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Results on Expansion Maps in Fuzzy Menger Space via Property-(E.A) and (E.A)-like Property

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Abstract

The main goal of this paper is to establish two results in fuzzy menger space by using property-(E.A), (E.A) like property and occasionally weakly compatible mappings. Furthermore these results are justified with proper examples.These are generalization of the theorem proved by Diwan and others.

Keywords: Property-(E.A); (E.A)-Like property; Occasionally weakly compatible mappings; fuzzy menger space.
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1 Introduction

Menger space is one of the metric space generalizations. It is coined by Menger and Sklar [1959]. After words the effort of Karl Menger and Sklar [1960] the topology is generated with the neighborhood concept in menger space. It is rapidly growing and attracted many researchers. After introducing the concept of fuzziness by Zadeh [1978] we get various definitions on fuzzy metric space among one definition is popular which is proposed by George and Veeraman [1994] et al. By inserting the probabilistic notion into fuzzy metric space leads to generation of fuzzy menger space coined by Shrivastav and Dhagat [2013] et al. which is well suited for investigating physical quantities in different fields like partial differential equations, random differential, integral equations and integro-differential equations. Because of various applications, usefulness and simplicity it becomes very popular tool to solve existence problems in pure applied sciences. Some results in fuzzy menger space generated by Anil Goyal [2015] et al. using the Property (E.A). Recently Vijayabaskerreddy and Srinivasi [2021] using (E.A)- Like property generated some outcomes of fixed point results in metric space.Pathak and Rashmi [2014] et al. used the concepts of (P-1) compatible mappings to produce some of the fixed points in fuzzy menger space. Some of these outcomes can be witnessed like Ruchi Singh and Dhagat [2015], Y. Singh and Devi [2015]. S.D. Diwan and Raja [2016] and others applying the concepts of expansion mappings as well as CLR_S- property extracted fixed points in fuzzy menger space.

In this paper we established two results in fuzzy menger space by employing the notions (E.A) -like property, occasionally weakly compatible mappings and justified by suitable examples. These are generalization of the theorem proved by S.D. Diwan and Raja [2016] and others.

2 Preliminaries

Definition 2.1. *A non empty set $\mathcal{X}$ and a mapping F_α from $\mathcal{X} \times \mathcal{X}$ into the collection of all fuzzy distribution functions $F_\alpha \in \Re,\ \forall\, \alpha \in [0,1]$ make up a fuzzy probabilistic metric space ($\mathcal{FPM}$-space) $(\mathcal{X}, F_\alpha)$ S.D. Diwan and Raja [2016] and the fuzzy distribution function is expressed by $F_\alpha(p,q)$and $F_{\alpha(a,b)}(\mu)$ is the expansion of $F_{\alpha(p,q)}$ at $\mu \in \Re$.*
The function $F_{\alpha(p,q)}\, \forall\, \alpha \in [0,1]$ is to satisfy the following properties
(a) $F_{\alpha(p,q)}(\mu) = 1 \Longleftrightarrow p = q,$
(b) $F_{\alpha(p,q)}(0) = 0$
(c) $F_{\alpha(p,q)}(\mu) = F_{\alpha(q,p)}(\mu)$
(d) If $F_{\alpha(p,q)}(u_1) = 1$ and $F_{\alpha(q,r)}(u_2) = 1 \implies F_{\alpha(p,r)}(u_1 + u_2) = 1$
$\forall\, p, q, r \in \mathcal{X}$ and $u_1, u_2 > 0.$

Definition 2.2. *A binary relation* $t_\phi\colon [0,1]\times[0,1]\to[0,1]$ *is stated as t- norm S.D. Diwan and Raja [2016] if*
(i) $t_\phi(p,1)=p, t_\phi(0,0)=0$
(ii) $t_\phi(p,q)=t_\phi(q,p)$
(iii) $t_\phi(p,t_\phi(q,r))=t_\phi(t_\phi(p,q),r)$
(iv) If $r\geq p, s\geq q \implies t_\phi(r,s)\geq t_\phi(p,q)$
$forall\, p,q,r,s\in[0,1]$.

