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New approximate fixed point results for rational contraction mappings

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Abstract

In this paper, we investigate approximate fixed point results for rational contraction mappings in a metric space. This manuscript's intention is to demonstrate approximate fixed point results and the diameter of the approximate fixed point results on metric spaces. Particularly, we use some rational contraction mappings, which were mainly discussed in Dass and Gupta [1975] and Jaggi [1977]. A few examples are included to illustrate our results. Also, we discuss some applications of approximate fixed point results in the field of mathematics rigorously.
Keywords: fixed point, approximate fixed point, rational contraction, diameter approximate fixed point.
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1 Introduction

Researchers from several fields have contributed to the growth of science and technology by employing fixed point theory. Large-scale problems requiring fixed point theory are highly esteemed for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing fixed point theory approaches and have provided various useful techniques for discovering fixed points in complex issues. These are currently crucial in many mathematics related areas and applications, including economics, astronomy, dynamical systems, decision theory, and parameter estimation. The father of fixed point theory, mathematician Brouwer [1911], proposed fixed point theorems for continuous mappings on finite dimensional spaces. In 1922, Banach [1922] established and confirmed the renowned Banach cotraction principle. Then, using contractive mappings on metric spaces, various experts extended the Banach principle and proved more complicated fixed point results. Kannan (refer, Kannan [1968], Kannan [1969]) established a fixed point results for operators that are not required to be continuous. Chatterjea [1972] has investigated and presented a similar type of fixed point results. The author, Zamfirescu [1972], combined the above two contractions and derived his fixed point results. Ciric [1971] invented generalized contractions and found some fixed point theorems by using them. Further, Reich [1971], introduced his contraction operator and proposed fixed point findings. Subsequently, authors Hardy and Rogers [1973] introduced a new contraction known as the Hardy and Rogers contraction, and they derived some fixed point results. Similarly, Bianchini [1972] proved fixed point theorem by using another contraction mapping. The unique fixed point theorem for weakly B-contraction mapping was proved by Marudai and Bright [2015]. Additionally, some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters were proposed by Istratescu [1982] and Istratescu [1983]. Recently, many authors expaneded these results and proved some innovative theorems (see, Mitrović et al. [2019], Shahrazad et al. [2011]). The first scholars to investigate a generalisation of the Banach fixed point theorem while simultaneously using a contractive condition of the rational type were Dass and Gupta [1975]. In the similar manner, Jaggi [1977] used a contractive condition of the rational type to prove a fixed point theorem in complete metric space. Then, Harjani et al. [2010] expanded Jaggi's findings to partially ordered metric spaces. Later, some results that focused on coupled random fixed point theorems for mappings satisfying a contraction condition of rational type on partially ordered metric spaces were derived by Ciric et al. [2012]. Rational contraction conditions have been heavily employed in both the fixed point and common fixed point locations. Due to the complexity of finding an exact fixed point, one can show the results of an approximate fixed point. Because an exact fixed point has overly strict limitations. This is the main reason to find an approximate fixed point (ϵ-fixed point). Assume that a selfmap, $T : K \to K$, has an approximate fixed point (called, t_0). In which case, the point Kt_0 is "very near" to the point t_0. Here, the distance is less than ϵ, that is., $d(Kt_0, t_0) < \epsilon$. An approximate fixed point is a point that is nearly located at its respective fixed point. Originally, Tis et al. [2003] established the existence of approximate fixed points, which turn out to be still guaranteed under various weakened versions of the well-known fixed point theorems of Brouwer, Kakutani, and Banach. Moreover, they studied approximate fixed point results for contraction and nonexpansive mappings (refer to Theorems 2.1, 2.2, 3.1 and 4.1, respectively). Following that, Berinde [2006] proposed approximate fixed point results (Qualitative theorems) by using various operators on metric spaces (not necessarily complete). Further, he derived the diameter of the approximate fixed point results (Quantitative theorems) by using two main lemmas (refer, Berinde [2002] and Berinde [2003]). Also, Dey and Saha [2012] extended these results, and they explained that the diameter of approximate fixed points for the Reich operator tends to zero when ϵ approaches zero. Further, the authors (refer, Mohsenialhosseini [2017], Mohsenalhosseini et al. [2011]) considered some new approximate fixed point results for cyclical contraction mappings. Additionally, Mohsenalhosseini and Saheli [2021] extended these results for a family of contraction mappings and showed the existence of a common fixed point for Mohseni-Saheli mapping. Also, the authours Tijani and Olayemi [2021] proved approximate fixed point results by using some rational contraction mappings. In this manuscript, we derive some results, which include approximate fixed point theorems and diameters of approximate fixed point theorems on metric spaces (not necessarily complete), by using some rational contraction mappings, which were discussed mainly in Dass and Gupta [1975] and Jaggi [1977].

The remaining parts of this manuscript are presented as follows: In Section 2, some definitions and lemmas are recalled from previous literature. In Section 3, we propose the main results of this work, where the existence and diameter of approximate fixed points are rigorously discussed. In Section 4, we discuss and prove some applications related to approximate fixed points in the field of applied mathematics. Finally, in Section 5, we present some conclusions.

