# Welfare Analyses of Spatially Dependent Policies for Retail Agglomeration: Market Failures and Spatial Distributions of Retail Stores and Residents 

by

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# Welfare Analyses of Spatially Dependent Policies for Retail Agglomeration: Market Failures and Spatial Distributions of Retail Stores and Residents 

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#### Abstract

Shopping is an indispensable daily activity in our lives. The hollowing-out of urban commercial centers has been a growing economic-geographical problem over the past several decades. Local governments in Japan regard the hollowing-out as a severe urban problem because it is hard for them to promote compact cities where the hollowing-out is ongoing. In order to promote socially efficient compact cities with urban policies for shopping, we need to elucidate which policies agglomerate firms into downtown areas and increase social welfare.

Market mechanism for locations of retail stores has been explored for almost a century since Hotelling (1929). The model developed by Hotelling (1929) is called the spatial competition model. Various spatial competition models have been developed to capture some unique economic mechanisms.

The prototype of the spatial competition model, however, does not consider that consumers purchase several goods from stores in a marketplace. This behavior is called multipurpose shopping. Multipurpose shopping is ubiquitous in the real world. Spatial price competition models with multipurpose shopping have been developed. In these models, there are marketplaces where retail stores operate, and several goods are sold in the marketplaces, unlike the spatial competition model developed by Hotelling (1929). These marketplaces are interpreted as department stores, shopping streets, or shopping malls.


Turning our attention to urban policies, we can regard most urban policies as spatially dependent policies. Examples of spatially dependent policies are road improvement policies and subsidizing retail stores operating in the downtown area of a city. Both of the policies have been applied in the real world. It is essential to investigate the welfare impacts of spatially dependent policies. Nevertheless, spatially dependent policies have been beyond the scope of theoretical analysis with multipurpose shopping.

This motivates the theoretical study of the welfare impacts of spatially dependent policies for retail agglomeration. The present thesis aims to elucidate how local governments should apply spatially dependent policies that drive the agglomeration of retail stores in cities. Hence, the present thesis fills the gap regarding policy analysis between theoretical and empirical research in terms of analyzing the welfare impacts of spatially dependent policies for retail agglomeration.

Chapter 1 is the introduction that summarizes theoretical background and the contributions of the present thesis.

Chapter 2 investigates where retail stores agglomerate in a road network with radial roads and a ring road in a two-dimensional space. We conduct equilibrium analysis with symmetry of the geographical space. Results show 1) how a difference in improvement sequences in the radial and ring roads generates a difference in the agglomeration patterns with different welfare levels and 2) how the two-dimensional geographical position of shopping agglomerations ensuring the highest welfare level differs from that in market equilibrium.

Chapter 3 introduces an example of analysis with symmetry of a geographical space. Conducting equilibrium analysis with a New Economic Geography model, we investigate theoretically where such satellite regions emerge in a two-dimensional economic space in which discrete locations are evenly distributed in a regular-hexagonal domain. To elucidate this emergence, we introduce two viewpoints with symmetry: (1) the bifurcation mechanism of the full agglomeration at the geographical center in this domain
(mono-center), which produces satellite regions around this center, and (2) the existence of invariant patterns, which are equilibria for any value of the transport cost parameter.

Chapter 4 focuses on place-based policies to revitalize decayed shopping areas in downtown. Results show that, whether or not place-based policies are efficient depends on the recipients to whom the policies give benefits, even if the policies promote retail agglomeration in downtown. Specifically, subsidizing consumers residing near downtown is inevitably harmful from the viewpoint of welfare, whereas subsidizing retail stores can be efficient. Moreover, we show these results hold for any geographical space.

Chapter 5 relates to the theoretical analysis of Chapter 4. We build a quantitative multipurpose shopping model in Chapter 5. Chapter 5 quantitatively evaluates the welfare impacts of the place-based policies focused in Chapter 4. Results show the welfare impacts are qualitatively the same as the theoretical results shown in Chapter 4.

Chapter 6 summarizes the main results of the present thesis and suggests a direction of future research.

Keywords: Agglomeration; Bifurcation; Monopolistic competition; Multipurpose shopping; Place-based policy; Spatially dependent policy.

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## 1. Introduction

### 1.1. Background

Shopping is an indispensable daily activity in our lives. The hollowing-out of urban commercial centers has been a growing economic-geographical problem over the past several decades. For example, in Toyama in Japan, the number of retail stores in the downtown area decreased by more than $50 \%$ from 2007 to $2016 .{ }^{1}$ Kanemoto and Fujiwara (2016) summarize causes of the hollowing-out. One is to promote operating for retail stores in suburbs (e.g., deregulation of Act on the Measures by Large-Scale Retail Stores for Preservation of Living Environment in Japan). The other is progress on mortalization and decreases in travel costs generated by road improvements. However, few studies investigate how policy makers should apply urban policies to revitalize the downtown area and increase social welfare.

Local governments in Japan regard the hollowing-out as a severe urban problem because it is hard for them to promote compact cities where the hollowing-out is ongoing. Compact cities can reduce public investment and total $\mathrm{CO}_{2}$ emissions in cities. In order to promote socially efficient compact cities with urban policies for shopping, we need to elucidate which policies agglomerate firms into downtown areas and increase social welfare.

Market mechanism for locations of retail stores has been explored for almost a century since Hotelling (1929). The pioneering work by Hotelling investigates the locations of two stores. The model developed by Hotelling is called the spatial competition model. Various spatial competition models have been developed to capture some unique economic mechanisms. For example, the framework of the Cournot competition (e.g., Hamilton et al., 1994; Guo and Lai, 2015) and the Bertrand competition (e.g.,

[^0]d'Aspremont et al., 1979; de Palma et al., 1985) is introduced to the spatial competition model. ${ }^{2}$

The prototype of the spatial competition model, however, does not consider that consumers purchase several goods from stores in a marketplace. This behavior is called multipurpose shopping. Multipurpose shopping is ubiquitous in the real world. This shopping behavior is the market failure generated by the shopping externality (O'Sullivan, 1993). Hence, urban policies that affect the market equilibrium relating to multipurpose shopping may increase social welfare.

Spatial price competition models with multipurpose shopping have been developed. In these models, there are marketplaces where retail stores operate, and several goods are sold in the marketplaces, unlike the spatial competition model developed by Hotelling (1929). These marketplaces are interpreted as department stores, shopping streets, or shopping malls. For example, the Bertrand competition among department stores has been explored (e.g., Lal and Matutes, 1989; Smith and Hay, 2005; Brandão et al., 2014). Competition among shopping streets (or malls) also has been explored with a geographical space expressed by a line segment (e.g., Henkel et al., 2000; Tabuchi, 2009; Ushchev et al., 2015). In these models, retail stores operating in each marketplace are expressed with monopolistic competition.

In contrast to the above theoretical analyses, policy analyses with spatial competition models have been conducted (Lai and Tsai, 2004; Chen and Lai, 2008; Matsumura and Matsushima, 2012). These analyses focus on zoning policy. For example, Chen and Lai (2008) explore optimal zoning policy with the Cournot competition between two retail stores. Zoning policies are spatially dependent since zoning is not uniformly adopted in geographical space.

Most urban policies, including zoning policies, are spatially dependent. The present

[^1]thesis calls a policy that can be applied in a part of geographical space, "spatially dependent policy". Other examples of spatially dependent policies are road improvement policies and subsidizing retail stores operating in the downtown area of a city. Both of the policies have been applied in the real world. For example, Albuquerque in the U.S.A. subsidizes retail stores operating in the downtown area. Toyama in Japan subsidizes consumers who migrate from outside to an area around the downtown area on cities.

It is essential to investigate the welfare impacts of spatially dependent policies. Nevertheless, spatially dependent policies have been beyond the scope of theoretical analysis with multipurpose shopping.

Spatially dependent policies have been investigated in empirical research. For example, Shi et al. (2015) empirically show that a shopping mall (i.e., an agglomeration of retail stores) was built around a railway station in Shanghai after the railway station had been built. Moreover, impacts of place-based policies on retail stores have been empirically investigated (e.g., Givord et al., 2013; Neumark and Simpson, 2015; Iwata and Kondo, 2021). For example, Givord et al. (2013) empirically show that the agglomeration of retail stores in a targeted area has been promoted by a place based policy, which indicates that place-based policies can revitalize the downtown areas. In these empirical analyses, however, the welfare impacts are not focused on. The place-based policy does not ensure that social welfare increases because it can produce deadweight losses in the policy-implemented market, and can cause a decline in the number of retail stores in other areas. Hence, we have to evaluate how the place-based policy affects social welfare.

One of the reasons why welfare analyses have not been conducted would be the lack of data sets. For example, the effects of dynamic transport improvements on the spatial distribution of retail stores have not been clarified yet. In fact, since there is no exact record of dynamic transport improvements and shopping agglomerations, it is
hard to empirically show a dynamic relationship between transport improvements and shopping agglomerations. This indicates that it is hard for us to empirically analyze the welfare impact of spatially dependent policies.

This motivates the theoretical study of the welfare impacts of spatially dependent policies for retail agglomeration. The present thesis aims to elucidate how local governments should apply spatially dependent policies in cities. This thesis focuses on spatially dependent policies that drive the agglomeration of retail stores. Hence, the present thesis fills the gap regarding policy analysis between theoretical and empirical research in terms of analyzing the welfare impacts of spatially dependent policies for retail agglomeration.

### 1.2. Purpose, Research Strategy, and Summary of Contributions

## Purpose

The present thesis aims to elucidate what spatial dependent policies local governments should apply in order to increase social welfare and promote retail agglomeration in the downtown area. The present thesis aims to address the following issues. Spatially dependent policies focused on the issues are applied worldwide.

- How does an improvement sequence on a road network affect the agglomeration patterns of retail stores and social welfare?
- Which place-based policies increase social welfare, and which decrease social welfare?


## Research Strategy

The present thesis focuses on the first nature (Cronon, 1991) of geographical spaces, which is observed in cities: downtown areas and suburbs. There is a downtown area in each city; the downtown has the advantage of being convenient for consumers to visit. Such first nature is generated by road networks embedded in a two-dimensional space
and is ubiquitous in cities worldwide. In fact, it is observed that the center of the road network is the downtown area, and the suburbs are in peripheral zones.

The importance of the first nature is recognized in Spatial Economics, in particular, New Economic Geography (NEG) pioneered by Krugman (1991). In NEG, a setup of two regions where workers can reside has often been employed to theoretically elucidate agglomeration mechanisms. The Two-region setup, however, cannot express the diversity of population agglomeration observed in the real world, as Behrens and Thisse (2007) point out. Examples of the multi-region setups are a star economy, where there is a center with several regions connected to it, and a hexagonal lattice with a boundary (e.g., Barbero and Zofío, 2016; Ikeda et al., 2017b). These setups express the first nature since the centers of the geographical spaces have a geographical advantage. ${ }^{3}$

Some theoretical properties of the location patterns on hexagonal domains have been clarified (e.g., Ikeda et al., 2017b, 2018a, 2019a). The present thesis applies methodologies for two-dimensional geographical spaces with symmetry applied in NEG to analyze market equilibrium in spatial price competition models with multipurpose shopping.

## Summary of Contributions

The contributions of the present thesis are summarized as follows.
Chapter 2 investigates where retail stores agglomerate in a road network with radial roads and a ring road in a two-dimensional space. We conduct equilibrium analysis with symmetry of the geographical space in order to investigate retail agglomeration with road improvements among numerous market equilibria as the analysis in Chapter 2. Per-distance travel cost on the radial roads can be different from that on the ring road. The transition of the two-dimensional agglomeration patterns of retail stores

[^2]is investigated with decreases in the travel costs. We show that retail agglomeration develops around improved roads in the road network. This result corresponds to the empirical result by Shi et al. (2015), which is that a shopping mall was built around a railway station after the station had been built. Main results show 1) how a difference in improvement sequences in the radial and ring roads generates a difference in the agglomeration patterns with different welfare levels and 2) how the two-dimensional geographical position of shopping agglomerations ensuring the highest welfare level differs from that in market equilibrium.

Chapter 3 introduces an example of analysis with symmetry of a geographical space. We conduct the analysis in order to investigate population agglomeration in a twodimensional space with a change in transport cost among numerous market equilibria. Conducting equilibrium analysis with NEG model, we investigate theoretically where such satellite regions emerge in a two-dimensional economic space in which discrete locations are evenly distributed in a regular-hexagonal domain. To elucidate this emergence, we introduce two viewpoints with symmetry: (1) the bifurcation mechanism of the full agglomeration at the geographical center in this domain (mono-center), which produces satellite regions around this center, and (2) the existence of invariant patterns, which are equilibria for any value of the transport cost parameter. Theoretically-predicted agglomeration patterns are sure to exist as stable equilibria for a spatial economic model proposed by Forslid and Ottaviano (2003). We theoretically find one large central city surrounded by hexagonal satellite regions. This transition is an intrinsic feature observed in the two-dimensional spatial platform with the geographical center. Moreover, the theoretical results of Chapter 3 indicate that the spatial setting of a geographical space can determine equilibrium. Since the setting can affect the allocation at equilibrium, this setting also affects the results of policy analyses depending on the allocation.

Chapter 4 focuses on place-based policies to revitalize decayed shopping areas in the downtown areas. Developing a multipurpose shopping model, we evaluate the
welfare impacts of place-based policies for retail agglomeration in the downtown area. In the model, retail stores are under monopolistic competition, and consumers are free to choose where to reside. Results show that whether or not place-based policies are efficient depends on the recipients to whom the policies give benefits, even if the policies promote retail agglomeration in the downtown area. Specifically, subsidizing consumers residing near the downtown area is inevitably harmful from the viewpoint of welfare, whereas subsidizing retail stores can be efficient. As the results of Chapter 3 indicate, geographical space affects the allocation at equilibrium. Although the results of our welfare analyses depend on the allocation in general, we show that the results of Chapter 4 hold for any geographical space.

Chapter 5 relates to the theoretical analysis of Chapter 4. Chapter 5 quantitatively evaluates the welfare impacts of the place-based policies focused on Chapter 4. In order to conduct the analysis, we develop a quantitative multipurpose shopping model. Results show that the welfare impacts are qualitatively the same as the results in Chapter 4.

Chapter 6 summarizes the main results of the present thesis and suggests a direction of future research.

## 2. Two-dimensional Geographical Position as a Factor in Determining the Growth and Decline of Retail Agglomeration

### 2.1. Introduction

Shopping is an indispensable daily activity in our lives. The hollowing-out of urban commercial centers has been an economic geographical progressing problem over the past several decades. One of the factors driving the hollowing-out is a decrease in travel costs caused by automobility and road improvements. Road improvements, however, provide social benefits to consumers. If the hollowing-out harms social welfare, it is an urban problem. Hence, it is essential to elucidate how road improvements affect social welfare related to the agglomeration of retail stores in a downtown and suburbs. We explore how road improvements in a two-dimensional road network affect the agglomeration pattern and social welfare.

The location of retail stores has been studied for almost a century since Hotelling (1929). One feature of Hotelling's framework is a simplified urban space: a line segment where consumers are distributed uniformly. Although several studies extend this feature to capture some unique economic mechanisms, ${ }^{4}$ urban spaces in the real world are more complex than the spaces employed by those studies. One realistic factor increasing complexity is a road network embedded in a two-dimensional space. The road network generates geographical heterogeneity, such as a center and suburbs. In order to explore recent urban problems (e.g., the hollowing-out of the center), it is essential to differentiate the center from the suburbs.

Some studies focus on the differentiation of the center from the suburbs. For example, Braid (1993) explores price competition among retail stores on the Manhattan

[^3]roadway grid. Similarly, Braid (2013) investigates the optimal locations of retail stores in a city with a central intersection and radial roadways extending from a center to suburbs. Kishi, Kono and Nozoe (2015) analyze a spatial price model à la Capozza and Van Order (1977) in a similar space. Guo and Lai (2015) analyze the Cournot competition in a circle with a diameter as a main street. These studies differentiate the center from the suburbs by embedding a two-dimensional road network. On the other hand, Ushchev, Sloev and Thisse (2015) analyze competition between retail stores in a downtown (i.e., a center) and a shopping mall in a suburb in a line segment.

In contrast to these previous studies, we focus on heterogeneity in road networks observed in the real world. Actually, per-distance travel cost near a center is different from those near suburbs in a real network. Moreover, roads in a city are not simultaneously improved by a local government. ${ }^{5}$ Transport improvements generate the agglomeration of retail stores. For example, Shi et al. (2015) empirically show that a shopping mall (i.e., an agglomeration of retail stores) was built around a railway station in Shanghai after the railway station had been built. However, the effects of dynamic transport improvements on the spatial distribution of retail stores has not been clarified yet. In fact, since there is no exact record of dynamic transport improvements and shopping agglomerations, it is hard to empirically show a dynamic relationship between transport improvements and shopping agglomerations. Focusing on how an improvement sequence on a road network affects the agglomeration patterns of retail stores and social welfare, we theoretically investigate where retail stores are located in such a heterogeneous road network.

We build on the spatial competition model proposed by Tabuchi (2009). This model comprises a homogeneous space, monopolistic competition among retail stores, ${ }^{6}$ and a

[^4]dynamical system that describes changes in the sizes of marketplaces where the retail stores are located. Tabuchi (2009) shows that the self-organization of the retail stores, which can be interpreted as the emergence of subcenters, occurs as a result of their competition in the homogeneous space.

Our chapter differs from Tabuchi (2009) in a space where retail stores can be located. We employ a regular-hexagonal shape with one center and six suburbs (Figure 2.1), which are potential marketplaces for retail stores. In the real world, road networks in cities are constructed around the central business district. Most cities have radial roads and ring roads in the network. Hence, the regular-hexagonal shape is a simplified description of real road networks for our theoretical analysis. ${ }^{7}$

Actually, hexagonal domains have recently been employed as a two-dimensional spatial platform for New Economic Geography models. Ikeda et al. (2014), for example, explore where and how population agglomeration takes place in a hexagonal domain by bifurcation analysis. Some theoretical properties of the location patterns on hexagonal domains have been clarified (e.g., Ikeda et al., 2017b, 2018a, 2019a). Conducting bifurcation analysis introduced by Ikeda et al. (2014), we investigate market equilibria.

Moreover, we relax the uniform per-distance travel cost assumption employed in many spatial competition models. In our model, the per-distance travel cost on the radial roads can be different from that on the ring road. Such a relaxation captures one of the features of road networks in the real world. Combining the spatial platform and this relaxation, we investigate how improvement sequences in the road network affect the agglomeration pattern of retail stores and social welfare. In particular, we explore where retail stores should be located from the viewpoint of social welfare.

The contribution of this chapter is twofold. First, we show that a difference in

[^5]

Figure 2.1: City shape. Black lines: the road network in the city; node 0 : the center in the city; nodes $1, \ldots, 6$ : the suburbs.
improvement sequences in the road network generates a difference in agglomeration patterns in equilibrium even for the same travel costs parameters. Conducting bifurcation analysis to explore market equilibria, we demonstrate that all the retail stores agglomerate in the center if the radial roads are improved first. In contrast, the stores are located in the center as well as in several suburbs if the ring road is improved first.

Second, we show that the scale of agglomeration of retail stores in each marketplace as well as the two-dimensional location pattern of marketplaces in which stores operate at a market equilibrium differ from those at the first-best situation particularly when the travel costs are low. This implies that policymakers should guide stores to form an appropriate location pattern with policies such as land-use regulation. ${ }^{8}$

The rest of this chapter is organized as follows. A spatial competition model is introduced in Section 2. Agglomeration patterns of retail stores are explored in Section 3. Our theoretical results are verified with numerical comparative statics analysis of the distribution of retail stores in Section 4. Section 5 concludes this chapter.

[^6]
### 2.2. Model

### 2.2.1. City and goods

We consider a city composed of seven potential marketplaces labeled $0,1, \ldots, 6$. Marketplace 0 and marketplaces $1, \ldots, 6$ are in the center and suburbs, respectively. The center is connected to the suburbs by radial roads, whereas the suburbs are located on a ring road. We consider that the radial roads and the ring road form a regularhexagonal road network as shown in Figure 2.1. For simplicity, the length of all the line segments between the marketplaces is assumed to be one.

We consider two types of goods: horizontally differentiated goods and an outside good. The differentiated goods are supplied by a large number of profit-maximizing retail stores in the marketplaces. The outside good is supplied by perfectly competitive firms and chosen as a numéraire good.

### 2.2.2. Consumers

Consumers in the city are uniformly distributed over the road network with the density normalized to 1 . Let $\mathcal{L}$ denote all the positions on the road network. The utility of consumers residing at $\ell \in \mathcal{L}$ and visiting marketplace $j$ is given by $U(\ell, j)=$ $\ln M_{j}(\ell)+A(\ell)$, where $M_{j}(\ell)=\left(\int_{0}^{n_{j}} q(\ell, k)^{\frac{\sigma-1}{\sigma}} \mathrm{~d} k\right)^{\frac{\sigma}{\sigma-1}} \cdot q(\ell, k)$ is the consumption of the $k$ th variety, $n_{j}$ is the mass of varieties supplied in marketplace $j, \sigma(>1)$ is the elasticity of substitution between any two varieties, and $A(\ell)$ is the consumption of the outside good.

If consumers choose to visit marketplace $j$, then the budget constraint is given by $\int_{0}^{n_{j}} p_{j}(k) q(\ell, k) \mathrm{d} k+t(\ell, j)+A(\ell)=W$, where $p_{j}(k)$ is the price of the $k$ th variety in marketplace $j, W$ is the income, and $t(\ell, j)$ is the travel cost paid by the consumers. ${ }^{9}$

[^7]Solving the utility maximization problem, we obtain demand functions:

$$
\begin{align*}
& q(\ell, k)=p_{j}(k)^{-\sigma} R_{j}^{-1}  \tag{2.1}\\
& A(\ell)=W-t(\ell, j)-1 \tag{2.2}
\end{align*}
$$

where $R_{j}=\int_{0}^{n_{j}} p_{j}(k)^{1-\sigma} \mathrm{d} k$. We assume that income $W$ is high so that $A(\ell)$ is positive in equilibria.

### 2.2.3. Retail stores

Retail stores are located in marketplaces. These stores share the same marginal production cost $c$ and the same fixed cost $f$. We assume that retail stores in the same marketplace are under monopolistic competition. The total number of retail stores at each marketplace is determined by free entry.

Let $\pi_{i}(k)$ be the profit of the retail store producing the $k$ th variety at marketplace i. $\pi_{i}(k)$ is given by

$$
\begin{equation*}
\pi_{i}(k)=\left(p_{i}(k)-c\right) Q_{i}(k)-f, \tag{2.3}
\end{equation*}
$$

where $Q_{i}(k)$ is the total demand for the $k$ th variety at marketplace $i$. Each retail store has a negligible impact on the prices of other goods in the marketplace because its supply is very small compared to the total supply of all the stores. That is, $R_{i}$ does not change, as in Dixit and Stiglitz (1977). Using (2.1), we obtain the profit-maximizing prices, which are the same across all the varieties and marketplaces: $p_{i}(k)=p^{*}(\forall i, k)$, where $p^{*}=c \sigma /(\sigma-1)$. We regard $\pi_{i}(k)$ as $\pi_{i}$ because each firm at the same marketplace can be treated symmetrically.

### 2.2.4. Market area

Consumers are assumed to visit one marketplace where they can obtain the highest utility. Hence, total demand for a retail store $Q_{i}(k)$ is determined by consumers'
behavior. To obtain $Q_{i}(k)$, we introduce 'market area', which is all the residential locations of the consumers visiting the same marketplace. We classify the two-dimensional agglomeration patterns of retail stores with the market area in Section 2.3.

Substituting $p^{*}$ into (2.1), we obtain demand for the $k$ th variety (i.e., $q(\ell, k)$ ) for consumers at $\ell(\in \mathcal{L})$ visiting marketplace $j: q(\ell, k)=1 /\left(p^{*} n_{j}\right)$. Substituting this function and (2.2) into the utility function, we obtain the indirect utility of the consumers:

$$
\begin{equation*}
V(\ell, j)=\sigma_{-1} \ln n_{j}-t(\ell, j)+V_{D} \tag{2.4}
\end{equation*}
$$

where $\sigma_{-1}=(\sigma-1)^{-1}, V_{D}=-\ln p^{*}+W-1$. We define the set of the indirect utilities that the consumers can obtain by visiting a marketplace:

$$
\mathcal{V}(\ell)=\{V(\ell, 0), V(\ell, 1), \ldots, V(\ell, 6)\}, \quad \ell \in \mathcal{L}
$$

Using $\mathcal{V}(\ell)$, we mathematically define the market area.
Definition 2.1. The market area of marketplace $i(i=0,1, \ldots, 6)$ is the following set:

$$
\begin{equation*}
\mathcal{M}_{i}=\{\ell \in \mathcal{L} \mid \max \mathcal{V}(\ell)=V(\ell, i)\} \tag{2.5}
\end{equation*}
$$

We can obtain $Q_{i}(k)$ using the defined market area. Let $\mu\left(\mathcal{M}_{i}\right)$ denote the total length of market area $\mathcal{M}_{i}$. Using demand function $q(\ell, k)$ and $\mu\left(\mathcal{M}_{i}\right)$, we obtain the total demand:

$$
Q_{i}(k)= \begin{cases}\mu\left(\mathcal{M}_{i}\right) /\left(p^{*} n_{i}\right) & \left(n_{i}>0\right)  \tag{2.6}\\ 0 & \left(n_{i}=0\right)\end{cases}
$$

### 2.2.5. Market equilibrium

We introduce the market equilibrium condition of the size of marketplaces. Let $\boldsymbol{n}=\left(n_{0}, \ldots, n_{6}\right)^{\top}$ denote the distribution of the retail stores across the marketplaces in the city. The market equilibrium condition for $\boldsymbol{n}$ is the following condition:

$$
\left\{\begin{array}{l}
\pi_{i}=0 \quad \text { if } \quad n_{i}>0,  \tag{2.7}\\
\pi_{i} \leq 0 \quad \text { if } \quad n_{i}=0
\end{array} \quad i=0,1, \ldots, 6\right.
$$

Condition (2.7) implies that retail stores have no incentive to locate at marketplace $i$ if the profit they obtain at marketplace $i$ is not positive. Note that profit $\pi_{i}$ is a function of $\boldsymbol{n}$ because $Q_{i}(k)$ in (2.6) depends on $\boldsymbol{n}$.

We employ a dynamical system to investigate the stability of equilibria. We assume that $\boldsymbol{n}$ gradually evolves in proportion to both profit $\boldsymbol{\pi}$ and state $\boldsymbol{n}$ itself as follows: ${ }^{10}$

$$
\begin{equation*}
\frac{\mathrm{d} \boldsymbol{n}}{\mathrm{~d} t}=\boldsymbol{F}(\boldsymbol{n}), \tag{2.8}
\end{equation*}
$$

where $\boldsymbol{F}(\boldsymbol{n})=\left(F_{0}(\boldsymbol{n}), F_{1}(\boldsymbol{n}), \ldots, F_{6}(\boldsymbol{n})\right)^{\top}$ and $F_{i}(\boldsymbol{n})=n_{i} \pi_{i}(i=0,1, \ldots, 6)$. Since dynamics (2.8) implies that the growth rate of $n_{i}$ per unit time is equal to profit $\pi_{i}$, retail stores are attracted to marketplaces where they can obtain profits. This dynamics has an advantage shown by the following lemma. ${ }^{11}$

Lemma 2.1. $\boldsymbol{n}$ is the market equilibrium iff $\boldsymbol{n}$ is a stationary point of dynamics (2.8).

Proof. See Appendix A.1.

We investigate the market equilibria by finding stationary points of dynamics (2.8). A stationary point is linearly-stable if every eigenvalue of Jacobian matrix $\partial \boldsymbol{F} / \partial \boldsymbol{n}$ has a negative real part. We call linearly-stable stationary points stable equilibria. We also investigate transitions from unstable equilibria under dynamics (2.8) in Section 2.4.

[^8]
### 2.2.6. Travel cost

Consumers have several route choices to the marketplaces. The consumers choose the route with the lowest travel cost. We mathematically define $\mathcal{L}$ to express travel costs, which are determined by the distance between consumers and a marketplace. Let $D$ and $S$ denote the radial roads and the ring road in the city, respectively. Since we assume that the length of each road between the marketplaces is 1 , we can represent $\mathcal{L}$ by $\mathcal{L}=\mathcal{A} \times \mathcal{P} \times X$, where $\mathcal{A}=\{D, S\}, \mathcal{P}=\{1,2, \ldots, 6\}$, and $X=(0,1) .(D, i, x) \in$ $\{D\} \times \mathcal{P} \times X$ is equal to position $x$ distant from the center on the radial road between the center and suburb $i$ (e.g., see $(D, 1, x)$ in Figure 2.1). Similarly, $(S, i, y) \in\{S\} \times \mathcal{P} \times X$ is equal to position $y$ distant from suburb $i$ on the ring road between suburb $i$ and $j(\equiv i+1 \bmod 6)$ (e.g., see $(S, 1, y)$ in Figure 2.1). Therefore, $\{D\} \times \mathcal{P} \times X$ is all the positions on the radial roads, whereas $\{S\} \times \mathcal{P} \times X$ is that on the ring road.

For consumers residing at $\ell=(D, i, x) \in\{D\} \times \mathcal{P} \times X$ (i.e., consumers residing along the radial roads), the travel cost is given by

$$
t(\ell, j)= \begin{cases}\phi x & (j=0)  \tag{2.9}\\ \min \left\{\phi(1+x), \phi(1-x)+\tau L_{i j}\right\} & (j \in \mathcal{P})\end{cases}
$$

where $\phi$ is the per-distance travel cost on the radial roads, $\tau$ is the per-distance travel cost on the ring road, and $L_{i j}=\min \{|i-j|, 6-|i-j|\} . \phi x$ is the travel cost when the consumers visit the center; $\phi(1+x)$ is when the consumers visit suburb $j(\in \mathcal{P})$ via the center; $\phi(1-x)+\tau L_{i j}$ is when via the suburbs. On the other hand, for consumers residing at $\ell=(S, i, x) \in\{S\} \times \mathcal{P} \times X$ (i.e., consumers residing along the ring road), the travel cost is given by

$$
t(\ell, j)= \begin{cases}\phi+\tau(1 / 2-|x-1 / 2|) & (j=0)  \tag{2.10}\\ \tau \times \min \{|i+x-j|, 6-|i+x-j|\} & (j \in \mathcal{P})\end{cases}
$$

$\phi+\tau(\cdots)$ is the travel cost when the consumers visit the center via the nearest suburb; $\tau \times \min \{\cdots\}$ is the travel cost when the consumers visit marketplace $j$ via the shortest


Figure 2.2: The dispersion. Black area: $\mathcal{M}_{0}$; green: $\mathcal{M}_{1} ;$ red: $\mathcal{M}_{2}$; sky blue: $\mathcal{M}_{3} ;$ pink: $\mathcal{M}_{4}$; yellowgreen: $\mathcal{M}_{5}$; brown: $\mathcal{M}_{6}$; the size of $\bigcirc$ : the number of retail stores.
route along the ring road.

### 2.2.7. Welfare

We measure the efficiency of the distribution of retail stores. Since the retail stores' profits are zero in the equilibria by condition (2.7), social welfare $S W$ is total consumer utility (in monetary terms).

### 2.3. Agglomeration patterns of retail stores

We focus on some agglomeration patterns of retail stores. These patterns are possible market equilibria, which are investigated in Section 2.4.

### 2.3.1. A simple agglomeration pattern

We focus on the agglomeration pattern of retail stores in which every marketplace has a market area (i.e., $\left.\mathcal{M}_{i} \neq \emptyset(i=0,1, \ldots, 6)\right)$. We define this market area pattern as market pattern (D), and the equilibria that forms market pattern (D) as the dispersion (Figure 2.2). See Appendix A.2.2 for the details of these definitions.

A symmetric assumption for an equilibrium is often employed when the change of an agglomeration pattern with the change in an exogenous parameter is investigated (e.g., Ikeda et al., 2014). Assuming $n_{1}=n_{2}=\cdots=n_{6}$, we investigate how decreases in per-distance travel cost $\phi$ and $\tau$ affect the dispersion. For $\boldsymbol{n}=\left(n_{0}, n_{1}, \ldots, n_{1}\right)$, we can obtain dynamics (2.8) as follows.

Lemma 2.2. For $\boldsymbol{n}=\left(n_{0}, n_{1}, \ldots, n_{1}\right)$, dynamics (2.8) under market pattern (D) is

$$
\begin{align*}
& F_{0}(\boldsymbol{n})=\frac{3}{\sigma}\left(\frac{\ln \left(n_{0} / n_{1}\right)}{\phi(\sigma-1)}+1\right)-f n_{0}  \tag{2.11}\\
& F_{i}(\boldsymbol{n})=\frac{1}{2 \sigma}\left(-\frac{\ln \left(n_{0} / n_{1}\right)}{\phi(\sigma-1)}+3\right)-f n_{1}, \quad i=1,2, \ldots, 6 . \tag{2.12}
\end{align*}
$$

Proof. See Appendix A.2.3.
Let $\boldsymbol{n}_{d} \equiv\left(n_{0}, n_{1}, \ldots, n_{1}\right)$ be a symmetric equilibrium of the dispersion (i.e., $F_{i}\left(\boldsymbol{n}_{d}\right)=$ $0(i=0,1, \ldots, 6))$. First, we investigate the change in $\boldsymbol{n}_{d}$ and the emergence of another agglomeration pattern with a decrease in $\phi$. This change is summarized as follows.

Lemma 2.3. If $n_{0}>n_{1}$ holds in linearly-stable $\boldsymbol{n}_{d}$, then $n_{0}$ and $n_{1}$ in the equilibria monotonously change with an increase in $\phi$ :

$$
\begin{equation*}
\frac{\mathrm{d} n_{0}}{\mathrm{~d} \phi}<0, \quad \frac{\mathrm{~d} n_{1}}{\mathrm{~d} \phi}>0 \tag{2.13}
\end{equation*}
$$

If $n_{0}<n_{1}$ holds in linearly-stable $\boldsymbol{n}_{d}$, then

$$
\begin{equation*}
\frac{\mathrm{d} n_{0}}{\mathrm{~d} \phi}>0, \quad \frac{\mathrm{~d} n_{1}}{\mathrm{~d} \phi}<0 \tag{2.14}
\end{equation*}
$$

Proof. See Appendix A.2.3.
Monotonicity (2.13) in Lemma 2.3 indicates that the full agglomeration of retail stores in the center is a possible market equilibrium.

Next, we investigate the change in $\boldsymbol{n}_{d}$ with a decrease in $\tau$ and the emergence of another agglomeration pattern. Since $\tau$ is not included in (2.11) or (2.12), $\tau$ does not affect the change in $n_{0}$ and $n_{1}$ of $\boldsymbol{n}_{d}$. A decrease in $\tau$ can affect the linear stability.

We briefly investigate the change in linearly-unstable $\boldsymbol{n}_{d}$ at a certain level of $\tau$. Solutions starting near unstable $\boldsymbol{n}_{d}$ under dynamics (2.8) are classified into 1) the solution diverging from $\boldsymbol{n}_{d}$ and 2) the solution converging to $\boldsymbol{n}_{d}$. In particular, near $\boldsymbol{n}_{d}$, the motion of any solution diverging from $\boldsymbol{n}_{d}$ is almost equal to a linear combination of


Figure 2.3: The change from $\boldsymbol{n}_{d}$ at a certain level of $\tau$ by a decrease in $\tau$. The size of $\bigcirc$ : the number of retail stores.
the eigenvectors for the eigenvalues of $\partial \boldsymbol{F} / \partial \boldsymbol{n}$ that has a positive real part. ${ }^{12}$ Hence, the linear combination is the most likely change from $\boldsymbol{n}_{d}$ after it is not linearly-stable.

Lemma 2.4. Just after stationary point $\boldsymbol{n}_{d}$ is unstable at a certain level of $\tau$ by a decrease in $\tau$, the eigenvector for the eigenvalues of $\partial \boldsymbol{F} / \partial \boldsymbol{n}$ that has a positive real part is $w(0,1,-1,1,-1,1,-1)^{\top}(w \in \mathbb{R})$.

Proof. See Appendix A.2.3.

Lemma 2.4 indicates that the dispersion changes into an agglomeration pattern where large agglomerations and small agglomerations alternately emerge on the ring road (See Figure 2.3). ${ }^{13}$

### 2.3.2. Corner equilibria

We next focus on the equilibria in which some marketplaces have no market area. We call these equilibria corner equilibria. Various symmetric corner equilibria can hold because the geometrical symmetry of the road network generates symmetric market area patterns. Among the corner equilibria, we investigate four corner equilibria (Figure

[^9]

Figure 2.4: Corner equilibria under investigation. (a): the full agglomeration; (b) the triangle pattern; (c) the asymmetric pattern; (d) the linear pattern. $\bigcirc$ : the number of retail stores. Black area: $\mathcal{M}_{0}$; green: $\mathcal{M}_{1} ;$ sky blue: $\mathcal{M}_{3} ;$ pink: $\mathcal{M}_{4} ;$ yellowgreen: $\mathcal{M}_{5}$.
$2.4)$ : the full agglomeration $\left(\mathcal{M}_{0} \neq \emptyset\right)$, the triangle pattern $\left(\mathcal{M}_{0}, \mathcal{M}_{1}, \mathcal{M}_{3}, \mathcal{M}_{5} \neq \emptyset\right)$, the asymmetric pattern $\left(\mathcal{M}_{0}, \mathcal{M}_{1}, \mathcal{M}_{3} \neq \emptyset\right)$, and the linear pattern $\left(\mathcal{M}_{0}, \mathcal{M}_{1}, \mathcal{M}_{4} \neq \emptyset\right)$. These equilibria are possible agglomeration patterns into which the dispersion changes with decreases in $\phi$ and $\tau$.

Note that these corner equilibria hold under some inequality conditions. We investigate these conditions with the definitions of market area patterns. We define a market area pattern for the full agglomeration as market pattern (F), a market area pattern for the triangle pattern as market pattern $(\mathrm{P})$, a market area pattern for the asymmetric pattern as market pattern (A), and a market area pattern for the linear pattern as market pattern (L). See Appendix A. 3 for these detailed explanations.

It is most likely that the full agglomeration and the triangle pattern ${ }^{14}$ are corner equilibria into which the dispersion changes (Lemmas 2.3 and 2.4). Moreover, we can

[^10]infer from Lemma 2.3 that the number of retail stores in marketplace 5 under the triangle pattern decreases with a decrease in $\tau$. Hence, the asymmetric pattern is a possible equilibrium into which the triangle pattern changes.

On the other hand, the linear pattern seems to be the most efficient equilibrium in the equilibria in which retail stores are located in three marketplaces. However, the linear pattern is not a corner equilibrium pattern into which the triangle pattern or the asymmetric pattern changes under dynamics (2.8).

Proposition 2.1. Neither any distribution $\boldsymbol{n}$ in market pattern $(P)$ nor that in (A) changes into any distribution in market pattern (L) under dynamics (2.8).

