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NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES WITH VARIABLE TIME-DELAY IN TIME UNBOUNDED DOMAINS

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Abstract. We research the well-posedness of the problem without initial condition for nonlinear parabolic variational inequalities with variable time-delay. To justify our results, we impose some assumptions on the solution behavior and growth of the data-in as time variable tends to $-\infty$. Also, we obtain estimates for weak solutions of this problem.

Key words: evolution variational inequality, evolution subdifferential inclusion, time delay, Fourier problem, unbounded domain.

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1. Introduction

In this paper we consider the problem without initial condition or, in other words, Fourier problem for evolution variational inequalities (inclusions) with a time-depended delay. Let us introduce an example of the problem being studied here.

Let $p \geq 2$, Ω be a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$), $\partial\Omega$ be the boundary of Ω. We put $Q := \Omega \times (-\infty, 0]$, $\Sigma := \partial\Omega \times (-\infty, 0]$, $\Omega_t := \Omega \times \{t\}$ $\forall\, t \in \mathbb{R}$. Let $L^p(\Omega)$ and $L^p(Q)$ be the standard Lebesgue spaces. Denote by $W^{1,p}(\Omega) := \{v \in L^p(\Omega) \mid v_{x_i} \in L^p(\Omega),\ i = \overline{1,n}\}$ the standard Sobolev space with the norm $||v||_{W^{1,p}(\Omega)} := \Big(\int_\Omega [|\nabla v|^p + |v|^p]\Big)^{1/p}$, where $\nabla v := (v_{x_1}, \dots, v_{x_n})$.

Let K be a convex closed set in $W^{1,p}(\Omega)$ which contains 0. Let us consider the problem of finding a function $u \in L^p(Q)$ such that $u_{x_i} \in L^p(Q)$, $i = \overline{1,n}$, $u_t \in L^2(Q)$, and, for a.e. $t \in (-\infty, 0]$, $u(\cdot, t) \in K$ and

$$\int_{\Omega_t} \Big[u_t(v-u) + |\nabla u|^{p-2}\nabla u \nabla(v-u) + |u|^{p-2}u(v-u) + \widehat{b}u(v-u) + \widehat{c}(v-u)\int_{t-\tau(t)}^{t} u(x,s)\,ds \Big]\,dx \geq \int_{\Omega_t} f(v-u)\,dx \quad \forall\, v \in K, \tag{1.1}$$

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where $\widehat{b}, \widehat{c}$ are positive constants, $f \in L^2(Q)$, $\tau \in C((-\infty, 0])$, $\tau(t) \geq 0$ $\forall t \in (-\infty, 0]$, $\tau^+ := \sup_{t\in(-\infty,0]} \tau(t) < \infty$.

As it will be shown below, this problem, which we will call Problem (1.1), has a unique solution, when $\widehat{b} - \widehat{c}\tau^+ > 0$.

Note that Problem (1.1) can be written in more abstract way. Indeed, after an appropriate identification of functions and functionals, we have continuous and dense imbedding

$$W^{1,p}(\Omega) \subset L^2(\Omega) \subset (W^{1,p}(\Omega))',$$

where $(W^{1,p}(\Omega))'$ is dual to $W^{1,p}(\Omega)$ space. Clearly, for any $h \in L^2(\Omega)$ and $v \in W^{1,p}(\Omega)$ we have $\langle h, v\rangle = (h, v)$, where $\langle \cdot, \cdot\rangle$ is the notation for scalar product on the dual pair $\big[(W^{1,p}(\Omega))', W^{1,p}(\Omega)\big]$, and $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$. Thus, we will use the notation $(\cdot, \cdot)$ instead of $\langle \cdot, \cdot\rangle$.

Now, we denote $S := (-\infty, 0]$, $V := W^{1,p}(\Omega)$, $H := L^2(\Omega)$ and define an operator $A : V \to V'$ as follows

$$(A(v), w) = \int_\Omega \big[|\nabla v|^{p-2}\nabla v \nabla w + |v|^{p-2} v w + \widehat{b} v w\big]\, dx, \quad v, w \in V.$$

Then, Problem (1.1) becomes equivalent to the next problem: to find a function $u \in L^p(S; V)$ such that $u' \in L^2(S; H)$, and, for a.e. $t \in S$, $u(t) \in K$ and

$$(u'(t) + A(u(t)) + \widehat{c}\int_{t-\tau(t)}^{t} u(s)\, ds, v - u(t)) \geq (f(t), v - u(t)) \quad \forall\, v \in K. \quad (1.2)$$

Here $f \in L^2(S; H)$, τ is as above.

Note that variational inequality (1.2) can be written as a subdifferential inclusion. For this purpose, we put $I_K(v) := 0$ if $v \in K$, and $I_K(v) := +\infty$ if $v \in V \setminus K$, and also

$$\Phi(v) := \int_\Omega \big[p^{-1}|\nabla v|^p + p^{-1}|v|^p + 2^{-1}\widehat{b}|v|^2\big]\, dx + I_K(v), \quad v \in V.$$

It is easy to verify that the functional $\Phi : V \to \mathbb{R}_\infty := (-\infty; +\infty]$ is proper, convex and semi-lower-continuous. By the known results (see, e.g., [22, p. 83]), it follows that the problem of finding a solution of variational inequality (1.2) can be written as the following subdifferential inclusion: to find a function $u \in L^p(S; V)$ such that $u' \in L^2(S; H)$ and, for a.e. $t \in S$, $u(t) \in D(\partial\Phi)$ and

$$u'(t) + \partial\Phi(u(t)) + \widehat{c}\int_{t-\tau(t)}^{t} u(s)\, ds \ni f(t) \quad \text{in} \quad H, \quad (1.3)$$

where $\partial\Phi : V \to 2^{V'}$ is a subdifferential of Φ, $D(\partial\Phi)$ is a domain of $\partial\Phi$ ($\partial\Phi$ and $D(\partial\Phi)$ will be defined of later).

The aim of this paper is to investigate problems for inclusions of type (1.3).

Let us mention that initial-value problems for evolution inclusions with constant delay were studied in [19], [25], [26] and others. Many results on such problems were obtained by using the semi-group theory. Refer to [25] for more comments and citations. In [19], [26] the fixed point theorems were used.

Problem without initial conditions for evolution equations arise in modeling different nonstationary processes in nature, that started long time ago and initial conditions do not affect on them in the actual time moment. Thus, we can assume that the initial time is $-\infty$, while 0 is the final time, and initial conditions can be replaced with the behaviour of the solution as time variable turns to $-\infty$. Such problems appear in modeling in many fields of science such as ecology, economics, physics, cybernetics, etc. The research of the problem without initial conditions for the evolution equations and variational inequalities (without delay) were conducted in the monographs [15], [17], [22], and the papers [3], [4], [5], [7], [10], [14], [16], [18], [23] and others. In particular, R.E. Showalter in the paper [21] proved the existence of unique solution $u \in \mathrm{e}^{2\omega\cdot}H^1(S;H)$, where H is a Hilbert space, of the problem without initial condition

$$u'(t) + \mu u(t) + A\big(u(t)\big) \ni f(t), \quad t \in S,$$

for $\omega + \mu > 0$ and $f \in \mathrm{e}^{2\omega\cdot}H^1(S;H)$ in case when $A : H \to 2^H$ is a maximal monotone operator such that $0 \in A(0)$. Moreover, if $A = \partial\varphi$, where $\varphi : H \to (-\infty, +\infty]$ is proper, convex and lower-semi-continuous functional such that $\varphi(0) = 0 = \inf\{\varphi(v) \mid v \in H\}$, then this problem has a unique solution for each $\mu > 0$, $f \in L^2(S;H)$ and $\omega = 0$.

Note that the uniqueness of the solutions of problem without initial conditions for linear parabolic equations and variational inequalities is possible only under some restrictions on the behavior of solutions when time variable tends to $-\infty$. For the first time it was strictly justified by A.N. Tikhonov [24] in the case of heat equation. However, as it was shown by M.M. Bokalo [3], problem without initial conditions for some nonlinear parabolic equations has a unique solution in the class of functions without behavior restriction as time variable tends to $-\infty$. Similar results were also obtained for evolutionary variational inequalities in the paper [4].

Previously, problems without initial conditions of evolution equations with constant delay were studied in [6], [11], and with variable delay, as far as we know, only in [13]. Let us note that problems without initial conditions for variational inequalities or inclusions with delay have not been considered in the literature, which serves as one of the motivations for the study of such problems.

The outline of this paper is as follows. In Section 2, we give notations, definitions of function spaces and auxiliary results. In Section 3, we formulate the problem and main result. In Section 4, we prove the main result.

2. Preliminaries

We set, as above, $S := (-\infty, 0]$. Let V be a reflexive and separable Banach space with norm $\|\cdot\|$, and H be a separable Hilbert spaces with the scalar product

$(\cdot,\cdot)$ and norm $|\cdot|$. Suppose that $V \subset H$ with dense, continuous and compact injection.