2 Preliminaries

In this section, we present some fundamental definitions and lemmas of approximate fixed point results, which are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. *Berinde [2006] Let (K, d) be a metric space and $T : K \to K$, $\epsilon > 0$. Then $t \in K$ is said to be an approximate fixed point (ϵ-fixed point) of K if*

$$d(t, Kt) < \epsilon.$$

Let $F_\epsilon(K) = \{t \in K : d(t, Kt) < \epsilon\}$ denotes the set of all ϵ-fixed points of K for a given $\epsilon > 0$.

Definition 2.2. *Berinde [2006] Let $T : K \to K$. Then K has the approximate fixed point property (a.f.p.p) if for all $\epsilon > 0$,*

$$F_\epsilon(M) \neq \emptyset.$$

Lemma 2.1. *Berinde [2006] Let (K, d) be a metric space, $T : K \to K$ such that K is asymptotic regular, that is., $d(K^n(t), K^{n+1}(t)) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $t \in K$. Then, for all $\epsilon > 0$,*

$$F_\epsilon(K) \neq 0.$$

Definition 2.3. *Berinde [2006] Let (K, d) be a metric space, $T : K \to K$ a operator and $\epsilon > 0$. We define the diameter of the set $F_\epsilon(K)$, that is.,*

$$\Delta(F_\epsilon(K)) = sup\{d(t, r) : t, r \in F_\epsilon(K)\}.$$

Lemma 2.2. *Berinde [2006] Let (K, d) be a metric space, $T : K \to K$ an operator and $\epsilon > 0$. We assume that:*

(i) $F_\epsilon(K) \neq \emptyset$;

(ii) *for all $\theta > 0$, there exists $\phi(\theta) > 0$ such that $d(t, r) - d(Kt, Kr) \leq \theta$ implies that $d(t, r) \leq \phi(\theta)$, for all $t, r \in F_\epsilon(K)$.*

Then;

$$\Delta(F_\epsilon(K)) \leq \phi(2\epsilon).$$

Definition 2.4. *Tijani and Olayemi [2021] Consider a selfmap $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then φ is said to be a comparison mapping if it satisfies the following conditions:*

(i) *φ is monotone increasing mapping, and*

(ii) *$\varphi^n(t)$ converges to 0 as $n \longrightarrow +\infty$, for all $t \in \mathbb{R}_+$.*

Furthermore, we prove the following result, which is necessary to demonstrate the main findings of this manuscript:

Theorem 2.1. *Let (K, d) be a metric space and let $T : K \to K$ be a contraction. Then K has an approximate fixed point (ϵ-fixed point).*

Proof. Fix $k_0 \in K$ and a sequence $\{k_n\}$ is defined by $k_{n+1} = Kk_n$, for all $n \geq 0$. This implies $\{k_n\}$ is a Cauchy sequence. Thus, for all $\epsilon > 0$ there exist $k_0 \in \mathbb{N}$ such that for every $t, r \geq k_0$ implies $d(k_t, k_r) < \epsilon$. In particular, if $n \geq k_0, d(k_n, k_{n+1}) < \epsilon$. That is, $d(k_n, Kk_n) < \epsilon$. Therefore, $k_n \in F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. Hence, K has an approximate fixed point (ϵ-fixed point). □

Example 2.1. *Let $K = (0, 1)$ and a selfmap $T : K \to K$ is defined by $Kk = k/2$. Then, K is a contraction. Since $(0, 1)$ is not complete and hence K does not have a fixed point. But $\{k_n\}$ is defined by $k_{n+1} = Kk_n$ with $k_0 = 1/2$. Note that $k_{n+1} = 1/2^{n+2}$. Then, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. Hence, K has an ϵ-fixed point.*

3 Main Results

In this section, we prove some approximate fixed point results for various rational type contraction mappings on metric spaces.

Theorem 3.1. *Let (K, d) be a metric space and $T : K \to K$. Then there exists $\nu, \mu \in [0, 1)$ with $\nu + \mu < 1$ such that $d(t, Kt) + d(r, Kr) \neq 0$ and for all $t, r \in K$ such that*

$$d(Kt, Kr) \leq \frac{\nu[d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)]}{d(t, Kr) + d(r, Kt)} + \mu d(t, r).$$

Prove that K has an approximate fixed point (ϵ-fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + \mu^2 + 6\nu - 6\mu - 2\nu\mu + 9) + \epsilon(\nu + \mu + 1)}{2(1 - \mu)}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0, t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n,$ for all $n \geq 0$. Consider,

$$\begin{aligned}
d(K^n t, K^{n+1} t) &= d(K(K^{n-1}t), K(K^n t)) \\
&\leq \nu \left[\frac{d(K^{n-1}t, K^n t)d(K^{n-1}t, K^{n+1}t) + d(K^n t, K^{n+1}t)d(K^n t, K^n t)}{d(K^{n-1}t, K^{n+1}t) + d(K^n t, K^n t)} \right] \\
&\quad + \mu d(K^{n-1}t, K^n t) \\
&\leq \nu d(K^{n-1}t, K^n t) + \mu d(K^{n-1}t, K^n t) \\
&\leq \lambda d(K^{n-1}t, K^n t), \text{ where } \lambda = (\nu + \mu) \\
&\leq \lambda^2 d(K^{n-2}t, K^{n-1}t) \\
&\dots \\
&\leq \lambda^n d(t, Kt)
\end{aligned}$$