Proof. See Appendix A. 4.

Proposition 2.1 shows that neither the triangle pattern nor the asymmetric pattern changes into the linear pattern with decreases in $\phi$ and $\tau$. In other words, improvements in the road network do not change the triangle pattern (the asymmetric pattern) into the linear pattern.

Moreover, while one may intuitively consider that the dispersion tends to change into the linear pattern, Lemma 2.4 indicates that such a result does not occur. This is verified in Section 2.4.

### 2.4. Two-dimensional geographical positions of retail stores

We explore how road improvements affect equilibria and social welfare. In this chapter, we regard road improvements as decreases in travel costs ( $\phi$ and $\tau$ ). Conducting bifurcation analysis to explore market equilibria, we show how road improvement sequences affect the equilibrium. We set exogenous variables $\sigma$ and $f$ at 6.0 and 20, respectively

### 2.4.1. Dependency of stable agglomeration patterns on travel costs

Since there are numerous road improvement sequence patterns in our model as well as in the real world, we focus on which road improvement sequence can generate a difference in equilibria and social welfare. Such a difference can occur with travel costs parameters for which multiple stable equilibria exist. Hence, we examine whether or not multiple stable equilibria exist with travel costs parameters.

The stability of equilibria introduced in Section 2.3 in the space of $(\phi, \tau) \in(0,1) \times$ $(0,1)$ was investigated and the zones in which they were stable are enclosed by solid lines in Figure 2.5(a). According to the result, the dispersion tends to be stable with relatively high $\phi$ and $\tau$. On the other hand, the triangle, the asymmetric, and the linear pattern tend to be stable with relatively higher $\phi$ than $\tau$. The full agglomeration is always stable in the space. ${ }^{15}$

As Figure 2.5(a) shows, multiple stable equilibria exist in the space. Hence, an equilibrium forming an agglomeration pattern on specific travel costs is likely to change into an equilibrium forming another agglomeration pattern with decreases in travel costs. For example, the dispersion at stage $\alpha$ marked by $\diamond$ in Figure 2.5(a) is likely to change into the full agglomeration or the asymmetric pattern on stage $\gamma$. This example indicates that the road improvement sequence affects which agglomeration pattern emerges on stage $\gamma$.

We investigate which agglomeration pattern is most efficient in terms of the social welfare in the space. In Figure 2.5(b), the color of each zone (black, blue, orange, and red) represents the most efficient agglomeration pattern in that zone. ${ }^{16}$ For example, the full agglomeration is the most efficient agglomeration pattern at stage $\gamma$.

The linear pattern is the most efficient agglomeration pattern in a part of the zone

[^11]
(a)

Dispersion

(b)

Figure 2.5: (a) Zones of stable equilibria in $(\phi, \tau) \in(0,1) \times(0,1)$. The zone bounded by black lines: the dispersion; blue: the triangle pattern; green: the asymmetric pattern; orange: the linear pattern; red: the full agglomeration. (b) The most efficient agglomeration pattern in terms of the social welfare. Black zone: the dispersion; blue zone: the triangle pattern; orange zone: the linear pattern; red zone: the full agglomeration.
where multiple equilibria exist. This pattern, however, is predicted not to emerge from the dispersion, the triangle pattern, or the asymmetric pattern with decreases in travel costs (Lemma 2.4 and Proposition 2.1). Hence, the result indicates that roads improvements do not generate an efficient agglomeration pattern even in terms of the locations of marketplaces where retail stores are located.

### 2.4.2. Dependency of agglomeration patterns on improvement sequences in the road network

Conducting the numerical comparative statics analysis of equilibria, we verify that road improvement sequences generate differences in the agglomeration patterns and the social welfare in equilibrium. We show two main findings through the numerical comparative statics analysis of the equilibrium for the travel costs:

- main finding 1: a difference in improvement sequences in the road network finally generates a difference in the equilibria.
- main finding 2: the welfare of the linear pattern is higher than that of the asymmetric pattern while the market system does not produce the linear pattern.

We investigate the transition of the stable dispersion from stage $\alpha$ to stage $\gamma$ shown in Figure 2.5. Among various road improvement sequence patterns to stage $\gamma$, we focus on two simple improvement sequence patterns: (1) the radial roads are improved first and (2) the ring road is improved first. ${ }^{17}$

We investigate the following two cases of changes in the travel costs:

[^12]- The radial-roads first case:

$$
(\phi, \tau)=\underbrace{(1.0,1.0)}_{\text {Stage } \alpha} \stackrel{\text { Transition } 1}{\longrightarrow} \underbrace{(0.35,1.0)}_{\text {Stage } \beta_{1}} \xrightarrow{\text { Transition } 2} \underbrace{(0.35,0.17)}_{\text {Stage } \gamma} .
$$

- The ring-road first case:

$$
(\phi, \tau)=\underbrace{(1.0,1.0)}_{\text {Stage } \alpha} \stackrel{\text { Transition } 1}{\longrightarrow} \underbrace{(1.0,0.17)}_{\text {Stage } \beta_{2}} \xrightarrow{\text { Transition } 2} \underbrace{(0.35,0.17)}_{\text {Stage } \gamma}
$$

The radial-roads first case is that the radial roads are improved first, and the ring road is improved next. ${ }^{18}$ The ring-road first case is that the ring road is improved first.

First, we focus on the result of the radial-roads first case shown in Figure 2.6. Figure 2.6(a-1) is the comparative statics analysis with a decrease in $\phi$, which is equal to Stage 1. Solid lines $A_{1} B_{1}$ and $A_{2} B_{2}$ are the stable dispersion and the stable full agglomeration, respectively. Both dispersion and the full agglomeration exist for large $\phi$ ( $>0.37$ ). In the dispersion, number of retail stores in the center $n_{0}$ increases and market area of the center $\mathcal{M}_{0}$ expands with a decrease in $\phi$ (Lemma 2.3). For small $\phi(=0.37), \mathcal{M}_{0}$ entirely covers the radial roads.

We investigate how a point in a neighborhood of $B_{1}$ changes under dynamics (2.8). Let $\widehat{B_{1}}$ denote the point. ${ }^{19}$ The solution starting at $\widehat{B_{1}}$ under dynamics (2.8) is shown in Figure 2.6(c-1). ${ }^{20}$ This solution converges at the square marker ( $\square$ ). The point marked

[^13]

Figure 2.6: The radial-roads first case. (a) The market equilibria with decreases in the travel costs. Solid line: stable equilibria. (b) Social welfare of stable equilibria in (a). (c-1) The solution starting at a point in the neighborhood of point $B_{1}$. Dashed-dotted line: the solution under market pattern (F).
by the square marker shows the full agglomeration. In summary, the dispersion changes into the full agglomeration when the radial roads are improved. ${ }^{21}$

The full agglomeration is always stable (Figure 2.6(a-1)). The red point in Figure 2.6(a-1) is equal to the agglomeration pattern at stage $\beta_{1}$ (i.e., the full agglomeration).

We focus on stage $\gamma$ of the radial-roads first case. Figure 2.6(a-2) is the comparative statics analysis with a decrease in $\tau$, which is equal to Transition 2 . The green point in Figure 2.6(a-2) shows the agglomeration pattern of the final stage (i.e., the full agglomeration). Hence, the radial-roads first case results in the full agglomeration emerging from the dispersion.

Next, we focus on the result of the ring-road first case shown in Figure 2.7. Figure 2.7(a-1) is the comparative statics analysis with a decrease in $\tau$, which is equal to Stage 1. Solid line $A_{1} B_{1}$ is the stable dispersion; $A_{3} B_{3}$ is the triangle pattern; $A_{4} B_{4}$ is the linear pattern; $A_{5} B_{5}$ is the asymmetric pattern. The dispersion becomes unstable at point $B_{1}$, which is a bifurcation point. Three unstable equilibria emerge at this point. When the dispersion is unstable, a small perturbation to this state generates an agglomeration pattern where large agglomerations and small agglomerations alternately emerge on the ring road, as shown in Figure 2.3 (Lemma 2.4).

To investigate the change from the unstable equilibrium, we investigate how a point in a neighborhood of bifurcation point $B_{1}$ changes under dynamics (2.8). The solution starting at this point under dynamics (2.8) is shown in Figure 2.7(c-1). ${ }^{22}$ The solutions shown with the blue line and the pink line in Figure 2.7(c-1) are obtained under market areas $(\mathrm{D})$ and (P), respectively. ${ }^{23}$ The solution converges at the point marked by the

[^14]square marker ( $\square$ ). This result shows that the dispersion changes into the triangle pattern. Moreover, in the triangle pattern, $n_{1}$ increases with a decrease in $\tau$. This pattern disappears at point $B_{3}$ (Figure 2.7(a-1)).

To elucidate the change from point $B_{3}$, we investigate how a point in the neighborhood of point $B_{3}$ changes under dynamics (2.8). The solution starting at this point is shown in Figure 2.7(d-1). ${ }^{24}$ Near $\widehat{B_{3}}$, we obtained this solution with dynamics (2.8) in market pattern $(\mathrm{P})$. For small $n_{5}$, the solution was obtained under that in market pattern (A)..$^{25}$ This result shows that the triangle pattern changes into the asymmetric pattern. Hence, the asymmetric pattern emergin at stage $\beta_{2}$ is the red point in Figure 2.7(a-1).

We focus on stage $\gamma$. Figure $2.7(\mathrm{a}-2)$ is the comparative statics analysis with a decrease in $\phi$, which is equal to Stage 2. In both asymmetric pattern and the linear pattern, $n_{1}$ decreases with a decrease in $\phi$. The green point in Figure 2.7(a-2) is equal to the agglomeration pattern of stage $\gamma$ (i.e., the asymmetric pattern). This pattern is not the agglomeration pattern that emerges in the radial-roads first case. In summary, the improvement sequences in the road network finally generate the difference in the agglomeration pattern. This observation is main finding 1.
color of the line changes. This point is at the boundary between market area conditions (D) and (P) (See Lemmas A. 1 in Appendix A. 2 and A. 5 in Appendix A.3). We obtained the solution to the square marker by following the solution starting at a point satisfying market area condition (P) in the neighborhood of this boundary.
${ }^{24} B_{3}$ is $\left(\tau, n_{1}\right)=\left(0.188,3.36 \times 10^{-2}\right)$ and $\widehat{B_{3}}$ is $\left(\tau, n_{1}\right)=\left(0.187,3.36 \times 10^{-2}\right)$.
${ }^{25} \mathrm{We}$ obtained this result by the same procedure as we did for Figure 2.7(c-1).


Figure 2.7: The ring-road first case. (a) The market equilibria with decreases in the travel costs. Solid line: stable equilibria; dashed line: unstable equilibria (b) Social welfare of stable equilibria in (a). (c-1) The solution of the dynamics starting at a point in a neighborhood of $B_{1}$. Blue dashed-dotted line: the solution under market pattern (D); pink: market pattern (P). (d-1) The solution starting at a point in a neighborhood of $B_{3}$. Pink: market pattern (P); brown: market pattern (A).

### 2.4.3. Welfare analysis

We discuss the welfare analysis results shown in Figures 2.6 and 2.7. First, we compare the welfare of stage $\gamma$ of the radial-roads first case and that of the ring-road first case. The welfare is 11.7 in the radial-roads first case (Figure 2.6(b-2)), whereas the welfare is 10.8 in the ring-road first case (Figure 2.7(b-2)). These demonstrations show that the radial-roads first case is more effective than the ring-road first case.

Next, we focus on the welfare of the linear pattern shown in the ring-road first case. In the ranges of travel costs $(0.01<\tau<0.34$ in Figure $2.7(\mathrm{~b}-1)$ and $0.35<\phi \leq 1.00$ in Figure 2.7(b-2)), the welfare in the linear pattern is higher than that in the other patterns in the ring-road first case. In particular, the social welfare of the linear pattern is higher than that of the asymmetric pattern at the same travel costs. However, not the linear pattern but the asymmetric pattern emerges from the dispersion in the market system. That is, the two-dimensional shape of the location in the market system is not that of the first-best location. This result is main finding 2 , which indicates that policies that change the locations of marketplaces are needed (e.g., land-use regulations).

### 2.5. Conclusion

We have investigated how improvement sequences on a two-dimensional road network affect the agglomeration patterns of retail stores and social welfare. We have two main findings: (1) the improvement sequence in the road network finally generates the difference in agglomeration patterns and (2) the two-dimensional shape of the locations in the market system differs from that in the first-best location. Main finding (2) indicates that policies that change the locations of marketplaces are needed (e.g., land-use regulations). In particular, the asymmetric pattern emerges if the ring road is improved first. This result contrasts with the main result of Tabuchi (2009), which is the emergence of the Christaller-Lösch system of hexagonal market area in a two-dimensional homogeneous space. The improvement sequence in the road network generates this
contrast.
Our model is specific, but more realistic assumptions can be considered with this model. We would like to review our three assumptions one by one in the following.

First, we assume so-called one-stop shopping, in contrast to two-stop shopping models which have been developed in recent years (Kim and Serfes, 2006; Brandão et al., 2014; Ushchev et al., 2015; Anderson et al., 2017). The assumption of two-stop shopping, however, makes the analysis more complex. Moreover, the results of the agglomeration of retail stores are similar to that of one-stopping shopping. One-stop shopping thus has a benefit to simply investigate the agglomeration patterns of retail stores, which is suitable for accomplishing our objective.

Second, we assume a uniform consumers-distribution in our model. This distribution is observed in local cities in the real world. Our model mainly targets the storeagglomeration mechanism in these cities. On the other hand, non-uniform distribution or endogenous consumer distribution has been considered in spatial competition models (e.g., Tabuchi and Thisse, 1995; Fujita and Thisse, 1986). Our analysis focuses on the symmetry of the road network. The assumption of the non-uniform distribution would not qualitatively affect our result unless the symmetry of the distribution differed. The assumption of exogenous consumers-distribution thus has a benefit to simply investigate the agglomeration mechanism of retail stores in a city. However, if we particularly investigate the interaction between consumers distribution and the location of shopping centers, it is necessary to consider endogenous consumer distribution.

Third, in our model, we assume that there are six suburbs in a ring road. In order to elaborately investigate how the spatial structure of a road network affects our results, we need to relax the assumption of the number of suburbs. One research direction is to investigate how n suburbs affect our results. Moreover, with more general preferences than the CES, we can investigate how pro-competitive effects and road improvement affect equilibrium and social welfare.


Mono-center


Twin cities


Three cities


Racetrack cities

Figure 3.1: Spatial agglomeration patterns that are found to be superior in stability in this chapter. •: the location where population distribute.
3. Satellite Region Formation for Spatial Economic Models: Bifurcation Mechanism in a Hexagonal Domain

### 3.1. Introduction

Agglomeration patterns of one large city surrounded by satellite regions are observed worldwide. An emergence of a core-place surrounded by a satellite (periphery) place is shown for a two-place economy (Krugman, 1991). This chapter aims to elucidate the mechanism of the formation of such satellite regions in a two-dimensional space for spatial economic models. As a major theoretical finding of this chapter, we demonstrate how a core-satellite pattern (the downtown area surrounded by hexagonal satellite regions) emerges from mono-center and racetrack regions in the two-dimensional space (see Fig. 3.1) as a transport cost changes.

There are several studies of the emergence of cities in spatial economics. The emergence of satellite regions around a single large city is explored in a linear space (Mori, 1997; Fujita and Mori, 1997; Fujita et al., 1999a). The transition from a central monocenter in three regions in a linear space is investigated to show a hub city formation (Ago et al., 2006). Various agglomeration patterns are numerically observed by changing an agglomeration force and a transport cost in discrete places in a line segment with more cities (Ikeda et al., 2017a). The economic spaces in these studies, however,
are restricted to be one-dimensional. ${ }^{26}$
That said, we aim to theoretically elucidate how and where satellite regions emerge in a two-dimensional space. We employ a regular-hexagonal domain where discrete locations are evenly distributed. ${ }^{27}$ We introduce two viewpoints: (1) the bifurcation mechanism of the full agglomeration at the center in this domain (mono-center), which produces satellite regions around this center, and (2) the existence of invariant patterns (Ikeda et al., 2018b), which are equilibria for any value of transport cost parameter. We focus on various patterns of satellite region formations that one-dimensional spatial platforms cannot express completely.

The first viewpoint is associated with the bifurcation from a sustain point, which was first studied in a two-region economy under replicator dynamics (Krugman, 1991; Fujita et al., 1999b; Baldwin et al., 2003). The analysis of sustain points is explored in an equidistant economy (Aizawa et al., 2020; Gaspar et al., 2021). The emergence of satellite regions from a large central city is studied in a line segment economy (Ikeda et al., 2020). It is customary to start from the uniform state in a two-dimensional economy. Nowadays, however, it would be far more important to investigate where satellite regions emerge than to investigate the self-organization of cities in a flat land envisaged in central place theory. We investigate the bifurcation mechanism of the emergence of satellite regions from the state of mono-center as the transport cost changes. The development of this bifurcation mechanism is a major theoretical contribution of this chapter.

[^15]The second viewpoint is to set forth invariant patterns as the candidates of stable agglomeration patterns. Invariant patterns have come to be employed in the description of the agglomeration mechanism of several kinds of spatial platforms such as an equidistant economy (Gaspar et al., 2018), a racetrack economy (Takayama et al., 2020), and a square lattice economy (Ikeda et al., 2018b). Characteristic agglomeration patterns of economic interest have successfully been captured. In this chapter, we adapt the methodology in Ikeda et al. (2019a) for a hexagonal lattice with periodic boundary conditions to the hexagonal lattice with boundary employed in this chapter. We investigate exhaustively stable invariant patterns in this space.

The contribution of this chapter is twofold. First, the city system comprising one large central city and satellite regions (core-satellite pattern) is theoretically found and shown to be stable by comparative static analysis for the spatial economic model proposed by Forslid and Ottaviano (2003) (the FO model). ${ }^{28}$ The transitions of stable equilibria as the transport cost changes are observed.

Second, we demonstrate that invariant patterns of mono-center, twin cities, three cities, and racetrack cities in Fig. 3.1 are predominant stable equilibria in the twodimensional space. It is noteworthy that these patterns are those which have been studied extensively in spatial economics. The twin cities, three cities, and racetrack cities are absorbed into the mono-center as the transport cost decreases, thereby displaying the progress of the formation of a large city at the geographical center. In particular, it is demonstrated that only the core-satellite pattern is stable in this progress. Most of the conventional spatial platforms, such as two place (Krugman, 1991), three places (Castro et al., 2012), and racetrack (Ikeda et al., 2019b) have no geographical

[^16]center and cannot express such central city formation. These transitions are an intrinsic feature observed in the two-dimensional spatial platform with the geographical center.

This chapter is organized as follows. The spatial economic model with the replicator dynamics is introduced in Section 3.2. Invariant patterns in the regular-hexagonal domain are presented in Section 3.3. Bifurcation from the mono-center is studied in Section 3.4. Section 3.5 concludes.

### 3.2. Spatial equilibrium and replicator dynamics

A spatial economic model with the replicator dynamics is presented. Our theoretical framework of this chapter, shown in Sections 3.3 and 3.4, is applicable to canonical spatial economic models which express population migration. We introduce the FO model, which is to be used in the investigation of the stability of agglomeration patterns in Sections 3.3 and 3.4.

### 3.2.1. Basic modeling

We briefly introduce a spatial equilibrium of the economy comprising $K(\geq 2)$ places, in which workers (or consumers) are allowed to migrate among the places.

Let $P=\{1, \ldots, K\}$ be the set of places. We focus on indirect utilities that workers can obtain by residing in a place. We assume that prices and income are functions of the population distribution and model parameters (e.g., trade freeness). This assumption implies that indirect utility is also a function of these parameters. We define the indirect utility function vector $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{\lambda}, \phi) \in \mathbb{R}^{K}$ as a continuous function of a workers' distribution vector $\boldsymbol{\lambda}\left(\lambda_{i} \geq 0 ; i \in P\right)$ and the trade freeness parameter $\phi$. The workers reside in one of $K$ places and the utility depends on the spatial distribution of them in the economy. An equilibrium is defined as the workers' spatial distribution $\boldsymbol{\lambda}$ that satisfies the following conditions:

$$
\left\{\begin{array}{l}
v^{*}-v_{i}=0 \quad \text { if } \quad \lambda_{i}>0  \tag{3.1}\\
v^{*}-v_{i} \geq 0
\end{array} \quad \text { if } \quad \lambda_{i}=0, ~ \$\right.
$$

and $\sum_{i=1}^{K} \lambda_{i}=1$, where $v^{*}$ denotes the equilibrium utility level.
We consider the replicator dynamics:

$$
\begin{equation*}
\frac{d \boldsymbol{\lambda}}{d t}=\boldsymbol{F}(\boldsymbol{\lambda}, \phi), \tag{3.2}
\end{equation*}
$$

where $\boldsymbol{F}(\boldsymbol{\lambda}, \phi)=\left(F_{i}(\boldsymbol{\lambda}, \phi) \mid i \in P\right)$, and

$$
\begin{equation*}
F_{i}(\boldsymbol{\lambda}, \phi)=\lambda_{i}\left(v_{i}(\boldsymbol{\lambda}, \phi)-\bar{v}(\boldsymbol{\lambda}, \phi)\right), \quad i \in P . \tag{3.3}
\end{equation*}
$$

Here, $\bar{v}=\sum_{i \in P} \lambda_{i} v_{i}$ represents the weighted average utility.
We restate the problem of obtaining a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics (Sandholm, 2010). Stationary points (rest points) $(\boldsymbol{\lambda}, \phi)$ are defined as solutions to the static governing equation:

$$
\begin{equation*}
\boldsymbol{F}(\boldsymbol{\lambda}, \phi)=\mathbf{0} . \tag{3.4}
\end{equation*}
$$

Using the eigenvalues of the Jacobian matrix $J(\boldsymbol{\lambda}, \phi)=\partial \boldsymbol{F} / \partial \boldsymbol{\lambda}$, a stationary point is defined as linearly stable if every eigenvalue has a negative real part. We investigate stable equilibria through finding stable stationary points in Sections 3.3 and 3.4.

### 3.2.2. Modeling of a spatial economy

As a representative of spatial economic models, a multi-regional version of the analytically solvable core-periphery model proposed by Forslid and Ottaviano (2003) (the FO model) is briefly introduced, whereas details are presented in Appendix B.1.

The economy of this model comprises $K(\geq 2)$ places, two factors of production (skilled and unskilled workers), and two sectors (manufacturing M, and agriculture A). The agricultural production is constant returns to scale.

The $H$ skilled and $L$ unskilled workers consume final goods of two types: manufacturing sector goods and an agricultural sector good. Workers supply one unit of each type of labor inelastically. Skilled workers are mobile among places. The num-
ber of skilled workers in place $i \in P$ is denoted by $\lambda_{i}$ under the normalizing constraint $\sum_{i \in P} \lambda_{i}=1$. Unskilled workers are immobile and distributed equally across all places with $L / K$. The mobile skilled labor is used as the fixed input in manufacturing production, while the immobile unskilled labor is used as the variable input of both manufacturing and agricultural production. ${ }^{29}$

Preferences $U$ over the M-sector and A-sector goods are identical across individuals. The utility of an individual in a place $i$ is

$$
\begin{equation*}
U\left(C_{i}^{\mathrm{M}}, C_{i}^{\mathrm{A}}\right)=\mu \ln C_{i}^{\mathrm{M}}+(1-\mu) \ln C_{i}^{\mathrm{A}} \quad(0<\mu<1) \tag{3.5}
\end{equation*}
$$

where $\mu$ is the constant expenditure share of manufacturing sector goods, $C_{i}^{\mathrm{A}}$ stands for the consumption of the A-sector product in the place $i$, and $C_{i}^{\mathrm{M}}$ represents the manufacturing aggregate in the place $i$, defined as $C_{i}^{\mathrm{M}} \equiv\left(\sum_{j \in P} \int_{0}^{n_{j}} q_{j i}(\ell)^{(\sigma-1) / \sigma} d \ell\right)^{\sigma /(\sigma-1)}$, where $q_{j i}(\ell)$ represents the consumption in the place $i \in P$ of a variety $\ell \in\left[0, n_{j}\right]$ produced in the place $j \in P, n_{j}$ stands for the number of produced varieties at the place $j$, and $\sigma>1$ denotes the constant elasticity of substitution between any two varieties.

The transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from a place $i$ to a place $j \neq i$, only a fraction $1 / \tau_{i j}<1$ actually arrives $\left(\tau_{i i}=1\right)$. It is assumed that $\tau_{i j}=\exp (\tau m(i, j) \tilde{L})$ is a function of a transport cost parameter $\tau>0$, where $m(i, j)$ is an integer expressing the road distance between the places $i$ and $j$ and $\tilde{L}$ is the distance unit. We further introduce the trade freeness:

$$
\begin{equation*}
\phi=\exp [-\tau(\sigma-1) \tilde{L}] \in(0,1) \tag{3.6}
\end{equation*}
$$

[^17]$\phi$ is to be employed in the analysis in Sections 3.3 and 3.4. The spatial discounting factor $d_{j i}=\tau_{j i}^{1-\sigma}=\phi^{m(i, j)}$ represents friction between the places $j$ and $i$ that decays in proportion to the transportation distance. In our formulation, which relies on $d_{j i}$, the distance unit $\tilde{L}$ need not be specified.

The wage vector for the skilled workers $\boldsymbol{w}=\left(w_{1}, \ldots, w_{K}\right)$ can be obtained analytically ((B.10) in Appendix B.1). The indirect utility $v_{i}$ for them in the place $i$ is expressed in terms of $w_{i}$ and $\Delta_{i}=\sum_{k \in P} d_{k i} \lambda_{k}$ as

$$
\begin{equation*}
v_{i}=\frac{\mu}{\sigma-1} \ln \Delta_{i}+\ln w_{i} . \tag{3.7}
\end{equation*}
$$

### 3.3. Invariant patterns in the regular-hexagonal domain

We theoretically investigate symmetric agglomeration patterns for the basic model in Section 3.2.1. The replicator dynamics with the symmetric spatial platform accommodates the so-called invariant patterns which are stationary points of the dynamics for any values of the trade freeness $\phi$ and microeconomic parameters (Ikeda et al., 2018b, 2019a). The invariant patterns in a regular-hexagonal domain are obtained as candidates of stable economic agglomeration patterns. The theoretically predicted invariant patterns are the candidates for the basic model. Some of them actually are shown to be stable for the FO model (Section 3.3.4).

### 3.3.1. A hexagonal lattice and the orbit decomposition of the places

We employ, as a spatial platform, a regular-hexagonal domain with a set of $K$ places at the nodal points where workers locate (Fig. 3.2). In the description of agglomeration patterns in this domain, it is essential to resort to its regular-hexagonal symmetry labeled by the dihedral group:

$$
\begin{equation*}
\mathrm{G}=\mathrm{D}_{6}=\left\{e, r, \ldots, r^{5}, s, s r, \ldots, s r^{5}\right\} \tag{3.8}
\end{equation*}
$$

where $e$ is the identity transformation, $s$ is the reflection $y \mapsto-y$, and $r^{j}$ is a counterclockwise rotation about the center of the circle at an angle of $\pi j / 3(j=0,1, \ldots, 5)$.


Figure 3.2: The regular-hexagonal domain. Nodal points: the place at which workers locate.

We introduce the orbit decomposition of places in the hexagonal domain. The $K$ nodal points are decomposed into subsets, called orbits. Each orbit is a set of places and has some geometrical symmetry. ${ }^{30}$

We consider subgroups of G :

$$
\begin{array}{ll}
\mathrm{D}_{6}=\left\{e, r, r^{2}, r^{3}, r^{4}, r^{5}, s, s r, s r^{2}, s r^{3}, s r^{4}, s r^{5}\right\}, & \text { hexagonal symmetry }, \\
\mathrm{D}_{3}=\left\{e, r^{2}, r^{4}, s, s r^{2}, s r^{4}\right\}, & \text { triangle symmetry }, \\
\mathrm{D}_{2}=\left\{e, r^{3}, s, s r^{3}\right\}, & \text { rectangle symmetry }, \\
\mathrm{D}_{1}=\{e, s\}, & \text { bilateral symmetry }, \\
\mathrm{C}_{6}=\left\{e, r, r^{2}, r^{3}, r^{4}, r^{5}\right\}, & (\pi / 3) \text {-rotation symmetry }, \\
\mathrm{C}_{3}=\left\{e, r^{2}, r^{4}\right\}, & (2 \pi / 3) \text {-rotation symmetry }, \\
\mathrm{C}_{2}=\left\{e, r^{3}\right\}, & \pi \text {-rotation symmetry }, \\
E=\mathrm{C}_{1}=\{e\}, & \text { asymmetry }
\end{array}
$$

We can decompose the set of places $P$ into disjoint orbits with respect to a subgroup $\mathrm{G}^{\prime}$ of G :

$$
\begin{equation*}
P=\bigcup_{l \in \mathcal{L}} P_{l}, \tag{3.9}
\end{equation*}
$$

where $P_{l}$ is an orbit of places with the symmetry labeled by $\mathrm{G}^{\prime}$ and $\mathcal{L}$ is the whole set of orbits. We define $N_{l}$ as the number of elements in the orbit $P_{l}$. Orbit decompositions

[^18]with respect to subgroups other than $E$ are depicted in Fig. 3.3, while each node becomes an orbit for $\mathrm{G}^{\prime}=E$. The same symbols (such as $\bigcirc$ and $\triangle$ ) in Fig. 3.3 imply that they belong to the same orbit. These orbits are proved to correspond to invariant patterns, which express diverse agglomeration patterns such as hexagons, triangles, rectangles, twins, and monos (Section 3.3.3).

### 3.3.2. The symmetry of indirect utilities with the geometrical symmetry of the hexagonal lattice

The mathematical mechanism of occurrence of places with the same level of $v_{i}$ is explained below. We assume that the symmetry of the regular-hexagonal domain corresponds to that of the indirect utility function:

$$
\begin{equation*}
T(g) \boldsymbol{v}(\boldsymbol{\lambda}, \phi)=\boldsymbol{v}(T(g) \boldsymbol{\lambda}, \phi), \quad g \in \mathrm{G} \tag{3.10}
\end{equation*}
$$

where $T(g)$ is a matrix representation of G that permutes place numbers. This assumption is called equivariance. Equivariance has a pivotal role of investigating the invariant patterns and the bifurcation patterns from the mono-center in our theoretical analysis (Sections 3.3.3 and 3.4.1).

Lemma 3.1. $v_{i}$ in the same orbit for a subgroup takes the same value when the population distribution is symmetric with respect to this subgroup.

Proof. See Appendix B.3.1 for the proof.


Figure 3.3: Orbit decompositions of the regular-hexagonal domain with respect to subgroups of G.

### 3.3.3. Theory of invariant patterns

The theory of invariant patterns of the replicator dynamics is introduced. Let $\boldsymbol{\lambda}_{P_{l}}$ be the identical complete agglomeration to the places in the orbit $P_{l}$ :

$$
\begin{equation*}
\boldsymbol{\lambda}_{P_{l}}=\left\{\lambda_{i}=\frac{1}{m} \text { for } i \in P_{l}, \lambda_{i}=0 \text { for } i \notin P_{l}\right\} \tag{3.11}
\end{equation*}
$$

where $m=N_{l}$ is the number of nodes belonging to the orbit $P_{l}$. By the symmetry, all places in the orbit have the same level of indirect utility for $\boldsymbol{\lambda}=\boldsymbol{\lambda}_{P_{l}}$.

Lemma 3.2. If $\boldsymbol{\lambda}=\boldsymbol{\lambda}_{P_{l}}$, we have $v_{i}=\bar{v}$ for every $i \in P_{l}$.

Proof. See Appendix B.3.2 for the proof.

This pattern $\boldsymbol{\lambda}_{P_{l}}$ is an invariant pattern as explained in the following proposition. Thus a complete and identical agglomeration to the places in each orbit shown in Fig. 3.3 is an invariant pattern.

Proposition 3.1. The pattern $\boldsymbol{\lambda}_{P_{l}}$ in (3.11) is an invariant pattern.

Proof. See Appendix B.3.3 for the proof.

In addition, the full agglomeration $\boldsymbol{\lambda}_{i}^{\mathrm{FA}}$ of population at the place $i$ :

$$
\begin{equation*}
\boldsymbol{\lambda}_{i}^{\mathrm{FA}}=\left\{\lambda_{i}=1, \lambda_{j}=0 \text { for } j \neq i\right\} \tag{3.12}
\end{equation*}
$$

which corresponds to the case of $m=1$ in (3.11), is an invariant pattern. ${ }^{31}$

Proposition 3.2. The full agglomeration $\boldsymbol{\lambda}_{i}^{\mathrm{FA}}$ at any place $i$ is an invariant pattern.

Proof. See Appendix B.3.4 for the proof.

[^19]
### 3.3.4. Stable invariant patterns for the FO model

By the theory in Section 3.3.3, it is possible to obtain invariant patterns exhaustively. We investigate invariant patterns that are superior in stability as candidates of twodimensional agglomerations of economic interest. For this purpose, we consider the FO model on the regular-hexagonal domain with $K=37$ places. This domain has a total of 45 invariant patterns that are depicted in Fig. 3.4. The value of the constant elasticity of substitution between any two industrial varieties $\sigma$ and that of the constant expenditure share on them $\mu$ for the FO model are chosen as $(\sigma, \mu)=(6.0,0.4)$.

The stability of these 45 invariant patterns was investigated for the trade freeness $\phi \in(0,1)$. Only four patterns are stable in some ranges of $\phi$ shown by the solid lines in Fig. 3.5. These four stable patterns are the mono-center, twin cities, triangular cities and racetrack cities. Note that twin cities, triangular (three) cities, and racetrack cities are employed as spatial platforms in spatial economics. ${ }^{32}$ These four patterns have some symmetry. Symmetric invariant patterns thus tend to be superior in stability. In contrast, the remaining 41 patterns were all unstable for any $\phi \in(0,1)$; in most of these patterns, populated places are located too closely each other or to the border and apparently look inferior in stability.

The mono-center is most superior in stability and is a unique stable invariant pattern for a wide range of $\phi(>0.49)$. The twin cities $(0.46<\phi<0.48)$ and the triangular cities $(0.42<\phi<0.49)$ coexist as stable invariant patterns with the mono-center. The racetrack cities is a unique stable invariant pattern for relatively small $\phi(0.348<\phi<$ 0.355). There are no stable invariant patterns for small $\phi(<0.348)$. As $\phi$ increases, there tend to be fewer and larger agglomerations in line with the observation for spatial economic models in the literature (e.g., Fujita et al., 1999b).

[^20]
(a) Mono city

(b) Twin cities

(c) Triangular cities

(d) Rectangular cities

(e) Racetrack cities

Figure 3.4: Invariant patterns for a regular-hexagonal domain with $K=37$ places. •: the places with population.


Figure 3.5: Stable invariant patterns for the regular-hexagonal domain with 37 places for $\phi \in(0,1) .-$ : stable equilibrium; ---: unstable equilibrium; $\bigcirc$ : bifurcation point; $\bullet$ : the places with population; $(\sigma, \mu)=(6.0,0.4)$.

There are bifurcation points at both end points of the stable ranges of $\phi$ for the invariant patterns, except for the end point $\phi=1$ for the mono-center. To search for possible connectivity between these four stable invariant patterns, bifurcating solutions from these points are obtained in Section 4.2.

### 3.4. Bifurcation from the mono-center

The theoretical bifurcation analysis from the mono-center is conducted (Section 3.4.1). This bifurcation produces a state of a large central city surrounded by satellite regions of various kinds. Such a state is of keen interest of spatial economics to answer a question: "how and where satellite regions form?". In particular, a core-satellite pattern is shown to be superior in stability by comparative static analysis for the FO model (Section 3.4.2).

### 3.4.1. Theoretical bifurcation analysis

Let $\boldsymbol{\lambda}^{\mathrm{FA}}=(1,0, \ldots, 0)$ be the mono-center (the full agglomeration to the place at the center). A bifurcation from the mono-center can occur when the Jacobian matrix is singular. For the mono-center, this matrix takes the following form (see Appendix B. 2 for details):

$$
J\left(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi\right)=\left(\begin{array}{cc}
-v_{1} & J_{+0}  \tag{3.13}\\
& J_{0}
\end{array}\right)
$$

where

$$
\begin{equation*}
J_{+0}=\left(-v_{2}, \ldots,-v_{K}\right), \quad J_{0}=\operatorname{diag}\left(v_{2}-v_{1}, \ldots, v_{K}-v_{1}\right) \tag{3.14}
\end{equation*}
$$

In preparation for the discussion of the bifurcating solutions, we carry out the orbit decomposition with respect to hexagonal symmetry $D_{6}$. Each orbit other than the center in the domain comprises racetracks with six or twelve places (see Fig. 3.3(a)). We denote these orbits by

$$
\begin{equation*}
S=\left\{c, \alpha 1, \ldots, \alpha n_{1}, \beta 1, \ldots, \beta n_{2}\right\} \tag{3.15}
\end{equation*}
$$

where $c$ denotes an orbit comprising only the place at the center, $\alpha i\left(i=1, \ldots, n_{1}\right)$ denotes orbits with six places $\left(N_{\ell}=6\right)$, and $\beta i\left(i=1, \ldots, n_{2}\right)$ denotes orbits with twelve places $\left(N_{\ell}=12\right)$. For example, Fig. 3.6 depicts orbits for $K=37$ places.


Figure 3.6: Orbits representing invariant patterns for a regular-hexagonal domain with 37 places. •: the places with population.

By Lemma 3.1, the indirect utility $v_{i}$ at the places in each orbit other than $c$ takes the same value. We denote such value as by

$$
\begin{array}{lll}
v_{\alpha 1}, \ldots, v_{\alpha n_{1}} & \text { for } \alpha i & \left(i=1, \ldots, n_{1}\right)  \tag{3.16}\\
v_{\beta 1}, \ldots, v_{\beta n_{2}} & \text { for } \beta i \quad\left(i=1, \ldots, n_{2}\right) .
\end{array}
$$

Therefore, the condition of the bifurcation from the mono-center is given by the following proposition.

Proposition 3.3. A bifurcation solution in the space $\sum_{j=1}^{K} \lambda_{j}=1$ emerges from the mono-center if one of the following conditions is satisfied:

$$
\begin{array}{ll}
v_{\alpha i}-v_{1}=0, & i=1, \ldots, n_{1} \\
v_{\beta i}-v_{1}=0, & i=1, \ldots, n_{2} \tag{3.18}
\end{array}
$$

Proof. See Appendix B.3.5 for the proof.

We investigate bifurcation solutions from the mono-center when $v_{\alpha i}-v_{c}=0$ and $v_{\beta i}-v_{c}=0$ hold. Let $\phi_{l}^{\mathrm{c}}$ be the trade freeness at $v_{l}-v_{c}=0(l \in S-\{c\})$. In the analysis of bifurcating solutions from a critical point $\left(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{l}^{\mathrm{c}}\right)$, we employ the so-called bifurcation equation (e.g., Golubitsky et al., 1988).

