L^{(p^+)'}(Q_T;\mathbb{R}^N)$ *to the flux* $A(t,x,\nabla u)$.

For our further analysis, we make also use of the following well-known results.

Lemma 2.2 ([57])**.** *Let* Ψ *be a class of integrands* $F(t,x,\xi)$ *that are convex with respect to* $\xi \in \mathbb{R}^N$, *measurable with respect to* $(t,x) \in Q_T$, *and satisfy the estimate*

$$c_1|\xi|^{p^-} \leqslant F(t,x,\xi) \leqslant c_2|\xi|^{p^+}, \quad 1 < p^- \leqslant p^+ < \infty,\ c_1, c_2 > 0.$$

Suppose that F_k *and* F *belong to the class* Ψ *and the following condition holds:*

$$\lim_{k\to\infty} F_k(t,x,\xi) = F(t,x,\xi) \quad \text{for a.e. } (t,x) \in Q_T \text{ and any } \xi \in \mathbb{R}^N.$$

Then the following lower semicontinuity property is valid:
if $v_k \rightharpoonup v$ *in* $L^1(Q_T;\mathbb{R}^N)$ *then*

$$\liminf_{k\to\infty} \int_{Q_T} F_k(t,x,v_k)\,dxdt \geqslant \int_{Q_T} F(t,x,v)\,dxdt. \tag{2.34}$$

Lemma 2.3 ([59]). *Let $A_k(t,x,\xi)$ and $A(t,x,\xi)$ be $\mathbb{R}^N$-valued Carathéodory functions with properties (C1)–(C3). Assume that*

$$v_k \rightharpoonup v \quad \text{and} \quad w_k = A_k(t,x,v_k) \rightharpoonup w \quad \text{in } L^1(Q_T;\mathbb{R}^N) \text{ as } k \to \infty,$$

and $(w,v) \in L^1(Q_T)$. Then

$$\liminf_{k\to\infty} \int_{Q_T} (A_k(t,x,v_k), v_k)\, dxdt \geqslant \int_{Q_T} (w,v)\, dxdt. \tag{2.35}$$

Lemma 2.4 ([4]). *Let ε be a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. Assume that the following conditions*

(*i*) $p_\varepsilon, p \in C(\overline{Q_T})$, $\quad p_\varepsilon \to p$ *in* $C(\overline{Q_T})$ *as* $\varepsilon \to 0$,

(*ii*) $v_\varepsilon \in L^1(Q_T;\mathbb{R}^N)$, $\quad \displaystyle\int_{Q_T} \left[|v_\varepsilon|^{p_\varepsilon} + \varepsilon |v_\varepsilon|^{p^+}\right] dxdt \leqslant K < \infty$ *for each* $\varepsilon > 0$,

(*iii*) $|v_\varepsilon|^{p_\varepsilon - 2} v_\varepsilon + \varepsilon |v_\varepsilon|^{p^+ - 2} v_\varepsilon \rightharpoonup z$ *in* $L^{(p^+)'}(Q_T;\mathbb{R}^N)$,
$(p^+)' = p^+/(p^+ - 1)$ *as* $\varepsilon \to 0$

hold true with some p^- and p^+ such that $1 < p^- \leqslant p_\varepsilon(t,x) \leqslant p^+ < \infty$ for all $\varepsilon > 0$ and $(t,x) \in Q_T$. Then $z \in L^{p'(\cdot)}(Q_T;\mathbb{R}^N)$.

3. Existence Result for a Class of Parabolic Equations with Variable Nonlocal Exponent

The main object of our consideration in this section is the following initial-boundary value problem (IBVP)

$$\frac{\partial u}{\partial t} - \operatorname{div} A_u(t,x,\nabla u) + \kappa u = \kappa(f - v) \quad \text{in } Q_T, \tag{3.1}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{3.2}$$

$$u(0,\cdot) = f_0 \quad \text{in } \Omega. \tag{3.3}$$

Here,

$$A_w(t,x,\nabla u) := |R_\eta \nabla u|^{p_w(t,x)-2} R_\eta \nabla u, \tag{3.4}$$

the exponent $p_w : Q_T \to (1,2]$ is given by the rule (1.7), the linear operator R_η is defined in (2.13), ∂_ν stands for the outward normal derivative, $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$ are given distributions, $v \in \mathcal{V}_{ad}$ stands for the control, and the class of admissible controls $\mathcal{V}_{ad}$ is defined as

$$\mathcal{V}_{ad} = \left\{ v \in L^2(0,T;L^1(\Omega)) \;:\; v_a(x) \leqslant v(t,x) \leqslant v_b(x), \text{ a.e. in } Q_T \right\}. \tag{3.5}$$

As follows from (3.4), (2.14), and Lemma 2.1, for each fixed function $w \in C([0,T];L^2(\Omega))$, the mapping $(t,x,\xi) \mapsto A_w(t,x,\xi)$ is a Carathéodory vector function, that is, $A_w(t,x,\xi)$ is continuous in $\xi\in\mathbb{R}^2$ and is measurable with respect to (t,x) for each $\xi\in\mathbb{R}^2$. Moreover, the following monotonicity, coerciveness and boundedness conditions hold for a.e. $(t,x)\in Q_T$ [58]:

$$\Big(A_w(t,x,\xi)-A_w(t,x,\zeta),\xi-\zeta\Big)\geqslant 0,\quad \forall\,\xi,\zeta\in\mathbb{R}^2, \tag{3.6}$$

$$\begin{aligned}(A_w(t,x,\xi),\xi)&=|R_\eta\xi|^{p_w(t,x)-2}\left(R_\eta\xi,R_\eta^{-1}R_\eta\xi\right)\\ &\overset{\text{by (2.15)}}{\geqslant}\left(1-\eta^2\right)^{p_w(t,x)}|\xi|^{p_w(t,x)}\geqslant\left(1-\eta^2\right)^2|\xi|^{p_w(t,x)},\ \forall\,\xi\in\mathbb{R}^2,\end{aligned} \tag{3.7}$$

$$|A_w(t,x,\xi)|^{p_w'(t,x)}\leqslant|\xi|^{p_w(t,x)},\quad \forall\,\xi\in\mathbb{R}^2, \tag{3.8}$$

However, in general, the principle operator $-\operatorname{div}\,A_u(t,x,\nabla u)+\kappa u$ provides an example of a strongly non-linear, non-monotone, and non-coercive operator in divergence form.

It is worth mentioning here that if the exponent $p=p(t,x)$ is a given function (i.e., it does not depend on the unknown solution u) and $p\in C^{0,\delta}(\overline{Q_T})$, with some $\delta\in(0,1]$, then for every $f\in L^2(Q_T)$, $f_0\in L^2(\Omega)$, and $v\in\mathcal{V}_{ad}$, problem (3.1), (3.3) with $R_\eta=I$ and with zero Dirichlet boundary conditions (instead of the Neumann one (3.2)) admits a weak solution $u\in C([0,T];L^2(\Omega))$ such that $\int_{Q_T}|\nabla u|^{p(t,x)}\,dxdt<+\infty$ (see, e.g. [10, Ch.4]). In this case the time derivative of the weak solution is a distribution u_t which may not belong to any Lebesgue space $L^s(Q_T)$ with $s>1$. Moreover, the issue of uniqueness for the weak solutions remains, apparently, an open question for nowadays [35, Chapter III].

As for the case of Dirichlet problem for the equation (3.1) with $p_u(t,x)$ given by (1.7), its regularity (see Lemma 2.1) is insufficient for the convergence of the sequence of Galerkin's approximations to a weak solution. To overcome this difficulty, it was recently proposed in [11] to construct the strong solutions with the extra regularity property $u_t\in L^2(Q_T)$. However, the existence of a strong solution and its uniqueness to the Dirichlet problem for the equation (3.1) has been proven in [11] if only the following condition for the range of the exponent $p_u(t,x)$ holds true

$$\frac{2N}{2+N}<p^-\leqslant p^+<2,\quad \text{where}\ \ N=\dim\,\Omega.$$

Since the fulfillment of this condition is rather questionable in our case (see Lemma 2.1), our prime interest in this section is to study the solvability issues for Cauchy-Neumann initial-boundary value problem (3.1)–(3.3) with $p_u(t,x)$ given by (1.7). We recall that a challenging feature of the equation (3.1) is that it cannot be interpreted as a duality relation in a fixed Banach space. Because of this, we can not write down the weak formulation of (3.1)–(3.3) as some equality in terms of duality. In particular, sequences of solutions u_k to this problem that correspond to different exponents p_{u_k}, belong to possible distinct Sobolev spaces. Mainly because of this, we specify the notion of weak solution as follows:

Definition 3.1. We say that, for given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $\theta \in L^\infty(\Omega;\mathbb{R}^2)$, and $v \in \mathcal{V}_{ad}$, a function u is a weak solution to the problem (3.1)–(3.3) if $u \in W_u(Q_T)$, i.e.,

$$\begin{gathered} u \in L^2(Q_T),\ u(t,\cdot) \in W^{1,1}(\Omega) \text{ for a.e. } t \in [0,T], \\ \int_{Q_T} |R_\eta \nabla u|^{p_u(t,x)}\,dxdt < +\infty, \end{gathered} \tag{3.9}$$

and the integral identity

$$\int_{Q_T} \left(-u\frac{\partial \varphi}{\partial t} + (A_u(t,x,\nabla u), \nabla\varphi) + \kappa u\varphi \right) dxdt \\ = \kappa \int_{Q_T} (f - v)\varphi\,dxdt + \int_\Omega f_0 \varphi|_{t=0}\,dx \tag{3.10}$$

holds true for any function $\varphi \in \Phi$, where $\Phi = \left\{\varphi \in C^\infty(\overline{Q}_T)\ :\ \varphi|_{t=T} = 0\right\}$.

To clarify the sense in which the initial value $u(0,\cdot) = f_0$ is assumed for the weak solutions, we give the following assertion (for the proof, we refer to [41, Proposition 2.2]).

Proposition 3.1. Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $\theta \in L^\infty(\Omega;\mathbb{R}^2)$, and $v \in \mathcal{V}_{ad}$ be given distributions. Let $u \in W_u(Q_T)$ be a weak solution to the problem (3.1)–(3.3) in the sense of Definition 3.1. Then, for any $\eta \in C^\infty(\overline{\Omega})$, the scalar function $h(t) = \int_\Omega u(t,x)\eta(x)\,dx$ belongs to $W^{1,1}(0,T)$ and $h(0) = \int_\Omega f_0(x)\eta(x)\,dx$.

Let us show that the problem (3.1)–(3.3) admits at least one weak solution. With that in mind, we make use of the perturbation technique and a classical fixed point theorem of Schauder [49] (we refer to [25, 33, 39, 42, 45] where the similar technique has been used).

We begin with the following auxiliary results. Following result is crucial in this section.

Theorem 3.1. *For given functions $w \in L^\infty(0,T;L^2(\Omega))$ and $\theta \in L^\infty(\Omega;\mathbb{R}^2)$, let the exponent $p_w : Q_T \to \mathbb{R}$ and the linear operator R_η be defined by the rules* (1.7) *and* (2.13)*, respectively. Then there exists a positive constant Λ such that, for a.a $(t,x) \in Q_T$ and for $\varepsilon > 0$ small enough, the following inequality holds true*

$$(A_w^\varepsilon(t,x,\nabla u), \nabla u) \geqslant \begin{cases} \Lambda |\nabla u|^{p_w(t,x)}, & \text{if } |\nabla u| \geqslant 1, \\ \Lambda \left(|\nabla u|^{p_w(t,x)} - 4 \right), & \text{if } |\nabla u| < 1, \end{cases} \quad \text{a.e. in } Q_T, \tag{3.11}$$

where ε is a small positive value and

$$A_w^\varepsilon(t,x,\nabla u) := (|R_\eta \nabla u| + \varepsilon)^{p_w(t,x)-2}\, R_\eta \nabla u.$$

Proof. Taking into account that (see (2.15))

$$(1-\eta^2)|\zeta| \leqslant |R_\eta \zeta| \leqslant |\zeta|, \quad \forall\, \zeta \in \mathbb{R}^2,\ \forall\,(t,x) \in Q_T,$$

we make use of the following chain of inequalities

$$\begin{aligned}(A_w^\varepsilon(t,x,\nabla u),\nabla u) &= (|R_\eta \nabla u| + \varepsilon)^{p_w(t,x)-2}\,(R_\eta \nabla u, \nabla u)\\ &\overset{\text{by (2.14)}}{\geqslant} \left(1-\eta^2\right)\frac{|R_\eta \nabla u|^2}{(|R_\eta \nabla u|+\varepsilon)^{2-p_w(t,x)}}\\ &\overset{\text{by (2.15)}}{\geqslant} \left(1-\eta^2\right)^3\frac{|\nabla u|^2}{(|\nabla u|+\varepsilon)^{2-p_w(t,x)}} \quad \text{a.e. in } \ Q_T. \end{aligned} \tag{3.12}$$

To deduce the proof, it remains to distinguish two cases $|\nabla u| \geqslant 1$ and $\nabla u| < 1$ (see Lemma 1 in [56, Lemma 1]). As a result, we see that, for all $\varepsilon > 0$,

$$\begin{aligned}(A_w^\varepsilon(t,x,\nabla u),\nabla u) &\geqslant \frac{\left(1-\eta^2\right)^3}{2^{2-p_w(t,x)}}|\nabla u|^{p_w(t,x)}\\ &\geqslant \frac{\left(1-\eta^2\right)^3}{2}|\nabla u|^{p^-}, \quad \text{if } \ |\nabla u| \geqslant 1. \end{aligned} \tag{3.13}$$

At the same time, if $|\nabla u| < 1$, then we get

$$\begin{aligned}(A_w^\varepsilon(t,x,\nabla u),\nabla u) &= \left(A_w^\varepsilon(t,x,\nabla u), \left[I-\eta^2\theta\otimes\theta\right]^{-1} R_\eta \nabla\right)\\ &\geqslant \left(1-\eta^2\right)|\nabla u|^2\,(1+|\nabla u|)^{p_w(t,x)-2}\\ &= \left(1-\eta^2\right)(|\nabla u|+1-1)^2\,(1+|\nabla u|)^{p_w(t,x)-2}\\ &\geqslant \left(1-\eta^2\right)\left(|\nabla u|^{p_w(t,x)} - 2\,(1+|\nabla u|)^{p_w(t,x)-1}\right)\\ &\geqslant \left(1-\eta^2\right)\left(|\nabla u|^{p_w(t,x)} - 4\right) \quad \text{a.e. in } \ Q_T. \end{aligned} \tag{3.14}$$

□

Theorem 3.2. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$ be given distributions, and $\theta \in L^\infty(\Omega;\mathbb{R}^2)$ is some vector field. Then, for each positive value $\varepsilon > 0$, the Cauchy-Neumann problem*

$$\frac{\partial u}{\partial t} - \varepsilon\Delta u - \operatorname{div} A_u^\varepsilon(t,x,\nabla u) + \kappa u = \kappa(f-v) \quad \text{in } \ Q_T := (0,T)\times\Omega, \tag{3.15}$$

$$\partial_\nu u = 0 \quad \text{on } \ (0,T)\times\partial\Omega, \tag{3.16}$$

$$u(0,\cdot) = f_0 \quad \text{in } \ \Omega, \tag{3.17}$$

has a weak solution $u_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ verifying (3.15)–(3.17) in the sense of distributions.