Let V' and H' be the dual spaces to V and H, respectively. We suppose (after appropriate identification of functionals), that the space H' is a subspace of V'. By the Riesz-Fréchet representation theorem, identifying the spaces H and H', we obtain the dense and continuous embeddings

$$V \subset H \subset V'. \tag{2.1}$$

Note that in this case $\langle g, v\rangle_V = (g, v)$ for every $v \in V, g \in H$, where $\langle\cdot,\cdot\rangle_V$ is the scalar product for the dual pair $\big[V', V\big]$. Thus, further we will be using notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle_V$.

We introduce some spaces of functions and distributions. Let X be an arbitrary Banach space with the norm $\|\cdot\|_X$. By $C(S;X)$ we mean the linear space of continuous functions defined on S with values in X. We say that $w_m \longrightarrow_{m\to\infty} w$ in $C(S;X)$ if for each $t_1, t_2 \in S,\ t_1 < t_2,$ the sequence of the restrictions of the functions $\{w_m\}_{m=1}^{\infty}$ to segment $[t_1, t_2]$ converges in $C([t_1, t_2]; X)$ to the restriction of w to the same segment.

Let $q \in [1, \infty]$, q' be dual to q, i.e., $1/q + 1/q' = 1$. Denote by $L^q_{\rm loc}(S;X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[t_1, t_2] \subset S$ belong to the space $L^q(t_1, t_2; X)$. We say that a sequence $\{w_m\}$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to w) in $L^q_{\rm loc}(S;X)$, if for each $t_1, t_2 \in S,\ t_1 < t_2,$ the sequence of restrictions of $\{w_m\}$ to the segment $[t_1, t_2]$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to the restriction of w to segment $[t_1, t_2]$) in $L^q(t_1, t_2; X)$.

By $D'(-\infty, 0; V')$ we mean the space of continuous linear functionals on $D(-\infty, 0)$ with values in V'_w. Hereafter $D(-\infty, 0)$ is a space of test functions, that is, the space of infinitely differentiable on $(-\infty, 0)$ functions with compact supports, equipped with the corresponding topology, and V'_w is the linear space V' equipped with weak topology. It is easy to see (using (2.1)), that spaces $L^q_{\rm loc}(S;V)$, $L^2_{\rm loc}(S;H)$, $L^{q'}_{\rm loc}(S;V')$ can be identified with the corresponding subspaces of $D'(-\infty, 0; V')$. In particular, this allows us to talk about derivatives w' of functions w from $L^q_{\rm loc}(S;V)$ or $L^2_{\rm loc}(S;H)$ in the sense of distributions $D'(-\infty, 0; V')$ and belonging of such derivatives to $L^{q'}_{\rm loc}(S;V')$ or $L^2_{\rm loc}(S;H)$.

Let us define the spaces

$$H^1(S;H) := \{w \in L^2(S;H) \,|\, w' \in L^2(S;H)\},$$

$$W^1_{q,\rm loc}(S;V) := \{w \in L^q_{\rm loc}(S;V) \,|\, w' \in L^{q'}_{\rm loc}(S;V')\}, \quad q > 1.$$

From known results (see., for example, [12, p. 177-179]) it follows that $H^1(S;H) \subset C(S;H)$ and $W^1_{q,\rm loc}(S;V) \subset C(S;H)$. Moreover, for every w from $H^1(S;H)$ or $W^1_{q,\rm loc}(S;V)$ the function $t \to |w(t)|^2$ is absolutely continuous on any segment of the interval S and the following equality holds

$$\frac{d}{dt}|w(t)|^2 = 2(w'(t), w(t)) \quad \text{for a.e.} \quad t \in S. \tag{2.2}$$

Remark 2.1. For $w \in L^2(S;H)$, we have

$$\lim_{\sigma\to-\infty}\int_{\sigma-1}^{\sigma}|w(t)|^2dt = 0.$$

If $w \in L^2(S;H) \cap C(S;H)$ then there exists a sequence $\{t_k\}_{k=0}^{\infty} \subset S$ such that $t_k \to -\infty$ for $k \to +\infty$ and

$$\lim_{k\to+\infty}|w(t_k)|^2 = 0.$$

In this paper we use the following well-known facts.

Proposition 2.1 (Cauchy-Schwarz-Bunjakovsky inequality; see, for example, [12, p. 158])**.** Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, and X is a Hilbert space with the scalar product $(\cdot,\cdot)_X$. Then, if $v \in L^2(t_1,t_2;X)$ and $w \in L^2(t_1,t_2;X)$, we have $(w(\cdot),v(\cdot))_X \in L^1\big(t_1,t_2\big)$ and

$$\int_{t_1}^{t_2}(w(t),v(t))_X\,dt \leqslant \|w\|_{L^2(t_1,t_2;X)}\|v\|_{L^2(t_1,t_2;X)}.$$

Proposition 2.2 ([27, p. 173,179])**.** Let Y be a Banach space with the norm $\|\cdot\|_Y$, and $\{v_k\}_{k=1}^{\infty}$ be a sequence of elements of Y, which is weakly or $*$-weakly convergent to v in Y. Then, $\varliminf_{k\to\infty}\|v_k\|_Y \geqslant \|v\|_Y$.

Proposition 2.3 (Aubin theorem [1], [2, p. 393])**.** Let $q > 1, r > 1$, $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, and B_0, B_1, B_2 are Banach spaces such that $B_0 \subset^c B_1 \subset B_2$ (here $\subset^c$ means compact embedding and $\subset$ means continuous embedding). Then

$$\{w \in L^q(t_1,t_2;B_0) \mid w' \in L^r(t_1,t_2;B_2)\} \overset{c}{\subset} \big(L^q(t_1,t_2;B_1) \cap C([t_1,t_2];B_2)\big). \quad (2.3)$$

Note that we understand the embedding (2.3) as follows: if a sequence $\{w_m\}$ is bounded in the space $L^q(t_1,t_2;B_0)$ and the sequence $\{w_m'\}_{m\in\mathbb{N}}$ is bounded in the space $L^r(t_1,t_2;B_2)$, then there exist a function $w \in L^q(t_1,t_2;B_1) \cap C([t_1,t_2];B_2)$ and a subsequence $\{w_{m_j}\}$ of the sequence $\{w_m\}$ such that $w_{m_j} \longrightarrow_{j\to\infty} w$ in $C([t_1,t_2];B_2)$ and strongly in $L^q(t_1,t_2;B_1)$.

Lemma 2.1. *If a sequence $\{w_m\}$ is bounded in the space $L^q_{\rm loc}(S;V), q > 1$, and the sequence $\{w_m'\}$ is bounded in the space $L^2_{\rm loc}(S;H)$, then there exist a function $w \in L^q_{\rm loc}(S;V)$, $w' \in L^2_{\rm loc}(S;H)$, and a subsequence $\{w_{m_j}\}$ of the sequence $\{w_m\}$ such that $w_{m_j} \longrightarrow_{j\to\infty} w$ in $C(S;H)$ and weakly in $L^q_{\rm loc}(S;V)$, and, $w_{m_j}' \longrightarrow_{j\to\infty} w'$ weakly in $L^2_{\rm loc}(S;H)$.*

Proof. The Proposition 2.3 for $r = 2$, $B_0 = V$, $B_1 = B_2 = H$ and the reflexiveness of V and H yields, for every $t_1, t_2 \in S$, $t_1 < t_2$, from the sequence of restrictions of the elements $\{w_m\}$ to the segment $[t_1,t_2]$ one can choose a subsequence which is convergent in $C([t_1,t_2];H)$ and weakly in $L^q(t_1,t_2;V)$, and the sequence of derivatives of the elements of this subsequence is weakly convergent in $L^2(t_1, t_2; H)$. For each $k \in \mathbb{N}$, we choose a subsequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$ of the given sequence, which is convergent in $C([-k, 0]; H)$ and weakly in $L^q(-k, 0; V)$ to some function $\widehat{w}_k \in C([-k, 0]; H) \cap L^q(-k, 0; V)$, and the sequence $\{w'_{m_{k,j}}\}_{j=1}^{\infty}$ is weakly convergent to the derivative $\widehat{w}'_k$ in $L^2(-k, 0; H)$. Making this choice we ensure that the sequence $\{w_{m_{k+1,j}}\}_{j=1}^{\infty}$ was a subsequence of the sequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$. Now, according to the diagonal process, we select the desired subsequence as $\{w_{m_{j,j}}\}_{j=1}^{\infty}$, and we define the function w as follows: for each $k \in \mathbb{N}$ we take $w(t) := \widehat{w}_k(t)$ for $t \in (-k, -k+1]$. □

In the sequel the Cauchy inequality of the following form will be used

$$ab \leq \varepsilon a^2 + (4\varepsilon)^{-1} b^2 \quad \forall a, b \in \mathbb{R}, \forall \varepsilon > 0. \tag{2.4}$$