The bifurcation equation for Type $\alpha i$ orbit takes a special form given in the following lemma.

(a) Type $\alpha i$ orbit (b) Type $\beta i$ orbit

Figure 3.7: Definition of node numbers within orbits. $\bigcirc$ : the places within a orbit.

Lemma 3.3. For a bifurcation point associated with an orbit comprising six places (Type $\alpha i$ orbit for some $i$ ), the bifurcation equation is six-dimensional and is expressed as

$$
\begin{align*}
& \tilde{F}_{1}\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=x_{1} R\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=0 \\
& \tilde{F}_{2}\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=x_{2} R\left(x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{1}, \psi\right)=0 \\
& \tilde{F}_{3}\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=x_{3} R\left(x_{3}, x_{4}, x_{5}, x_{6}, x_{1}, x_{2} \psi\right)=0 \\
& \tilde{F}_{4}\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=x_{4} R\left(x_{4}, x_{5}, x_{6}, x_{1}, x_{2}, x_{3}, \psi\right)=0  \tag{3.19}\\
& \tilde{F}_{5}\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=x_{5} R\left(x_{5}, x_{6}, x_{1}, x_{2}, x_{3}, x_{4}, \psi\right)=0 \\
& \tilde{F}_{6}\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=x_{6} R\left(x_{6}, x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, \psi\right)=0
\end{align*}
$$

where $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{6}\right)=\left\{\lambda_{j} \mid j \in \alpha i\right\}$ (Fig. 3.7(a)) and $R$ is a function with the symmetry condition:

$$
\begin{equation*}
R\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=R\left(x_{1}, x_{6}, x_{5}, x_{4}, x_{3}, x_{2}, \psi\right) . \tag{3.20}
\end{equation*}
$$

Proof. See Appendix B.3.6 for the proof.

This bifurcation equation has the following three properties:

- Migration of agents from the center to some places in the Type $\alpha i$ orbit.
- The product form of the replicator dynamics in (3.3).
- The $\mathrm{D}_{6}$-symmetry (regular-hexagonal symmetry) inherited to the bifurcation equation.



Twin-II




Twin-III Mono-I

Figure 3.8: Geometrical configurations of bifurcating solutions for the Type $\alpha i$ orbit $\left(N_{\ell}=6\right)$.

- : the population size at the place.

Solving this bifurcation equation, we can show the emergence of a series of bifurcating solutions from the bifurcation point associated with Type $\alpha i$ orbit.

Proposition 3.4. A bifurcation point associated with Type $\alpha i$ orbit has the following seven kinds of bifurcating solutions:

$$
\boldsymbol{x}=\left\{\begin{align*}
w(1,1,1,1,1,1): & \text { Racetrack-I, }  \tag{3.21}\\
w(1,1,0,1,1,0): & \text { Rectangle-I } \\
w(1,0,1,0,1,0): & \text { Triangle-I, } \\
w(1,1,0,0,0,0): & \text { Twin-I, } \\
w(1,0,1,0,0,0): & \text { Twin-II, } \\
w(1,0,0,1,0,0): & \text { Twin-III, } \\
w(1,0,0,0,0,0): & \text { Mono-I }
\end{align*}\right.
$$

for some $w>0$.
Proof. See Appendix B.3.7 for the proof.

These seven bifurcating solutions can be classified into Racetrack, Rectangle, Triangle, Twin and Mono in the view of their geometrical symmetry. The geometrical patterns for these solutions are illustrated in Fig. 3.8.

The bifurcation equation for Type $\beta i$ orbit can be obtained similarly (see Lemma B. 1 in Appendix B.3.8). By solving this bifurcation equation, we can show the emergence of a series of bifurcation solutions from a bifurcation point associated with the orbit $\beta i$ comprising twelve places.

Proposition 3.5. The critical point for Type $\beta i$ orbit has the following 16 kinds of bifurcating solutions:

$$
\boldsymbol{x}= \begin{cases}w(1,1,1,1,1,1,1,1,1,1,1,1): & \text { Racetrack-II, }  \tag{3.22}\\ w(1,0,1,0,1,0,1,0,1,0,1,0): & \text { Racetrack-III, } \\ w(1,1,0,0,1,1,0,0,1,1,0,0): & \text { Triangle-II, } \\ w(1,0,0,1,1,0,0,1,1,0,0,1): & \text { Triangle-III, } \\ w(1,0,0,1,0,0,1,0,0,1,0,0): & \text { Rectangle-II, } \\ w(1,0,0,0,0,1,1,0,0,0,0,1): & \text { Rectangle-III, } \\ w(1,1,0,0,0,0,1,1,0,0,0,0): & \text { Rectangle-IV, } \\ w(1,0,0,0,1,0,0,0,1,0,0,0): & \text { Triangle-IV, } \\ w(1,1,0,0,0,0,0,0,0,0,0,0): & \text { Twin-IV, } \\ w(1,0,0,1,0,0,0,0,0,0,0,0): & \text { Twin-V, } \\ w(1,0,0,0,0,1,0,0,0,0,0,0): & \text { Twin-VI, } \\ w(1,0,0,0,0,0,1,0,0,0,0,0): & \text { Twin-VII, } \\ w(1,0,0,0,0,0,0,1,0,0,0,0): & \text { Twin-VIII, } \\ w(1,0,0,0,0,0,0,0,0,1,0,0): & \text { Twin-VIIII, } \\ w(1,0,0,0,0,0,0,0,0,0,0,1): & \text { Twin-X, } \\ w(1,0,0,0,0,0,0,0,0,0,0,0): & \text { Mono-II }\end{cases}
$$

for some $w>0$.

Proof. The proof is similar to that of Proposition 3.4.

These 16 bifurcating solutions can be classified in the view of their geometrical symmetry. The geometrical patterns for these solutions are illustrated in Fig. 3.9.

### 3.4.2. Bifurcating patterns from the sustain point for the FO model

We apply Propositions 3.4 and 3.5 in Section 3.4.1 to the investigation of bifurcating solutions from the mono-center for the FO model with 37 places in Fig. 3.6. Parameter
Rectangle-II Rectangle-III Rectangle-IV Triangle-IV



Racetrack-II Racetrack-III




Twin-V


Twin-VIII


Twin-IV



Twin-VI


Twin-X


Triangle-II Triangle-III




Twin-VII


Mono-II

Figure 3.9: Geometrical configurations of bifurcating solutions for Type $\beta i$ orbit ( $N_{\ell}=12$ ). - orbit: the population size at the place.
values are chosen as $(\sigma, \mu)=(6.0,0.4)$ similarly in Section 3.3.4. The 37 places can be decomposed into six kinds of orbits:

$$
\begin{cases}c, & \text { the mono-center, }  \tag{3.23}\\ \alpha 1, \ldots, \alpha 4, & 6 \text { places } \\ \beta 1, & 12 \text { places, }\end{cases}
$$

and there are five kinds of bifurcation points associated with Type $\alpha 1, \ldots, \alpha 4$, and $\beta 1$ orbits, whereas the orbit $c$ is not associated with bifurcation.

Among these five bifurcation points, we focus on the bifurcating solutions from the sustain point at which the system becomes unsustainable as $\phi$ decreases from 1. The sustain point was associated with the orbit $\alpha 2$ for $(\sigma, \mu)=(6.0,0.4)$. The bifurcating solutions from the sustain point are depicted in Fig. 3.10(a).

(a) Bifurcation diagram

(Racetrack-I)
(b) The progress of stable equilibria observed as $\phi$ increases $\left(A_{1} \rightarrow Q_{\alpha 2}\right)$

(Rectangle-I) (Triangle-I) (Twin-III) (Twin-II) (Twin-I) (Mono-I)
(c) Unstable bifurcating solutions

Figure 3.10: Bifurcation diagram from the sustain point of the mono-center for 37 regions. - : a stable solution; -- : an unstable solution; $\bigcirc$ : a bifurcation point; $\bullet$ the size of population at the place.

Among seven bifurcating solutions $Q_{\alpha 2} A_{1}, \ldots, Q_{\alpha 2} A_{7}$ in (3.21), there is only one stable bifurcating solution $Q_{\alpha 2} A_{1}$ shown by the solid curve, which has an agglomeration pattern with one huge city and hexagonal satellite regions, called a core-satellite patterns. ${ }^{33}$ This agglomeration pattern resembles the hexagonal pattern in central place theory (Christaller, 1933). As the trade freeness $\phi$ decreases from the sustain point $Q_{\alpha 2}$, the agglomeration pattern of this bifurcating solution shifts to the hexagonal racetracklike satellite regions. Conversely, as $\phi$ increases from 0.27, we can see that the satellite regions are absorbed into the city at the center, thereby engendering the mono-center at the point $Q_{\alpha 2}$ (Fig. 3.10(b)). Other bifurcating solutions the point $Q_{\alpha 2} A_{j}(j=2, \ldots, 7)$ in Fig. 3.10(c) were all unstable. Because the change of agglomeration pattern is qualitatively the same as the result for the square lattice (Kogure and Ikeda, 2022), the change is robust in terms of the spatial structure of a geographical space.

### 3.4.3. Comparison of spatial platforms with and without the geographical center

In preparation for the discussion of the difference of agglomerations in various kinds of spatial platforms, we investigate the transitions (via bifurcating solutions) from the three kinds of agglomeration patterns of twin cities, triangular cities, and racetrack cities in Fig. 3.5(a) by the comparative static analysis with respect to $\phi$. The bifurcating solutions for the twin cities branching from $I_{1}^{\prime}$ and the triangular cities branching from $I_{2}^{\prime}$ both reached the stable mono-center although these solutions were unstable throughout (Figs. 3.11(a) and (b)). In contrast, the bifurcating solution for the racetrack cities ( $\alpha 4$ ) from $I_{3}^{\prime}$ is stable at first, loses stability on the way, and reaches the mono-center at the point $Q_{\alpha 4} \cdot{ }^{34}$ At point C, there emerges a stable core-satellite pattern, which is a two-

[^21]dimensional counterpart of the core-periphery pattern (Krugman, 1991).

(a) The bifurcation diagram for twin cities (Twin-III for $\alpha 1$ )

(b) The bifurcation diagram for triangular cities (Triangle-I for $\alpha 1$ )

(c) The bifurcation diagram for racetrack cities (Racetrack-I for $\alpha 4$ )

Figure 3.11: Bifurcation solutions from stable invariant patterns for 37 regions. - : stable equilibria; - - : : unstable equilibria; $\bigcirc$ : bifurcation point; •: the size of population at the place.


Two places (Krugman, 1991)


Three places (Castro et al., 2012)


Racetrack (Ikeda et al., 2019b)
(a) Some spaces without the center

Two places


Three places


Racetrack (this paper and Ikeda et al., 2017b)

Figure 3.12: The progress of agglomeration patterns for several kinds of spatial platforms as the trade freeness increases. •: the size of population at the place.

With reference to these results, we discuss the difference of agglomerations in spatial platforms with and without the geographical center. In the two place, three place, and racetrack, every potential city is identical geographically and there is no apparent geographical center. The uniform population distribution prevails for large transport cost and encounters the bifurcation (Krugman, 1991; Castro et al., 2012; Ikeda et al., 2019b) to engender an asymmetric agglomeration pattern as the trade freeness increases (Fig. 3.12(a)). As the trade freeness $\phi$ increases, one of the identical cities grow into a mono-center. On the other hand, in spatial platforms with the geographical center (e.g., our spatial platform), the city located at the center absorbs all population in satellite

(a) 3 places (Ago et al., 2006) (b) 17 places (Ikeda et al., 2017a)

Figure 3.13: The progress of agglomeration patterns in a line segment as the trade freeness increases. •: the size of population at the place.
regions as the trade freeness increases (Fig. 3.12(b)). ${ }^{35}$ Such emergence of the city at the geographical center is what most of the conventional spatial platforms cannot express. An exception is a linear or line segment economy, in which the emergence of the mono-center from the state of the predominant twin cities was observed as the trade freeness increases (Ago et al., 2006; Ikeda et al., 2017a) (Fig. 3.13).

### 3.5. Conclusions

We have investigated where satellite regions emerge around the mono-center in a two-dimensional space, in which discrete locations are evenly spread over a regularhexagonal domain. We have proposed several candidates of stable economic agglomeration patterns: invariant patterns and bifurcating patterns engendered from the monocenter. It is to be emphasized that the proposed theory is applicable to general spatial economic models with the replicator dynamics.

As an contribution of this chapter, the core-satellite pattern was highlighted as a stable equilibria that emerges from the mono-center as the trade freeness $\phi$ decreases for the FO model. This core-satellite pattern is a two-dimensional counterpart of the core-periphery pattern (Krugman, 1991). Conversely, as $\phi$ increases, satellite regions

[^22]were absorbed into the city at the center, engendering the mono-center.
We have demonstrated that spatial patterns of the mono-center, twin cities, three cities, and racetrack cities are predominant stable equilibria in the two-dimensional space. These are, in fact, spatial patterns that have been studied in spatial economics, for example, in a two places economy (Krugman, 1991), three places economy (Castro et al., 2012), and racetrack economy (Takayama et al., 2020). This chapter thus has a close relation with the previous studies for other spatial platforms and can present an insight on economic agglomerations. Moreover, the two-dimensional space treated in this chapter can represent various patterns of satellite region formations that onedimensional spatial platforms cannot express completely.

In this chapter, the stability of equilibrium was investigated only for the FO models. It will be a future topic to extend the horizon of such investigation. For example, application of the bifurcation theory in our chapter is limited to an economy with one industry. There, however, are many industries in the real world. Some researches explain spatial agglomeration patterns of population based on central place theory (e.g., Tabuchi and Thisse, 2011). One of future works is to apply the bifurcation theory to such researches.
4. How should place-based policies be designed for efficiently promoting retail agglomeration?

### 4.1. Introduction

Shopping is an indispensable daily activity in our lives. The decline of retail stores operating in downtown areas has been regarded as an urban problem over the past several decades. Local governments have recently implemented place-based policies in order to make retail stores agglomerate in downtown areas. A feature of placebased policies is that stores and/or consumers in a targeted area are subsidized. For example, the city of Albuquerque in the U.S.A. subsidizes retail stores operating in the downtown area. Toyama in Japan subsidizes consumers who migrate from outside to an area around the downtown area.

Impacts of place-based policies on retail stores have been empirically investigated (e.g., Givord et al., 2013; Neumark and Simpson, 2015; Iwata and Kondo, 2021). For example, Givord et al. (2013) empirically show that, in France, the government has promoted the agglomeration of retail stores by a place-based policy, which indicates that place-based policies can revitalize downtown areas. However, the place-based policy does not ensure that social welfare increases because it can produce deadweight losses in the policy-implemented market, and can cause a decline in the number of retail stores in other areas. We theoretically clarify which place-based policies increase social welfare, and which decrease social welfare.

Since agglomeration of retail stores generally involves market failures, a place-based policy may increase social welfare. Examples of such market failures are the shopping externality generated by multipurpose shopping (O'Sullivan, 1993), which is purchasing goods from stores on a single trip, and price distortions caused by imperfect competition among stores. Arentze et al. (2005) empirically show that agglomeration of retail stores relates to multipurpose shopping.

General equilibrium models in which consumers engage in multipurpose shopping with imperfect competition in a marketplace (e.g., shopping streets and shopping malls) have been developed (Henkel et al., 2000; Arakawa, 2006; Tabuchi, 2009; Ushchev et al., 2015). Most multipurpose shopping models have two features. One feature is that retail stores operating in marketplaces are under monopolistic competition. The other feature is that the spatial distribution of consumers is exogenous.

In order to evaluate place-based policies, we need to consider the endogenous spatial distribution of consumers rather than the exogenous spatial distribution. Some studies develop spatial competition models in which consumers and firms compete in the land market (e.g., Fujita and Thisse, 1986; Fujita, 1988; Liu and Fujita, 1991). However, these studies do not answer how place-based policies affect social welfare. One of the place-based policies is to subsidize consumers to reside around downtown areas. This policy intends to agglomerate retail stores in downtown areas by encouraging more consumers to reside close to the downtown areas and visit the downtown areas for shopping. This policy can be adopted in a sprawled city to revitalize the center of the city. Another place-based policy is direct subsidies for stores to agglomerate.

We evaluate the welfare impacts of place-based policies for retail agglomeration by developing a multipurpose shopping model. In the model, retail stores are under monopolistic competition, and consumers are free to choose where to reside. We focus on two place-based policies which have been adopted by local governments. One is location subsidies to consumers, and the other is location subsidies to stores. We evaluate the welfare impacts of these policies in terms of social surplus.

The welfare impacts of adopting place-based policies can be decomposed into three terms, according to Harberger's welfare change measurement formula (Harberger, 1971). The first term is the total change in deadweight losses caused by the price distortions of the varieties supplied in marketplaces. The second term is the net social benefit generated by so-called variety distortion (e.g., Kanemoto, 2013a). The third term is
the migration fiscal externality generated by income transfer inefficiency by a placebased policy (Boadway and Flatters, 1982; Kono et al., 2007).

Our investigation finds that whether or not place-based policies are socially efficient depends on the recipients of the subsidies, even if the policies promote downtown retail agglomeration. Specifically, with the constant elasticity of substitution (CES) utility function of varieties, location subsidies to consumers is harmful from the viewpoint of welfare, whereas location subsidies to stores is desirable. Location subsidies to stores is desirable because positive net benefits generated by variety distortions necessarily exceed the negative deadweight losses caused by price distortions. On the other hand, location subsidies to consumers is harmful because they cause negative net benefits.

In order to validate the theoretical results, we numerically evaluate the welfare impacts with the CES utility function. We show that the numerical results are the same as the theoretical results. Moreover, we conduct numerical analyses with a variable elasticity of substitution (VES) utility function in order to examine whether or not relaxing the assumption regarding the elasticity affects the welfare impacts. We show that the welfare impacts are qualitatively the same as the results of the CES function. In the numerical analyses as well as the theoretical derivation, we decompose the welfare change into net benefits generated by the price distortion and the variety distortion by using Harberger's welfare measurement formula. With the location subsidies to residents, both net benefits are negative. With subsidizing retail stores, the former and the latter are negative and positive, respectively, and the latter exceeds the former.

Our paper is organized as follows. Basic assumptions are introduced in Section 2. The formula to evaluate the welfare impact of place-based policies is introduced in Section 3. Welfare analysis is conducted in Section 4. Section 5 concludes our paper.

### 4.2. Model

### 4.2.1. Basic assumptions

The model city is a closed city where $\bar{N}$ homogeneous consumers reside. This city consists of residential zones and marketplaces. Let $\mathcal{I} \equiv\{1, \ldots, I\}$ and $\mathcal{J} \equiv$ $\{1, \ldots, J\}$ denote the sets of the residential zones and marketplaces, respectively $(I, J>$ 2). Consumers reside in a residential zone and visit a marketplace for shopping. They can choose where to reside.

### 4.2.2. Consumers

We explain the utility and the budget constraint of consumers residing in residential zone $i(\in \mathcal{I})$ and visiting marketplace $j(\in \mathcal{J})$ for shopping. Consumers in the city derive utility from differentiated goods, housing measured in floor area, and a composite good. The utility of consumers residing in residential zone $i$ is given by $U_{i}\left(M_{i}, h_{i}, a_{i}\right)$, where $M_{i}$ is the composite index of the consumption of differentiated goods, $h_{i}$ is the consumption of housing measured by floor area, and $a_{i}$ is the consumption of the composite good. $M_{i}$ is assumed to be an additively separable function over the varieties (i.e., the differentiated goods) supplied in a marketplace:

$$
\begin{equation*}
M_{i}=\int_{0}^{m_{j}} u\left(q_{i}(k)\right) \mathrm{d} k \tag{4.1}
\end{equation*}
$$

where $q_{i}(k)$ is the consumption of the $k$ th variety and $m_{j}$ is the mass of varieties supplied in the $j$ th marketplace. ${ }^{36}$

The budget constraint of the consumers is given by

$$
\begin{equation*}
\int_{0}^{m_{j}} p_{j}^{M}(k) q_{i}(k) \mathrm{d} k+p_{i}^{H} h_{i}+a_{i}=y_{i} \tag{4.2}
\end{equation*}
$$

where $p_{j}^{M}(k)$ is the price of $k$ th variety supplied in the $j$ th marketplace, $p_{i}^{H}$ is the price per square foot of housing in residential zone $i$, and $y_{i}$ is the net income of consumers residing in residential zone $i$. The composite good is assumed to be the numéraire.

[^23]We assume public ownership of land and firms for simplicity. Consumers' net income $y_{i}$ is composed of common income $y$, travel cost to the marketplace $t_{i}$, equal share of profits and rents $\Pi$, and subsidy (or tax) $s_{i}(s): y_{i}=\widetilde{y}_{i}(s) \equiv y-t_{i}+\Pi+s_{i}(s)$. Each placebased policy determines $s_{i}(s)$ and $s(\in \mathbb{R})$ expresses the level of policy implemented. We call $s$ the policy instrument.

Our paper considers two place-based policies: location subsidies to stores, and location subsidies to consumers. ${ }^{37}$ Consumers (retail stores) in the same zone can receive the same amount of subsidy with these policies. Let $n_{i}$ and $s_{j}^{M}(s)$ denote the total number of consumers residing in residential zone $i$ and the total subsidy provided to retail stores in marketplace $j$, respectively. The formal definitions for the place-based policies are as follows.

Definition 4.1. Let $N_{\widehat{\mathcal{I}}}$ denote the total population in target area $\widehat{\mathcal{I}}(\subset \mathcal{I})$ (i.e., $\left.N_{\widehat{\mathcal{I}}}=\sum_{a \in \widehat{\mathcal{I}}} n_{a}\right)$. Location subsidies to consumers in target area $\widehat{\mathcal{I}}$ is the place-based policy such that the following equations hold.

$$
s_{i}(s)= \begin{cases}\left(\bar{N}-N_{\widehat{\mathcal{I}}}\right) s / N_{\widehat{\mathcal{I}}} & (i \in \widehat{\mathcal{I}}), \quad s_{j}^{M}(s)=0 \quad \forall j \in \mathcal{J} .  \tag{4.3}\\ -s & (i \notin \widehat{\mathcal{I}}),\end{cases}
$$

Definition 4.2. Location subsidies to stores is the place-based policy such that the following equations hold.

$$
s_{i}(s)=-s / \bar{N} \quad \forall i \in \mathcal{I}, \quad s_{j}^{M}(s)=\left\{\begin{array}{cc}
s & (j=1)  \tag{4.4}\\
0 & (j \neq 1)
\end{array}\right.
$$

Location subsidies to consumers implies that consumers residing in a target area are subsidized. Location subsidies to stores implies that retail stores operating in a

[^24]marketplace are subsidized. The subsidies with the policies are paid by consumers:
\[

$$
\begin{equation*}
\sum_{i \in \mathcal{I}} n_{i} s_{i}(s)+\sum_{j \in \mathcal{J}} s_{j}^{M}(s)=0 \tag{4.5}
\end{equation*}
$$

\]

In order to evaluate the welfare impacts of the place-based policies with Harberger's welfare change measurement formula (Harberger, 1971), we solve the following expenditure minimization problem:

$$
\begin{equation*}
\min _{\left\{q_{i}(k)\right\}_{k}, h_{i}, a_{i}} \int_{0}^{m_{j}} p_{j}^{M}(k) q_{i}(k) \mathrm{d} k+p_{i}^{H} h_{i}+a_{i} \quad \text { s.t. } \quad \text { Eq. (4.1) and } U_{i}=\bar{U}, \tag{4.6}
\end{equation*}
$$

where $\bar{U}$ denote the target utility. We decompose the expenditure minimization problem into two problems regarding two-stage budgeting. The conditional demands are functions of $k,\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}$, and $M_{i}$ :

$$
\begin{equation*}
q_{i}^{*}(k)=\widetilde{q_{i}^{*}}\left(\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}, M_{i}\right) \quad \forall k \in\left[0, m_{j}\right], \tag{4.7}
\end{equation*}
$$

where superscript "*" and tilde " $\bullet$ " denote the optimal solution and a function that maps arguments onto " •", respectively. We assume that all the consumers consume all the varieties (i.e., $\left.q_{i}^{*}(k)>0\left(\forall k \in\left[0, m_{j}\right]\right)\right)$. The demand functions are functions of $\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}, p_{i}^{H}$, and $\bar{U}$ :

$$
\begin{aligned}
& M_{i}^{*}=\widetilde{M_{i}^{*}}\left(\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}, p_{i}^{H}, \bar{U}\right), \\
& h_{i}^{*}=\widetilde{h_{i}^{*}}\left(\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}, p_{i}^{H}, \bar{U}\right), \\
& a_{i}^{*}=\widetilde{a_{i}^{*}}\left(\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}, p_{i}^{H}, \bar{U}\right) .
\end{aligned}
$$

Substituting $M_{i}^{*}$ into $q_{i}^{*}(k)$ yields

$$
\begin{equation*}
q_{i}^{*}(k)=\widetilde{q_{i}^{*}}\left(\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}, \widetilde{M_{i}^{*}}\left(\left\{p_{j}^{M}(k)\right\}_{k}, m_{j}, p_{i}^{H}, \bar{U}\right)\right) . \tag{4.8}
\end{equation*}
$$

See Appendix C.1.1 for detailed derivation of the demand functions.

### 4.2.3. Retail stores

Retail stores supply differentiated goods in marketplaces. Each retail store supplies a variety in a marketplace. They are under monopolistic competition. Hence, the total mass of retail stores in each marketplace is endogenously determined by free entry. They rent units of land in marketplaces.

All the retail stores incur the same marginal production cost $c$ to supply varieties. The retail store that supplies the $k$ th variety incurs $k+r_{j}(k)$ for the fixed cost, where $k$ also represents the fixed cost that depends on varieties, and $r_{j}(k)$ is land rent of a constant unit of land for a store. Some retail stores can receive subsidies, as shown in Definitions 4.1 and 4.2.

Let $Q_{j}(k)$ and $\pi_{j}^{M}(k)$ denote the supply of the $k$ th variety and the profit of the retail store supplying the $k$ th variety in marketplace $j$, respectively. $\pi_{j}^{M}(k)$ is given by

$$
\begin{equation*}
\pi_{j}^{M}(k)=\left(p_{j}^{M}(k)-c\right) Q_{j}(k)-k+\frac{s_{j}^{M}(s)}{m_{j}}-r_{j}(k) \quad \forall k \in\left[0, m_{j}\right] \tag{4.9}
\end{equation*}
$$

We assume that each store pays the bid rent. Using profit (4.9) yields the maximum land rent that each store can pay:

$$
\begin{equation*}
r_{j}(k)=\max _{p_{j}^{M}(k)}\left(\left(p_{j}^{M}(k)-c\right) Q_{j}(k)-k+\frac{s_{j}^{M}(s)}{m_{j}}\right) . \tag{4.10}
\end{equation*}
$$

Eq. (4.10) implies that the more demand for a variety in a marketplace, the larger the bid rent. Hence, if the prices of a variety supplied in some marketplaces are the same, then a retail store operating in a larger marketplace can propose a higher bid rent.

Since the total demand depends on the number of customers, we have to determine how many consumers visit each marketplace in order to solve maximization problem (4.10). Our paper focuses on an equilibrium such that all consumers in the same residential zone visit the same marketplace. In order to express such an equilibrium, we introduce market area. Let $\mathcal{I}_{j}(\subset \mathcal{I})$ denote the residential zones where consumers

(a) A line segment city with two symmetric marketplaces.

(b) A city with a large marketplace and a small marketplace.

Figure 4.1: Two examples of the model city with two marketplaces given market areas. Circles: residential zones in the city; Triangles: marketplaces in the city.
visit the $j$ th marketplace for shopping. ${ }^{38} \mathcal{I}_{j}$ is the market area represented by residential zones. ${ }^{39}$ Figure 4.1(a) and (b) represent examples of geographical patterns of residential zones and marketplaces with market areas. If each residential zone is small and the zones densely line up, shown in Figure 4.1, then we can interpret the geographical setting in our model as a discrete version of a continuous geographical space employed by most multipurpose shopping models (Tabuchi, 2009; Ushchev et al., 2015).

[^25]The total supply (or demand) is given by

$$
\begin{equation*}
Q_{j}(k)=\sum_{a \in \mathcal{I}_{j}} n_{a} q_{a}^{*}(k) \tag{4.11}
\end{equation*}
$$

The first order condition for maximization problem (4.10) is given by

$$
\begin{equation*}
\frac{Q_{j}(k)}{p_{j}^{M}(k)}\left(p_{j}^{M}(k)+\left(p_{j}^{M}(k)-c\right) \eta_{j}^{M}(k)\right)=0 \tag{4.12}
\end{equation*}
$$

where $\eta_{j}^{M}(k)$ is the price elasticity of the total demand: $\eta_{j}^{M}(k)=\partial \ln Q_{j}(k) / \partial \ln p_{j}^{M}(k)$. For simplicity, we focus on a symmetric price equilibrium ${ }^{40}$ such that the following holds:

$$
\begin{equation*}
\eta_{j}^{M}(k)=\eta_{j}^{M}\left(k^{\prime}\right) \quad \forall k, k^{\prime} \in\left[0, m_{j}\right] . \tag{4.13}
\end{equation*}
$$

We can obtain the above equation with subutility function $M_{i}$ employed in Section 4.4 (e.g., CES function). Using the first order condition (4.12) and Eq. (4.13) yields the prices of varieties supplied in marketplace $j$ :

$$
\begin{equation*}
p_{j}^{M}(k)=\widetilde{p_{j}^{M}}\left(\left\{n_{i}\right\}_{i \in \mathcal{I}_{j}}, m_{j},\left\{p_{i}^{H}\right\}_{i \in \mathcal{I}_{j}}, \bar{U}\right) \quad \forall k \in\left[0, m_{j}\right] . \tag{4.14}
\end{equation*}
$$

Since the prices do not depend on $k$, we express $p_{j}^{M}(k)$ as $p_{j}^{M}$. Furthermore, under the symmetric price equilibrium, the total demand for varieties supplied in the same marketplace are the same: $Q_{j}(k)=Q_{j}\left(k^{\prime}\right)\left(\forall k, k^{\prime} \in\left[0, m_{j}\right]\right)$. Hence, we express $Q_{j}(k)$ as $Q_{j}$.

### 4.2.4. Developers

Developers are assumed to be perfectly competitive and homogeneous. They supply residential buildings in residential zones.

Following Brueckner (2007) and Domon et al. (2022), we specify developers as follows. Buildings are produced by combining one unit of land and housing capital (or

[^26]building materials). The area of land in each residential zone is assumed to be one unit. The building output per unit of land is expressed as $g(b)$, where $g$ is the housing production function and $b$ is the capital-to-land ratio. Let $\pi_{i}^{H}$ and $H_{i}$ denote the developers' net profit in residential zone $i$ and the housing output, respectively. $\pi_{i}^{H}$ is given by
\[

$$
\begin{equation*}
\pi_{i}^{H}=p_{i}^{H} H_{i}-g^{-1}\left(H_{i}\right)-R_{i}^{H}, \tag{4.15}
\end{equation*}
$$

\]

where $g^{-1}$ is the inverse function of $g$ and $R_{i}^{H}(i \in \mathcal{I})$ is the total land rent in residential zone $i$.

We assume that developers pay the bid land rent. Using profit (4.15) yields the maximum land rent that developers can pay:

$$
\begin{equation*}
R_{i}^{H}=\max _{H_{i}}\left(p_{i}^{H} H_{i}-g^{-1}\left(H_{i}\right)\right) . \tag{4.16}
\end{equation*}
$$

The first order condition for maximization problem (4.16) is

$$
\begin{equation*}
p_{i}^{H}-\frac{\partial g^{-1}\left(H_{i}\right)}{\partial H_{i}}=0 \quad \forall i \in \mathcal{I} \tag{4.17}
\end{equation*}
$$

Using this condition, we can obtain $H_{i}^{*}=\widetilde{H_{i}^{*}}\left(p_{i}^{H}\right)$. Hence, the bid rent is expressed as

$$
\begin{equation*}
R_{i}^{H}=p_{i}^{H} H_{i}^{*}-g^{-1}\left(H_{i}^{*}\right) \quad \forall i \in \mathcal{I} \tag{4.18}
\end{equation*}
$$

### 4.2.5. Market equilibrium condition

We introduce market equilibrium condition. In the equilibrium, given the spatial distribution of consumers (i.e., $\left.\left(n_{i}\right)_{i \in \mathcal{I}}\right)$, the market clearing condition of housing holds and the mass of retail stores is determined.

The market clearing condition for housing is given by

$$
\begin{equation*}
\widetilde{H_{i}^{*}}\left(p_{i}^{H}\right)=n_{i} \widetilde{h_{i}^{*}}\left(p_{j(i)}^{M}, m_{j(i)}, p_{i}^{H}, \bar{U}\right) \quad \forall i \in \mathcal{I}, \tag{4.19}
\end{equation*}
$$

where $j(i)(\in \mathcal{J})$ denotes the marketplace that consumers in residential zone $i$ visit for shopping. ${ }^{41}$ Next, we focus on the mass of retail stores (i.e., $m_{j}$ ). Since $p_{j}^{M}$ and $Q_{j}$ do not depend on $k,\left(p_{j}^{M}-c\right) Q_{j}+s_{j}^{M}(s) / m_{j}$ also does not depend on $k$. Land rent $r_{j}(k)$, shown by Eq. (4.10), monotonously decreases with an increase in $k$. Using this monotonicity, $r_{j}(k) \geq 0\left(\forall k \in\left[0, m_{j}\right]\right)$, and Eq. (4.10), we obtain the following condition for the mass of stores $m_{j}$ :

$$
\begin{equation*}
r_{j}\left(m_{j}\right)=\left(p_{j}^{M}-c\right) Q_{j}-m_{j}+\frac{s_{j}^{M}(s)}{m_{j}}=0 \quad \forall j \in \mathcal{J} \tag{4.20}
\end{equation*}
$$

Eq. (4.20) implies that sales are equals to the cost for the store supplying variety $m_{j}$.
Let $\boldsymbol{n} \equiv\left(n_{i}\right)_{i \in \mathcal{I}}$ denote the spatial distribution of the consumers in the city. The total number of equations, which are Eqs. (4.19) and (4.20), is equal to that of endogenous variables, which are $m_{j}$ and $p_{i}^{H}$. Using these equations, we can obtain these variables as functions of spatial distribution $\boldsymbol{n}$, target utility $\bar{U}$, and policy instrument $s$ :

$$
m_{j}=\widetilde{m_{j}}(\boldsymbol{n}, \bar{U}, s), \quad p_{i}^{H}=\widetilde{p_{i}^{H}}(\boldsymbol{n}, \bar{U}, s) .
$$

Substituting these functions into $\widetilde{p_{j}^{M}}$ (i.e., Eq. (4.14)), we obtain $\widetilde{p_{j}^{M}}$ as a function of $\boldsymbol{n}, \bar{U}$, and $s$. Since the prices and the mass are functions of $\boldsymbol{n}, \bar{U}$, and $s$, the demand functions are also functions of $\boldsymbol{n}, \bar{U}$, and $s$ in the equilibrium.

### 4.3. Marginal welfare impacts of place-based policies

### 4.3.1. Allais surplus

We investigate the welfare impact of place-based policies. In this paper, we measure the welfare impact in terms of the Allais surplus (Allais, 1977). The Allais surplus is

[^27]defined as a surplus of goods that can be taken up with a policy to keep the utility levels constant. There are two advantages of employing the Allais surplus when we evaluate welfare impact. First, we can evaluate the welfare impact in terms of the compensation criterion. Second, we can interpret the welfare impact in terms of distortions generated by market failure; our paper focuses on this advantage.

Following Wheaton (1977) and Kono and Kishi (2018), we define the Allais surplus with population migration. The Allais surplus is the weighted sum of income minus the expenditure function of consumers with the weights being the number of consumers. We obtain the surpluses that are equal among the residential zones. ${ }^{42}$

The substructed income of consumers residing in zone $i$ with the expenditure is given by $y_{i}-e_{i}=y-t_{i}+\Pi+s_{i}(s)-e_{i}$, where $e_{i}$ is the expenditure function of consumers residing zone $i$. Using the assumption of the public ownership, we obtain equal share of profits and rents $\Pi$ :

$$
\begin{equation*}
\Pi=\bar{N}^{-1}\left(\sum_{i \in \mathcal{I}}\left(\pi_{i}^{H}+R_{i}^{H}\right)+\sum_{j \in \mathcal{J}}\left(\int_{0}^{m_{j}} \pi_{j}^{M}(k) \mathrm{d} k+\int_{0}^{m_{j}} r_{j}(k) \mathrm{d} k\right)\right) \tag{4.21}
\end{equation*}
$$

Substituting Eqs. (4.9) and (4.18) into the above $\Pi$ yields

$$
\begin{equation*}
\Pi=\bar{N}^{-1}\left(\sum_{i \in \mathcal{I}}\left(p_{i}^{H} H_{i}^{*}-g^{-1}\left(H_{i}^{*}\right)\right)+\sum_{j \in \mathcal{J}}\left(\left(p_{j}^{M}-c\right) Q_{j} m_{j}-\frac{m_{j}^{2}}{2}+s_{j}^{M}(s)\right)\right) . \tag{4.22}
\end{equation*}
$$

The condition for the equal surpluses among the residential zones is given by

$$
\begin{equation*}
y-t_{i}+\Pi+s_{i}(s)-e_{i}=\bar{E} \quad \forall i \in \mathcal{I}, \tag{4.23}
\end{equation*}
$$

where $\bar{E}(\in \mathbb{R})$ is the surplus level in each residential zone. Moreover, population constraint condition holds in the closed city:

$$
\begin{equation*}
\sum_{i \in \mathcal{I}} n_{i}=\bar{N} \tag{4.24}
\end{equation*}
$$

[^28]Using Eqs. (4.23) and (4.24), we can obtain spatial distribution $\boldsymbol{n}$ as a function of policy instrument $s: \boldsymbol{n}=\boldsymbol{n}(s)$.

Let $A S$ denote the Allais surplus. We can obtain $A S$ as a function of $s$, keeping the utility level at $\bar{U}$ :

$$
\begin{equation*}
A S(\boldsymbol{n}(s), s, \bar{U})=\sum_{i \in \mathcal{I}} n_{i}\left(y-t_{i}+\Pi+s_{i}(s)-e_{i}\right)=\bar{N} \times \bar{E} \tag{4.25}
\end{equation*}
$$

### 4.3.2. Marginal change in Allais surplus $A S$ with a change in policy instrument $s$

Investigating marginal change in $A S$ with an increase in policy instrument $s$, we evaluate the welfare impact of adopting a place-based policy. In order to evaluate the welfare impact, we need to determine target utility level $\bar{U}$. We set the target utility level such that all the consumers maximize their utility under the equilibrium prices with no policy. We focus on target utility level $\overline{U^{*}}$ and spatial distribution $\boldsymbol{n}^{*}$ such that following equation holds:

$$
\begin{equation*}
y-t_{i}+\Pi+s_{i}(0)-e_{i}=0 \quad \forall i \in \mathcal{I} \tag{4.26}
\end{equation*}
$$

which is condition (4.23) at $s=0$ and $\bar{E}=0$. This condition implies that under the equilibrium prices and $\overline{U^{*}}$, all the consumers maximize their utility because expenditure $e_{i}$ equals net income $y_{i}$ (See Eq. (4.2)).