Proof. We introduce the space

$$W(0,T) = \left\{ w \in L^2(0,T; W^{1,2}(\Omega)),\ \frac{dw}{dt} \in L^2(0,T; \left[W^{1,2}(\Omega)\right]') \right\}.$$

This space is a Hilbert space with respect to the graph norm. Let us fix an arbitrary function $w \in W(0,T) \cap L^\infty(0,T; L^2(\Omega))$ such that

$$\left.\begin{aligned} \|w\|_{L^2(0,T;W^{1,2}(\Omega))} &\leqslant C_1, \\ \|w\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \sqrt{2\kappa} C_1, \\ \|\tfrac{\partial w}{\partial t}\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\leqslant C_3, \\ w(0,\cdot) &= f_0 \ \text{ in } \ \Omega, \end{aligned}\right\} \tag{3.18}$$

with

$$C_1 = \sqrt{\|f - v\|^2_{L^2(Q_T)} + \frac{2}{\kappa}\|f_0\|^2_{L^2(\Omega)}}\,,$$
$$C_2 = \left(\Lambda^{-1}\kappa C_1^2 + 5\right)^{1/p^-},$$

where the constant Λ comes from inequality (3.11), and C_3 is defined in (3.31).

We divide the proof onto several steps.

Step 1. Let us associate with w the following variational problem: Find $u = U_\varepsilon(w) \in W(0,T)$ satisfying

$$\left\langle \frac{\partial u(t)}{\partial t}, \psi \right\rangle + \int_\Omega \left[\varepsilon\left(\nabla u(t), \nabla \psi\right) + \left(A_w^\varepsilon(t,x,\nabla u(t)), \nabla\psi\right) + \kappa u(t)\psi\right] dx$$
$$= \kappa \int_\Omega \left(f(t) - v(t)\right)\psi\, dx, \quad \forall\, \psi \in W^{1,2}(\Omega) \quad \text{a.e. in } [0,T], \tag{3.19}$$
$$u(0) = f_0. \tag{3.20}$$

Since

$$\text{the condition } w \in W(0,T) \text{ implies } w \in C([0,T]; L^2(\Omega)) \tag{3.21}$$

(see [29, Chapter XVIII]), it follows from Lemma 2.1 that the corresponding exponent

$$p_w := 1 + g\left(\frac{1}{h}\int_{t-h}^t \left|(\nabla G_\sigma * w(\tau,\cdot))\right|^2 d\tau\right)$$

is such that $p_w \in C^{0,1}(Q_T)$ and $1 < p^- \leqslant q(\cdot,\cdot) \leqslant p^+$ in $\overline{Q_T}$, with $p^+ = 2$ and $p^- = 1 + \delta$, where (see the proof of Lemma 2.1)

$$\delta = g\left(\|G_\sigma\|_{C^1(\overline{\Omega - \Omega})}\frac{1}{h}\|w\|_{L^1(0,T;L^1(\Omega))}\right).$$

Taking into account that the anisotropic diffusion tensor R_η satisfies the two-side inequality (2.14), it is easy to deduce that, for a given value $\varepsilon > 0$, the principle operator $B : L^2(0,T;W^{1,2}(\Omega)) \to L^2(0,T;\left(W^{1,2}(\Omega)\right)')$, defined by the rule

$$\langle Bu, q\rangle = \int_{Q_T} (\varepsilon\nabla u + A^\varepsilon_w(t,x,\nabla u), \nabla q)\, dxdt + \kappa\int_{Q_T} uq\, dxdt,$$

is coercive, monotone, and hemicontinuous, where the hemicontinuity property means the continuity of the scalar function

$$\begin{aligned} z(\lambda) &= \langle B(u+\lambda q), \varphi\rangle \\ &= \int_{Q_T} (\varepsilon(\nabla u + \lambda\nabla q) + A^\varepsilon_w(t,x,\nabla u + \lambda\nabla q), \nabla\varphi)\, dxdt \\ &\quad + \kappa\int_{Q_T} (u+\lambda q)\varphi\, dxdt, \quad \forall\, u, q, \varphi \in L^2(0,T;W^{1,2}(\Omega)) \end{aligned}$$

at the point $\lambda = 0$. Since A^ε_w is a Carathéodory functions, this property can be easily derived with the Lebesgue theorem and the following estimate

$$\begin{aligned} &|A^\varepsilon_w(t,x,\nabla u + \lambda\nabla q)||\nabla\varphi| \\ &\quad\leqslant \frac{1}{p'_w(t,x)}|A^\varepsilon_w(t,x,\nabla u+\lambda\nabla q)|^{p'_w(t,x)} + \frac{1}{p_w(t,x)}|\nabla\varphi|^{p_w(t,x)} \\ &\quad\overset{\text{by (2.2)}}{\leqslant} \frac{c_0}{2}\left(|\nabla u + \lambda\nabla q| + \varepsilon\right)^{p_w(t,x)-2}|\nabla u + \lambda\nabla q|^2 + \frac{1}{p^-}|\nabla\varphi|^{p_w(t,x)} \\ &\quad\leqslant c_1\left(|\nabla u|^{p_w(t,x)} + |\nabla q|^{p_w(t,x)} + 1\right) + \frac{1}{p^-}|\nabla\varphi|^{p_w(t,x)} \in L^1(Q_T). \end{aligned} \tag{3.22}$$

Hence, by the classical results on parabolic equations [46] (see also results of Alkhutov and Zhikov [4, 5]), we deduce that the problem (3.19)–(3.20) has a unique weak solution $U_\varepsilon(w) \in W(0,T)$ in the sense of distributions. Since the integral identity (3.19) is valid for all test functions $\psi = \psi(t,x)$ which are stepwise with respect to variable t, it follows that this identity remains true for all $\psi \in L^2(0,T;W^{1,2}(\Omega))$, and hence for all $\psi \in W^{1,2}(Q_T)$ such that $\psi(T,\cdot) = 0$. So, after integration by parts, one can easily deduces from (3.19) that the solution $U_\varepsilon(w)$ satisfies both the integral identity

$$\begin{aligned} &\int_{Q_T}\left(-U_\varepsilon(w)\frac{\partial\varphi}{\partial t} + (\varepsilon\nabla U_\varepsilon(w) + A^\varepsilon_w(t,x,\nabla U_\varepsilon(w)), \nabla\varphi) + \kappa U_\varepsilon(w)\varphi\right) dxdt \\ &\qquad = \kappa\int_{Q_T}(f-v)\varphi\, dxdt + \int_\Omega f_0\varphi|_{t=0}\, dx \quad \forall\,\varphi\in\Phi \end{aligned} \tag{3.23}$$

and the energy equality

$$\begin{aligned} &\frac{1}{2}\int_\Omega U^2_\varepsilon(w)\, dx \\ &\quad + \int_0^t\int_\Omega \left(\varepsilon|\nabla U_\varepsilon(w)|^2 + (A^\varepsilon_w(s,x,\nabla U_\epsilon(w)), \nabla U_\varepsilon(w)) + \kappa U^2_\varepsilon(w)\right) dxds \\ &\quad = \kappa\int_0^t\int_\Omega (f-v)U_\varepsilon(w)\, dxds + \int_\Omega f_0^2\, dx, \quad \forall\, t\in[0,T], \end{aligned} \tag{3.24}$$

where, in view of (3.21), the first term in (3.24) is well defined for each $t \in [0,T]$.
Step 2. Using (3.24), we see that

$$\frac{1}{2}\int_\Omega U_\varepsilon^2(w)\,dx + \kappa\int_0^t\int_\Omega U_\varepsilon^2(w)\,dxds \leqslant \frac{\kappa}{2}\|f-v\|^2_{L^2(Q_T)} + \frac{\kappa}{2}\|U_\epsilon(w)\|^2_{L^2(Q_T)} + \|f_0\|^2_{L^2(\Omega)}.$$

From this, (3.24), and (3.11), we derive the following estimates:

$$\|U_\varepsilon(w)\|^2_{L^2(Q_T)} \leqslant \|f-v\|^2_{L^2(Q_T)} + \frac{2}{\kappa}\|f_0\|^2_{L^2(\Omega)} =: C_1^2, \tag{3.25}$$

$$\begin{aligned}\|\nabla U_\varepsilon(w)\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)} &\overset{\text{by (2.21)}}{\leqslant} \left(\int_{Q_T}|\nabla U_\varepsilon(w)|^{p_w(t,x)}\,dxdt + 1\right)^{1/p^-}\\ &\leqslant \left(\Lambda^{-1}\left(\|f_0\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|f-v\|^2_{L^2(Q_T)} + \frac{\kappa}{2}\|U_\epsilon(w)\|^2_{L^2(Q_T)}\right)+5\right)^{1/p^-}\\ &\overset{\text{by (3.25)}}{\leqslant} \left(\Lambda^{-1}\kappa C_1^2+5\right)^{1/p^-} =: C_2,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\|U_\varepsilon(w)\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \sqrt{2\left(\|f_0\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|f-v\|^2_{L^2(Q_T)} + \frac{\kappa}{2}\|U_\varepsilon(w)\|^2_{L^2(Q_T)}\right)}\\ &\leqslant \sqrt{2\kappa}C_1,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\|\nabla U_\varepsilon(w)\|_{L^2(Q_T;\mathbb{R}^2)} &\leqslant \frac{1}{\sqrt{\varepsilon}}\sqrt{\|f_0\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|f-v\|^2_{L^2(Q_T)} + \frac{\kappa}{2}\|U_\varepsilon(w)\|^2_{L^2(Q_T)}}\\ &\leqslant \sqrt{\frac{\kappa}{\varepsilon}}C_1,\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\int_{Q_T}|A_w^\varepsilon(t,x,\nabla U_\varepsilon(w))|^{p'_w(t,x)}\,dxdt\\ &\qquad\overset{\text{by (3.25)}}{\leqslant} \left(\sqrt{2}\right)^{p'_w(t,x)}\int_{Q_T}\left(|\nabla U_\varepsilon(w)|+\varepsilon\right)^{p_w(t,x)}\,dxdt\\ &\qquad\overset{\text{by (3.26)}}{<} +\infty.\end{aligned} \tag{3.29}$$

We also notice that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}\left|\left\langle\frac{\partial U_\varepsilon(w)}{\partial t},\psi\right\rangle\right| &\overset{\text{by (3.19)}}{\leqslant} \sqrt{\varepsilon}\|\nabla U_\varepsilon(w)\|_{L^2(Q_T;\mathbb{R}^2)}\|\nabla\psi\|_{L^2(Q_T;\mathbb{R}^2)}\\ &\quad + 2\|A_w^\varepsilon(t,x,\nabla U_\varepsilon(w))\|_{L^{p'_w(\cdot)}(Q_T;\mathbb{R}^2)}\|\nabla\psi\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)}\\ &\quad + \kappa\|U_\varepsilon\|_{L^2(Q_T)}\|\psi\|_{L^2(Q_T)} + \kappa\|f-v\|_{L^2(Q_T)}\|\psi\|_{L^2(Q_T)}\end{aligned}$$

$$\overset{\text{by }(2.24)}{\leqslant} \Big[\sqrt{\varepsilon}\|\nabla U_\varepsilon(w)\|_{L^2(Q_T;\mathbb{R}^2)} + \kappa\|U_\varepsilon\|_{L^2(Q_T)} + \kappa\|f-v\|_{L^2(Q_T)}\Big]$$
$$\times \|\psi\|_{L^2(0,T;W^{1,2}(\Omega))}$$
$$+ \left(1 + \int_{Q_T} |A^\varepsilon_w(t,x,\nabla U_\varepsilon(w))|^{p'_w(t,x)}\, dxdt\right)^{1/2}$$
$$\times \left(1+T|\Omega|\right)^{1/2} \|\psi\|_{L^2(Q_T)}$$
$$\overset{\text{by }(3.25)\text{–}(3.29)}{\leqslant} C_3\|\psi\|_{L^2(0,T;W^{1,2}(\Omega))}, \quad \forall\, \psi \in L^2(0,T;W^{1,2}(\Omega)). \tag{3.30}$$

Hence,

$$\left\|\frac{\partial U_\varepsilon(w)}{\partial t}\right\|_{L^2(0,T;(W^{1,2}(\Omega))')} \leqslant C_3. \tag{3.31}$$

Taking into account these estimates, we introduce the following subset W_0 of the space $W(0,T)$

$$W_0 = \left\{ z \in W(0,T) \;\middle|\; \begin{array}{c} \|z\|_{L^2(0,T;W^{1,2}(\Omega))} \leqslant \left(1+\sqrt{\frac{\kappa}{\varepsilon}}\right) C_1, \\ \|z\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \sqrt{2\kappa} C_1, \\ \|\frac{\partial z}{\partial t}\|_{L^2(0,T;(W^{1,2}(\Omega))')} \leqslant C_3, \\ z(0,\cdot) = f_0 \end{array} \right\}$$

In view of estimates (3.25)–(3.31) and condition (3.18), it is clear that $w \in W_0$ and, hence, U_ϵ can be interpreted as a mapping from W_0 into W_0. Moreover, we see that W_0 is a nonempty, convex, and weakly compact subset of $W(0,T)$. Moreover, in view of the fact that the embedding of $W^{1,2}(\Omega)$ in $L^2(\Omega)$ is compact, a refinement of Aubin's lemma (see, e.g. [53, Section 8, Corollary 4] ensures that any bounded subset of $W(0,T)$ is relatively compact in $L^2(Q_T)$. So, in order to apply the Schauder fixed-point theorem, it remains to show that the mapping U_ε is weakly continuous from W_0 into W_0. As a result, the Schauder fixed-point theorem will provide the existence of element u_ε in W_0 such that $u_\varepsilon = U_\epsilon(u_\varepsilon)$.