3. Setting of the problem and main result

Let $\Phi : V \to (-\infty, +\infty]$ be a proper functional, i.e., $\mathrm{dom}(\Phi) := \{v \in V : \Phi(v) < +\infty\} \neq \emptyset$, which satisfies such conditions:

$(\mathcal{A}_1)$ $\quad \Phi\big(\alpha v + (1-\alpha) w\big) \leqslant \alpha \Phi(v) + (1-\alpha)\Phi(w) \quad \forall\, v, w \in V,\ \forall\, \alpha \in [0, 1],$

i.e., the functional Φ is *convex*,

$(\mathcal{A}_2)$ $\quad v_k \longrightarrow_{k\to\infty} v$ in $V \quad \Longrightarrow \quad \underline{\inf}_{k\to\infty} \Phi(v_k) \geq \Phi(v),$

i.e., the functional Φ is *lower semicontinuous.*

Recall that the *subdifferential* of functional Φ is a mapping $\partial\Phi : V \to 2^{V'}$, defined as follows

$$\partial\Phi(v) := \{v^* \in V' \mid \Phi(w) \geqslant \Phi(v) + (v^*, w - v) \quad \forall\, w \in V\},$$

for any $v \in V$, and the *domain* of the subdifferential $\partial\Phi$ is the set $D(\partial\Phi) := \{v \in V \mid \partial\Phi(v) \neq \emptyset\}$. We identify the subdifferential $\partial\Phi$ with its graph, assuming that $[v, v^*] \in \partial\Phi$ if and only if $v^* \in \partial\Phi(v)$, i.e., $\partial\Phi = \{[v, v^*] \mid v \in D(\partial\Phi),\ v^* \in \partial\Phi(v))\}$. R. Rockafellar (see [20, Theorem A]) proves that the subdifferential $\partial\Phi$ is a *maximal monotone operator*, that is,

$$(v_1^* - v_2^*, v_1 - v_2) \geqslant 0 \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi,$$

and for every element $[v_1, v_1^*] \in V \times V'$ we have the implication

$$(v_1^* - v_2^*, v_1 - v_2) \geqslant 0 \quad \forall\, [v_2, v_2^*] \in \partial\Phi \implies [v_1, v_1^*] \in \partial\Phi.$$

Additionally, assume that the following conditions hold:

($\mathcal{A}_3$) there exist constants $p \geq 2$, $K_1 > 0$ such that

$$\Phi(v) \geqslant K_1 \|v\|^p \quad \forall\, v \in \operatorname{dom}(\Phi);$$

moreover, $\Phi(0) = 0$;

($\mathcal{A}_4$) there exists a constant $K_2 > 0$ such that

$$(v_1^* - v_2^*, v_1 - v_2) \geqslant K_2 |v_1 - v_2|^2 \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi.$$

Remark 3.1. Condition $\Phi(0) = 0$ (see ($\mathcal{A}_3$)) implies that $\Phi(v) \geq \Phi(0) + (0, v - 0)$ $\forall v \in V$, hence, $[0, 0] \in \partial\Phi$. From this and condition ($\mathcal{A}_4$) we have

$$(v^*, v) \geq K_2 |v|^2 \quad \forall\, [v, v^*] \in \partial\Phi. \tag{3.1}$$

Let $\tau : S \to \mathbb{R}$ be a function such that

($\mathcal{T}$) $\tau \in C(S)$, $\tau(t) \geq 0$ for all $t \in S$, $\tau^+ := \sup_{t\in S} \tau(t) < \infty$.

Let $c : \Pi_\tau \times H \to H$, where $\Pi_\tau := \{(t, s) \,|\, t \leq 0,\ t - \tau(t) \leq s \leq t\}$ and τ satisfies condition ($\mathcal{T}$), be a function which satisfies the condition:

($\mathcal{C}$) for any $v \in H$ the mapping $c(\cdot, \cdot, v) : \Pi_\tau \to H$ is measurable, and there exists a constant $L \geq 0$ such that following inequality holds

$$|c(t, s, v_1) - c(t, s, v_2)| \leq L |v_1 - v_2|$$

for a.e. $(t, s) \in \Pi_\tau$ and for all $v_1, v_2 \in H$; in addition, $c(t, s, 0) = 0$ for a.e. $(t, s) \in \Pi_\tau$.

Remark 3.2. From the condition ($\mathcal{C}$), it follows that, for a.e. $(t, s) \in \Pi_\tau$ and for every $v \in H$, the following estimate is valid:

$$|c(t, s, v)| \leq L|v|. \tag{3.2}$$

Remark 3.3. Conditions ($\mathcal{T}$), ($\mathcal{C}$) and remark 3.2 yield, for any function $w \in L^2(S; H)$ the function $t \mapsto \int_{t-\tau(t)}^{t} c(t, s, w(s))\, ds : S \to H$ belongs to $L^2(S; H)$.

Indeed, by (3.2), assuming that $w(t) = 0$ for all $t > 0$, using Cauchy-Schwarz-Bunjakovsky inequality and changing the order of integration, we have

$$\begin{aligned} \int_\sigma^0 \Big| \int_{t-\tau(t)}^{t} c(t, s, w(s))\, ds \Big|^2 dt &\leq L^2 \tau^+ \int_\sigma^0 \int_{t-\tau^+}^{t} |w(s)|^2\, ds dt \\ \leq L^2 \tau^+ \int_{\sigma-\tau^+}^{0} |w(s)|^2\, ds \int_s^{s+\tau^+} dt &= (L\tau^+)^2 \int_{\sigma-\tau^+}^{0} |w(s)|^2\, ds \\ &\leq (L\tau^+)^2 ||w||^2_{L^2(S;H)} \end{aligned} \tag{3.3}$$

for each $\sigma \in S$. Thus, the function $t \mapsto \int_{t-\tau(t)}^{t} c(t, s, w(s))\, ds$ belongs to $L^2(S; H)$.

Let us consider the evolutionary variational inequality

$$u'(t) + \partial\Phi\big(u(t)\big) + \int_{t-\tau(t)}^{t} c(t,s,u(s))\,ds \ni f(t), \quad t \in S, \tag{3.4}$$

where $f : S \to V'$ is a given measurable function, and $u : S \to V$ is an unknown function.

Definition 3.1. Let conditions $(\mathcal{A}_1)$ – $(\mathcal{A}_3)$, $(\mathcal{T})$, $(\mathcal{C})$ hold, and $f \in L^{p'}_{\mathrm{loc}}(S;V')$. The *solution* of variational inequality (3.4) is a function $u : S \to V$ that satisfies the following conditions:

1) $u \in W^1_{p,\mathrm{loc}}(S;V)$;

2) $u(t) \in D(\partial\Phi)$ for a.e. $t \in S$;

3) there exists a function $g \in L^{p'}_{\mathrm{loc}}(S;V')$ such that, for a.e. $t \in S$, $g(t) \in \partial\Phi\big(u(t)\big)$ and

$$u'(t) + g(t) + \int_{t-\tau(t)}^{t} c(t,s,u(s))\,ds = f(t) \quad \text{in} \ \ V'.$$

We consider the problem of finding a solution u of variational inequality (3.4) for given Φ, c, τ and f such that

$$\int_S |u(t)|^2\,dt < +\infty, \quad \text{that is} \ \ u \in L^2(S;H). \tag{3.5}$$

This problem is called the *problem* $\mathbf{P}(\Phi,\tau,c,f)$, and the function u is called its *solution*.

Theorem 3.1. *Let conditions* $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$, $(\mathcal{T})$, $(\mathcal{C})$ *hold,* $f \in L^2(S;H)$, *and*

$$K_2 - L\tau^+ > 0. \tag{3.6}$$

Then the problem $\boldsymbol{P}(\Phi,\tau,c,f)$ *has a unique solution, it belongs to the space* $L^\infty(S;V) \cap L^p(S;V) \cap H^1(S;H)$ *and satisfies the estimate*

$$\operatorname*{ess\,sup}_{t\in S} \|u(t))\|^p + \int_S \big(\|u(t)\|^p + |u(t)|^2 + |u'(t)|^2\big)\,dt$$

$$+ \int_S \Phi(u(t))dt \leqslant C_1 \int_S |f(t)|^2\,dt, \tag{3.7}$$

where C_1 *is a positive constant depending on* $K_1, K_2, L,$ *and* τ^+ *only.*

Remark 3.4. The problem $\mathbf{P}(\Phi,\tau,c,f)$ can be replaced by the following one. Let K be a convex and closed set in V, $A : V \to V'$ be a monotone, bounded and semicontinuous operator such that $(A(v),v) \geq \widetilde{K}_1\|v\|^p \quad \forall v \in V$, where $p \geq 2, \widetilde{K}_1 = \text{const} > 0$. The problem is to find a function $u \in W^1_{p,\text{loc}}(S;V) \cap L^2(S;H)$ such that for a.e. $t \in S$, $u(t) \in K$ and

$$(u'(t) + A(u(t)) + \int_{t-\tau(t)}^{t} c(t,s,u(s))\,ds, v - u(t)) \geq (f(t), v - u(t)) \quad \forall\, v \in K.$$

4. Proof of the main result

We divide the proof of Theorem 3.1 into seven steps.

