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Research article

A comprehensive subclass of bi-univalent functions defined by a linear combination and satisfying subordination conditions

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Abstract: In this article, we derive some estimates for the Taylor-Maclaurin coefficients of functions that belong to a new general subclass $\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$ of bi-univalent functions in an open unit disk, which is defined by using the Ruscheweyh derivative operator and the principle of differential subordination between holomorphic functions. Our results are more accurate than the previous works and they generalize and improve some outcomes that have been obtained by other researchers. Under certain conditions, the derived bounds are smaller than those in the previous findings. Furthermore, if we specialize the parameters, several repercussions of this generic subclass will be properly obtained.

Keywords: holomorphic functions; bi-univalent functions; coefficient estimates; starlike functions; convex functions; Ruscheweyh derivative; subordination
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1. Introduction

If we make the assumption that $\mathfrak{D}$ represents an open unit disk in the entire complex plane, defined as $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$, and if we consider $\mathcal{A}$ as a collection that contains functions denoted as f, which are holomorphic within $\mathfrak{D}$ and satisfy the normalization forms $f(0) = 0 = f'(0) - 1$, then we can obtain the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathfrak{D}). \tag{1.1}$$

If f is a function in $\mathcal{A}$ that is defined by (1.1), then the Ruscheweyh derivative of order n (see [13]) can be defined by

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n,k) a_k z^k, \quad (z \in \mathfrak{D})$$

such that $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$$\sigma(n,k) := \frac{\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}.$$

Furthermore, let us suppose that $\mathcal{S}$ is the sub-collection for all functions in $\mathcal{A}$ that are univalent and satisfy (1.1) in $\mathfrak{D}$, which converts $\mathcal{S}$ to be a subclass of $\mathcal{A}$. Besides, let us assume that $\mathcal{P}$ denotes all functions $\mathfrak{p}(z)$ characterized by the property that within the domain $\mathfrak{D}$, $\mathfrak{p}$ is holomorphic and its real part is positive. These functions take the following form:

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} \mathfrak{p}_n z^n, \quad (z \in \mathfrak{D}). \tag{1.2}$$

The most significant and major well-studied subclasses of functions belonging to class $\mathcal{S}$ are the starlike functions $\mathcal{S}^*(\vartheta)$ and convex functions $\mathcal{K}(\vartheta)$ of order ϑ $(0 \leq \vartheta < 1)$. So, we can find the following for $z \in \mathfrak{D}$, by definition:

$$\mathcal{S}^*(\vartheta) = \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \vartheta \right\},$$

and

$$\mathcal{K}(\vartheta) = \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \vartheta \right\}.$$

We note that

$$\mathcal{K}(\vartheta) \subset \mathcal{S}^*(\vartheta),$$

$$f(z) \in \mathcal{K}(\vartheta) \iff zf'(z) \in \mathcal{S}^*(\vartheta)$$

and

$$f(z) \in \mathcal{S}^*(\vartheta) \iff \int_0^z \frac{f(s)}{s} ds = f(z) \in \mathcal{K}(\vartheta).$$

Assume that f and g are two holomorphic functions which are defined in $\mathfrak{D}$. So, the function f is considered to be subordinate to g, i.e.,

$$f(z) \prec g(z) \quad (z \in \mathfrak{D}),$$

when we can identify a Schwarz function, referred to as ω, that exhibits holomorphic properties within the domain $\mathfrak{D}$, as follows:

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1,$$

and

$$f(z) = g(\omega(z)).$$

Particularly, when g stands for a univalent function in $\mathfrak{D}$, the below equivalence is achieved (see [12]):

$$f \prec g \iff f(0) = g(0) \text{ and } f(\mathfrak{D}) \subseteq g(\mathfrak{D}).$$