Since $d(K^n t, K^{n+1} t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t, r \in K$. This implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. That is, K has an ϵ-fixed point. Clearly condition (i) of Lemma 2.2 is proved. Now only to prove, the condition (ii) of Lemma 2.2. For that, take $\theta > 0$ and $t, r \in F_\epsilon(K)$. Assume also that $d(t,r) - d(Kt, Kr) \leq \theta$. Show that $\phi(\theta) > 0$ exists. Consider,

$$\begin{aligned}
d(t,r) &\leq d(Kt,Kr) + \theta \\
&\leq \nu \left[\frac{d(t,Kt)d(t,Kr) + d(r,Kr)d(r,Kt)}{d(t,Kr) + d(r,Kt)} \right] + \mu d(t,r) + 2\epsilon \\
&= \nu \left[\frac{d(t,Kt)[d(t,r) + d(r,Kr)] + d(r,Kr)[d(t,r) + d(t,Kt)]}{d(t,r) + d(r,Kr) + d(t,r) + d(t,Kt)} \right] + \mu d(t,r) + 2\epsilon \\
&= \nu \left[\frac{\epsilon[d(t,r) + \epsilon] + \epsilon[d(t,r) + \epsilon]}{2d(t,r) + 2\epsilon} \right] + \mu d(t,r) + 2\epsilon \\
&= \frac{2\nu\epsilon d(t,r) + 2\nu\epsilon^2 + 2\mu\epsilon d(t,r) + 2\mu[d(t,r)]^2 + 4\epsilon d(t,r) + 4\epsilon^2}{2d(t,r) + 2\epsilon}
\end{aligned}$$

Which implies that

$$\begin{aligned}
&2[d(t,r)]^2 + 2\epsilon d(t,r) \\
&\qquad = 2\nu\epsilon d(t,r) + 2\nu\epsilon^2 + 2\mu\epsilon d(t,r) + 2\mu[d(t,r)]^2 + 4\epsilon^2 + 4\epsilon d(t,r)
\end{aligned}$$

That is,

$$\begin{aligned}
2[d(t,r)]^2 - 2\mu[d(t,r)]^2 + 2\epsilon d(t,r) - 2\nu\epsilon d(t,r) - 2\mu\epsilon d(t,r) - 4\epsilon d(t,r) \\
= 2\nu\epsilon^2 + 4\epsilon^2
\end{aligned}$$

$$(2 - 2\mu)[d(t,r)]^2 - (2\epsilon + 2\mu\epsilon + 2\nu\epsilon)d(t,r) = 2\epsilon^2(\nu + 2)$$

Thus, we have $a = (2 - 2\mu)$, $b = -(2\epsilon + 2\mu\epsilon + 2\nu\epsilon)$ and $c = -2\epsilon^2(\nu + 2)$. Therefore,

$$\begin{aligned}
d(t,r) &\leq \frac{2\epsilon + 2\mu\epsilon + 2\nu\epsilon \pm \sqrt{(2\epsilon + 2\mu\epsilon + 2\nu\epsilon)^2 + 4(2 - 2\mu)(2\nu\epsilon^2 + 4\epsilon^2)}}{4 - 4\mu} \\
&= \frac{2\epsilon + 2\mu\epsilon + 2\nu\epsilon + \sqrt{\begin{aligned}&4\epsilon^2 + 4\mu^2\epsilon^2 + 4\nu^2\epsilon^2 + 8\mu\epsilon^2 + 8\nu\mu\epsilon^2 + 8\nu\epsilon^2 \\ &\qquad + 16\nu\epsilon^2 + 32\epsilon^2 - 16\nu\mu\epsilon^2 - 32\mu\epsilon^2\end{aligned}}}{4(1 - \mu)} \\
&= \frac{2\epsilon + 2\mu\epsilon + 2\nu\epsilon + \sqrt{36\epsilon^2 + 4\mu^2\epsilon^2 + 4\nu^2\epsilon^2 - 24\mu\epsilon^2 - 8\nu\mu\epsilon^2 + 24\nu\epsilon}}{4(1 - \mu)} \\
&= \frac{2(\epsilon + \mu\epsilon + \nu\epsilon) + 2\sqrt{9\epsilon^2 + \mu^2\epsilon^2 + \nu^2\epsilon^2 - 6\mu\epsilon^2 - 2\nu\mu\epsilon^2 + 6\nu\epsilon^2}}{4(1 - \mu)}
\end{aligned}$$

$$< \frac{\epsilon + \mu\epsilon + \nu\epsilon + 9\epsilon^2 + \mu^2\epsilon^2 + \nu^2\epsilon^2 - 6\mu\epsilon^2 - 2\nu\mu\epsilon^2 + 6\nu\epsilon^2}{2(1-\mu)}$$
$$= \frac{\epsilon^2(9 + \mu^2 + \nu^2 - 6\mu - 2\nu\mu + 6\nu) + \epsilon(\nu + \mu + 1)}{2(1-\mu)}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + \mu^2 + 6\nu - 6\mu - 2\nu\mu + 9) + \epsilon(\nu + \mu + 1)}{2(1-\mu)}, \text{ for all } \epsilon > 0.$$

□

Example 3.1. *Let (K, d) be a metric space. Let $K = (0, 1/2]$ be endowed with usual metric. Let $T : K \to K$ be defined by $Kt = t/2$, for all $t \in K$. For that, choose $\nu = 0.25, \mu = 0.2$ and $\epsilon = 0.5$. Also $t = 0.5, r = 0.25 \in K$. Then, $d(t, Kt) = 0.333 < 0.5$, $d(r, Kr) = 0.125 < 0.5$. We have $d(t, r) = 0.25$ and*

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + \mu^2 + 6\nu - 6\mu - 2\nu\mu + 9) + \epsilon(\nu + \mu + 1)}{2(1-\mu)} = 1.907.$$

Hence, Theorem3.1 is satisfied.