Definition 2.3. *A fuzzy menger space* $(\mathcal{X},F_\alpha,t_\phi)$ *Ruchi Singh and Dhagat [2015] is formed by* $(\mathcal{X},F_\alpha)\mathcal{FPM}$*-space ,* t_ϕ *where* t_ϕ *is t- norm and satisfy triangle inequality*
$F_{\alpha(p,r)}(u_1+u_2)\geq t_\phi(F_{\alpha(p,q)}(u_1),F_{\alpha(q,r)}(u_2))\,\forall p,q,r\in\mathcal{X}$ *and* $u_1,u_2>0$ *and* $\alpha\in[0,1]$.

2.1 Example

If $(\mathcal{X},\rho)$ is metric space then $F_\alpha\colon\mathcal{X}\times\mathcal{X}\to L$ given by
$(i)\,F_{\alpha(p,q)}=H_{\alpha(x-\rho(p,q))}, p,q\in\mathcal{X}\quad\forall\alpha\in[0,1]$.
and if $t_\phi\colon[0,1]\times[0,1]\to[0,1]$ is given by
$(ii)\,t_\phi(r,s)=min\{r,s\}$, then $(\mathcal{X},F_\alpha,t_\phi)$ forms fuzzy menger space. Further it is complete whenever $(\mathcal{X},\rho)$ is complete.

Definition 2.4. *In fuzzy menger space* $(\mathcal{X},F_\alpha,t_\phi)$ *two mappings*
$\gamma,\delta\colon\mathcal{X}\to\mathcal{X}$ *are compatible S.D. Diwan and Raja [2016]*
if $F_{\alpha(\gamma\delta a_n,\delta\gamma a_n)}(t_\phi)\to 1\quad\forall\, t_\phi>0$
whenever $(a_n)\in\mathcal{X}$ *in order for* $\gamma a_n,\delta a_n\to\mu$ *for some* $\mu\in\mathcal{X}$.

Definition 2.5. *Let* $(\mathcal{X},F_\alpha,t_\phi)$ *is fuzzy menger space.*
Then the mappings $\gamma,\delta\colon\mathcal{X}\to\mathcal{X}$ *satisfy*
(a) CLR_δ *-property S.D. Diwan and Raja [2016] if* $\exists\,(c_n)\in\mathcal{X}$ *in order for* $\gamma c_n,\delta c_n\to\delta\eta$ *for some* $\eta\in\mathcal{X}$
(b) the (E.A) like property Vijayabaskerreddy and Srinivasi [2021] if there existing a sequence $(c_n)\in\mathcal{X}$ *in order for* $\gamma c_n,\delta c_n\to\eta$ *where* $\eta\in\gamma(\mathcal{X})\cup\delta(\mathcal{X})$
(c) the property (E.A) Vijayabaskerreddy and Srinivasi [2021] if there exists a sequence $(c_n)\in\mathcal{X}$ *in order for*
$\gamma c_n,\delta c_n\to\eta$ *for some* $\eta\in\mathcal{X}$

From the definitions we can deduce that CLR_δ -property implies (E.A) like property and (E.A) like property implies the property (E.A). But the converse need not be true.
We are discussing through counter examples.

2.2 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$
def1ne the mapping $\gamma, \delta \colon \mathcal{X} \to \mathcal{X}$ as, where $\mathcal{X} = [0,1]$

$$\gamma(a) = \sin^2(\pi x), \forall a \in [0,1] \tag{1}$$

$$\delta(a) = \begin{cases} \cos^2(\pi x) & \text{if } a \in [0,1] - \{\frac{1}{4}\} \\ \frac{1}{\sqrt{2}}, & \text{if } a = \frac{1}{4}. \end{cases} \tag{2}$$

From (1) and (2)
$\gamma(\mathcal{X}) = [0,1], \delta(\mathcal{X}) = [0,1] - \{\frac{1}{2}\}$.
choose $(c_m) = \frac{1}{4} - \frac{2}{m} \quad \forall m \geq 1$. Then
$\lim_{m\to\infty} \gamma(c_m) = \lim_{m\to\infty} \gamma(\frac{1}{4} - \frac{2}{m}) = \lim_{m\to\infty} \sin^2(\pi(\frac{1}{4} - \frac{2}{m})) = \frac{1}{2}$.
$\lim_{m\to\infty} \delta(c_m) = \lim_{m\to\infty} \delta(\frac{1}{4} - \frac{2}{m}) = \lim_{m\to\infty} \cos^2(\pi(\frac{1}{4} - \frac{2}{m})) = \frac{1}{2}$.
$\lim_{m\to\infty} \gamma(c_m) = \lim_{m\to\infty} \delta(c_m) = \frac{1}{2}$.
$\frac{1}{2} \in \gamma(\mathcal{X}) \cup \delta(\mathcal{X})$ but not $\frac{1}{2}$ in $\delta(\mathcal{X})$.
Consequences the mappings γ, δ satisfy (E.A) like property but do not satisfy $CLR_\delta-$ property.