Assuming that φ is a univalent function in $\mathfrak{D}$ with a positive real part, $\varphi(\mathfrak{D})$ is symmetric for the real axis, φ is a starlike function under the condition of $\varphi(0) = 1$ and $\varphi'(0) > 0$. The subclasses $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ were introduced by Ma and Minda [11] and contain all functions $f \in \mathcal{S}$ which satisfy

$$\frac{zf'(z)}{f(z)} \prec \varphi(z)$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),$$

which, as special cases, consist of some various well-known subclasses, where all of the various subclasses of starlike and convex functions are consistently represented. For instance, when

$$\varphi(z) = \frac{1 + \mathrm{A}z}{1 + \mathrm{B}z} \quad (-1 \leq \mathrm{B} < \mathrm{A} \leq 1),$$

the subclass $\mathcal{S}^*(\varphi)$ reduces to Janowski's class $\mathcal{S}^*[\mathrm{A}, \mathrm{B}]$ (see [9]). Let $(0 \leq \eta < 1)$; for

$$\varphi(z) = \frac{1 + (1 - 2\eta)z}{1 - z}$$

the subclass

$$\mathcal{S}^*_\eta := \mathcal{S}^*\left(\frac{1 + (1 - 2\eta)z}{1 - z}\right)$$

represents the starlike function class of order η, and the subclass

$$\mathcal{K}_\eta := \mathcal{K}\left(\frac{1 + (1 - 2\eta)z}{1 - z}\right)$$

represents the convex function class of order η. Further, the class

$$\mathcal{S}^*_\zeta := \mathcal{S}^*\left(\frac{1 + z}{1 - z}\right)^\zeta$$

consists of the class of strongly starlike functions of order ζ $(0 < \zeta \leq 1)$.

According to the Koebe 1/4 theorem (see [6]) the image of $\mathfrak{D}$, under each univalent function comprises a disk with a radius of 1/4. As a consequence, any function f in $\mathcal{S}$ has the inverse represented by f^{-1} such that

$$f^{-1}(f(z)) = z \quad (z \in \mathfrak{D})$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f), r_0(f) \geq \ 1/4)\,.$$

We note that the inverse of the function f, i.e., $g = f^{-1}$, has the following known form:

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots\,. \tag{1.3}$$

If the function $f \in \mathcal{A}$ and its inverse are both univalent within the disk $\mathfrak{D}$, then we say that f is a bi-univalent function in $\mathcal{A}$. In this article, the collective set of bi-univalent functions within the domain $\mathfrak{D}$ is denoted as Σ. The following presents some examples of functions in the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z) \text{ and } \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

with the corresponding inverse functions:

$$\frac{e^w - 1}{e^w}, \quad \frac{w}{1+w} \text{ and } \quad \frac{e^{2w}-1}{e^{2w}+1},$$

respectively.

The functions class Σ were the focus of many recent studies, whose aim was to identify sharp coefficient bounds of the initial Taylor-Maclaurin coefficients of f in Σ, i.e., $|a_2|$ and $|a_3|$. For an initial history, as well as further significant instances in the class Σ refer to [18]. The recent ground-breaking work by Srivastava et al. [18], which has been extensively cited, promoted the analysis of univalent and bi-univalent functions to a specialized level and led to further studies about the topic (see, for example, [1, 2, 5, 10]). Brannan and Clunie's conjecture [4] was further investigated [16], and subordination properties were also obtained for certain subclasses of bi-univalent functions [15]. However, the bound of $|a_n|$ $(n \in \mathbb{N}\backslash\{2, 3\})$, known as the coefficient estimation problem, has not been settled yet. In some works, and under certain conditions, the bounds of the higher-order coefficients are determined by applying the Faber polynomial method (see, for instance, [3, 7]).

The primary aim of this article is to introduce a novel subclass, denoted as $\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$, within the broader universal class Σ, to and provide estimations for the upper bounds of the coefficients a_2 and a_3 for all functions contained in this subclass. Here, we introduce this subclass by using a linear combination of functions:

$$\frac{f(z)}{z}, \quad f'(z) \quad \text{and} \quad zf''(z).$$

Our findings serve to generalize and improve several previous findings. Furthermore, if we specialize the parameters, several repercussions of this generic class will be properly obtained.

In fact, we have successfully estimated the bound for the first two coefficients of the functions contained in this new subclass by using a Ruscheweyh operator. The findings given in this paper are more accurate than some related works by other researchers, and we have improved them. In addition, several repercussions of this general subclass are correctly noted as a consequence of parameter specialization.