Theorem 3.2. *Let (K, d) be a metric space and $T : K \to K$. Then there exists $\nu, \mu \in [0, 1)$ with $\nu + \mu < 1$ such that $1 + d(t, r) \neq 0$ and for all $t, r \in K$ implies that*

$$d(Kt, Kr) \leq \frac{\nu d(r, Kr)[1 + d(t, Kt)]}{1 + d(t, r)} + \mu d(t, r).$$

Prove that K has an approximate fixed point (ϵ-fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{2\epsilon^2(\nu - \nu\mu + 1) + \epsilon(2\nu - 2\mu - 2\nu\mu + 3)}{1 - \mu}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$, $t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n$, for all $n \geq 0$. Consider,

$$\begin{aligned}
d(K^n t, K^{n+1} t) &= d(K(K^{n-1} t), K(K^n t)) \\
&\leq \nu \left[\frac{d(K^n t, K^{n+1} t)[1 + d(K^{n-1} t, K^n t)]}{1 + d(K^{n-1} t, K^n t)} \right] + \mu d(K^{n-1} t, K^n t) \\
&\leq \nu d(K^n t, K^{n+1} t) + \mu d(K^{n-1} t, K^n t) \\
&\leq \lambda d(K^{n-1} t, K^n t), \text{ where } \lambda = \frac{\mu}{1 - \nu} \\
&\leq \lambda^2 d(K^{n-2} t, K^{n-1} t) \\
&\cdots \\
&\leq \lambda^n d(t, Kt)
\end{aligned}$$

Since $d(K^n t, K^{n+1}t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t, r \in K$. This implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset,$ for all $\epsilon > 0$. That is, K has an ϵ-fixed point. Here, as in the previous Theorem 3.1, we have

$$\begin{aligned}
d(t,r) &\leq d(Kt,Kr) + \theta \\
&= \frac{\nu d(r,Kr)[1 + d(t,Kt)]}{1 + d(t,r)} + \mu d(t,r) + 2\theta \\
&= \frac{\nu\epsilon[1+\epsilon]}{1+d(t,r)} + \mu d(t,r) + 2\epsilon \\
&= \frac{\nu\epsilon + \alpha\epsilon^2 + \mu d(t,r) + \mu[d(t,r)]^2 + 2\epsilon d(t,r) + d(t,r)}{1 + d(t,r)}
\end{aligned}$$

Which implies that

$$d(t,r) + [d(t,r)]^2 \leq \nu\epsilon + \nu\epsilon^2 + \mu d(t,r) + \mu[d(t,r)]^2 + 2\epsilon d(t,r) + 2\epsilon$$

$$d(t,r) + [d(t,r)]^2 - \mu d(t,r) - \mu[d(t,r)]^2 - 2\epsilon d(t,r) \leq \nu\epsilon^2 + \nu\epsilon + 2\epsilon$$

$$(1-\mu)[d(t,r)]^2 + (1 - \mu - 2\epsilon)d(t,r) \leq \nu\epsilon^2 + \nu\epsilon + 2\epsilon$$

Thus, we have $a = (1-\mu)$, $b = (1-\mu-2\epsilon)$ and $c = -(\nu\epsilon^2 + \nu\epsilon + 2\epsilon)$. Therefore,

$$\begin{aligned}
d(t,r) &\leq \frac{-(1-\mu-2\epsilon) \pm \sqrt{(1-\mu-2\epsilon)^2 + 4(1-\mu)(\nu\epsilon^2 + \nu\epsilon + 2\epsilon)}}{2(1-\mu)} \\
&\leq \frac{-1 + \mu + 2\epsilon + \sqrt{\begin{array}{r}1 - \mu^2 + 4\epsilon^2 - 2\mu + 4\mu\epsilon - 4\epsilon + 4\nu\epsilon^2 + 4\nu\epsilon \\ + 8\epsilon - 4\nu\mu\epsilon^2 - 4\nu\mu\epsilon - 8\mu\epsilon\end{array}}}{2(1-\mu)} \\
&\leq \frac{-1 + \mu + 2\epsilon + \sqrt{\begin{array}{r}1 - \mu^2 - 2\mu + 4\epsilon^2 + 4\nu\epsilon^2 + 4\epsilon \\ + 4\nu\epsilon + 4\nu\mu\epsilon^2 - 4\nu\mu\epsilon - 4\mu\epsilon\end{array}}}{2(1-\mu)} \\
&< \frac{\begin{array}{c}-1 + \mu + 2\epsilon + 1 + \mu^2 - 2\mu + 4\epsilon^2 + 4\nu\epsilon^2 + 4\epsilon \\ + 4\nu\epsilon - 4\nu\mu\epsilon^2 - 4\nu\mu\epsilon - 4\mu\epsilon\end{array}}{2(1-\mu)} \\
&= \frac{\mu^2 - \mu + 4\epsilon^2(1 + \nu - \nu\mu) + 2\epsilon(3 + 2\nu - 2\nu\mu - 2\mu)}{2(1-\mu)} \\
&< \frac{4\epsilon^2(1 + \nu - \nu\mu) + 2\epsilon(2\nu - 2\mu - 2\nu\mu + 3)}{2(1-\mu)}
\end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{2\epsilon^2(\nu - \nu\mu + 1) + \epsilon(2\nu - 2\mu - 2\nu\mu + 3)}{1-\mu}, \text{ for all } \epsilon > 0.$$