2.3 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$
def1ne the mapping $\gamma, \delta \colon \mathcal{X} \to \mathcal{X}$ as, where $\mathcal{X} = [-1,0]$

$$\gamma(a) = \begin{cases} \sin(\frac{\pi}{2} x) & \text{if } a \in [-1,0] - \{\frac{-1}{2}\} \\ \frac{-1}{3}, & \text{if } a = \frac{-1}{2}. \end{cases} \tag{3}$$

$$\delta(a) = \begin{cases} -\cos(\frac{\pi}{2} x) & \text{if } a \in [-1,0] - \{\frac{-1}{2}\} \\ \frac{-1}{2}, & \text{if } a = \frac{-1}{2}. \end{cases} \tag{4}$$

From (3) and (4)
$\gamma(\mathcal{X}) = [-1,0] - \{\frac{-1}{\sqrt{2}}\}, \delta(\mathcal{X}) = [-1,0] - \{\frac{-1}{\sqrt{2}}\}$
choose $(c_m) = \frac{-1}{2} + \frac{3}{m} \quad \forall m \geq 1$. Then
$\lim_{m\to\infty} \gamma(c_m) = \lim_{m\to\infty} \gamma(\frac{-1}{2} + \frac{3}{m}) = \lim_{m\to\infty} \sin(\frac{\pi}{2}(\frac{-1}{2} - \frac{2}{m})) = -\frac{1}{\sqrt{2}}$.
$\lim_{m\to\infty} \delta(c_m) = \lim_{m\to\infty} \delta(\frac{-1}{2} - \frac{3}{m}) = \lim_{m\to\infty} \cos(\frac{\pi}{2}(\frac{-1}{4} - \frac{3}{m})) = -\frac{1}{\sqrt{2}}$.
$\lim_{m\to\infty} \gamma(c_m) = \lim_{m\to\infty} \delta(c_m) = -\frac{1}{\sqrt{2}} \in \mathcal{X}$.
But $-\frac{1}{\sqrt{2}}$ not in $\delta(\mathcal{X}) \cup \gamma(\mathcal{X})$.
Consequences the mappings γ, δ satisfy (E.A) property but neither satisfy $CLR_\delta-$ property nor satisfy (E.A) like property

Note

(a) If $\delta(\mathcal{X})$ is closed then (E.A) property and $CLR_\delta-$ property are coincide.

(b) Further if $\delta(\mathcal{X})$ is closed as well as $\gamma(\mathcal{X})$ is closed then all the three properties coincide.

Definition 2.6. *Two mappings $\gamma, \delta\colon \mathcal{X} \to \mathcal{X}$ are weakly compatible Y. Singh and Devi [2015] in fuzzy menger space if these are commuting at their coincidence points.*

Definition 2.7. *Two mappings $\gamma, \delta\colon X \to X$ are occasionally weakly compatible Y. Singh and Devi [2015] in fuzzy menger space if there is a coincidence point at which the mapping commutes.*

The following example demonstrates that owc not necessarily weakly compatible.

2.4 Example

Define the mappings are $\gamma(a) = \sin^2(\pi x), \forall a \in [0, \frac{1}{2}]$.
$\delta(a) = \frac{1}{\sqrt{2}}\sin(\pi x), \forall a \in [0, \frac{1}{2}]$.
Then the mappings coincide at $a = \frac{1}{4}, 1, 0$ however at $a = \frac{1}{4}$
$\gamma(\frac{1}{4}) = \sin^2(\pi\frac{1}{4}) = \frac{1}{2}$.
$\delta(\frac{1}{4}) = \frac{1}{\sqrt{2}}\sin(\pi\frac{1}{4}) = \frac{1}{2}$,
$\gamma(\frac{1}{4}) = \delta(\frac{1}{4}) = \frac{1}{2}$.
$\gamma\delta(\frac{1}{4}) = \gamma(\frac{1}{2}) = \sin^2(\pi\frac{1}{2}) = 1$,
$\delta\gamma(\frac{1}{4}) = \delta(\frac{1}{2}) = \frac{1}{\sqrt{2}}\sin(\pi\frac{1}{2}) = \frac{1}{\sqrt{2}}$.
At $a = 1$,
$\gamma(1) = \sin^2(\pi 1) = 0$.
$\delta(1) = \frac{1}{\sqrt{2}}\sin(\pi 1) = 0$,
$\gamma(1) = \delta(1) = 0$.
$\gamma\delta(1) = \gamma(0) = 0$,
$\delta\gamma(1) = \delta(0) = \sin(\pi 0) = 0$.
Resulting $\gamma\delta(1) = \delta\gamma(1)$ but $\gamma\delta(\frac{1}{4}) \neq \delta\gamma(\frac{1}{4})$. Hence the result.