2. The subclass $\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$

In the current section, we use the concept of subordination to define a new general class $\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$ that contains the functions that are bi-univalent in $\mathfrak{D}$. This is done to obtain results that are more accurate than those of previous works and to improve upon outcomes obtained by other researchers.

Assume that φ is a holomorphic function with $\mathrm{Re}(\varphi) > 0$ in $\mathfrak{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathfrak{D})$ is symmetric with respect to the real axis. A function of this form has the following power series expansion:

$$\varphi(z) = 1 + \mathfrak{J}_1 z + \mathfrak{J}_2 z^2 + \mathfrak{J}_3 z^3 + \cdots \tag{2.1}$$

$$(\mathfrak{J}_1 > 0 \quad \text{and} \quad \mathfrak{J}_2, \mathfrak{J}_3 \quad \text{is any real number}) .$$

Furthermore, let us assume that in this section, f and g are defined by expansions (1.1) and (1.3), respectively, with z and w belonging to $\mathfrak{D}$.

Definition 2.1. *A function f, which belongs to Σ, is considered to be a member of the class $\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$ if it meets the following criteria:*

$$1 + \frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(z)}{z} + (\rho\tau + \rho + \tau)(\mathcal{R}^n f(z))' + \rho\tau\left(z(\mathcal{R}^n f(z))'' - 2\right) - 1\Big] \prec \varphi(z), \tag{2.2}$$

and

$$1 + \frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n g(w)}{w} + (\rho\tau + \rho + \tau)(\mathcal{R}^n g(w))' + \rho\tau\left(w(\mathcal{R}^n g(w))'' - 2\right) - 1\Big] \prec \varphi(w), \tag{2.3}$$

where $\delta \in \mathbb{C}\backslash\{0\}, \rho \geq 0, 0 \leq \tau \leq 1, n \in \mathbb{N}_0$ and the function φ is given by (2.1).

There are many options for φ and the parameters δ, ρ, τ, n that would provide and generate interesting subclasses of the class $\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$. Let us present some examples.

Example 2.1. *By putting*

$$\varphi(z) = \frac{1 + \mathrm{A}z}{1 + \mathrm{B}z} = 1 + (\mathrm{A} - \mathrm{B})\, z - \mathrm{B}\,(\mathrm{A} - \mathrm{B})\, z^2 + \cdots \quad (-1 \leq \mathrm{B} < \mathrm{A} \leq 1)\,,$$

the class

$$\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$$

reduces to

$$\Upsilon_\Sigma(\delta, \rho, \tau, n; \mathrm{A}, \mathrm{B}),$$

which is defined by assuming that $f \in \Sigma$,

$$1 + \frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(z)}{z} + (\rho\tau + \rho + \tau)(\mathcal{R}^n f(z))' + \rho\tau\left(z(\mathcal{R}^n f(z))'' - 2\right) - 1\Big] \prec \frac{1 + \mathrm{A}z}{1 + \mathrm{B}z}$$

and

$$1+\frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n g(w)}{w}+(\rho\tau+\rho+\tau)(\mathcal{R}^n g(w))'+\rho\tau\left(w(\mathcal{R}^n g(w))''-2\right)-1\Big]\prec\frac{1+\mathrm{A}w}{1+\mathrm{B}w}.$$

Example 2.2. *By putting*

$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\zeta}=1+2\zeta z+2\zeta^2 z+\cdots\quad(0<\zeta\leq 1),$$

the class

$$\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi)$$

reduces to

$$\mathcal{Q}_{\Sigma_1}(\delta,\rho,\tau,n;\zeta),$$

which is defined by assuming that $f\in\Sigma$,

$$\left|\arg\left(1+\frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(z)}{z}+(\rho\tau+\rho+\tau)\left(\mathcal{R}^n f(z)\right)'+\rho\tau\left(z\left(\mathcal{R}^n f(z)\right)''-2\right)-1\Big]\right)\right|<\frac{\zeta\pi}{2}$$

and

$$\left|\arg\left(1+\frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n g(w)}{w}+(\rho\tau+\rho+\tau)\left(\mathcal{R}^n g(w)\right)'+\rho\tau\left(w\left(\mathcal{R}^n g(w)\right)''-2\right)-1\Big]\right)\right|<\frac{\zeta\pi}{2}.$$

The above example ensures that

$$\mathcal{Q}_{\Sigma_1}(\delta,\rho,\tau,n;\zeta)\subset\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi),$$

and that the class $\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi)$ is not empty.