Step 3. Let $\{w_j\}_{j\in\mathbb{R}}$ be a sequence in W_0 converging weakly in W_0 to some $w \in W_0$. Setting $u_{\varepsilon,j} = U_\varepsilon(w_j)$ and utilizing the weak compactness of the set W_0 and the Aubin's lemma, we see that $\{u_{\varepsilon,j}\}_{j\in\mathbb{R}}$ contains a subsequence such that

$$u_{\varepsilon,j} \rightharpoonup u_\varepsilon \quad \text{weakly in} \quad L^2(0,T;W^{1,2}(\Omega)), \tag{3.32}$$
$$u_{\varepsilon,j} \to u_\varepsilon \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \tag{3.33}$$
$$\frac{\partial u_{\varepsilon,j}}{\partial t} \rightharpoonup \frac{\partial u_\varepsilon}{\partial t} \quad \text{weakly in} \quad L^2(0,T;\left(W^{1,2}(\Omega)\right)'), \tag{3.34}$$
$$u_{\varepsilon,j} \to u_\varepsilon \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)) \text{ and a.e. in } Q_T, \tag{3.35}$$
$$\frac{\partial u_{\varepsilon,j}}{\partial x_i} \rightharpoonup \frac{\partial u_\varepsilon}{\partial x_i} \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)), \tag{3.36}$$
$$w_j \to w \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.37}$$

Then Lemma 2.1 implies that

$$p_{w_j}(t,x) \to p_w(t,x) \quad \text{uniformly in } \overline{Q_T} \text{ as } j \to \infty. \tag{3.38}$$

Moreover, taking into account that

$$\begin{aligned}\|A^{\varepsilon}_{w_j}(t,x,\nabla u_{\varepsilon,j})\|^{(p^+)'}_{L^{(p^+)'}(Q_T;\mathbb{R}^2)} &\overset{\text{by (2.23)}}{\leqslant} (1+T|\Omega|)\,\|A^{\varepsilon}_{w_j}(t,x,\nabla u_{\varepsilon,j})\|^{(p^+)'}_{L^{p'_{w_j}(\cdot)}(Q_T;\mathbb{R}^2)} \\ &\overset{\text{by (2.21)}}{\leqslant} (1+T|\Omega|)\left(1+\int_{Q_T}|A_{w_j}(t,x,\nabla u_{\varepsilon,j})|^{p'_{w_j}(t,x)}\,dxdt\right) \\ &\leqslant C\left(1+\int_{Q_T}|\nabla u_{\varepsilon,j}|^{p_{w_j}}\,dxdt\right) \overset{\text{by (3.26)}}{<} \infty,\end{aligned} \tag{3.39}$$

we deduce from (3.28) and (2.17) that the sequence $\left\{\varepsilon\nabla u_{\varepsilon,j} + A^{\varepsilon}_{w_j}(t,x,\nabla u_{\varepsilon,j})\right\}_{j\in\mathbb{R}}$ is bounded in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$. Hence, we can suppose that there exists an element $z \in L^{(p^+)'}(Q_T;\mathbb{R}^2)$ such that

$$\varepsilon\nabla u_{\varepsilon,j} + A^{\varepsilon}_{w_j}(t,x,\nabla u_{\varepsilon,j}) \rightharpoonup z \quad \text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \text{ as } j\to\infty. \tag{3.40}$$

Utilizing this fact together with the properties

$$\begin{aligned} u_{\varepsilon,j} \rightharpoonup u_\varepsilon \ \text{ in } L^{p^-}(0,T;W^{1,p^-}(\Omega)) \text{ with } p^- = 1+\delta \text{ (by (3.32))}, \\ \{u_{\varepsilon,j}\}_{j\in\mathbb{N}} \ \text{ are bounded in } L^\infty(0,T;L^2(\Omega)) \text{ (by (3.27))}, \\ u_{\varepsilon,j} \in L^{p^+}(0,T;W^{1,p^+}(\Omega)) \ \forall\, j\in\mathbb{N} \ \text{ by (3.28)}, \\ \sup_{j\in\mathbb{N}} \|\left(A^{\varepsilon}_{w_j}(t,x,\nabla u_{\varepsilon,j}),\nabla u_{\varepsilon,j}\right)\|_{L^1(Q_T)} < \infty \text{ (by (3.22))}, \end{aligned} \tag{3.41}$$

and taking into account that $1 < 1+\delta = p^- < p^+ = 2 < 2p^-$, we deduce from Theorem 2.1 that the flow $A^{\varepsilon}_{w_j}(t,x,\nabla u_{\epsilon,j})$ weakly converges in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ to the flow $A^{\varepsilon}_{w}(t,x,\nabla u_\varepsilon)$, i.e., $z = A^{\varepsilon}_{w}(t,x,\nabla u_\varepsilon)$.

Then we can pass to the limit in relations (3.19)–(3.20) with $u = u_{\varepsilon,j}$ and $w = w_j$ as $j\to\infty$. This yields

$$\left\langle \frac{\partial u_\varepsilon(t)}{\partial t},\psi\right\rangle + \int_\Omega \left[\varepsilon\left(\nabla u_\varepsilon(t),\nabla\psi\right) + \left(A^{\varepsilon}_{w}(t,x,\nabla u_\varepsilon(t)),\nabla\psi\right) + \kappa u_\varepsilon(t)\psi\right]dx$$

$$= \kappa\int_\Omega \left(f(t)-v(t)\right)\psi\,dx, \quad \forall\,\psi\in W^{1,2}(\Omega) \quad \text{a.e. in } [0,T], \tag{3.42}$$

$$u_\varepsilon(0) = f_0, \tag{3.43}$$

i.e., $u_\varepsilon = U_\varepsilon(w)$. Moreover, since variational problem (3.42)–(3.43) has a unique solution, it follows that the entire sequence $\{u_{\varepsilon,j}\}_{j\in\mathbb{R}}$ converges weakly in $W(0,T)$ to $u_\varepsilon = U_\varepsilon(w)$.

Thus, the mapping $U_\varepsilon : W_0 \mapsto W_0$ is weakly continuous and, hence, by the Schauder fixed point theorem, u_ε is a weak solution of the perturbed problem (3.15)–(3.17).

To the end of this proof, let us make use of the following observation: if u_ε is a weak solution to (3.15)–(3.17), then arguing as at the Step 1 and using the integration by parts formula, it is easily to deduce from (3.19) that u_ε satisfies the integral identity

$$\int_{Q_T}\left(-u_\varepsilon\frac{\partial\varphi}{\partial t}+\varepsilon\left(\nabla u_\varepsilon,\nabla\varphi\right)+\left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon),\nabla\varphi\right)+\kappa u_\varepsilon\varphi\right)dxdt$$
$$=\kappa\int_{Q_T}(f-v)\varphi\,dxdt+\int_\Omega f_0\varphi|_{t=0}\,dx\quad\forall\,\varphi\in\Phi.\quad(3.44)$$

□

Let us specify some extra properties of the weak solutions u_ε, given by Theorem 3.2.

Corollary 3.1. *Let $f\in L^2(Q_T)$, $f_0\in L^2(\Omega)$, $v\in\mathcal{V}_{ad}$, $\theta\in L^\infty(\Omega;\mathbb{R}^2)$, and $\varepsilon>0$ be given. Let $u_\varepsilon\in W(0,T)$ be a weak solution of* (3.15)–(3.17) *in the sense of distributions given by Theorem 3.2. Then $u_\varepsilon\in W_{u_\varepsilon}(Q_T)$ and the following energy equality holds*

$$\frac{1}{2}\int_\Omega u_\varepsilon^2\,dx+\int_0^t\int_\Omega\left(\varepsilon|\nabla u_\varepsilon|^2+\left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon),\nabla u_\varepsilon\right)+\kappa u_\varepsilon^2\right)dxds$$
$$=\kappa\int_0^t\int_\Omega(f-v)u_\varepsilon\,dxds+\int_\Omega f_0^2\,dx\quad\forall\,t\in[0,T].\quad(3.45)$$

Proof. Taking into account the definition of the space $W_{u_\varepsilon}(Q_T)$ (see (3.9)), let us show that

$$\int_{Q_T}|R_\eta\nabla u_\varepsilon|^{p_{u_\varepsilon}(t,x)}\,dxdt<+\infty\ \text{ for a.e. } t\in[0,T].$$

Since u_ε is a solution of (3.15)–(3.17) given by Theorem 3.2, it follows that there exists a sequence $\{u_{\varepsilon,j}\}_{j\in\mathbb{R}}\in W_0$ with properties (3.32)–(3.36) and such that $u_{\varepsilon,j}=U_\varepsilon(u_{\varepsilon,j-1})$, for $j=2,3,\dots$. Moreover, this sequence possesses the properties (3.40)–(3.41). Hence, $\nabla u_{\varepsilon,j}\in L^1(Q_T;\mathbb{R}^2)$ by (3.32), and

$$\int_{Q_T}|\nabla u_{\varepsilon,j}|^{p_{u_{\varepsilon,j}}}\,dxdt=\frac{1}{\Lambda}\int_{Q_T}\Lambda|\nabla u_{\varepsilon,j}|^{p_{u_{\varepsilon,j}}}\,dxdt>\infty$$
$$\leqslant\frac{1}{\Lambda}\sup_{j\in\mathbb{N}}\int_{Q_T}\left(\left|\left(A^\varepsilon_{u_{\varepsilon,j-1}}(t,x,\nabla u_{\varepsilon,j}),\nabla u_{\varepsilon,j}\right)\right|+4\right)dxdt$$
$$\overset{\text{by (3.41)}}{<}\infty.\quad(3.46)$$

Therefore,

$$\int_{Q_T} |R_\eta \nabla u_{\varepsilon,j}|^{p_{u_{\varepsilon,j}}(t,x)}\, dxdt \leqslant \int_{Q_T} |\nabla u_{\varepsilon,j}|^{p_{u_{\varepsilon,j}}(t,x)}\, dxdt < +\infty. \tag{3.47}$$

Then, by Proposition 2.2 and property $(3.41)_1$, we have:

$$\int_{Q_T} |R_\eta \nabla u_\varepsilon|^{p_{u_\varepsilon}(t,x)}\, dxdt \leqslant \liminf_{j\to\infty} \int_{Q_T} |R_\eta \nabla u_{\varepsilon,j}|^{p_{u_{\varepsilon,j}}(t,x)}\, dxdt < \infty.$$

Thus, $u_\varepsilon \in W_{u_\varepsilon}(Q_T)$.

It remains to prove the energy equality (3.45). Since u_ε is in $W(0,T)$ and the set of test functions $C^\infty([0,T]; C_c^\infty(\mathbb{R}^N))$ is dense in $L^2(0,T;W^{1,2}(\Omega))$, it follows that there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset C^\infty([0,T]; C_c^\infty(\mathbb{R}^N))$ such that

$$\varphi_j \to u_\varepsilon \quad \text{in } L^2(0,T;W^{1,2}(\Omega)) \quad \text{as } j \to \infty. \tag{3.48}$$

Taking into account that, for each $j \in \mathbb{N}$, the integral identity

$$\begin{aligned}&\int_0^t \int_\Omega \left[\varepsilon \left(\nabla u_\varepsilon(t), \nabla \varphi_j(t)\right) + \left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon(t)), \nabla\varphi_j(t)\right) + \kappa u_\varepsilon(t)\varphi_j(t)\right] dxdt \\ &+\int_0^t \left\langle \frac{\partial u_\varepsilon(t)}{\partial t}, \varphi_j(t)\right\rangle dt = \kappa \int_0^t \int_\Omega \left(f(t) - v(t)\right)\varphi_j(t)\, dxdt, \quad \forall\, j\in\mathbb{N},\ \forall\, t\in[0,T]\end{aligned} \tag{3.49}$$

holds true, we can pass to the limit in (3.49) as $j\to\infty$. To do so, we notice that

$$\begin{aligned}&\int_0^t \int_\Omega \left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon(t)), \nabla\varphi_j(t)\right) dxdt \\ &\qquad = \int_0^t \int_\Omega \left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon(t)), \nabla u_\varepsilon(t)\right) dxdt \\ &\qquad\qquad + \int_0^t \int_\Omega \left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon(t)), \nabla\varphi_j(t) - \nabla u_\varepsilon(t)\right) dxdt,\end{aligned}$$

where $\nabla\varphi_j - \nabla u_\varepsilon \to 0$ a.e. in Q_T by (3.48), and

$$\left|\left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon), \nabla\varphi_j - \nabla u_\varepsilon\right)\right| \leqslant c_1 |\nabla u_\varepsilon|^{p_{u_\varepsilon}} + \frac{1}{p^-}|\nabla\varphi_j - \nabla u_\varepsilon|^{p_{u_\varepsilon}} + c_1 \in L^1(Q_T)$$

by (3.22). Hence, by the Lebesgue dominated theorem, the limit passage in (3.49) leads to the equality

$$\begin{aligned}&\int_0^t \int_\Omega \left[\varepsilon \left(\nabla u_\varepsilon(t), \nabla u_\varepsilon(t)\right) + \left(A^\varepsilon_{w}(t,x,\nabla u_\varepsilon(t)), \nabla u_\varepsilon(t)\right) + \kappa u^2_\varepsilon(t)\right] dxdt \\ &\quad + \int_0^t \left\langle \frac{\partial u_\varepsilon(t)}{\partial t}, u_\varepsilon(t)\right\rangle dt = \kappa \int_0^t \int_\Omega \left(f(t) - v(t)\right) u_\varepsilon(t)\, dxdt, \quad \forall\, t\in[0,T].\end{aligned} \tag{3.50}$$

Thus, to obtain the energy equality (3.45), it remains to apply the integration by parts formula. □

For our further analysis, we make use of the following result.

Lemma 3.1. *Let $\{u_\varepsilon\}_{\varepsilon\to 0}\subset W(0,T)$ be a sequence such that*

$$\sup_{\varepsilon\to 0}\left(\varepsilon\int_{Q_T}|\nabla u_\varepsilon|^2\,dxdt\right)<+\infty. \tag{3.51}$$

Then $\varepsilon\nabla u_\varepsilon\rightharpoonup 0$ in $L^2(Q_T;\mathbb{R}^N)$.

Proof. Let $\varphi\in C_0^\infty(Q_T)$ be an arbitrary vector-function. Then

$$\left|\int_{Q_T}(\varepsilon\nabla u_\varepsilon,\varphi)\,dxdt\right|\leqslant\sqrt{\varepsilon}\left(\int_{Q_T}\varepsilon|\nabla u_\varepsilon|^2\,dxdt\right)^{1/2}\left(\int_{Q_T}|\varphi|^2\,dxdt\right)^{1/2}.$$

Hence, the sequence $\{\varepsilon\nabla u_\varepsilon\}_{\varepsilon\to 0}$ is bounded in $L^2(Q_T;\mathbb{R}^N)$. As a result, we have

$$\left|\int_{Q_T}(\varepsilon\nabla u_\varepsilon,\varphi)\,dxdt\right|\overset{\text{by (3.51)}}{\leqslant}C\sqrt{\varepsilon}\left(\int_{Q_T}\varepsilon|\nabla u_\varepsilon|^2\,dxdt\right)^{1/2}\leqslant\widehat{C}\sqrt{\varepsilon}\to 0.$$

The proof is complete. □

We are now in a position to prove the main result of this section.