The following theorem proved by S.D. Diwan and Raja [2016] et al.

Theorem 2.1. *"Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t)$ suppose that*
(i) (A, S) satisfies CLR_S-property or (B, T) satisfies CLR_T-property,
(ii) $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$
(iii) (A, S) and (B, T) are weakly compatible
(iv) one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$
(v) there exists a constant $k > 1$ such that

$$F_{\alpha(Ax,By)}(kt_\phi) \leq F_{\alpha(Sx,Ty)}(t_\phi)$$

$\forall x, y \in \mathcal{X}$,for all $\alpha \in [0, 1]$ and $t_\phi > 0$.
Then A, B, S and T have a unique common fixed point in $\mathcal{X}$."

The above theorem generalized as following by using weaker conditions.

3 Main results

Theorem 3.1. *Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t_\phi)$ suppose that*
(i) (A, S) satisfied (E.A) property or (B, T) satisfies(E.A) property,
(ii) $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$
(iii) (A, S) and (B, T) are occasionally weakly compatible
(iv) one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$
(v) There exists a constant $k > 1$ such that

$$F_{\alpha(Ax,By)}(kt_\phi) \leq F_{\alpha(Sx,Ty)}(t_\phi) \tag{5}$$

$\forall x, y \in \mathcal{X}$,for all $\alpha \in [0, 1]$ and $t_\phi > 0$.
Then A, B, S and T having unique common fixed point in $\mathcal{X}$.

Proof. Assume that the pair (A, S) satisfy (E.A)- Property so that $\exists (c_m) \in \mathcal{X}$ as

$$\lim_{m\to\infty} Ac_m = \lim_{m\to\infty} Sc_m = \mu, \mu \in \mathcal{X}. \tag{6}$$

By using $A(\mathcal{X}) \subseteq T(\mathcal{X})$ we can obtain sequence (d_m) in order for $Ac_m = Td_m$. Taking limit on both sides we get

$$\lim_{m\to\infty} Ac_m = \lim_{m\to\infty} Td_m \tag{7}$$

By (5)

$$F_{\alpha(Ac_m,Bd_m)}(kt_\phi) \leq F_{\alpha(Sc_m,Td_m)}(t_\phi) \tag{8}$$

as $m \to \infty$ using (6) and (7) in (8) we get

$$F_{\alpha(\mu,Bd_m)}(kt_\phi) \leq F_{\alpha(\mu,\mu)}(t_\phi) = 1 \tag{9}$$

as $m \to \infty$ implies

$$\lim_{m\to\infty} Bd_m = \mu. \tag{10}$$

From (6),(7) and (10) we have

$$\lim_{m\to\infty} Ac_m = \lim_{m\to\infty} Sc_m = \lim_{m\to\infty} Td_m = \lim_{m\to\infty} Bd_m = \mu. \tag{11}$$

Assume $A(\mathcal{X})$ is closed sub set of $\mathcal{X}$.
Then $\mu = \lim_{m\to\infty} Ac_m \in A(\mathcal{X})$ and $A(\mathcal{X}) \subseteq T(\mathcal{X})$ implies $\mu \in T(\mathcal{X})$
Then there existing a point v in $\mathcal{X}$ as $\mu = Tv$. By (5)

$$F_{\alpha(Ac_m,Bv)}(kt_\phi) \le F_{\alpha(Sc_m,Tv)}(t_\phi) \tag{12}$$

as $m \to \infty$ using (6) $\mu = Tv$ implies

$$F_{\alpha(\mu,Bv)}(kt_\phi) \le F_{\alpha(\mu,\mu)}(t_\phi) = 1 \tag{13}$$

This gives $\mu = Bv$ implies $\mu = Tv = Bv$.
$\mu = Bv \in B(\mathcal{X}) \subset S(\mathcal{X})$.
Then there existing w in $\mathcal{X}$ as $\mu = Tv = Bv = Sw$
Claim $Aw = Sw$. By (5)

$$F_{\alpha(Aw,Bv)}(kt_\phi) \le F_{\alpha(Sw,Ty)}(t_\phi) = 1 \tag{14}$$

implies $Aw = Bv$ implies $Aw = Sw$.
Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \tag{15}$$

