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HOW CAN WE MANAGE REPAIRING A BROKEN FINITE VIBRATING STRING? FORMULATIONS OF THE PROBLEM

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Abstract. Three approaches to solve a well-known IBVP posed for vibrating composite string with piece-wise constant properties have been applied. The main issue of the stufy is the number of the matching conditions to be imposed for the solution to the IBVP to be obtained.

Key words: separation of variables, the Laplace transform, eigenvalues and eigenfunctions, the energy equation, the transmission and matching conditions.

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1. Introduction and the problem formulation

The current study was inspired by a series of our previous publications [1–5] dealing with an IBVP for the 1D degenerate wave equation. Interpreting the IBVP as that of a vibrating 'string', driven by: *a*) initial disturbances of the shape and velocity of 'the string', and *b*) known external controls imposed at both ends of 'the string', is convenient to discuss the IBVP formulation and approaches to solve it. Since the degeneracy was related to the coefficient function of the wave equation vanishing in an interior point of the spatial segment, being interpreted as the undisturbed position of 'the string', the point of degeneracy, in its turn, can be interpreted as a hinge, where the known transmission conditions (continuity of 'the string' and the transverse component of the longitudinal tension, usually referred to as the flux) hold.

We studied different approaches to solve the IBVP for the 1D degenerate wave equation based on separation of variables (SV). Some of them imply that we pose and solve separately associated IBVPs for both regular parts of 'the string' and then match the obtained solutions at the degeneracy point using the transmission conditions. Physically this means that we: *a*) 'cut' (or, 'break') 'the string' at the degeneracy point, *b*) observe vibrations of both parts of 'the string', and then *c*) match both parts again at any instant to restore 'the entire string' (or, in other

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words, 'to repair' it) by forcing some corrections to positions and inclinations of both regular parts for the transmission conditions to hold. We found out that these conditions, generally speaking, are not sufficient for matching (or, for 'repairing' the broken 'degenerate string'), therefore some other conditions are necessary to be applied. To clarify the situation, we have decided to transfer our approaches to the entire regular string composed of two parts with different elastic properties. Since the way to solve the IBVP of the vibrating composite string is known, we expect that will have ideal opportunities for studying our previous approaches and comparing them with those known for the composite string.

Thus, we currently deal with the following well-known initial boundary value problem (referred to below as the *original* IBVP, or the IBVPO for short) for the 1D homogeneous wave-equation in closed space-time rectangle $[0,T]\times[0,l]$

$$\begin{cases} Q\,[W(t,x)] = 0\,, & (t,x)\in G_0\,,\\ \left.\begin{aligned} \frac{\partial W(0,x)}{\partial t} &= \overset{**}{W}(x)\\ W(0,x) &= \overset{*}{W}(x)\end{aligned}\right\}, & x\in[0,l]\,,\\ \left.\begin{aligned} W(t,0) &= \chi_1(t)\\ W(t,\,l) &= \chi_2(t)\end{aligned}\right\}, & t\in[0,T]\,,\end{cases} \tag{1.1}$$

where the second order differential operator

$$Q\,[W(t,x)] = \frac{\partial^2 W(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(c(x)\,\frac{\partial W(t,x)}{\partial x}\right) \tag{1.2}$$

is defined in $G_0 = G_1 \bigcup G_2$, $G_1 = (0,T]\times(0,x_0)$, $G_2 = (0,T]\times(x_0,l)$, $0<x_0<l$, for functions $W(t,x)\in\mathscr{C}^{(2,2)}(G_0)$; the coefficient function is piece-wise constant

$$c(x) = \begin{cases} c_1 > 0\,, & x\in[0,x_0)\,,\\ c_2 > 0\,, & x\in(x_0,\,l]\,,\end{cases} \tag{1.3}$$

being defined non-uniquely at the jump discontinuity: $c_1 \leqslant c(x_0) \leqslant c_2$; the given *a*) control $\chi_1(t),\chi_2(t)\in\mathscr{C}^1[0,T]\bigcap\mathscr{C}^2(0,T]$ and *b*) initial $\overset{*}{W}(x),\overset{**}{W}(x)\in\mathscr{C}^1[0,l]$ functions obey the compatibility conditions

$$\begin{cases} \chi_1(0) = \overset{*}{W}(0)\,, & \overset{*}{W}(l) = \chi_2(0)\,,\\ \chi_1'(0) = \overset{**}{W}(0)\,, & \overset{**}{W}(l) = \chi_2'(0)\,.\end{cases} \tag{1.4}$$

Along the dividing segment $[0,T]\times\{x_0\}$ of subrectangles G_1, G_2 the required solution $W(t,x)$ to the IBVPO obeys the transmission conditions

$$\begin{cases} W(t,x_0-0) = W(t,x_0+0)\,,\\ \mathcal{F}\,(t,x_0-0) = \mathcal{F}\,(t,x_0+0)\,,\end{cases} \quad t\in[0,T]\,, \tag{1.5}$$

the second condition being expressed in terms of the flux

$$\mathcal{F}(t,x) = c(x)\,\frac{\partial W(t,x)}{\partial x}\,. \tag{1.6}$$

The only solution $W(t,x)$ to the IBVPO is known can be interpreted as the distributed over segment $[0,l]$ displacements of a string, subject to known external controls $\chi_1(t)$ and $\chi_2(t)$, imposed at both ends of the string as the Dirichlet boundary conditions, and having the initial distributed displacements $\overset{*}{W}(x)$ and velocities $\overset{**}{W}(x)$. Therefore, the first transmission condition (1.5) expresses the continuity of the string, whereas the second one expresses the continuity of the transverse component of the tension produced in the string.

The original IBVP, posed in $\bar{G}_0 = [0,T]\times[0,l]$, is known can be solved by applying SV [6,8], but our concern related to the original IBVP is based on the following underlying ideas: *a*) reformulating the original IBVP into two associated IBVPs in the above closed space-time subrectangles $\bar{G}_1$, $\bar{G}_2$; *b*) adding conditions of smooth matching the solutions to the associated IBVPs along the dividing segment; *c*) solving such a *composite* IBVP.

For the sake of convenience (and to establish a relation with [1–5] dealing with the IBVP similar to (1.1) – (1.5) for a 1D degenerate wave equation), we introduce the following transformation of the independent variables

$$\begin{cases} t = \dfrac{l}{\sqrt{c_0}}\,\underline{t}\,, & \underline{t} \in [0,\underline{T}]\,,\\[2mm] x = l_1\,\underline{x} + l_1, & \underline{x} \in K_1\,,\\[2mm] x = l_2\,\underline{x} + l_1\,, & \underline{x} \in K_2\,, \end{cases} \qquad \begin{cases} \underline{t} = \dfrac{\sqrt{c_0}}{l}\,t\,, & t \in [0,T]\,,\\[2mm] \underline{x} = \dfrac{x-l_1}{l_1}\,, & x \in [0,\,l_1]\,,\\[2mm] \underline{x} = \dfrac{x-l_1}{l_2}\,, & x \in [l_1,\,l]\,, \end{cases} \tag{1.7}$$

where $K_1 := [-1,0]$, $K_2 := [0,+1]$, $c_0 = \frac{1}{2}\,(c_1 + c_2)$, $l_1 = x_0$, $l_2 = l - l_1$; then replace the given in (1.1) functions with the following ones

$$\begin{cases} \overset{**}{u}(\underline{x}) = \dfrac{\sqrt{c_0}}{l}\,\overset{**}{W}(x)\big|_{x\to\underline{x}}\,,\\[2mm] \overset{*}{u}(\underline{x}) = \qquad \overset{*}{W}(x)\big|_{x\to\underline{x}}\,, \end{cases} \qquad \begin{cases} h_1(\underline{t}) = \chi_1(t)\big|_{t\to\underline{t}}\,,\\[2mm] h_2(\underline{t}) = \chi_2(t)\big|_{t\to\underline{t}}\,. \end{cases}$$

Finally, to simplify notation, we drop the bars under the new independent variables and present the IBVPO in the following form

$$\begin{cases} S\,[u(t,x)] = 0\,, & (t,x) \in D_0\,,\\[2mm] \left.\begin{aligned} \dfrac{\partial u(0,x)}{\partial t} &= \overset{**}{u}(x)\\ u(0,x) &= \overset{*}{u}(x) \end{aligned}\right\}, & x \in K_0\,,\\[2mm] \left.\begin{aligned} u\,(t,-1) &= h_1(t)\\ u\,(t,+1) &= h_2(t) \end{aligned}\right\}, & t \in [0,T]\,, \end{cases} \tag{1.8}$$

where: $a)$ $K_0 \coloneqq [-1,+1]$; $b)$ the second order differential operator

$$S\,[u(t,x)] = \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x)\,\frac{\partial u(t,x)}{\partial x}\right) \tag{1.9}$$

is defined in $D_0 = D_1 \bigcup D_2$, $D_1 = (0,T] \times I_1$, $D_2 = (0,T] \times I_2$, $I_1 \coloneqq (-1,0)$, and $I_2 \coloneqq (0,+1)$; $c)$ the piece-wise constant coefficient function is as follows

$$a(x) = \begin{cases} a_1 = \dfrac{c_1}{c_0}\left(\dfrac{l}{l_1}\right)^2 > 0\,, & x \in J_1\,, \\[2ex] a_2 = \dfrac{c_2}{c_0}\left(\dfrac{l}{l_2}\right)^2 > 0\,, & x \in J_2\,, \end{cases} \tag{1.10}$$

$J_1 \coloneqq [-1,0)$, $J_2 \coloneqq (0,+1]$; $d)$ the given 1) control $h_1(t), h_2(t)$ and 2) initial $\overset{*}{u}(x)$, $\overset{**}{u}(x)$ functions obey the compatibility conditions

$$\begin{cases} h_1(0) = \overset{*}{u}(-1)\,, & \overset{*}{u}(+1) = h_2(0)\,, \\ h_1'(0) = \overset{**}{u}(-1)\,, & \overset{**}{u}(+1) = h_2'(0)\,; \end{cases} \tag{1.11}$$

$e)$ the transmission conditions read

$$\begin{cases} u(t,0-0) = u(t,0+0)\,, \\ f(t,0-0) = f(t,0+0)\,, \end{cases} \qquad t \in [0,T]\,, \tag{1.12}$$

where the flux is defined similarly to that of (1.6)

$$f(t,x) = a(x)\,\frac{\partial u(t,x)}{\partial x}\,\cdot \tag{1.13}$$

The above formulation of the IBVPO below is referred to as the *derived* IBVP, or shortly as the IBVPD. We believe, that confusing the independent variables (t,x), being dimensional in the IBVPO and non-dimensional in the IBVPD, is impossible, due to the context they will be used in.