Theorem 3.3. *Let $f\in L^2(Q_T)$, $f_0\in L^2(\Omega)$, and $\theta\in L^\infty(\Omega;\mathbb{R}^2)$ be given distributions. Then, for each $v\in\mathcal{V}_{ad}$, the initial-boundary value problem* (3.1)–(3.3) *admits at least one weak solution $u\in W_u(Q_T)$.*

Proof. Let ε be a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. Let $\{u_\varepsilon\in W(0,T)\}_{\varepsilon\to 0}$ be a sequence of weak solutions to the approximating problem (3.15)–(3.17) given by Theorem 3.2. Then, for each $\varepsilon>0$, u_ε satisfies the energy equality (3.45). Hence, we can deduce from (3.45) the following estimates

$$\sup_{\varepsilon>0}\|u_\varepsilon\|^2_{L^2(Q_T)}\leqslant C_1^2=\|f-v\|^2_{L^2(Q_T)}+\frac{2}{\kappa}\|f_0\|^2_{L^2(\Omega)}, \tag{3.52}$$

$$\begin{aligned}\sup_{\varepsilon>0}\|\nabla u_\varepsilon\|_{L^{p_{u_\varepsilon}(\cdot)}(Q_T;\mathbb{R}^2)}&\overset{\text{by (2.21)}}{\leqslant}\sup_{\varepsilon>0}\left(\int_{Q_T}|\nabla u_\varepsilon|^{p_{u_\varepsilon}(t,x)}\,dxdt+1\right)^{1/p^-}\\&\overset{\text{by (3.45)}}{\leqslant}\sup_{\varepsilon>0}\left(\Lambda^{-1}\left(\|f_0\|^2_{L^2(\Omega)}+\frac{\kappa}{2}\|f-v\|^2_{L^2(Q_T)}+\frac{\kappa}{2}\|u_\varepsilon\|^2_{L^2(Q_T)}\right)+5\right)^{1/p^-}\\&\overset{\text{by (3.52)}}{\leqslant}C_2=\left(\Lambda^{-1}\kappa C_1^2+5\right)^{1/p^-},\end{aligned} \tag{3.53}$$

$$
\begin{aligned}
\sup_{\varepsilon>0}\|\nabla u_\varepsilon\|_{L^{p^-}(Q_T;\mathbb{R}^2)} &\overset{\text{by (2.23)}}{\leqslant} \sup_{\varepsilon>0}\left(1+T|\Omega|\right)^{1/p^-}\|\nabla u_\varepsilon\|_{L^{p_{u_\varepsilon}(\cdot)}(Q_T;\mathbb{R}^N)}\\
&\overset{\text{by (3.53)}}{\leqslant} \left(1+T|\Omega|\right)^{1/p^-}C_2, \qquad (3.54)\\
\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \sup_{\varepsilon>0}\sqrt{2\left(\|f_0\|^2_{L^2(\Omega)}+\frac{\kappa}{2}\|f-v\|^2_{L^2(Q_T)}+\frac{\kappa}{2}\|u_\varepsilon\|^2_{L^2(Q_T)}\right)}\\
&\leqslant \sqrt{2\kappa}C_1, \qquad (3.55)\\
\|\nabla u_\varepsilon\|_{L^2(Q_T;\mathbb{R}^2)} &\leqslant \frac{1}{\sqrt{\varepsilon}}\sqrt{|f_0\|^2_{L^2(\Omega)}+\frac{\kappa}{2}\|f-v\|^2_{L^2(Q_T)}+\frac{\kappa}{2}\|u_\varepsilon\|^2_{L^2(Q_T)}}\\
&\leqslant \sqrt{\frac{\kappa}{\varepsilon}}C_1. \qquad (3.56)
\end{aligned}
$$

Taking this into account, we see that the sequence $\{u_\varepsilon\}_{\varepsilon\to 0}$ is bounded in the spaces $L^\infty(0,T;L^2(\Omega))$ and $L^{p^-}(0,T;W^{1,p^-}(\Omega))$. Therefore, there exists an element

$$
u \in L^{p^-}(0,T;W^{1,p^-}(\Omega))\cap L^\infty(0,T;L^2(\Omega)) \tag{3.57}
$$

such that, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $L^{p^-}(0,T;W^{1,p^-}(\Omega))$ as $\varepsilon\to 0$. Moreover, the uniform boundedness of the fluxes $\left\{A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon)\right\}_{\varepsilon\to 0}$ in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ with respect to $\varepsilon>0$ implies that this sequence is sequentially weakly compact in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ (for arguments see (3.39)). Hence, we may admit the existence of a vector-function w such that $w_\varepsilon = A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon) \rightharpoonup w$ in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ as $\varepsilon\to 0$. Arguing as in the proof of Theorem 3.2, it can be shown that $\sup_{\varepsilon\to 0}\|\left(w_\varepsilon,\nabla u_\varepsilon\right)\|_{L^1(Q_T)}<\infty$. As a result, Theorem 2.1 implies that the flow $A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon)$ weakly converges in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ to the flow $w = A_u(t,x,\nabla u)$. Then, utilizing Lemma 3.1, we see that the passage to the limit in the integral identity (3.44) leads to a similar identity for equation (3.1). It remains to take into account Lemma 2.4 and relations (3.46)–(3.47) in order to deduce that $\displaystyle\int_{Q_T}|R_\eta\nabla u|^{p_u(t,x)}\,dxdt<+\infty$. Thus, u is an element of the space $W_u(Q_T)$ and, as a consequence, u is a weak solution to the problem (3.1)–(3.3). □

Before proceeding further, it is worth to notice that the uniqueness of weak solutions to the perturbed problem (3.15)–(3.17) and, hence, for the original one (3.1)–(3.3), seems to be an open question. In view of this, we adopt the following concept:

Definition 3.2. We say that a weak solution $u\in W_u(Q_T)$ to the problem (3.1)–(3.3), for given distributions $f\in L^2(Q_T)$, $f_0\in L^2(\Omega)$, $\theta\in L^\infty(\Omega;\mathbb{R}^2)$, and $v\in\mathcal{V}_{ad}$, is W_0-attainable if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero as $n\to\infty$ and such that

$$
\left.\begin{aligned}
u_n &\rightharpoonup u \ \text{ in } \ L^{p^-}(0,T;W^{1,p^-}(\Omega)),\\
A^{\varepsilon_n}_{u_{n-1}}(t,x,\nabla u_n) &\rightharpoonup A_u(t,x,\nabla u) \ \text{ in } \ L^{(p^+)'}(Q_T;\mathbb{R}^2)
\end{aligned}\right\}\ \text{as } n\to\infty, \tag{3.58}
$$

where, for each $n \in \mathbb{N}$, u_n is the weak solution to the following perturbed problem

$$\frac{\partial u}{\partial t} - \varepsilon_n \Delta u - \operatorname{div} A^{\varepsilon_n}_{u_{n-1}}(t, x, \nabla u) + \kappa u = \kappa(f - v) \quad \text{in} \quad Q_T, \tag{3.59}$$

$$\partial_\nu u = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \tag{3.60}$$

$$u(0, \cdot) = f_0 \quad \text{in} \quad \Omega. \tag{3.61}$$

We can supplement the result on Theorem 3.3 with the following assertions.

Corollary 3.2. *Let $u \in W_u(Q_T)$ be a weak solution to the problem* (3.1)–(3.3) *that has been obtained as a cluster point of the weak solutions $\{u_\epsilon \in W(0, T)\}$ to the approximating problems* (3.15)–(3.17)*. Then the following energy inequality*

$$\frac{1}{2} \int_\Omega u^2\, dx + \int_0^t \int_\Omega \left((A_u(t, x, \nabla u), \nabla u) + \kappa u^2 \right) dxdt \leqslant \kappa \int_0^t \int_\Omega (f - v) u \, dxdt + \int_\Omega f_0^2 \, dx \tag{3.62}$$

holds true for almost all $t \in [0, T]$.

Proof. To deduce this inequality, we make use of the estimate (3.31) and the celebrated Aubin's lemma. As a result, we can supplement the properties (3.58) by the following one: $u_\varepsilon \to u$ in $L^2(0, T; L^2(\Omega))$ as $\varepsilon \to 0$. So, without loss of generality, we can suppose that $u_\varepsilon(t, x) \to u(t, x)$ almost everywhere in Q_T. Hence,

$$\|u_\varepsilon(t, \cdot)\|^2_{L^2(\Omega)} \to \|u(t, \cdot)\|^2_{L^2(\Omega)} \text{ for a.a } \ t \in [0, T]. \tag{3.63}$$

Taking this fact into account and passing to the limit in relation (3.45) as $\varepsilon \to 0$ using the weak convergence $u_\varepsilon \rightharpoonup u$ in $L^{p^-}(0, T; W^{1,p^-}(\Omega))$, Lemma 2.3, and the weak convergence of fluxes to flux (Theorem 2.1), we arrive at the announced inequality (3.62). □

Remark 3.1. It is worth to emphasize that Theorem 3.3 can be now specified as follows: For given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $\theta \in L^\infty(\Omega; \mathbb{R}^2)$, and $v \in \mathcal{V}_{ad}$, the initial-boundary value problem (3.1)–(3.3) admits at least one W_0-attainable weak solution $u \in W_u(Q_T)$ for which the energy inequality (3.62) holds true for all $t \in [0, T]$. Moreover, as follows from estimates (3.52)–(3.55), this solution in bounded in $L^{p^-}(0, T; W^{1,p^-}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. However, the mapping $t \mapsto \|u(t, \cdot)\|_{L^2(\Omega)}$ is not necessary continuous. Because of that the second term $\frac{1}{2} \int_\Omega |u(T) - f_0|^2 \, dx$ in the cost functional is not well defined (see Proposition 3.1 for the details). This means that the original OCP (1.1)–(1.5) is generally ill posed, and some its relaxation is required.

4. Setting of the Relaxed Optimal Control Problem and Existence Result

As was pointed out in the previous section, the operator $-\operatorname{div}\, A_u(t,x,\nabla u)+\kappa u$ provides an example of a non-linear operator in divergence form which is neither monotone nor coercive. In this case (see Theorem 3.3) the initial-boundary value problem (3.1)–(3.3) admits a W_0-attainable weak solution that satisfies the energy inequality (3.62). However, it is unknown whether under some admissible control $v\in\mathcal{V}_{ad}$ this solution is unique and belongs to the space $C([0,T];L^2(\Omega))$. Moreover, it is an open question whether all weak solutions to (3.1)–(3.3) satisfy energy inequality (3.62) that plays a crucial role for derivation of a priori estimates like (3.52)–(3.55).

Our prime interest in this section is to study the existence issues for the following relaxed version of the original optimal control problem

$$\begin{aligned}&\text{Minimize}\ \ J(v,u)=\|v\|^2_{L^2(0,T;L^1(\Omega))}+\frac{\mu}{2\omega}\int_{T-\omega}^{T}\|u(t,\cdot)-f_0(\cdot)\|^2_{L^2(\Omega)}\,dt\\&\qquad\qquad\text{subject to the constraints (1.2)–(1.4), (3.5),}\end{aligned}\tag{4.1}$$

where ω is a small positive value such that $T-\omega\gg 0$, $f\in L^2(\Omega)$, $f_0\in L^2(\Omega)$, $v\in\mathcal{V}_{ad}$, and $\theta\in L^\infty(\Omega;\mathbb{R}^2)$ are given distributions.

In image processing, the distributions $f\in L^2(\Omega)$ and $f_0\in L^2(\Omega)$ are usually related to some noise-corrupted image. For instance, $f\in L^2(\Omega)$ is the original gray-scale image with noise, whereas $f_0\in L^2(\Omega)$ is the pre-denoised image by applying a median filter to f. In this case $\theta\in L^\infty(\Omega;\mathbb{R}^2)$ can be stood for the vector field of unit normals to the topographic map of a smoothed version of function f_0. So, instead of E_I in (2.11), we can take

$$\left(G_\sigma * f_0\right)(x)=\int_\Omega G_\sigma(x-y)f_0(y)\,dy,\quad \forall\, x\in\Omega.$$

We say that $(v.u)$ is a feasible pair to OCP (4.1) if:

$$\left.\begin{array}{c} v\in\mathcal{V}_{ad},\quad u\in W_u(Q_T),\quad J(v,u)<+\infty,\\ (v,u)\ \text{are related by integral identity (3.10) and energy inequality (3.62)},\\ \text{and } u \text{ is a } W_0\text{-attainable weak solution to (3.1)–(3.3) for the given } v. \end{array}\right\}\tag{4.2}$$

Let $\Xi\subset L^2(Q_T)\times W_u(Q_T)$ be the set of all feasible solutions to the problem (4.1). Then Theorem 3.3 implies that $\Xi\neq\emptyset$. Since the structure of the set Ξ and its main topological properties are unknown, we begin with the following observation.

Theorem 4.1. *For given distributions $f\in L^2(Q_T)$, $f_0\in L^2(\Omega)$, and $\theta\in L^\infty(\Omega;\mathbb{R}^2)$, the set Ξ is sequentially closed with respect to the weak topology of $L^2(Q_T)\times L^{p^-}(0,T;W^{1,p^-}(\Omega))$.*

Proof. Let $\{(v_k, u_k)\}_{k\in\mathbb{N}} \subset \Xi$ be a sequence such that

$$v_k \rightharpoonup v \text{ in } L^2(Q_T), \quad u_k \rightharpoonup u \text{ in } L^{p^-}(0,T;W^{1,p^-}(\Omega)). \tag{4.3}$$

Since the set $\mathcal{V}_{ad}$ is convex and closed, it follows by Mazur's theorem that $\mathcal{V}_{ad}$ is sequentially closed with respect to the weak topology of $L^2(Q_T)$. Therefore, $v \in \mathcal{V}_{ad}$. Our aim is to show that $(v,u) \in \Xi$. We will do it in several steps.