□

Theorem 3.3. *Let (K, d) be a metric space and $T : K \to K$. Then there exists $\nu \in [0, 1)$ such that $d(t, Kt) + d(r, Kr) \neq 0$ and for all $t, r \in K$ implies that*

$$d(Kt, Kr) \leq \frac{d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)}{d(t, Kr) + d(r, Kt)}.$$

Prove that K has an approximate fixed point (ϵ-fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + 6\nu + 9) + \epsilon(\nu + 1)}{2}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0, t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n,$ for all $n \geq 0$. Consider,

$$\begin{aligned}
d(K^n t, K^{n+1}t) &= d(K(K^{n-1}t), K(K^n t)) \\
&\leq \nu \frac{d(K^{n-1}t, K^n t)d(K^{n-1}t, K^{n+1}t) + d(K^n t, K^{n+1}t)d(K^n t, K^n t)}{d(K^{n-1}, K^{n+1}t) + d(K^n t, K^n t)} \\
&= \nu \frac{d(K^{n-1}t, K^n t)d(K^{n-1}t, K^{n+1}t)}{d(K^{n-1}t, K^{n+1}t)} \\
&= \nu d(K^{n-1}t, K^n t) \\
&\leq \nu^2 d(K^{n-2}t, K^{n-1}t) \\
&\dots \\
&\leq \nu^n d(t, Kt)
\end{aligned}$$

Since $d(K^n t, K^{n+1}t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t, r \in K$. Which implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset,$ for all $\epsilon > 0$. That is, K has an ϵ-fixed point. Here, as same as in previous Theorem 3.1, we have

$$\begin{aligned}
d(t, r) &\leq d(Kt, Kr) + \theta \\
&\leq \frac{\nu[d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)]}{d(t, Kr) + d(r, Kt)} + \theta \\
&= \frac{\nu\epsilon[\epsilon + d(t, r)] + \nu\epsilon[\epsilon + d(t, r)]}{2\epsilon + 2d(t, r)} + 2\epsilon \\
&= \frac{\nu\epsilon^2 + \nu\epsilon d(t, r) + \nu\epsilon^2 + \nu d(t, r) + 4\epsilon^2 + 4\epsilon d(t, r)}{2\epsilon + 2d(t, r)}
\end{aligned}$$

Which implies that

$$2[d(t,r)]^2 + 2\epsilon d(t,r) - 2\nu\epsilon d(t,r) - 4\epsilon d(t,r) = 2\nu\epsilon^2 + 4\epsilon^2$$

$$2[d(t,r)]^2 - 2\epsilon(\nu+1)d(t,r) = 2\epsilon^2(\nu+2)$$

Thus, we have $a = 2$, $b = -2\epsilon(\nu+1)$ and $c = -2\epsilon^2(\nu+2)$. Therefore,

$$\begin{aligned}
d(t,r) &\leq \frac{2\epsilon(\nu+1) \pm \sqrt{4\epsilon^2(\nu+1)^2 + 16\epsilon^2(\nu+2)}}{4} \\
&= \frac{2\epsilon(\nu+1) + \sqrt{4\epsilon^2(\nu^2+1+2\nu) + 16\epsilon^2\nu + 32\epsilon^2}}{4} \\
&= \frac{2\epsilon(\nu+1) + \sqrt{4\epsilon^2\nu^2 + 4\epsilon^2 + 8\epsilon^2\nu + 16\epsilon^2\nu + 32\epsilon^2}}{4} \\
&= \frac{2\epsilon(\nu+1) + \sqrt{4\epsilon^2\nu^2 + 36\epsilon^2 + 24\epsilon^2\nu}}{4} \\
&= \frac{2\epsilon(\nu+1) + 2\sqrt{\epsilon^2\nu^2 + 9\epsilon^2 + 6\epsilon^2\nu}}{4} \\
&< \frac{\epsilon\nu + \epsilon + \epsilon^2\nu^2 + 9\epsilon^2 + 6\epsilon^2\nu}{2} \\
&= \frac{\epsilon^2(\nu^2+6\nu+9) + \epsilon(\nu+1)}{2}
\end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2+6\nu+9) + \epsilon(\nu+1)}{2}, \text{ for all } \epsilon > 0.$$

□

Corolary 3.1. *Let* (K,d) *be a metric space and* $T : K \to K$*. Then there exists* $\nu \in [0,1)$ *such that* $d(t,Kt) + d(r,Kr) \neq 0$ *and for all* $t, r \in K$ *implies that*

$$d(Kt,Kr) \leq \nu.max\left\{\frac{d(t,Kt)d(t,Kr) + d(r,Kr)d(r,Kt)}{d(t,Kr) + d(r,Kt)}, d(t,r)\right\}.$$