Example 2.3. *Upon setting*

$$\varphi(z)=\frac{1+(1-2\eta)z}{1-z}=1+2(1-\eta)z+2(1-\eta)z^2+\cdots\quad(0\leq\eta<1),$$

the class

$$\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi)$$

reduces to

$$\Theta_{\Sigma_1}(\delta,\rho,\tau,n;\eta),$$

which is defined by assuming that $f\in\Sigma$,

$$\mathrm{Re}\left\{1+\frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(z)}{z}+(\rho\tau+\rho+\tau)\left(\mathcal{R}^n f(z)\right)'+\rho\tau\left(z\left(\mathcal{R}^n f(z)\right)''-2\right)-1\Big]\right\}>\eta$$

and

$$\mathrm{Re}\left\{1+\frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n g(w)}{w}+(\rho\tau+\rho+\tau)\left(\mathcal{R}^n g(w)\right)'+\rho\tau\left(w\left(\mathcal{R}^n g(w)\right)''-2\right)-1\Big]\right\}>\eta.$$

This example ensures that

$$\Theta_{\Sigma_1}(\delta,\rho,\tau,n;\eta)\subset\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi).$$

The subclasses $\mathcal{Q}_{\Sigma_1}(\delta,\rho,\tau,n;\zeta)$ and $\Theta_{\Sigma_1}(\delta,\rho,\tau,n;\eta)$ have been researched recently by Sabir [14].

Example 2.4. *If we take $n=0$ and*

$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\zeta},$$

then the class

$$\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi)$$

reduces to

$$\mathcal{WS}_{\Sigma}(\delta,\rho,\tau;\zeta),$$

which is defined by assuming that $f\in\Sigma$,

$$\left|\arg\left(1+\frac{1}{\delta}\left[(1-\rho)(1-\tau)\frac{f(z)}{z}+(\rho\tau+\rho+\tau)f'(z)+\rho\tau\left(zf''(z)-2\right)-1\right]\right)\right|<\frac{\zeta\pi}{2}$$

and

$$\left|\arg\left(1+\frac{1}{\delta}\left[(1-\rho)(1-\tau)\frac{g(w)}{w}+(\rho\tau+\rho+\tau)g'(w)+\rho\tau\left(wg''(w)-2\right)-1\right]\right)\right|<\frac{\zeta\pi}{2}.$$

Example 2.5. *If we let $n=0$ and*

$$\varphi(z)=\frac{1+(1-2\eta)z}{1-z},$$

then the class

$$\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi)$$

reduces to

$$\mathcal{WS}^{*}_{\Sigma}(\delta,\rho,\tau;\eta),$$

which is defined by assuming that $f\in\Sigma$,

$$\operatorname{Re}\left\{1+\frac{1}{\delta}\left[(1-\rho)(1-\tau)\frac{f(z)}{z}+(\rho\tau+\rho+\tau)f'(z)+\rho\tau\left(zf''(z)-2\right)-1\right]\right\}>\eta$$

and

$$\operatorname{Re}\left\{1+\frac{1}{\delta}\left[(1-\rho)(1-\tau)\frac{g(w)}{w}+(\rho\tau+\rho+\tau)g'(w)+\rho\tau\left(wg''(w)-2\right)-1\right]\right\}>\eta.$$

The subclasses $\mathcal{WS}_{\Sigma}(\delta,\rho,\tau;\zeta)$ and $\mathcal{WS}^{*}_{\Sigma}(\delta,\rho,\tau;\eta)$ have been investigated by Srivastava and Wanas [19].