Step 1. By the initial assumptions, for each $k \in \mathbb{N}$, the pair (v_k, u_k) satisfy the energy inequality (3.62), and u_k is a W_0-attainable weak solution of (3.1)–(3.3). Hence, we may suppose that there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset W(0,T)$ such that $\{u_{k,n}\}_{n\in\mathbb{N}}$ are the weak solutions (in the sense of distributions) of (3.59)–(3.61) with $\varepsilon_n = 1/n$ and $v = v_k$. and

$$u_{k,n} \rightharpoonup u_k \text{ in } L^{p^-}(0,T;W^{1,p^-}(\Omega)), \quad \text{as } n \to \infty, \tag{4.4}$$

$$A^{1/n}_{u_{k,n-1}}(t,x,\nabla u_{k,n}) \rightharpoonup A_{u_k}(t,x,\nabla u_k) \text{ in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \quad \text{as } n \to \infty, \tag{4.5}$$

Moreover, the fact that the energy equality

$$\frac{1}{2}\int_\Omega u_{k,n}^2\,dx + \int_0^t\int_\Omega \left(\frac{1}{n}|\nabla u_{k,n}|^2 + \left(A^{1/n}_{u_{k,n-1}}(t,x,\nabla u_{k,n}), \nabla u_{k,n}\right) + \kappa u_{k,n}^2\right)dxdt$$
$$= \kappa\int_{Q_T}(f - v_k)u_{k,n}\,dxdt + \int_\Omega f_0^2\,dx, \quad \forall\, t\in[0,T] \tag{4.6}$$

holds true for all $n,k \in \mathbb{N}$, implies the boundedness of the sequence $\{u_{k,k}\}_{k\in\mathbb{N}}$ in the space $L^{p^-}(0,T;W^{1,p^-}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Hence, combining this fact with (4.4) and (4.3), we deduce

$$u_{k,k} \rightharpoonup u \text{ in } L^{p^-}(0,T;W^{1,p^-}(\Omega)), \quad \text{as } k \to \infty, \tag{4.7}$$

$$u_{k,k} \rightharpoonup u \text{ in } L^2(0,T;L^2(\Omega)), \quad \text{as } k \to \infty. \tag{4.8}$$

Step 2. Utilizing the energy equality (4.6) and arguing as in (3.25)–(3.27), we can derive the following a priori estimates

$$\|u_{k,k}\|^2_{L^2(Q_T)} \leqslant 2\|f\|^2_{L^2(Q_T)} + 2\sup_{k\in\mathbb{N}}\|v_k\|^2_{L^2(Q_T)} + \frac{2}{\kappa}\|f_0\|^2_{L^2(\Omega)} =: S_1^2, \tag{4.9}$$

$$\begin{aligned}
&\|\nabla u_{k,k}\|^{p^-}_{L^{p_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)} \\
&\quad\leqslant \Lambda^{-1}\left(\|f_0\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|f - v_k\|^2_{L^2(Q_T)} + \frac{\kappa}{2}\|u_{k,k}\|^2_{L^2(Q_T)}\right) + 5 \\
&\quad\leqslant \Lambda^{-1}\left(2\|f_0\|^2_{L^2(\Omega)} + 2\kappa\|f\|^2_{L^2(Q_T)} + 2\kappa\|v_k\|^2_{L^2(Q_T)}\right) + 5 \\
&\quad= \frac{\Lambda^{-1}}{\kappa}S_1^2 + 5 =: S_2^{p^-},
\end{aligned} \tag{4.10}$$

$$\|\nabla u_{k,k}\|_{L^{p^-}(Q_T;\mathbb{R}^2)} \leqslant (1 + T|\Omega|)^{1/p^-} S_2, \tag{4.11}$$

$$
\begin{aligned}
\|u_{k,k}\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \sqrt{2\left(\|f_0\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|f-v_k\|^2_{L^2(Q_T)} + \frac{\kappa}{2}\|u_{k,k}\|^2_{L^2(Q_T)}\right)}\\
&\leqslant \sqrt{\frac{2}{\kappa}}S_1, \qquad (4.12)\\
\|\nabla u_{k,k}\|_{L^2(Q_T;\mathbb{R}^2)} &\leqslant \sqrt{k}\sqrt{\|f_0\|^2_{L^2(\Omega)} + \kappa\|f-v_k\|_{L^2(Q_T)}\|u_{k,k}\|_{L^2(Q_T)}}\\
&\overset{\text{by (4.9)}}{\leqslant} \sqrt{\frac{k}{\kappa}}S_1. \qquad (4.13)
\end{aligned}
$$

for all $k \in \mathbb{N}$, where

$$
\sup_{k\in\mathbb{N}} \|v_k\|_{L^2(Q_T)} \leqslant \sqrt{T}\|v_b\|_{L^2(\Omega)} < +\infty. \qquad (4.14)
$$

Let us show that the following asymptotic property

$$
\frac{1}{k}\nabla u_{k,k} \rightharpoonup 0 \quad \text{in} \quad L^2(Q_T;\mathbb{R}^2) \qquad (4.15)
$$

holds true.

Indeed, for any vector-valued test function $\varphi \in C_0^\infty(Q_T)$, we have

$$
\left|\int_{Q_T}\left(\frac{1}{k}\nabla u_{k,k},\varphi\right)dxdt\right| \leqslant \frac{1}{\sqrt{k}}\left(\int_{Q_T}\frac{1}{k}|\nabla u_{k,k}|^2\,dxdt\right)^{1/2}\left(\int_{Q_T}|\varphi|^2\,dxdt\right)^{1/2}.
$$

Hence, the sequence $\left\{\frac{1}{k}\nabla u_{k,k}\right\}_{k\in\mathbb{N}}$ is bounded in $L^2(Q_T;\mathbb{R}^2)$. As a result, we obtain

$$
\left|\int_{Q_T}\left(\frac{1}{k}\nabla u_{k,k},\varphi\right)dxdt\right| \overset{\text{by (4.13)}}{\leqslant} S_1\frac{1}{\sqrt{k\kappa}}\left(\int_{Q_T}|\varphi|^2\,dxdt\right)^{1/2} \to 0 \quad \text{as} \quad k\to\infty.
$$

Step 3. At this step we prove that the flux $\frac{1}{k}\nabla u_{k,k} + A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k})$ weakly converges in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ to the flux $A_u(t,x,\nabla u)$ as $k\to\infty$. With that in mind, we show that all preconditions (C1)–(C7) of Theorem 2.1 are fulfilled.

To begin with, we notice that the conclusion, similar to (4.7), can be also made with respect to the sequence $\{u_{k,k-1}\}_{k\in\mathbb{N}}$. Then Lemma 2.1 implies that

$$
p_{u_{k,k-1}}(t,x) \to p_u(t,x) \text{ uniformly in } \overline{Q_T} \text{ as } k\to\infty. \qquad (4.16)
$$

Moreover, we deduce from (3.8) and (4.10) that the sequence

$$
\left\{\frac{1}{k}\nabla u_{k,k} + A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k})\right\}_{k\in\mathbb{R}}
$$

is bounded in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$. Hence, we can suppose that there exists an element $z \in L^{(p^+)'}(Q_T;\mathbb{R}^2)$ such that

$$
\frac{1}{k}\nabla u_{k,k} + A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k}) \rightharpoonup z \quad \text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \text{ as } k\to\infty. \qquad (4.17)
$$

We also make use of the following observation: the sequence

$$\left\{\frac{1}{k}|\nabla u_{k,k}|^2+\left(A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\right)\right\}_{k\in\mathbb{N}} \tag{4.18}$$

is uniformly bounded in $L^1(Q_T)$. Indeed, this inference is a direct consequence of estimates (4.13), (4.10), and (3.22). Utilizing this fact together with the properties (4.16), (4.17), (3.6), (4.7) and

$$u_{k,k}\in L^{p^+}(0,T;W^{1,p^+}(\Omega))\ \forall\, k\in\mathbb{N}\ \text{ by (4.9),(4.13)},$$

and taking into account that $1<1+\delta=p^-<p^+=2<2p^-$, we see that all preconditions of Theorem 2.1 hold true. Hence, in view of the property (4.15), the assertion (4.17) can be rewritten as follows

$$\frac{1}{k}\nabla u_{k,k}+A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k})\rightharpoonup A_u(t,x,\nabla u)$$
$$\text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \text{ as } k\to\infty. \tag{4.19}$$

Step 4. The standard formulation of the Aubin-Lions lemma states that if U is a bounded set in $L^p(0,T;X)$ and $\partial U/\partial t=\{\partial u/\partial t\,:\,u\in U\}$ is bounded in $L^r(0,T;Y)$, $r\geqslant 1$, then U is relatively compact in $L^p(0,T;B)$, under the conditions that

$$X\hookrightarrow B\ \text{ compactly},\quad B\hookrightarrow Y\ \text{continuously}.$$

Setting $U=\{u_{k,k}\}_{k\in\mathbb{N}}$, we deduce from (4.9)–(4.12) that

$$\{u_{k,k}\}_{k\in\mathbb{N}}\ \text{ is bounded in } L^{p^-}(0,T;W^{1,p^-}(\Omega)\cap L^2(\Omega)). \tag{4.20}$$

Since, by the Sobolev embedding Theorem, $W^{1,p^-}(\Omega)\hookrightarrow L^{p^-}(\Omega)$ compactly, it follows from the Lebesgue dominated Theorem that the following embeddings are compact as well

$$W^{1,p^-}(\Omega)\cap L^2(\Omega)\hookrightarrow L^2(\Omega),\quad L^2(\Omega)\hookrightarrow \left(W^{1,2}(\Omega)\right)'\ \text{(by the duality arguments)}. \tag{4.21}$$

Further, having in mind the fact that for each $k\in\mathbb{N}$, the functions $u_{k,k}$ are the solutions in $W(0,T)$ of the variational problem

$$\left\langle\frac{\partial u_{k,k}(t)}{\partial t},\varphi\right\rangle_{(W^{1,2}(\Omega))';W^{1,2}(\Omega)}+\int_\Omega\left[\frac{1}{k}\left(\nabla u_{k,k}(t),\nabla\varphi\right)\right]dx$$
$$+\int_\Omega\left[\left(A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k}(t)),\nabla\varphi\right)+\kappa u_{k,k}(t)\varphi\right]dx$$
$$=\kappa\int_\Omega(f(t)-v_k(t))\varphi\,dx,\quad\forall\,\varphi\in W^{1,2}(\Omega)\quad\text{a.e. in }[0,T], \tag{4.22}$$
$$u_{k,k}(0)=f_0. \tag{4.23}$$

we derive from this the following estimate

$$\begin{aligned}\left|\left\langle \frac{\partial u_{k,k}}{\partial t}, \varphi\right\rangle\right| &\leqslant \frac{1}{k}\|\nabla u_{k,k}\|_{L^2(Q_T;\mathbb{R}^2)}\|\nabla\varphi\|_{L^2(Q_T;\mathbb{R}^2)} \\ &\quad + 2\|A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k})\|_{L^{p'_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)}\|\nabla\varphi\|_{L^{p_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)} \\ &\quad + \kappa\|u_{k,k}\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)} + \kappa\|f - v_k\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)} \\ &\leqslant (\text{by (4.9)–(4.13)}) \\ &\leqslant \Big[\frac{1}{\sqrt{\kappa}}S_1 + \kappa S_1 + \kappa\|f\|_{L^2(Q_T)} + \kappa\sup_{k\in\mathbb{N}}\|v_k\|_{L^2(Q_T)}\Big]\|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))} \\ &\quad + \left(1 + \int_{Q_T}|A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k})|^{p'_{u_{k,k-1}}(t,x)}\,dxdt\right)^{1/2}(1+T|\Omega|)^{1/2}\|\varphi\|_{L^2(Q_T)} \\ &\overset{\text{by (4.10),(3.8)}}{\leqslant} \text{const}\|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))}, \quad \forall\,\varphi \in L^2(0,T;W^{1,2}(\Omega)).\end{aligned}$$

Hence,

$$\left\|\frac{\partial u_{k,k}}{\partial t}\right\|_{L^2(0,T;(W^{1,2}(\Omega))')} < +\infty. \tag{4.24}$$

Utilizing this fact together with (4.20) and (4.21), we deduce from the Aubin-Lions lemma that the set $U = \{u_{k,k}\}_{k\in\mathbb{N}}$ is relatively compact in $L^{p^-}(0,T;L^2(\Omega))$. Hence, we can suppose that $u_{k,k} \to u$ strongly in $L^{p^-}(0,T;L^2(\Omega))$ as $k\to\infty$. Since, U is bounded in $L^\infty(0,T;L^2(\Omega))$, it leads to the conclusion

$$u_{k,k} \to u \ \text{ strongly in } \ L^2(0,T;L^2(\Omega)), \quad \text{as } k\to\infty. \tag{4.25}$$

Step 5. At this stage we show that the limit pair (v,u) is related by the integral identity (3.10). First we notice that $u_{k,k}$ is a weak solution (in the sense of distributions) of (3.59)–(3.61) with $n = k$, $\varepsilon_n = 1/k$ and $v = v_k$. Hence, $u_{k,k}$ satisfies the integral identity

$$\begin{aligned}\int_{Q_T}\left(-u_{k,k}\frac{\partial\varphi}{\partial t} + \frac{1}{k}(\nabla u_{k,k},\nabla\varphi) + \left(A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla\varphi\right) + \kappa u_{k,k}\varphi\right)dxdt \\ = \kappa\int_{Q_T}(f - v_k)\varphi\,dxdt + \int_\Omega f_0\varphi|_{t=0}\,dx \quad \forall\,\varphi\in\Phi. \end{aligned} \tag{4.26}$$

Then, utilizing the properties (4.19), (4.7), and (4.3), and passing to the limit in (4.26) as $k\to\infty$, we immediately arrive at the announced identity (3.10).