We focus on marginal change in $A S$ from $\left(\boldsymbol{n}^{*}, \overline{U^{*}}\right)$ with an increase in $s$ from zero along the equilibrium path. We can obtain $\mathrm{d} A S / \mathrm{d} s$ with the market distortions generated by monopolistic competition and place-based policies, which is consistent with Harberger's welfare change measurement formula (Harberger, 1971).

Lemma 4.1. For any given market area $\left\{\mathcal{I}_{j}\right\}_{j \in \mathcal{J}}$, the following holds for $s \geq 0$ :

$$
\begin{equation*}
\frac{\mathrm{d} A S}{\mathrm{~d} s}=P D H+P D+V D+F D \tag{4.27}
\end{equation*}
$$

where

$$
\begin{aligned}
& P D H=\sum_{i \in \mathcal{I}}[(\underbrace{p_{i}^{H}-\frac{\partial g^{-1}}{\partial H_{i}}}_{=0}) \frac{\mathrm{d} H_{i}^{*}}{\mathrm{~d} s}], \quad P D \equiv \sum_{j \in \mathcal{J}} \underbrace{\left(p_{j}^{M}-c\right) m_{j}}_{\geq 0} \frac{\mathrm{~d} Q_{j}}{\mathrm{~d} s} \\
& V D \equiv \sum_{j \in \mathcal{J}}(\underbrace{\sum_{a \in \mathcal{I}_{j}}\left(\frac{n_{a} p_{j}^{M} u\left(q_{a}\right)}{u^{\prime}\left(q_{a}\right)}\right)-c Q_{j}-m_{j}}_{\geq 0}) \frac{\mathrm{d} m_{j}}{\mathrm{~d} s}, \quad F D=\sum_{i \in \mathcal{I}}\left(-s_{i}\right) \frac{\mathrm{d} n_{i}}{\mathrm{~d} s} .
\end{aligned}
$$

Note that the variables in these equations are obtained with the Hicksian demands.

Proof. See Appendix C.1.2.

Eq. (4.27) shows that $\mathrm{d} A S / \mathrm{d} s$ is decomposed into four parts. ${ }^{43} P D_{H}$ and $P D$ express the total change in deadweight losses in the housing markets and the differentiated goods markets, respectively. $V D$ is caused by so-called variety distortion (Kanemoto, 2013a,b; Behrens et al., 2015). FD is caused by uneven income transfer among residential zones with a place-based policy. ${ }^{44}$ FD indicates that place-based policies distort market allocation and decrease surplus. For example, if population in residential zone $i$ where consumers can receive subsidy increases by a place-based policy, then the city loses $s_{i} \times \mathrm{d} n_{i} / \mathrm{d} s$ of surplus.

We can decompose $P D$ and $V D$ into two effects, which are employed in Section 4.4. One is the effect generated by population migration, whereas the other is the effect generated by only subsidy. Let $E P$ and $E S$ denote the former effect and the latter

[^29]effect, respectively. Using $P D$ and $V D$, we can express $E P$ and $E S$ as
$$
E P=P D_{P}+V D_{P}, \quad E S=P D_{S}+V D_{S}
$$
where
\[

$$
\begin{align*}
P D_{P} & =\sum_{j \in \mathcal{J}}\left[\left(p_{j}^{M}-c\right) m_{j} \sum_{a \in \mathcal{I}} \frac{\partial Q_{j}}{\partial n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} s}\right],  \tag{4.28}\\
P D_{S} & =\sum_{j \in \mathcal{J}}\left[\left(p_{j}^{M}-c\right) m_{j} \frac{\partial Q_{j}}{\partial s}\right],  \tag{4.29}\\
V D_{P} & =\sum_{j \in \mathcal{J}}\left[\left(\sum_{a \in \mathcal{I}_{j}}\left(\frac{n_{a} p_{j}^{M} u\left(q_{a}\right)}{u^{\prime}\left(q_{a}\right)}\right)-c Q_{j}-m_{j}\right)\left(\sum_{a \in \mathcal{I}} \frac{\partial m_{j}}{\partial n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} s}\right)\right],  \tag{4.30}\\
V D_{S} & =\sum_{j \in \mathcal{J}}\left(\sum_{a \in \mathcal{I}_{j}}\left(\frac{n_{a} p_{j}^{M} u\left(q_{a}\right)}{u^{\prime}\left(q_{a}\right)}\right)-c Q_{j}-m_{j}\right) \frac{\partial m_{j}}{\partial s} . \tag{4.31}
\end{align*}
$$
\]

We can interpret $V D$ as follows. Using one of the first-order conditions for utility maximization (4.6), we can interpret $u^{\prime}\left(q_{a}\right) / p_{j}^{M}\left(a \in \mathcal{I}_{j}\right)$ as the marginal utility of shopping expenditure. This interpretation implies that $p_{j}^{M} u\left(q_{a}\right) / u^{\prime}\left(q_{a}\right)$ is the benefit that a consumer can obtain by consuming an additional variety of goods supplied in marketplace $j$. Hence, the first term of $V D$ is the total benefit that the consumers in the city can obtain. Furthermore, since $c Q_{j}+m_{j}$ is the cost that new retail stores entering in marketplace $j$ must incur, the second term is the total cost caused by a place-based policy. That is, we can interpret $V D$ as the total benefit subtracted by the total cost.

Since developers are under perfect competition, the price of housing and the marginal cost are the same. This implies $P D H=0 ; \mathrm{d} A S / \mathrm{d} s$ is composed of $P D, V D$, and $F D$. This equation is similar to welfare change measurement formulae with monopolistic competition (Kanemoto, 2013a,b; Behrens et al., 2015). In contrast to these studies, we focus on income transfer among consumers and retail stores by a place-based policy. FD, which does not appear in Kanemoto (2013a,b) and Behrens et al. (2015), is
added to the welfare change measurement formula because our model takes account of place-based policies generating migration fiscal distortions.

To explore welfare analyses of policies, we explain the signs of coefficients in $P D$ and $V D$. Since each retail store operating in a marketplace supplies a good at a price larger than marginal cost $c$, we have $\left(p_{j}^{M}-c\right) m_{j} \geq 0(\forall j \in \mathcal{J})$.

We can obtain the sign of the coefficient of $\mathrm{d} m_{j} / \mathrm{d} s$ as follows. Using the love of variety condition (e.g., Behrens and Murata, 2007; Behrens et al., 2015) yields $u\left(q_{i}\right) / u^{\prime}\left(q_{i}\right) \geq q_{i}(\forall i \in \mathcal{I})$. Using this inequality, the definition of total demand, and Eq. (4.20) yields

$$
\begin{equation*}
\sum_{a \in \mathcal{I}_{j}}\left(\frac{n_{a} p_{j}^{M} u\left(q_{a}\right)}{u^{\prime}\left(q_{a}\right)}\right)-c Q_{j}-m_{j} \geq p_{j}^{M} Q_{j}-c Q_{j}-m_{j}=0 \tag{4.32}
\end{equation*}
$$

The signs of the coefficients of $\mathrm{d} Q_{j} / \mathrm{d} s$ and $\mathrm{d} m_{j} / \mathrm{d} s$ are non-negative, whereas those of $\mathrm{d} Q_{j} / \mathrm{d} s$ and $\mathrm{d} m_{j} / \mathrm{d} s$ depend on place-based policies. We can intuitively predict that policies promoting marketplace $j_{1}$ generate $\mathrm{d} Q_{j_{1}} / \mathrm{d} s, \mathrm{~d} m_{j_{1}} / \mathrm{d} s>0$ and $\mathrm{d} Q_{j} / \mathrm{d} s, \mathrm{~d} m_{j} / \mathrm{d} s \leq 0\left(j \neq j_{1}\right)$. We, however, cannot determine $\mathrm{d} A S / \mathrm{d} s>0$ for such a policy since all the coefficients are non-negative. Hence, the welfare impact of a place-based policy depends on how we specify the utility function and the place-based policy. Specifying the utility function in Section 4.4, we investigate the welfare impact of place-based policies.
4.4. Welfare analysis of place-based policies with the constant elasticity of substitution and the variable elasticity of substitution cases

We evaluate the welfare impact of adopting place-based policies. We focus on two place-based policies shown in Definitions 4.1 and 4.2 (i.e., location subsidies to consumers and location subsidies to stores).

### 4.4.1. Model specification with the constant elasticity of substitution

Specifying the utility function and the housing production function, we demonstrate how place-based policies improve social welfare with the Allais surplus defined in Section 4.3.1.

Most multipurpose shopping models in which retail stores are under monopolistic competition represent consumers' love of variety with constant elasticity of substitution (CES) function (e.g., Henkel et al., 2000; Tabuchi, 2009; Ushchev et al., 2015). We evaluate the welfare impact with the following utility function:

$$
\begin{equation*}
U_{i}=\frac{\sigma \mu}{\sigma-1} \ln M_{i}+(1-\mu) \ln h_{i}+a_{i}, \quad 0<\mu<1 \tag{4.33}
\end{equation*}
$$

where $M_{i}=\int_{0}^{m_{j}} q_{j}(k)^{(\sigma-1) / \sigma} \mathrm{d} k . \sigma$ and $\mu$ are the elasticities of substitution between any two varieties and the shopping expenditure, respectively. ${ }^{45}$ In addition to the above specification for consumers' preference, we specify the housing production function employed by urban economics models (e.g., Brueckner, 2007; Kono et al., 2019; Domon et al., 2022):

$$
\begin{equation*}
g(b)=\theta b^{\beta} \quad(0<\theta, 0<\beta<1) . \tag{4.34}
\end{equation*}
$$

## Properties of $\mathrm{d} A S / \mathrm{d} s$

We show properties of $\mathrm{d} A S / \mathrm{d} s$ with the specification in order to discuss the welfare impacts of the place-based policies. We can obtain the variables to express $E P$ and $E S$ with the market equilibrium conditions (see Appendix C.2.1 for the derivation):

$$
\begin{align*}
m_{j} & =\left(\frac{\mu}{\sigma} \sum_{a \in \mathcal{I}_{j}} n_{a}+s_{j}^{M}(s)\right)^{1 / 2} \quad \forall j \in \mathcal{J}  \tag{4.35}\\
q_{i}^{*} & =\mu\left(p^{M} m_{j(i)}\right)^{-1} \quad \forall i \in \mathcal{I}  \tag{4.36}\\
Q_{j} & =\sum_{a \in \mathcal{I}_{j}} n_{a} q_{a}^{*} \quad \forall j \in \mathcal{J} \tag{4.37}
\end{align*}
$$

[^30]where $p^{M}=c \sigma /(\sigma-1)$. The following lemma holds with the above model specification.

Lemma 4.2. If the utility function and the production function are expressed by (4.33) and (4.34) respectively, then $P D_{P}=V D_{P}=0$ holds at the market equilibrium (i.e., $(\boldsymbol{n}, s, \bar{U})=\left(\boldsymbol{n}^{*}, 0, \overline{U^{*}}\right)$ ).

Proof. See Appendix C.2.3.

Lemma 4.2 shows that $P D=P D_{S}, V D=V D_{S}$, and $\mathrm{d} A S / \mathrm{d} s=E S$ hold. This result would be obtained because all the retail stores in the city supply varieties at the same price. Such pricing is caused when we assume the CES preference because the CES preference causes the price elasticity of the total demand to be constant (i.e., $\sigma$ ).

## Location subsidies to consumers

We focus on location subsidies to consumers. If a place-based policy does not subsidize retail stores, then mass of variety $m_{j}$ is not affected by policy instrument $s$ (see Eq. (4.35)). Hence, $\mathrm{d} A S / \mathrm{d} s=E S=0$ holds for any location subsidies to consumers. This result indicate $A S$ at $(\boldsymbol{n}, s, \bar{U})=\left(\boldsymbol{n}^{*}, 0, \overline{U^{*}}\right)$ is locally maximized.

Formulating a maximization problem for $A S$, we examine whether or not $A S$ is locally maximized for place-based policies that do not generate $E S$. The maximization problem of $A S$ is defined as follows:

$$
\begin{array}{ll}
\max _{\boldsymbol{n}} & A S  \tag{4.38}\\
\text { s.t. } & \gamma_{i}(\boldsymbol{n}) \equiv-n_{i} \leq 0 \quad(i \in \mathcal{I}), \quad \Gamma(\boldsymbol{n}) \equiv \bar{N}-\sum_{i \in \mathcal{I}} n_{i}=0 .
\end{array}
$$

We analyze this maximization problem with the Karush-Kuhn-Tucker (KKT) condition. The results regarding the first-order necessary conditions and the second-order sufficient conditions are as follows.

Lemma 4.3. At the market equilibrium (i.e., $(\boldsymbol{n}, s, \bar{U})=\left(\boldsymbol{n}^{*}, 0, \overline{U^{*}}\right)$ ), $\boldsymbol{n}^{*}$ satisfies the KKT conditions of maximization problem (4.38).

## Proof. See Appendix C.2.3.

Lemma 4.4. At the market equilibrium (i.e., $(\boldsymbol{n}, s, \bar{U})=\left(\boldsymbol{n}^{*}, 0, \overline{U^{*}}\right)$ ), $\boldsymbol{n}^{*}$ satisfies the second-order sufficient conditions of maximization problem (4.38) if $\mu /(1-\mu)<$ $2(\sigma-1)(1-\beta)$.

Proof. See Appendix C.2.3.

Even though there is a market failure generated by monopolistic competition (i.e., imperfect competition), Lemma 4.4 implies that any inner market equilibrium is locally maximized if the expenditure share of differentiated goods and housing is lower than $2(\sigma-1)(1-\beta)$. Low $\sigma$ implies that consumers love variety, whereas high $\beta$ implies that developers are more productive. Lemma 4.4 implies that any policy that generates population migration (i.e., change in $\boldsymbol{n}$ ), as well as location subsidies to consumers, decrease the Allais surplus. For example, adopting land-use regulation decreases the Allais surplus. We restate Lemma 4.4 in the following proposition:

Proposition 4.1. The inner market equilibria are locally efficient regarding the spatial distribution of consumers, even though there are price distortions and the variety distortions generated by monopolistic competition.

Proposition 4.1 is similar to one of the results shown by Dhingra and Morrow (2019). Dhingra and Morrow (2019) compare the allocation at market equilibrium with that at the socially optimal state in an economy that consists of workers and firms under monopolistic competition. In particular, they show that if workers' demands for varieties are expressed by the CES preference, then the allocation at the market equilibrium is socially optimal in a non-space economy with no migration. We show that, in an economy with population migration and monopolistic competition, allocations determined by $\boldsymbol{n}$ at the equilibrium are locally efficient, in contrast to the results of Dhingra and Morrow.

Regarding Proposition 4.1, we should note that at the equilibrium, mass of variety $m_{j}$ is determined by $\boldsymbol{n}$ (see Eq. (4.35)). Since policymakers can choose the level of the mass, Proposition 4.1 does not ensure that the equilibrium is first-best, which is a difference between Proposition 4.1 and the results of Dhingra and Morrow (2019). For example, place-based policies that generate positive direct benefit (i.e., $M S>0$ ) increase the Allais surplus. ${ }^{46}$

## Location subsidies to stores

We explore location subsidies to stores. This place-based policy is an example in which $E S \neq 0$ could hold. Using Eq. (4.4) in Definition 4.2 and Eq. (4.35) yields $\partial m_{1} / \partial s \neq 0$. Furthermore, we obtain the following result.

Lemma 4.5. At the market equilibrium (i.e., $(\boldsymbol{n}, s, \bar{U})=\left(\boldsymbol{n}^{*}, 0, \overline{U^{*}}\right)$ ), the following holds.

$$
\frac{\mathrm{d} A S}{\mathrm{~d} s}=\underbrace{P D}_{<0}+\underbrace{V D}_{>0}>0 .
$$

Proof. See Appendix C.2.3.

Lemma 4.5 shows that adopting location subsidies to stores marginally increases the Allais surplus. While the deadweight loss generated by the price distortion decreases $A S$, the total net benefit generated by the variety distortion exceeds the loss.

In this subsection, we have shown that subsidizing retail stores operating in a marketplace (e.g., the downtown area in a city) is desirable from the viewpoint of welfare, whereas subsidizing consumers residing near the marketplace is harmful. Hence, whether or not place-based policies are socially efficient depends on the recipients of the subsidies, even if the policies promote retail agglomeration in the downtown area.

[^31]
### 4.4.2. Numerical examples

Conducting numerical analysis of the equilibrium and the Allais surplus on the equilibrium for $s \geq 0$, we demonstrate how the surplus changes on the equilibrium. We consider the model city shown in Figure 4.1(b). That is, this city consists of the downtown area and the suburb. The downtown area and the suburb have one marketplace (i.e., $J=2$ ). There are more residential zones in the downtown area than the zones in the suburb. We represent the assumption as $\mathcal{I}=\{1,2, \ldots, 8\}$, $\mathcal{I}_{1}=\{1,2, \ldots, 5\}$, and $\mathcal{I}_{2}=\{6,7,8\}$. The travel costs to the marketplaces are the same: $t_{i}=10(\forall i \in \mathcal{I})$. We set common income of consumers $y$ at 1000 . Hence, $1 \%$ of the common income is the travel cost to the marketplace. $\bar{N}$ is set at $1 ; n_{i}$ is interpreted as the ratio to the total population in the city.

## How the place-based policies change the Allais surplus

We conduct numerical analysis with utility function (4.33) and production function (4.34). There are five exogenous parameters: $\theta, \beta, \mu, \sigma$, and $c$. Referring to the empirical results shown by Domon et al. (2022), we set $\theta$ and $\beta$ at 0.0028 and 0.75 , respectively. We set $\mu$ at 0.4 , which means that the ratio of the shopping expenditure to the housing expenditure is about $66 \% . \sigma$ and $c$ are set at 6.0 and 1.0, respectively.

We numerically evaluate the two place-based policies defined in Section 4.2. One is location subsidies to consumers residing in the downtown area:

$$
s_{i}(s)=\left\{\begin{array}{ll}
\left(\left(\sum_{a \in \mathcal{I}_{1}} n_{a}\right)^{-1}-1\right) s & \left(i \in \mathcal{I}_{1}\right),  \tag{4.39}\\
-s & \left(i \in \mathcal{I}_{2}\right),
\end{array} \quad s_{j}^{M}(s)=0 \quad \forall j \in \mathcal{J}\right.
$$

which is the case for $\widehat{\mathcal{I}}=\mathcal{I}_{1}$ in Definition 4.1. The other is location subsidies to stores operating in the downtown area:

$$
s_{i}(s)=-s, \quad s_{j}^{M}(s)= \begin{cases}s & (j=1)  \tag{4.40}\\ 0 & (j=2)\end{cases}
$$

which is the case for $J=2$ in Definition 4.2.


Figure 4.2: Population and the Allais surplus on the equilibria with utility function (4.33) and production function (4.34) . Left: population in residential zone 1. Right: the Allais surplus. Red dashed-dotted line: the result obtained for policy function (4.40); blue dashed line: the result obtained for policy function (4.39).

We investigate the equilibrium and the Allais surplus for $0 \leq s \leq 10$. Figure 4.2 shows the population in residential zone 1 (i.e., $n_{1}$ ) and the Allais surplus (i.e., $A S$ ), which changes as policy instrument $s$ changes. The Allais surplus increases with the place-based policy expressed by Eq. (4.40) and decreases with the place-based policy expressed by Eq. (4.39) from $s=0$. Both results are consistent with the theoretical results shown in Section 4.4 .1 (i.e., Lemmas 4.4 and 4.5). We also check that the Allais surplus monotonously decreases for $0.04 \leq s \leq 10$. In order to clearly show that the Allais surplus increases for location subsidies to the retail stores, the results for the range are not shown.

## Relaxing the assumption regarding the elasticity of substitution between varieties

We show the theoretical results under the constant elasticity of substitution between varieties in Section 4.4.1. In this section, relaxing this assumption, we explore how the welfare impacts of adopting the place-based policies change. We employ the Constant Absolute Risk Aversion (CARA) utility function as a utility function that represents the


Figure 4.3: Population and the Allais surplus on the equilibria with utility function (4.41) and production function (4.34). Left: population in residential zone 1. Right: the Allais surplus. Red dashed-dotted line: the result obtained for policy function (4.40); blue dashed line: the result obtained for policy function (4.39).
variable elasticity of substitution between varieties. Behrens and Murata (2007) show that a pro-competitive effect emerges when we employ this function. In our model, the pro-competitive effect implies that the price of varieties in a marketplace $p_{j}^{M}$ decreases with an increase in the mass of the varieties $m_{j}$.

We investigate the welfare impacts of adopting the place-based policies with the following utility function:

$$
\begin{equation*}
U_{i}=\mu_{1} \ln M_{i}+\mu_{2} \ln h_{i}+\mu_{3} \ln a_{i} \tag{4.41}
\end{equation*}
$$

where $\mu_{1}, \mu_{2}, \mu_{3}>0, \mu_{1}+\mu_{2}+\mu_{3}=1$, and

$$
\begin{equation*}
M_{i}=\int_{0}^{m_{j}} 1-\alpha \exp \left(-\omega q_{i}(k)\right) \mathrm{d} k, \quad \alpha, \omega>0 \tag{4.42}
\end{equation*}
$$

We employ the same production function (i.e., Eq. (4.34)). Because the specification is so intractable that we cannot obtain even the closed forms of the expenditure function and the indirect utility, we resort to conducting only numerical analysis for the welfare impacts. The set of parameters $\alpha=1, \omega=1, \mu_{1}=0.1, \mu_{2}=0.3$, and $\mu_{3}=0.6$ is employed to investigate the welfare impacts.

Table 4.1: The composition of $\mathrm{d} A S / \mathrm{d} s$.

|  | $P D$ |  |  | $V D$ |  |  | Total |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $P D_{P}$ | $P D_{S}$ | Total | $V D_{P}$ | $V D_{S}$ | Total |  |
| Location subsidies to stores | -0.00570 | -0.0450 | -0.0507 | -0.00536 | 0.321 | 0.316 | 0.27 |
| Location subsidies to consumers | -0.00645 | 0 | -0.00645 | -0.00607 | 0 | -0.00607 | -0.013 |

Notes: $P D_{P}, P D_{S}, V D_{P}$, and $V D_{S}$ are given by Eqs. (4.28)-(4.31). For the location subsidies to consumers, policy instrument $s$ does not affect the Hicks demands and the equilibrium conditions; this value affects only population migration. Hence, we obtain $P D_{S}=V D_{S}=0$ for the policy.

Figure 4.3 shows population in residential zone 1 and the Allais surplus on the equilibria with the utility function and the production function for $0 \leq s \leq 10$. The Allais surplus increases with the place-based policy expressed by Eq. (4.40) and decreases with the place-based policy expressed by Eq. (4.39) from $s=0$. Hence, these results are qualitatively the same as the CES case.

Table 4.1 shows the composition of $\mathrm{d} A S / \mathrm{d} s$ at the market equilibrium for $s=0$. Table 4.1 shows that $P D_{P}$ and $V D_{P}$ are negative for both policies unlike the CES case (Lemma 4.2). With location subsidies to consumers, both welfare changes generated by the price and the variety distortions are negative. With location subsidies to stores, the former and the latter are negative and positive, respectively, and the latter exceeds the former.

### 4.5. Conclusion

We have evaluated how place-based policies affect social welfare. Conducting theoretical analyses with constant and variable elasticity of substitution between varieties supplied in marketplaces, we obtain two main findings: (1) subsidizing retail stores operating in downtown areas is desirable from the viewpoint of welfare, and (2) sub-
sidizing consumers residing near the downtown areas is harmful. The main reason for the difference is the level of variety distortion generated by a place-based policy. Since directly subsidizing retail stores generates a positive net benefit with the variety distortion, we obtain these results. Furthermore, we have shown that adopting policies that change the spatial distribution of consumers (e.g., land-use regulation) is harmful with the constant elasticity of substitution, even though there are market distortions generated by monopolistic competition.

Our model can be extended in the following manner. By developing a structural model that expresses the agglomeration of retail stores in marketplaces, we can quantitatively evaluate the benefit of place-based policies. For example, combining the so-called Quantitative Spatial Economics model (e.g., Redding and Rossi-Hansberg, 2017) and our multipurpose shopping model will enable us to evaluate such a benefit.

## 5. Quantitative Welfare Analyses of Place-based Policies

### 5.1. Introduction

Place-based policies have been applied worldwide. Empirical research to evaluate the policy impacts of place-based policies has been performed. ${ }^{47}$ For example, Iwata and Kondo (2021) investigate the policy impact of a place-based policy applied in Toyama, which aims to revitalize the downtown area. However, little attention has been given to the welfare evaluation of place-based policies for retail agglomeration.

In this chapter, we aim to quantitatively evaluate the welfare impacts of placebased policies to agglomerate retail stores in the downtown area of a city. Building the multipurpose shopping model developed in Chapter 4, we quantitatively evaluate the following place-based policies:

- Location subsidies to households residing in the downtown area.
- Location subsidies to retail stores operating in the downtown area.

The quantitative results show that the welfare impacts are qualitatively the same as the theoretical results shown in Chapter 4.

This chapter relates to quantitative research that focuses on what drives retail agglomeration (e.g., Davis, 2006; Koster et al., 2019). For example, Koster et al. (2019) show the existence of shopping externality with the data for the number of pedestrians that pass shops in shopping streets. They evaluate the welfare impact measured by the profits of retail stores of a retail policy that subsidizes retail stores. In contrast to Koster et al. (2019), we focus on households' location choices as well as the stores' location choices in order to evaluate the welfare impacts of place-based policies. Our welfare evaluation, moreover, is based on a general equilibrium framework.

[^32]Some quantitative studies evaluate the welfare impacts of place-based policies (e.g., Busso et al., 2013) with quantitative spatial equilibrium models. These studies focus on spatial consumers' commuting and residing patterns in order to evaluate the welfare impact of policies to promote the growth of a business area. In contrast to these studies, we focus on spatial households' residing patterns affected by retail agglomeration in order to evaluate retail place-based policies. In particular, we focus on the agglomeration economy driven by the agglomeration of retail stores in marketplaces.

The rest of this chapter is organized as follows. Basic assumptions are introduced in Section 5.2. The results of the quantitative analyses are shown in Section 5.3. Section 5.4 concludes this chapter.

### 5.2. Model

### 5.2.1. Basic assumptions

The basic structure of our model is as follows. We consider a closed city where homogeneous $\bar{N}$ households reside. This city consists of the downtown area and homogeneous suburbs. Let $\mathcal{I} \equiv\{0,1,2, \ldots, I\}$ denote the set of residential zones. $I$ is the number of the suburbs in the city. We regard residential zone 0 as the downtown area and zone $i(=1, \ldots, I)$ as the suburb. There is one marketplace in each residential zone. In the marketplaces, retail stores supply goods; households choose where to reside.

### 5.2.2. Households

Households in the city derive utility from differentiated goods, housing measured in floor area, and a composite good. The utility of households residing in residential zone $i(\in \mathcal{I})$ is given by

$$
\begin{equation*}
U_{i}\left(M_{i}, h_{i}, a_{i}\right)=\mu_{1} \ln M_{i}+\mu_{2} \ln h_{i}+\mu_{3} \ln a_{i}+\bar{A}_{i}, \quad \mu_{1}+\mu_{2}+\mu_{3}=1 \tag{5.1}
\end{equation*}
$$

where $M_{i}$ is the composite index of the consumption of differentiated goods, $h_{i}$ is the consumption of housing measured by floor space, $a_{i}$ is the consumption of the
composite good, and $\bar{A}_{i}$ is the level of amenities in zone $i$, which is a constant term. $\bar{A}_{1}=\bar{A}_{2}=\cdots=\bar{A}_{I}$ holds since the suburbs are homogeneous. $M_{i}$ is assumed to be the constant elasticity of substitution function:

$$
\begin{equation*}
M_{i}=\left(\int_{0}^{m_{i}} q_{i}(k)^{(\sigma-1) / \sigma} \mathrm{d} k\right)^{\frac{\sigma}{\sigma-1}} \tag{5.2}
\end{equation*}
$$

where $q_{i}(k)$ is the consumption of the $k$ th variety, $m_{i}$ is the mass of varieties supplied in zone $i$.

Households residing in the downtown area do not need land for housing. In the downtown area, floor space is supplied with residential buildings. On the other hand, households residing in the suburb need land for housing. In the suburb, floor space is supplied with housing and land. Let $\phi$ denote the floor-area ratio in the suburb for housing. Households need $1 / \phi$ unit of land for one unit of floor space in the suburb. We assume that households residing in the suburb periodically pay land rent. We regard land consumption as a flow.

The budget constraint of the households residing in zone $i$ is given by

$$
\begin{cases}\int_{0}^{m_{0}} p_{0}^{M}(k) q_{0}(k) \mathrm{d} k+p_{0}^{H} h_{0}+a_{0}=y_{0} & (i=0)  \tag{5.3}\\ \int_{0}^{m_{i}} p_{i}^{M}(k) q_{i}(k) \mathrm{d} k+p_{i} h_{i}+\left(R_{i}^{H} / \psi\right) h_{i}+a_{i}=y_{i} & (i=1, \ldots, I)\end{cases}
$$

where $p_{i}^{M}(k)$ is the price of the $k$ th variety supplied in zone $i, p_{0}^{H}$ is the price per square foot of housing in the downtown area, and $y_{i}$ is the net income of households. $p_{i}$ and $R_{i}^{H}(i=1, \ldots, I)$ are the price of housing and land rent per square foot in the suburb, respectively. $p_{i} h_{i}$ and $\left(R_{i}^{H} / \psi\right) h_{i}$ are the total housing cost and land rent in the suburb, respectively. We summarize the prices with $p_{i}^{H} \equiv p_{i}+R_{i}^{H} / \psi(i=1, \ldots, I)$, which is the housing price per square foot in the suburb. The composite good is assumed to be the numéraire.

We assume public ownership of land and firms for simplicity. Households' net income $y_{i}$ is composed of common income $y$, equal share of the sum of profits and rents $\Pi$, and subsidy (or tax) $s_{i}(s): y_{i}=\widetilde{y}_{i}(s) \equiv y+\Pi+s_{i}(s)$. Each place-based policy determines
$s_{i}(s)$ and $s(\in \mathbb{R})$ expresses the level of policy implemented. We call $s$ the policy instrument.

This chapter considers two place-based policies: location subsidies to stores, and location subsidies to households. These policies are the same as the policies focused on in Chapter 4. Households (retail stores) in the same zone can receive the same amount of subsidy for the policies. Let $n_{i}$ and $s_{i}^{M}(s)$ denote the total number of households and the total subsidy provided to retail stores in residential zone $i$, respectively. The formal definitions for the place-based policies are as follows.

Definition 5.1. Location subsidies to households in the downtown area is the placebased policy such that the following equations hold.

$$
s_{i}(s)=\left\{\begin{array}{ll}
\left(\bar{N}-n_{0}\right) s / n_{0} & (i=0),  \tag{5.4}\\
-s & (i=1, \ldots, I),
\end{array} \quad s_{i}^{M}(s)=0 \quad(i \in \mathcal{I})\right.
$$

Definition 5.2. Location subsidies to stores in the downtown area is the place-based policy such that the following equations hold.

$$
s_{i}(s)=-s / \bar{N} \quad(i \in \mathcal{I}), \quad s_{i}^{M}(s)=\left\{\begin{align*}
s & (i=0)  \tag{5.5}\\
0 & (i=1, \ldots, I)
\end{align*}\right.
$$

"Location subsidies to households" imply that households residing in the downtown are subsidized. "Location subsidies to stores" imply that retail stores operating in the downtown are subsidized. The subsidies with the policies are paid by households:

$$
\begin{equation*}
\sum_{i \in \mathcal{I}}\left(n_{i} s_{i}(s)+s_{i}^{M}(s)\right)=0 \tag{5.6}
\end{equation*}
$$

We solve the following utility maximization problem:

$$
\begin{equation*}
\max _{\left\{q_{i}(k)\right\}_{k}, h_{i}, a_{i}} U_{i}\left(M_{i}, h_{i}, a_{i}\right) \quad \text { s.t. } \quad \text { Eqs. (5.2) and (5.3). } \tag{5.7}
\end{equation*}
$$

We decompose the utility maximization problem into two problems regarding two-stage budgeting. The conditional demands are given by

$$
q_{i}^{*}(k)=p_{i}^{M}(k)^{-\sigma} P_{i}^{\sigma} M_{i} \quad \forall k \in\left[0, m_{i}\right],
$$

where superscript " * " denotes the optimal solution and $P_{i}$ is the price index for the varieties supplied in residential zone $i$ :

$$
P_{i}=\left(\int_{0}^{m_{i}} p_{i}^{M}(k)^{1-\sigma} \mathrm{d} k\right)^{1 /(1-\sigma)}
$$

The demand functions are given by

$$
M_{i}^{*}=\mu_{1} y_{i} / P_{i}, \quad h_{i}^{*}=\mu_{2} y_{i} / p_{i}^{H}, \quad a_{i}^{*}=\mu_{3} y_{i} .
$$

Substituting $M_{i}^{*}$ into $q_{i}^{*}(k)$ yields

$$
q_{i}^{*}(k)=\mu_{1} p_{i}^{M}(k)^{-\sigma} y_{i} / P_{i}^{1-\sigma} .
$$

Let $V_{i}$ denote the indirect utility of households residing in residential zone $i$. Substituting the demand functions into the indirect utility yields

$$
\begin{equation*}
V_{i}=\ln y_{i}-\mu_{1} \ln P_{i}-\mu_{2} \ln p_{i}^{H}+\bar{A}_{i}+\xi, \tag{5.8}
\end{equation*}
$$

where $\xi=\mu_{1} \ln \mu_{1}+\mu_{2} \ln \mu_{2}+\mu_{3} \ln \mu_{3}$.

### 5.2.3. Retail stores

Retail stores supply differentiated goods in marketplaces. Each retail store supplies a variety in a marketplace. They are under monopolistic competition (Dixit and Stiglitz, 1977). Hence, the total mass of retail stores in each marketplace is endogenously determined by free entry. They rent units of land in marketplaces.

All the retail stores incur the same marginal production cost $c$ to supply varieties. The retail store that supplies the $k$ th variety incur $k+r_{i}(k)$ for the fixed cost, where $k$ also represents the fixed cost that depends on varieties, and $r_{i}(k)$ is land rent of a
constant unit of land for a store. Some retail stores can receive subsidies, as shown in Definitions 4.1 and 4.2.

Let $Q_{i}(k)$ and $\pi_{i}^{M}(k)$ denote the supply of the $k$ th variety and the profit of the retail store supplying the $k$ th variety in residential zone $i$, respectively. $\pi_{i}^{M}(k)$ is given by

$$
\begin{equation*}
\pi_{i}^{M}(k)=\left(p_{i}^{M}(k)-c\right) Q_{i}(k)-k-r_{i}(k)+\frac{s_{i}^{M}(s)}{m_{i}} \quad \forall k \in\left[0, m_{i}\right] . \tag{5.9}
\end{equation*}
$$

We assume that each store pays the bid rent. Using the profit yields the maximum land rent that each store can pay:

$$
\begin{equation*}
r_{i}(k)=\max _{p_{i}^{M}(k)}\left(\left(p_{i}^{M}(k)-c\right) Q_{i}(k)-k+\frac{s_{i}^{M}(s)}{m_{i}}\right) . \tag{5.10}
\end{equation*}
$$

This equation implies that the more demand for a variety in a marketplace, the higher the bid rent. Hence, if the prices of a variety supplied in some marketplaces are the same, then a retail store operating in a larger marketplace can propose a higher bid rent.

The total supply (or demand) is given by

$$
\begin{equation*}
Q_{i}(k)=n_{i} q_{i}^{*}(k) \tag{5.11}
\end{equation*}
$$

Using this equation and solving maximization problem (5.10) yields the prices of varieties supplied in zone $i$.

$$
p_{i}^{M}(k)=c \sigma /(\sigma-1), \quad \forall j \in \mathcal{J}, \quad \forall k \in\left[0, m_{i}\right] .
$$

Since the prices do not depend on $i$ and $k$, we express $p_{i}^{M}(k)$ as $p^{M}$. Under the symmetric price equilibrium, the demand and the total demand for varieties are given by

$$
\begin{align*}
& q_{i}^{*}(k)=\mu_{1} y_{i} /\left(p^{M} m_{i}\right) \quad \forall k \in\left[0, m_{i}\right],  \tag{5.12}\\
& Q_{i}(k)=\mu_{1} n_{i} y_{i} /\left(p^{M} m_{i}\right) \quad \forall k \in\left[0, m_{i}\right] . \tag{5.13}
\end{align*}
$$

### 5.2.4. Firms supplying floor space

Floor space is supplied by developers and house builders. Developers and house builders supply floor space in the downtown area and the suburb, respectively. They are under perfect competition.

## Developers

Following Brueckner (2007) and Domon et al. (2022), we specify developers as follows. Residential buildings are produced by combining land and housing capital (or building materials). The area of land in the downtown area is $\bar{L}_{0}$. The building output measured in height per unit of land is expressed as $g(b)=\theta b^{\beta} \quad(0<\theta, 0<\beta<1)$, where $b$ is the capital-to-land ratio. Let $\pi_{0}^{H}$ and $H_{0}$ denote the developers' net profit in the downtown area and the height of buildings, respectively. $\pi_{0}^{H}$ is given by

$$
\begin{equation*}
\pi_{0}^{H}=p_{0}^{H} \bar{L}_{0} H_{0}-\bar{L}_{0} g^{-1}\left(H_{0}\right)-\bar{L}_{0} R_{0}^{H} \tag{5.14}
\end{equation*}
$$

where $g^{-1}$ is the inverse function of $g$ and $R_{0}^{H}$ is the land rent per unit of land in the downtown area.

We assume that developers pay the bid land rent. Using the profit yields the maximum land rent that developers can pay:

$$
\begin{equation*}
R_{0}^{H}=\max _{H_{0}}\left(p_{0}^{H} H_{0}-g^{-1}\left(H_{0}\right)\right) \tag{5.15}
\end{equation*}
$$

Solving this maximization problem yields the height of buildings, the aggregated profits, and the bid rent:

$$
\begin{align*}
& H_{0}^{*}=\theta^{1 /(1-\beta)}\left(\beta p_{0}^{H}\right)^{\beta /(1-\beta)}  \tag{5.16}\\
& \pi_{0}^{H}=\bar{L}_{0}\left[\theta^{1 /(1-\beta)}\left(\beta^{\beta /(1-\beta)}-\beta^{1 /(1-\beta)}\right)\left(p_{0}^{H}\right)^{1 /(1-\beta)}-R_{0}^{H}\right]  \tag{5.17}\\
& R_{0}^{H}=p_{0}^{H} H_{0}^{*}-g^{-1}\left(H_{0}^{*}\right)=\theta^{1 /(1-\beta)}\left(\beta^{\beta /(1-\beta)}-\beta^{1 /(1-\beta)}\right)\left(p_{0}^{H}\right)^{1 /(1-\beta)} \tag{5.18}
\end{align*}
$$

## House builders

House builders supply floor area in the suburb with constant marginal cost $c^{H}$ and no fixed cost.