Example 2.6. *If we put* $n = 0, \tau = 0$ *and*

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\zeta},$$

then the class

$$\Upsilon_{\Sigma}(\delta, \rho, \tau, n; \varphi)$$

reduces to

$$\mathcal{B}_{\Sigma_1}(\delta, \rho; \zeta),$$

which is defined by assuming that $f \in \Sigma$,

$$\left|\arg\left(1 + \frac{1}{\delta}\left[(1-\rho)\frac{f(z)}{z} + \rho f'(z) - 1\right]\right)\right| < \frac{\zeta\pi}{2}$$

and

$$\left|\arg\left(1 + \frac{1}{\delta}\left[(1-\rho)\frac{g(w)}{w} + \rho g'(w) - 1\right]\right)\right| < \frac{\zeta\pi}{2},$$

Example 2.7. *If we set* $n = 0, \tau = 0$ *and*

$$\varphi(z) = \frac{1 + (1-2\eta)z}{1-z},$$

then the class

$$\Upsilon_{\Sigma}(\delta, \rho, \tau, n; \varphi)$$

reduces to

$$\mathcal{B}_{\Sigma_1}(\delta, \rho; \eta),$$

which is defined by assuming that $f \in \Sigma$,

$$\mathrm{Re}\left(1 + \frac{1}{\delta}\left[(1-\rho)\frac{f(z)}{z} + \rho f'(z) - 1\right]\right) > \eta$$

and

$$\mathrm{Re}\left(1 + \frac{1}{\delta}\left[(1-\rho)\frac{g(w)}{w} + \rho g'(w) - 1\right]\right) > \eta.$$

Two subclasses $\mathcal{B}_{\Sigma_1}(\delta, \rho; \zeta)$ and $\mathcal{B}_{\Sigma_1}(\delta, \rho; \eta)$ have been considered by Srivastava et al. [17].

Example 2.8. *By putting* $n = 0, \tau = 0, \delta = 1$ *and*

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\zeta},$$

the class

$$\Upsilon_{\Sigma}(\delta, \rho, \tau, n; \varphi)$$

reduces to

$$\mathcal{B}_{\Sigma}(\zeta,\rho),$$

which is defined by assuming that $f \in \Sigma$*,*

$$\left|\arg\left((1-\rho)\frac{f(z)}{z}+\rho f'(z)\right)\right| < \frac{\zeta\pi}{2}$$

and

$$\left|\arg\left((1-\rho)\frac{g(w)}{w}+\rho g'(w)\right)\right| < \frac{\zeta\pi}{2}.$$

Example 2.9. *If we set* $n = 0, \tau = 0, \delta = 1$ *and*

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z},$$

then the class

$$\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi)$$

reduces to

$$\mathcal{B}_{\Sigma}(\eta,\rho),$$

which is defined by assuming that $f \in \Sigma$*,*

$$\operatorname{Re}\left\{(1-\rho)\frac{f(z)}{z}+\rho f'(z)\right\} > \eta$$

and

$$\operatorname{Re}\left\{(1-\rho)\frac{g(w)}{w}+\rho g'(w)\right\} > \eta.$$

The subclasses $\mathcal{B}_{\Sigma}(\zeta,\rho)$ and $\mathcal{B}_{\Sigma}(\eta,\rho)$ have been introduced and investigated by Frasin and Aouf [8].

Example 2.10. *If we set* $n = 0, \tau = 0, \delta = 1, \rho = 1$ *and*

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\zeta},$$

then the class

$$\Upsilon_{\Sigma}(\delta,\rho,\tau,n;\varphi)$$

reduces to

$$\mathcal{H}_{\Sigma}(\zeta),$$

which is defined by assuming that $f \in \Sigma$*,*

$$\left|\arg\left(f'(z)\right)\right| < \frac{\zeta\pi}{2}$$

and

$$\left|\arg\left(g'(w)\right)\right| < \frac{\zeta\pi}{2}.$$

Example 2.11. *If we set* $n = 0, \tau = 0, \delta = 1, \rho = 1$ *and*

$$\varphi(z) = \frac{1 + (1 - 2\eta)z}{1 - z},$$

then the class

$$\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$$

reduces to

$$\mathcal{H}_\Sigma(\eta),$$

which is defined by assuming that $f \in \Sigma$,

$$\mathrm{Re}\,\{f'(z)\} > \eta$$

and

$$\mathrm{Re}\,\{g'(w)\} > \eta.$$

The subclasses $\mathcal{H}_\Sigma(\zeta)$ and $\mathcal{H}_\Sigma(\eta)$ have been defined and studied by Srivastava et al. [18].