Prove that K *has an approximate fixed point (*ϵ*-fixed point) and*

$$\Delta(F_\epsilon(K)) < max\left\{\frac{\epsilon^2(\nu^2+6\nu+9) + \epsilon(\nu+1)}{2}, \frac{2\epsilon}{1-\nu}\right\}, \textit{ for all } \epsilon > 0.$$

Proof. The same proof of above Theorem 3.3 completes this corollary. □

Example 3.2. *Let (K, d) be a metric space. Let $K = (0, 1/2]$ be endowed with usual metric. Let $T : K \to K$ be defined by $Kt = t/22$, for all $t \in K$. To show that the corollary 3.1 is satisfied. For that choose $\nu = 0.25$ and $\epsilon = 0.5$. Also $t = 0.5, r = 0.25 \in M$. Then, $d(t, Kt) = 0.333 < 0.5; d(r, Kr) = 0.125 < 0.5$. We have $d(t, r) = 0.25$ and*

$$\Delta(F_\epsilon(K)) = 0.25 < \frac{\epsilon^2(\nu^2 + 6\nu + 9) + \epsilon(\nu + 1)}{2} = 1.633.$$

Hence, Theorem 3.3 and Corollary 3.1 are satisfied.

Theorem 3.4. *Let (K, d) be a metric space and $T : K \to K$. Then there exists $\nu, \mu \in [0, 1)$ and for all $t, r \in K$ such that*

$$d(Kt, Kr) \le \nu d(t, r) + \mu \frac{d(t, Kt)d(r, Kr)}{d(t, r)}.$$

Prove that K has an approximate fixed point (ϵ-fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\mu - \nu\mu + 1) + \epsilon}{1 - \nu}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0, t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n$, for all $n \ge 0$. Consider,

$$\begin{aligned} d(K^n t, K^{n+1} t) &= d(K(K^{n-1} t), K(K^n t)) \\ &\le \nu d(K^{n-1} t, K^n t) + \frac{\mu d(k^{n-1}, K^n t)(K^n t, K^{n+1} t)}{d(K^{n-1} t, K^n t)} \\ &= \nu d(K^{n-1} t, K^n t) + \mu d(K^n t, K^{n+1} t) \\ &= \lambda d(K^{n-1} t, K^n t), \text{ where } \lambda = \frac{\nu}{1 - \mu} \\ &\le \lambda^2 d(K^{n-2} t, K^{n-1} t) \\ &\dots \\ &\le \lambda^n d(t, Kt) \end{aligned}$$

Since $d(K^n t, K^{n+1} t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t, r \in K$. Which implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \ne \emptyset$, for all $\epsilon > 0$. That is, K has an ϵ-fixed point. Here, as in the previous Theorem 3.1, we have

$$\begin{aligned} d(t, r) &\le d(Kt, Kr) + \theta \\ &= \nu d(t, r) + \mu \frac{d(t, Kt)d(r, Kr)}{d(t, r)} + \theta \\ &= \nu d(t, r) + \frac{\mu\epsilon^2}{d(t, r)} + 2\epsilon \\ &= \frac{\nu[d(t, r)]^2 + \mu\epsilon^2 + 2\epsilon d(t, r)}{d(t, r)} \end{aligned}$$

On simplifying,

$$[d(t,r)]^2 = \nu[d(t,r)]^2 + \mu\epsilon^2 + 2\epsilon d(t,r)$$

$$[d(t,r)]^2 - \nu[d(t,r)]^2 - 2\epsilon d(t,r) = \mu\epsilon^2$$

$$(1-\nu)[d(t,r)]^2 - 2\epsilon d(t,r) = \mu\epsilon^2$$

Which implies that $a = (1-\nu)$, $b = -2\epsilon$ and $c = -\mu\epsilon^2$. Therefore,

$$\begin{aligned} d(t,r) &= \frac{2\epsilon \pm \sqrt{4\epsilon^2 + 4(1-\nu)\mu\epsilon^2}}{2(1-\nu)} \\ &= \frac{2\epsilon + \sqrt{4\epsilon^2 + 4\mu\epsilon^2 - 4\nu\mu\epsilon^2}}{2(1-\nu)} \\ &= \frac{2\epsilon + 2\sqrt{\epsilon^2 + \mu\epsilon^2 - \nu\mu\epsilon^2}}{2(1-\nu)} \\ &< \frac{\epsilon + \epsilon^2 + \mu\epsilon^2 - \nu\mu\epsilon^2}{(1-\nu)} \end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\mu - \nu\mu + 1) + \epsilon}{1-\nu}, \text{ for all } \epsilon > 0.$$

□

Corolary 3.2. *Let (K,d) be a metric space and $T : K \to K$. Then there exists $\nu \in [0,1)$ and for all $t, r \in K$ such that*

$$d(Kt, Kr) \leq \frac{\mu d(t,Kt) d(r,Kr)}{d(t,r)}.$$

Prove that K has an approximate fixed point (ϵ-fixed point) and

$$\Delta(F_\epsilon(K)) < \epsilon^2(1+\mu) + \epsilon, \textit{ for all } \epsilon > 0.$$