From (15) the pair (A, S) & (B, T) having coincidence points w and v respectively.
Further occasionally weakly compatibility of (A, S) & (B, T) resulting

$$A\mu = S\mu, B\mu = T\mu.. \tag{16}$$

By (5)

$$F_{\alpha(A\mu,B\mu)}(kt_\phi) \le F_{\alpha(S\mu,T\mu)}(t_\phi). \tag{17}$$

Using (16) in (17) we get

$$F_{\alpha(A\mu,B\mu)}(kt_\phi) \le F_{\alpha(A\mu,B\mu)}(t_\phi). \tag{18}$$

Then we have

$$A\mu = B\mu. \tag{19}$$

Claim $A\mu = \mu$. By (5) we have

$$F_{\alpha(A\mu,Bv)}(kt_\phi) \le F_{\alpha(S\mu,Tv)}(t_\phi). \tag{20}$$

From (15) and (16)

$$F_{\alpha(A\mu,\mu)}(kt_\phi) \le F_{\alpha(A\mu,\mu)}(t_\phi). \tag{21}$$

Then we have

$$A\mu = \mu. \tag{22}$$

From(16), (19) and (16) resulting

$$A\mu = S\mu = T\mu = B\mu = \mu. \tag{23}$$

Thus μ is combined fixed point for four self mappings.
In similar way when $S(\mathcal{X}), T(\mathcal{X})$ and $B(\mathcal{X})$ closed we can obtain fixed point.

Uniqueness
Let μ_1 be other similar point satisfy (23) implies

$$A\mu_1 = S\mu_1 = T\mu_1 = B\mu_1 = \mu_1. \tag{24}$$

By (5) we have

$$F_{\alpha(A\mu,B\mu_1)}(kt_\phi) \leq F_{\alpha(S\mu,T\mu_1)}(t_\phi) \tag{25}$$

From (23) and (24)

$$F_{\alpha(\mu,\mu_1)}(kt_\phi) \leq F_{\alpha(\mu,\mu_1)}(t_\phi) \tag{26}$$

Hence $\mu = \mu_1$
Consequently the four self mappings are having unique common fixed point in $\mathcal{X}$.
□

Now we validate our theorem by discussing suitable example.

3.1 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$
def1ne the mapping A, B, S and $T\colon \mathcal{X} \to \mathcal{X}$ as, where $\mathcal{X} = (-3, 0]$

$$A(a) = B(a) = \begin{cases} -e^{1-|a|} & \text{if } a \in (-3,-1] \\ -1 - \frac{a}{2}, & \text{if } a \in (-1, 0] \end{cases} \tag{27}$$

$$S(a) = T(a) = \begin{cases} -e^{-(1+a)^2} & \text{if } a \in (-3,-1] \\ \frac{a}{3}, & \text{if } a \in (-1, 0]. \end{cases} \tag{28}$$

From (27) and (28)
$A(\mathcal{X}) = [-1, -\frac{1}{e^2}) = B(\mathcal{X}) \& T(\mathcal{X}) = [-1, 0] = S(\mathcal{X})$
This implies $A(\mathcal{X}) \subseteq T(\mathcal{X}) \& B(\mathcal{X}) \subseteq S(\mathcal{X})$.
Moreover the mappings A, S have coincidence the points at $a = -2, -1$.
At $a = -1, A(-1) = S(-1) = -1$ and $AS(-1) = SA(-1) = -1$.
But at $a = -2, A(-2) = S(-2) = -\frac{1}{e}$ and
$AS(-2) = A(-\frac{1}{e}) = -1 + \frac{1}{2e}, SA(-2) = S(-\frac{1}{e}) = -\frac{1}{3e}$.
Consequences both the pairs (A, S) & (B, T) satisfy OWC but are not weakly compatible. Define the sequence $(c_m) = -1 - \frac{3}{m} \quad \forall m \geq 1$. Then