3. Coefficient estimates for the class $\Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$

To obtain the key results of this study, we must first recall the following lemma.

Lemma 3.1. (see [6]) *If* $\mathfrak{p}$ *belongs to* $\mathcal{P}$, *with* $\mathfrak{p}(z)$ *given by* (1.2), *then* $|\mathfrak{p}_n| \leq 2$ *for any* $n \in \mathbb{N}$.

Theorem 3.1. *Let* $f(z) \in \Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$ *be of the form* (1.1). *Then,*

$$|a_2| \leq \min\Bigg[\frac{|\delta|\mathfrak{I}_1}{(1+\rho+\tau+5\rho\tau)(n+1)}, \sqrt{\frac{2|\delta|(\mathfrak{I}_1 + |\mathfrak{I}_1 - \mathfrak{I}_2|)}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}}, \frac{|\delta|\mathfrak{I}_1\sqrt{2\mathfrak{I}_1}}{\sqrt{|2(\mathfrak{I}_1 - \mathfrak{I}_2)(1+\rho+\tau+5\rho\tau)^2(n+1)^2 + \delta\mathfrak{I}_1^2(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)|}}\Bigg]$$

and

$$|a_3| \leq \min\Bigg[\frac{|\delta|^2\mathfrak{I}_1^2}{(1+\rho+\tau+5\rho\tau)^2(n+1)^2} + \frac{2|\delta|\mathfrak{I}_1}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}, \frac{2|\delta|(\mathfrak{I}_1 + |\mathfrak{I}_1 - \mathfrak{I}_2|)}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}\Bigg],$$

where the coefficients $\mathfrak{I}_1$ *and* $\mathfrak{I}_2$ *are as in* (2.1).

Proof. For $f \in \Upsilon_\Sigma(\delta, \rho, \tau, n; \varphi)$, there exist Schwarz functions $\omega_1, \omega_2 : \mathfrak{D} \longrightarrow \mathfrak{D}$ that are holomorphic in $\mathfrak{D}$, i.e.,

$$\omega_1(0) = 0 = \omega_2(0) \text{ and } \max\{|\omega_1(z)|, |\omega_2(w)|\} < 1,$$

and satisfying

$$1+\frac{1}{\delta}\left[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(z)}{z}+(\rho\tau+\rho+\tau)(\mathcal{R}^n f(z))'+\rho\tau\left(z(\mathcal{R}^n f(z))''-2\right)-1\right]=\varphi(\omega_1(z)) \tag{3.1}$$

and

$$1+\frac{1}{\delta}\left[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(w)}{w}+(\rho\tau+\rho+\tau)(\mathcal{R}^n f(w))'+\rho\tau\left(w(\mathcal{R}^n f(w))''-2\right)-1\right]=\varphi(\omega_2(w)). \tag{3.2}$$

The main part of the proof depends on the definitions of two functions $\mathfrak{q}_1(z)$ and $\mathfrak{q}_2(w)$, which are respectively defined by

$$\mathfrak{q}_1(z)=\frac{1+\omega_1(z)}{1-\omega_1(z)}=1+b_1z+b_2z^2+\cdots \tag{3.3}$$

and

$$\mathfrak{q}_2(w)=\frac{1+\omega_2(w)}{1-\omega_2(w)}=1+c_1w+c_2w^2+\cdots. \tag{3.4}$$