Step 6. In order to show that the limit pair (v,u) satisfies the energy inequality (3.62), we have to realize the limit passage as $k\to\infty$ in the relation (see [41])

$$\begin{aligned}\frac{1}{2}\int_\Omega u^2_{k,k}\,dx + \int_0^t\int_\Omega\left(\frac{1}{k}|\nabla u_{k,k}|^2 + \left(A^{1/k}_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\right) + \kappa u^2_{k,k}\right)dxdt \\ = \kappa\int_{Q_T}(f - v_k)u_{k,k}\,dxdt + \int_\Omega f_0^2\,dx \quad \forall\,t\in[0,T]. \end{aligned} \tag{4.27}$$

that can be viewed as the energy equality for the weak solutions of the problem (3.59)–(3.61) with $n = k$, $\varepsilon_n = 1/k$ and $v = v_k$. To this end, we notice that the strong convergence in (4.25) implies the pointwise convergence

$$u_{k,k}^2(t,\cdot) \to u^2(t,\cdot) \quad \text{a.e. in } Q_T.$$

Then, in view of estimate (4.12), we have (by the Lebesgue dominated Theorem) the strong convergence $u_{k,k}^2(t,\cdot) \to u^2(t,\cdot)$ in $L^1(\Omega)$ for a.a. $t \in (0,T)$, and, therefore,

$$\frac{1}{2}\lim_{k\to\infty}\int_\Omega u_{k,k}^2(t,x)\,dx = \frac{1}{2}\int_\Omega u^2(t,x)\,dx \quad \text{for a.a. } t \in (0,T). \tag{4.28}$$

Moreover, taking into account that the $L^2(Q_T)$-norm is continuous with respect to the strong convergence (4.25), we see that

$$\lim_{k\to\infty}\int_0^t\int_\Omega u_{k,k}^2\,dxdt = \int_0^t\int_\Omega u^2\,dxdt. \tag{4.29}$$

We also notice that due to the properties (4.15), (4.19), and (4.7), we have

$$\nabla u_{k,k} \rightharpoonup \nabla u \;\text{ and }\; A_{u_{k,k-1}}^{1/k}(t,x,\nabla u_{k,k}) \rightharpoonup A_u(t,x,\nabla u) \;\text{ in } L^1(Q_T;\mathbb{R}^2) \text{ as } k\to\infty.$$

Since $(A_u(t,x,\nabla u),\nabla u) \in L^1(Q_T)$ (see (3.22)), it follows from Lemma 2.3 (see also Proposition 2.2) that

$$\begin{aligned}\lim_{k\to\infty}\int_0^t\int_\Omega\Big[\frac{1}{k}|\nabla u_{k,k}|^2 + \Big(A_{u_{k,k-1}}^{1/k}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\Big)\Big]\,dxdt&\\ \geqslant \lim_{k\to\infty}\int_0^t\int_\Omega\Big[\frac{1}{k}|\nabla u_{k,k}|^2\Big]\,dxdt&\\ + \liminf_{k\to\infty}\int_0^t\int_\Omega\Big(A_{u_{k,k-1}}^{1/k}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\Big)\,dxdt&\\ \overset{\text{by (4.13)}}{\geqslant}\int_0^t\int_\Omega\left(A_u(t,x,\nabla u),\nabla u\right)\,dxdt.&\end{aligned} \tag{4.30}$$

So, in order to pass to the limit in (4.27), it remains to notice that the term

$$\int_{Q_T}(f - v_k)u_{k,k}\,dxdt$$

is the product of weakly and strongly convergent sequences in $L^2(0,T;L^2(\Omega))$. As a result, we have

$$\lim_{k\to\infty}\int_{Q_T}(f - v_k)u_{k,k}\,dxdt = \int_{Q_T}(f - v)u\,dxdt. \tag{4.31}$$

Thus, utilizing the obtained collection of properties (see (4.28), (4.29), (4.30), and (4.31)), we can pass to the limit in (4.27) as $k\to\infty$. As a result, we arrive at the energy inequality (3.62).

Step 7. To conclude the proof, it remains to notice that, due to the properties (3.9), that were established at the previous steps, we have: $J(v,u) < +\infty$ and $u \in W_u(Q_T)$. Moreover, in this case the sequence $\{u_{k,k}\}_{k\in\mathbb{N}}$ satisfies all requirements that were mentioned in Definition 3.2. Hence, $u \in W_u(Q_T)$ is a W_0-attainable weak solution to the problem (3.1)–(3.3). The proof is complete. □

Taking this result into account, it is easy to show that the original optimal control problem (4.1) has a solution. Indeed, this issue immediately follows from Theorem 4.1 and the facts that the set of feasible solutions Ξ is bounded in $L^2(Q_T) \times L^{p^-}(0,T; W^{1,p^-}(\Omega))$ (see estimates (4.9)–(4.13), (4.14)), and the objective functional $J(v,u)$ is lower semicontinuous with respect to the weak topology of $L^2(Q_T) \times \Big(L^{p^-}(0,T; W^{1,p^-}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))\Big)$. So, as a direct consequence, we can finalize this inference as follows:

Theorem 4.2. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $\theta \in L^\infty(\Omega;\mathbb{R}^2)$, and $v_a, v_b \in L^2(\Omega)$, $v_a(x) \leqslant v_b(x)$ a.e. in Ω, be given distributions, and let $\kappa > 0$ and $\mu > 0$ be some constants. Then, for each $0 < \omega < T$ the optimal control problem* (4.1) *admits at least one solution $(v^0, u^0) \in \Xi$.*

5. Approximation of the Relaxed OCP

Let ε, as usual, be a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. In order to find out whether some optimal pairs to the original OCP (4.1) can be attained in an appropriate topology, we make use the basic ideas coming from the perturbation theory and variational convergence of minimization problems [26,39,42]. With that in mind, we introduce the following family of perturbed OCPs

$$\text{Minimize} \quad J_\varepsilon(v,u) \quad \text{provided} \quad (v,u) \in \Xi_\varepsilon, \tag{5.1}$$

where

$$J_\varepsilon(v,u) = \|v\|^2_{L^2(0,T;L^1(\Omega))} + \frac{\mu}{2\omega}\int_{T-\omega}^T \int_\Omega |u(t,x) - f_0(x)|^2 \, dxdt \tag{5.2}$$

and $\Xi_\varepsilon \subset L^2(0,T;L^1(\Omega)) \times L^2(0,T;W^{1,p^-}(\Omega))$ stands for the set of feasible solutions which we define as follows: $(v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon$ if

$$\left\{ \begin{array}{c} v_\varepsilon \in \mathcal{V}_{ad}, \quad u_\varepsilon \in W_{u_\varepsilon}(Q_T), \quad J_\varepsilon(v_\varepsilon, u_\varepsilon) < +\infty, \\ u_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)) \text{ is a } W_0\text{-attainable} \\ \text{weak solution of the problem (3.15)–(3.17) for the given } v_\varepsilon. \end{array} \right\} \tag{5.3}$$

Taking into account Theorem 3.2 and arguing as in Theorem 4.2, it can be proven the following result.

Theorem 5.1. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $\theta \in L^\infty(\Omega;\mathbb{R}^2)$, and $v_a, v_b \in L^2(\Omega)$, $v_a(x) \leqslant v_b(x)$ a.e. in Ω, be given distributions, and let $\kappa > 0$ and $\mu > 0$ be some constants. Then, for each $\varepsilon > 0$ and $0 < \omega < T$, there exists at least one solution $(v^0_\varepsilon, u^0_\varepsilon)$ of optimal control problem (5.1).*

The primary goal of this section is to show that the relaxed OCP (4.1) can be successfully approximated by the OCPs (5.1). In means that there is a pair $(v^0, u^0) \in \Xi$ such that

$$J(v^0,u^0) = \inf_{(v,u)\in\Xi} J(v,u),$$
$$\lim_{\varepsilon\to 0} J(v^0_\varepsilon, u^0_\varepsilon) = \lim_{\varepsilon\to 0} \inf_{(v,u)\in\Xi_\varepsilon} J_\varepsilon(v,u) = J(v^0,u^0),$$
$$(v^0_\varepsilon, u^0_\varepsilon) \to (v^0,u^0) \text{ as } \varepsilon \to 0 \text{ in some appropriate topology.}$$

We begin with a couple of auxiliaries lemmas.

Lemma 5.1. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $\theta \in L^\infty(\Omega;\mathbb{R}^2)$ be given distributions. Let $\{(v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\to 0}$ be a sequence of feasible pairs such that $\{v_\varepsilon\}_{\varepsilon\to 0}$ is bounded in $L^2(Q_T)$. Then there exists a constant $C > 0$ such that*

$$\sup_{\varepsilon\to 0}\left[\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u_\varepsilon\|_{L^{p^-}(Q_T;\mathbb{R}^2)}\right] \leqslant C. \tag{5.4}$$

Proof. The fact that $\{(v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\to 0}$ is a collection of feasible pairs to the corresponding problems (5.1), implies (see Theorem 3.2 and Corollary 3.1) that, for each $\varepsilon > 0$, they are related by the integral identity

$$\int_{Q_T} \left(-u_\varepsilon \frac{\partial \varphi}{\partial t} + \varepsilon\left(\nabla u_\varepsilon, \nabla\varphi\right) + \left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon), \nabla\varphi\right) + \kappa u_\varepsilon \varphi\right) dxdt$$
$$= \kappa \int_{Q_T} (f - v_\varepsilon)\varphi\, dxdt + \int_\Omega f_0 \varphi|_{t=0}\, dx \quad \forall\, \varphi \in \Phi. \tag{5.5}$$

and satisfy the energy equality

$$\frac{1}{2}\int_\Omega u^2_\varepsilon\, dx + \int_0^t \int_\Omega \left(\varepsilon|\nabla u_\varepsilon|^2 + \left(A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon), \nabla u_\varepsilon\right) + \kappa u^2_\varepsilon\right) dxds$$
$$= \kappa \int_0^t \int_\Omega (f - v_\varepsilon) u_\varepsilon\, dxds + \int_\Omega f_0^2\, dx \quad \text{for a.a } \ t \in [0,T]. \tag{5.6}$$

Then arguing as in the proof of Theorem 3.3, we deduce from (5.6)

$$\left.\begin{array}{c} \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \sqrt{2\kappa} C_1, \\ \sup_{\varepsilon>0} \|\nabla u_\varepsilon\|_{L^{p^-}(Q_T;\mathbb{R}^2)} \leqslant (1 + T|\Omega|)^{1/p^-}\left(\Lambda^{-1}\kappa C_1^2 + 5\right)^{1/p^-}, \\ \sup_{\varepsilon>0} \|\nabla u_\varepsilon\|_{L^2(Q_T;\mathbb{R}^2)} \leqslant \frac{1}{\sqrt{\varepsilon\kappa}} C_1 \end{array}\right\} \tag{5.7}$$

with $C_1 = \|f\|_{L^2(Q_T)} + \frac{2}{\kappa}\|f_0\|_{L^2(\Omega)} + \sup_{\varepsilon>0}\|v_\varepsilon\|_{L^2(Q_T)}$. As a result, we arrive at the estimate (5.4). □

Taking this result into account and arguing as in Theorem 4.1, it can be shown the weak $L^2(Q_T)$-compactness of admissible controls for the perturbed OCPs (5.1) implies some compactness properties for the corresponding sequence of feasible solutions.

Lemma 5.2. *Let $\{(v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\to 0}$ be a sequence of feasible pairs to the OCPs (5.1). Assume that $v_\varepsilon \rightharpoonup v$ in $L^2(Q_T)$. Then, for given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $\theta \in L^\infty(\Omega;\mathbb{R}^2)$, we have*

$$u_\varepsilon \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \tag{5.8}$$

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad L^{p^-}(0,T;W^{1,p^-}(\Omega)), \tag{5.9}$$

$$\varepsilon\nabla u_\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad L^2(Q_T;\mathbb{R}^2), \tag{5.10}$$

$$p_{u_\varepsilon}(t,x) \to p_u(t,x) \ \text{uniformly in} \ \overline{Q_T}, \tag{5.11}$$

$$\varepsilon\nabla u_\varepsilon + A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon) \rightharpoonup A_u(t,x,\nabla u) \quad \text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2), \tag{5.12}$$

where $(v,u) \in \Xi$.

Proof. Since for each $\varepsilon > 0$ the function u_ε is a W_0-attainable weak solution to the problems (3.15)–(3.17) with $v = v_\varepsilon$, it follows that we can utilize the a priori estimates (3.52)–(3.56). Hence, the existence of element u with properties (5.9)–(5.12) follows from Theorem 3.3. To establish the fact that the pair (v,u) is feasible to the relaxed OCP 4.1, we can apply the arguments of the proof of Theorem 4.1. It remains to notice that in order to deduce the strong convergence property (5.8), it is enough to take into account the boundedness of the sequence $\{u_\varepsilon\}_{\varepsilon\to 0}$ in $L^{p^-}(0,T;W^{1,p^-}(\Omega))$ (see (5.9)), the estimate

$$\begin{aligned}
\left|\left\langle \frac{\partial u_\varepsilon}{\partial t}, \varphi \right\rangle\right| &\leqslant \varepsilon\|\nabla u_\varepsilon\|_{L^2(Q_T;\mathbb{R}^2)}\|\nabla\varphi\|_{L^2(Q_T;\mathbb{R}^2)} \\
&\quad + 2\|A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon)\|_{L^{p'_{u_\varepsilon}(\cdot)}(Q_T;\mathbb{R}^2)}\|\nabla\varphi\|_{L^{p_{u_\varepsilon}(\cdot)}(Q_T;\mathbb{R}^2)} \\
&\quad + \kappa\|u_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)} + \kappa\|f - v_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)} \\
&\leqslant (\text{by } (5.7)) \\
&\leqslant \Big[\text{const} + \kappa\|f\|_{L^2(Q_T)} + \kappa \sup_{k\in\mathbb{N}}\|v_k\|_{L^2(Q_T)}\Big]\|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))} \\
&\quad + \left(1 + \int_{Q_T} |A^\varepsilon_{u_\varepsilon}(t,x,\nabla u_\varepsilon)|^{p'_{u_\varepsilon}(t,x)}\,dxdt\right)^{1/2}(1+T|\Omega|)^{1/2}\|\varphi\|_{L^2(Q_T)} \\
&\leqslant \text{const}\|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))}, \quad \forall\,\varphi \in L^2(0,T;W^{1,2}(\Omega)).
\end{aligned}$$

and then (5.8) immediately follows from the Aubin-Lions lemma. □

The main question we are going to discuss the convergence of minima of (5.1) to minima of (4.1) as ε tends to zero. In other words, our aim is to show that some optimal solutions to (4.1) can be approximated by the solutions of (5.1). To this end, we make use of the basic results of the variational convergence of minimization problems [36–38, 42, 44]. We begin with some preliminaries.

Lemma 5.3. *Let* $\{(v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\to 0}$ *be a sequence of feasible pairs such that* $\{v_\varepsilon\}_{\varepsilon\to 0}$ *is bounded in* $L^2(Q_T)$*. Then there exists a function* $u \in W_u(Q_T)$ *with properties* (5.8)–(5.12) *such that*

$$(v,u) \in \Xi \quad \text{and} \quad J(v,u) \leqslant \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon, u_\varepsilon). \tag{5.13}$$

Proof. This assertion immediately follows from Lemma 5.2 the lower semicontinuity of the $L^2(Q_T)$ and $L^2(0,T;L^1(\Omega))$-norms with respect to weak convergence in $L^2(Q_T) \times L^{p^-}(0,T;W^{1,p^-}(\Omega))$. As a result, we have

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|^2_{L^2(0,T;L^1(\Omega))} \geqslant \|v\|^2_{L^2(0,T;L^1(\Omega))},$$
$$\lim_{\varepsilon\to 0} \int_{T-\omega}^{T} \|u_\varepsilon(t,\cdot) - f_0\|^2_{L^2(\Omega)}\, dt = \int_{T-\omega}^{T} \|u(t,\cdot) - f_0\|^2_{L^2(\Omega)}\, dt.$$

□

Before proceeding further, we note that, the initial-boundary value problem (3.15)–(3.17) may have a non-unique solution under a fixed control (see Theorem 3.2). In view of this we define the binary relation $\langle L; \Xi_\varepsilon\rangle$ on each of the sets Ξ_ε following the rule: $(v_\varepsilon, u_\varepsilon)\, L\, (\widehat{v}_\varepsilon, \widehat{u}_\varepsilon)$ if and only if $v_\varepsilon = \widehat{v}_\varepsilon$ a.e. in Q_T. It is easily seen that $\langle L; \Xi_\varepsilon\rangle$ is an equivalence relation. So, hereinafter we will not distinguish the triplets belonging to the same class of equivalence.