$\lim_{m\to\infty} A(c_m) = \lim_{m\to\infty} A(-1-\frac{3}{m}) = \lim_{m\to\infty} -e^{1-|-1-\frac{3}{m}|} = -1$,
$\lim_{m\to\infty} S(c_m) = \lim_{m\to\infty} S(-1-\frac{3}{m}) = \lim_{m\to\infty} -e^{-(1+(-1-\frac{3}{m}))^2} = -1$.
$\lim_{m\to\infty} A(c_m) = \lim_{m\to\infty} S(c_m) = -1$.
This implies the mappings A, S satisfy Property-(E.A) and $S(\mathcal{X})$ is closed.
Further at point $a = -1$ the values are $A(-1) = B(-1) = S(-1) = T(-1) = -1$
Resulting the mappings A, B, S and T having unique common fixed point $a = -1$.
Hence all the norms of the theorem satisfied.

Now we have one more generalization of the Theorem(2.1).

Theorem 3.2. *Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t_\phi)$ suppose that*
(i) (A, S) satisfies (E.A) like property or (B, T) satisfies (E.A) like property,
(ii) $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$
(iii) (A, S) & (B, T) are occasionally weakly compatible
(iv) one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$
(v) there exists a constant $k > 1$ such that

$$F_{\alpha(Ax,By)}(kt_\phi) \leq F_{\alpha(Sx,Ty)}(t_\phi) \tag{29}$$

$\forall x, y \in \mathcal{X}$ *,for all $\alpha \in [0, 1]$ and $t_\phi > 0$.*
Then A,B,S and T having unique common fixed point in $\mathcal{X}$.

Proof.
Case(i) Assuming that the pair (A,S) has (E.A) -Like property implies there existing a sequence $(c_m) \in \mathcal{X}$ as

$$\lim_{m\to\infty} Ac_m = \lim_{m\to\infty} Sc_m = \mu, \tag{30}$$

where $\mu \in A(\mathcal{X}) \cup S(\mathcal{X})$.
$\mu \in A(\mathcal{X}) \cup S(\mathcal{X}) \implies \mu \in A(\mathcal{X})$ or $\mu \in S(\mathcal{X})$.
Sub case (i) If $\mu \in A(X)$ and $A(\mathcal{X}) \subseteq T(\mathcal{X})$ implies $\mu \in T(\mathcal{X})$
Then there existing v in in $\mathcal{X}$ as $\mu = Tv$. By (29)

$$F_{\alpha(Ac_m,Bv)}(kt_\phi) \leq F_{\alpha(Sc_m,Tv)}(t_\phi) \tag{31}$$

as $m \to \infty$ using (30) $\mu = Tv$ implies

$$F_{\alpha(\mu,Bv)}(kt_\phi) \leq F_{\alpha(\mu,\mu)}(t_\phi) = 1 \tag{32}$$

This gives $\mu = Bv$ implies $\mu = Tv = Bv$.
$\mu = Bv \in B(\mathcal{X}) \subseteq S(\mathcal{X})$.

Then there exists w in X with $\mu = Tv = Bv = Sw$
Claim $Aw = Sw$. By (29)

$$F_{\alpha(Aw,Bv)}(kt_\phi) \leq F_{\alpha(Sw,Ty)}(t_\phi) = 1 \tag{33}$$

implies $Aw = Bv$ implies $Aw = Sw$. Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \tag{34}$$

From (34) the pairs (A,S) & (B,T) are having coincidence points w and v respectively.
Further occasionally weakly compatibility of (A,S) & (B,T) resulting

$$A\mu = S\mu, B\mu = T\mu.. \tag{35}$$

By (29)

$$F_{\alpha(A\mu,B\mu)}(kt_\phi) \leq F_{\alpha(S\mu,T\mu)}(t_\phi). \tag{36}$$

Using (34) in (35) we get

$$F_{\alpha(A\mu,B\mu)}(kt_\phi) \leq F_{\alpha(A\mu,B\mu)}(t_\phi). \tag{37}$$

By lemma we have

$$A\mu = B\mu. \tag{38}$$

Claim $A\mu = \mu$. By (29) we have

$$F_{\alpha(A\mu,Bv)}(kt_\phi) \leq F_{\alpha(S\mu,Tv)}(t_\phi). \tag{39}$$

From (34) and (35)