Step 1 (*uniqueness of solution*). Assume the contrary. Let u_1, u_2 be two solutions of the problem $\mathbf{P}(\Phi,\tau,c,f)$. Then for every $i \in \{1,2\}$ there exists function $g_i \in L^{p'}_{\text{loc}}(S;V')$ such that, for a.e. $t \in S$, we have $g_i(t) \in \partial\Phi\big(u_i(t)\big)$ and

$$u_i'(t) + g_i(t) + \int_{t-\tau(t)}^{t} c(t,s,u_i(s))\,ds = f(t) \quad \text{in } V', \quad i = 1,2. \tag{4.1}$$

We put $w := u_1 - u_2$. From equalities (4.1), for a.e. $t \in S$, we obtain

$$w'(t) + g_1(t) - g_2(t) + \int_{t-\tau(t)}^{t} \big(c(t,s,u_1(s)) - c(t,s,u_2(s))\big)ds = 0 \ \text{ in } \ V'. \tag{4.2}$$

Let $t_1, t_2 \in S$ be arbitrary numbers such that $t_1 < t_2$. Multiplying equality (4.2) by $w(t)$ and integrating from t_1 to t_2, we have

$$\int_{t_1}^{t_2} (w'(t), w(t))\,dt + \int_{t_1}^{t_2} \big(g_1(t) - g_2(t), u_1(t) - u_2(t)\big)\,dt \\ + \int_{t_1}^{t_2} \Big(\int_{t-\tau(t)}^{t} (c(t,s,u_1(s)) - c(t,s,u_2(s)))\,ds, w(t)\Big)\,dt = 0. \tag{4.3}$$

Consider the third term from left-hand side of equality (4.3). By condition $(\mathcal{T})$, $(\mathcal{C})$, the Fubini Theorem and the Cauchy-Schwarz-Bunjakovsky inequality, we get

$$\begin{aligned} \Big|\int_{t_1}^{t_2} \Big(\int_{t-\tau(t)}^{t} \big(c(t,s,u_1(s)) - c(t,s,u_2(s))\big)\,ds, w(t)\Big)dt\Big| \\ \leq \int_{t_1}^{t_2} \Big(\int_{t-\tau(t)}^{t} \big|c(t,s,u_1(s)) - c(t,s,u_2(s))\big|ds\Big)|w(t)|\,dt \\ \leq L\int_{t_1}^{t_2} \Big(\int_{t-\tau^+}^{t} |w(s)|\,ds\Big)\,|w(t)|\,dt \\ \leq L\sqrt{\tau^+}\Big(\int_{t_1}^{t_2} |w(t)|^2\,dt\Big)^{1/2}\Big(\int_{t_1}^{t_2}\Big(\int_{t-\tau^+}^{t} |w(s)|^2\,ds\Big)\,dt\Big)^{1/2}. \end{aligned} \tag{4.4}$$

Changing the order of integration, we have

$$\int_{t_1}^{t_2}\Big(\int_{t-\tau^+}^{t}|w(s)|^2ds\Big)\,dt \le \int_{t_1-\tau^+}^{t_2}|w(s)|^2\,ds\int_{s}^{s+\tau^+}dt$$
$$= \tau^+\Big(\int_{t_1}^{t_2}|w(s)|^2\,ds+\int_{t_1-\tau^+}^{t_1}|w(s)|^2\,ds\Big). \quad (4.5)$$

Substituting in (4.4) the last term from relations chain (4.5) instead of the first one, and using inequalities: $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$, $\sqrt{a}\sqrt{b}\le\varepsilon a+(4\varepsilon)^{-1}b$, $a\ge 0$, $b\ge 0$, $\varepsilon>0$, we obtain

$$\Big|\int_{t_1}^{t_2}\Big(\int_{t-\tau(t)}^{t}\big(c(t,s,u_1(s))-c(t,s,u_2(s))\big)\,ds, w(t)\Big)\,dt\Big|$$
$$\le L\tau^+\Big((1+\varepsilon)\int_{t_1}^{t_2}|w(t)|^2\,dt+(4\varepsilon)^{-1}\int_{t_1-\tau^+}^{t_1}|w(t)|^2\,dt\Big), \quad (4.6)$$

where $\varepsilon>0$ is an arbitrary.

By equality (2.2), inequality (4.6), condition $(\mathcal{A}_4)$ and the fact that $g_i(t)\in\partial\Phi(u_i(t))$ for a.e. $t\in S$, $i=1,2$, from (4.3) for a.e. $t\in S$, we obtain

$$\frac{1}{2}\int_{t_1}^{t_2}\big(|w(t)|^2\big)'\,dt$$
$$+\big(K_2-(1+\varepsilon)L\tau^+\big)\int_{t_1}^{t_2}|w(t)|^2\,dt-(4\varepsilon)^{-1}L\tau^+\int_{t_1-\tau^+}^{t_1}|w(t)|^2dt\le 0. \quad (4.7)$$

Using the integration-by-parts formula, we have

$$|w(t)|^2\Big|_{t_1}^{t_2}+2\big(K_2-(1+\varepsilon)L\tau^+\big)\int_{t_1}^{t_2}|w(t)|^2\,dt$$
$$\le(2\varepsilon)^{-1}L\tau^+\int_{t_1-\tau^+}^{t_1}|w(t)|^2\,dt. \quad (4.8)$$

By inequality (3.6) and taking $\varepsilon>0$ such that $K_2-(1+\varepsilon)L\tau^+\ge 0$, from (4.8) we obtain

$$|w(t_2)|^2\le|w(t_1)|^2+C_3\int_{t_1-\tau^+}^{t_1}|w(t)|^2dt, \quad (4.9)$$

where $C_3>0$ is a constant independent of t_1,t_2.

Let us fix an arbitrary $t_2\in S$. Since $w\in L^2(S;H)\cap C(S;H)$, according to Remark 2.1 there exists a sequence $\{t_{1,k}\}_{k=1}^{\infty}\subset S$ such that $t_{1,k}<t_2$ for all $k\in\mathbb{N}$, $t_{1,k}\longrightarrow_{k\to+\infty}-\infty$, and

$$|w(t_{1,k})|^2+C_3\int_{t_{1,k}-\tau^+}^{t_{1,k}}|w(t)|^2\,dt\underset{k\to+\infty}{\to}0.$$

Taking $t_{1,k}$ ($k \in \mathbb{N}$) instead of t_1 in (4.9) and passing to the limit as $k \to +\infty$ we obtain $|w(x,t_2)|^2 = 0$. Since $t_2 \in S$ is an arbitrary number, we have $w(t) = 0$ for a.e. $t \in S$, this contradicts our assumption. Therefore, a solution of the problem $\mathbf{P}(\Phi, \tau, c, f)$ is unique.

Step 2 (*auxiliary statements*). We define the functional $\Phi_H : H \to (-\infty, +\infty]$ by the rule: $\Phi_H(v) := \Phi(v)$, if $v \in V$, and $\Phi_H(v) := +\infty$ otherwise. Note that conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, Lemma IV.5.2 and Proposition IV.5.2 of the monograph [22] imply that Φ_H is a proper, convex, and lower-semi-continuous functional on H, $\mathrm{dom}(\Phi_H) = \mathrm{dom}(\Phi) \subset V$ and $\partial\Phi_H = \partial\Phi \cap (V \times H)$, where $\partial\Phi_H : H \to 2^H$ is the subdifferential of the functional Φ_H. Moreover, condition $(\mathcal{A}_3)$ yields $0 \in \partial\Phi_H(0)$ (see Remark 3.1).

The following statements will be used in the sequel.