Or, in other words,

$$\omega_1(z)=\frac{\mathfrak{q}_1(z)-1}{\mathfrak{q}_1(z)+1}=\frac{1}{2}\left(b_1z+\left(b_2-\frac{b_1^2}{2}\right)z^2+\cdots\right), \tag{3.5}$$

and

$$\omega_2(w)=\frac{\mathfrak{q}_2(w)-1}{\mathfrak{q}_2(w)+1}=\frac{1}{2}\left(c_1w+\left(c_2-\frac{c_1^2}{2}\right)w^2+\cdots\right). \tag{3.6}$$

Subsequently, $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are holomorphic functions within the domain $\mathfrak{D}$, characterized by positive real parts, and both $\mathfrak{q}_1(0)$ and $\mathfrak{q}_2(0)$ equal to 1. Therefore, applying Lemma 3.1, we have that $|b_n|\leq 2$ and $|c_n|\leq 2$ for all $n\in\mathbb{N}$. Now, since

$$\frac{\mathcal{R}^n f(z)}{z}=1+\sigma(n,2)a_2z+\sigma(n,3)a_3z^2+\cdots, \tag{3.7}$$

$$(\mathcal{R}^n f(z))'=1+2\sigma(n,2)a_2z+3\sigma(n,3)a_3z^2+\cdots, \tag{3.8}$$

$$z(\mathcal{R}^n f(z))''=2\sigma(n,2)a_2z+6\sigma(n,3)a_3z^2+\cdots, \tag{3.9}$$

$$\frac{\mathcal{R}^n g(w)}{w}=1-\sigma(n,2)a_2w+\sigma(n,3)(2a_2^2-a_3)w^2-\cdots, \tag{3.10}$$

$$(\mathcal{R}^n g(w))'=1-2\sigma(n,2)a_2w+3\sigma(n,3)(2a_2^2-a_3)w^2-\cdots \tag{3.11}$$

and

$$w(\mathcal{R}^n g(w))'' = -2\sigma(n,2)a_2 w + 6\sigma(n,3)(2a_2^2 - a_3)w^2 + \cdots, \tag{3.12}$$

clearly, we get

$$\begin{aligned} 1 + \frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(z)}{z} + (\rho\tau+\rho+\tau)(\mathcal{R}^n f(z))' + \rho\tau\left(z(\mathcal{R}^n f(z))'' - 2\right) - 1\Big] \\ = 1 + \frac{(1+\rho+\tau+5\rho\tau)}{\delta}\sigma(n,2)a_2 z + \frac{(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)a_3 z^2 + \cdots \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} 1 + \frac{1}{\delta}\Big[(1-\rho)(1-\tau)\frac{\mathcal{R}^n f(w)}{w} + (\rho\tau+\rho+\tau)(\mathcal{R}^n f(w))' + \rho\tau\left(w(\mathcal{R}^n f(w))'' - 2\right) - 1\Big] \\ = 1 - \frac{(1+\rho+\tau+5\rho\tau)}{\delta}\sigma(n,2)a_2 w + \frac{(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)(2a_2^2 - a_3)w^2 - \cdots. \end{aligned} \tag{3.14}$$

We also find that

$$\varphi(\omega_1(z)) = \varphi\left(\frac{\mathfrak{q}_1(z)-1}{\mathfrak{q}_1(z)+1}\right) = 1 + \frac{1}{2}J_1 b_1 z + \left(\frac{1}{2}J_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}J_2 b_1^2\right)z^2 + \cdots \tag{3.15}$$

and

$$\varphi(\omega_2(w)) = \varphi\left(\frac{\mathfrak{q}_2(w)-1}{\mathfrak{q}_2(w)+1}\right) = 1 + \frac{1}{2}J_1 c_1 w + \left(\frac{1}{2}J_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}J_2 c_1^2\right)w^2 + \cdots. \tag{3.16}$$