$$F_{\alpha(A\mu,\mu)}(kt_\phi) \leq F_{\alpha(A\mu,\mu)}(t_\phi). \tag{40}$$

By lemma we have

$$A\mu = \mu. \tag{41}$$

From(35), (38) and (41) resulting

$$A\mu = S\mu = T\mu = B\mu = \mu. \tag{42}$$

Thus μ is common fixed point for four self mappings.
Sub case (ii) If $\mu \in S(\mathcal{X})$ then there existing v in in $\mathcal{X}$ as $\mu = Sw$. By (29)

$$F_{\alpha(Aw,Bc_m)}(kt_\phi) \leq F_{\alpha(Sw,Tc_m)}(t_\phi) \tag{43}$$

as $m \to \infty$ using (30) $\mu = Sw$ implies

$$F_{\alpha(Aw,\mu)}(kt_\phi) \leq F_{\alpha(\mu,\mu)}(t_\phi) = 1. \tag{44}$$

This gives $\mu = Aw$ implies $\mu = Aw = Sw$.
$\mu = Aw \in A(\mathcal{X}) \subseteq T(\mathcal{X})$.
So that there existing v in $\mathcal{X}$ with $\mu = Tv = Aw = Sw$
Claim $Bv = Tv$. By (29)

$$F_{\alpha(Aw,Bv)}(kt_\phi) \leq F_{\alpha(Sw,Tv)}(t_\phi) = 1 \tag{45}$$

implies $Aw = Bv$ implies $Bv = Tv$. Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \tag{46}$$

From here on words leads to sub case(i).
Consequently the four self mappings have unique common fixed point in $\mathcal{X}$.

Case(ii) Assuming that the pair (B,T) satisfy (E.A)- Like property so that there existing a sequence $(e_m) \in \mathcal{X}$ as

$$\lim_{m\to\infty} Be_m = \lim_{m\to\infty} Te_m = \mu, \tag{47}$$

$\mu \in B(\mathcal{X}) \cup T(\mathcal{X}) \implies \mu \in B(\mathcal{X})$ or $\mu \in T(\mathcal{X})$.
Sub case (i) If $\mu \in B(X)$ and $B(\mathcal{X}) \subseteq S(\mathcal{X})$ implies $\mu \in S(\mathcal{X})$
Implies there existing v in $\mathcal{X}$ such that $\mu = Sw$. By (29)

$$F_{\alpha(Aw,Be_m)}(kt_\phi) \leq F_{\alpha(Sw,Te_m)}(t_\phi) \tag{48}$$

as $m \to \infty$ using (30) $\mu = Sw$ implies

$$F_{\alpha(Aw,\mu)}(kt_\phi) \leq F_{\alpha(\mu,\mu)}(t_\phi) = 1 \tag{49}$$

This gives $\mu = Aw$ implies $\mu = Aw = Sw$.
$\mu = Aw \in A(\mathcal{X}) \subseteq T(\mathcal{X})$.
Then there existing w in X as $\mu = Tv = Aw = Sw$
Claim $Bv = Tv$. By (29)

$$F_{\alpha(Aw,Bv)}(kt_\phi) \leq F_{\alpha(Sw,Tv)}(t_\phi) = 1 \tag{50}$$

implies $Aw = Bv$ implies $Bv = Tv$. Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \tag{51}$$

Here on words leads to sub case(i)
Sub case (ii) If $\mu \in T(\mathcal{X})$ hence there existing w in $\mathcal{X}$ such that $\mu = Tv$. By (29)

$$F_{\alpha(Af_m,Bv)}(kt_\phi) \leq F_{\alpha(Sf_m,Tv)}(t_\phi) \tag{52}$$

as $m \to \infty$ using $\mu = Tv$ implies

$$F_{\alpha(\mu,Bv)}(kt_\phi) \leq F_{\alpha(\mu,\mu)}(t_\phi) = 1. \tag{53}$$

This gives $\mu = Bv$ implies $\mu = Bv = Tv$.
$\mu = Bv \in B(\mathcal{X}) \subset S(\mathcal{X})$.
Then there existing v in $\mathcal{X}$ as $\mu = Sw = Bv = Tv$
Claim $Aw = Sw$. By (29)

$$F_{\alpha(Aw,Bv)}(kt_\phi) \leq F_{\alpha(Sw,Tv)}(t_\phi) = 1 \tag{54}$$

implies $Aw = Bv$ implies $Aw = Sw$.
Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \tag{55}$$

From here on words leads to the sub case (i).
□

Now our theory substantiated by following example.