Proof. The same proof of Theorem 3.4 completes this corollary when $\nu = 0$. □

Theorem 3.5. *Let (K,d) be a metric space and $T : K \to K$. Then there exists $\nu, \mu \in [0,1)$ with $d(r,Kr) + d(t,r) \neq 0$ and for all $t, r \in K$ such that*

$$d(Kt, Kr) \leq \nu \frac{d(t,Kt) d(t,Kr) d(r,Kr)}{d(r,Kr) + d(t,r)} + \mu d(t,r).$$

Prove that K has an approximate fixed point (ϵ-fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^4\nu^2 + \epsilon^3(6\nu - 6\mu - 2\nu\mu + 8) + \epsilon^2(\mu^2 + \nu + 1) + \epsilon(\mu + 1)}{2(1-\mu)}, \forall \epsilon > 0.$$

Proof. Let $\epsilon > 0, t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n,$ for all $n \geq 0$. Consider,

$$\begin{aligned} d(K^{n+1}t, K^n t) &= d(K(K^n t), K(K^{n-1}t)) \\ &\leq \nu \frac{d(K^n t, K^{n+1}t)d(K^n t, K^n t)d(K^{n-1}t, K^n t)}{d(K^{n-1}, K^{n+1}t)} + \mu d(K^n t, K^{n-1}t) \\ &= \mu d(K^n t, K^{n-1}t) \\ &\leq \mu^2 d(K^{n-1}t, K^{n-2}t) \\ &\dots \\ &\leq \mu^n d(Kt, t) \end{aligned}$$

Since $d(K^n t, K^{n+1}t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t, r \in K$. Which implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset,$ for all $\epsilon > 0$. That is, K has an ϵ-fixed point. Here, as in the previus Theorem 3.1, we have

$$\begin{aligned} d(t, r) &\leq d(Kt, Kr) + \theta \\ &\leq \frac{d(t, Kt)d(t, Kr)d(r, Kr)}{d(r, Kr) + d(t, r)} + \mu d(t, r) + \theta \\ &= \frac{\nu\epsilon d(t, r) + \nu\epsilon^3}{\epsilon + d(t, r)} + \mu d(t, r) + 2\epsilon \\ &= \frac{\nu\epsilon^2 d(t, r) + \nu\epsilon^3 + \mu d(t, r) + \mu[d(t, r)]^2 + 2\epsilon^3 + 2\epsilon d(t, r)}{\epsilon + d(t, r)} \end{aligned}$$

That is,

$$(1 - \mu)[d(t, r)]^2 = (\nu\epsilon^2 + \mu\epsilon + \epsilon)d(t, r) + \nu\epsilon^3 + 2\epsilon^3$$

Which implies that $a = (1 - \mu)$, $b = -(\nu\epsilon^2 + \mu\epsilon + \epsilon)$ and $c = -(\mu\epsilon^3 + 2\epsilon^3)$. Therefore,

$$\begin{aligned} d(t, r) &= \frac{\nu\epsilon^2 + \mu\epsilon + \epsilon \pm \sqrt{(\nu\epsilon^2 + \mu\epsilon + \epsilon)^2 + 4(1 - \mu)(\nu\epsilon^3 + 2\epsilon^3)}}{2(1 - \mu)} \\ &= \frac{\nu\epsilon^2 + \mu\epsilon + \epsilon + \sqrt{\begin{aligned}&\nu^2\epsilon^4 + \mu^2\epsilon^2 + \epsilon^2 + 2\nu\mu\epsilon^3 + 2\mu\epsilon^2 \\ &\quad + 2\nu\epsilon^3 + 4\nu\epsilon^3 + 8\epsilon^3 - 4\nu\mu\epsilon^3 - 8\mu\epsilon^3\end{aligned}}}{2(1 - \mu)} \\ &= \frac{\nu\epsilon^2 + \mu\epsilon + \epsilon + \sqrt{\nu^2\epsilon^4 + \mu^2\epsilon^2 + \epsilon^2 - 2\nu\mu\epsilon^3 - 6\mu\epsilon^3 + 6\nu\epsilon^3 + 8\epsilon^3}}{2(1 - \mu)} \\ &< \frac{\nu\epsilon^2 + \mu\epsilon + \epsilon + \nu^2\epsilon^4 + \mu^2\epsilon^2 + \epsilon^2 - 2\nu\mu\epsilon^3 - 6\mu\epsilon^3 + 6\nu\epsilon^3 + 8\epsilon^3}{2(1 - \mu)} \end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^4\nu^2 + \epsilon^3(6\nu - 6\mu - 2\nu\mu + 8) + \epsilon^2(\mu^2 + \nu + 1) + \epsilon(\mu + 1)}{2(1-\mu)}, \forall \epsilon > 0.$$

□

Corolary 3.3. *Let* (K, d) *be a metric space and* $T : K \to K$*. Then there exist* $\nu \in [0, 1)$ *and* φ*, a comparison mapping satisfying* $\varphi(t) < t$*, for all* $t > 0$*. Also,* $d(r, Kr) + d(t, r) \neq 0$ *and for all* $t, r \in K$ *such that*

$$d(Kt, Kr) \leq \nu \frac{d(t, Kt)d(t, Kr)d(r, Kr)}{d(r, Kr) + d(t, r)} + \varphi(\mu d(t, r)).$$