Proposition 4.1 ([22, Lemma IV.4.3])**.** Let $-\infty < a < b < +\infty$, $w \in H^1(a,b;H)$, and $g \in L^2(a,b;H)$ such that $w(t) \in D(\partial\Phi_H)$ and $g(t) \in \partial\Phi_H\big(w(t)\big)$ for a.e. $t \in (a,b)$. Then the function $\Phi_H\big(w(\cdot)\big)$ is absolutely continuous on the interval $[a,b]$ and for any function $h : [a,b] \to H$ such that $h(t) \in \partial\Phi_H\big(w(t)\big)$ for a.e. $t \in [a,b]$ the following equality holds

$$\frac{d}{dt}\Phi_H\big(w(t)\big) = (h(t), w'(t)) \quad \text{for a.e. } t \in [a,b].$$

Proposition 4.2 ([8, Proposition 3.12], [22, Proposition IV.5.2])**.** Let $T > 0$, $\widetilde{f} \in L^2(0,T;H)$ and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function $w \in H^1(0,T;H)$ such that $w(0) = w_0$ and, for a.e. $t \in (0,T]$, we have $w(t) \in D(\partial\Phi_H)$ and

$$w'(t) + \partial\Phi_H\big(w(t)\big) \ni \widetilde{f}(t) \quad \text{in } H. \tag{4.10}$$

i.e., there exists a function $\widetilde{g} \in L^2(0,T;H)$ such that, for a.e. $t \in (0,T]$, we have $\widetilde{g}(t) \in \partial\Phi_H(w(t))$ and

$$w'(t) + \widetilde{g}(t) = \widetilde{f}(t) \quad \text{in } H. \tag{4.11}$$

Lemma 4.1. *Let $t_0 < 0$, $\widetilde{f} \in L^2(t_0,0;H)$, and $w_0 \in C([\tau_0,t_0];H)$, $w_0(t) \in \mathrm{dom}(\Phi)$ for every $t \in [\tau_0,t_0]$, where $\tau_0 := \min_{t\in[t_0,0]}(t-\tau(t))$ (if $\tau_0 = t_0$ then $[\tau_0,t_0] = \{t_0\}$). Then there exists a unique function $w \in C([\tau_0,0];H) \cap H^1(t_0,0;H)$ such that $w(t) = w_0(t)$ for every $t \in [\tau_0,t_0]$, and, for a.e. $t \in (t_0,0]$, we have $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) + \int_{t-\tau(t)}^{t} c(t,s,w(s))\,ds \ni \widetilde{f}(t) \quad \text{in } H, \tag{4.12}$$

that is, there exists function $\widetilde{g} \in L^2(t_0,0;H)$ such that, for a.e. $t \in (t_0,0]$ we have $\widetilde{g}(t) \in \partial\Phi_H(w(t))$ and

$$w'(t) + \widetilde{g}(t) + \int_{t-\tau(t)}^{t} c(t,s,w(s))\,ds = \widetilde{f}(t) \quad \text{in } H. \tag{4.13}$$

Proof of Lemma 4.1. Let $M := \{w \in C([\tau_0, 0]; H) \mid w(t) = w_0(t)\ \forall t \in [\tau_0, t_0]\}$ be a set with the metric

$$\rho(w_1, w_2) = \max_{t\in[t_0,0]} \big[e^{-\alpha(t-t_0)}|w_1(t) - w_2(t)|\big], \quad w_1, w_2 \in M,$$

where $\alpha > 0$ is an arbitrary fixed number. It is obvious that the metric space (M, ρ) is complete. Now let us consider an operator $A : M \to M$ defined as follows: for any given function $\widetilde{w} \in M$, it defines a function $\widehat{w} \in M \cap H^1(t_0, 0; H)$ such that, for a.e. $t \in [t_0, 0]$, we have $\widehat{w}(t) \in D(\partial\Phi_H)$ and

$$\widehat{w}\,'(t) + \partial\Phi_H(\widehat{w}(t)) \ni \widetilde{f}(t) - \int_{t-\tau(t)}^{t} c(t, s, \widetilde{w}(s))\, ds \quad \text{in} \quad H. \tag{4.14}$$

Clearly, variational inequality (4.14) coincides with variational inequality (4.10) after replacing $[0, T]$ by $[t_0, 0]$, $\widetilde{f}(t)$ by $\widetilde{f}(t) - \int_{t-\tau(t)}^{t} c(t, s, \widetilde{w}(s))\, ds$ and the condition $w(0) = w_0$ by the condition $\widehat{w}(0) = w_0(t_0)$. Thus, using Proposition 4.2, we get that operator A is well-defined. Let us show that the operator A is a contraction for some $\alpha > 0$. Indeed, let $\widetilde{w}_1, \widetilde{w}_2$ be arbitrary functions from M and $\widehat{w}_1 := A\widetilde{w}_1$, $\widehat{w}_2 = A\widetilde{w}_2$. According to (4.14) (see (4.11)) there exist functions $\widehat{g}_1$, $\widehat{g}_2$ from $L^2(t_0, 0; H)$ such that, for each $k \in \{1, 2\}$ and a.e. $t \in (t_0, 0]$, we have $\widehat{g}_k(t) \in \partial\Phi_H(\widehat{w}_k(t))$ and

$$\widehat{w}_k'(t) + \widehat{g}_k(t) = \widetilde{f}(t) - \int_{t-\tau(t)}^{t} c(t, s, \widetilde{w}_k(s))\, ds \quad \text{in} \quad H, \tag{4.15}$$

while $\widehat{w}_k(t) = w_0(t)$ for a.e. $t \in [\tau_0, t_0]$.

Subtracting identity (4.15) for $k = 2$ from identity (4.15) for $k = 1$, and, for a.e. $t \in (t_0, 0]$, multiplying the obtained identity by $\widehat{w}_1(t) - \widehat{w}_2(t)$, we get

$$\begin{aligned}\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) + (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t))\\ = -\Big(\int_{t-\tau(t)}^{t} \big(c(t, s, \widetilde{w}_1(s)) - c(t, s, \widetilde{w}_2(s))\big)\, ds, \widehat{w}_1(t) - \widehat{w}_2(t)\Big),\end{aligned} \tag{4.16}$$

$$\widehat{w}_1(t) - \widehat{w}_2(t) = 0 \quad \text{for a.e.} \quad t \in [\tau_0, t_0]. \tag{4.17}$$

We integrate equality (4.16) by t from t_0 to $\sigma \in [t_0, 0]$, taking into account that for a.e. $t \in (t_0, 0]$ we have

$$\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) = \frac{1}{2}\frac{d}{dt}|\widehat{w}_1(t) - \widehat{w}_2(t)|^2.$$

As a result, we get the equality

$$\begin{aligned}\frac{1}{2}|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + \int_{t_0}^{\sigma} (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t))\, dt\\ = -\int_{t_0}^{\sigma} \Big(\int_{t-\tau(t)}^{t} \big(c(t, s, \widetilde{w}_1(s)) - c(t, s, \widetilde{w}_2(s))\big)\, ds, \widehat{w}_1(t) - \widehat{w}_2(t)\Big) dt.\end{aligned} \tag{4.18}$$

By condition $(\mathcal{A}_4)$, for a.e. $t \in (t_0, 0]$, we have the inequality

$$(\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t)) \geqslant K_2|\widehat{w}_1(t) - \widehat{w}_2(t))|^2. \tag{4.19}$$

Taking into account conditions $(\mathcal{T})$, $(\mathcal{C})$ and the Cauchy inequality (2.4), for a.e. $t \in (t_0, 0]$, we obtain

$$\begin{aligned}
&\Big|\Big(\int_{t-\tau(t)}^{t} \big(c(t,s,\widetilde{w}_1(s)) - c(t,s,\widetilde{w}_2(s))\big)\,ds, \widehat{w}_1(t) - \widehat{w}_2(t)\Big)\Big| \\
&\le \Big(\int_{t-\tau(t)}^{t} \big|c(t,s,\widetilde{w}_1(s)) - c(t,s,\widetilde{w}_2(s))\big|\,ds\Big)\,|\widehat{w}_1(t) - \widehat{w}_2(t)| \\
&\le L\Big(\int_{t-\tau^+}^{t} |\widetilde{w}_1(s) - \widetilde{w}_2(s)|\,ds\Big)|\widehat{w}_1(t) - \widehat{w}_2(t)| \\
&\le \varepsilon|\widehat{w}_1(t) - \widehat{w}_2(t)|^2 + (4\varepsilon)^{-1}L^2\Big(\int_{t-\tau^+}^{t} |\widetilde{w}_1(s) - \widetilde{w}_2(s)|\,ds\Big)^2 \\
&\le \varepsilon|\widehat{w}_1(t) - \widehat{w}_2(t)|^2 + (4\varepsilon)^{-1}L^2\tau^+\int_{t-\tau^+}^{t} |\widetilde{w}_1(s) - \widetilde{w}_2(s)|^2\,ds,
\end{aligned} \tag{4.20}$$

where $\varepsilon > 0$ is an arbitrary number, $\widetilde{w}_1(s) - \widetilde{w}_2(s) := 0 \ \forall s \le \tau_0$.

From (4.18), according to (4.19) and (4.20), we have

$$\begin{aligned}
&|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + 2(K_2 - \varepsilon)\int_{t_0}^{\sigma} |\widehat{w}_1(t) - \widehat{w}_2(t)|^2\,dt \\
&\qquad\le (2\varepsilon)^{-1}L^2\tau^+\int_{t_0}^{\sigma}\Big(\int_{t-\tau^+}^{t} |\widetilde{w}_1(s) - \widetilde{w}_2(s)|^2\,ds\Big)\,dt.
\end{aligned} \tag{4.21}$$

Let us consider the right-hand side of the inequality (4.21). Using the assumption that $\widetilde{w}_1(s) - \widetilde{w}_2(s) = 0$ for $s \le t_0$ and $s \ge 0$, we obtain

$$\int_{t_0}^{\sigma}\Big(\int_{t-\tau^+}^{t} |\widetilde{w}_1(s) - \widetilde{w}_2(s)|^2\,ds\Big)\,dt \le t_0\int_{t_0}^{\sigma} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2\,dt. \tag{4.22}$$

From (4.21) and (4.22), choosing $\varepsilon = K_2$, we get

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 \leqslant C_2\int_{t_0}^{\sigma} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2\,dt, \quad \sigma \in (t_0, 0], \tag{4.23}$$

where $C_2 > 0$ is a constant depending on L, K_2, τ^+, and t_0 only.