Following the underlying idea of the current study, we formulate two IBVPs, respectively in the closures $\bar{D}_1$ and $\bar{D}_2$ of subrectangles D_1 and D_2, associated with the derived IBVPD (1.8) – (1.12). The left associated IBVP_1 yields to

$$\begin{cases} S\,[u_1(t,x)] = 0\,, & (t,x) \in D_1\,, \\[1ex] \left.\begin{aligned} \frac{\partial u_1(0,x)}{\partial t} &= \overset{**}{u}(x) \\ u_1(0,x) &= \overset{*}{u}(x) \end{aligned}\right\}, & x \in K_1\,, \\[3ex] \left.\begin{aligned} u_1(t,-1) &= h_1(t) \\ |u_1(t,\;\;0)| &< \infty \end{aligned}\right\}, & t \in [0,T]\,, \end{cases} \tag{1.14}$$

supplemented with the following compatibility conditions

$$\begin{cases} h_1(0) = \overset{*}{u}(-1)\,, & u_1(0,0) = \overset{*}{u}(0)\,; \\ h_1'(0) = \overset{**}{u}(-1)\,, & \dfrac{\partial u_1(0,0)}{\partial t} = \overset{**}{u}(0)\,; \end{cases} \tag{1.15}$$

whereas the right associated IBVP$_2$ yields to

$$\begin{cases} S\left[u_2(t,x)\right] = 0\,, & (t,x) \in D_2\,, \\ \left.\begin{aligned} \frac{\partial u_2(0,x)}{\partial t} &= \overset{**}{u}(x) \\ u_2(0,x) &= \overset{*}{u}(x) \end{aligned}\right\}, & x \in K_2\,, \\ \left.\begin{aligned} |u_2(t,\ \ 0)| &< \infty \\ u_2(t,+1) &= h_2(t) \end{aligned}\right\}, & t \in [0,T]\,, \end{cases} \tag{1.16}$$

supplemented with the following compatibility conditions

$$\begin{cases} \overset{*}{u}(0) = \ u_2(0,0)\ , & \overset{*}{u}(+1) = h_2(0)\,; \\ \overset{**}{u}(0) = \dfrac{\partial u_2(0,0)}{\partial t}\,, & \overset{**}{u}(+1) = h_2'(0)\,. \end{cases} \tag{1.17}$$

For the composite solution

$$u(t,x) = \begin{cases} u_1(t,x)\,, & (t,x) \in \bar{D}_1\,, \\ u_2(t,x)\,, & (t,x) \in \bar{D}_2\,, \end{cases} \tag{1.18}$$

to be one-valued, continuous and to have the continuous flux, the following matching conditions, inheriting the transmission conditions (1.12), are imposed along the dividing segment

$$\begin{cases} u_1(t,0) = u_2(t,0)\,, \\ f_1(t,0) = f_2(t,0)\,, \end{cases} \qquad t \in [0,T]\,, \tag{1.19}$$

where both fluxes $f_1(t,x), f_2(t,x)$ are defined similarly to that of (1.13).

To answer the question of how to match the solutions to the above associated IBPVs (1.14), (1.16) we will take the following steps.

In Sect. 2 we refer to the energy equation and recall some its properties used further. In Sect. 3 we consider in some detail a classical approach [6], based on SV, to solve the IBVPO (1.8). The solution obtained will be used further not only as the master solution, but also to demonstrate continuous differentiability of the flux at the midpoint $x_0 = 0$, where the coefficient function $a(x)$ is discontinuous.

In Sect. 4 we apply the Laplace transformation (LT) to the associated IBVPs, obtain the solutions to the images of the IBVPs followed by matching the obtained solutions. We consider this approach to be classical as well, since applying the integral transformation of the associated IBVPs precedes matching their solutions. The key issue we are interested is how many conditions we need to match the solutions, whereas the inversion of the image solutions will be postponed to the next publication on the subject.

In Sect. 5 we apply SV to the associated IBVPs to obtain their solutions, then match the obtained solutions followed by applying LT to the matching conditions. Again we are interested in the number of conditions necessary for the matching and show that the matching conditions (1.19) should be supplemented with two more conditions, one of which is local, similarly to (1.19), and the other is non-local and presented by the energy equation.

In Sect. 6 we briefly summarize our observations concerning the matching procedures and the number of the matching conditions.

2. The energy equation

In case of treating the string as a whole, being obeyed the IBVPD (1.8), (1.12), (1.11), the energy equation can be obtained following a well known procedure: 1) multiplying the wave equation by the local velocity of the transverse motion, and 2) integrating the product in x over the whole segment K_0

$$\int_{K_0} \frac{\partial u}{\partial t}\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a\,\frac{\partial u}{\partial x}\right)\right] \mathrm{d}x = 0\,. \tag{2.1}$$

Integration is then performed by parts for the first

$$\int_{K_0} \frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t^2}\,\mathrm{d}x = \int_{K_0} \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2\right] \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{K_0}\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2\right] \mathrm{d}x}_{\Omega(t)}$$

and second

$$\int_{K_0} \frac{\partial u}{\partial t}\,\frac{\partial}{\partial x}\left(a\,\frac{\partial u}{\partial x}\right)\,\mathrm{d}x = \underbrace{\left[\frac{\partial u}{\partial t}\left(a\,\frac{\partial u}{\partial x}\right)\right]\Bigg|_{x=-1}^{x=+1}}_{\mathrm{A}(t)} - \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{K_0}\left[\frac{a}{2}\left(\frac{\partial u}{\partial x}\right)^2\right] \mathrm{d}x}_{\Pi(t)}$$

terms in (2.1) respectively. The resulting equation for the total energy rate reads

$$\Theta'(t) = \mathrm{A}(t)\,, \qquad t \in [0, T]\,, \tag{2.2}$$

where $\Theta(t)$ is the total energy, composed of the kinetic $\Omega(t)$ and potential $\Pi(t)$ one, and $\mathrm{A}(t)$ is the power of the external forces acting on the ends of the 'string' as the known controls $h_1(t)$, $h_2(t)$. Integrating the above equation in t yields to the required total energy equation

$$\Theta(t) = \Theta(0) + \int_0^t \mathrm{A}(\tau)\,\mathrm{d}\tau\,, \qquad t \in [0, T]\,, \tag{2.3}$$

where the integral over $[0,t]$ represents the work done by the external forces.

In case of treating the string as consisting of two parts, being obeyed the left and the right associated IBVPs of Sect. 1, (1.14), (1.15) and (1.16), (1.17) respectively, one finds the following a) total energy rate equations

$$\Theta_j'(t) = \mathrm{A}_j(t)\,, \qquad t \in [0,T]\,, \tag{2.4}$$

and b) total energy equations

$$\Theta_j(t) = \Theta_j(0) + \int_0^t \mathrm{A}_j(\tau)\,\mathrm{d}\tau\,, \qquad t \in [0,T]\,, \tag{2.5}$$

where $j = 1, 2$ (for the left and right parts of the string respectively), and the powers of the external forces acting on the ends of both parts of the string read

$$\mathrm{A}_1(t) = \left[\frac{\partial u_1}{\partial t}\left(a_1\frac{\partial u_1}{\partial x}\right)\right]\Bigg|_{x=-1}^{x=\ 0}, \quad \mathrm{A}_2(t) = \left[\frac{\partial u_2}{\partial t}\left(a_2\frac{\partial u_2}{\partial x}\right)\right]\Bigg|_{x=\ 0}^{x=+1}, \quad t \in [0,T]\,.$$

Both equations (2.2) and (2.3) can be readily derived from (2.4) and (2.5) respectively, since: a) the total energy $\Theta(t)$ is additive; b) the following equality

$$\left[\frac{\partial u_1}{\partial t}\left(a_1\frac{\partial u_1}{\partial x}\right)\right]\Bigg|_{x=0} = \left[\frac{\partial u_2}{\partial t}\left(a_2\frac{\partial u_2}{\partial x}\right)\right]\Bigg|_{x=0}, \qquad t \in [0,T]\,,$$

holds, due to the matching conditions (1.19).

3. A classical approach to solve the IBVPD based on SV

3.1. Preliminaries to SV

A classical approach to solve the IBVPD (1.8) – (1.12) is based on: a) reducing the former to an auxiliary IBVP with the homogeneous boundary conditions and b) invoking the anzatz

$$\underset{\smile}{v}(t,x) = \underset{\smile}{Q}(t)\,\underset{\smile}{X}(x) \tag{3.1}$$

for finding particular solutions to the auxiliary IBVP by applying SV as follows

$$\frac{\underset{\smile}{Q}''(t)}{\underset{\smile}{Q}(t)} = \frac{\underset{\smile}{F}'(x)}{\underset{\smile}{X}(x)} = -\underset{\smile}{\lambda} = \texttt{const}\,, \qquad \underset{\smile}{\lambda} > \texttt{0}\,, \tag{3.2}$$

where $\underset{\smile}{F}(x) = a(x)\,\underset{\smile}{X}'(x)$ is 'the flux' of $\underset{\smile}{X}(x)$. The key issue in invoking (3.1) is to build the complete set of functions $\underset{\smile}{X}(x)$ that satisfy (3.2) and the proper boundary and transmission conditions. According to [6] (a collection of problems, supplementing the textbook [8]; refer, for example, to problems 164 – 166 on p. 37, problem 57 on p. 128, etc.), building the required set is based on

Proposition 3.1. Let the following composite BVP be given

$$\begin{cases} \utilde{F}'(x) + \utilde{\lambda}\utilde{X}(x) = 0\,, & x \in I_1 \bigcup I_2\,, \\ \utilde{X}(-1) = \utilde{X}(+1) = 0\,, \end{cases} \tag{3.3}$$

supplemented with the transmission conditions

$$\begin{cases} \utilde{X}(0-0) = \utilde{X}(0+0)\,, \\ \utilde{F}(0-0) = \utilde{F}(0+0)\,, \end{cases} \tag{3.4}$$

then: *a*) the complete countable set of the continuous and piece-wise smooth eigenfunctions of the problem is determined as follows

$$\utilde{X}_\mu(x) = \begin{cases} \left[\sin\left(\dfrac{\utilde{\alpha}_\mu}{\sqrt{a_1}}\right)\right]^{-1} \sin\left(\dfrac{\utilde{\alpha}_\mu}{\sqrt{a_1}}\,(1+x)\right), & x \in J_1\,, \\ \left[\sin\left(\dfrac{\utilde{\alpha}_\mu}{\sqrt{a_2}}\right)\right]^{-1} \sin\left(\dfrac{\utilde{\alpha}_\mu}{\sqrt{a_2}}\,(1-x)\right), & x \in J_2\,, \end{cases} \tag{3.5}$$

whereas the eigenvalues $\{\utilde{\lambda}_\mu\}_{\mu=1}^{\infty} \equiv \{\utilde{\alpha}_\mu^2\}_{\mu=1}^{\infty}$ associated with (3.5) are determined as the set of the squared roots of the following transcendental equation wrt $\utilde{\alpha}_\mu$

$$\sqrt{a_1}\,\cot\left(\frac{\utilde{\alpha}_\mu}{\sqrt{a_1}}\right) + \sqrt{a_2}\,\cot\left(\frac{\utilde{\alpha}_\mu}{\sqrt{a_2}}\right) = 0\,; \tag{3.6}$$

b) the eigenfunctions (3.5) are orthogonal in $\mathscr{L}_{2,0} \coloneqq \mathscr{L}_2\,(K_0)$, that is

$$\left(\utilde{X}_\mu\,, \utilde{X}_\gamma\right)_0 = \int_{K_0} \utilde{X}_\mu(x)\,\utilde{X}_\gamma(x)\,\mathrm{d}x = \|\utilde{X}_\mu\|_0^2\,\delta_{\mu,\gamma}\,, \tag{3.7}$$

$(p,q)_0$ is the inner product of two elements $p(x), q(x) \in \mathscr{L}_{2,0}$, $\|r\|_0$ is the norm of an element $r(x) \in \mathscr{L}_{2,0}$, $\mu, \gamma \in \mathbb{N}$.

We note, in addition to the properties specified in Prop. 3.1, that the eigenfunctions $\utilde{X}_\mu(x)$ (3.5) have one more remarkable and easily verified property of continuous differentiability of their fluxes $\utilde{F}_\mu(x) = a(x)\,\utilde{X}'_\mu(x)$ over the segment K_0, including the midpoint: $\utilde{F}'_\mu(x)(0-0) = \utilde{F}'_\mu(0+0)$, where the piece-wise constant coefficient function $a(x)$ is discontinuous. We will apply this property in Sect. 5 to solve the IBVPD using a non-classical approach.

To clarify building $\utilde{X}_\mu(x)$ (3.5), we consider 3 cases, indicated in Tbl. 1 (the origin of the cases is explained by taking $c_1 = 1$, $c_2 = 4$, $l = 2$ and $x_0 = 0.25$, 0.50, 1.00 in (1.3) and then applying (1.10)). The infinite series of the roots $\utilde{\alpha}_\mu$ of (3.6) for the cases are formed by periodic shifts of the 8-tuples, 4-tuples, and 2-tuples (the octuples, quadruples, and couples), respectively, placed in Tbl. 1. Some eigenfunctions for the cases are shown in Figs. 3.1, 3.2, 3.3.