### 5.2.5. Market equilibrium condition

Considering the short-run equilibrium and the long-run equilibrium, we obtain the market equilibrium. In the short-run equilibrium, given the spatial distribution of households (i.e., $\left(n_{i}\right)_{i \in \mathcal{I}}$ ), the market clearing condition of housing holds and the mass of retail stores is determined. In the long-run equilibrium, the spatial distribution is determined. We focus on the market equilibrium at which the numbers of households residing in the suburbs are the same (i.e., $n_{1}=n_{2}=\cdots=n_{I}$ ).

We focus on the market clearing conditions for housing. Since the marginal cost of house builders is constant, $p_{1}=c^{H}$ holds. The other market clearing conditions regarding housing are the market clearing condition for floor space in the downtown area and land in the suburb:

$$
\begin{align*}
& n_{0} h_{0}^{*}=\bar{L}_{0} H_{0}^{*}  \tag{5.19}\\
& n_{1} h_{1}^{*}=\psi \bar{L}_{1} . \tag{5.20}
\end{align*}
$$

Using Eq. (5.19), we obtain the floor area price in the residential zone:

$$
\begin{equation*}
p_{0}^{H}=\left[\mu_{2} \theta^{-1 /(1-\beta)} \beta^{-\beta /(1-\beta)} n_{0} y_{0} / \bar{L}_{0}\right]^{1-\beta} \tag{5.21}
\end{equation*}
$$

Using Eq. (5.20), we obtain the land rent in the residential zone:

$$
\begin{equation*}
R_{1}^{H}=\psi\left(\mu_{2} n_{1} y_{1} /\left(\psi \bar{L}_{1}\right)-c^{H}\right), \tag{5.22}
\end{equation*}
$$

where $\bar{L}_{1}$ is the area of land in the suburb. Substituting $c^{H}$ and Eq. (5.22) into total housing price $p_{1}^{H}$ yields $p_{1}^{H}=\mu_{2} n_{1} y_{1} /\left(\psi \bar{L}_{1}\right)$.

Next, mass of retail stores $m_{i}$ is determined as follows. Since $p^{M}$ and $Q_{i}$ do not depend on $k$, $\left(p^{M}-c\right) Q_{i}+s_{i}^{M}(s) / m_{i}$ also does not depend on $k$. Land rent $r_{i}(k)$, shown by Eq. (5.10), monotonously decreases with an increase in $k$. Using this monotonicity, $r_{i}(k) \geq 0\left(\forall k \in\left[0, m_{i}\right]\right)$, and Eq. (5.10), we obtain the following condition for mass of stores $m_{i}$ :

$$
\begin{equation*}
r_{i}\left(m_{i}\right)=\left(p_{i}^{M}-c\right) Q_{i}-m_{i}+\frac{s_{i}^{M}(s)}{m_{i}}=0 \tag{5.23}
\end{equation*}
$$

This equation implies that sales equals the cost for the store supplying variety $m_{j}$. Substituting Eq. (5.13) into Eq. (5.23) yields

$$
\begin{equation*}
m_{i}=\sqrt{\frac{\mu_{1}}{\sigma} n_{i} y_{i}+s_{i}^{M}(s)} \tag{5.24}
\end{equation*}
$$

We focus on the net income of households (i.e., $y_{i}$ ). With the assumption of the public ownership, the profits and rents are equally divided among households:

$$
\begin{equation*}
\Pi=\bar{N}^{-1}\left(\pi_{0}^{H}+\bar{L}_{0} R_{0}^{H}+I \bar{L}_{1} R_{1}^{H}+\int_{0}^{m_{0}} \pi_{0}^{M}(k)+r_{0}(k) \mathrm{d} k+I \int_{0}^{m_{1}} \pi_{1}^{M}(k)+r_{1}(k) \mathrm{d} k\right) \tag{5.25}
\end{equation*}
$$

Using the market clearing conditions regarding housing yields

$$
\begin{align*}
& \bar{L}_{0} R_{0}^{H}+\pi_{0}^{H}=\mu_{2} n_{0} y_{0}(1-\beta)  \tag{5.26}\\
& \bar{L}_{1} R_{1}^{H}=\mu_{2} n_{1} y_{1}-c^{H} \psi \bar{L}_{1} \tag{5.27}
\end{align*}
$$

Using Eqs. (5.9) and (5.24) yields

$$
\begin{align*}
\int_{0}^{m_{0}} \pi_{0}^{M}(k)+r_{0}(k) \mathrm{d} k+I & \int_{0}^{m_{1}} \pi_{1}^{M}(k)+r_{1}(k) \mathrm{d} k= \\
& \frac{1}{2}\left(\frac{\mu_{1}}{\sigma} n_{0} y_{0}+\frac{\mu_{1}}{\sigma} I n_{1} y_{1}+s_{0}^{M}(s)+I s_{1}^{M}(s)\right) . \tag{5.28}
\end{align*}
$$

Substituting Eqs. (5.26)-(5.28) into Eq. (5.25) yields

$$
\Pi=\frac{1}{\bar{N}}\left[n_{0} y_{0}\left(\frac{\mu_{1}}{2 \sigma}+\mu_{2}(1-\beta)\right)+\operatorname{In}_{1} y_{1}\left(\frac{\mu_{1}}{2 \sigma}+\mu_{2}\right)+\frac{s_{0}^{M}+I s_{1}^{M}}{2}-c^{H} \psi I \bar{L}_{1}\right] .
$$

We can solve $y_{i}=y+\Pi+s_{i}(s)$ for $i=0,1$ :

$$
\begin{align*}
& y_{0}=\widetilde{y}_{0}\left(n_{0}, n_{1}, s\right) \equiv \phi\left(y+\frac{s_{0}^{M}+I s_{1}^{M}}{2 \bar{N}}-\frac{c^{H} \psi I \bar{L}_{1}}{\bar{N}}+(1-b) s_{0}+b s_{1}\right),  \tag{5.29}\\
& y_{1}=\widetilde{y}_{1}\left(n_{0}, n_{1}, s\right) \equiv \phi\left(y+\frac{s_{0}^{M}+I s_{1}^{M}}{2 \bar{N}}-\frac{c^{H} \psi I \bar{L}_{1}}{\bar{N}}+a s_{0}+(1-a) s_{1}\right), \tag{5.30}
\end{align*}
$$

where $\phi=(1-a-b)^{-1}, a=\bar{N}^{-1} n_{0}\left(\mu_{1} /(2 \sigma)+\mu_{2}(1-\beta)\right), b=\bar{N}^{-1} n_{1}\left(\mu_{1} /(2 \sigma)+\mu_{2}\right)$.
Let $\boldsymbol{n} \equiv\left(n_{0}, n_{1}\right)$ denote the spatial distribution of the households in the downtown and the suburb. Using $\widetilde{y}_{i}(\boldsymbol{n}, s)$, we obtain the prices, the masses, and the net income
as functions of $\boldsymbol{n}$ and $s$. Hence, the indirect utilities are also functions of $\boldsymbol{n}, s$, and exogenous variables:

$$
\begin{align*}
& V_{0}(\boldsymbol{n}, s)=\ln \widetilde{y}_{0}+\frac{\mu_{1}}{2(\sigma-1)} \ln \left(\frac{\mu_{1} n_{0} \widetilde{y_{0}}}{\sigma}+s_{0}^{M}\right)-\mu_{2}(1-\beta) \ln \left(n_{0} \widetilde{y_{0}}\right)+\Psi_{0}+\kappa,  \tag{5.31}\\
& V_{1}(\boldsymbol{n}, s)=\ln \widetilde{y_{1}}+\frac{\mu_{1}}{2(\sigma-1)} \ln \left(\frac{\mu_{1} n_{1} \widetilde{y_{1}}}{\sigma}+s_{1}^{M}\right)-\mu_{2} \ln \left(n_{1} \widetilde{y_{1}}\right)+\Psi_{1}+\kappa, \tag{5.32}
\end{align*}
$$

where

$$
\begin{aligned}
& \Psi_{0}=\bar{A}_{0}+\mu_{2}(1-\beta) \ln \bar{L}_{0}+\mu_{2} \beta \ln \mu_{2}+\mu_{2} \ln \left(\theta \beta^{\beta}\right), \quad \Psi_{1}=\bar{A}_{1}+\mu_{2} \ln \left(\psi \bar{L}_{1}\right), \\
& \kappa=-\mu_{1} \ln p^{M}-\mu_{2} \ln \mu_{2}+\xi
\end{aligned}
$$

In the long-run equilibrium, the spatial distribution is determined. $\boldsymbol{n}$ is an equilibrium iff $\bar{V}$ exists such that the following conditions hold:

$$
\left\{\begin{array}{lll}
V_{i}(\boldsymbol{n}, s)=\bar{V} & \text { if } \quad n_{i}>0  \tag{5.33}\\
V_{i}(\boldsymbol{n}, s)<\bar{V} & \text { if } \quad n_{i}=0
\end{array} \quad(i=0,1)\right.
$$

and

$$
\begin{equation*}
n_{0}+I n_{1}=\bar{N} \tag{5.34}
\end{equation*}
$$

### 5.3. How much benefit the place-based policies generate

### 5.3.1. Parameter calibration

In order to quantitatively evaluate the welfare impact, we calibrate exogenous parameters. We calibrate number of total households, $\bar{N}$; number of households in residential zone $i, \bar{n}_{i}$; number of the suburbs, $I$; expenditure share, $\mu_{j}(j=1,2,3)$; common income, $y$; marginal cost that house builders incur, $c^{h}$; elasticities of substitution, $\sigma$; land area in zone $i, \bar{L}_{i}$; marginal cost that retail stores incur, $c$; exogenous parameters regarding constructing buildings in the downtown area, $\beta$ and $\theta$; and amenities level in zone $i, A_{i}$.

Using the data of Sendai in Japan, we calibrate the parameters. The center of the CBD is at Sendai Station. We regard the downtown area as the area within a range of

2 km from the CBD. We regard a suburb as the area within a range of 2 km from the center of Izumi-Park town, which is a suburban town in Sendai. We consider that the center is at Sendai-Izumi Premium Outlet.

The calibration procedures are as follows.

## Number of total households $\bar{N}$, and number of households in zones $\bar{n}_{i}$, number of suburbs $I$

We use the data provided by Population Census 2005 of Japan and GIS. In our quantitative analyses, we focus on the behavior of nuclear families and benefits to the families with place-based policies. Hence, we assume that each household consists of three people (i.e., two parents with a child). We calibrate $\bar{n}_{i}$ by dividing the population in each area by three. We set $\bar{n}_{1}$ and $\bar{n}_{2}$ at 32,131 and 7,253 , respectively. Because there are Izumi-Park town and a suburban town around Nagamachi station as suburbs around the Sendai Station, We set $I$ at 2.

## Land area $\bar{L}_{i}$

We use the data provided by the Population Census 2005 of Japan and floor area ratio determined by Sendai City Government. We obtain the total floor area from the data. The total floor space in the downtown area, employed by our calibration, is the space in residential buildings, whereas the total floor space in the suburb is the space in houses. Referring to the data regarding urban planning provided by Sendai City Government, we set floor area ratio in the downtown area and the suburb at 4.0 and 0.6 , respectively. ${ }^{48}$ We calibrate $\bar{L}_{i}$ by dividing the total floor space by the floor area ratio. We set $\bar{L}_{1}$ and $\bar{L}_{2}$ at $675,977\left(\mathrm{~m}^{2}\right)$ and $1,646,686\left(\mathrm{~m}^{2}\right)$, respectively.

Common income $y$ and expenditure share $\mu_{j}(j=1,2,3)$
We use the data provided by the National Family Expenditure Structure Survey 2019

[^33]of Japan. We set common income $y$ at 3, 923, 988 (JPY/year). We assume that the shopping expenditure of households $\mu_{1}$ consists of the expenditure for food, manufactured goods, and clothes. We set $\mu_{1}, \mu_{2}$, and $\mu_{3}$ at $0.280,0.224$, and 0.496 , respectively. Marginal cost $c^{h}$ and $c$

We use the data provided by the Statistical Survey of Construction Starts 2021 of Japan. We use the average production cost of houses per square foot of floor space in the data. Since housing consumption is regarded as a flow in our model, we use the average production cost transformed into the present value. We assume that households pay the total housing price over 30 years. Using discount rate per year set at $2 \%$, we set $c^{h}$ at 6,566 (JPY/year).

Since marginal cost that retail stores incur $c$ affects neither the market equilibrium nor the welfare analyses, the level of $c$ does not matter. Hence, we set $c$ at 1.0.

Parameters regarding construction of residential buildings $\beta, \theta$
We set $\beta$ and $\sigma$ at 0.70 and 0.0028 so that the consumption levels of floor space in the downtown area and the suburb are roughly $85\left(\mathrm{~m}^{2}\right)$ and $135\left(\mathrm{~m}^{2}\right)$, respectively. Domon et al. (2022) estimate $\beta$ and $\sigma$ as 0.75 and 0.0028 , respectively. Hence, the calibrated parameters are close to the parameters estimated by Domon et al. (2022).

## Elasticities of substitution $\sigma$

In our model, $\sigma$ is equal to price elasticity for the varieties since we employ the CES function. DellaVigna and Gentzkow (2019) show the histogram of price elasticities for goods that retail stores supply. Based on the histogram, we set $\sigma$ (i.e., the price elasticity) at 2.5 , which is similar to the elasticity taking the maximum value of the histogram.

## Amenities level $A_{i}$

We set $A_{2}$ at 0 for the purpose of normalization. We calibrate amenities level $A_{1}$ with long-run equilibrium condition (5.33). Using the calibrated parameters, we set $A_{1}$

Table 5.1: Calibrated parameters

| Total households $\bar{N}$ | 46,687 | Elasticities of substitution $\sigma$ | 2.5 |
| :--- | :--- | :--- | :--- |
| Households in the downtown area $\bar{n}_{0}$ | 32,181 | Land area in the downtown area $\bar{L}_{1}$ | 675,977 |
| Households in the suburb $\bar{n}_{1}$ | 7,253 | Land area in the suburb $\bar{L}_{2}$ | $1,646,686$ |
| Expenditure share for shopping $\mu_{1}$ | 0.280 | Marginal cost $c$ | 1.0 |
| Expenditure share for housing $\mu_{2}$ | 0.224 | Parameter regarding buildings $\beta$ | 0.70 |
| Expenditure share for other goods $\mu_{3}$ | 0.496 | Parameter regarding buildings $\theta$ | 0.0028 |
| Common income $y$ | $3,923,988$ | Amenities level in the downtown $A_{1}$ | -0.034351 |
| Marginal cost $c^{h}$ | 6,566 | Amenities level in the suburb $A_{2}$ | 0 |
| Floor area ratio in the suburb $\psi$ | 0.6 | Number of the suburbs $I$ | 2 |

at -0.034351 .
Table 5.1 shows the results of the calibration.

### 5.3.2. Quantitative result

## Simulation setting

We conduct equilibrium and welfare analyses for the place-based policies shown in Definitions 4.1 and 4.2. In order to evaluate the welfare of the market equilibrium after the place-based policies are applied, we evaluate the welfare impact of the place-based policies with the equivalent variation. Let $E_{i}\left(P_{i}, p_{i}^{H}, U\right)$ denote the expenditure function for goods supplied in residential zone $i$. In our analysis, the prices are the functions of spatial distribution of households $\boldsymbol{n}: P_{i}=P_{i}(\boldsymbol{n}), p_{i}^{H}=p_{i}^{H}(\boldsymbol{n})$. Let $E V_{i}$ denote the equivalent variation for the households residing in zone $i . E V_{i}$ is given by

$$
\begin{equation*}
E V_{i} \equiv E_{i}\left(P_{i}\left(\boldsymbol{n}_{\text {before }}\right), p_{i}^{H}\left(\boldsymbol{n}_{\text {before }}\right), U_{\text {after }}\right)-E_{i}\left(P_{i}\left(\boldsymbol{n}_{\text {before }}\right), p_{i}^{H}\left(\boldsymbol{n}_{\text {before }}\right), U_{\text {before }}\right), \tag{5.35}
\end{equation*}
$$

where variables with subscripts "before" and "after" denote variables before and after place-based policies are applied, respectively. Since we regard $\boldsymbol{n}_{\text {before }}$ as the calibrated spatial households distribution, $\boldsymbol{n}_{\text {before }}=\left(\bar{n}_{1}, \bar{n}_{2}\right)=(32,181,7,253)$ holds. Let $n_{i, \text { after }}$

Table 5.2: Equivalent variation for applying the place-based policies

|  |  | $(1)$ |  |
| :--- | :---: | :---: | :---: |
|  |  | Place-based policy |  |
|  |  | Location subsidies <br> to households | Location subsidies <br> to stores |
| Income transfer (JPY / year) |  | 100,000 | 98,700 |
| Spatial distribution | $n_{1, \text { after }}$ | 35,007 | 34,345 |
| of households | $n_{2, \text { after }}$ | 5,840 | 6,171 |
| EV per household | $E V_{1}$ | $-12,118$ | 7,328 |
| (JPY / year) | $E V_{2}$ | $-12,118$ | 7,328 |
| Total EV (106 JPY / year) | $S E V$ | -566 | 342 |

denote the population in zone $i$. The aggregated equivalent variation is given by $S E V \equiv$ $n_{1, \text { after }} \times E V_{1}+I \times n_{2, \text { after }} \times E V_{2}$. We employ $S E V$ to evaluate the welfare impacts.

We conduct welfare analyses for each place-based policy by changing policy instrument $s$. In order to restrict income transfer among households with place-based policies to be applicable in the real world, we restrict $s$ to satisfy the condition that the income transfer is within 100, 000 (JPY / year). Conducting the analyses, we elucidate the efficient level of the policy instrument. If applying a place-based policy decreases welfare, we calculate the size of the decrease in welfare by applying the policy that generates income transfer with 100, 000 (JPY / year).

## Results

Table 5.2 shows the result of the welfare analyses. Column (1) shows the result of applying location subsidies to households. Since $n_{1 \text { after }}>\bar{n}_{1}$ holds, this policy promotes the flourishing of the downtown area. We check whether this policy monotonously decreases the welfare. The result shown in Column (1) is the result where the policy
is applied to generate income transfer with 100,000 Japanese yen. As $S E V$ shows, a negative benefit being equal to $566 \times 10^{6}$ Japanese yen occurs for each year.

Column (2) shows that of applying location subsidies to stores. The sign of the result regarding welfare is the opposite of the location subsidies to households. The efficient level of income transfer is 98,700 Japanese yen. Since this policy agglomerates households in the downtown area and increases the welfare, this policy is a desirable place-based policy.

### 5.4. Conclusion

We have quantitatively evaluated how place-based policies affect social welfare. We obtain two main findings: (1) subsidizing retail stores operating in downtown is desirable from the viewpoint of welfare, and (2) subsidizing consumers residing near the downtown is harmful. These results are qualitatively the same as the results in Chapter 4. The results indicate that policy makers should apply place-based policies that not indirectly but directly agglomerate retail stores in the downtown area.

Our model can be extended in the following manner. In this chapter, quantitative analysis with the CES preference is conducted. It will be a future topic to quantitatively evaluate welfare impacts with variable elasticity of substitution (VES) preferences. In spatial Economics, quantitative models with VES preferences have recently been developed (e.g., Bertoletti et al., 2018; Arkolakis et al., 2019). Quantitative methods employed by such studies will enable us to evaluate more elaborate quantitative analyses.

## 6. Concluding Remarks

The present thesis introduces a methodology with symmetry of geographical spaces for spatial economic models and elucidates which spatially dependent policies are socially efficient.

Chapter 2 investigates where retail stores agglomerate in a road network with radial roads and a ring road in a two-dimensional space. We show 1) how a difference in improvement sequences in the radial and ring roads generates a difference in the agglomeration patterns with different welfare levels and 2) how the two-dimensional geographical position of shopping agglomerations ensuring the highest welfare level differs from that in market equilibrium. These results indicate that policy makers should regulate the locations of marketplaces in order to generate the socially efficient geographical positions of retail stores.

Chapter 3 introduces an example of analysis with symmetry of a geographical space. We introduce two viewpoints with symmetry: (1) the bifurcation mechanism of the full agglomeration at the geographical center in this domain (mono-center) and (2) the existence of invariant patterns, which are equilibria for any value of the transport cost parameter. We theoretically find one large central city surrounded by hexagonal satellite regions with a spatial economic model proposed by Forslid and Ottaviano (2003) with the regular-hexagonal domain. This transition is an intrinsic feature observed in the two-dimensional spatial platform with a geographical center.

Chapters 4 and 5 focus on place-based policies to revitalize decayed shopping areas in the downtown area. We show that, whether or not place-based policies are efficient depends on the recipients to whom the policies give benefits, even if the policies promote retail agglomeration in the downtown area. Specifically, subsidizing consumers residing near the downtown area is inevitably harmful from the viewpoint of welfare, whereas subsidizing retail stores is efficient. These results indicate that policy makers should
not indirectly but directly agglomerate retail stores in the downtown area.
There are many research directions which we have not investigated in the present thesis. One direction is to develop multipurpose shopping models that are analytically tractable and are applied for empirical tests. In order to precisely evaluate the welfare impacts of spatially dependent policies, we need to conduct welfare analyses with such a model. Since we obtain several analytical results with the CES preference in the present thesis, one extension for the shopping models of the present thesis is to build the models which are analytically tractable with variable elasticity of substitution (VES) preferences. Using theoretical results for VES preferences (e.g., Parenti et al., 2017; Fally, 2022) may enable us to conduct the extension.

## A. Appendices for Chapter 2

## A.1. Proof of Lemma 2.1

$(\Rightarrow)$ It is obvious. $(\Leftarrow)$ Let $\boldsymbol{n}^{*}=\left(n_{0}^{*}, n_{1}^{*}, \ldots, n_{6}^{*}\right)$ be the stationary point of dynamics (2.8). Using $\boldsymbol{n}^{*}$ and dynamics (2.8), we obtain $n_{i}^{*} \pi_{i}\left(\boldsymbol{n}^{*}\right)=0(i=0,1, \ldots, 6)$. $n_{i}^{*} \pi_{i}\left(\boldsymbol{n}^{*}\right)=0$ holds if and only if

$$
\begin{equation*}
n_{i}^{*}=0 \text { or } \pi_{i}\left(\boldsymbol{n}^{*}\right)=0 . \tag{A.1}
\end{equation*}
$$

We check that market equilibria condition (2.7) holds at $\boldsymbol{n}^{*}$. If $n_{i}^{*}>0$, then we obtain $\pi_{i}\left(\boldsymbol{n}^{*}\right)=0$ by condition (A.1). On the other hand, if $n_{i}^{*}=0, Q_{i}=0$ holds by Eq. (2.6). Therefore, $\pi_{i}\left(\boldsymbol{n}^{*}\right)=-f<0$ holds.

## A.2. Theoretical properties of the dispersion

## A.2.1. Market boundary

We focus on market boundaries. A market boundary is a position at which consumers obtain the same indirect utility across multiple marketplaces. ${ }^{49}$ Let $t_{i}$ denote the market boundary between the center and suburb $i$ and $T_{i}$ denote the market boundary between suburb $i$ and $j(\equiv i+1 \bmod 6)$. Since the length of all the roads between the marketplaces is one, $t_{i}, T_{i} \in(0,1)(i=0,1, \ldots, 6)$ hold.

Using the market boundaries, we express the market areas. To express the market areas as subsets of all the positions on the road network $\mathcal{L}$, we define the following sets: $\mathcal{D}_{i}(Y)=\{(D, i, x) \in \mathcal{L} \mid x \in Y\}, \mathcal{S}_{i}(Y)=\{(S, i, x) \in \mathcal{L} \mid x \in Y\}(i \in \mathcal{P})$. $\mathcal{D}_{i}(Y)$ denotes an area on the radial road between the center and suburb $i$, whereas $\mathcal{S}_{i}(Y)$ denotes an area on the ring road between suburb $i$ and $j(\equiv i+1 \bmod 6)$. These subsets are employed in Appendix A.2.2 and A.3.

[^34]
## A.2.2. The definitions of market pattern (D) and the dispersion

We define market pattern (D) with the market boundaries.

Definition A.1. Market pattern (D) is market areas given by

$$
\begin{align*}
& \mathcal{M}_{0}=\cup_{m \in \mathcal{P}} \mathcal{D}_{m}\left(\left(0, t_{m}\right]\right),  \tag{A.2}\\
& \mathcal{M}_{i}=\mathcal{D}_{i}\left(\left[t_{i}, 1\right)\right) \cup \mathcal{S}_{i}\left(\left(0, T_{i}\right]\right) \cup \mathcal{S}_{j}\left(\left[T_{j}, 1\right)\right), \quad i, j \in \mathcal{P}, j \equiv i-1 \bmod 6 \tag{A.3}
\end{align*}
$$

The definition of market pattern (D) implies that every marketplace has a market area nearby. We can obtain the market boundaries as follows.

Lemma A.1. In market pattern ( $D$ ), market boundaries $t_{i}, T_{i}(i \in \mathcal{P})$ are given by

$$
\begin{align*}
t_{i} & =\frac{1}{2}\left(\frac{\ln \left(n_{0} / n_{i}\right)}{\phi(\sigma-1)}+1\right),  \tag{A.4}\\
T_{i} & =\frac{1}{2}\left(\frac{\ln \left(n_{i} / n_{j}\right)}{\tau(\sigma-1)}+1\right), \quad j \equiv i+1 \bmod 6 \tag{A.5}
\end{align*}
$$

Proof. See Appendix A.5.1.

We can obtain dynamics (2.8) with $t_{i}$ and $T_{i}$ in market pattern (D).

Lemma A.2. Dynamics (2.8) in market pattern (D) is given by

$$
\begin{align*}
& F_{0}(\boldsymbol{n})=n_{0}\left(\frac{1}{\sigma n_{0}} \sum_{m=1}^{6} t_{m}-f\right)  \tag{A.6}\\
& F_{i}(\boldsymbol{n})=n_{i}\left(\frac{1}{\sigma n_{i}}\left(\left(1-t_{i}\right)+T_{i}+\left(1-T_{j}\right)\right)-f\right), \quad i, j \in \mathcal{P}, \quad j \equiv i-1 \bmod 6 \tag{A.7}
\end{align*}
$$

Proof. See Appendix A.5.2.

The dispersion is the stationary points of dynamics (2.8) given by (A.6) and (A.7).

## A.2.3. Proofs of Lemmas in Section 2.3.1

## Proof of Lemma 2.2

Substituting $\boldsymbol{n}=\left(n_{0}, n_{1}, \ldots, n_{1}\right)$ into (A.6) and (A.7), we can obtain (2.11) and (2.12).

## Brief proof of Lemma 2.3

We use the implicit function theorem. Using the Jacobian matrix of $\widetilde{\boldsymbol{F}}=\left(F_{0}, F_{1}\right)^{\top}$ with respect to $\widetilde{\boldsymbol{n}}=\left(n_{0}, n_{1}\right)$ and the linearly-stable condition of the dispersion, we can prove the Lemma. Since the proof is long, see Appendix A.5.4 for details.

## Brief proof of Lemma 2.4

We can analytically obtain the eigenvalues of $\partial \boldsymbol{F} / \partial \boldsymbol{n}$ at $\boldsymbol{n}_{d}$. Obtaining the eigenvector for the largest of these eigenvalues, we can prove the Lemma. Since the proof is long, see Appendix A.5.5 for details.

## A.3. Mathematical explanation of the corner equilibria in Section 2.3.2

In this appendix, we show the definitions of market area patterns and corner equilibria shown in Section 2.3.2. To express $\boldsymbol{F}(\boldsymbol{n})$ as a matrix, we appropriately permute the components of $\boldsymbol{F}(\boldsymbol{n})$ as follows:

$$
\begin{equation*}
\widehat{\boldsymbol{F}}(\boldsymbol{n})=\binom{\boldsymbol{F}^{+}(\boldsymbol{n})}{\boldsymbol{F}^{0}(\boldsymbol{n})} \tag{A.8}
\end{equation*}
$$

where

$$
\begin{aligned}
& \boldsymbol{F}^{+}(\boldsymbol{n})=\left(F_{i_{1}}(\boldsymbol{n}), \ldots, F_{i_{m}}(\boldsymbol{n})\right)^{\top}, \quad\left(i_{1}<\cdots<i_{m}, \quad \mathcal{M}_{i_{1}}, \ldots, \mathcal{M}_{i_{m}} \neq \phi\right), \\
& \boldsymbol{F}^{0}(\boldsymbol{n})=\left(F_{j_{1}}(\boldsymbol{n}), \ldots, F_{j_{k}}(\boldsymbol{n})\right)^{\top}, \quad\left(j_{1}<\cdots<j_{k}, \quad \mathcal{M}_{j_{1}}=\cdots=\mathcal{M}_{j_{k}}=\phi\right) .
\end{aligned}
$$

$i_{r}(r=1, \ldots, m)$ denotes an index assigned to a marketplace with a market area, whereas $j_{r}(r=1, \ldots, k)$ denotes an index assigned to one with no market area.

## A.3.1. The full agglomeration

We focus on market pattern (F) and the full agglomeration.

Definition A.2. Market pattern ( $F$ ) is market areas given by $\mathcal{M}_{0}=\mathcal{L}, \mathcal{M}_{i}=$ $\emptyset(i \in \mathcal{P})$.

The definition of market pattern (F) implies that the center has the market area entirely covering the entire city. We can obtain inequality conditions in this market pattern.

Lemma A.3. Market pattern $(F)$ holds if and only if the following inequality holds.

$$
\begin{equation*}
\phi \leq \sigma_{-1} \ln \left(n_{0} / n_{i}\right), \quad i \in \mathcal{P} \tag{A.9}
\end{equation*}
$$

where $\sigma_{-1}=(\sigma-1)^{-1}$.

Proof. See Appendix A.6.1.
In market pattern $(\mathrm{F}), i_{1}=0$ and $\left(j_{1}, \ldots, j_{6}\right)=(1, \ldots, 6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this market pattern as follows.

Lemma A.4. In market pattern $(F), \boldsymbol{F}^{+}(\boldsymbol{n})$ and $\boldsymbol{F}^{0}(\boldsymbol{n})$ are given by

$$
\begin{equation*}
\boldsymbol{F}^{+}(\boldsymbol{n})=\frac{12}{\sigma}-f n_{0}, \quad \boldsymbol{F}^{0}(\boldsymbol{n})=-f \boldsymbol{n}^{0} \tag{A.10}
\end{equation*}
$$

where $\boldsymbol{n}^{0}=\left(n_{1}, \ldots, n_{6}\right)^{\top}$.

Proof. The proof is similar to that of Lemma A.2.

The full agglomeration is the stationary points of dynamics (2.8) given by (A.10). Note that the full agglomeration is always linearly-stable because the eigenvalues are $-f$, which is negative.

## A.3.2. The triangle pattern

We focus on market pattern (P) and the triangle pattern.

Definition A.3. Market pattern $(P)$ is market areas given by

$$
\begin{align*}
& \mathcal{M}_{0}=\cup_{m \in \mathcal{P}} \mathcal{D}_{m}\left(\left(0, t_{m}\right]\right),  \tag{A.11}\\
& \mathcal{M}_{1}=\left(\cup_{m \in\{1,2,6\}} \mathcal{D}_{m}\left(\left[t_{m}, 1\right)\right)\right) \cup\left(\cup_{m \in\{1,6\}} \mathcal{S}_{m}(X)\right) \cup \mathcal{S}_{2}\left(\left(0, T_{2}\right]\right) \cup \mathcal{S}_{5}\left(\left[T_{5}, 1\right)\right), \tag{A.12}
\end{align*}
$$

$$
\begin{align*}
& \mathcal{M}_{3}=\left(\cup_{m \in\{3,4\}} \mathcal{D}_{m}\left(\left[t_{m}, 1\right)\right)\right) \cup \mathcal{S}_{3}(X) \cup \mathcal{S}_{2}\left(\left[T_{2}, 1\right)\right) \cup \mathcal{S}_{4}\left(\left(0, T_{4}\right]\right)  \tag{A.13}\\
& \mathcal{M}_{5}=\mathcal{D}_{5}\left(\left[t_{5}, 1\right)\right) \cup \mathcal{S}_{4}\left(\left[T_{4}, 1\right)\right) \cup \mathcal{S}_{5}\left(\left(0, T_{5}\right]\right)  \tag{A.14}\\
& \mathcal{M}_{2}=\mathcal{M}_{4}=\mathcal{M}_{6}=\emptyset \tag{A.15}
\end{align*}
$$

The definition of market pattern (P) has three features: (1) the center has a market area only on the radial roads, (2) suburbs 1,3 , and 5 have a market area on both radial roads and the ring road, and (3) $\mu\left(\mathcal{M}_{5}\right)<\mu\left(\mathcal{M}_{3}\right)<\mu\left(\mathcal{M}_{1}\right)$ always holds. We can obtain inequality conditions in this market pattern.

Lemma A.5. Market pattern ( $P$ ) holds if and only if the following inequalities hold.

$$
\begin{align*}
-\phi & <\sigma_{-1} \ln \left(n_{i} / n_{j}\right)<\phi-\tau, \quad(i, j)=(0,1),(0,3),  \tag{A.16}\\
-\phi & <\sigma_{-1} \ln \left(n_{0} / n_{5}\right)<\phi,  \tag{A.17}\\
0 & <\sigma_{-1} \ln \left(n_{i} / n_{j}\right)<2 \tau, \quad(i, j)=(1,3),(3,5),(1,5),  \tag{A.18}\\
\tau & <\sigma_{-1} \ln \left(n_{i} / n_{j}\right), \quad(i, j)=(1,2),(3,4),(1,6) . \tag{A.19}
\end{align*}
$$

Proof. See Appendix A.6.2.

In market pattern $(\mathrm{P}),\left(i_{1}, i_{2}, i_{3}, i_{4}\right)=(0,1,3,5)$ and $\left(j_{1}, j_{2}, j_{3}\right)=(2,4,6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this market pattern as follows.

Lemma A.6. In market pattern $(P), \boldsymbol{F}^{+}(\boldsymbol{n})$ and $\boldsymbol{F}^{0}(\boldsymbol{n})$ are given by

$$
\begin{equation*}
\boldsymbol{F}^{+}(\boldsymbol{n})=\frac{1}{2 \sigma}\left(A_{P} \boldsymbol{z}_{P}^{+}+\boldsymbol{b}_{P}\right)-f \boldsymbol{n}_{P}^{+}, \quad \boldsymbol{F}^{0}(\boldsymbol{n})=-f \boldsymbol{n}_{P}^{0} \tag{A.20}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{P}=\frac{1}{\sigma-1}\left(\begin{array}{c|c}
6 \phi^{-1} & -\phi^{-1} \boldsymbol{c}_{P}^{\top} \\
\hline-\phi^{-1} \boldsymbol{c}_{P} & \phi^{-1} B_{P}+\tau^{-1} C_{P}
\end{array}\right) \\
& \boldsymbol{z}_{P}^{+}=\left(\ln n_{0}, \ln n_{1}, \ln n_{3}, \ln n_{5}\right)^{\top}, \quad \boldsymbol{c}_{P}=(3,2,1)^{\top}, \quad B_{P}=\operatorname{diag}(3,2,1)
\end{aligned}
$$

$C_{P}=\left(\begin{array}{ccc}2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2\end{array}\right), \boldsymbol{b}_{P}=\binom{3 \tau \phi^{-1}+6}{\boldsymbol{b}_{P 1}}, \begin{aligned} & \boldsymbol{b}_{P 1}=\left(-2 \tau \phi^{-1}+7,-\tau \phi^{-1}+6,5\right)^{\top}, \\ & \boldsymbol{n}_{P}^{0}=\left(n_{2}, n_{4}, n_{6}\right)^{\top} .\end{aligned}$
Proof. The proof is similar to that of Lemma A.2.

The triangle pattern is the stationary points of dynamics (2.8) given by (A.20).

## A.3.3. The asymmetric pattern

We focus on market pattern (A) and the asymmetric pattern.
Definition A.4. Market pattern (A) is market areas given by

$$
\begin{align*}
& \mathcal{M}_{0}=\cup_{m \in \mathcal{P}} \mathcal{D}_{m}\left(\left(0, t_{m}\right]\right),  \tag{A.21}\\
& \mathcal{M}_{1}=\left(\cup_{m \in\{1,2,5,6\}} \mathcal{D}_{m}\left(\left[t_{m}, 1\right)\right)\right) \cup\left(\cup_{m \in\{1,5,6\}} \mathcal{S}_{m}(X)\right) \cup \mathcal{S}_{2}\left(\left(0, T_{2}\right]\right) \cup \mathcal{S}_{4}\left(\left[T_{4}, 1\right)\right), \tag{A.22}
\end{align*}
$$

$$
\begin{equation*}
\mathcal{M}_{3}=\left(\cup_{m \in\{3,4\}} \mathcal{D}_{m}\left(\left[t_{m}, 1\right)\right)\right) \cup \mathcal{S}_{3}(X) \cup \mathcal{S}_{2}\left(\left[T_{2}, 1\right)\right) \cup \mathcal{S}_{4}\left(\left(0, T_{4}\right]\right), \tag{A.23}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{M}_{2}=\mathcal{M}_{4}=\mathcal{M}_{5}=\mathcal{M}_{6}=\emptyset \tag{A.24}
\end{equation*}
$$

The definition of market pattern (A) has three features: (1) the center has a market area only on the radial roads, (2) suburbs 1 and 3 each have a market area on both radial roads and the ring road, and (3) $\mu\left(\mathcal{M}_{3}\right)<\mu\left(\mathcal{M}_{1}\right)$ always holds. We can obtain inequality conditions as follows.

Lemma A.7. Market pattern (A) holds if and only if the following inequalities hold.

$$
\begin{align*}
-\phi & <\sigma_{-1} \ln \left(n_{0} / n_{1}\right)<\phi-2 \tau  \tag{A.25}\\
-\phi & <\sigma_{-1} \ln \left(n_{0} / n_{3}\right)<\phi-\tau  \tag{A.26}\\
0 & <\sigma_{-1} \ln \left(n_{1} / n_{3}\right)<2 \tau  \tag{A.27}\\
\tau & <\sigma_{-1} \ln \left(n_{i} / n_{j}\right), \quad(i, j)=(1,2),(1,6),(3,4),  \tag{A.28}\\
2 \tau & <\sigma_{-1} \ln \left(n_{1} / n_{5}\right) \tag{A.29}
\end{align*}
$$

Proof. The proof is similar to that of Lemma A.5.

In market pattern $(\mathrm{A}),\left(i_{1}, i_{2}, i_{3}\right)=(0,1,3)$ and $\left(j_{1}, j_{2}, j_{3}, j_{4}\right)=(2,4,5,6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this market pattern as follows.