Lemma 5.4. *For every class of equivalence* $\Xi/L(v)$ *with* $v \in \mathcal{V}_{ad}$ *there can be found a pair* $(v,u) \in \Xi/L(v)$ *and a sequence* $\{(v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *with properties* (5.8)–(5.12) *such that*

$$v_\varepsilon \rightharpoonup v \quad \text{in} \quad L^2(Q_T), \quad \text{and} \quad J(v,u) \geqslant \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon, u_\varepsilon). \tag{5.14}$$

Proof. Let $v^* \in \mathcal{V}_{ad}$ be an arbitrary admissible control. Let

$$\Xi/L(v^*) = \{(v^*, u) \in \Xi\}$$

be the corresponding class on equivalence.

We define the sequence $\{(v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ as follows $v_\varepsilon \equiv v^*$ for each $\varepsilon > 0$, and u_ε is a weak solution to the problem (3.1)–(3.3) with $v = v^*$ in the sense of Definition 3.1. Then arguing as in Lemma 5.3 and Theorems 3.2 and 3.3, it can be shown that there exists a W_0-attainable solution $u^* \in W_{u^*}(Q_T)$ to the problem (3.1)–(3.3) such that $(v^*, u^*) \in \Xi$ and $u_\varepsilon \to u^*$ strongly in $L^2(0,T;L^2(\Omega))$ (see properties (5.8)–(5.12)). It is clear now that $(v^*, u^*) \in \Xi/L$ and the following relations

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|^2_{L^2(0,T;L^1(\Omega))} = \|v^*\|^2_{L^2(0,T;L^1(\Omega))},$$
$$\lim_{\varepsilon\to 0} \int_{T-\omega}^{T} \|u_\varepsilon(t,\cdot) - f_0\|^2_{L^2(\Omega)}\, dt = \int_{T-\omega}^{T} \|u^*(t,\cdot) - f_0\|^2_{L^2(\Omega)}\, dt$$

hold true. From this (5.14) follows. □

We are now in a position to prove the main result of this section.

Theorem 5.2. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $v_a, v_b \in L^2(\Omega)$, and $\theta \in L^\infty(\Omega;\mathbb{R}^2)$ be given distributions. Let $\left\{(v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal pairs to the corresponding OCPs* (5.1)*. Then there exists a pair $(v^0, u^0) \in \Xi$ such that, up to a subsequence, $(v^0_\varepsilon, u^0_\varepsilon) \to (v^0, u^0)$ in the sense of convergences* (5.8)–(5.12) *and*

$$\inf_{(v,u)\in\Xi_\varepsilon} J(v,u) = J(v^0,u^0) = \lim_{\varepsilon\to 0} J_\varepsilon(v^0_\varepsilon, u^0_\varepsilon) = \lim_{\varepsilon\to 0} \inf_{(v,u)\in\Xi_\varepsilon} J_\varepsilon(v,u). \tag{5.15}$$

Proof. Let $\left\{(v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a given sequence of optimal pairs to the OCPs (5.1). Taking into account the definition of the set of admissible controls $\mathcal{V}_{ad}$ and the a priori estimates (3.52)–(3.56), it can be show that (we refer to Lemma 5.2 for the details) the sequence $\left\{(v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ is relatively compact with respect to the convergence (5.8)–(5.12). So, we may suppose that there exist a subsequence $\left\{(v^0_{\varepsilon_k}, u^0_{\varepsilon_k}) \in \Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ of the sequence of optimal solutions and a pair (v^*, u^*), such that $(v^0_{\varepsilon_k}, u^0_{\varepsilon_k}) \longrightarrow (v^*, u^*)$ as $\varepsilon_k \to 0$ in the sense of (5.8)–(5.12). Then, by Lemma 5.2, we deduce that $(v^*, y^*) \in \Xi$, and

$$\begin{aligned} \liminf_{k\to\infty} \min_{(v,v)\in\Xi_{\varepsilon_k}} J_{\varepsilon_k}(v,u) &= \liminf_{k\to\infty} J_{\varepsilon_k}(v^0_{\varepsilon_k}, u^0_{\varepsilon_k}) \\ &\geqslant J(v^*,u^*) \geqslant \min_{(v,u)\in\Xi} J(v,u) = J(v^0,u^0), \end{aligned} \tag{5.16}$$

where (v^0, u^0) is an optimal pair to (5.1).

Let $\Xi/L(v^0) = \left\{(v^0, u) \in \Xi\right\}$ be the corresponding class on equivalence. It is clear that $(v^0, u^0) \in \Xi/L(v^0)$. Since u^0 is a W_0-attainable weak solution to the problems (3.15)–(3.17) with $v = v^0$ (in fact, (v^0, u^0) is the limit of a minimizing sequence), it follows from Lemma 5.4 that there exists a sequence $\left\{(v^0, \widehat{u}_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ with properties (5.8)–(5.12) such that

$$J(v^0,u^0) \geqslant \limsup_{\varepsilon\to 0} J_\varepsilon(v^0, \widehat{u}_\varepsilon).$$

Using this fact, we have

$$\begin{aligned} \min_{(v,u)\in\Xi} J(v,u) &= J(v^0,u^0) \\ &\geqslant \limsup_{\varepsilon\to 0} J_\varepsilon(v^0,\widehat{u}_\varepsilon) \geqslant \limsup_{\varepsilon\to 0} \min_{(v,u)\in\Xi_\varepsilon} J_\varepsilon(v,u) \\ &\geqslant \limsup_{k\to\infty} \min_{(v,u)\in\Xi_{\varepsilon_k}} J_{\varepsilon_k}(v,u) = \limsup_{k\to\infty} J_{\varepsilon_k}(v^0_{\varepsilon_k}, u^0_{\varepsilon_k}). \end{aligned} \tag{5.17}$$

From this and (5.16) we deduce that

$$\liminf_{k\to\infty} J_{\varepsilon_k}(v^0_{\varepsilon_k}, u^0_{\varepsilon_k}) \geqslant \limsup_{k\to\infty} J_{\varepsilon_k}(v^0_{\varepsilon_k}, u^0_{\varepsilon_k}).$$

Then, combining (5.16) and (5.17), we get

$$J(v^*, u^*) = J(v^0, u^0) = \min_{(v,u)\in\Xi} J(v, u) = \lim_{k\to\infty} \min_{(v,u)\in\Xi_{\varepsilon_k}} J_{\varepsilon_k}(v, u). \tag{5.18}$$

Using these relations and the fact that the problem(5.4) has a nonempty set of solutions, we may suppose that $v^* = v^0$ and u^* together with u^0 belong to the same class of equivalence $\Xi/L(v^0)$. Since equality (5.18) holds for the limits of all subsequences of $\left\{(v_\varepsilon^0, u_\varepsilon^0)\right\}_{\varepsilon>0}$, it follows that these limits coincide and, therefore, (v^0, u^0) is the limit of the whole sequence $\left\{(v_\varepsilon^0, u_\varepsilon^0)\right\}_{\varepsilon>0}$. Then, using the same argument for the entire sequence of minimizers, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} \min_{(v,u)\in\Xi_\varepsilon} J_\varepsilon(v, u) &= \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon^0, u_\varepsilon^0) \geqslant J(v^0, u^0) = \min_{(v,u)\in\Xi} J(v, u)\\ &\geqslant \limsup_{\varepsilon\to 0} J_\varepsilon(v^0, \widehat{u}_\varepsilon) \geqslant \limsup_{\varepsilon\to 0} \min_{(v,u)\in\Xi_\varepsilon} J_\varepsilon(v, u)\\ &= \limsup_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon^0, u_\varepsilon^0)\end{aligned}$$

and this concludes the proof. □

6. On Optimality Conditions for Approximating OCPs

For the sake of simplicity, we assume that the function $g : [0, \infty) \to (0, 1]$ in (1.7) is defined by the rule

$$g(s) = \delta + \frac{a^2(1-\delta)}{a^2 + s^2}, \quad \forall\, s \in [0, +\infty), \tag{6.1}$$

where $0 < \delta \ll 1$ is a given threshold, and $a \in [0, \infty)$ is a tuning parameters. We also assume that the variance σ in the Gaussian kernel G_σ (see (1.9)) is small enough. It means that, for each function $u \in W^{1,1}(\Omega)$, its texture index $p_u(t, x)$ can be approximately defined as

$$p_u(t, x) = 1 + g(|\nabla u|). \tag{6.2}$$

It is worth to notice that from practical point of view, the above mentioned simplification is not very strong because we can always suppose that in this case the restriction of exponent $p_u(t, x)$ given by (6.2) on any grid will be almost the same the restriction on this grid of the Lipschitz-continuous function $\widehat{p}_u(t, x) := 1 + g\left(\frac{1}{h}\int_{t-h}^{t} |(\nabla G_\sigma * \widetilde{u}(\tau, \cdot))\,(x)|\; d\tau\right)$ provided $\sigma > 0$ and $h > 0$ are small enough.

Further, for each $\varepsilon > 0$, we associate with the OCP (5.1) the following Lagrange functional

$$\begin{aligned}\mathcal{L}_\varepsilon\,(v, u, \lambda, z) := \lambda J_\varepsilon(v, u) + \int_{Q_T} \left[-\varepsilon\Delta u - \operatorname{div} A_u^\varepsilon(t, x, \nabla u)\right] z\, dxdt\\ + \int_{Q_T} \left[\frac{\partial u}{\partial t} + \kappa(u - f + v)\right] z\, dxdt,\end{aligned} \tag{6.3}$$

where $\lambda \in \mathbb{R}_+$, $z \in L^2(0,T;W^{1,p^-}(\Omega))$, and

$$J_\varepsilon(v,u) = \|v\|^2_{L^2(0,T;L^1(\Omega))} + \frac{\mu}{2\omega}\int_{T-\omega}^{T}\int_\Omega |u(t,x) - f_0(x)|^2\,dxdt, \tag{6.4}$$

$$A^\varepsilon_u(t,x,\nabla u) := (|R_\eta \nabla u| + \varepsilon)^{p_u(t,x)-2}\, R_\eta \nabla u. \tag{6.5}$$

Let $(v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\varepsilon$ be an optimal pair to the problem (5.1). To characterize this solution, we make use of the celebrated Ioffe-Tikhomirov theorem [34]. With that in mind, let us show that the mappings

$$u \mapsto J_\varepsilon(v,u), \tag{6.6}$$

$$u \mapsto \frac{\partial u}{\partial t} - \varepsilon\Delta u + \kappa(u - f + v), \tag{6.7}$$

$$u \mapsto -\operatorname{div} A^\varepsilon_u(t,x,\nabla u) \tag{6.8}$$

are continuously differentiable in some neighborhood of the point u_ε. Since this property is obviously true for the mappings (6.6)–(6.7), we establish it for the $u \mapsto -\operatorname{div} A^\varepsilon_u(t,x,\nabla u)$.

Let $F'(u)[h]$ stands for the directional derivative of a functional $F : X \to Y$ at the point $u \in X$ along a vector $h \in X$, i.e.,

$$F'(u)[h] = \lim_{\sigma\to 0} \frac{F(u+\sigma h) - F(u)}{\sigma}.$$

Then, arguing as in [6], we can obtain the following result:

Proposition 6.1. Let $p : Q_T \to [p^-, p^+] \subset (1,2]$, with $p^\pm = \text{const}$, be a given exponent and let

$$\widetilde{F}_1(u) = -\operatorname{div}\left[(|\nabla u| + \varepsilon)^{p(x)-2}\,\nabla u\right], \quad \forall\, u \in W_u(Q_T).$$

Then, for each $u \in W_u(Q_T)$, we have

$$\widehat{F}_1'(u)[h] = \widehat{\mathbb{F}}_{11}(u)[h] + \widehat{\mathbb{F}}_{12}(u)[h], \tag{6.9}$$

where

$$\widehat{\mathbb{F}}_{11}(u)[h] = -\operatorname{div}\left[(|\nabla u| + \varepsilon)^{p-2}\,\nabla h\right],$$

$$\widehat{\mathbb{F}}_{12}(u)[h] = -\operatorname{div}\left[(p-2)\,(|\nabla u| + \varepsilon)^{p-4}\,\nabla u\,(\nabla u, \nabla h)\right].$$

Then, applying the similar arguments and taking into account the representation (2.13), we can generalize the previous proposition as follows.