Table 1. Roots of transcendental equation (3.6) (cases 1 – 3)

No	a_1	a_2	α_μ
1	25.6	2.0897959183...	4.11705742307002802074... 7.36681251991832973695... 9.95011848102124236679... 13.80472706515471812453... 17.98595538539280582565... 21.84056396952628158339... 24.42386993062919421323... 27.67362502747749592944...
2	6.4	2.8(4)	3.30228352255807050061... 6.16702110866456655154... 9.72832011660919542354... 12.59305770271569147447...
3	1.6	6.4	2.90995823927664691729... 5.03771237336023407025...

In case 4, with the coefficients $a_1 = a_2 = 4$ (originated from $c_1 = c_2 = 1$, $l = 2$, and $x_0 = 1$ in (1.3)), the roots of (3.6) and the eigenfunctions (3.5) are reduced to

$$\frac{\alpha_\mu}{\sqrt{a_0}} = \mu\pi - \frac{\pi}{2} \equiv \hat{\alpha}_\mu\,, \qquad X_\mu(x) = \cos\left(\frac{\alpha_\mu}{\sqrt{a_0}}\,x\right) = \cos\frac{(2\mu-1)\,\pi x}{2}\,\cdot \tag{3.8}$$

On the other hand, independent of Prop. 3.1 considering identically constant coefficient function (1.10) (case 5), that is $a(x) \equiv a_0$, is based on

Proposition 3.2. Let the following BVP be given

$$\begin{cases} a_0 X''(x) + \lambda X(x) = 0\,, & x \in I_0\,, \\ X(-1) = X(+1) = 0\,, \end{cases} \tag{3.9}$$

where $I_0 := (-1, +1)$, then: *a*) the countable sets of the eigenvalues and the eigenfunctions of the problem of two kinds are determined as follows (see Fig. 3.3)

$$\lambda_{1,\mu} \equiv \sigma_{1,\mu}^2 = a_0\,(\mu\pi)^2, \qquad \lambda_{2,\mu} \equiv \sigma_{2,\mu}^2 = a_0\Big(\mu\pi - \frac{\pi}{2}\Big)^2, \tag{3.10}$$

$$\begin{cases} X_{1,\mu}(x) = \sin\left(\dfrac{\sigma_{1,\mu}}{\sqrt{a_0}}\,x\right) = \sin\left(\mu\pi x\right), \\ X_{2,\mu}(x) = \cos\left(\dfrac{\sigma_{2,\mu}}{\sqrt{a_0}}\,x\right) = \cos\dfrac{(2\mu-1)\,\pi x}{2}\,; \end{cases} \tag{3.11}$$

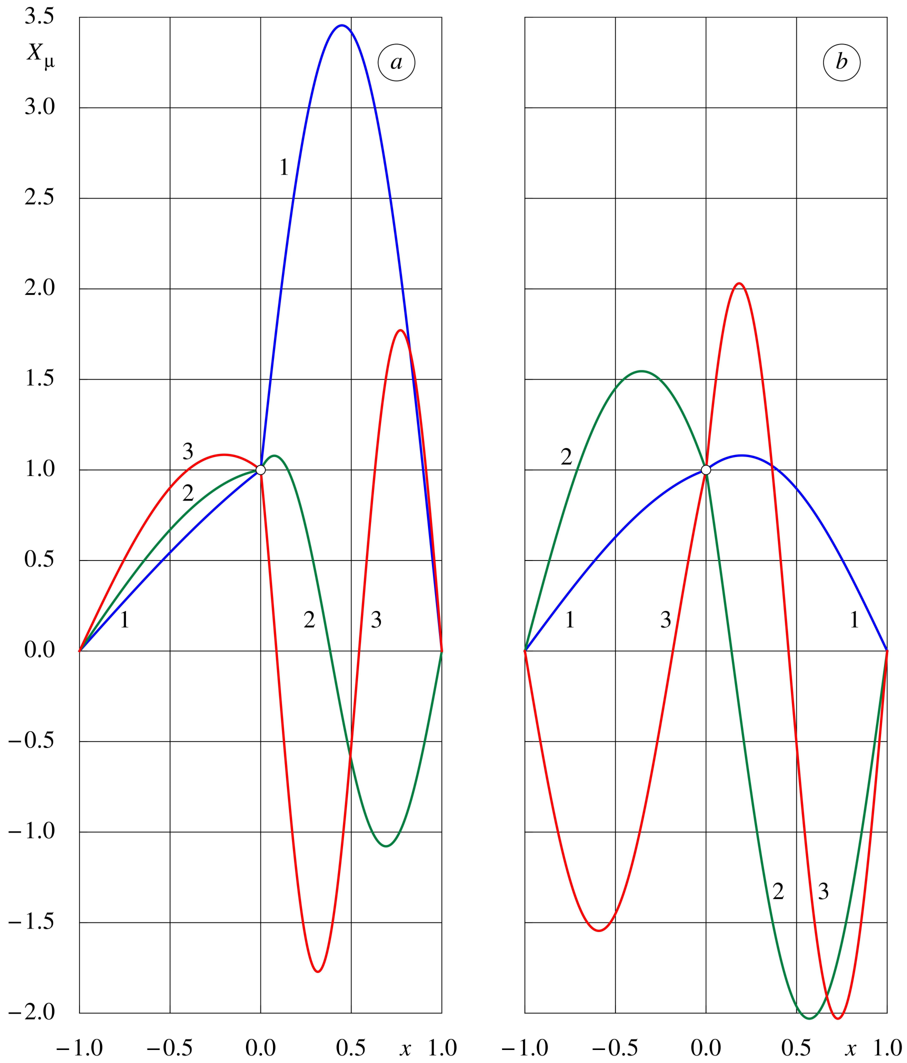

Fig. 3.1. Eigenfunctions (3.5): case 1 (a), case 2 (b); $\mu = 1, 2, 3$

b) eigenfunctions (3.11): 1) of each kind are orthogonal in $\mathscr{L}_{2,0}$; 2) of both kinds are biorthogonal in $\mathscr{L}_{2,0}$, that is

$$\big(X_{k,\mu}\,, X_{k,\gamma}\big)_0 = \|X_{k,\mu}\|_0^2\,\delta_{\mu,\gamma} = \delta_{\mu,\gamma}\,, \qquad \big(X_{1,\mu}\,, X_{2,\gamma}\big)_0 = 0\,. \tag{3.12}$$

We note that the eigenfunctions (3.8) are evidently to be the same as those (3.11) of the second kind (cf. Figs. 3.2, b and 3.3, b).

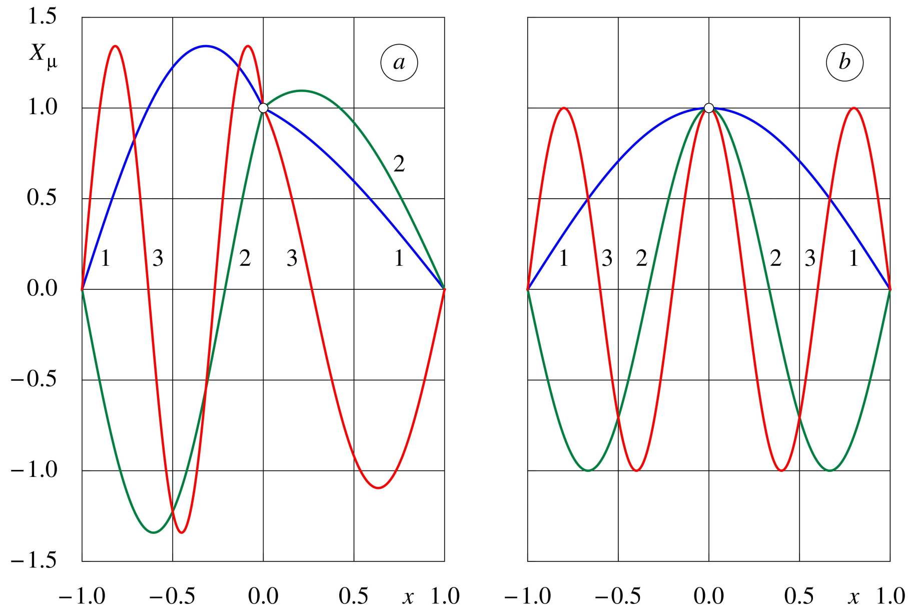

Fig. 3.2. Eigenfunctions (3.5): case 3 (a), and eigenfunctions (3.8): case 4 (b); $\mu = 1, 2, 3$

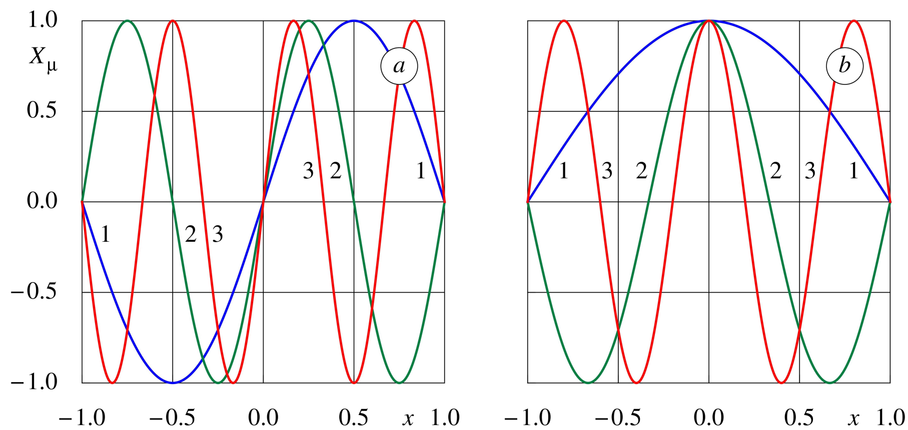

Fig. 3.3. Eigenfunctions (3.11), case 5: of the first (a) and second (b) kinds; $\mu = 1, 2, 3, 4$

3.2. Applying SV to the IBVPD

The ansatz for the solution to the derived IBVPD (1.8) – (1.12) read

$$\underset{\smile}{u}(t, x) = \underset{\smile}{v}(t, x) + \underset{\smile}{w}(t, x)\,, \tag{3.13}$$

where: a) function $\underset{\smile}{v}(t, x)$ is required; b) function $\underset{\smile}{w}(t, x)$ is given as follows

$$\underset{\smile}{w}(t, x) = \underset{\smile}{\phi}_1(x)\, h_1(t) + \underset{\smile}{\phi}_2(x)\, h_2(t)\,; \tag{3.14}$$

c) blending functions $\underset{\smile}{\phi}_1(x), \underset{\smile}{\phi}_2(x)$ satisfy the following conditions

$$\begin{cases} \underset{\smile}{\phi}_1(-1) = 1\,; & \underset{\smile}{\phi}_1(x) \equiv 0\,, \quad x \in K_1\,; \quad \underset{\smile}{\phi}''_1(x) \in \mathscr{C}\,(K_0)\,; \\ \underset{\smile}{\phi}_2(+1) = 1\,; & \underset{\smile}{\phi}_2(x) \equiv 0\,, \quad x \in K_2\,; \quad \underset{\smile}{\phi}''_2(x) \in \mathscr{C}\,(K_0)\,; \end{cases} \tag{3.15}$$

whereas their fluxes $\underset{\smile}{\varphi}_j(x) = a(x)\, \underset{\smile}{\phi}'_j(x)$ vanish on the dividing segment: $\underset{\smile}{\varphi}_j(0) = 0$, to meet the transmission conditions (1.12).