Lemma A.8. In market pattern $(A), \boldsymbol{F}^{+}(\boldsymbol{n})$ and $\boldsymbol{F}^{0}(\boldsymbol{n})$ are given by

$$
\begin{equation*}
\boldsymbol{F}^{+}(\boldsymbol{n})=\frac{1}{2 \sigma}\left(A_{A} \boldsymbol{z}_{A}^{+}+\boldsymbol{b}_{A}\right)-f \boldsymbol{n}_{A}^{+}, \quad \boldsymbol{F}^{0}(\boldsymbol{n})=-f \boldsymbol{n}_{A}^{0} \tag{A.30}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{A}=\frac{1}{\sigma-1}\left(\begin{array}{c|c}
6 \phi^{-1} & -\phi^{-1} \boldsymbol{c}_{A}^{\top} \\
\hline-\phi^{-1} \boldsymbol{c}_{A} & \phi^{-1} B_{A}+\tau^{-1} C_{A}
\end{array}\right), \\
& \boldsymbol{z}_{A}^{+}=\left(\ln n_{0}, \ln n_{1}, \ln n_{3}\right)^{\top}, \quad \boldsymbol{c}_{A}=(4,2)^{\top}, \quad B_{A}=\operatorname{diag}(4,2), \\
& C_{A}=2\left(\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right), \quad \boldsymbol{b}_{A}=\binom{5 \tau \phi^{-1}+6}{\boldsymbol{b}_{A 1}}, \quad \begin{array}{l}
\boldsymbol{b}_{A 1}=\left(-4 \tau \phi^{-1}+10,-\tau \phi^{-1}+8\right)^{\top}, \\
\boldsymbol{n}_{A}^{0}=\left(n_{2}, n_{4}, n_{5}, n_{6}\right)^{\top} .
\end{array}
\end{aligned}
$$

Proof. The proof is similar to that of Lemma A.2.

The asymmetric pattern is the stationary points of dynamics (2.8) given by (A.30).

## A.3.4. The linear pattern

We focus on market pattern ( L ) and the linear pattern.

Definition A.5. Market pattern (L) is market areas given by

$$
\begin{align*}
& \mathcal{M}_{0}=\cup_{i=1}^{6} \mathcal{D}_{i}\left(\left(0, t_{i}\right]\right),  \tag{A.31}\\
& \mathcal{M}_{1}=\left(\cup_{i \in\{1,2,6\}} \mathcal{D}_{i}\left(\left[t_{i}, 1\right)\right)\right) \cup\left(\cup_{i \in\{1,6\}} \mathcal{S}_{i}(X)\right) \cup \mathcal{S}_{2}\left(\left(0, T_{2}\right]\right) \cup \mathcal{S}_{5}\left(\left[T_{5}, 1\right)\right),  \tag{A.32}\\
& \mathcal{M}_{4}=\left(\cup_{i \in\{3,4,5\}} \mathcal{D}_{i}\left(\left[t_{i}, 1\right)\right)\right) \cup\left(\cup_{i \in\{3,4\}} \mathcal{S}_{i}(X)\right) \cup \mathcal{S}_{2}\left(\left[T_{2}, 1\right)\right) \cup \mathcal{S}_{5}\left(\left(0, T_{5}\right]\right),  \tag{А.33}\\
& \mathcal{M}_{2}=\mathcal{M}_{3}=\mathcal{M}_{5}=\mathcal{M}_{6}=\emptyset . \tag{A.34}
\end{align*}
$$

The definition of market pattern (L) has two features: (1) the center has a market area only on the radial roads, (2) suburbs 1 and 4 each have a market area on both radial roads and the ring road. We can obtain inequality conditions as follows.

Lemma A.9. Market pattern ( $L$ ) holds if and only if the following inequalities hold.

$$
\begin{align*}
-\phi & <\sigma_{-1} \ln \left(n_{0} / n_{j}\right)<\phi-\tau, \quad j=1,4  \tag{A.35}\\
-\tau & <\sigma_{-1} \ln \left(n_{1} / n_{4}\right)<\tau  \tag{A.36}\\
\tau & <\sigma_{-1} \ln \left(n_{i} / n_{j}\right), \quad(i, j)=(1,2),(1,6),(4,3),(4,5) \tag{A.37}
\end{align*}
$$

Proof. The proof is similar to that of Lemma A.5.

In market pattern $(\mathrm{L}),\left(i_{1}, i_{2}, i_{3}\right)=(0,1,4)$ and $\left(j_{1}, j_{2}, j_{3}, j_{4}\right)=(2,3,5,6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this pattern as follows.

Lemma A.10. In market pattern $(L), \boldsymbol{F}^{+}(\boldsymbol{n})$ and $\boldsymbol{F}^{0}(\boldsymbol{n})$ are given by

$$
\begin{equation*}
\boldsymbol{F}^{+}(\boldsymbol{n})=\frac{1}{2 \sigma}\left(A_{L} \boldsymbol{z}_{L}^{+}+\boldsymbol{b}_{L}\right)-f \boldsymbol{n}_{L}^{+}, \quad \boldsymbol{F}^{0}(\boldsymbol{n})=-f \boldsymbol{n}_{L}^{0} \tag{A.38}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{L}=\frac{1}{\sigma-1}\left(\begin{array}{c|c}
6 \phi^{-1} & -\phi^{-1} \mathbf{3}_{2}^{\top} \\
\hline-\phi^{-1} \mathbf{3}_{2} & 3 \phi^{-1} I_{2}+\tau^{-1} C_{L}
\end{array}\right), \quad \boldsymbol{z}_{L}^{+}=\left(\ln n_{0}, \ln n_{1}, \ln n_{4}\right)^{\top} \\
& C_{L}=2\left(\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right), \quad \boldsymbol{b}_{L}=\binom{4 \tau \phi^{-1}+6}{\boldsymbol{b}_{L 1}}, \quad \begin{array}{l}
\boldsymbol{b}_{L 1}=\left(-2 \tau \phi^{-1}+9,-2 \tau \phi^{-1}+9\right)^{\top} \\
\boldsymbol{n}_{L}^{0}=\left(n_{2}, n_{3}, n_{5}, n_{6}\right)^{\top}
\end{array}
\end{aligned}
$$

Proof. The proof is similar to that of Lemma A.2.

The linear pattern is the stationary points of dynamics (2.8) given by (A.38).

## A.4. Proof of Proposition 2.1

Let $\mathcal{A}_{p}$ be the closure of the set of $\boldsymbol{n}$ satisfying the inequality conditions for market pattern $(\mathrm{P}), \mathcal{A}_{a}$ be the closure for market pattern (A), and $\mathcal{A}_{l}$ be the closure for market pattern (L). $\mathcal{A}_{p}, \mathcal{A}_{a}$, and $\mathcal{A}_{l}$ are given by

$$
\begin{align*}
& \mathcal{A}_{p}=\operatorname{cl}\left\{\boldsymbol{n} \in \mathbb{R}_{+}^{7} \mid(A .16)-(A .19)\right\}  \tag{A.39}\\
& \mathcal{A}_{a}=\operatorname{cl}\left\{\boldsymbol{n} \in \mathbb{R}_{+}^{7} \mid(A .25)-(A .29)\right\} \tag{A.40}
\end{align*}
$$

$$
\begin{equation*}
\mathcal{A}_{l}=\operatorname{cl}\left\{\boldsymbol{n} \in \mathbb{R}_{+}^{7} \mid(A .35)-(A .37)\right\}, \tag{A.41}
\end{equation*}
$$

where $\operatorname{cl}\{\cdot\}$ is the closure of $\{\cdot\}$. Since the closure of (A.19) and the closure of (A.37) are disjoint sets. $\mathcal{A}_{p} \cap \mathcal{A}_{l}=\emptyset$ thus holds. Similarly, $\mathcal{A}_{a} \cap \mathcal{A}_{l}=\emptyset$ holds. Therefore, the solution starting at any point in market pattern (P) (or market pattern (A)) under dynamics (2.8) does not go to any state in market pattern (L).

## A.5. Proofs in Appendix A.2

## A.5.1. Proof of Lemma A. 1

For any $i \in \mathcal{P}$, the following conditions hold in market pattern (D):

$$
\begin{align*}
& \ell=\left(D, i, t_{i}\right) \Rightarrow V(\ell, 0)=V(\ell, i)  \tag{A.42}\\
& \ell=\left(S, i, T_{i}\right), j \equiv i+1 \bmod 6 \Rightarrow V(\ell, i)=V(\ell, j) . \tag{A.43}
\end{align*}
$$

Using (2.4) and (A.42), we can obtain (A.4). On the other hand, using (2.4) and (A.43), we can obtain (A.5).

## A.5.2. Proof of Lemma A. 2

Using market boundaries $t_{i}$ and $T_{i}(i \in \mathcal{P})$, we obtain $\mu\left(\mathcal{M}_{0}\right)$ and $\mu\left(\mathcal{M}_{i}\right)$ in market pattern (D):

$$
\begin{align*}
& \mu\left(\mathcal{M}_{0}\right)=\sum_{m=1}^{6} t_{m}  \tag{A.44}\\
& \mu\left(\mathcal{M}_{i}\right)=\left(1-t_{i}\right)+T_{i}+\left(1-T_{j}\right), \quad i \in \mathcal{P}, \quad j \equiv i-1 \bmod 6 \tag{A.45}
\end{align*}
$$

Substituting (A.44) (or (A.45)) into (2.6), we obtain $Q_{j}(j=0,1, \ldots, 6)$ in market pattern (F). Since $\pi_{j}$ in (2.3) is determined by $Q_{j}$, we can obtain $n_{j} \pi_{j}$ in dynamics (2.8), which is equal to (A.6) (or (A.7)).

## A.5.3. The eigenvalues of the Jacobian matrix of $F(n)$ for $n_{d}$

To obtain the eigenvalues of the Jacobian matrix $\partial \boldsymbol{F} / \partial \boldsymbol{n}$ in market pattern (D), we rewrite $\boldsymbol{F}(\boldsymbol{n})$ as a matrix:

$$
\begin{equation*}
\boldsymbol{F}(\boldsymbol{n})=\frac{1}{2 \sigma}(A \boldsymbol{z}+\boldsymbol{b})-f \boldsymbol{n} \tag{A.46}
\end{equation*}
$$

where

$$
\begin{aligned}
& A=\frac{1}{\sigma-1}\left(\begin{array}{cc|c}
6 a_{1} & -a_{1} \mathbf{1}_{6}^{\top} \\
\hline-a_{1} \mathbf{1}_{6} & a_{1} I_{6}+a_{2} B
\end{array}\right), \\
& B=\left(\begin{array}{ccccc}
2 & -1 & & & -1 \\
-1 & 2 & -1 & & \\
& -1 & 2 & -1 & \\
& & -1 & 2 & -1 \\
\\
& & -1 & 2 & -1 \\
-1 & & & -1 & 2
\end{array}\right), \quad \begin{array}{l}
\boldsymbol{z}=\left(\ln \left(n_{0}\right), \ln \left(n_{1}\right), \ldots, \ln \left(n_{6}\right)\right)^{\top}, \\
\boldsymbol{b}=(6,3,3,3,3,3,3)^{\top} .
\end{array}
\end{aligned}
$$

$\boldsymbol{k}_{6}$ is all- $k 6$-dimensional column vector whereas $I_{k}$ is $k \times k$ identity matrix.
The following Lemma is employed for proofs of Lemmas 2.3 and 2.4.

Lemma A.11. For $\boldsymbol{n}=\boldsymbol{n}_{d}$, the eigenvalues of Jacobian matrix $\partial \boldsymbol{F} / \partial \boldsymbol{n}$ are given by

$$
\begin{cases}\lambda_{1}=-f  \tag{A.47}\\ \lambda_{2}=\frac{1}{2 \phi \sigma(\sigma-1)}\left(\frac{6}{n_{0}}+\frac{1}{n_{1}}\right)-f \\ \lambda_{3}=\frac{1}{2 n_{1} \sigma(\sigma-1)}\left(\frac{1}{\phi}+\frac{4}{\tau}\right)-f \\ \lambda_{4}=\frac{1}{2 n_{1} \sigma(\sigma-1)}\left(\frac{1}{\phi}+\frac{3}{\tau}\right)-f & (\text { repeated twice }) \\ \lambda_{5}=\frac{1}{2 n_{1} \sigma(\sigma-1)}\left(\frac{1}{\phi}+\frac{1}{\tau}\right)-f \quad & \text { (repeated twice) }\end{cases}
$$

Proof. Let $J(\boldsymbol{n})$ denote Jacobian matrix $\partial \boldsymbol{F} / \partial \boldsymbol{n}$. Using (A.46), we obtain $J\left(\boldsymbol{n}_{d}\right)$ as follows:

$$
J\left(\boldsymbol{n}_{d}\right)=\frac{1}{2 \sigma}\left(A \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}}\left(\boldsymbol{n}_{d}\right)\right)-f I_{7}=\left(\begin{array}{c|c}
6 d_{1}-f & -d_{2} \mathbf{1}_{6}^{\top}  \tag{A.48}\\
\hline-d_{1} \mathbf{1}_{6} & \left(d_{2}-f\right) I_{6}+d_{3} B
\end{array}\right)
$$

where $d_{1}=d_{0} / \phi n_{0}, d_{2}=d_{0} / \phi n_{1}, d_{3}=d_{0} / \tau n_{1}, d_{0}=1 / 2 \sigma(\sigma-1)$. Let $\lambda$ be an eigenvalue of $J\left(\boldsymbol{n}_{d}\right)$. Using elementary transformation of matrices, we obtain the determinant of $J-\lambda I_{7}$ :

$$
\begin{aligned}
\operatorname{det}\left(J\left(\boldsymbol{n}_{d}\right)-\lambda I_{7}\right) & =\operatorname{det}\left(\begin{array}{c|c}
6 d_{1}-f-\lambda & -d_{2} \mathbf{1}_{6}^{\top} \\
\hline-d_{1} \mathbf{1}_{6} & \left(d_{2}-f-\lambda\right) I_{6}+d_{3} B
\end{array}\right) \\
& =\operatorname{det}\left(\begin{array}{c|c}
-f-\lambda & \mathbf{0}_{6}^{\top} \\
\hline-d_{1} \mathbf{1}_{6} & J_{1}
\end{array}\right) \\
& =(-f-\lambda) \operatorname{det} J_{1},
\end{aligned}
$$

where $J_{1}=d_{1} \mathbf{1}_{6} \mathbf{1}_{6}^{\top}+\left(d_{2}-f-\lambda\right) I_{6}+d_{3} B$.
Next, we calculate det $J_{1}$ with the property of orthogonal matrices. We consider the following orthogonal matrix:

$$
Q=\left(\begin{array}{cccccc}
1 / \sqrt{6} & 1 / \sqrt{6} & -1 / \sqrt{12} & 1 / \sqrt{4} & 1 / \sqrt{12} & 1 / \sqrt{4}  \tag{A.49}\\
1 / \sqrt{6} & -1 / \sqrt{6} & -1 / \sqrt{12} & -1 / \sqrt{4} & 2 / \sqrt{12} & 0 \\
1 / \sqrt{6} & 1 / \sqrt{6} & 2 / \sqrt{12} & 0 & 1 / \sqrt{12} & -1 / \sqrt{4} \\
1 / \sqrt{6} & -1 / \sqrt{6} & -1 / \sqrt{12} & 1 / \sqrt{4} & -1 / \sqrt{12} & -1 / \sqrt{4} \\
1 / \sqrt{6} & 1 / \sqrt{6} & -1 / \sqrt{12} & -1 / \sqrt{4} & -2 / \sqrt{12} & 0 \\
1 / \sqrt{6} & -1 / \sqrt{6} & 2 / \sqrt{12} & 0 & -1 / \sqrt{12} & 1 / \sqrt{4}
\end{array}\right) .
$$

Since $Q$ is an orthogonal matrix, $\operatorname{det}\left(J_{1}\right)=\operatorname{det}\left(Q^{\top} J_{1} Q\right)$ holds. We carry out orthogonal transformation to matrices in $J_{1}: Q^{\top} \mathbf{1}_{6} \mathbf{1}_{6}^{\top} Q=\operatorname{diag}(6,0,0,0,0,0), Q^{\top} I_{6} Q=$ $I_{6}, Q^{\top} B Q=\operatorname{diag}(0,4,3,3,1,1)$. Using these matrices, we obtain $\operatorname{det}\left(J\left(\boldsymbol{n}_{d}\right)-\lambda I_{7}\right)=$ $(-f-\lambda) e_{1} e_{2} e_{3}^{2} e_{4}^{2}$, where $e_{1}=6 d_{1}+d_{2}-f-\lambda, e_{2}=d_{2}+4 d_{3}-f-\lambda, e_{3}=d_{2}+3 d_{3}-$ $f-\lambda, e_{4}=d_{2}+d_{3}-f-\lambda$. Therefore, the eigenvalues of Jacobian matrix $J\left(\boldsymbol{n}_{d}\right)$ are given by (A.47).

## A.5.4. Detailed proof of Lemma 2.3

Let $\widetilde{J}$ denote $\partial \widetilde{\boldsymbol{F}} / \partial \widetilde{\boldsymbol{n}}$. Using (2.11) and (2.12), we can obtain $\widetilde{J}$ :

$$
\widetilde{J}=\left(\begin{array}{cc}
6 \alpha_{1}-f & -6 \alpha_{2}  \tag{A.50}\\
-\alpha_{1} & \alpha_{2}-f
\end{array}\right),
$$

where $\alpha_{1}=\alpha / n_{0}, \alpha_{2}=\alpha / n_{1}, \alpha=1 / 2 \phi \sigma(\sigma-1)$. We can obtain $\operatorname{det} \widetilde{J}$ with one of eigenvalues (A.47):

$$
\operatorname{det} \widetilde{J}=f\left(f-6 \alpha_{1}-\alpha_{2}\right)=-f \lambda_{2} .
$$

Since we assume that $\boldsymbol{n}_{d}$ is linearly-stable, $\lambda_{2}<0$ (i.e., $\lambda_{2} \neq 0$ ) holds. Hence, we can apply the implicit function theorem:

$$
\begin{equation*}
\left(\partial n_{0} / \partial \phi \quad \partial n_{1} / \partial \phi\right)^{\top}=\lambda_{2}^{-1} \gamma(6 \quad-1)^{\top} . \tag{A.51}
\end{equation*}
$$

where $\gamma=\left(2 \phi^{2} \sigma(\sigma-1)\right)^{-1} \ln \left(n_{0} / n_{1}\right)$. Since the sign of $\partial n_{0} / \partial \phi$ and $\partial n_{1} / \partial \phi$ are determined by that of $\ln \left(n_{0} / n_{1}\right)$, we can obtain (2.13) and (2.14).

## A.5.5. Detailed proof of Lemma 2.4

We focus on the signs of the eigenvalues in (A.47). $\lambda_{1}$ is negative because $f$ is positive. Moreover, if an equilibrium is stable, $\lambda_{2}$ is always negative regardless of $\tau$. It is obvious that $\lambda_{4}, \lambda_{5}<\lambda_{3}$. Therefore, if the equilibrium becomes unstable with a decrease in $\tau, \lambda_{3}$ becomes positive.

We define $\xi=\frac{1}{\sqrt{6}}(0,1,-1,1,-1,1,-1)^{\top}$. Since $J\left(\boldsymbol{n}_{d}\right) \xi=\lambda_{3} \xi$ holds, $\xi$ is the eigenvector for $\lambda_{3}$.

## A.6. Proofs in Appendix A. 3

## A.6.1. Proof of Lemma A. 3

$V(\ell, 0)>V(\ell, i)(\forall \ell \in \mathcal{L}, \forall i \in \mathcal{P})$ holds if and only if market pattern (F) holds. By Eq. (2.4), this inequality is equivalent to the following inequalities:

$$
\left\{\begin{array}{l}
\sigma_{-1} \ln \left(n_{0} / n_{j}\right)>t((D, 0, x), 0)-t((D, i, x), j)  \tag{A.52}\\
\sigma_{-1} \ln \left(n_{0} / n_{j}\right)>t((S, 0, x), 0)-t((S, i, x), j)
\end{array} \quad \forall i, j \in \mathcal{P}, \forall x \in X\right.
$$

By Eq. (2.9), one of the inequalities in (A.52) is equivalent to the following inequalities:

$$
\sigma_{-1} \ln \left(n_{0} / n_{j}\right)>t((D, 0, x), 0)-t((D, i, x), j) \quad \forall i, j \in \mathcal{P}, \forall x \in X
$$

$$
\begin{align*}
& \Leftrightarrow \sigma_{-1} \ln \left(n_{0} / n_{j}\right)>\phi(2 x-1) \quad \forall j \in \mathcal{P}, \forall x \in X \\
& \Leftrightarrow \sigma_{-1} \ln \left(n_{0} / n_{j}\right) \geq \phi \quad \forall j \in \mathcal{P} . \tag{A.53}
\end{align*}
$$

On the other hand, by Eq. (2.10), the other inequality is equivalent to the followings:

$$
\begin{align*}
& \sigma_{-1} \ln \left(n_{0} / n_{j}\right)>t((S, 0, x), 0)-t((S, i, x), j) \quad \forall i, j \in \mathcal{P}, \forall x \in X \\
& \quad \Leftrightarrow \sigma_{-1} \ln \left(n_{0} / n_{j}\right)>\phi+\tau\left(\frac{1}{2}-\left|x-\frac{1}{2}\right|-\min \{x, 1-x\}\right) \quad \forall j \in \mathcal{P}, \forall x \in X \\
& \quad \Leftrightarrow \sigma_{-1} \ln \left(n_{0} / n_{j}\right)>\phi \quad \forall j \in \mathcal{P} . \tag{A.54}
\end{align*}
$$

By (A.53) and (A.54), (A.52) is equivalent to (A.9).

## A.6.2. Proof of Lemma A. 5

We rewrite inequalities (A.16)-(A.19) to concisely show our proof:

$$
\begin{align*}
-\phi & <\sigma_{-1} \ln \left(n_{0} / n_{1}\right)<\phi-\tau,  \tag{A.55}\\
-\phi & <\sigma_{-1} \ln \left(n_{0} / n_{3}\right)<\phi-\tau,  \tag{A.56}\\
-\phi & <\sigma_{-1} \ln \left(n_{0} / n_{5}\right)<\phi,  \tag{A.57}\\
0 & <\sigma_{-1} \ln \left(n_{1} / n_{3}\right)<2 \tau,  \tag{A.58}\\
0 & <\sigma_{-1} \ln \left(n_{3} / n_{5}\right)<2 \tau,  \tag{A.59}\\
0 & <\sigma_{-1} \ln \left(n_{1} / n_{5}\right)<2 \tau,  \tag{A.60}\\
\tau & <\sigma_{-1} \ln \left(n_{1} / n_{2}\right)  \tag{A.61}\\
\tau & <\sigma_{-1} \ln \left(n_{3} / n_{4}\right)  \tag{A.62}\\
\tau & <\sigma_{-1} \ln \left(n_{1} / n_{6}\right) . \tag{A.63}
\end{align*}
$$

$(\Rightarrow)$ We check that (A.55)-(A.63) hold when market pattern (P) holds. In other words, using (A.11)-(A.15), we prove that inequalities (A.55)-(A.63) hold. First, since $\mathcal{M}_{0} \cap \mathcal{M}_{1}=\left\{\left(D, 1, t_{1}\right),\left(D, 2, t_{2}\right),\left(D, 6, t_{6}\right)\right\}$ holds by (A.11) and (A.12), we can obtain $t_{1}, t_{2}$ and $t_{6}$ :
$t_{1}=\frac{1}{2}\left[\frac{\ln \left(n_{0} / n_{1}\right)}{\phi(\sigma-1)}+1\right], t_{2}=\frac{1}{2}\left[\frac{\ln \left(n_{0} / n_{1}\right)}{\phi(\sigma-1)}+1+\frac{\tau}{\phi}\right], t_{6}=\frac{1}{2}\left[\frac{\ln \left(n_{0} / n_{1}\right)}{\phi(\sigma-1)}+1+\frac{\tau}{\phi}\right]$.

Since $0<t_{1}, t_{2}, t_{6}<1$, we obtain (A.55). Similarly, using (A.11), (A.13) and (A.14), we obtain (A.56) and (A.57).

Since $\mathcal{M}_{1} \cap \mathcal{M}_{3}=\left\{\left(S, 2, T_{2}\right)\right\}$ holds by (A.12) and (A.13), we can obtain $T_{2}=$ $\sigma_{-1}(2 \tau)^{-1} \ln \left(n_{1} / n_{3}\right)$. Since $0<T_{2}<1$, we obtain (A.58). Similarly, using (A.12)(A.14), we obtain (A.59) and (A.60).

Next, since $\mathcal{S}_{2}\left(\left(0, T_{2}\right]\right) \subset \mathcal{M}_{1}$ holds by (A.12), we obtain the following inequality:

$$
\begin{equation*}
V((S, 2, x), 1)>V((S, 2, x), 2) \quad \forall x \in\left(0, T_{2}\right] . \tag{A.64}
\end{equation*}
$$

Using (A.64), we can obtain (A.61):

$$
\begin{aligned}
(A .64) & \Leftrightarrow \sigma_{-1} \ln n_{2}-\tau x<\sigma_{-1} \ln n_{1}-\tau(x+1) \quad \forall x \in\left(0, T_{2}\right] \\
& \Rightarrow \tau<\sigma_{-1} \ln \left(n_{1} / n_{2}\right) .
\end{aligned}
$$

Similarly, using (A.12)-(A.14), we can obtain (A.62) and (A.63).
$(\Leftarrow)$ We check that market pattern $(\mathrm{P})$ holds when (A.55)-(A.63) hold. We first prove $\mathcal{M}_{2}=\mathcal{M}_{4}=\mathcal{M}_{6}=\emptyset$ (i.e., (A.15)). We prove $\mathcal{M}_{2}=\emptyset$. The following holds by Eq. (2.10):

$$
\begin{equation*}
t(\ell, 1)-t(\ell, 2) \leq \tau \quad \forall \ell \in \cup_{m \in \mathcal{P}} S_{m}(X) \tag{A.65}
\end{equation*}
$$

Using (2.4), (A.61) and (A.65), we obtain

$$
\begin{equation*}
V(\ell, 1)-V(\ell, 2) \geq \sigma_{-1} \ln \left(n_{1} / n_{2}\right)-\tau>\tau-\tau=0 \quad \forall \ell \in \cup_{m \in \mathcal{P}} S_{m}(X) \tag{A.66}
\end{equation*}
$$

For any $i \in \mathcal{P}$, the following holds by Eqs. (2.4) and (2.9):

$$
\begin{equation*}
V(\ell, 2) \in\left\{v_{a 1}, v_{a 2}\right\} \quad \forall \ell \in D_{i}(X) \tag{A.67}
\end{equation*}
$$

where $v_{a 1}=\sigma_{-1} \ln n_{2}-\phi(1+x)+V_{D}, v_{a 2}=\sigma_{-1} \ln n_{2}-\phi(1-x)-\tau L_{i 2}+V_{D}$. By (A.55) and (A.61), the following holds:

$$
\begin{equation*}
V(\ell, 0)-v_{a 1}=\sigma_{-1} \ln \left(n_{0} / n_{2}\right)+\phi>\tau>0 \quad \forall \ell \in D_{i}(X) . \tag{A.68}
\end{equation*}
$$

Moreover, since $V(\ell, 1) \geq \sigma_{-1} \ln n_{1}-\phi(1-x)-\tau L_{i 1}+V_{D}\left(\forall \ell \in D_{i}(X)\right)$ holds by (2.9), the following holds by (A.61):

$$
\begin{equation*}
V(\ell, 1)-v_{a 2} \geq \sigma_{-1} \ln \left(n_{1} / n_{2}\right)+\tau L_{i 2}-\tau L_{i 1}>0 \quad \forall \ell \in D_{i}(X) \tag{A.69}
\end{equation*}
$$

By (A.66)-(A.69), either $V(\ell, 2)<V(\ell, 0)$ or $V(\ell, 2)<V(\ell, 1)$ holds $(\forall \ell \in \mathcal{L})$, which implies $\mathcal{M}_{2}=\emptyset$. Similarly, by (A.55), (A.56), (A.62) and (A.63), $\mathcal{M}_{4}=\mathcal{M}_{6}=\emptyset$ holds.

Next, we prove (A.11)-(A.14). Note that $\mathcal{M}_{0}, \mathcal{M}_{1}, \mathcal{M}_{3}, \mathcal{M}_{5}$ are determined by the relationship only among $V(\ell, 0), V(\ell, 1), V(\ell, 3)$ and $V(\ell, 5)$ because $\mathcal{M}_{2}=\mathcal{M}_{4}=$ $\mathcal{M}_{6}=\emptyset$.

We focus on the market areas on the ring road (i.e., $\left.\ell \in \cup_{m \in \mathcal{P}} S_{m}(X)\right)$. Using (2.4), (2.10), and (A.55), we can obtain the following inequality:

$$
\begin{equation*}
V(\ell, 0)<V(\ell, 1) \quad \forall \ell \in \mathcal{S}_{1}(X) \cup \mathcal{S}_{2}((0,1 / 2]) \cup \mathcal{S}_{5}([1 / 2,1)) \cup \mathcal{S}_{6}(X) \tag{A.70}
\end{equation*}
$$

Similarly, by (A.56) and (A.57), the followings hold:

$$
\begin{align*}
& V(\ell, 0)<V(\ell, 3) \quad \forall \ell \in \mathcal{S}_{2}([1 / 2,1)) \cup \mathcal{S}_{3}(X) \cup \mathcal{S}_{4}((0,1 / 2]),  \tag{A.71}\\
& V(\ell, 0)<V(\ell, 5) \quad \forall \ell \in \mathcal{S}_{4}([1 / 2,1)) \cup \mathcal{S}_{5}((0,1 / 2]) . \tag{A.72}
\end{align*}
$$

By (A.70)-(A.72), $\mathcal{M}_{0} \cap\left(\cup_{m \in \mathcal{P}} S_{m}(X)\right)=\emptyset$ and $\cup_{m \in \mathcal{P}} S_{m}(X) \subset \mathcal{M}_{1} \cup \mathcal{M}_{3} \cup \mathcal{M}_{5}$ hold.
We compare $V(\ell, 1), V(\ell, 3)$ and $V(\ell, 5)$ for $\ell \in \cup_{m \in \mathcal{P}} S_{m}(X)$. These functions are shown in Table A.1. By the results in Table A. 1 and (A.58)-(A.60), there exist $T_{2}, T_{4}, T_{5} \in X$ such that

$$
\begin{array}{r}
\mathcal{S}_{1}(X) \cup \mathcal{S}_{2}\left(\left(0, T_{2}\right]\right) \cup \mathcal{S}_{5}\left(\left(T_{5}, 1\right)\right) \cup \mathcal{S}_{6}(X) \subset \mathcal{M}_{1}, \\
\mathcal{S}_{2}\left(\left(T_{2}, 1\right)\right) \cup \mathcal{S}_{3}(X) \cup \mathcal{S}_{4}\left(\left(0, T_{4}\right)\right) \subset \mathcal{M}_{3}, \\
\mathcal{S}_{4} \times\left(\left(T_{4}, 1\right)\right) \cup \mathcal{S}_{5}\left(\left(0, T_{5}\right)\right) \subset \mathcal{M}_{5} . \tag{A.75}
\end{array}
$$

Next, we focus on the market areas on the radial roads (i.e., $\ell \in \cup_{m \in \mathcal{P}} D_{m}(X)$ ). By (A.55)-(A.57) and (A.73)-(A.75), a similar argument to the derivation of (A.67)-(A.69)

Table A.1: The list of the indirect utilities for $\ell \in \cup_{m \in \mathcal{P}} S_{m}(X)$.

|  | $V(\ell, 1)$ | $V(\ell, 3)$ | $V(\ell, 5)$ |
| :--- | :--- | :--- | :--- |
| $\ell \in S_{1}(X)$ | $\beta_{1}-\tau x$ | $\beta_{2}-\tau(2-x)$ | $\beta_{3}-\tau(2+x)$ |
| $\ell \in S_{2}(X)$ | $\beta_{1}-\tau(1+x)$ | $\beta_{2}-\tau(1-x)$ | $\beta_{3}-\tau(3-x)$ |
| $\ell \in S_{3}(X)$ | $\beta_{1}-\tau(2+x)$ | $\beta_{2}-\tau x$ | $\beta_{3}-\tau(2-x)$ |
| $\ell \in S_{4}(X)$ | $\beta_{1}-\tau(3-x)$ | $\beta_{2}-\tau(1+x)$ | $\beta_{3}-\tau(1-x)$ |
| $\ell \in S_{5}(X)$ | $\beta_{1}-\tau(2-x)$ | $\beta_{2}-\tau(2+x)$ | $\beta_{3}-\tau x$ |
| $\ell \in S_{6}(X)$ | $\beta_{1}-\tau(1-x)$ | $\beta_{2}-\tau(3-x)$ | $\beta_{3}-\tau(1+x)$ |

Notes: $\beta_{1}=\sigma_{-1} \ln n_{1}+V_{D} ; \beta_{2}=\sigma_{-1} \ln n_{3}+V_{D} ; \beta_{3}=\sigma_{-1} \ln n_{5}+V_{D}$.
shows that there exist $t_{i} \in X(\forall i \in \mathcal{P})$ such that

$$
\left.\begin{array}{rl}
\cup_{m=1}^{6} \mathcal{D}_{m}\left(\left(0, t_{i}\right]\right) & \subset \mathcal{M}_{0} \\
\mathcal{D}_{1}\left(\left[t_{1}, 1\right)\right) \cup \mathcal{D}_{2}\left(\left[t_{2}, 1\right)\right) \cup \mathcal{D}_{6}\left(\left[t_{6}, 1\right)\right) & \subset \mathcal{M}_{1} \\
\mathcal{D}_{3}\left(\left[t_{3}, 1\right)\right) \cup \mathcal{D}_{4}\left(\left[t_{4}, 1\right)\right) & \subset \mathcal{M}_{3} \\
& \mathcal{D}_{5}\left(\left[t_{5}, 1\right)\right) \tag{A.79}
\end{array}\right) \subset \mathcal{M}_{5} .
$$

(A.73)-(A.79) are equal to (A.11)-(A.14) .

## B. Appendices for Chapter 3

## B.1. Details of modeling of the FO model

The fundamental logic and the governing equation of a multi-regional version of the model by Forslid and Ottaviano (2003) are presented (see Akamatsu et al., 2012). The budget constraint is given as

$$
\begin{equation*}
p_{i}^{\mathrm{A}} C_{i}^{\mathrm{A}}+\sum_{j \in P} \int_{0}^{n_{j}} p_{j i}(\ell) q_{j i}(\ell) d \ell=Y_{i} \tag{B.1}
\end{equation*}
$$

where $p_{i}^{\mathrm{A}}$ represents the price of the A -sector good in place $i, C_{i}^{\mathrm{A}}$ is the consumption of A-sector goods in place $i, P=\{1, \ldots, K\}, n_{j}$ is the number of varieties produced in region $j, p_{j i}(\ell)$ denotes the price of a variety $\ell$ in place $i$ produced in place $j, q_{j i}(\ell)$ is the consumption of variety $\ell \in\left[0, n_{j}\right]$ in place $i$ produced in place $j$, and $Y_{i}$ is the income of an individual in place $i$. The incomes (wages) of skilled workers and unskilled workers are represented respectively by $w_{i}$ and $w_{i}^{\mathrm{L}}$.

An individual at place $i$ maximizes the utility in (3.5) subject to the budget constraint in (B.1). This maximization yields the following demand functions

$$
C_{i}^{\mathrm{A}}=(1-\mu) \frac{Y_{i}}{p_{i}^{\mathrm{A}}}, \quad C_{i}^{\mathrm{M}}=\mu \frac{Y_{i}}{\rho_{i}}, \quad q_{j i}(\ell)=\mu \frac{\rho_{i}^{\sigma-1} Y_{i}}{p_{j i}(\ell)^{\sigma}},
$$

where $\rho_{i}$ denotes the price index of the differentiated products in place $i$, and is given by

$$
\begin{equation*}
\rho_{i}=\left(\sum_{j \in P} \int_{0}^{n_{j}} p_{j i}(\ell)^{1-\sigma} d \ell\right)^{1 /(1-\sigma)} \tag{B.2}
\end{equation*}
$$

Because the total income in place $i$ is $w_{i} \lambda_{i}+w_{i}^{\mathrm{L}}$, the total demand $Q_{j i}(\ell)$ in place $i$ for a variety $\ell$ produced in place $j$ is given as

$$
\begin{equation*}
Q_{j i}(\ell)=\mu \frac{\rho_{i}^{\sigma-1}}{p_{j i}(\ell)^{\sigma}}\left(w_{i} \lambda_{i}+w_{i}^{\mathrm{L}}\right) \tag{B.3}
\end{equation*}
$$

The A-sector is perfectly competitive and produces homogeneous goods under constant-returns-to-scale, and requires one unit of unskilled labor per unit of output. The

A-sector good is traded freely across locations and is chosen as the numéraire. In equilibrium, $p_{i}^{\mathrm{A}}=w_{i}^{\mathrm{L}}=1$ for each $i$.

The M-sector output is produced under increasing-returns-to-scale and Dixit-Stiglitz monopolistic competition. A firm incurs a fixed input requirement of $\alpha$ units of skilled labor and a marginal input requirement of $\beta$ units of unskilled labor. An M-sector firm located in place $i$ chooses $\left(p_{i j}(\ell) \mid j \in P\right)$ that maximizes its profit

$$
\begin{equation*}
\Pi_{i}(\ell)=\sum_{j \in P} p_{i j}(\ell) Q_{i j}(\ell)-\left(\alpha w_{i}+\beta x_{i}(\ell)\right), \tag{B.4}
\end{equation*}
$$

where $x_{i}(\ell)$ denotes the total supply of variety $\ell$ produced in place $i$ and $\alpha w_{i}+\beta x_{i}(\ell)$ signifies the cost function introduced by Flam and Helpman (1987).

With the use of the iceberg form of the transport cost, we have

$$
\begin{equation*}
x_{i}(\ell)=\sum_{j \in P} \tau_{i j} Q_{i j}(\ell) \tag{B.5}
\end{equation*}
$$

Then the profit function of the M-sector firm in place $i$, given in (B.4) above, can be rewritten as

$$
\Pi_{i}(\ell)=\sum_{j \in P} p_{i j}(\ell) Q_{i j}(\ell)-\left(\alpha w_{i}+\beta \sum_{j \in P} \tau_{i j} Q_{i j}(\ell)\right)
$$

which is maximized by the firm. The first-order condition for this profit maximization yields the following optimal price

$$
\begin{equation*}
p_{i j}(\ell)=\frac{\sigma \beta}{\sigma-1} \tau_{i j} . \tag{B.6}
\end{equation*}
$$

This result implies that $p_{i j}(\ell), Q_{i j}(\ell)$, and $x_{i}(\ell)$ are independent of $\ell$. Therefore, the argument $\ell$ is suppressed subsequently.