Multiplying (4.23) by $e^{-2\alpha(\sigma-t_0)}$, we obtain

$$\begin{aligned}
&e^{-2\alpha(\sigma-t_0)}|\widehat{w}_1(\sigma)-\widehat{w}_2(\sigma)|^2\\
&\qquad\leqslant C_2e^{-2\alpha(\sigma-t_0)}\int_{t_0}^{\sigma}e^{2\alpha(t-t_0)}e^{-2\alpha(t-t_0)}|\widetilde{w}_1(t)-\widetilde{w}_2(t)|^2\,dt\\
&\qquad\leqslant C_2e^{-2\alpha(\sigma-t_0)}\max_{t\in[t_0,0]}\big[e^{-\alpha(t-t_0)}|\widetilde{w}_1(t)-\widetilde{w}_2(t)|\big]^2\int_{t_0}^{\sigma}e^{2\alpha(t-t_0)}\,dt\\
&\qquad=\frac{C_2}{2\alpha}\big(1-e^{-2\alpha(\sigma-t_0)}\big)\big[\rho(\widetilde{w}_1,\widetilde{w}_2)\big]^2\\
&\qquad\leqslant\frac{C_2}{2\alpha}\big[\rho(\widetilde{w}_1,\widetilde{w}_2)\big]^2,\quad \sigma\in[t_0,0]. \qquad (4.24)
\end{aligned}$$

From (4.24) it follows

$$\rho(\widehat{w}_1,\widehat{w}_2)\leqslant\sqrt{C_2/(2\alpha)}\rho(\widetilde{w}_1,\widetilde{w}_2).$$

This, choosing $\alpha>0$ such that $C_2/(2\alpha)<1$, yields, operator A is a contraction. Hence, we may apply the Banach fixed-point theorem (in other words, the contraction mapping principle; see, for example, [9, Theorem 5.7]) and deduce that there exists a unique function $w\in M$ such that $Aw=w$, i.e., we have proved our proposition. □

Step 3 (*solution approximations*). We construct a sequence of functions which, in some sense, approximate the solution of the problem $\mathbf{P}(\Phi,\tau,c,f)$.

Let $\{\varkappa_k\}_{k=1}^{\infty}$ be a monotonically decreasing sequence of numbers from S such that $\varkappa_1<0$ and $\lim_{k\to\infty}\varkappa_k=-\infty$. Denote $\widehat{f}_k(t):=f(t)$ for $t\in[\varkappa_k,0]$, $\tau_k:=\min_{t\in[\varkappa_k,0]}(t-\tau(t))$, $k\in\mathbb{N}$.

For each $k\in\mathbb{N}$ consider the problem of finding a function $\widehat{u}_k\in C([\tau_k,0];H)\cap H^1(\varkappa_k,0;H)$ such that, for a.e. $t\in(\varkappa_k,0]$, we have $\widehat{u}_k(t)\in D(\partial\Phi_H)$ and

$$\widehat{u}_k'(t)+\partial\Phi_H\big(\widehat{u}_k(t)\big)+\int_{t-\tau(t)}^{t}c(t,s,\widehat{u}_k(s))\,ds\ni\widehat{f}_k(t)\quad\text{in } H, \qquad (4.25)$$

and

$$\widehat{u}_k(t)=0,\quad t\in[\tau_k,\varkappa_k]. \qquad (4.26)$$

Inclusion (4.25) means that there exists a function $\widehat{g}_k\in L^2(\varkappa_k,0;H)$ such that, for a.e. $t\in(\varkappa_k,0]$, we have $\widehat{g}_k(t)\in\partial\Phi_H(\widehat{u}_k(t))$ and

$$\widehat{u}_k'(t)+\widehat{g}_k(t)+\int_{t-\tau(t)}^{t}c(t,s,\widehat{u}_k(s))\,ds=\widehat{f}_k(t)\quad\text{in } H. \qquad (4.27)$$

Lemma 4.1 implies the existence and uniqueness of solution of the problem (4.25), (4.26). Since $D(\partial\Phi_H)\subset\mathrm{dom}(\Phi_H)=\mathrm{dom}(\Phi)$ then $\widehat{u}_k(t)\in V$ for a.e. $t\in[\varkappa_k,0]$. According to the definition of the subdifferential of a functional and the fact that $\widehat{g}_k(t)\in\partial\Phi_H(\widehat{u}(t))$ for a.e. $t\in(\varkappa_k,0]$, we have

$$\Phi_H(0)\geq\Phi_H(\widehat{u}_k(t))+(\widehat{g}_k(t),0-\widehat{u}_k(t))\quad\text{for a.e. } t\in(\varkappa_k,0].$$

This and condition $(\mathcal{A}_3)$ yield that for a.e. $t \in (\varkappa_k, 0]$ we have

$$(\widehat{g}_k(t), \widehat{u}_k(t)) \geq \Phi(\widehat{u}_k(t)) \geq K_1 \|\widehat{u}_k(t)\|^p. \tag{4.28}$$

Since the left side of this chain of inequalities belongs to $L^1(S_k)$ then $\widehat{u}_k$ belongs to $L^p(\varkappa_k, 0; V)$.

For each $k \in \mathbb{N}$ we extend functions $\widehat{f}_k, \widehat{u}_k$ and $\widehat{g}_k$ by zero for the entire interval S, and denote these extensions by f_k, u_k and g_k, respectively. From the above it follows that for each $k \in \mathbb{N}$ the function u_k belongs to $L^p(S; V)$, its derivative u_k' belongs to $L^2(S; H)$ and for a.e. $t \in S$ the inclusion $g_k(t) \in \partial\Phi_H\big(u_k(t)\big)$ and the following equality (see (4.27)) hold

$$u_k'(t) + g_k(t) + \int_{t-\tau(t)}^{t} c(t, s, u_k(s))\, ds = f_k(t) \quad \text{in} \quad H. \tag{4.29}$$

In order to show the convergence $\{u_k\}_{k=1}^{+\infty}$ to the solution of the problem $\mathbf{P}(\Phi, \tau, c, f)$ we need some estimates of the functions u_k, $k \in \mathbb{N}$.

Step 4 (*first order estimates of solution approximations*).

Let $t_1,\ t_2 \in S$ be arbitrary numbers such that $t_1 < t_2$. Multiplying identity (4.29) for a.e. $t \in S$ by $u_k(t)$ and integrating by t from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2} (u_k'(t), u_k(t))\, dt + \int_{t_1}^{t_2} (g_k(t), u_k(t))\, dt + \int_{t_1}^{t_2} \Big(\int_{t-\tau(t)}^{t} c(t, s, u_k(s))\, ds, u_k(t) \Big)\, dt = \int_{t_1}^{t_2} (f_k(t), u_k(t))\, dt. \tag{4.30}$$

Equality (2.2) yield

$$\int_{t_1}^{t_2} (u_k'(t), u_k(t)) dt = \int_{t_1}^{t_2} \frac{d}{dt} |u_k(t)|^2 dt = \frac{1}{2} (|u_k(t_2)|^2 - |u_k(t_1)|^2). \tag{4.31}$$

From Remark 3.1, it follows

$$(g_k(t), u_k(t)) \geq K_2 |u_k(t)|^2 \quad \text{for a.e.} \quad t \in S. \tag{4.32}$$

By inequalities (4.28) and (4.32), for a.e. $t \in S$, we have

$$\begin{aligned} (g_k(t), u_k(t)) &\geq \delta (g_k(t), u_k(t)) + (1-\delta)(g_k(t), u_k(t)) \\ &\geq \delta K_2 |u_k(t)|^2 + \frac{1}{2}(1-\delta) K_1 \|u_k(t)\|^p + \frac{1}{2}(1-\delta)\Phi(u_k(t)), \end{aligned} \tag{4.33}$$

where $\delta \in (0, 1)$ is an arbitrary number.