Combining (3.13) – (3.15) with the IBVPD formulation (1.8), (1.11) yields to: a) the initial conditions for the required function $\underset{\smile}{v}(t,x)$

$$\begin{cases} \dfrac{\partial \underset{\smile}{v}(0,x)}{\partial t} = \dfrac{\partial \underset{\smile}{u}(0,x)}{\partial t} - \dfrac{\partial \underset{\smile}{w}(0,x)}{\partial t} = \overset{**}{u}(x) - \dfrac{\partial \underset{\smile}{w}(0,x)}{\partial t} \equiv \overset{**}{\underset{\smile}{v}}(x)\,, \\ \underset{\smile}{v}(0,x) = \underset{\smile}{u}(0,x) - \underset{\smile}{w}(0,x) = \overset{*}{u}(x) - \underset{\smile}{w}(0,x) \equiv \overset{*}{\underset{\smile}{v}}(x)\,, \end{cases} \tag{3.16}$$

and b) reformulation of the IBVPD into the following auxiliary IBVPA wrt $\underset{\smile}{v}(t,x)$

$$\begin{cases} S\,[\underset{\smile}{v}(t,x)] = \underset{\smile}{g}(t,x)\,, & (t,x) \in D_0\,, \\ \left.\begin{array}{r} \dfrac{\partial \underset{\smile}{v}(0,x)}{\partial t} = \overset{**}{\underset{\smile}{v}}(x) \\ \underset{\smile}{v}(0,x) = \overset{*}{\underset{\smile}{v}}(x) \end{array}\right\}, & x \in K_0\,, \\ \left.\begin{array}{r} \underset{\smile}{v}(t,-1) = 0 \\ \underset{\smile}{v}(t,+1) = 0 \end{array}\right\}, & t \in [0,T]\,, \end{cases} \tag{3.17}$$

where $\underset{\smile}{g}(t,x) = -S\,[\underset{\smile}{w}(t,x)]$ is the right hand side of the above non-homogeneous wave equation, and the compatibility conditions hold: $\overset{*}{\underset{\smile}{v}}(-1) = \overset{*}{\underset{\smile}{v}}(+1) = 0$, $\overset{**}{\underset{\smile}{v}}(-1) = \overset{**}{\underset{\smile}{v}}(+1) = 0$.

Then we: a) expand $\underset{\smile}{g}(t,x)$ and $\overset{*}{\underset{\smile}{v}}(x)$, $\overset{**}{\underset{\smile}{v}}(x)$ into the series wrt $\underset{\smile}{X}_\mu(x)$ (3.5)

$$\underset{\smile}{g}(t,x) = \sum_{\mu=1}^{\infty} \underset{\smile}{g}_\mu(t)\, \underset{\smile}{X}_\mu(x)\,, \qquad \overset{*}{\underset{\smile}{v}}(x) = \sum_{\mu=1}^{\infty} \overset{*}{\underset{\smile}{v}}_\mu\, \underset{\smile}{X}_\mu(x)\,, \quad \overset{**}{\underset{\smile}{v}}(x) = \sum_{\mu=1}^{\infty} \overset{*}{\underset{\smile}{v}}_\mu\, \underset{\smile}{X}_\mu(x)\,,$$

where the coefficients are determined directly by integration: $\|\underset{\smile}{X}_\mu\|_0^2\, y_\mu = \big(y, \underset{\smile}{X}_\mu\big)_0$, for y being $\underset{\smile}{g}(t,x)$, $\overset{*}{\underset{\smile}{v}}(x)$, and $\overset{**}{\underset{\smile}{v}}(x)$, respectively; b) account for Prop. 3.1 and apply ansatz (3.1) for the required solution to the IBVPA (3.17) as follows

$$\underset{\smile}{v}(t,x) = \sum_{\mu=1}^{\infty} \underset{\smile}{Q}_\mu(t)\, \underset{\smile}{X}_\mu(x)\,; \tag{3.18}$$

c) obtain the following Cauchy problems wrt the coefficient functions of (3.18)

$$\begin{cases} \underset{\smile}{Q}''_\mu(t) + \underset{\smile}{\alpha}_\mu^2\, \underset{\smile}{Q}_\mu(t) = \underset{\smile}{g}_\mu(t)\,, & t \in (0,T]\,, \\ \left.\begin{array}{r} \underset{\smile}{Q}'_\mu(0) = \overset{**}{\underset{\smile}{v}}_\mu \\ \underset{\smile}{Q}_\mu(0) = \overset{*}{\underset{\smile}{v}}_\mu \end{array}\right\}, & \end{cases} \qquad \mu \in \mathbb{N}\,. \tag{3.19}$$

The solutions to the above Cauchy problems

$$\underset{\smile}{Q}_\mu(t) = \overset{*}{\underset{\smile}{v}}_\mu \cos\big(\underset{\smile}{\alpha}_\mu t\big) + \underset{\smile}{\alpha}_\mu^{-1}\, \overset{**}{\underset{\smile}{v}}_\mu \sin\big(\underset{\smile}{\alpha}_\mu t\big) + \underset{\smile}{\alpha}_\mu^{-1} \int_0^t \underset{\smile}{g}_\mu(\theta)\, \sin\big[\underset{\smile}{\alpha}_\mu (t-\theta)\big]\, \mathrm{d}\theta$$

can be presented shortly as

$$\underset{\smile}{Q}_\mu(t) = \overset{*}{\underset{\smile}{v}}_\mu \cos\big(\underset{\smile}{\alpha}_\mu t\big) + \underset{\smile}{\alpha}_\mu^{-1}\, \overset{**}{\underset{\smile}{v}}_\mu \sin\big(\underset{\smile}{\alpha}_\mu t\big) + \underset{\smile}{\alpha}_\mu^{-1}\, \underset{\smile}{g}_\mu(t) * \sin\big(\underset{\smile}{\alpha}_\mu t\big)\,, \tag{3.20}$$

by invoking the notion of convolution.

Finally, the solution to the IBVPD (1.8), (1.11) reads

$$\underset{\smile}{u}(t,x) = \sum_{\mu=1}^{\infty} \underset{\smile}{Q}_\mu(t)\, \underset{\smile}{X}_\mu(x) + \underset{\smile}{\phi}_1(x)\, h_1(t) + \underset{\smile}{\phi}_2(x)\, h_2(t)\,. \tag{3.21}$$

We note, that the flux of the above solution

$$\underset{\smile}{f}(t,x) = a(x)\, \frac{\partial \underset{\smile}{u}(t,x)}{\partial x} = \sum_{\mu=1}^{\infty} \underset{\smile}{Q}_\mu(t)\, \underset{\smile}{F}_\mu(x) + \underset{\smile}{\varphi}_1(x)\, h_1(t) + \underset{\smile}{\varphi}_2(x)\, h_2(t) \tag{3.22}$$

is continuous and continuously differentiable wrt x over the segment K_0, due to continuity and continuous differentiabilty of the fluxes $\underset{\smile}{F}_\mu(x)$ (refer to p. 96).

4. A classical approach to solve the IBVPD based on LT

4.1. Preliminaries to LT

For a function $p(t)$, $t \in [0,\infty)$, its Laplace transform [7] is defined as follows

$$P(\tau) = \mathfrak{L}\,[p(t)] := \int_0^\infty p(t)\, \mathsf{e}^{-\tau t}\, \mathrm{d}t\,, \qquad \tau = \xi + i\eta \in \mathbb{C}\,, \tag{4.1}$$

provided the original function $p(t)$ satisfies the known sufficient conditions for the image function $P(\tau)$ to exist.

Applying the Laplace transformation for solving second order partial differential equations and integro-differential equations of convolution type is based on:

a) the rule for the images of the first and second derivatives of the function $p(t)$

$$\mathfrak{L}\big[p'(t)\big] = P(\tau)\,\tau - p(0)\,, \quad \mathfrak{L}\big[p''(t)\big] = P(\tau)\,\tau^2 - p(0)\,\tau - p'(0)\,;$$

b) the convolution theorem

$$\mathfrak{L}\,[p(t) * q(t)] = \mathfrak{L}\,[p(t)] \cdot \mathfrak{L}\,[q(t)] = P(\tau) \cdot Q(\tau)\,.$$

If the image function $P(\tau)$ for an original function $p(t)$ is known, then applying the inverse Laplace transformation [7], sometimes referred to as the Bromwich integral, yields to the required original function as follows

$$p(t) = \mathfrak{L}^{-1}\,[P(\tau)] = \frac{1}{2\pi i} \int_{\xi^*-i\infty}^{\xi^*+i\infty} P(\tau)\, \mathsf{e}^{+t\tau}\, \mathrm{d}\tau\,, \tag{4.2}$$

where $\Re\,\tau = \xi^*$ is a vertical straight line on the whole τ-plane lying to the right of all the singularities of the image $P(\tau)$.

4.2. Applying LT to the $\mathbf{BVP}_1$ and $\mathbf{BVP}_2$

Applying (4.1) to the associated IBVP_j (1.14), (1.16) yields to their images being the following BVP_j

$$\begin{cases} \dfrac{\mathrm{d}F_1(\tau,x)}{\mathrm{d}x} - \tau^2 U_1(\tau,x) = -\tau\,\overset{*}{u}(x) - \overset{**}{u}(x)\,, \qquad x \in I_1\,, \\ U_1(\tau,-1) = H_1(\tau)\,, \end{cases} \tag{4.3}$$

$$\begin{cases} \dfrac{\mathrm{d}F_2(\tau,x)}{\mathrm{d}x} - \tau^2 U_2(\tau,x) = -\tau\,\overset{*}{u}(x) - \overset{**}{u}(x)\,, \qquad x \in I_2\,, \\ U_2(\tau,+1) = H_2(\tau)\,, \end{cases} \tag{4.4}$$

wrt the images $U_j(\tau,x)$ of the solutions $u_j(t,x)$ to the IBVP_j, where

$$F_j(\tau,x) = a_j\,\frac{\mathrm{d}U_j(\tau,x)}{\mathrm{d}x} \tag{4.5}$$

are the fluxes of the images $U_j(\tau,x)$ (or, it is the same, the images of the fluxes $f_j(t,x)$ of the solutions $u_j(t,x)$). Both BVP_j are supplemented with the images of the matching conditions (1.19)

$$\begin{cases} U_1(\tau,0) = U_2(\tau,0)\,, \\ F_1(\tau,0) = F_2(\tau,0)\,. \end{cases} \tag{4.6}$$

4.3. Finding the image functions $\boldsymbol{U_j(\tau,x)}$

The equations of the BVP_j (4.3), (4.4) are linear non-homogeneous ordinary differential equations of the second order, and their 2-parameter solutions (usually referred to as the general solutions) can be readily written as

$$\begin{cases} U_1(\tau,x) = A_1(\tau,x)\,\mathrm{e}^{-\frac{\tau x}{s_1}} + B_1(\tau,x)\,\mathrm{e}^{+\frac{\tau x}{s_1}}\,, \\ U_2(\tau,x) = A_2(\tau,x)\,\mathrm{e}^{-\frac{\tau x}{s_2}} + B_2(\tau,x)\,\mathrm{e}^{+\frac{\tau x}{s_2}}\,, \end{cases} \tag{4.7}$$

where: *a*) the 1-parameter coefficient functions $A_j(\tau,x)$, $B_j(\tau,x)$ read

$$\begin{cases} A_1(\tau,x) = A_{1,0}(\tau) + \dfrac{\Lambda_1^+(\tau;-1,x)}{s_1\tau}\,, \\ B_1(\tau,x) = B_{1,0}(\tau) - \dfrac{\Lambda_1^-(\tau;-1,x)}{s_1\tau}\,, \end{cases} \tag{4.8}$$

$$\begin{cases} A_2(\tau,x) = A_{2,0}(\tau) - \dfrac{\Lambda_2^+(\tau;x,+1)}{s_2\tau}\,, \\[2ex] B_2(\tau,x) = B_{2,0}(\tau) + \dfrac{\Lambda_2^-(\tau;x,+1)}{s_2\tau}\,; \end{cases} \tag{4.9}$$

b) the auxiliary functions involved in (4.8), (4.9) are as follows

$$\Lambda_j^{\mp}(\tau;a,b) = \frac{1}{2}\int_a^b [\tau\,\overset{*}{u}(\xi) - \overset{**}{u}(\xi)]\ \mathsf{e}^{\mp\frac{\tau\xi}{s_j}}\,\mathrm{d}\xi\,; \tag{4.10}$$

c) the parameter functions $A_{j,0}(\tau)$, $B_{j,0}(\tau)$ are undetermined; and d) $s_j^2 = a_j$.