In the short run, skilled workers are immobile between places, i.e., their spatial distribution $\boldsymbol{\lambda}=\left(\lambda_{i} \mid i \in P\right)$ is assumed to be given. The market equilibrium conditions consist of three conditions: the M-sector goods market clearing condition, the zeroprofit condition attributable to the free entry and exit of firms, and the skilled labor
market clearing condition. The first condition is written as (B.5) above. The second one requires that the operating profit of a firm, given in (B.4), be absorbed entirely by the wage bill of its skilled workers. This gives

$$
\begin{equation*}
w_{i}=\frac{1}{\alpha}\left\{\sum_{j \in P} p_{i j} Q_{i j}-\beta x_{i}\right\} . \tag{B.7}
\end{equation*}
$$

The third condition is expressed as $\alpha n_{i}=\lambda_{i}$ and the price index $\rho_{i}$ in (B.2) can be rewritten using (B.6) as

$$
\begin{equation*}
\rho_{i}=\frac{\sigma \beta}{\sigma-1}\left(\frac{1}{\alpha} \sum_{j \in P} \lambda_{j} d_{j i}\right)^{1 /(1-\sigma)} \tag{B.8}
\end{equation*}
$$

The market equilibrium wage $w_{i}$ in (B.7) can be represented as

$$
\begin{equation*}
w_{i}=\frac{\mu}{\sigma} \sum_{j \in P} \frac{d_{i j}}{\Delta_{j}}\left(w_{j} \lambda_{j}+1\right) \tag{B.9}
\end{equation*}
$$

using $d_{j i}=\tau_{j i}^{1-\sigma}=\phi^{m(i, j)}$, (B.3), (B.5), (B.6), and (B.8). Here, $\Delta_{j}=\sum_{k \in P} d_{k j} \lambda_{k}$. Equation (B.9) can be rewritten, using $\boldsymbol{w}=\left(w_{1}, \ldots, w_{K}\right)$, as $\boldsymbol{w}=\frac{\mu}{\sigma} D \Delta^{-1}(\Lambda \boldsymbol{w}+\mathbf{1})$, which is solved for $\boldsymbol{w}$ as

$$
\begin{equation*}
\boldsymbol{w}=\frac{\mu}{\sigma}\left(I-\frac{\mu}{\sigma} D \Delta^{-1} \Lambda\right)^{-1} D \Delta^{-1} \mathbf{1} \tag{B.10}
\end{equation*}
$$

with $I$ being the identity matrix, $\mathbf{1}=(1, \ldots, 1)^{\top}$, and

$$
\begin{equation*}
D=\left(d_{i j}\right), \quad \Delta=\operatorname{diag}\left(\Delta_{1}, \ldots, \Delta_{K}\right), \quad \Lambda=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{K}\right) \tag{B.11}
\end{equation*}
$$

## B.2. Classifications of stationary points

The mono-center is classified into corner solutions, for which some places have zero population. We can appropriately permute the components of $\boldsymbol{\lambda}$, without loss of generality, to arrive at

$$
\hat{\boldsymbol{\lambda}}=\left(\boldsymbol{\lambda}_{+}, \mathbf{0}_{K-m}\right),
$$

where all components of $\boldsymbol{\lambda}_{+}=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ are positive and $\mathbf{0}_{K-m}$ is the $(K-m)$ dimensional zero vector. $\mathbf{0}_{K-m}$ is present for corner solutions $(K>m)$. For corner solutions, the governing equation (3.4) and associated Jacobian matrix can be rearranged, respectively, as (Ikeda et al., 2018b)

$$
\hat{\boldsymbol{F}}=\binom{\boldsymbol{F}_{+}(\hat{\boldsymbol{\lambda}}, \phi)}{\boldsymbol{F}_{0}(\hat{\boldsymbol{\lambda}}, \phi)}, \quad \hat{J}=\frac{\partial \hat{\boldsymbol{F}}}{\partial \hat{\boldsymbol{\lambda}}}=\left(\begin{array}{cc}
J_{+} & J_{+0} \\
O & J_{0}
\end{array}\right)
$$

where

$$
\begin{aligned}
& J_{+}=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m}\right) \times\left(\left.\frac{\partial\left(v_{i}-\bar{v}\right)}{\partial \lambda_{j}} \right\rvert\, i, j=1, \ldots, m\right) \\
& J_{+0}=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m}\right) \times\left(\left.\frac{\partial\left(v_{i}-\bar{v}\right)}{\partial \lambda_{j}} \right\rvert\, i=1, \ldots, m ; j=m+1, \ldots, K\right), \\
& J_{0}=\operatorname{diag}\left(v_{m+1}-\bar{v}, \ldots, v_{K}-\bar{v}\right)
\end{aligned}
$$

and $\operatorname{diag}(\cdots)$ denotes a diagonal matrix with the entries in parentheses.
The stability condition of a stable corner solution is decomposed into two conditions:

$$
\begin{cases}\text { Stability condition for } \boldsymbol{\lambda}_{+}: & \text {all eigenvalues of } J_{+} \text {are negative. } \\ \text { Sustainability condition: } & \text { all diagonal entries of } J_{0} \text { are negative. }\end{cases}
$$

Both of these two conditions are satisfied if and only if all eigenvalues of $\hat{J}$ are negative. ${ }^{50}$
Critical points are those which have one or more zero eigenvalue(s) of the Jacobian matrix $\hat{J}$. Critical points are classified into a bifurcation point with singular $J_{+}$or $J_{0}$, and a limit point of $\phi$ with singular $J_{+}$. We classify bifurcation points into a break bifurcation point with singular $J_{+}$and a corner bifurcation point with singular $J_{0}$. The corner bifurcation from the mono-center is theoretically investigated in this chapter.

[^35]
## B.3. Theoretical details

## B.3.1. Proof of Lemma 3.1

Since $T(g) \boldsymbol{\lambda}=\boldsymbol{\lambda}$ for a subgroup $G^{\prime}$ of $G$, we have $T(g) \boldsymbol{v}(\boldsymbol{\lambda}, \phi)=\boldsymbol{v}(\boldsymbol{\lambda}, \phi)$ for $g \in G^{\prime}$ by (3.10). This means that $v_{i}$ in the same orbit is permutable. This suffices for the proof.

## B.3.2. Proof of Lemma 3.2

We can rearrange the components of $\boldsymbol{\lambda}_{P_{l}}$ to arrive at $\hat{\boldsymbol{\lambda}}=\left(\frac{1}{m} \mathbf{1}_{m}, \mathbf{0}_{K-m}\right)$, where $\mathbf{1}_{m}$ is the $m$-dimensional all-one vector, and $\mathbf{0}_{K-m}$ is the $(K-m)$-dimensional zero vector. The $m\left(=N_{l}\right)$ places belonging to $\boldsymbol{\lambda}_{+}=\frac{1}{m} \mathbf{1}_{m}$ are permuted each other by the geometrical transformation by an element of a subgroup of $G$ and $\boldsymbol{\lambda}_{+}$is invariant with respect to the permutation. By equivariance (3.10), we have $v_{1}=\cdots=v_{m}$, as well as $\lambda_{1}=\cdots=\lambda_{m}=1 / m$. Thus we have $v_{i}=\bar{v}\left(i \in P_{l}\right)$.

## B.3.3. Proof of Proposition 3.1

By Lemma 3.2, we have $v_{i}-\bar{v}=0(i=1, \ldots, m)$. Thus, $\boldsymbol{F}_{+}\left(\frac{1}{m} \mathbf{1}_{m}, \mathbf{0}_{K-m}, \phi\right)=\mathbf{0}_{m}$ is satisfied. For $K-m$ places with no population, we have $\lambda_{j}=0$, thereby satisfying $\boldsymbol{F}_{0}\left(\frac{1}{m} \mathbf{1}_{m}, \mathbf{0}_{K-m}, \phi\right)=\mathbf{0}_{K-m}$. This shows that $\left(\boldsymbol{\lambda}_{+}, \boldsymbol{\lambda}_{0}, \phi\right)=\left(\frac{1}{m} \mathbf{1}_{m}, \mathbf{0}_{K-m}, \phi\right)$ serves as a solution to (3.4) for any $\phi$.

## B.3.4. Proof of Proposition 3.2

By choosing $G=E$, which leaves each place unchanged, each place forms an orbit. Therefore, the full agglomeration at any place $i$ is an invariant pattern by Proposition 3.1.

## B.3.5. Proof of Proposition 3.3

The bifurcating conditions of the Jacobian matrix (3.13) are given by as follows:

$$
\begin{equation*}
v_{1}=0, \tag{B.12}
\end{equation*}
$$

$$
\begin{array}{ll}
v_{\alpha i}-v_{c}=0, & i=1, \ldots, n_{1} \\
v_{\beta i}-v_{c}=0, & i=1, \ldots, n_{2} \tag{B.14}
\end{array}
$$

However, no bifurcation solution emerges in the space $\sum_{i=1}^{K} \lambda_{i}=1$ because the direction of this solution is $(1,0, \ldots, 0)$. Therefore, only (B.13) and (B.14) are the bifurcating conditions from the mono-center.

## B.3.6. Proof of Lemma 3.3

By the product form of the replicator dynamics in (3.3), $\tilde{F}_{i}(\boldsymbol{x}, \psi)$ takes the product form: $\tilde{F}_{i}(\boldsymbol{x}, \psi)=x_{i} G_{i}(\boldsymbol{x}, \psi)\left(i \in P_{l}\right)$. Since the group $\mathrm{D}_{6}$ is generated by the elements $r$ and $s$, it suffices to consider the symmetry condition $T(g) \tilde{\boldsymbol{F}}(\boldsymbol{x})=\tilde{\boldsymbol{F}}(T(g) \boldsymbol{x})$ for $g=r, s$. This condition for $g=r$ gives the form (3.19) for some function $R(\boldsymbol{x}, \psi)$ and that for $g=s$ gives the symmetry condition (3.20).

## B.3.7. Proof of proposition 3.4

Proof of Proposition 3.4: To begin with, we consider $\boldsymbol{x}=w(1,1,1,1,1,1)$. Then (3.19) reduced to a single condition $w R(w, w, w, w, w, w, \psi)=0$. Since $w=0$ corresponds to the pre-bifurcation solution, we focus on a relation $R(w, w, w, w, w, w, \psi)=0$. Because this relation in general has a solution of the form $\psi=a w+$ (higher order terms) for some real constant $a$ in the neighborhood of the bifurcation point, the bifurcating solution of the form $\boldsymbol{x}=w(1,1,1,1,1,1)$ exists. The other five cases can be treated similarly.

## B.3.8. The detail of the bifurcation equation for the orbit of Type $\beta \boldsymbol{\beta i}$

Lemma B.1. For the orbit of Type $\beta i$ Bifurcation equation for Type $\beta i$ orbit is derived as follow: some $i$, which comprises twelve points, the bifurcation equation becomes:

$$
\tilde{\boldsymbol{F}}_{1}(\boldsymbol{x}, \psi)=x_{1} R\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}, x_{10}, x_{11}, x_{12}, \psi\right)=0
$$

$$
\begin{aligned}
\tilde{\boldsymbol{F}}_{2}(\boldsymbol{x}, \psi) & =x_{2} R\left(x_{2}, x_{1}, x_{12}, x_{11}, x_{10}, x_{9}, x_{8}, x_{7}, x_{6}, x_{5}, x_{4}, x_{3}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{3}(\boldsymbol{x}, \psi) & =x_{3} R\left(x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}, x_{10}, x_{11}, x_{12}, x_{1}, x_{2}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{4}(\boldsymbol{x}, \psi) & =x_{4} R\left(x_{4}, x_{3}, x_{2}, x_{1}, x_{12}, x_{11}, x_{10}, x_{9}, x_{8}, x_{7}, x_{6}, x_{5}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{5}(\boldsymbol{x}, \psi) & =x_{5} R\left(x_{5}, x_{6}, x_{7}, x_{8}, x_{9}, x_{10}, x_{11}, x_{12}, x_{1}, x_{2}, x_{3}, x_{4}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{6}(\boldsymbol{x}, \psi) & =x_{6} R\left(x_{6}, x_{5}, x_{4}, x_{3}, x_{2}, x_{1}, x_{12}, x_{11}, x_{10}, x_{9}, x_{8}, x_{7}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{7}(\boldsymbol{x}, \psi) & =x_{7} R\left(x_{7}, x_{8}, x_{9}, x_{10}, x_{11}, x_{12}, x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{8}(\boldsymbol{x}, \psi) & =x_{8} R\left(x_{8}, x_{7}, x_{6}, x_{5}, x_{4}, x_{3}, x_{2}, x_{1}, x_{12}, x_{11}, x_{10}, x_{9}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{9}(\boldsymbol{x}, \psi) & =x_{9} R\left(x_{9}, x_{10}, x_{11}, x_{12}, x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{10}(\boldsymbol{x}, \psi) & =x_{10} R\left(x_{10}, x_{9}, x_{8}, x_{7}, x_{6}, x_{5}, x_{4}, x_{3}, x_{2}, x_{1}, x_{12}, x_{11}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{11}(\boldsymbol{x}, \psi) & =x_{11} R\left(x_{11}, x_{12}, x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}, x_{10}, \psi\right)=0 \\
\tilde{\boldsymbol{F}}_{12}(\boldsymbol{x}, \psi) & =x_{12} R\left(x_{12}, x_{11}, x_{10}, x_{9}, x_{8}, x_{7}, x_{6}, x_{5}, x_{4}, x_{3}, x_{2}, x_{1}, \psi\right)=0
\end{aligned}
$$

where $\boldsymbol{x}=\left(x_{1}, \ldots, x_{12}\right)=\left\{\lambda_{i} \mid i \in \beta i\right\}$ (Fig. 3.7(b)) and $\psi=\phi-\phi_{l}^{\text {c }}$.

Proof. By the product form of the replicator dynamics in (3.3), $\tilde{F}_{i}(\boldsymbol{x}, \psi)$ takes the product form: $\tilde{F}_{i}(\boldsymbol{x}, \psi)=x_{i} R_{i}(\boldsymbol{x}, \psi)\left(i \in P_{\beta i}\right)$. Since the group $\mathrm{D}_{6}$ is generated by the elements $r$ and $s$, it suffice to consider the symmetry condition $T(g) \tilde{\boldsymbol{F}}(\boldsymbol{x})=\tilde{\boldsymbol{F}}(T(g) \boldsymbol{x})$ for $g=r, s$. This condition for $g=r$ gives the form for $\tilde{\boldsymbol{F}}_{i}(i=1,3, \ldots, 11)$ for some function $R(\boldsymbol{x}, \psi)$ and that for $g=s$ gives the form for $\tilde{\boldsymbol{F}}_{i}(i=2,4, \ldots, 12)$.

## B.4. Robustness check with 91 regions case

Conducting bifurcation analysis for stable agglomeration patterns for $K=91$, we examine whether or not the number of regions qualitatively affects the change of agglomeration pattern with a change in the transportation cost. Fig. B. 1 shows a transition from the agglomeration pattern of racetrack cities by the comparative static analysis with a change in $\phi$. The bifurcating solution for the racetrack cities branching from $I_{\beta}^{\prime}$ reached the stable mono-center. At point C, there emerges a stable core-satellite

(a) The bifurcation diagram for twin cities (Twin-III for $\alpha 1$ )

Figure B.1: Bifurcation solutions from stable invariant patterns for 91 regions $((\sigma, \mu)=$ $6.0,0.4)$. - : stable equilibria; -- - : unstable equilibria; $\bigcirc$ : bifurcation point; $\bullet$ : the size of population at the place.
pattern. Because the direction of the local unstable manifold at $I_{\beta}^{\prime}$ is the mono-center, the racetrack cities dynamically changes to the stable mono-center near $\phi=0.565$. Hence, the result of the comparative static analysis is qualitatively the same as the results shown in Section 3.4.3.

## C. Appendices for Chapter 4

## C.1. Theoretical details of Section 4.3

We show theoretical details given market area $\left\{\mathcal{I}_{j}\right\}_{j \in \mathcal{J}}$ in this Appendix. Using the market area, we focus on equilibrium such that consumers in residential zone $i(\in \mathcal{I})$ visit marketplace $j(i)(\in \mathcal{J})$ for shopping.

## C.1.1. First order conditions for the expenditure minimization problem

We solve the following expenditure minimization problem:

$$
\begin{equation*}
\min _{\left\{q_{i}(k)\right\}_{k}} \int_{0}^{m_{j(i)}} p_{j(i)}^{M}(k) q_{i}(k) \mathrm{d} k, \quad \text { s.t. } M_{i}=\int_{0}^{m_{j(i)}} u\left(q_{i}(k)\right) \mathrm{d} k . \tag{C.1}
\end{equation*}
$$

The first order condition for the optimization of problem (C.1) is given by

$$
\begin{align*}
& p_{j(i)}^{M}(k)=\rho_{1} u^{\prime}\left(q_{i}(k)\right) \quad \forall k,  \tag{C.2}\\
& M_{i}=\int_{0}^{m_{j(i)}} u\left(q_{i}(k)\right) \mathrm{d} k, \tag{C.3}
\end{align*}
$$

where $\rho_{1}$ is the Lagrange multiplier. Solving this problem with the above first order condition, we can obtain conditional demand (4.7), as shown in Section 4.2:

$$
q_{i}^{*}(k)=\widetilde{q_{i}^{*}}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, M_{i}\right) .
$$

Let $e_{i}^{M}$ be expenditure function regarding the above conditional demands. This is given by

$$
e_{i}^{M}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, M_{i}\right)=\int_{0}^{m_{j(i)}} p_{j(i)}^{M}(k) q_{i}^{*}(k) \mathrm{d} k
$$

Next, we solve the following expenditure minimization problem:

$$
\begin{equation*}
\min _{M_{i}, h_{i}, a_{i}} p_{i}^{H} h_{i}+e_{i}^{M}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, M_{i}\right)+a_{i}, \quad \text { s.t. } U_{i}=\bar{U} . \tag{C.4}
\end{equation*}
$$

The first order condition for the above optimization problem is given by

$$
\begin{equation*}
p_{i}^{H}=\rho_{2} \frac{\partial U_{i}}{\partial h_{i}} \tag{C.5}
\end{equation*}
$$

$$
\begin{align*}
\frac{\partial e_{i}^{M}}{\partial M_{i}} & =\rho_{2} \frac{\partial U_{i}}{\partial M_{i}}  \tag{C.6}\\
1 & =\rho_{2} \frac{\partial U_{i}}{\partial a_{i}}  \tag{C.7}\\
U_{i} & =\bar{U} \tag{C.8}
\end{align*}
$$

where $\rho_{2}$ is the Lagrange multiplier. Solving this problem with the above first order condition, we can obtain the Hicksian demand functions:

$$
\begin{aligned}
& M_{i}^{*}=\widetilde{M_{i}^{*}}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, p_{i}^{H}, \bar{U}\right), \\
& h_{i}^{*}=\widetilde{h_{i}^{*}}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, p_{i}^{H}, \bar{U}\right), \\
& a_{i}^{*}=\widetilde{a_{i}^{*}}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, p_{i}^{H}, \bar{U}\right) .
\end{aligned}
$$

Substituting $M_{i}^{*}$ into conditional demand $q_{i}^{*}(k)$ yields

$$
q_{i}^{*}(k)=\widetilde{q_{i}^{*}}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, \widetilde{M_{i}^{*}}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, p_{i}^{H}, \bar{U}\right)\right)
$$

Using the Hicksian demands, we obtain expenditure function for consumers residing in zone $i$ :

$$
\begin{equation*}
e_{i}=p_{i}^{H} h_{i}^{*}+e_{i}^{M}\left(\left\{p_{j(i)}^{M}(k)\right\}_{k}, m_{j(i)}, M_{i}^{*}\right)+a_{i}^{*} \tag{C.9}
\end{equation*}
$$

## C.1.2. Proof of Lemma 4.1

We focus on a marginal change in the Allais surplus at $\left(\boldsymbol{n}(s), \overline{U^{*}}\right)$ with respect to $s$. We have

$$
\begin{equation*}
\frac{\mathrm{d} A S}{\mathrm{~d} s}=\sum_{i \in \mathcal{I}} n_{i} \frac{\mathrm{~d}}{\mathrm{~d} s}\left(y-t_{i}+\Pi+s_{i}(s)-e_{i}\right)+\left(y-t_{i}+\Pi+s_{i}(s)-e_{i}\right) \frac{\mathrm{d} n_{i}}{\mathrm{~d} s} \tag{C.10}
\end{equation*}
$$

Since $y-t_{i}+\Pi+s_{i}(s)-e_{i}=\bar{E}$ and $\sum_{i \in \mathcal{I}} \mathrm{~d} n_{i} / \mathrm{d} s=0$ hold by conditions (4.23) and (4.24), the second term is zero. Furthermore, $y$ and $t_{i}$ are not functions of $s$. Hence, we have

$$
\begin{equation*}
\frac{\mathrm{d} A S}{\mathrm{~d} s}=\sum_{i \in \mathcal{I}} n_{i} \frac{\mathrm{~d}}{\mathrm{~d} s}\left(s_{i}(s)+\Pi-e_{i}\right) \tag{C.11}
\end{equation*}
$$

We focus on the derivative of the expenditure function. Under the price equilibrium of varieties, the prices of the varieties supplied in a marketplace are the same as shown in (4.14). Hence, the derivative of expenditure function (C.9) is given by

$$
\begin{equation*}
\frac{\mathrm{d} e_{i}}{\mathrm{~d} s}=h_{i}^{*} \frac{\mathrm{~d} p_{i}^{H}}{\mathrm{~d} s}+p_{i}^{H} \frac{\mathrm{~d} h_{i}^{*}}{\mathrm{~d} s}+\frac{\partial e_{i}^{M}}{\partial p_{j(i)}^{M}} \frac{\mathrm{~d} p_{j(i)}^{M}}{\mathrm{~d} s}+\frac{\partial e_{i}^{M}}{\partial m_{j(i)}} \frac{\mathrm{d} m_{j(i)}}{\mathrm{d} s}+\frac{\partial e_{i}^{M}}{\partial M_{i}} \frac{\mathrm{~d} M_{i}^{*}}{\mathrm{~d} s}+\frac{\mathrm{d} a_{i}^{*}}{\mathrm{~d} s} . \tag{C.12}
\end{equation*}
$$

Substituting the Hicksian demands into the utility function yields $U_{i}\left(M_{i}^{*}, h_{i}^{*}, a_{i}^{*}\right)=$ $\overline{U^{*}}$. The derivative of the utility is given by

$$
\begin{equation*}
\frac{\mathrm{d} U_{i}}{\mathrm{~d} s}=\frac{\partial U_{i}}{\partial M_{i}} \frac{\mathrm{~d} M_{i}^{*}}{\mathrm{~d} s}+\frac{\partial U_{i}}{\partial h_{i}} \frac{\mathrm{~d} h_{i}^{*}}{\mathrm{~d} s}+\frac{\partial U_{i}}{\partial a_{i}} \frac{\mathrm{~d} a_{i}^{*}}{\mathrm{~d} s}=0 . \tag{C.13}
\end{equation*}
$$

Using first order conditions (C.5)-(C.7) for expenditure minimization problem (C.4) yields

$$
\begin{align*}
\frac{\partial e_{i}^{M}}{\partial M_{i}} & =\left(\frac{\partial U_{i}}{\partial M_{i}}\right)\left(\frac{\partial U_{i}}{\partial a_{i}}\right)^{-1}  \tag{C.14}\\
p_{i}^{H} & =\left(\frac{\partial U_{i}}{\partial h_{i}}\right)\left(\frac{\partial U_{i}}{\partial a_{i}}\right)^{-1} \tag{C.15}
\end{align*}
$$

Multiplying both sides of Eq. (C.13) by $\left(\partial U_{i} / \partial a_{i}\right)^{-1}$ and substituting (C.14) and (C.15) into the equation yields

$$
\begin{equation*}
\frac{\partial e_{i}^{M}}{\partial M_{i}} \frac{\mathrm{~d} M_{i}^{*}}{\mathrm{~d} s}+p_{i}^{H} \frac{\mathrm{~d} h_{i}^{*}}{\mathrm{~d} s}+\frac{\mathrm{d} a_{i}^{*}}{\mathrm{~d} s}=0 . \tag{C.16}
\end{equation*}
$$

In the equilibrium, using first order conditions (C.2) and (C.3) for expenditure minimization problem (C.1) yields

$$
\begin{align*}
\frac{\partial e_{i}^{M}}{\partial p_{j(i)}^{M}} & =m_{j(i)} q_{i}^{*}  \tag{C.17}\\
\frac{\partial e_{i}^{M}}{\partial m_{j(i)}} & =-\frac{p_{j(i)}^{M} u\left(q_{i}^{*}\right)}{u^{\prime}\left(q_{i}^{*}\right)}+p_{j(i)}^{M} q_{i}^{*} \tag{C.18}
\end{align*}
$$

Substituting Eqs. (C.16)-(C.18) into (C.12) yields

$$
\begin{equation*}
\frac{\mathrm{d} e_{i}}{\mathrm{~d} s}=h_{i}^{*} \frac{\mathrm{~d} p_{i}^{H}}{\mathrm{~d} s}+m_{j(i)} q_{i}^{*} \frac{\mathrm{~d} p_{j(i)}^{M}}{\mathrm{~d} s}+\left(-\frac{p_{j(i)}^{M} u\left(q_{i}^{*}\right)}{u^{\prime}\left(q_{i}^{*}\right)}+p_{j(i)}^{M} q_{i}^{*}\right) \frac{\mathrm{d} m_{j(i)}}{\mathrm{d} s} . \tag{C.19}
\end{equation*}
$$

Substituting the derivative of total profit (4.22) and Eq. (C.19) into (C.11) and using Eq. (4.5) yields

$$
\begin{align*}
\frac{\mathrm{d} A S}{\mathrm{~d} s}= & \sum_{i \in \mathcal{I}}\left[\left(H_{i}^{*}-n_{i} h_{i}^{*}\right) \frac{\mathrm{d} p_{i}^{H}}{\mathrm{~d} s}+\left(p_{i}^{H}-\frac{\partial g^{-1}}{\partial H_{i}}\right) \frac{\mathrm{d} H_{i}^{*}}{\mathrm{~d} s}\right] \\
+ & \sum_{j \in \mathcal{J}}\left[\left(Q_{j} m_{j}-\sum_{a \in \mathcal{I}_{j}} n_{a} m_{j} q_{a}^{*}\right) \frac{\mathrm{d} p_{j}^{M}}{\mathrm{~d} s}+\left(p_{j}^{M}-c\right) m_{j} \frac{\mathrm{~d} Q_{j}}{\mathrm{~d} s}\right. \\
& \left.+\left(\left(p_{j}^{M}-c\right) Q_{j}-m_{j}+\sum_{a \in \mathcal{I}_{j}} n_{a}\left(\frac{p_{j(a)}^{M} u\left(q_{a}^{*}\right)}{u^{\prime}\left(q_{a}^{*}\right)}-p_{j}^{M} q_{a}^{*}\right)\right) \frac{\mathrm{d} m_{j}}{\mathrm{~d} s}\right] \\
+ & \left(\sum_{i \in \mathcal{I}} n_{i} \frac{\mathrm{~d} s_{i}}{\mathrm{~d} s}\right)+\left(\sum_{j \in \mathcal{J}} \frac{\mathrm{~d} s_{j}^{M}}{\mathrm{~d} s}\right) . \tag{C.20}
\end{align*}
$$

Using equilibrium conditions (4.11) and (4.19), and Eq. (4.5) yields

$$
\begin{align*}
& \sum_{i \in \mathcal{I}}\left(H_{i}^{*}-n_{i} h_{i}^{*}\right) \frac{\mathrm{d} p_{i}^{H}}{\mathrm{~d} s}=0  \tag{C.21}\\
& \sum_{j \in \mathcal{J}}\left(Q_{j} m_{j}-\sum_{a \in \mathcal{I}_{j}} n_{a} m_{j} q_{a}^{*}\right) \frac{\mathrm{d} p_{j}^{M}}{\mathrm{~d} s}=0  \tag{C.22}\\
& \sum_{i \in \mathcal{I}}\left(n_{i} \frac{\mathrm{~d} s_{i}}{\mathrm{~d} s}+s_{i} \frac{\mathrm{~d} n_{i}}{\mathrm{~d} s}\right)+\sum_{j \in \mathcal{J}} \frac{\mathrm{~d} s_{j}^{M}}{\mathrm{~d} s}=0 \tag{C.23}
\end{align*}
$$

Substituting Eqs. (C.21)-(C.23) into Eq. (C.20) and using equilibrium condition (4.11) yields Eq. (4.27).

## C.2. Theoretical details of Section 4.4

## C.2.1. Endogenous variables in the equilibrium with the specification in <br> Section 4.4.1

Following the discussion in Section 4.2, we obtain endogenous variables in the equilibrium and the Allais surplus with the specification in Section 4.4.1.

Solving (C.1) with $M_{i}=\int_{0}^{m_{j(i)}} q_{j(i)}(k)^{(\sigma-1) / \sigma} \mathrm{d} k$, we obtain the conditional demand:

$$
\begin{equation*}
q_{j(i)}^{*}(k)=p_{j(i)}^{M}(k)^{-\sigma} P_{j(i)}^{\sigma} M_{i}^{\sigma /(\sigma-1)}, \tag{C.24}
\end{equation*}
$$

where $P_{j}=\left(\int_{0}^{m_{j}} p_{j}^{M}(k)^{1-\sigma} \mathrm{d} k\right)^{1 /(1-\sigma)}$ is the price index of differentiated goods supplied in marketplace $j$. We can obtain the expenditure function as a function of the price index and the composite index: $e_{i}^{M}=\int_{0}^{m_{j(i)}} p_{j(i)}^{M}(k) q_{j(i)}(k) \mathrm{d} k=P_{j(i)} M_{j(i)}^{\sigma /(\sigma-1)}$. Solving (C.4) gives us the Hicksian demands:

$$
\begin{align*}
& M_{i}^{*}=\left(\mu P_{j(i)}^{-1}\right)^{(\sigma-1) / \sigma}  \tag{C.25}\\
& h_{i}^{*}=(1-\mu) / p_{i}^{H}  \tag{C.26}\\
& a_{i}^{*}=\bar{U}+\mu \ln \left(P_{j(i)}\right)+(1-\mu) \ln \left(p_{i}^{H}\right)-\mu \ln (\mu)-(1-\mu) \ln (1-\mu) . \tag{C.27}
\end{align*}
$$

Substituting Eq. (C.25) into Eq. (C.24) yields $q_{i}^{*}(k)=\mu p_{j(i)}^{M}(k)^{-\sigma} P_{j(i)}^{\sigma-1}$. The expenditure function is given by

$$
\begin{equation*}
e_{i}=\bar{U}+\mu \ln \left(P_{j(i)}\right)+(1-\mu) \ln \left(p_{i}^{H}\right)-\mu \ln (\mu)-(1-\mu) \ln (1-\mu)+1 \tag{C.28}
\end{equation*}
$$

We focus on retail stores and developers. Using $q_{i}^{*}(k)$ gives us the total demand:

$$
Q_{j}(k)=\mu p_{j}^{M}(k)^{-\sigma} P_{j}^{\sigma-1} \sum_{i \in \mathcal{I}_{j}} n_{i} \quad(j \in \mathcal{J})
$$

The price elasticity of the total demand is $\eta_{j}^{M}(k)=-\sigma(j \in \mathcal{J})$. Using Eq. (4.12) gives us the equilibrium price: $p_{j}^{M}(k)=c \sigma /(\sigma-1)(\forall j, k)$. We express $p_{j}^{M}(k)$ as $p^{M}$. Applying Eq. (4.17) to $g(b)=\theta b^{\beta}$ gives us the profit maximizing supply as a function of the price: $H_{i}^{*}=\theta^{1 /(1-\beta)}\left(\beta p_{i}^{H}\right)^{\beta /(1-\beta)}(\forall i \in \mathcal{I})$. Using this function gives us the bid rent in the residential zones:

$$
R_{i}^{H}=\theta^{1 /(1-\beta)}\left(\beta^{\beta /(1-\beta)}-\beta^{1 /(1-\beta)}\right)\left(p_{i}^{H}\right)^{1 /(1-\beta)} .
$$

We focus on the short-run equilibrium. Under the equilibrium price for the retail stores, we have

$$
\begin{align*}
& q_{i}^{*}(k)=\mu\left(p^{M} m_{j(i)}\right)^{-1},  \tag{C.29}\\
& Q_{j}(k)=\mu\left(p^{M} m_{j}\right)^{-1} \sum_{i \in \mathcal{I}_{j}} n_{i} . \tag{C.30}
\end{align*}
$$

Substituting $Q_{j}(k)$ into equilibrium condition (4.20) yields the equilibrium mass:

$$
\begin{equation*}
m_{j}=\left(\frac{\mu}{\sigma} \sum_{a \in \mathcal{I}_{j}} n_{a}+s_{j}^{M}(s)\right)^{1 / 2} \quad \forall j \in \mathcal{J} \tag{C.31}
\end{equation*}
$$

The market clearing condition regarding housing (4.19) gives us the equilibrium price: $p_{i}^{H}=\left(\theta \beta^{\beta}\right)^{-1}\left((1-\mu) n_{i}\right)^{1-\beta} \quad(i \in \mathcal{I})$.

Next, we will obtain the Allais surplus. Substituting $p_{i}^{H}$ and $H_{i}^{*}$ into the first term of (4.22) yields

$$
\begin{equation*}
\bar{N}^{-1} \sum_{i \in \mathcal{I}}\left(p_{i}^{H} H_{i}^{*}-g^{-1}\left(H_{i}^{*}\right)\right)=\bar{N}^{-1}(1-\beta)(1-\mu) \sum_{i \in \mathcal{I}} n_{i}=(1-\beta)(1-\mu) . \tag{C.32}
\end{equation*}
$$

Substituting $p^{M}, Q_{j}(k)$, and $m_{j}$ into the second term of (4.22) yields

$$
\begin{equation*}
\bar{N}^{-1} \sum_{j \in \mathcal{J}}\left(\left(p_{j}^{M}-c\right) Q_{j} m_{j}-\frac{m_{j}^{2}}{2}+s_{j}^{M}(s)\right)=\frac{\mu}{2 \sigma}+\bar{N}^{-1} \sum_{j \in \mathcal{J}} s_{j}^{M}(s) . \tag{С.33}
\end{equation*}
$$

Substituting (C.32) and (C.33) into (4.22) yields

$$
\begin{equation*}
\widetilde{\Pi}(\boldsymbol{n}, s, \bar{U})=(1-\beta)(1-\mu)+\mu /(2 \sigma)+\bar{N}^{-1} \sum_{j \in \mathcal{J}} s_{j}^{M}(s) . \tag{C.34}
\end{equation*}
$$

In addition, we can obtain expenditure function (C.28) with equilibrium mass $m_{j}$, price of varieties $p^{M}$, and housing price $p_{i}^{H}$ :

$$
\begin{equation*}
\widetilde{e}_{i}(\boldsymbol{n}, s, \bar{U})=\bar{U}-\zeta_{1} \ln \left(\frac{\mu}{\sigma} \sum_{a \in \mathcal{I}_{j(i)}} n_{a}+s_{j(i)}^{M}(s)\right)+\zeta_{2} \ln n_{i}+\Psi, \tag{C.35}
\end{equation*}
$$

where $\zeta_{1}, \zeta_{2}$, and $\Psi$ are constant values:

$$
\begin{aligned}
& \zeta_{1}=\frac{\mu}{2(\sigma-1)}, \quad \zeta_{2}=(1-\mu)(1-\beta) \\
& \Psi=\mu \ln p^{M}-\mu \ln \mu-\beta(1-\mu) \ln (1-\mu)-(1-\mu)(\ln \theta+\beta \ln \beta)+1
\end{aligned}
$$

Substituting Eqs. (C.34) and (C.35) into Eq. (4.25) yields the Allais surplus.

## C.2.2. Lemmas proving Lemmas 4.3 and 4.4

We introduce two lemmas employed to prove Lemmas 4.3 and 4.4 in Section 4.4. These lemmas are related to algebraic properties of the model.

We can express Allais surplus (4.25) with matrices:

$$
\begin{equation*}
A S=\boldsymbol{n}^{\top} \boldsymbol{Y} \tag{C.36}
\end{equation*}
$$

where $\boldsymbol{Y}=\boldsymbol{s}+\boldsymbol{y}+\widetilde{\Pi} \cdot \mathbf{1}_{I}-\widetilde{\boldsymbol{e}}, \boldsymbol{s}=\left(s_{i}(s)\right)_{i \in \mathcal{I}}, \boldsymbol{y}=\left(y-t_{i}\right)_{i \in \mathcal{I}}, \widetilde{\boldsymbol{e}}=\left(\widetilde{e_{i}}(\boldsymbol{n}, \bar{U}, s)\right)_{i \in \mathcal{I}}$, and $\mathbf{1}_{I}$ is the $I$ dimensional vector with each component equaling one. $\widetilde{\boldsymbol{e}}$ has a symmetric property expressed by the following lemma.

Lemma C.1. For $s=0, \partial \widetilde{\boldsymbol{e}} / \partial \boldsymbol{n}$ is a symmetric matrix and the following holds:

$$
\begin{align*}
\frac{\partial \widetilde{\boldsymbol{e}}}{\partial \boldsymbol{n}} & =-\zeta_{1} E_{1}+\zeta_{2} E_{2},  \tag{C.37}\\
\left(\frac{\partial \widetilde{\boldsymbol{e}}}{\partial \boldsymbol{n}}\right)^{\top} \boldsymbol{n} & =-\left(\zeta_{1}-\zeta_{2}\right) \mathbf{1}_{I} \tag{C.38}
\end{align*}
$$

where

$$
E_{1}=\left(\begin{array}{llll}
\left(\sum_{a \in \mathcal{I}_{1}} n_{a}\right)^{-1} \mathbf{1}_{I_{1}} \mathbf{1}_{I_{1}}^{\top} & & & \\
& \left(\sum_{a \in \mathcal{I}_{2}} n_{a}\right)^{-1} \mathbf{1}_{I_{2}} \mathbf{1}_{I_{2}}^{\top} & & \\
& & \ddots & \\
& & & \\
& & \left.\sum_{a \in \mathcal{I}_{J}} n_{a}\right)^{-1} \mathbf{1}_{I_{J}} \mathbf{1}_{I_{J}}^{\top}
\end{array}\right),
$$

$E_{2}=\operatorname{diag}\left(n_{1}^{-1}, n_{2}^{-1}, \ldots, n_{I}^{-1}\right)$.

Proof. See Supplement C.3.1.

The following lemma has an important role in proving Lemma 4.4.