Let us estimate the second term of the left-hand side of equality (4.30) by using (4.33), in this way

$$\int_{t_1}^{t_2} (g_k(t), u_k(t))\, dt \geq \frac{1}{2} \int_{t_1}^{t_2} \Big((1-\delta)\Phi\big(u_k(t)\big) + (1-\delta) K_1 \|u_k(t)\|^p + 2\delta K_2 |u_k(t)|^2 \Big)\, dt. \tag{4.34}$$

We estimate the third term on the left-hand side of equality (4.30) by using the Cauchy-Schwartz-Bunjakovsky inequality and (3.2). As the result, we obtain

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2}\Big(\int_{t-\tau(t)}^{t} c(t,s,u_k(s))\,ds, u_k(t)\Big)\,dt\Big| \\
&\qquad \le \int_{t_1}^{t_2}\Big(\int_{t-\tau(t)}^{t} \big|c(t,s,u_k(s))\big|\,ds\Big)|u_k(t)|\,dt \\
&\qquad \le L\int_{t_1}^{t_2}\Big(\int_{t-\tau^+}^{t} |u_k(s)|\,ds\Big)\,|u_k(t)|\,dt \\
&\qquad \le L\sqrt{\tau^+}\Big(\int_{t_1}^{t_2}|u_k(t)|^2\,dt\Big)^{1/2}\Big(\int_{t_1}^{t_2}\Big(\int_{t-\tau^+}^{t}|u_k(s)|^2\,ds\Big)\,dt\Big)^{1/2}. \qquad (4.35)
\end{aligned}$$

Now, let us estimate the last item on the inequality chain above. Changing the order of integration, we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\Big(\int_{t-\tau^+}^{t}|u_k(s)|^2 ds\Big)\,dt \\
&\qquad\qquad \le \int_{t_1-\tau^+}^{t_2}|u_k(s)|^2\,ds\int_{s}^{s+\tau^+}dt = \tau^+\int_{t_1-\tau^+}^{t_2}|u_k(t)|^2\,dt. \quad (4.36)
\end{aligned}$$

From (4.35), (4.36) with $t_1 < \varkappa_k$, and definition of u_k, it follows

$$\Big|\int_{t_1}^{t_2}\Big(\int_{t-\tau(t)}^{t} c(t,s,u_k(s))\,ds, u_k(t)\Big)dt\Big| \le L\tau^+\int_{t_1}^{t_2}|u_k(t)|^2 dt. \qquad (4.37)$$

Now we estimate the first term of the right-hand side of equality (4.30) by using the Cauchy-Schwartz-Bunjakovsky and Cauchy inequalities (2.4). As the result, we obtain

$$\int_{t_1}^{t_2}(f_k(t),u_k(t))\,dt \leqslant \varepsilon\int_{t_1}^{t_2}|u_k(t)|^2\,dt + (4\varepsilon)^{-1}\int_{t_1}^{t_2}|f_k(t)|^2\,dt, \qquad (4.38)$$

where $\varepsilon > 0$ is an arbitrary number.

From (4.30), taking into account (4.31), (4.34), (4.37) and (4.38), for any $t_1, t_2 \in S$ such that $t_1 < \min\{\varkappa_k, t_2\}$, we obtain

$$\begin{aligned}
&|u_k(t_2)|^2 + (1-\delta)\int_{t_1}^{t_2}\Phi\big(u_k(t)\big)dt + (1-\delta)K_1\int_{t_1}^{t_2}\|u_k(t)\|^p\,dt \\
&\qquad\qquad + 2[\delta K_2 - L\tau^+ - \varepsilon]\int_{t_1}^{t_2}|u_k(t)|^2\,dt \le (2\varepsilon)^{-1}\int_{t_1}^{t_2}|f_k(t)|^2\,dt.
\end{aligned}$$

First we choose $\delta \in (0,1)$ such that $\delta K_2 - L\tau^+ > 0$ (see (3.6)). Then take $\varepsilon = (\delta K_2 - L\tau^+)/2 > 0$. As a result, we obtain

$$|u_k(t_2)|^2 + \int_{t_1}^{t_2}\Phi\big(u_k(t)\big)\,dt + \int_{t_1}^{t_2}\big[\|u_k(t)\|^p + |u_k(t)|^2\big]\,dt \le C_4\int_{t_1}^{t_2}|f_k(t)|^2\,dt,$$

where C_4 is a positive constant depended on K_1, K_2, L, and τ^+ only.

Since $t_2 \in S$ is an arbitrary, by the definition of f_k, we have

$$\sup_{t\in S} |u_k(t)|^2 + \int_S \Phi\big(u_k(t)\big)\, dt + \int_S \big[\|u_k(t)\|^p + |u_k(t)|^2\big]\, dt \leqslant C_5 \int_S |f(t)|^2\, dt, \quad (4.39)$$

where $C_5 > 0$ is a positive constant depended on K_1, K_2, L, and τ^+ only.

From (4.39) it follows

$$\{u_k\}_{k=1}^{+\infty} \text{ is bounded in } L^\infty(S; H) \cap L^p(S; V) \cap L^2(S; H). \quad (4.40)$$

Step 5 (*second order estimates of solution approximations*). Now we shell estimate the functions u_k', $k \in \mathbb{N}$. Let t_1, t_2 be arbitrary numbers such that t_1, $t_2 \in S$, $t_1 < t_2$. For almost every $t \in [t_1, t_2]$ we multiply equality (4.29) by the function $u_k'(t)$ (recall that $u_k' \in L^2(S; H)$) and integrate the resulting equality from t_1 to t_2. Then we obtain

$$\int_{t_1}^{t_2} |u_k'(t)|^2\, dt + \int_{t_1}^{t_2} (g_k(t), u_k'(t))\, dt$$
$$= \int_{t_1}^{t_2} (f_k(t), u_k'(t))\, dt - \int_{t_1}^{t_2} \Big(\int_{t-\tau(t)}^{t} c(t, s, u_k(s))\, ds, u_k'(t) \Big)\, dt. \quad (4.41)$$

Since $g_k \in L^2(S; H)$ and $g_k(t) \in \partial\Phi(u_k(t))$ for a.e. $t \in S$, Proposition 4.1 implies that the function $\Phi_H\big(u_k(\cdot)\big)$ is absolutely continuous on $[t_1, t_2]$ and

$$\frac{d}{dt}\Phi_H\big(u_k(t)\big) = (g_k(t), u_k'(t)) \quad \text{for a.e. } t \in (t_1, t_2). \quad (4.42)$$

By (4.42) we can rewrite the second term on the left-hand side of the equality (4.41) as follows

$$\int_{t_1}^{t_2} (g_k(t), u_k'(t))\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\Phi_H\big(u_k(t)\big)\, dt$$
$$= \Phi_H\big(u_k(t)\big)\Big|_{t_1}^{t_2} = \Phi\big(u_k(t)\big)\Big|_{t_1}^{t_2}. \quad (4.43)$$

By the Cauchy inequality (2.4), changing the order of integration (see 4.36) and (3.2), we have

$$\Big|\int_{t_1}^{t_2} (f_k(t), u_k'(t))\, dt\Big| \leq \int_{t_1}^{t_2} |f_k(t)||u_k'(t)|\, dt$$
$$\leq \int_{t_1}^{t_2} |f_k(t)|^2\, dt + \frac{1}{4}\int_{t_1}^{t_2} |u_k'(t)|^2\, dt. \quad (4.44)$$

$$
\begin{aligned}
\Big| \int_{t_1}^{t_2} & \Big(\int_{t-\tau(t)}^{t} c(t,s,u_k(s))\,ds, u_k'(t) \Big)\,dt \Big| \\
&\le \int_{t_1}^{t_2} \Big(\int_{t-\tau(t)}^{t} \big| c(t,s,u_k(s)) \big|\,ds \Big) |u_k'(t)|\,dt \\
&\le L \int_{t_1}^{t_2} \Big(\int_{t-\tau(t)}^{t} |u_k(s)|\,ds \Big) |u_k'(t)|\,dt \\
&\le L^2 \tau^+ \int_{t_1}^{t_2} \Big(\int_{t-\tau^+}^{t} |u_k(s)|^2\,ds \Big)\,dt + \frac{1}{4} \int_{t_1}^{t_2} |u_k'(t)|^2\,dt \\
&\le (L\tau^+)^2 \int_{t_1-\tau^+}^{t_2} |u_k(t)|^2 dt + \frac{1}{4} \int_{t_1}^{t_2} |u_k'(t)|^2\,dt, \qquad (4.45)
\end{aligned}
$$

By (4.43), (4.45), (4.44), from (4.41) we get

$$
\frac{1}{2} \int_{t_1}^{t_2} |u_k'(t)|^2\,dt + \Phi_H\big(u_k(t)\big)\Big|_{t_1}^{t_2} \le (L\tau^+)^2 \int_{t_1-\tau^+}^{t_2} |u_k(t)|^2 dt + \int_{t_1}^{t_2} |f_k(t)|^2\,dt. \qquad (4.46)
$$

By the definitions of u_k and f_k we pass to the limit in (4.46) when $t_1 \to -\infty$. Taking into account condition $\Phi(0) = 0$ (see $(\mathcal{A}_3)$) and estimate (4.39), from (4.46), taking $t_2 = \sigma \in S$, we have

$$
\Phi\big(u_k(\sigma)\big) + \int_{-\infty}^{\sigma} |u_k'(t)|^2\,dt \leqslant C_6 \int_{-\infty}^{\sigma} |f(t)|^2\,dt, \qquad (4.47)
$$

where $C_6 > 0$ is a positive constant depended on K_1, K_2, L, and τ^+ only.