To determine four functions $A_{j,0}(\tau)$, $B_{j,0}(\tau)$ we need four conditions, they are known to be: a) two boundary conditions of the BVP_1 (4.3) and the BVP_2 (4.4); b) two matching conditions (4.6).

Due to the known images (4.7), we have for their boundary values at $|x| = 1$ the following expressions

$$U_1(\tau,-1) = \Bigg[A_{1,0}(\tau) + \underbrace{\frac{\Lambda_1^+(\tau;-1,-1)}{s_1\tau}}_{0}\Bigg]\mathsf{e}^{+\frac{\tau}{s_1}} + \Bigg[B_{1,0}(\tau) - \underbrace{\frac{\Lambda_1^-(\tau;-1,-1)}{s_1\tau}}_{0}\Bigg]\mathsf{e}^{-\frac{\tau}{s_1}},$$

$$U_2(\tau,+1) = \Bigg[A_{2,0}(\tau) - \underbrace{\frac{\Lambda_2^+(\tau;+1,+1)}{s_2\tau}}_{0}\Bigg]\mathsf{e}^{-\frac{\tau}{s_2}} + \Bigg[B_{2,0}(\tau) + \underbrace{\frac{\Lambda_2^-(\tau;+1,+1)}{s_2\tau}}_{0}\Bigg]\mathsf{e}^{+\frac{\tau}{s_2}},$$

and after simplifications the above expressions reduce to

$$\begin{cases} U_1(\tau,-1) = A_{1,0}(\tau)\ \mathsf{e}^{+\frac{\tau}{s_1}} + B_{1,0}(\tau)\ \mathsf{e}^{-\frac{\tau}{s_1}}\,, \\[2ex] U_2(\tau,+1) = A_{2,0}(\tau)\ \mathsf{e}^{-\frac{\tau}{s_2}} + B_{2,0}(\tau)\ \mathsf{e}^{+\frac{\tau}{s_2}}\,. \end{cases} \tag{4.11}$$

Doing in the same way, we obtain: a) the values of the images $U_j(\tau,x)$ and b) their fluxes $F_j(\tau,x)$ at the midpoint $x_0 = 0$

$$\begin{cases} U_1(\tau,0) = \Bigg[A_{1,0}(\tau) + \dfrac{\Lambda_1^+(\tau;-1,0)}{s_1\tau}\Bigg] + \Bigg[B_{1,0}(\tau) - \dfrac{\Lambda_1^-(\tau;-1,0)}{s_1\tau}\Bigg], \\[2ex] U_2(\tau,0) = \Bigg[A_{2,0}(\tau) - \dfrac{\Lambda_2^+(\tau;0,+1)}{s_2\tau}\Bigg] + \Bigg[B_{2,0}(\tau) + \dfrac{\Lambda_2^-(\tau;0,+1)}{s_2\tau}\Bigg], \end{cases} \tag{4.12}$$

$$\begin{cases} F_1(\tau,0) = -s_1\tau\,A_{1,0}(\tau) - \Lambda_1^+(\tau;-1,0) + s_1\tau\,B_{1,0}(\tau) - \Lambda_1^-(\tau;-1,0)\,, \\[2ex] F_2(\tau,0) = -s_2\tau\,A_{2,0}(\tau) + \Lambda_2^+(\tau;0,+1) + s_2\tau\,B_{2,0}(\tau) + \Lambda_2^-(\tau;0,+1)\,. \end{cases} \tag{4.13}$$

Finally, substituting: a) the values (4.11) into the boundary conditions of the BVP_1 (4.3) and the BVP_2 (4.4) respectively; then b) the values (4.12), (4.13) into the first and the second matching conditions (4.6) respectively, yields to the following linear algebraic system wrt the required parameter functions

$$\begin{pmatrix} \mathsf{e}^{+\frac{\tau}{s_1}} & \mathsf{e}^{-\frac{\tau}{s_1}} & 0 & 0 \\ 0 & 0 & \mathsf{e}^{-\frac{\tau}{s_2}} & \mathsf{e}^{+\frac{\tau}{s_2}} \\ +s_1 s_2\,\tau & +s_1 s_2 \tau & -s_1 s_2\,\tau & -s_1 s_2\,\tau \\ +s_1\,\tau & -s_1\tau & -s_2\tau & +s_2\tau \end{pmatrix} \begin{pmatrix} A_{1,0}(\tau) \\ B_{1,0}(\tau) \\ A_{2,0}(\tau) \\ B_{2,0}(\tau) \end{pmatrix} = \begin{pmatrix} H_1(\tau) \\ H_2(\tau) \\ R_1(\tau) \\ R_2(\tau) \end{pmatrix}, \tag{4.14}$$

where the rhs of the two last equations read

$$\begin{cases} R_1(\tau) = -\, s_2\Lambda_1^+(\tau;-1,0) + s_2\Lambda_1^-(\tau;-1,0) \\ \qquad\qquad -\, s_1\Lambda_2^+(\tau;0,+1) + s_1\Lambda_2^-(\tau;0,+1)\,, \\ R_2(\tau) = -\quad \Lambda_1^+(\tau;-1,0) - \quad \Lambda_1^-(\tau;-1,0) \\ \qquad\qquad -\quad \Lambda_2^+(\tau;0,+1) - \quad \Lambda_2^-(\tau;0,+1)\,. \end{cases} \tag{4.15}$$

We present the expression for the determinant of the system (4.14), (4.15)

$$\Delta_0(\tau) = s_1 s_2\,\tau\Big[(s_1 - s_2)\left(\mathsf{e}^{-\theta_1\tau} - \mathsf{e}^{+\theta_1\tau}\right) + (s_1 + s_2)\left(\mathsf{e}^{-\theta_2\tau} - \mathsf{e}^{+\theta_2\tau}\right)\Big], \tag{4.16}$$

$$\theta_1 = \frac{(s_1 - s_2)\,\tau}{s_1 s_2}\,, \qquad \theta_2 = \frac{(s_1 + s_2)\,\tau}{s_1 s_2}\,,$$

since this expression is important both for: a) solving the system and b) choosing the method of inversion of the images $U_j(\tau,x)$ of the solutions $u_j(\tau,x)$ to the associated IBVP$_j$ (1.14), (1.16).

5. A non-classical approach to solve the IBVPD based on SV+LT

5.1. Preliminaries to applying SV

In case of identically constant coefficient function $a(x) \equiv a_0$ an alternative approach to SV (compared to that of (3.2))

$$\frac{1}{a_0}\,\frac{O''(t)}{O(t)} = \frac{X''(x)}{X(x)} = -\lambda\,, \qquad \lambda > 0\,, \tag{5.1}$$

yields to the following BVP wrt $X(x)$

$$\begin{cases} X''(x) + \lambda X(x) = 0\,, \qquad x \in I_0\,, \\ X(\mp 1) = 0\,. \end{cases} \tag{5.2}$$

not involving a_0 (cf. the BVP (3.9) and its eigenvalues (3.10) and eigenfunctions (3.11)). The above BVP is known to have the countable sets of the eigenvalues and eigenfunctions of two kinds

$$\begin{cases} \lambda_{1,\mu} \equiv \alpha_\mu^2 = (\mu\pi)^2\,, & Y_\mu(x) = \sin\big(\alpha_\mu x\big)\,, \\ \lambda_{2,\mu} \equiv \omega_\mu^2 = \Big(\mu\pi - \dfrac{\pi}{2}\Big)^2, & Z_\mu(x) = \cos\big(\omega_\mu x\big)\,. \end{cases} \tag{5.3}$$

The eigenfunctions (5.3): *a*) of each kind are orthogonal in $\mathscr{L}_{2,0}$; *b*) of both kinds are biorthogonal in $\mathscr{L}_{2,0}$. The eigenfunctions $Y_\mu(x)$ of the first kind (see Fig. 3.3, *a*) are responsible for the string inclination to the x-axis (the undisturbed string position) at the midpoint $x_0\!=\!0$ of the segment K_0, whereas the eigenfunctions $Z_\mu(x)$ of the second kind (see Fig. 3.3, *b*) are responsible for the string standoff from the x-axis at the midpoint.

For solving the IBVPD, using a non-classical approach, we cut the segment K_0 down the midpoint $x_0\!=\!0$ and obtain functions of two kinds: *a*) $Y_{1,\mu}(x)\!=\!Y_\mu(x)$, $Z_{1,\mu}(x)\!=\!Z_\mu(x)$ for the left subsegment K_1 and *b*) $Y_{2,\mu}(x)\!=\!Y_\mu(x)$, $Z_{2,\mu}(x)\!=\!Z_\mu(x)$ for the right subsegment K_2. The functions $\big\{Y_{j,\mu}(x)\big\}_{\mu=1}^\infty$, $\big\{Z_{j,\mu}(x)\big\}_{\mu=1}^\infty$ of each kind are orthogonal in $\mathscr{L}_{2,j}\!=\!\mathscr{L}_2(K_j)$, but they are not biorthogonal, that is

$$\big(Y_{j,\mu}, Y_{j,\gamma}\big)_j = \big(Z_{j,\mu}, Z_{j,\gamma}\big)_j = \frac{1}{2}\,\delta_{\mu,\gamma}\,, \qquad \big(Y_{j,\mu}, Z_{j,\gamma}\big)_j = \mp\frac{\alpha_\mu}{\alpha_\mu^2-\omega_\gamma^2} \neq 0\,, \tag{5.4}$$

where

$$(p_j, q_j)_j = \int_{K_j} p_j(x)\,q_j(x)\,\mathrm{d}x = \mp\int_0^{\mp 1} p_j(x)\,q_j(x)\,\mathrm{d}x \tag{5.5}$$

is the inner product of two elements $p_j(x), q_j(x) \in \mathscr{L}_{2,j}$, whereas $(r_j, r_j)_j = \|r_j\|_j^2$ is the norm of an element $r_j(x) \in \mathscr{L}_{2,j}$.

Due to completeness of $\big\{Y_{j,\mu}(x)\big\}_{\mu=1}^\infty$ and $\big\{Z_{j,\mu}(x)\big\}_{\mu=1}^\infty$ (separetely) on K_j, any function $r_j(x)$, smooth on K_j and vanishing at the end points of K_j, can be expanded in series of $\big\{Y_{j,\mu}(x)\big\}_{\mu=1}^\infty$ or $\big\{Z_{j,\mu}(x)\big\}_{\mu=1}^\infty$, uniformly convergent on K_j. But, for smooth functions $r_j(x)$ not vanishing at the midpoint $x_0\!=\!0$ (this is the case as for: *a*) the solutions $u_j(t,x)$ to the associated IBVP$_j$ of Sect. 1, as *b*) the solutions $v_j(t,x)$ to the auxiliary IBVPA$_j$ (5.11), (5.12), introduced below) expansions in series of $\big\{Y_{j,\mu}(x)\big\}_{\mu=1}^\infty$ are not valid. Therefore, choosing $\big\{Z_{j,\mu}(x)\big\}_{\mu=1}^\infty$ for representing on K_j smooth functions $r_j(x)$, we nevertheless account that 'the fluxes' $\Psi_{j,\mu}(x)\!=\!a_j\,Z'_{j,\mu}(x)$ vanish at the midpoint identically on μ and meet the second condition (1.19) only in a trivial way. To avoid this, we will introduce 'manually' correction terms $\phi_{j+4}(x)\,k_j(t)$ in the series solutions $u_j(t,x)$ for the string to have a non-vanishing inclination at the midpoint.