Proposition 6.2. Let $p : Q_T \to [p^-, p^+] \subset (1,2]$, with $p^\pm = \text{const}$, be a given exponent and let

$$F_1(u) = -\operatorname{div}\left[(|R_\eta \nabla u| + \varepsilon)^{p(x)-2}\, R_\eta \nabla u\right], \quad \forall\, u \in W_u(Q_T).$$

Then, for each $u \in W_u(Q_T)$, we have

$$F_1'(u)[h] = \mathbb{F}_{11}(u)[h] + \mathbb{F}_{12}(u)[h] + \mathbb{F}_{13}(u)[h] + \mathbb{F}_{14?}(u)[h], \tag{6.10}$$

where

$$\mathbb{F}_{11}(u)[h] = -\operatorname{div}\left[\left(|R_\eta \nabla u| + \varepsilon\right)^{p-2} \nabla h\right], \tag{6.11}$$

$$\mathbb{F}_{12}(u)[h] = -\operatorname{div}\left[(p-2)\left(|R_\eta \nabla u| + \varepsilon\right)^{p-4} R_\eta \nabla u\,(\nabla u, \nabla h)\right], \tag{6.12}$$

$$\mathbb{F}_{13}(u)[h] = \eta^2 \operatorname{div}\left[\left(|R_\eta \nabla u| + \varepsilon\right)^{p-2} (\theta \otimes \theta)\, \nabla h\right], \tag{6.13}$$

$$\begin{aligned}\mathbb{F}_{14}(u)[h] &= \eta^2 \operatorname{div}\left[(p-2)\left(|R_\eta \nabla u| + \varepsilon\right)^{p-4} R_\eta \nabla u\,(\theta, \nabla h)\,\theta\right] \\ &= \eta^2 \operatorname{div}\left[(p-2)\left(|R_\eta \nabla u| + \varepsilon\right)^{p-4} R_\eta \nabla u\,(\theta \otimes \theta)\, \nabla h\right].\end{aligned} \tag{6.14}$$

Proposition 6.3. Let $(v, u) \in \Xi_\varepsilon$ be a given feasible solution, let

$$p[\nabla u] := 1 + \delta + \frac{a^2(1-\delta)}{a^2 + |\nabla u|^2},$$

and let

$$F_2(u) = -\operatorname{div}\left(|R_\eta \nabla q|^{p[\nabla u]-2} R_\eta \nabla q\right), \quad \forall\, u \in L^2(0, T; W^{1,1+\delta}(\Omega)),$$

where $q \in L^2(0, T; W^{1,p[\nabla u]}(\Omega))$ is a given function. Then, for each element $q \in L^2(0, T; W^{1,p[\nabla u]}(\Omega))$ and for all $h \in L^2(0, T; W^{1,2}(\Omega))$, we have

$$F_2'(u)[h] = -\operatorname{div}\left(|R_\eta \nabla q|^{p[\nabla u]-2} \frac{2a^2(1-\delta)\log\left(|R_\eta \nabla q|\right)}{\left(a^2 + |\nabla u|^2\right)^2} \left(R_\eta \nabla q \otimes \nabla u\right) \nabla h\right). \tag{6.15}$$

Proof. The representation (6.15) immediately follows from definition of the directional derivative. □

Utilizing the representations (6.10)–(6.15), we see that

$$\begin{aligned}\left[I_\varepsilon(v_\varepsilon^0, u_\varepsilon^0)\right]_u' [h] = {} & \lambda \frac{\mu}{\omega} \int_{T-\omega}^{T} \int_\Omega \left(u_\varepsilon^0(t,x) - f_0(x)\right) h(t,x)\, dxdt \\ & + \int_{Q_T} \left[\frac{\partial h}{\partial t} + \kappa h\right] z\, dxdt \\ & + \int_{Q_T} \left[-\varepsilon \Delta h + \sum_{i=1}^{4} \mathbb{F}_{1i}(u_\varepsilon^0)[h]\right] z\, dxdt \\ & + \int_{Q_T} F_2'(u_\varepsilon^0)[h]\Big|_{q=u_\varepsilon^0} z\, dxdt, \quad \forall\, h \in .L^2(0, T; W^{1,2}(\Omega)).\end{aligned} \tag{6.16}$$

Taking into account that

$$\begin{aligned}&\int_{Q_T} F_2'(u_\varepsilon^0)[h]z\,dxdt\\&=\int_{Q_T}|R_\eta\nabla u_\varepsilon^0|^{p[\nabla u_\varepsilon^0]-2}\frac{2a^2(1-\delta)\log\left(|R_\eta\nabla u_\varepsilon^0|\right)}{(a^2+|\nabla u_\varepsilon^0|^2)^2}\left(R_\eta\nabla u_\varepsilon^0\otimes\nabla u_\varepsilon^0\right)(\nabla z,\nabla h)\;dxdt,\end{aligned}$$

we have

$$\begin{aligned}|R_\eta\nabla u_\varepsilon^0|^{p[\nabla u_\varepsilon^0]-2}&\frac{\log\left(|R_\eta\nabla u_\varepsilon^0|\right)}{(a^2+|\nabla u_\varepsilon^0|^2)^2}\left|\left(R_\eta\nabla u_\varepsilon^0\otimes\nabla u_\varepsilon^0\right)\right||(\nabla z,\nabla h)|\\&\leqslant(1-\eta^2)^{-2}|R_\eta\nabla u_\varepsilon^0|^{p[\nabla u_\varepsilon^0]}\frac{\log\left(|R_\eta\nabla u_\varepsilon^0|\right)}{(a^2+|\nabla u_\varepsilon^0|^2)^2}|\nabla z||\nabla h|,\end{aligned}$$

where

$$\begin{aligned}|R_\eta\nabla u_\varepsilon^0|^2\frac{|\log\left(|R_\eta\nabla u_\varepsilon^0|\right)|}{(a^2+|R_\eta\nabla u_\varepsilon^0|^2)^2}|\nabla z||\nabla h|&\leqslant\frac{|\log\left(|R_\eta\nabla u_\varepsilon^0|\right)|}{a^2+|R_\eta\nabla u_\varepsilon^0|^2}|\nabla z||\nabla h|\\&\leqslant\mathrm{const}|\nabla z||\nabla h|,\quad\text{as }|\nabla u_\varepsilon^0|\to\infty,\\|R_\eta\nabla u_\varepsilon^0|^2\frac{|\log\left(|R_\eta\nabla u_\varepsilon^0|\right)|}{(a^2+|R_\eta\nabla u_\varepsilon^0|^2)^2}&\leqslant\frac{1}{a^4}|R_\eta\nabla u_\varepsilon^0|^2|\log\left(|R_\eta\nabla u_\varepsilon^0|\right)|<+\infty\\&\text{as }|\nabla u_\varepsilon^0|\to 0\end{aligned}$$

by the L'Hôpital's rule.

Thus, from this we can deduce that

$$|R_\eta\nabla u_\varepsilon^0|^2\frac{|\log\left(|R_\eta\nabla u_\varepsilon^0|\right)|}{(a^2+|R_\eta\nabla u_\varepsilon^0|^2)^2}\in L^\infty(Q_T)$$

and there exists a constant $M>0$ such that

$$\begin{aligned}\Big|\int_{Q_T}F_2'(u_\varepsilon^0)[h]z\,dxdt\Big|&\leqslant 2a^2\frac{(1-\delta)}{(1-\eta^2)^2}\left\||R_\eta\nabla u_\varepsilon^0|^2\frac{|\log\left(|R_\eta\nabla u_\varepsilon^0|\right)|}{(a^2+|R_\eta\nabla u_\varepsilon^0|^2)^2}\right\|_{L^\infty(\Omega)}\\&\quad\times\int_{Q_T}|R_\eta\nabla u_\varepsilon^0|^{p(|\nabla u_\varepsilon^0|)-2}|\nabla z||\nabla h|\,dxdt\\&\leqslant\mathrm{const}\int_{Q_T}|\nabla u_\varepsilon^0|^{p(|\nabla u_\varepsilon^0|)-2}|\nabla z||\nabla h|\,dxdt\\&\leqslant\mathrm{const}\left(\int_{Q_T}|\nabla u_\varepsilon^0|^{p(|\nabla u_\varepsilon^0|)-2}|\nabla z|^2\,dxdt\right)^{1/2}\\&\quad\times\left(\int_{Q_T}|\nabla u_\varepsilon^0|^{p(|\nabla u_\varepsilon^0|)-2}|\nabla h|^2\,dxdt\right)^{1/2}\\&\leqslant\mathrm{const}\,\|z\|_{L^2(0,T;H^{p^-,u_\varepsilon^0}(\Omega))}\,\|h\|_{L^2(0,T;H^{p^-,u_\varepsilon^0}(\Omega))}\\&\leqslant M\|h\|_{L^2(0,T;W^{1,2}(\Omega))}\,\|z\|_{L^2(0,T;H^{p^-,u_\varepsilon^0}(\Omega))}\,.\end{aligned}\tag{6.17}$$

Here, $H^{p^-,u^0_\varepsilon}(\Omega)$ stands for the weighted Sobolev space which is defined as a completeness of $C^\infty_c(\mathbb{R}^2)$ with respect to the norm

$$\|z\|_{H^{p^-,u^0_\varepsilon}(\Omega)} = \int_\Omega \left[z^2 + \left(1+|\nabla u^0_\varepsilon|\right)^{p^--2}|\nabla z|^2\right] dx.$$

It is easy to check that $H^{p^-,u^0_\varepsilon}(\Omega)$ is a Hilbert space with the inner product

$$(z_1,z_2)_{H^{p^-,u^0_\varepsilon}(\Omega)} = \int_\Omega \left[z_1 z_2 + \left(1+|\nabla u^0_\varepsilon|\right)^{p^--2}(\nabla z_1,\nabla z_2)\right] dx.$$

Moreover, since $p^- = 1+\delta << 2$, it follows from the estimates

$$\int_\Omega \left(1+|\nabla u^0_\varepsilon|\right)^{p^--2}|\nabla z|^2\,dx \leqslant \int_\Omega |\nabla z|^2\,dx,$$

$$\begin{aligned}\int_\Omega |\nabla y|^{p^-}\,dx &= \int_\Omega \frac{|\nabla y|^{p^-}}{\left(1+|\nabla u^0_\varepsilon|\right)^{\frac{p^-(2-p^-)}{2}}}\left(1+|\nabla u^0_\varepsilon|\right)^{\frac{p^-\left(2-p^-\right)}{2}}\,dx\\ &\leqslant \left(\int_\Omega \left(1+|\nabla u^0_\varepsilon|\right)^{p^--2}|\nabla y|^2\,dx\right)^{\frac{p^-}{2}}\left(\int_\Omega \left(1+|\nabla u^0_\varepsilon|\right)^{p^-}\,dx\right)^{\frac{2-p^-}{2}},\end{aligned}$$

which hold true for each $z \in H^{p^-,u^0_\varepsilon}(\Omega)$ and $y \in W^{1,p^-}(\Omega)$, that

$$H^1(\Omega) \hookrightarrow H^{p^-,u^0_\varepsilon}(\Omega) \hookrightarrow W^{1,p^-}(\Omega)$$

with continuous embeddings.

Thus, in view of estimate (6.17), it is clear that, the directional derivative $F_2'(u)[h]$ with its representation (6.15) is the Gâteaux derivative of the operator F_2 and moreover, this derivative is a strongly continuous and bounded mapping. Arguing as in [6], it can be shown that, in fact, the mapping $u \mapsto \mathcal{L}_\varepsilon(v,u,\lambda,z)$ is continuously differentiable in some neighborhood of the optimal pair $(v^0_\varepsilon,u^0_\varepsilon) \in \Xi_\varepsilon\varepsilon$. Thus, in order to derive optimality conditions for the approximating OCP (5.1), it remains to repeat all arguments from [32] (see Section 2.8.2). As a result, we deduce that $\lambda = 1$ in (6.3), and the following result holds true.

Theorem 6.1. *Let for given distributions $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, $\theta \in L^\infty(\Omega;\mathbb{R}^2)$, and $v_a, v_b \in L^2(\Omega)$, and for given values of the small parameters $\varepsilon > 0$ and $\omega > 0$, $(v^0_\varepsilon,u^0_\varepsilon) \in \Xi_\varepsilon\varepsilon$ is an optimal pair to the OCP (5.1). Then there exists a unique $z_\varepsilon \in L^2(0,T;H^{p^-,u^0_\varepsilon}(\Omega))$ such that $\dot z_\varepsilon \in L^2(0,T;\left[H^{p^-,u^0_\varepsilon}(\Omega)\right]')$ and*

$$\left.\begin{aligned}\frac{\partial u^0_\varepsilon}{\partial t} - \varepsilon\Delta u^0_\varepsilon - \operatorname{div}\left[\left(|R_\eta\nabla u^0_\varepsilon|+\varepsilon\right)^{p_{u^0_\varepsilon}(t,x)-2}R_\eta\nabla u^0_\varepsilon\right] + \kappa u^0_\varepsilon&\\ = \kappa(f - v^0_\varepsilon) \quad \text{in} \ \ Q_T := (0,T)\times\Omega,&\\ \partial_\nu u^0_\varepsilon = 0 \quad \text{on} \ \ (0,T)\times\partial\Omega,&\\ u^0_\varepsilon(0,\cdot) = f_0 \quad \text{in} \ \ \Omega,&\end{aligned}\right\} \tag{6.18}$$

$$\left.\begin{aligned}
&-\frac{\partial z_\varepsilon}{\partial t} - \varepsilon\Delta z_\varepsilon - \operatorname{div}\left[\left(|R_\eta \nabla u^0_\varepsilon| + \varepsilon\right)^{p_{u^0_\varepsilon}(t,x)-2}\nabla z_\varepsilon\right] + \kappa z_\varepsilon\\
&\quad - \operatorname{div}\left[(p_{u^0_\varepsilon}(t,x)-2)\left(|R_\eta\nabla u^0_\varepsilon| + \varepsilon\right)^{p_{u^0_\varepsilon}(t,x)-4} R_\eta \nabla u^0_\varepsilon\left(\nabla u^0_\varepsilon, \nabla z_\varepsilon\right)\right]\\
&\quad + \eta^2 \operatorname{div}\left[\left(|R_\eta\nabla u^0_\varepsilon| + \varepsilon\right)^{p_{u^0_\varepsilon}(t,x)-2}(\theta\otimes\theta)\nabla z_\varepsilon\right]\\
&\quad + \eta^2 \operatorname{div}\left[(p_{u^0_\varepsilon}(t,x)-2)\left(|R_\eta\nabla u^0_\varepsilon| + \varepsilon\right)^{p_{u^0_\varepsilon}(t,x)-4} R_\eta\nabla u^0_\varepsilon(\theta\otimes\theta)\nabla z_\varepsilon\right]\\
&\quad - \operatorname{div}\left(|R_\eta\nabla u^0_\varepsilon|^{p_{u^0_\varepsilon}(t,x)-2}\frac{2a^2(1-\delta)\log\left(|R_\eta\nabla u^0_\varepsilon|\right)}{\left(a^2+|\nabla u^0_\varepsilon|^2\right)^2}\left(R_\eta\nabla u^0_\varepsilon\otimes\nabla u^0_\varepsilon\right)\nabla z_\varepsilon\right)\\
&\quad = -\frac{\mu}{\omega}(u^0_\varepsilon - f_0)\chi_{[T-\omega,T]}(t) \quad in \quad Q_T := (0,T)\times\Omega,\\
&\partial_\nu z_\varepsilon = 0 \quad on \quad (0,T)\times\partial\Omega,\\
&z_\varepsilon(T,\cdot) = 0 \quad in \quad \Omega,
\end{aligned}\right\} \tag{6.19}$$

$$\int_0^T\left[\int_\Omega \frac{2v^0_\varepsilon}{|v^0_\varepsilon|+\varepsilon}\int_\Omega |v^0_\varepsilon|\,dx + \kappa z_\varepsilon\left(v - v^0_\varepsilon\right)\,dx\right]dt \geqslant 0, \quad \forall\, v\in\mathcal{V}_{ad}. \tag{6.20}$$

Here, $\chi_{[T-\omega,T]}(t)$ stands for the characteristic function of the set $[T-\omega,T]$.

Finally, it is worth to notice that the elliptic operator in the principle part of the system 7.15b is coercive, monotone, and hemicontinuous for each $\varepsilon > 0$. Hence, by the classical results of the theory of linear PDE [31], a weak solution of the adjoint system (6.19) is unique in the weighted space

$$\left\{z \in z_\varepsilon \in L^2(0,T; H^{p^-,u^0_\varepsilon}(\Omega)), \quad \dot{z}_\varepsilon \in L^2(0,T;\left[H^{p^-,u^0_\varepsilon}(\Omega)\right]')\right\}.$$

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Received 17.12.2023