Lemma C.2. For $n \geq 3$ and $a_{1}, a_{2}, \ldots, a_{n}, b_{1}, b_{2}, \ldots, b_{n} \in \mathbb{R}$, the following holds:

$$
\left(\prod_{i=1}^{n} a_{i}\right)\left(\sum_{i=1}^{n} b_{i}\right)^{2}-\left(\sum_{i=1}^{n} a_{i}\right) \sum_{i=1}^{n}\left(b_{i}^{2} \prod_{j \in \mathcal{N} \backslash\{i\}} a_{j}\right)
$$

$$
\begin{equation*}
=-\sum_{i, j \in \mathcal{N}, i \neq j}\left(\frac{1}{2}\left(a_{i} b_{j}-a_{j} b_{i}\right)^{2} \prod_{k \in \mathcal{N} \backslash\{i, j\}} a_{k}\right) \tag{C.39}
\end{equation*}
$$

where $\mathcal{N}=\{1,2, \ldots, n\}$.
Proof. See Supplement C.3.2.

## C.2.3. Proofs of main lemmas shown in Section 4.4

## Proof of Lemma 4.2

We prove $P D_{P}=0$. Using Eq. (4.11) yields

$$
\begin{equation*}
\sum_{a \in \mathcal{I}_{j}} \frac{\partial Q_{j}}{\partial n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} s}=\sum_{a \in \mathcal{I}_{j}}\left(q_{a}^{*}+\sum_{b \in \mathcal{I}_{j}} n_{b} \frac{\partial q_{b}^{*}}{\partial n_{a}}\right) \frac{\mathrm{d} n_{a}}{\mathrm{~d} s} . \tag{C.40}
\end{equation*}
$$

Using Eqs. (C.24), (C.25), and (C.31) yields the derivative of $q_{b}^{*}$ and $m_{j}$ for $b \in \mathcal{I}_{j}$ and $s=0$ :

$$
\frac{\partial q_{b}^{*}}{\partial n_{a}}=-\frac{\mu}{p^{M} m_{j}^{2}} \frac{\partial m_{j}}{\partial n_{a}}, \quad \frac{\partial m_{j}}{\partial n_{a}}=\frac{\mu}{2 \sigma m_{j}}
$$

Using these equations yields $\left(\partial q_{b}^{*} / \partial n_{a}\right)=-\mu^{2}\left(2 \sigma p^{M} m_{j}^{3}\right)^{-1}$. Substituting (C.29) and this equation into (C.40) yields

$$
\begin{equation*}
\sum_{a \in \mathcal{I}_{j}} \frac{\partial Q_{j}}{\partial n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} s}=\frac{\mu}{2 p^{M} m_{j}} \sum_{a \in \mathcal{I}_{j}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} s} \tag{C.41}
\end{equation*}
$$

Because we have $\sum_{i \in \mathcal{I}}\left(\mathrm{~d} n_{i} / \mathrm{d} s\right)=0$ with Eq. (4.24), using this equation yields $P D_{P}=0$.
The proof of $V D_{P}=0$ is similar to the above proof:

$$
\begin{align*}
& \sum_{j \in \mathcal{J}}\left(\sum_{a \in \mathcal{I}_{j}}\left(\frac{n_{a} p_{j(a)}^{M} u\left(q_{a}\right)}{u^{\prime}\left(q_{a}\right)}\right)-c Q_{j}-m_{j}\right) \sum_{a \in \mathcal{I}_{j}} \frac{\partial m_{j}}{\partial n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} s}= \\
& \frac{\mu}{2 \sigma}\left(\frac{\sigma^{2}}{\sigma-1}-\frac{c \sigma}{p^{M}}-1\right) \sum_{i \in \mathcal{I}} \frac{\mathrm{~d} n_{i}}{\mathrm{~d} s}=0 . \tag{C.42}
\end{align*}
$$

## Proof of Lemma 4.3

Let $L$ denote the Lagrangian for maximization problem (4.38). $L$ is given by

$$
\begin{equation*}
L=A S-\sum_{i \in \mathcal{I}} \nu_{i} \gamma_{i}-\lambda \Gamma \tag{C.43}
\end{equation*}
$$

where $\nu_{i}$ and $\lambda$ are the Lagrange multipliers. The KKT conditions for this problem are as follows:

$$
\begin{align*}
& \frac{\partial A S}{\partial \boldsymbol{n}}=\sum_{i \in \mathcal{I}} \nu_{i} \frac{\partial \gamma_{i}(\boldsymbol{n})}{\partial \boldsymbol{n}}+\lambda \frac{\partial \Gamma(\boldsymbol{n})}{\partial \boldsymbol{n}}  \tag{C.44}\\
& \gamma_{i}(\boldsymbol{n}) \leq 0, \quad \Gamma(\boldsymbol{n})=0  \tag{C.45}\\
& \nu_{i} \geq 0, \quad \nu_{i} \gamma_{i}=0 \tag{C.46}
\end{align*}
$$

Using (C.36) gives us the derivative of the Allais surplus:

$$
\begin{equation*}
\frac{\partial A S}{\partial \boldsymbol{n}}=\boldsymbol{Y}+\mathbf{1}_{I}\left(\frac{\partial \Pi}{\partial \boldsymbol{n}}\right)^{\top} \boldsymbol{n}-\left(\frac{\partial \widetilde{\boldsymbol{e}}}{\partial \boldsymbol{n}}\right)^{\top} \boldsymbol{n} . \tag{C.47}
\end{equation*}
$$

We focus on the first term of (C.47). Since consumers maximize their utility for $s=0$, we have $V_{i}=\bar{U}$ and $\widetilde{e}_{i}\left(\boldsymbol{n}^{*}, \bar{U}, 0\right)=y-t_{i}+\widetilde{\Pi}\left(\boldsymbol{n}^{*}, \bar{U}, 0\right)(\forall i \in \mathcal{I})$. Hence, $\boldsymbol{Y}=\mathbf{0}$ holds for $s=0$. The second term is zero because $(\partial \Pi / \partial \boldsymbol{n})=0$ holds for $s=0$ by Eq. (C.34). In addition, we have $(\partial \widetilde{\boldsymbol{e}} / \partial \boldsymbol{n})^{\top} \boldsymbol{n}=-\left(\zeta_{1}-\zeta_{2}\right) \mathbf{1}_{I}$ by Lemma C.1. Hence we have

$$
\begin{equation*}
\frac{\partial A S}{\partial \boldsymbol{n}}=\left(\zeta_{1}-\zeta_{2}\right) \mathbf{1}_{I} \tag{C.48}
\end{equation*}
$$

Substituting (C.48) into (C.44) gives us $\boldsymbol{\nu}+\left(\zeta_{1}-\zeta_{2}+\lambda\right) \mathbf{1}_{I}=\mathbf{0}_{I}$, where $\boldsymbol{\nu}=\left(\nu_{i}\right)_{i \in \mathcal{I}}$. Since we focus on an inner equilibrium, $\gamma_{i}<0$ and $\Gamma=0$ hold. Furthermore, if we set $\boldsymbol{\nu}=\mathbf{0}$ and $\lambda=-\zeta_{1}+\zeta_{2}$, then the other conditions are satisfied.

## Proof of Lemma 4.4

The Hessian of Lagrangian (C.43) is given by

$$
\begin{equation*}
\frac{\partial^{2} L}{\partial \boldsymbol{n}^{2}}=\frac{\partial^{2}\left(\boldsymbol{n}^{\top} \boldsymbol{Y}\right)}{\partial \boldsymbol{n}^{2}}=\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{n}}+\frac{\partial}{\partial \boldsymbol{n}}\left(\left(\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{n}}\right)^{\top} \boldsymbol{n}\right) \tag{C.49}
\end{equation*}
$$

We have $(\partial \boldsymbol{Y} / \partial \boldsymbol{n})=-(\partial \widetilde{\boldsymbol{e}} / \partial \boldsymbol{n})$. Hence, Lemma C. 1 yields

$$
\begin{equation*}
\frac{\partial^{2} L}{\partial \boldsymbol{n}^{2}}=\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{n}}=\zeta_{1} E_{1}-\zeta_{2} E_{2} \tag{C.50}
\end{equation*}
$$

We focus on $I \geq 3$. We define the following:

$$
\mathcal{M}^{+} \equiv\left\{\boldsymbol{z} \in \mathbb{R}^{I} \left\lvert\,\left(\frac{\partial \gamma_{i}}{\partial \boldsymbol{n}}\right)^{\top} \boldsymbol{z}=0\right.\right\}=\left\{\boldsymbol{z} \in \mathbb{R}^{I} \mid \sum_{i \in \mathcal{I}} z_{i}=0\right\}
$$

Since we have $\zeta_{1}<\zeta_{2}$ by assumption of Lemma 4.4, this inequality and $E_{1}$ and $E_{2}$, shown in Lemma C.1, gives us the following for any $\boldsymbol{z} \in \mathcal{M}^{+} \backslash\{\mathbf{0}\}$ :

$$
\begin{equation*}
\boldsymbol{z}^{\top} \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{n}} \boldsymbol{z}=\zeta_{1} \boldsymbol{z}^{\top} E_{1} \boldsymbol{z}-\zeta_{2} \boldsymbol{z}^{\top} E_{2} \boldsymbol{z}<\zeta_{1} \sum_{j \in \mathcal{J}} Z_{j} \tag{C.51}
\end{equation*}
$$

where $Z_{j}=\left(\sum_{a \in \mathcal{I}_{j}} n_{a}\right)^{-1}\left(\sum_{a \in \mathcal{I}_{j}} z_{a}\right)^{2}-\sum_{a \in \mathcal{I}_{j}}\left(z_{a}^{2} / n_{a}\right)$. Because $Z_{j} \leq 0(\forall j \in \mathcal{J})$ holds by Lemma C.2, we obtain $\boldsymbol{z}^{\top}(\partial \boldsymbol{Y} / \partial \boldsymbol{n}) \boldsymbol{z}<0$.

A similar argument to the above discussion shows that $\boldsymbol{z}^{\top}(\partial \boldsymbol{Y} / \partial \boldsymbol{n}) \boldsymbol{z}<0$ holds for $I=2$.

## Proof of Lemma 4.5

Using (C.31) gives us the mass of stores for the given policy function:

$$
\begin{equation*}
m_{j}=\left(\frac{\mu}{\sigma}\left(\sum_{a \in \mathcal{I}_{j}} n_{a}\right)+\delta_{j} s\right)^{1 / 2} \quad \forall j \in \mathcal{J} \tag{C.52}
\end{equation*}
$$

where $\delta_{1}=1$ and $\delta_{j}=0(j \neq 1)$.
Lemma 4.2 yields $\mathrm{d} A S / \mathrm{d} s=P D_{S}+V D_{S}$. Furthermore, the following hold:

$$
\frac{\partial p_{1}^{M}}{\partial s}=\cdots=\frac{\partial p_{J}^{M}}{\partial s}=0, \quad \frac{\partial Q_{2}}{\partial s}=\cdots=\frac{\partial Q_{J}}{\partial s}=0, \quad \frac{\partial m_{2}}{\partial s}=\cdots=\frac{\partial m_{J}}{\partial s}=0
$$

Hence, we have

$$
P D_{S}=\left(p_{1}^{M}-c\right) m_{1} \frac{\partial Q_{1}}{\partial s}, \quad V D_{S}=\left(\sum_{a \in \mathcal{I}_{1}}\left(\frac{n_{a} p_{j(a)}^{M} u\left(q_{a}\right)}{u^{\prime}\left(q_{a}\right)}\right)-c Q_{1}-m_{1}\right) \frac{\partial m_{1}}{\partial s}
$$

Using Eqs. (C.30) and (C.52) yields for $s=0: P D_{S}=-1 / 2<0$, while using Eqs. (C.24) and (C.52) yields for $s=0: V D_{S}=\sigma /(2(\sigma-1))>0$. Substituting $P D_{S}$ and $V D_{S}$ into $\mathrm{d} A S / \mathrm{d} s$ yields $\mathrm{d} A S / \mathrm{d} s=(2(\sigma-1))^{-1}>0$.
C.3. Proofs of Lemmas shown in Appendix C. 2

## C.3.1. Proof of Lemma C. 1

We express $\widetilde{\boldsymbol{e}}$ as a matrix: $\widetilde{\boldsymbol{e}}=-\zeta_{1}\left(L o \circ G_{1}\right)(\boldsymbol{n})+\zeta_{2} L o(\boldsymbol{n})+(\bar{U}-\Psi) \mathbf{1}_{I}$, where

$$
L o\left(z_{1}, z_{2}, \ldots, z_{I}\right)=\left(\ln z_{1}, \ln z_{2}, \ldots, \ln z_{I}\right)^{\top}
$$

$$
G_{1}(\boldsymbol{n})=\left(\left(\frac{\mu}{\sigma} \sum_{a \in \mathcal{I}_{1}} n_{a}+s_{1}^{M}(s)\right) \mathbf{1}_{I_{1}}^{\top}, \ldots,\left(\frac{\mu}{\sigma} \sum_{a \in \mathcal{I}_{J}} n_{a}+s_{J}^{M}(s)\right) \mathbf{1}_{I_{J}}^{\top}\right)^{\top}
$$

$\partial \widetilde{\boldsymbol{e}} / \partial \boldsymbol{n}$ is given by

$$
\begin{equation*}
\frac{\partial \widetilde{\boldsymbol{e}}}{\partial \boldsymbol{n}}=-\zeta_{1} \frac{\partial\left(\left(L o \circ G_{1}\right)(\boldsymbol{n})\right)}{\partial \boldsymbol{n}}+\zeta_{2} \frac{\partial L o(\boldsymbol{n})}{\partial \boldsymbol{n}} \tag{C.53}
\end{equation*}
$$

Using the chain rule, we obtain the Jacobian matrix of $\left(L o \circ G_{1}\right)(\boldsymbol{n})$ and $L o(\boldsymbol{n})$ for $s=0:$

$$
\begin{align*}
& \frac{\partial\left(\left(L o \circ G_{1}\right)(\boldsymbol{n})\right)}{\partial \boldsymbol{n}}= \\
& \left(\begin{array}{cccc}
\left(\sum_{a \in \mathcal{I}_{1}} n_{a}\right)^{-1} \mathbf{1}_{I_{1}} \mathbf{1}_{I_{1}}^{\top} & & & \\
& \left(\sum_{a \in \mathcal{I}_{2}} n_{a}\right)^{-1} \mathbf{1}_{I_{2}} \mathbf{1}_{I_{2}}^{\top} & & \\
& & \ddots & \\
& & & \left(\sum_{a \in \mathcal{I}_{J}} n_{a}\right)^{-1} \mathbf{1}_{I_{J}} \mathbf{1}_{I_{J}}^{\top}
\end{array}\right) \tag{C.54}
\end{align*}
$$

$$
\begin{equation*}
\frac{\partial L o(\boldsymbol{n})}{\partial \boldsymbol{n}}=\operatorname{diag}\left(n_{1}^{-1}, n_{2}^{-1}, \ldots, n_{I}^{-1}\right) \tag{C.55}
\end{equation*}
$$

Since the sum of symmetric matrices is also a symmetric matrix, $\partial \boldsymbol{V} / \partial \boldsymbol{n}$ is a symmetric matrix. Furthermore, substituting (C.54) and (C.55) into (C.53), we obtain (C.37) and (C.38).

## C.3.2. Proof of Lemma C. 2

Using the mathematical induction, we prove the lemma.
We obtain (C.39) for $n=3$ by a simple deformation:

$$
\begin{aligned}
a_{1} a_{2} a_{3}\left(b_{1}\right. & \left.+b_{2}+b_{3}\right)^{2}-\left(a_{1}+a_{2}+a_{3}\right)\left(a_{2} a_{3} b_{1}^{2}+a_{3} a_{1} b_{2}^{2}+a_{1} a_{2} b_{3}^{2}\right) \\
& =-a_{3}\left(a_{2} b_{1}-a_{1} b_{2}\right)^{2}-a_{2}\left(a_{3} b_{1}-a_{1} b_{3}\right)^{2}-a_{1}\left(a_{2} b_{3}-a_{3} b_{2}\right)^{2}
\end{aligned}
$$

Next, we assume that Eq. (C.39) holds for $n$. We verify whether Eq. (C.39) holds for $n+1$. That is, we will show the following holds:

$$
\left(\prod_{i=1}^{n+1} a_{i}\right)\left(\sum_{i=1}^{n+1} b_{i}\right)^{2}-\left(\sum_{i=1}^{n+1} a_{i}\right) \sum_{i=1}^{n+1}\left(b_{i}^{2} \prod_{j \in \hat{\mathcal{N}} \backslash\{i\}} a_{j}\right)
$$

$$
\begin{equation*}
=-\sum_{i, j \in \widehat{\mathcal{N}}, i \neq j}\left(\frac{1}{2}\left(a_{i} b_{j}-a_{j} b_{i}\right)^{2} \prod_{k \in \widehat{\mathcal{N}} \backslash\{i, j\}} a_{k}\right) \tag{C.56}
\end{equation*}
$$

where $\widehat{\mathcal{N}}=\{1,2, \ldots, n+1\}$.
We define $B_{n}=\sum_{i=1}^{n} b_{i}$. We focus on the LHS of (C.56):

$$
\begin{align*}
(\mathrm{LHS})= & \left(\prod_{i=1}^{n+1} a_{i}\right)\left(B_{n}^{2}+2 B_{n} b_{n+1}+b_{n+1}^{2}\right) \\
& -\left(\sum_{i=1}^{n+1} a_{i}\right) \sum_{i=1}^{n}\left(b_{i}^{2} \prod_{j \in \hat{\mathcal{M}} \backslash\{i\}} a_{j}+b_{n+1}^{2} \prod_{j \in \hat{\mathcal{M}} \backslash\{n+1\}} a_{j}\right) \\
= & X+Y, \tag{C.57}
\end{align*}
$$

where

$$
\begin{aligned}
& X= B_{n}^{2}\left(\prod_{i=1}^{n+1} a_{i}\right)-\left(\sum_{i=1}^{n} a_{i}\right)\left(\sum_{i=1}^{n} b_{i}^{2} \prod_{j \in \hat{\mathcal{N}} \backslash\{i\}} a_{j}\right), \\
& Y=-a_{n+1}\left(\sum_{i=1}^{n} b_{i}^{2} \prod_{j \in \widehat{\mathcal{N}} \backslash\{i\}} a_{j}\right)+ \\
&\left(2 B_{n} b_{n+1}+b_{n+1}^{2}\right)\left(\prod_{i=1}^{n+1} a_{i}\right) \\
&-b_{n+1}^{2}\left(\sum_{i=1}^{n+1} a_{i}\right)\left(\prod_{j \in \widehat{\mathcal{N}} \backslash\{n+1\}} a_{j}\right) .
\end{aligned}
$$

Using the assumption, we deform $X$ as follows:

$$
X=-\sum_{i, j \in \mathcal{N}, i \neq j} \frac{1}{2}\left(a_{i} b_{j}-a_{j} b_{i}\right)^{2} \prod_{k \in \widehat{\mathcal{N}} \backslash\{i, j\}} a_{k}
$$

We deform $Y$ as follows:

$$
\begin{equation*}
Y=-a_{n+1} \sum_{i=1}^{n}\left(b_{i}^{2} \prod_{j \in \widehat{\mathcal{N}} \backslash\{i\}} a_{j}\right)+2 B_{n} b_{n+1} \prod_{i=1}^{n+1} a_{i}+Y_{1}, \tag{C.58}
\end{equation*}
$$

where

$$
\begin{equation*}
Y_{1}=b_{n+1}^{2}\left(\prod_{i=1}^{n+1} a_{i}-\left(\sum_{i=1}^{n+1} a_{i}\right) \prod_{j \in \widehat{\mathcal{N}} \backslash\{n+1\}} a_{j}\right) \tag{C.59}
\end{equation*}
$$

We deform $Y_{1}$ as follows:

$$
\begin{equation*}
Y_{1}=-b_{n+1}^{2}\left(\sum_{i=1}^{n} a_{i}\right) \prod_{i \in \widehat{\mathcal{N}} \backslash\{n+1\}} a_{i} . \tag{C.60}
\end{equation*}
$$

Using (C.60), we deform $Y$ as follows:

$$
\begin{align*}
Y=- & \sum_{i=1}^{n} \frac{1}{2}\left(a_{n+1} b_{i}-a_{i} b_{n+1}\right)^{2} \prod_{j \in \widehat{\mathcal{M}} \backslash\{i, n+1\}} a_{j} \\
& -\sum_{i=1}^{n} \frac{1}{2}\left(a_{i} b_{n+1}-a_{n+1} b_{i}\right)^{2} \prod_{j \in \widehat{\mathcal{N}} \backslash\{i, n+1\}} a_{j} . \tag{C.61}
\end{align*}
$$

Substituting $X$ and $Y$ into (C.57) implies that LHS of (C.56) equals the RHS of (C.56).

## C.4. The case of continuous space

In this section, we obtain the welfare impact of a place-based policy in a continuous space. We show that the difference between a discrete space and the continuous space is the marginal welfare change generated by a change in a market boundary. Assumptions in the continuous space model other than geographical space of the city are the same as the discrete space model. We assume that functions are so continuous that we can obtain derivatives.

## C.4.1. Model setting

The geographical space of the continuous space model is given by $[0, I](I>0)$. Consumers can reside in $(0, I)$ and retail stores can operate at 0 or $I(0, I \in[0, I])$. The utility of consumers who reside in $x(\in(0, I))$ and visit marketplace $j(\in\{0, I\})$ for shopping is given by $U\left(M_{j}(x), h(x), a(x)\right)$, where $M_{j}(x)=\int_{0}^{m_{j}} u(q(x, k)) \mathrm{d} k$. The budget constraint is given by:

$$
\begin{equation*}
\int_{0}^{m_{j}} p_{j}^{M}(k) q(x, k) \mathrm{d} k+p^{H}(x) h(x)+a(x)=y_{j}(x) \tag{C.62}
\end{equation*}
$$

where $y_{j}(x)=\widetilde{y}_{j}(x, s) \equiv y-t_{j}(x)+\Pi+s_{j}(x, s)$. The profit of retail store supplying the $k$ th variety in marketplace $j$ is given by

$$
\begin{equation*}
\pi_{j}^{M}(k)=\left(p_{j}^{M}(k)-c\right) Q_{j}(k)-k+\frac{s_{j}^{M}(s)}{m_{j}}-r_{j}(k) \quad \forall k \in\left[0, m_{j}\right] \tag{C.63}
\end{equation*}
$$

where $Q_{j}(k)=\int_{0}^{I} q(x, k) \mathrm{d} x$. The developers' net profit at $x(\in[0, x])$ is given by

$$
\begin{equation*}
\pi^{H}(x)=p^{H}(x) H(x)-g^{-1}(H(x))-R^{H}(x) . \tag{C.64}
\end{equation*}
$$

An equal division of the profits and rents is given by

$$
\Pi=\bar{N}^{-1}\left(\int_{0}^{I} \pi^{H}(x)+R^{H}(x) \mathrm{d} x+\sum_{j \in \mathcal{J}}\left(\int_{0}^{m_{j}} \pi_{j}^{M}(k) \mathrm{d} k+\int_{0}^{m_{j}} r_{j}(k) \mathrm{d} k\right)\right),
$$

where $\mathcal{J}=\{0, I\}$.
Equilibrium conditions are given by

$$
\begin{align*}
& H(x)=n(x, s) h(x) \quad \forall x \in(0, I)  \tag{C.65}\\
& \left(p_{j}^{M}-c\right) Q_{j}-m_{j}+\frac{s_{j}^{M}(s)}{m_{j}}=0 \quad \forall j \in \mathcal{J} \equiv\{0, I\}  \tag{C.66}\\
& y(x, s)=y-t_{j(x)}(x)+\Pi+s_{j(x)}(x, s) \quad \forall x \in[0, I] \tag{C.67}
\end{align*}
$$

where $j(x)(\in \mathcal{J})$ is the marketplace that consumers residing in $x(\in(0, I))$ visit for shopping. Let $e(x, j, s)$ denote the expenditure function of consumers who reside in $x$ and visit marketplace $j$. The equilibrium conditions for the Allais surplus are given by

$$
\begin{align*}
& y-t_{j(x)}+\Pi+s_{j(x)}(x, s)-e(x, j, s)=\bar{E} \quad \exists \bar{E} \in \mathbb{R} \forall x \in(0, I),  \tag{C.68}\\
& \int_{0}^{I} n(x, s) \mathrm{d} x=\bar{N} \tag{C.69}
\end{align*}
$$

## C.4.2. Welfare impact of a place-based policy for the continuous space model

We can obtain the differentiation of the Allais surplus with the continuous space model. Let $b(s) \in(0, I)$ denote the market boundary given policy instrument $s$ with equilibrium conditions (C.65)-(C.69). We focus on a utility level at which consumers residing in $x \in(0, b(s)]$ visit marketplace 0 and consumers residing in $x \in[b(s), I)$ visit marketplace $I$. At the utility level, we obtain the Allais surplus:

$$
A S=\int_{0}^{I} n(x, s) \bar{E} \mathrm{~d} x
$$

$$
\begin{equation*}
=\int_{0}^{I} n(x, s)\left(y-t_{j(x)}(x)+\Pi+s_{j(x)}(x, s)-e(x, j(x), s)\right) \mathrm{d} x . \tag{C.70}
\end{equation*}
$$

We differentiate the Allais surplus with respect to $s$ :

$$
\begin{aligned}
\frac{\mathrm{d} A S}{\mathrm{~d} s}= & \int_{0}^{I} n(x, s) \frac{\mathrm{d}}{\mathrm{~d} s}\left(y-t_{j(x)}(x)+\Pi+s_{j(x)}(x, s)-e(x, j(x), s)\right) \mathrm{d} x \\
& +\int_{0}^{I} \frac{\mathrm{~d} n(x, s)}{\mathrm{d} s}\left(y-t_{j(x)}(x)+\Pi+s_{j(x)}(x, s)-e(x, j(x), s)\right) \mathrm{d} x+B D
\end{aligned}
$$

where

$$
\begin{aligned}
B D & =n(b(s), s)\left(Y_{I}-Y_{0}\right) \frac{\mathrm{d} b(s)}{\mathrm{d} s} \\
Y_{0} & =t_{0}(b(s))+e(x, 0, s)-s_{0}(b(s), s), \quad Y_{I}=t_{I}(b(s))+e(x, I, s)-s_{I}(b(s), s) .
\end{aligned}
$$

When we evaluate the welfare impact with the continuous model, the welfare impact generated by the change in the market boundary $B D$ is added to the welfare measurement formula. The same discussion for the derivation of the derivative of the Allais surplus, shown in Appendix C.1.2, gives us

$$
\begin{equation*}
\frac{\mathrm{d} A S}{\mathrm{~d} s}=P D+V D+F D+B D \tag{C.71}
\end{equation*}
$$

where

$$
\begin{aligned}
P D \equiv & \sum_{j \in \mathcal{J}}\left(p_{j}^{M}-c\right) m_{j} \frac{\mathrm{~d} Q_{j}}{\mathrm{~d} s}, \\
V D \equiv & \left(\int_{0}^{b(s)} \frac{n(x, s) p_{0}^{M} u(q(x))}{u^{\prime}(q(x))} \mathrm{d} x-c Q_{0}-m_{0}\right) \frac{\mathrm{d} m_{0}}{\mathrm{~d} s} \\
& \quad+\left(\int_{b(s)}^{I} \frac{n(x, s) p_{I}^{M} u(q(x))}{u^{\prime}(q(x))} \mathrm{d} x-c Q_{I}-m_{I}\right) \frac{\mathrm{d} m_{I}}{\mathrm{~d} s}, \\
F D= & \int_{0}^{I}-s_{j(x)}(x, s) \frac{\mathrm{d} n(x, s)}{\mathrm{d} s} \mathrm{~d} x .
\end{aligned}
$$

Since $Y_{I}-Y_{0}=t_{I}(b(0))-t_{0}(b(0))$ holds for $s=0$, we have $B D=\left(t_{I}(b(0))-\right.$ $\left.t_{0}(b(0))\right) \mathrm{d} b(s) / \mathrm{d} s$. If the difference in the travel costs is small, then $B D$ is small. That is, $B D$ hardly affects the welfare impact of adopting a place-based policy.

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[^0]:    ${ }^{1}$ See https://www.city.toyama.toyama.jp/seisakushokai/shuyopuran.html (last accessed on 22 November 2022).

[^1]:    ${ }^{2}$ See Biscaia and Mota (2013) for a survey.

[^2]:    ${ }^{3}$ There are theoretical researches that exclude first nature (e.g., Tabuchi et al., 2005; Gaspar et al., 2018; Aizawa et al., 2020). In these researches, all the transport costs between regions are the same. Such a methodology enables us to investigate how second nature affects results in terms of population agglomeration.

[^3]:    ${ }^{4}$ For example, a circle (e.g., Mulligan, 1996), a line segment where consumers are distributed nonuniformly (Tabuchi and Thisse, 1995), a homogeneous two-dimensional space (Tabuchi, 1994), and a homogeneous $n$-dimensional space (Irmen and Thisse, 1998) are employed.

[^4]:    ${ }^{5}$ For example, Mun (1997) shows that the asymmetry of transport cost in a road network generates a difference in city size distribution.
    ${ }^{6}$ Recently, locations where firms operate under monopolistic competition have been investigated

[^5]:    (e.g., Ago, 2008; Ushchev et al., 2015).
    ${ }^{7}$ Since our analysis focuses on the symmetry of the location of suburbs, a different number of suburbs would not qualitatively affect our result unless the symmetry differed.

[^6]:    ${ }^{8}$ Land-use regulation can be practical alternatives to superior policies that are often politically infeasible. The effect of the land-use regulation on social welfare has been theoretically investigated by many chapters (e.g., Brueckner, 2009; Kono and Joshi, 2018, 2019).

[^7]:    ${ }^{9}$ We ignore commuting in order to focus on how decreases in the travel cost (i.e., road improvements) affect the equilibrium of shopping stores.

[^8]:    ${ }^{10}$ Such a modeling methodology is called the Boltzmann, Lotka and Volterra method, which has been applied to statistical physics as well as regional science (e.g., Harris and Wilson (1978); Wilson (2008); Osawa et al. (2017)).
    ${ }^{11}$ The dynamics assumed by Tabuchi (2009) implies that changes in the number of retail stores per unit time depend on profits only. The dynamics assumed by Tabuchi (2009) and us are qualitatively the same. Furthermore, the dynamics in this chapter can capture corner equilibria as stationary points of the dynamics.

[^9]:    ${ }^{12}$ Such a classification can be applied to stationary points of general dynamical systems. See, e.g., Kuznetsov (Chapter 2.2, 2004) for the theoretical details.
    ${ }^{13}$ This change is qualitatively the same result as the spatial triangle bifurcation, which is often observed in the New Economic Geography (e.g., Ikeda et al. (2012)).

[^10]:    ${ }^{14}$ Note that market areas $\mathcal{M}_{1}, \mathcal{M}_{3}$, and $\mathcal{M}_{5}$ are asymmetric in market pattern (P) (i.e., $\mu\left(\mathcal{M}_{1}\right)>$ $\left.\mu\left(\mathcal{M}_{3}\right)>\mu\left(\mathcal{M}_{5}\right)\right)$. When $\mu\left(\mathcal{M}_{1}\right)=\mu\left(\mathcal{M}_{3}\right)=\mu\left(\mathcal{M}_{5}\right)$ holds at $\tau$ less than $\phi$, some consumers residing along the radial roads are indifferent to choosing one of suburbs 1,3 , and 5 . If a retail store enters one of the suburbs in such an agglomeration pattern, the suburb attracts the consumers. Hence, the symmetry of the market areas in the suburbs breaks. We thus focus on market pattern (P) rather than symmetric patterns where suburbs 1,3 , and 5 each have a market area.

[^11]:    ${ }^{15}$ The full agglomeration is always linearly stable. See Appendix A.3.1 for details.
    ${ }^{16}$ Note that the asymmetric pattern is not the most efficient in $(0,1) \times(0,1)$.

[^12]:    ${ }^{17}$ One may think that radial roads are improved first in the real world. Figure 2.5 (a) indicates that even if the road improvement sequence is initial radial roads, ring road, and additional radial roads, the equilibrium generated by the sequence is the same as that by the ring-road first case.

[^13]:    ${ }^{18}$ With lower travel costs, other agglomeration patterns not shown in the previous section can emerge (e.g., one downtown and a marketplace in the suburbs). To accomplish our aim, we have only to focus on the agglomeration patterns shown in the previous section.
    ${ }^{19}$ In this case, at $B_{1},\left(\phi, n_{0}, n_{1}\right)=\left(0.366,4.88 \times 10^{-2}, 0.853 \times 10^{-2}\right)$. At $\widehat{B_{1}},\left(\phi, n_{0}, n_{1}\right)=$ $\left(0.365,4.89 \times 10^{-2}, 0.852 \times 10^{-2}\right)$. We obtained the solution from $\widehat{B_{1}}$ with dynamics (2.8) in market pattern (D) because the market area at $\widehat{B_{1}}$ have this pattern (Lemma A. 3 in Appendix A.3.1).
    ${ }^{20}$ We obtained the solution with the Runge-Kutta 4th order method.

[^14]:    ${ }^{21}$ This change is called boundary equilibrium bifurcation in the dynamical systems theory (see e.g., Bernardo et al., 2008).
    ${ }^{22}$ In this case, at $B_{1},\left(\tau, n_{1}\right)=\left(0.306,1.17 \times 10^{-2}\right)$. At $\widehat{B_{1}},\left(\tau, n_{1}\right)=\left(0.305,1.18 \times 10^{-2}\right)$.
    ${ }^{23}$ In this numerical analysis, the path-following of the solution stopped at the point at which the

[^15]:    ${ }^{26}$ For example, with a ring and heterogeneous star network topologies, Barbero and Zofío (2016) analyze the agglomeration and dispersion forces of the core-periphery model.
    ${ }^{27}$ Several studies of spatial agglomeration have been conducted on a square lattice (Clarke and Wilson, 1983; Weidlich and Haag, 1987; Weidlich and Munz, 1990). Moreover, Stelder (2005) carries out a simulation of agglomeration for cities in Europe using a grid of points. However, a hexagonal lattice is employed in this chapter since it has a finer spatial resolution than a square lattice.

[^16]:    ${ }^{28}$ The FO model has an analytical tractability and a close resemblance to seminal Core-Periphery model by (Krugman, 1991). Akamatsu et al. (2019) elucidate the bifurcation mechanism of the FO model in racetrack economy. This mechanism, however, is restricted to the bifurcation from the uniform state.

[^17]:    ${ }^{29}$ The transition of population agglomeration for the two-regional version of the FO model with change in transport cost is qualitatively the same as the transition for the core-periphery model proposed by Krugman (1991). Only mobile workers are input for manufacturing production in the Krugman model, whereas both mobile and inmobile workers are the input in the FO model. This assumption is the critical difference between the Krugman and FO models.

[^18]:    ${ }^{30}$ See, e.g, Golubitsky et al. (Sec. 13.3.1, 1988) for the theoretical detail of the orbit decomposition.

[^19]:    ${ }^{31}$ The full agglomeration was shown to be an invariant solution in a racetrack economy for any number of places in Castro et al. (2012).

[^20]:    ${ }^{32}$ To name a few, twin cities were studied by Krugman (1991), the triangular cities by Castro et al. (2012), and racetrack cities by Tabuchi and Thisse (2011).

[^21]:    ${ }^{33}$ Such pattern was also observed in Ikeda et al. (2017b) and was called a core-satellite pattern (Ikeda et al., 2020).
    ${ }^{34}$ Figs. 3.10(a) and 3.11(c) contain all stable bifurcating solutions from the mono-center.

[^22]:    ${ }^{35}$ The emergence of central city from racetrack cities was observed in a ad-hoc manner in Ikeda et al. (2017b).

[^23]:    ${ }^{36}$ See Zhelobodko et al. (2012) for the properties of additively separable functions.

[^24]:    ${ }^{37}$ Similar policies to both policies are adopted by local governments in the real world (e.g., Albuquerque in the U.S.A. and Toyama in Japan).

[^25]:    ${ }^{38} \mathrm{We}$ introduce market equilibrium conditions given market area $\left\{\mathcal{I}_{j}\right\}_{j \in \mathcal{J}}$ in Section 4.2.5. One may consider that the market area should be endogenous. Note that our aim is to investigate how place-based policies affect social welfare at market equilibrium. Hence, we can accomplish our aim by conducting welfare analysis for any given market area. We will conduct the theoretical analysis in Sections 4.3 and 4.4.
    ${ }^{39}$ We have $\mathcal{I}_{j} \neq \emptyset(\forall j \in \mathcal{J}), \mathcal{I}_{j_{1}} \cap \mathcal{I}_{j_{2}}=\emptyset\left(j_{1} \neq j_{2}\right)$, and $\mathcal{I}=\cup_{j=1}^{J} \mathcal{I}_{j}$.

[^26]:    ${ }^{40}$ DellaVigna and Gentzkow (2019) empirically show that retail stores in the U.S.A. charge nearly uniform prices across stores. Based on this result, we focus on a symmetric price equilibrium.

[^27]:    ${ }^{41}$ The formal definition of $j(i)$ is as follows. We define mappings $J_{1}: i \mapsto \mathcal{I}_{j}$, where $i \in \mathcal{I}_{j}$, and $J_{2}: \mathcal{I}_{j} \mapsto j$. Note that $J_{1}$ is well-defined since the definition of the market area determines unique $\mathcal{I}_{j}$ for each $i(\in \mathcal{I}) . j()$ is the composite mapping of $J_{1}$ and $J_{2}: j(i) \equiv\left(J_{2} \circ J_{1}\right)(i)$.

[^28]:    ${ }^{42}$ Our definition of the Allais surplus is called the Equalized- $\beta$ measure as shown in Kono and Kishi (2018).

[^29]:    ${ }^{43}$ If the geographical space of the city is continuous, then the welfare impact generated by change in a market boundary is added to the welfare measurement formula. In our model, the welfare impact is composed of a difference between travel costs from the market boundary to marketplaces. Hence, if the difference is small, then the welfare impact is almost the same as that of the discrete model.
    ${ }^{44} \mathrm{We}$ can interpret $F D$ as the migration fiscal externality generated by income transfer inefficiency by a place-based policy (Boadway and Flatters, 1982; Kono et al., 2007).

[^30]:    ${ }^{45}$ In addition, $1-\mu$ implies the housing expenditure share.

[^31]:    ${ }^{46}$ Note that although there is no income effect for the quasi-linear preference, any subsidy policy can affect the total demand. This is because the policy can affect the mass of variety, which generates a change in the total demand (see Eqs. (4.35)-(4.37)).

[^32]:    ${ }^{47}$ See Neumark and Simpson (2015) for a survey.

[^33]:    ${ }^{48}$ See http://www.city.sendai.jp/toshi-kekakuchose/kurashi/machi/kaihatsu/toshikekaku/service.html (last accessed on 1 December 2022).

[^34]:    ${ }^{49}$ Market boundaries between marketplaces $i$ and $j(i \neq j)$, for example, are $\mathcal{M}_{i} \cap \mathcal{M}_{j}$.

[^35]:    ${ }^{50}$ Since the solution space of the governing equation is the ( $K-1$ )-dimensional simplex with a constant total population, the eigenvector $\boldsymbol{\eta}^{*}=(1, \ldots, 1)$ and the associated eigenvalue $e^{*}$ must be excluded in the investigation of stability and sustainability of the solutions.