5.2. Applying SV to the IBVP$_1$ and the IBVP$_2$

The ansatze for the solutions to the associated IBVP$_j$ of Sect. 1 read

$$u_j(t,x) = v_j(t,x) + w_j(t,x)\,, \tag{5.6}$$

where: *a*) the functions $v_j(t,x)$ are required; *b*) the functions $w_j(t,x)$ are given as follows

$$w_j(t,x) = \phi_j(x)\,h_j(t) + \phi_{j+2}(x)\,h_{j+2}(t) + \phi_{j+4}(x)\,k_j(t)\,, \tag{5.7}$$

c) the blending functions $\phi_{j+\nu}(x)$, $\nu \in \{0,2,4\}$, satisfy the following boundary

$$\begin{cases} \phi_1(-1) = 1\,, & \phi_1(0) = 0\,, \\ \phi_3(-1) = 0\,, & \phi_3(0) = 1\,, \\ \phi_5(-1) = 0\,, & \phi_5(0) = 0\,, \end{cases} \qquad \begin{cases} \phi_2(0) = 0\,, & \phi_2(+1) = 1\,, \\ \phi_4(0) = 1\,, & \phi_4(+1) = 0\,, \\ \phi_6(0) = 0\,, & \phi_6(+1) = 0\,, \end{cases} \tag{5.8}$$

and regularity

$$\begin{cases} \psi_{1+\nu}(x) \equiv \varphi'_{1+\nu}(x) \equiv \big(a_1\,\phi'_{1+\nu}(x)\big)' \equiv a_1\,\phi''_{1+\nu}(x) \in \mathscr{C}[-1,0]\,, \\ \psi_{2+\nu}(x) \equiv \varphi'_{2+\nu}(x) \equiv \big(a_2\,\phi'_{2+\nu}(x)\big)' \equiv a_2\,\phi''_{2+\nu}(x) \in \mathscr{C}[0,+1]\,, \end{cases} \tag{5.9}$$

conditions, whereas their fluxes $\varphi_{j+\nu}(x) = a_j\,\phi'_{j+\nu}(x)$ obey the following conditions at the midpoint: $\varphi_j(0) = 0$, $\varphi_{j+2}(0) = 0$, $\varphi_{j+4}(0) = a_j$, $\psi_{j+4}(0) = 0$, and for convenience we denote $b_{j+2} := \psi_{j+2}(0)$; *d*) $h_{j+2}(t)$ and $k_j(t)$ are the required corrections to the standoff and inclination of the left ($j=1$) and right ($j=2$) parts of the string on the dividing segment to meet the matching conditions (1.19).

Assuming that $h_{j+2}(0) = \overset{*}{u}(0)$, $h'_{j+2}(0) = \overset{**}{u}(0)$ and combining (5.6) – (5.8) yield to: *a*) the initial conditions for $v_j(t,x)$

$$\begin{cases} v_j(0,x) \;=\; u_j(0,x) \;-\; w_j(0,x) \equiv \overset{*}{v}_j(x)\,, \\ \dfrac{\partial v_j(0,x)}{\partial t} = \dfrac{\partial u_j(0,x)}{\partial t} - \dfrac{\partial w_j(0,x)}{\partial t} \equiv \overset{**}{v}_j(x)\,; \end{cases} \tag{5.10}$$

b) reformulation of the IBVP$_1$ (1.14), (1.15) into the auxiliary IBVPA$_1$ wrt $v_1(t,x)$

$$\begin{cases} S\,[v_1(t,x)] = g_1(t,x)\,, & (t,x) \in D_1\,, \\ \left.\begin{aligned} \dfrac{\partial v_1(0,x)}{\partial t} &= \overset{**}{v}(x) \\ v_1(0,x) &= \overset{*}{v}(x) \end{aligned}\right\}, & x \in [-1,0]\,, \\ \left.\begin{aligned} v_1(t,-1) &= 0 \\ |v_1(t,\;\;0)| &< \infty \end{aligned}\right\}, & t \in [0,T]\,; \end{cases} \tag{5.11}$$

c) reformulation of the IBVP$_2$ (1.16), (1.17) into the auxiliary IBVPA$_2$ wrt $v_2(t,x)$

$$\begin{cases} S\left[v_2(t,x)\right] = g_2(t,x)\,, & (t,x) \in D_2\,, \\ \left.\begin{aligned} \frac{\partial v_2(0,x)}{\partial t} &= \overset{**}{v}(x) \\ v_2(0,x) &= \overset{*}{v}(x) \end{aligned}\right\}, & x \in [0,+1]\,, \\ \left.\begin{aligned} |v_2(t,\ \ 0)| &< \infty \\ v_2(t,+1) &= 0 \end{aligned}\right\}, & t \in [0,T]\,; \end{cases} \tag{5.12}$$

where

$$\begin{aligned} g_j(t,x) = -S\,w_j(t,x) = &-\phi_j(x)\,h_j''(t) - \phi_{j+2}(x)\,h_{j+2}''(t) - \phi_{j+4}(x)\,k_j''(t) \\ &+ \psi_j(x)\,h_j(t) + \psi_{j+2}(x)\,h_{j+2}(t) + \psi_{j+4}(x)\,k_j(t)\,. \end{aligned} \tag{5.13}$$

Then the right hand sides (5.13) of the above non-homogeneous wave equations $S\left[v_j(t,x)\right] = g_j(t,x)$ and the initial functions $\overset{*}{v}_j(x)$, $\overset{**}{v}_j(x)$ (5.10) are expanded into the series wrt $Z_{j,\mu}(x)$

$$g_j(t,x) = \sum_{\mu=1}^{\infty} g_{j,\mu}(t)\,Z_{j,\mu}(x)\,, \quad \overset{*}{v}_j(x) = \sum_{\mu=1}^{\infty} \overset{*}{v}_{j,\mu}\,Z_{j,\mu}(x)\,, \quad \overset{**}{v}_j(x) = \sum_{\mu=1}^{\infty} \overset{**}{v}_{j,\mu}\,Z_{j,\mu}(x)\,,$$

where the expanded form of $g_{j,\mu}(t)$ reads

$$\begin{aligned} g_{j,\mu}(t) = &-\phi_{j,\mu}\,h_j''(t) - \phi_{j+2,\mu}\,h_{j+2}''(t) - \phi_{j+4,\mu}\,k_j''(t) \\ &+ a_j\,\phi^\star_{j,\mu}\,h_j(t) + a_j\,\phi^\star_{j+2,\mu}\,h_{j+2}(t) + a_j\,\phi^\star_{j+4,\mu}\,k_j(t)\,, \end{aligned} \tag{5.14}$$

and the subcoefficients $\phi_{j+\nu,\mu}$, $\phi^\star_{j+\nu,\mu}$ are as follows

$$\left\|Z_{j,\mu}\right\|_j^2\,\phi_{j+\nu,\mu} = \left(\phi_{j+\nu}, Z_{j,\mu}\right)_j\,, \qquad \left\|Z_{j,\mu}\right\|_j^2\,\phi^\star_{j+\nu,\mu} = \left(\phi''_{j+\nu}, Z_{j,\mu}\right)_j\,. \tag{5.15}$$

Substituting the ansatze for the solutions to the IBVPA$_j$ (5.11), (5.12)

$$v_j(t,x) = \sum_{\mu=1}^{\infty} O_{j,\mu}(t)\,Z_{j,\mu}(x) \tag{5.16}$$

into the IBVPA$_j$ yields to the Cauchy problems wrt the coefficient functions $O_{j,\mu}(t)$

$$\begin{cases} O_{j,\mu}''(t) + \omega_\mu^2\,O_{j,\mu}(t) = g_{j,\mu}(t)\,, & t \in (0,T]\,, \\ \left.\begin{aligned} O_{j,\mu}'(0) &= \overset{**}{v}_{j,\mu} \\ O_{j,\mu}(0) &= \overset{*}{v}_{j,\mu} \end{aligned}\right\}, & \end{cases} \qquad \mu \in \mathbb{N}\,. \tag{5.17}$$

The resulting expressions for the coefficients can be readily presented in the convolution form as follows

$$O_{j,\mu}(t) = \overset{*}{v}_{j,\mu} \cos\big(\omega_\mu t\big) + \omega_\mu^{-1} \overset{**}{v}_{j,\mu} \sin\big(\omega_\mu t\big) + \omega_\mu^{-1} g_{j,\mu}(t) * \sin\big(\omega_\mu t\big)\,. \tag{5.18}$$

Finally, the ansatze (5.6) give the solutions to the IBVP$_j$

$$\begin{cases} u_j(t,x) = \displaystyle\sum_{\mu=1}^{\infty} O_{j,\mu}(t)\, Z_{j,\mu}(x) \\ \qquad\qquad + \phi_j(x)\, h_j(t) + \phi_{j+2}(x)\, h_{j+2}(t) + \phi_{j+4}(x)\, k_j(t)\,. \end{cases} \tag{5.19}$$

We need further the first order partial derivatives of the solutions (5.19) and the flux, calculated here in advance as follows

$$\begin{cases} \dfrac{\partial u_j(t,x)}{\partial t} = \displaystyle\sum_{\mu=1}^{\infty} O'_{j,\mu}(t)\, Z_{j,\mu}(x) \\ \qquad\qquad + \phi_j(x)\, h'_j(t) + \phi_{j+2}(x)\, h'_{j+2}(t) + \phi_{j+4}(x)\, k'_j(t)\,, \end{cases} \tag{5.20}$$

$$\begin{cases} \dfrac{\partial u_j(t,x)}{\partial x} = \displaystyle\sum_{\mu=1}^{\infty} O_{j,\mu}(t)\, Z'_{j,\mu}(x) \\ \qquad\qquad + \phi'_j(x)\, h_j(t) + \phi'_{j+2}(x)\, h_{j+2}(t) + \phi'_{j+4}(x)\, k_j(t)\,, \end{cases} \tag{5.21}$$

$$\begin{cases} f_j(t,x) = \displaystyle\sum_{\mu=1}^{\infty} O_{j,\mu}(t)\, \Psi_{j,\mu}(x) \\ \qquad\qquad + \varphi_j(x)\, h_j(t) + \varphi_{j+2}(x)\, h_{j+2}(t) + \varphi_{j+4}(x)\, k_j(t)\,, \end{cases} \tag{5.22}$$

where $\Psi_{j,\mu}(x) = a_j\, Z'_{j,\mu}(x)$ are the fluxes of $Z_{j,\mu}(x)$, and the first order differentiation of the coefficient functions (5.18)

$$O'_{j,\mu}(t) = -\omega_\mu \overset{*}{v}_{j,\mu} \sin\big(\omega_\mu t\big) + \overset{**}{v}_{j,\mu} \cos\big(\omega_\mu t\big) + g_{j,\mu}(t) * \cos\big(\omega_\mu t\big) \tag{5.23}$$

is performed accounting for the formula

$$\big(p(t) * q(t)\big)' = p(t)\, q(0) + p(t) * q'(t)\,. \tag{5.24}$$

5.3. Matching the solutions to the IBVP$_1$ and the IBVP$_2$

Matching the obtained solutions (5.19) to the IBVP$_1$ (1.14), (1.15) and IBVP$_2$ (1.16), (1.17) of Sect. 1 follows the procedure:

a) to substitute $u_j(t,x)$ into the matching conditions (1.19) on p. 93, as follows

$$\begin{cases} \displaystyle\sum_{\mu=1}^{\infty} O_{1,\mu}(t)\, Z_{1,\mu}(0) + \phi_1(0)\, h_1(t) + \phi_3(0)\, h_3(t) + \phi_5(0)\, k_1(t) \\ \displaystyle = \sum_{\mu=1}^{\infty} O_{2,\mu}(t)\, Z_{2,\mu}(0) + \phi_2(0)\, h_2(t) + \phi_4(0)\, h_4(t) + \phi_6(0)\, k_2(t)\,, \end{cases} \tag{5.25}$$

$$\begin{cases} \displaystyle\sum_{\mu=1}^{\infty} O_{1,\mu}(t)\, \Psi_{1,\mu}(0) + \varphi_1(0)\, h_1(t) + \varphi_3(0)\, h_3(t) + \varphi_5(0)\, k_1(t) \\ \displaystyle = \sum_{\mu=1}^{\infty} O_{2,\mu}(t)\, \Psi_{2,\mu}(0) + \varphi_2(0)\, h_2(t) + \varphi_4(0)\, h_4(t) + \varphi_6(0)\, k_2(t)\,; \end{cases} \tag{5.26}$$

b) to account for: 1) the values $Z_{j,\mu}(0)=1$, $\Psi_{j,\mu}(0)=a_j Z'_{j,\mu}(0)=a_j\,\omega_\mu$, and 2) the boundary conditions (5.8) imposed on the blending functions $\phi_{j+m}(x)$ and their fluxes $\varphi_{j+m}(x) = a_j\,\phi'_{j+m}(x)$, to obtain the following equations wrt $h_{j+2}(t)$ and $k_j(t)$

$$\begin{cases} \displaystyle\sum_{\mu=1}^{\infty} O_{1,\mu}(t) + h_3(t) = \sum_{\mu=1}^{\infty} O_{2,\mu}(t) + h_4(t)\,, \\ a_1 k_1(t) = a_2\, k_2(t)\,, \end{cases} \qquad t\in[0,T]\,. \tag{5.27}$$