3.2 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$
def1ne the mapping A, B, S and $T\colon \mathcal{X} \to \mathcal{X}$ as, where $\mathcal{X} = [-1, \frac{1}{2}]$

$$A(a) = B(a) = \begin{cases} -e^{-(1+a)} & \text{if } a \in [-1, 0] \\ -1, & \text{if } a \in (0, \frac{1}{2}] \end{cases} \tag{56}$$

$$S(a) = T(a) = \begin{cases} -e^{-(1+a)^3} & \text{if } a \in [-1, 0] \\ -(\frac{1}{2} - a), & \text{if } a \in (0, \frac{1}{2}]. \end{cases} \tag{57}$$

From (59) and (60)
$B(\mathcal{X}) = A(\mathcal{X}) = [-1, -\frac{1}{e}], T(\mathcal{X}) = S(\mathcal{X}) = [-1, 0)$
Hence $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$.
Moreover $a = 0, -1$ are the only coincidence points of the mappings A, S.
At the point $a = -1, A(-1) = S(-1) = -1$ and $AS(-1) = SA(-1) = -1$.
But at $a = 0,\ S(0) = A(0) = -\frac{1}{e}$ moreover
$AS(0) = A(-\frac{1}{e}) = -e^{-(1-\frac{1}{e})}$,
$SA(0) = S(-\frac{1}{e}) = -e^{-(1-\frac{1}{e})^3}$.
Consequences these pairs (A, S) as well as (B, T) are OWC but are not weakly

compatible.
Define $(c_m) = -1 + \frac{1}{2m} \quad \forall \quad m \geq 1$. Then
$\lim_{m\to\infty} A(c_m) = \lim_{m\to\infty} A(-1 + \frac{1}{2m}) = \lim_{m\to\infty} -e^{1-(-1+\frac{1}{2m})} = -1,$
$\lim_{m\to\infty} S(c_m) = \lim_{m\to\infty} S(-1 + \frac{1}{2m}) = \lim_{m\to\infty} -e^{-(1+(-1+\frac{3}{m}))^3} = -1.$
$\lim_{m\to\infty} A(c_m) = \lim_{m\to\infty} S(c_m) = -1.$
This implies the mappings A, S satisfying (E.A) property and $A(\mathcal{X})$ is closed.
Further at $a = -1$ values are $A(-1) = S(-1) = B(-1) = T(-1) = -1$
Thus the four mappings A, B, S and T having unique common fixed point $a = -1$.
Hence all the conditions of the Theorem (3.2) satisfied.

4 Conclusion

We generated two results in fuzzy menger space by employing
(i) the pairs (A, S) or (B, T) are satisfying Property-(E.A) and both the pairs are OWC instead of the pairs (A, S) or (B, T) are satisfying CLR_S-property and both the pairs are weakly compatible in Theorem (3.1) .
(ii) The pairs (A, S) or (B, T) are satisfying (E.A)-Like property and both the pairs are OWC instead of the pairs (A, S) or (B, T) satisfies CLR_S-property and both the pairs are weakly compatible in Theorem (3.2).
These are generalizations of the theorem proved by Diwan et al. Moreover these two Theorems are supported with appropriate examples.

5 Remark

Further there is possibility to define the intimate mappings in fuzzy menger space and chance to obtain new results like as.

Definition 5.1. *Let two mappings $\gamma, \delta \colon X \to X$ in fuzzy menger space. Then {A,S} is said to be A-intimate if and only if*
$\beta F_{\alpha(ASx_n, Ax_n)}(t_\phi) \leq \beta F_{\alpha(SSx_n, Sx_n)}(t_\phi)$
where $\beta = limitsupremum$ or $limitinfumum$, (x_n) is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \mu$, *for some $\mu \in X$.*

Theorem 5.1. *Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t_\phi)$ suppose that*
(i) (A, S) satisfied (E.A) property or (B, T) satisfies(E.A) property,
(ii) $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$
(iii) the pair (A, S) is S-intimate and (B, T) is T-intimate

(iv) one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$
(v) There exists a constant $k > 1$ such that

$$F_{\alpha(Ax,By)}(kt_{\phi}) \leq F_{\alpha(Sx,Ty)}(t_{\phi}) \tag{58}$$

$\forall x, y \in \mathcal{X}$ *,for all $\alpha \in [0, 1]$ and $t_{\phi} > 0$.*
Then A, B, S and T having unique common fixed point in $\mathcal{X}$.

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