According to the definition of the functional Φ_H, condition $(\mathcal{A}_3)$ (recall that $u_k(t) \in V$ for a.e. $t \in S$) from (4.47), we have

$$
\sup_{\sigma \in S} ||u_k(\sigma)||^p + \int_S |u_k'(t)|^2\,dt \leqslant C_7 \int_S |f(t)|^2\,dt, \qquad (4.48)
$$

where C_7 is a positive constant depended on K_1, K_2, L, and τ^+ only.

Estimate (4.48) implies, that

$$
\text{the sequence } \big\{u_k\big\}_{k=1}^{+\infty} \text{ is bounded in } L^\infty(S;V), \qquad (4.49)
$$

$$
\text{the sequence } \big\{u_k'\big\}_{k=1}^{+\infty} \text{ is bounded in } L^2(S;H). \qquad (4.50)
$$

Let us show that

$$
\text{the sequence } \{g_k\}_{k=1}^{+\infty} \text{ is bounded in } L^2(S;H). \qquad (4.51)
$$

Indeed, from (3.3) for $w = u_k$, using (4.39), we have

$$\int_S \Big| \int_{t-\tau(t)}^{t} c(t,s,u_k(s))\,ds \Big|^2 dt \le (L\tau^+)^2 \int_S |u_k(t)|^2\,dt \le C_8 \int_S |f(t)|^2\,dt, \quad (4.52)$$

where C_8 is a positive constant dependent on K_1, K_2, L, and τ^+ only.

From (4.29), (4.48), (4.52), and the definitions of u_k, f_k we obtain (4.51)

Step 6 (*passing to the limit*). From (4.40), (4.49)–(4.51) and Lemma 2.1 we have that there exist functions $u \in L^\infty(S;V) \cap L^p(S;V) \cap H^1(S;H)$, $g \in L^2(S;H)$ and a subsequence of the sequence $\{u_k, g_k\}_{k=1}^{+\infty}$ (still denoted by $\{u_k, g_k\}_{k=1}^{+\infty}$) such that

$$\begin{gathered} u_k \longrightarrow_{k\to\infty} u \quad *\text{-weakly in } L^\infty(S;V), \\ \text{weakly in } L^p(S;V), \text{ weakly in } H^1(S;H), \end{gathered} \quad (4.53)$$

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{in } C(S;H), \quad (4.54)$$

$$g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in } L^2(S;H). \quad (4.55)$$

Using condition $(\mathcal{T})$, $(\mathcal{C})$, (4.54), the Cauchy-Schwarz-Bunjakovsky inequality and changing the order of integration (see (4.36)), for any $t_1, t_2 \in S$, $t_1 < t_2$, we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \Big| \int_{t-\tau(t)}^{t} c(t,s,u_k(s))\,ds - \int_{t-\tau(t)}^{t} c(t,s,u(s))\,ds \Big|^2 dt \\ \le L^2\tau^+ \int_{t_1}^{t_2} \Big(\int_{t-\tau^+}^{t} |u_k(s)-u(s)|^2 ds \Big)\,dt \\ \le (L\tau^+)^2 \int_{t_1-\tau^+}^{t_2} |u_k(t)-u(t)|^2\,dt \underset{k\to\infty}{\longrightarrow} 0. \quad (4.56) \end{aligned}$$

Thus, we have

$$\int_{t-\tau(t)}^{t} c(t,s,u_k(s))\,ds \underset{k\to\infty}{\longrightarrow} \int_{t-\tau(t)}^{t} c(t,s,u(s))\,ds \quad \text{strongly in } L^2_{\text{loc}}(S;H). \quad (4.57)$$

Let $v \in H, \varphi \in D(-\infty,0)$ be an arbitrary. For a.e. $t \in S$ we multiply equality (4.29) by v, and then we multiply the obtained equality by φ and integrate in t on S. As a result, we obtain the equality

$$\begin{aligned} \int_S (u_k'(t), v\varphi(t))\,dt + \int_S (g_k(t), v\varphi(t))\,dt + \int_S \Big(\int_{t-\tau(t)}^{t} c(t,s,u_k(s))\,ds, v\varphi(t) \Big)\,dt \\ = \int_S (f_k(t), v\varphi(t))\,dt, \quad k \in \mathbb{N}. \quad (4.58) \end{aligned}$$

We pass to the limit in (4.58) as $k \to \infty$, taking into account (4.53), (4.55), (4.57) and convergence of $\{f_k\}_{k=1}^{\infty}$ to f in $L^2_{\text{loc}}(S;H)$. As a result, since $v \in H, \varphi \in D(-\infty,0)$ are arbitrary, for a.e. $t \in S$ we obtain the equality

$$u'(t) + g(t) + \int_{t-\tau(t)}^{t} c(t,s,u(s))\,ds = f(t) \quad \text{in } H.$$

Step 7 (*completion of the proof*). In order to complete the proof of the theorem it remains only to show that $u(t) \in D(\partial\Phi)$ and $g(t) \in \partial\Phi\big(u(t)\big)$ for a.e. $t \in S$.

Let $k \in \mathbb{N}$ be an arbitrary number. Since $u_k(t) \in D(\partial\Phi)$ and $g_k(t) \in \partial\Phi_H\big(u_k(t)\big)$ for every $t \in S \setminus \widetilde{S}_k$, where $\widetilde{S}_k \subset S$ is a set of measure zero, applying the monotonicity of the subdifferential $\partial\Phi_H$, we obtain that for every $t \in S \setminus \widetilde{S}_k$ the following equality holds

$$(g_k(t) - v^*, u_k(t) - v) \geqslant 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{4.59}$$

Let $\sigma \in S$, $h > 0$ be arbitrary numbers. We integrate (4.59) on $[\sigma - h, \sigma]$:

$$\int_{\sigma-h}^{\sigma} (g_k(t) - v^*, u_k(t) - v)\, dt \geqslant 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{4.60}$$

Now according to (4.54) and (4.55) we pass to the limit in (4.60) as $k \to \infty$. As a result we obtain

$$\int_{\sigma-h}^{\sigma} (g(t) - v^*, u(t) - v)\, dt \geq 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{4.61}$$

The monograph [27, Theorem 2, p. 192] and (4.61) imply that for every $[v, v^*] \in \partial\Phi_H$ there exists a set $R_{[v,v_*]} \subset S$ of measure zero such that

$$0 \leq \lim_{h\to+0} \frac{1}{h} \int_{\sigma-h}^{\sigma} \big(g(t) - v^*, u(t) - v\big)\, dt = \big(g(\sigma) - v^*, u(\sigma) - v\big) \quad \forall\, \sigma \in S \setminus R_{[v,v_*]}. \tag{4.62}$$

Let us show that there exists a set $R \subset S$ of measure zero such that for all $\sigma \in S \setminus R$ the following inequality holds

$$\big(g(\sigma) - v^*, u(\sigma) - v\big) \geq 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{4.63}$$

Since V and H are separable spaces, there exists a countable set $F \subset \partial\Phi_H \subset V \times H$ which is dense in $\partial\Phi_H$. Let us denote $R := \cup_{[v,v^*]\in F} R_{[v,v_*]}$. Since the set F is countable, and any countable union of sets of measure zero is a set of measure zero, R is a set of measure zero. Therefore, by (4.62) for any $\sigma \in S \setminus R$ inequality $\big(g(\sigma) - v^*, u(\sigma) - v\big) \geq 0$ holds for every $[v, v^*] \in F$. Let $[\widehat{v}, \widehat{v}^*]$ be an arbitrary element from $\partial\Phi_H$. Then from the density F in $\partial\Phi_H$ we have the existence of a sequence $\{[v_l, v_l^*]\}_{l=1}^{\infty}$ such that $v_l \to \widehat{v}$ in V, $v_l^* \to \widehat{v}^*$ in H and for all $\sigma \in S \setminus R$

$$(g(\sigma) - v_l^*, u(\sigma) - v_l) \geqslant 0 \quad \forall l \in \mathbb{N}. \tag{4.64}$$

Thus, passing to the limit in this equality as $l \to \infty$, we get (4.63). Therefore, for a.e. $t \in S$ we have

$$(g(t) - v^*, u(t) - v) \geqslant 0 \quad \forall\, [v, v^*] \in \partial\Phi_H.$$

From this, according to maximal monotonicity of $\partial\Phi_H$, we obtain that $[u(t), g(t)] \in \partial\Phi_H$ for a.e. $t \in S$.

Estimate (3.1) of the solution of the problem $\mathbf{P}(\Phi, \tau, c, f)$ follows directly from (4.39) and (4.48), (4.53), (4.54), and Proposition 2.2, Fatou's Lemma and the fact that Φ_H is lower semicontinuous in H.

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