Since the system (5.27) is incomplete (it involves two equations wrt four unknown functions $h_{j+2}(t)$ and $k_j(t)$), we supply it with the following condition

$$\frac{\partial f_1(t,0)}{\partial x} = \frac{\partial f_2(t,0)}{\partial x}\,, \qquad t\in[0,T]\,, \tag{5.28}$$

being the flux continuous differentiability on the dividing segment (the flux (3.22) of the solution (3.21) obtained in Sect. 3 this condition holds). Then:

a) substituting $u_j(t,x)$ (5.19) into the above condition

$$\begin{cases} \displaystyle\sum_{\mu=1}^{\infty} O_{1,\mu}(t)\, \Psi'_{1,\mu}(0) + \psi_1(0)\, h_1(t) + \psi_3(0)\, h_3(t) + \psi_5(0)\, k_1(t) \\ \displaystyle = \sum_{\mu=1}^{\infty} O_{2,\mu}(t)\, \Psi'_{2,\mu}(0) + \psi_2(0)\, h_2(t) + \psi_4(0)\, h_4(t) + \psi_6(0)\, k_2(t)\,; \end{cases} \tag{5.29}$$

b) accounting for: 1) the values $\Psi'_{j,\mu}(0)=-a_j\,\omega_\mu^2$; 2) the boundary conditions imposed on $\psi_{j+m}(x)=\varphi'_{j+m}(x)$, gives one more equation wrt $h_{j+2}(t)$ and $k_j(t)$

$$a_1\sum_{\mu=1}^{\infty}\omega_\mu^2\, O_{1,\mu}(t) + a_1 b_3\, h_3(t) = a_2\sum_{\mu=1}^{\infty}\omega_\mu^2\, O_{2,\mu}(t) + a_2 b_4\, h_4(t)\,, \quad t\in[0,T]\,. \tag{5.30}$$

Now the difference between the number of unknown functions $h_{j+2}(t)$, $k_j(t)$ and the number of equations (5.27), (5.30) is equal to one, so we need one more additional equation to obtain a complete system to find the required functions.

In contrast to the local equations (5.27), (5.30), valid on the dividing segment, we can choose: a) the nonlocal total energy rate equation (2.2)

$$\sum_{j=1}^{2}\Big[\Omega_j'(t) + \Pi_j'(t)\Big] = \mathrm{A}(t)\,, \quad t \in [0,T]\,, \tag{5.31}$$

or b) the nonlocal total energy equation (2.3)

$$\sum_{j=1}^{2}\Big[\Omega_j(t) + \Pi_j(t) - \Omega_j(0) - \Pi_j(0)\Big] = \int_0^t \mathrm{A}(t)\,, \quad t \in [0,T]\,, \tag{5.32}$$

as the required additional equation: 1) both (5.31), (5.32) are composed of those respective equations (2.4) and (2.5) for the parts of the string; 2) the kinetic and potential energy are presented as

$$\Omega_j(t) = \frac{1}{2}\int_{K_j}\left[\frac{\partial u_j(t,x)}{\partial t}\right]^2 \mathrm{d}x\,, \qquad \Pi_j(t) = \frac{a_j}{2}\int_{K_j}\left[\frac{\partial u_j(t,x)}{\partial x}\right]^2 \mathrm{d}x\,, \tag{5.33}$$

3) the power of the external forces acting on the ends of the 'string' reads

$$\mathrm{A}(t) = a_2\frac{\partial u_2(t,+1)}{\partial x}\,h_2'(t) - a_1\frac{\partial u_1(t,-1)}{\partial x}\,h_1'(t)\,, \tag{5.34}$$

and 4) the partial derivatives in (5.33), (5.34) are those given by (5.20), (5.21).

Thus, we have obtained the system involving four equations wrt four unknown functions $h_{j+2}(t)$ and $k_j(t)$, three equations of the system are given by (5.27), (5.30), whereas the fourth is one of (5.31), (5.32).

The second equation of (5.27) is a trivial algebraic relation, whereas the first one is a linear integro-differential equation of convolution type, its expanded form (due to the resulting expression (5.18) for $O_{j,\mu}(t)$) reads

$$\begin{cases} \quad -\,\underline{\phi}_1(t) * h_1''(t) \;-\; \underline{\phi}_3(t) * h_3''(t) \;-\; \underline{\phi}_5(t) * k_1''(t) + \overset{*}{\underline{v}}_1(t) + \overset{**}{\underline{v}}_1(t) \\ +\; a_1\,\underline{\phi}_1^{\star}(t) * h_1(t) + a_1\,\underline{\phi}_3^{\star}(t) * h_3(t) + a_1\,\underline{\phi}_5^{\star}(t) * k_1(t) + h_3(t) \\ =\; -\,\underline{\phi}_2(t) * h_2''(t) \;-\; \underline{\phi}_4(t) * h_4''(t) \;-\; \underline{\phi}_6(t) * k_2''(t) + \overset{*}{\underline{v}}_2(t) + \overset{**}{\underline{v}}_2(t) \\ +\; a_2\,\underline{\phi}_2^{\star}(t) * h_2(t) + a_2\,\underline{\phi}_4^{\star}(t) * h_4(t) + a_2\,\underline{\phi}_6^{\star}(t) * k_2(t) + h_4(t)\,, \end{cases} \tag{5.35}$$

where the once underlined functions of t are determined by the following series

$$\begin{cases} \overset{*}{\underline{v}}_j(t) = \displaystyle\sum_{\mu=1}^{\infty} \overset{*}{v}_{j,\mu}\cos\big(\omega_\mu t\big), & \underline{\phi}_{j+\nu}(t) = \displaystyle\sum_{\mu=1}^{\infty}\omega_\mu^{-1}\,\phi_{j+\nu,\mu}\,\sin\big(\omega_\mu t\big)\,, \\ \overset{**}{\underline{v}}_j(t) = \displaystyle\sum_{\mu=1}^{\infty}\omega_\mu^{-1}\,\overset{**}{v}_{j,\mu}\sin\big(\omega_\mu t\big), & \underline{\phi}_{j+\nu}^{\star}(t) = \displaystyle\sum_{\mu=1}^{\infty}\omega_\mu^{-1}\,\phi_{j+\nu,\mu}^{\star}\,\sin\big(\omega_\mu t\big)\,. \end{cases} \tag{5.36}$$

The same is true for the equation (5.30) written in its expanded form as follows

$$\begin{cases} -\, a_1\, \underline{\underline{\phi}}_1(t) * h_1''(t) - a_1\, \underline{\underline{\phi}}_3(t) * h_3''(t) - a_1\, \underline{\underline{\phi}}_5(t) * k_1''(t) + a_1\, \underline{\underline{\overset{*}{v}}}_1(t) + a_1\, \underline{\underline{\overset{**}{v}}}_1(t) \\ +\, a_1^2\, \underline{\underline{\phi}}_1^\star(t) * h_1(t) + a_1^2\, \underline{\underline{\phi}}_3^\star(t) * h_3(t) + a_1^2\, \underline{\underline{\phi}}_5^\star(t) * k_1(t) + \qquad b_3 a_1\, h_3(t) \\ = \\ -\, a_2\, \underline{\underline{\phi}}_2(t) * h_2''(t) - a_2\, \underline{\underline{\phi}}_4(t) * h_4''(t) - a_2\, \underline{\underline{\phi}}_6(t) * k_2''(t) + a_2\, \underline{\underline{\overset{*}{v}}}_2(t) + a_2\, \underline{\underline{\overset{**}{v}}}_2(t) \\ +\, a_2^2\, \underline{\underline{\phi}}_2^\star(t) * h_2(t) + a_2^2\, \underline{\underline{\phi}}_4^\star(t) * h_4(t) + a_2^2\, \underline{\underline{\phi}}_6^\star(t) * k_2(t) + \qquad b_4 a_2\, h_4(t)\,, \end{cases} \tag{5.37}$$

where the twice underlined functions of t are determined by the following series

$$\begin{cases} \underline{\underline{\overset{*}{v}}}_j(t) = \sum_{\mu=1}^{\infty} \omega_\mu^2\, \overset{*}{v}_{j,\mu} \cos\left(\omega_\mu t\right), \qquad \underline{\underline{\phi}}_{j+\nu}(t) = \sum_{\mu=1}^{\infty} \omega_\mu\, \phi_{j+\nu,\mu} \sin\left(\omega_\mu t\right), \\ \underline{\underline{\overset{**}{v}}}_j(t) = \sum_{\mu=1}^{\infty} \omega_\mu\, \overset{**}{v}_{j,\mu} \sin\left(\omega_\mu t\right), \qquad \underline{\underline{\phi}}_{j+\nu}^\star(t) = \sum_{\mu=1}^{\infty} \omega_\mu\, \phi_{j+\nu,\mu}^\star \sin\left(\omega_\mu t\right). \end{cases} \tag{5.38}$$

5.4. Finding the image functions $H_{j+2}(\tau)$

Applying the Laplace transformation (4.1) to the linear integro-differential equations (5.35) – (5.38) wrt the origins $h_{j+2}(t)$ and $k_j(t)$ yields to the following linear algebraic non-homogeneous equations wrt the images $H_{j+2}(\tau)$ and $K_j(\tau)$

$$\begin{cases} \left[1 - \hat{\underline{\Phi}}_4(\tau)\right] H_4(\tau) - \left[1 - \hat{\underline{\Phi}}_3(\tau)\right] H_3(\tau) = \underline{D}_2(\tau) - \underline{D}_1(\tau)\,, \\ a_2\left[b_4 - \hat{\underline{\underline{\Phi}}}_4(\tau)\right] H_4(\tau) - a_1\left[b_3 - \hat{\underline{\underline{\Phi}}}_3(\tau)\right] H_3(\tau) = a_2\, \underline{\underline{D}}_2(\tau) - a_1 \underline{\underline{D}}_1(\tau)\,, \end{cases} \tag{5.39}$$

where the right hand sides are calculated using the following generic functions

$$\begin{aligned} D_j(\tau) = -\, V_j(\tau) + \hat{\Phi}_{j+0}(\tau)\, H_j(\tau) - \Phi_{j+0}(\tau)\left[h_{j+0}(0)\,\tau + h_{j+0}'(0)\right] \\ -\, \Phi_{j+2}(\tau)\left[h_{j+2}(0)\,\tau + h_{j+2}'(0)\right] \\ +\, \hat{\Phi}_{j+4}(\tau)\, K_j(\tau) - \Phi_{j+4}(\tau)\left[k_j\ \ (0)\,\tau + k_j'\ \ (0)\right]. \end{aligned} \tag{5.40}$$