Prove that K *has an approximate fixed point (*ϵ*-fixed point) and*

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^4\nu^2 + \epsilon^3(6\nu - 6\mu - 2\nu\mu + 8) + \epsilon^2(\mu^2 + \nu + 1) + \epsilon(\mu + 1)}{2(1-\mu)}, \forall \epsilon > 0.$$

Proof. The same proof of Theorem 3.5 completes this corollary because comparison mapping $\varphi(t) < t$, for all $t > 0$. □

4 Applications

Approximate fixed point theory covers a wide range of applications in mathematics, particularly differential geometry, numerical analysis, and so on. By reading Debnath et al. [2021], Khuri and Louhichi [2018] and references therein, one can find a variety of applications involving approximate fixed point results in the field of mathematics. The examples below demonstrate how to apply approximate fixed point findings in differential equations.

Example 4.1. *Consider* $l''(k) = 6l^2(k), 0 \leq k \leq 1$ *subect to* $l(0) = 1/4, l(1) = 1/9$*. Exact solution is* $l_0(k) = -5k/36 + 1/4$*; Consider a mapping* $K : [0, 1] \to [0, 1]$ *is defined by*

$$\begin{aligned} K(l) &= l + \int_0^1 G(k, s)[l''(s) - \phi(s, l(s), l'(s))]ds \\ &= \frac{-5k}{36} + \frac{1}{4} - \int_0^1 G(k, s)\phi(s, l(s), l'(s))ds \\ &== \frac{-5k}{36} + \frac{1}{4} - \int_0^1 G(k, s)6l''(s)ds \end{aligned}$$

Consider,

$$\begin{aligned}|K(l_1) - K(l_2)| &= 6\left|-\int_0^1 G(k,s)l_1^2(s)ds + \int_0^1 G(k,s)l_2^2(s)ds\right| \\ &= 6\left(\int_0^1 |G(k,s)|^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{3}}\left(\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds\right)^{\frac{1}{2}} \\ &< \sup_{[0,1]} |l_1(s) - l_2(s)|\end{aligned}$$

Hence, K is a contraction operator. So, by Theorem 2.1, K has an ϵ-fixed point.

Example 4.2. *Consider $l''(s) = \dfrac{3l^2(s)}{2}, 0 \leq k \leq 1$ subect to $l(0) = 4, l(1) = 1$. Exact solution is $l(s) = \dfrac{4}{(1+s)^2}$; Consider a mapping $K:\ [0,1] \to [0,1]$ by*

$$K(l) = l + \int_0^1 G(k,s)[l''(s) - \phi(s, l(s))]ds \tag{1}$$

Consider, $l''(k) = 0$ which implies

$$l(k) = c_1 k + c_2 \tag{2}$$

By initial condition we have $c_2 = 4$ and $c_1 = -3$. Then (2) becomes $l(k) = -3k + 4$. Therefore, from (1), we get

$$\begin{aligned}K(l) &= -3k + 4 + \int_0^1 G(k,s)[l''(s) - \phi(s, l(s))]ds \\ &= -3k + 4 + \int_0^1 G(k,s)l''(s)ds - \int_0^1 G(k,s)\phi(s, l(s))ds \\ &= -3k + 4 + \int_0^1 G(k,s)\frac{3}{2}l^2(s)ds\end{aligned}$$

Consider,

$$
\begin{aligned}
|K(l_1)-K(l_2)| &= \left|-\int_0^1 G(k,s)\frac{3}{2}l_1^2(s)ds + \int_0^1 G(k,s)\frac{3}{2}l_2^2(s)ds\right| \\
&= \frac{3}{2}\left|\int_0^1 G(k,s)[l_2^2(s)-l_1^2(s)]ds\right| \\
&\leq \frac{3}{2}\left(\int_0^1 |G(k,s)|^2 ds\right)^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left(\int_0^k s^2(1-k)^2 ds + \int_k^1 k^2(1-s)^2 ds\right)^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left\{\frac{(1-k)^2k^3}{3} + \frac{k^2(1-k)^3}{3}\right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left\{\frac{(1-k)^2}{3}[k^3+k^2(1-k)]\right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right]^{\frac{1}{2}} \\
&\leq \frac{3}{2}\left\{\frac{(1-k)^2k^2}{3}\right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right]^{\frac{1}{2}} \\
&\leq \frac{3}{8\sqrt{3}} \left[\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right]^{\frac{1}{2}} \\
&\leq \frac{\sqrt{3}}{8} \left[\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right]^{\frac{1}{2}} \\
&\leq \frac{\sqrt{3}}{8} \sup_{[0,1]} |l_2(s)-l_1(s)| \\
&\leq \sup_{[0,1]} |l_2(s)-l_1(s)|
\end{aligned}
$$

Hence, K is a contraction operator. So, by Theorem 2.1, K has an ϵ-fixed point.

5 Conclusion

In this paper, some approximate fixed point theorems are established in metric space by using various types rational contractions mappings. It is worth observing that in the limiting case $\epsilon \longrightarrow 0$, all the results established in the present paper produces more restricted approximate fixed points. Since different findings delivered in the future might be shown in a smaller setting to ensure the existence of the approximate fixed points. The same time the idea of an approximate fixed points (ϵ-fixed points) are therefore no less important than that of exact fixed points.

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