For obtaining the once $\underline{D}_j(\tau)$ and twice $\underline{\underline{D}}_j(\tau)$ underlined functions on the rhs of (5.39) one should replace all the functions of τ in (5.40) with the corresponding once and twice underlined functions of τ, determined by the following series

$$\begin{aligned} \underline{\Phi}_{j+m}(\tau) = \sum_{\mu=1}^{\infty} \frac{\phi_{j+m,\mu}}{\tau^2 + \omega_\mu^2}\,, \qquad \underline{\underline{\Phi}}_{j+m}(\tau) = \sum_{\mu=1}^{\infty} \omega_\mu^2\, \frac{\phi_{j+m,\mu}}{\tau^2 + \omega_\mu^2}\,, \\ \underline{\Phi}_{j+m}^\star(\tau) = \sum_{\mu=1}^{\infty} \frac{\phi_{j+m,\mu}^\star}{\tau^2 + \omega_\mu^2}\,, \qquad \underline{\underline{\Phi}}_{j+m}^\star(\tau) = \sum_{\mu=1}^{\infty} \omega_\mu^2\, \frac{\phi_{j+m,\mu}^\star}{\tau^2 + \omega_\mu^2}\,, \end{aligned} \tag{5.41}$$

$$\underline{V}_j(\tau) = \sum_{\mu=1}^{\infty} \frac{\overset{*}{v}_{j,\mu}\,\tau + \overset{**}{v}_{j,\mu}}{\tau^2 + \omega_\mu^2}\,, \qquad \hat{\underline{\Phi}}_{j+m}(\tau) = \tau^2\, \underline{\Phi}_{j+m}(\tau) - a_j\, \underline{\Phi}^{\star}_{j+m}(\tau)\,,$$
$$\underline{\underline{V}}_j(\tau) = \sum_{\mu=1}^{\infty} \omega_\mu^2\, \frac{\overset{*}{v}_{j,\mu}\,\tau + \overset{**}{v}_{j,\mu}}{\tau^2 + \omega_\mu^2}\,, \qquad \hat{\underline{\underline{\Phi}}}_{j+m}(\tau) = \tau^2\, \underline{\underline{\Phi}}_{j+m}(\tau) - a_j\, \underline{\underline{\Phi}}^{\star}_{j+m}(\tau)\,. \tag{5.42}$$

Note that: a) the images $K_j(\tau)$ are placed on the right hand side of (5.39), as if the former images were known, to solve (5.39) wrt the images $H_{j+2}(\tau)$ in an iterative manner; b) the second equation of (5.27) is not used explicitly to retain (5.39) in symmetric form wrt $K_j(\tau)$.

Invoking the Cramer rule yields to the unique solution to (5.39) as follows

$$H_3(\tau) = \frac{\Delta_3(\tau)}{\Delta_0(\tau)}\,, \qquad H_4(\tau) = \frac{\Delta_4(\tau)}{\Delta_0(\tau)}\,, \tag{5.43}$$

where the determinants are

$$\Delta_0(\tau) = \begin{vmatrix} \Big[1 - \hat{\underline{\Phi}}_3(\tau)\Big] & \Big[1 - \hat{\underline{\Phi}}_4(\tau)\Big] \\ a_1\Big[b_3 - \hat{\underline{\underline{\Phi}}}_3(\tau)\Big] & a_2\Big[b_4 - \hat{\underline{\underline{\Phi}}}_4(\tau)\Big] \end{vmatrix}, \tag{5.44}$$

$$\Delta_3(\tau) = \begin{vmatrix} \Big[1 - \hat{\underline{\Phi}}_4(\tau)\Big] & \Big[\ \ \underline{D}_2(\tau) - \ \ \underline{D}_1(\tau)\Big] \\ a_2\Big[b_4 - \hat{\underline{\underline{\Phi}}}_4(\tau)\Big] & \Big[a_2\, \underline{\underline{D}}_2(\tau) - a_1 \underline{\underline{D}}_1(\tau)\Big] \end{vmatrix}, \tag{5.45}$$

$$\Delta_4(\tau) = \begin{vmatrix} \Big[1 - \hat{\underline{\Phi}}_3(\tau)\Big] & \Big[\ \ \underline{D}_2(\tau) - \ \ \underline{D}_1(\tau)\Big] \\ a_1\Big[b_3 - \hat{\underline{\underline{\Phi}}}_3(\tau)\Big] & \Big[a_2\, \underline{\underline{D}}_2(\tau) - a_1 \underline{\underline{D}}_1(\tau)\Big] \end{vmatrix}. \tag{5.46}$$

We present the extended expression for the determinant of the system (5.39)

$$\begin{aligned} \Delta_0(\tau) = a_1\Big(b_3 - \hat{\underline{\underline{\Phi}}}_3(\tau) - b_1\, \hat{\underline{\Phi}}_4(\tau) + \hat{\underline{\underline{\Phi}}}_3(\tau)\, \hat{\underline{\Phi}}_4(\tau)\Big) \\ - a_2\Big(b_4 - \hat{\underline{\underline{\Phi}}}_4(\tau) - b_2\, \hat{\underline{\Phi}}_3(\tau) + \hat{\underline{\Phi}}_3(\tau)\, \hat{\underline{\underline{\Phi}}}_4(\tau)\Big)\,, \end{aligned} \tag{5.47}$$

since this expression is important both for: a) solving the system and b) choosing the inversion method for the images $H_{j+2}(\tau)$.

5.5. Finding the functions $k_j(t)$

Substituting the known expressions (5.20), (5.21) into (5.33) yields to the following expressions for the kinetic and potential energy for both parts of the string

$$\begin{aligned} 2\,\Omega_j(t) &= \Omega_{j,0}\, k_j'^{\,2}(t) + 2\,\Omega_{j,1}(t)\, k_j'(t) + \Omega_{j,2}(t)\,, \\ a_j^{-1} 2\,\Pi_j(t) &= \Pi_{j,0}\, k_j^{\,2}\,(t) + 2\,\Pi_{j,1}(t)\, k_j(t) + \Pi_{j,2}(t)\,, \end{aligned} \tag{5.48}$$

where the leading coefficients are constant: $\Omega_{j,0} = \|\phi_{j+4}\|_j^2$, $\Pi_{j,0} = \|\phi'_{j+4}\|_j^2$, and the other coefficients read

$$\begin{aligned}\Omega_{j,1}(t) &= \sum_{\mu=1}^{\infty} \left(Z_{j,\mu}, \phi_{j+4}\right)_j O'_{j,\mu}(t) + \sum_{\nu=0}^{2} \left(\phi_{j+\nu}, \phi_{j+4}\right)_j h'_{j+\nu}(t)\,, \\ \Pi_{j,1}(t) &= \sum_{\mu=1}^{\infty} \left(Z'_{j,\mu}, \phi'_{j+4}\right)_j O_{j,\mu}(t) + \sum_{\nu=0}^{2} \left(\phi'_{j+\nu}, \phi'_{j+4}\right)_j h_{j+\nu}(t)\,,\end{aligned} \tag{5.49}$$

$$\begin{aligned}\Omega_{j,2}(t) &= \sum_{\mu=1}^{\infty} \|Z_{j,\mu}\|_j^2\, O'^2_{j,\mu}(t) + 2\sum_{\nu=0}^{2}\left(\sum_{\mu=1}^{\infty} \left(Z_{j,\mu}, \phi_{j+\nu}\right)_j O'_{j,\mu}(t)\right) h'_{j+\nu}(t) \\ &\quad + \sum_{\nu=0}^{2} \|\phi_{j+\nu}\|_j^2\, h'^2_{j+\nu}(t) + 2\left(\phi_j, \phi_{j+2}\right)_j h'_j(t)\, h'_{j+2}(t)\,, \\ \Pi_{j,2}(t) &= \sum_{\mu=1}^{\infty} \|Z'_{j,\mu}\|_j^2\, O^2_{j,\mu}(t) + 2\sum_{\nu=0}^{2}\left(\sum_{\mu=1}^{\infty} \left(Z'_{j,\mu}, \phi'_{j+\nu}\right)_j O_{j,\mu}(t)\right) h_{j+\nu}(t) \\ &\quad + \sum_{\nu=0}^{2} \|\phi'^2_{j+\nu}\|_j^2\, h^2_{j+\nu}(t) + 2\left(\phi'_j, \phi'_{j+2}\right)_j h_j(t)\, h_{j+2}(t)\,,\end{aligned} \tag{5.50}$$

where the inner products in $\mathscr{L}_{2,j}$ are defined in (5.5); whereas substituting(5.21) into (5.34) is performed straightforwardly and is not presented here.

Note that: *a*) the kinetic and potential energy (5.48) are presented as dependent on $k'_j(t)$ and $k_j(t)$, whereas the functions $h_{j+2}(t)$ and their derivatives are 'hidden' in the expressions of the coefficients (5.49), (5.50), as if $h_{j+2}(t)$ were known, to solve the total energy equation (5.32) or the total energy rate equation (5.31) wrt $k_j(t)$ in an iterative manner; *b*) the second equation of (5.27) is not used explicitly to retain both energy equations, (5.32) and (5.31), in symmetric form wrt $k_j(t)$ and their derivatives.

The total energy equation (5.32), (5.48) – (5.50) is a nonlinear second order integro-differential equation: *a*) the second order derivatives of the required functions $h_j(t)$, $k_j(t)$ are involved in $O_{j,\mu}(t)$ (5.18) and $O'_{j,\mu}(t)$ (5.24) through the convolution terms $g_{j,\mu}(t) * \sin\left(\omega_\mu t\right)$ and $g_{j,\mu}(t) * \cos\left(\omega_\mu t\right)$ respectively, where $g_{j,\mu}(t)$ are given in (5.14); and *b*) nonlinearity stems from the products and squares of the first order derivatives of the required functions outside the convolution terms.

The total energy rate equation (5.31), (5.48) – (5.50) is a nonlinear second order integro-differential equation as well as (5.32), but, in contrast to the latter, it: *a*) involves second order derivatives of the required functions $h_j(t)$, $k_j(t)$ outside the convolution terms; *b*) is linear wrt the former derivatives.

6. Conclusions

We have considered three approaches to solve the IBVPO (the original IBVP) for the composite string with piece-wise constant elastic properties based on:

a) SV applied directly to the IBVPO, not followed by matching, since the former is build-in into SV;

b) LT applied to the associated IBVPs posed for both parts of the string with constant properties, then solving the transformed IBVPs and matching the solutions to the transformed IBVPs;

c) SV applied to the associated IBVPs and followed by matching the solutions to the IBVPs, matching involves applying LT to three matching conditions and solving an ordinary differential equation for one matching condition.

In cases *a*) and *b*) matching needs two local conditions, being continuity of the solution and its flux; whereas in case *c*) matching needs two more conditions, one of which is local, being continuous differentiability of the flux, and the other is non-local, being the energy equation. Both cases *b*) and *c*) need applying the procedures of the inverse Laplace transformation, being sometimes quite sophisticated. Therefore, final comparing the cases *b*) and *c*) will be possible after completing the procedures of the inversion and will be presented in the next publication on the subject.

References

1. V. L. Borsch, P. I. Kogut, G. Leugering, *On an initial boundary-value problem for 1D hyperbolic equation with interior degeneracy: series solutions with the continuously differentiable fluxes*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **28** (1) (2020), 1 – 42.
2. V. L. Borsch, P. I. Kogut, *The exact bounded solution to an initial boundary value problem for 1D hyperbolic equation with interior degeneracy. I. Separation of Variables*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **28** (2) (2020), 2 – 20.
3. V. L. Borsch, P. I. Kogut, *Solutions to a simplified initial boundary value problem for 1D hyperbolic equation with interior degeneracy*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **29** (1) (2021), 1 – 31.
4. V. L. Borsch, P. I. Kogut, *Can a finite degenerate 'string' hear itself? The exact solution to a simplified IBVP*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **30** (1) (2022), 89 – 121.
5. V. L. Borsch, P. I. Kogut, *Can a finite degenerate 'string' hear itself? Numerical Solutions to a Simplified IBVP*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **31** (1) (2023), 95 – 100.
6. B. M. Budak, A. A. Samarskii, A. N. Tikhonov, *A Collection of Problems on Mathematical Physics*, Pergamon, London, 1964.
7. G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer, NY, 1974.
8. A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, Dover Publications, Inc., NY, 1963.

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