Now, by equating the coefficients in (3.13), (3.14), (3.15) and (3.16), we get

$$\frac{(1+\rho+\tau+5\rho\tau)}{\delta}\sigma(n,2)a_2 = \frac{1}{2}\mathfrak{I}_1 b_1, \tag{3.17}$$

$$\frac{(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)a_3 = \frac{1}{2}\mathfrak{I}_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\mathfrak{I}_2 b_1^2, \tag{3.18}$$

$$-\frac{(1+\rho+\tau+5\rho\tau)}{\delta}\sigma(n,2)a_2 = \frac{1}{2}\mathfrak{I}_1 c_1 \tag{3.19}$$

and

$$\frac{(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)(2a_2^2 - a_3) = \frac{1}{2}\mathfrak{I}_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\mathfrak{I}_2 c_1^2. \tag{3.20}$$

From (3.17) and (3.19), it follows that

$$b_1 = -c_1 \tag{3.21}$$

and

$$\frac{2(1+\rho+\tau+5\rho\tau)^2}{\delta^2}(\sigma(n,2))^2 a_2^2 = \frac{1}{4}\mathfrak{I}_1^2(b_1^2+c_1^2). \tag{3.22}$$

In this step, we take the absolute value of (3.22) and apply Lemma 3.1 for b_1 and c_1 to deduce that

$$|a_2| \leq \frac{|\delta|\mathfrak{I}_1}{(1+\rho+\tau+5\rho\tau)(n+1)}. \tag{3.23}$$

By adding (3.18) and (3.20), we have

$$\frac{2(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)a_2^2 = \frac{1}{2}\mathfrak{I}_1(b_2+c_2) - \frac{1}{4}\mathfrak{I}_1(b_1^2+c_1^2) + \frac{1}{4}\mathfrak{I}_2(b_1^2+c_1^2) \tag{3.24}$$

Through further computations on (3.24), and by using Lemma 3.1 for the coefficients b_1, b_2, c_1 and c_2, we get that

$$|a_2| \leq \sqrt{\frac{2|\delta|(\mathfrak{I}_1+|\mathfrak{I}_1-\mathfrak{I}_2|)}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}}. \tag{3.25}$$

On the other hand, by applying (3.22) in (3.24) we obtain

$$\frac{2(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)a_2^2 = \frac{1}{2}\mathfrak{I}_1(b_2+c_2) - \frac{2(\mathfrak{I}_1-\mathfrak{I}_2)((1+\rho+\tau+5\rho\tau)^2(\sigma(n,2))^2a_2^2}{\mathfrak{I}_1^2\delta^2},$$

or, equivalently,

$$a_2^2 = \frac{2\mathfrak{I}_1^3\delta^2(b_2+c_2)}{[2(\mathfrak{I}_1-\mathfrak{I}_2)(1+\rho+\tau+5\rho\tau)^2(n+1)^2+\delta\mathfrak{I}_1^2(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)]}. \tag{3.26}$$

Next, we take the absolute value of (3.26) and, once again, utilize Lemma 3.1 for the coefficients b_2 and c_2 to derive

$$|a_2| \leq \frac{|\delta|\mathfrak{I}_1\sqrt{2\mathfrak{I}_1}}{\sqrt{|2(\mathfrak{I}_1-\mathfrak{I}_2)(1+\rho+\tau+5\rho\tau)^2(n+1)^2+\delta\mathfrak{I}_1^2(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)|}}. \tag{3.27}$$

Now from (3.23), (3.25) and (3.27), we can find the bound for $|a_2|$.

We now need to determine an upper bound for $|a_3|$. To do this, we subtract equation (3.20) from equation (3.18) and apply equation (3.21), resulting in

$$\frac{2(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)a_3 - \frac{2(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)a_2^2 = \frac{1}{2}\mathfrak{I}_1(b_2-c_2) \tag{3.28}$$

Now, we substitute the value of a_2^2 from (3.22) into (3.28), to get

$$a_3 = \frac{\Im_1^2\delta^2(b_1^2+c_1^2)}{8(1+\rho+\tau+5\rho\tau)^2(n+1)^2} + \frac{\Im_1\delta(b_2-c_2)}{4(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}, \tag{3.29}$$

Once more, we calculate the absolute value of (3.29) and utilize Lemma 3.1 for the coefficients b_1, b_2, c_1, and c_2 to deduce that

$$|a_3| \le \frac{|\delta|^2\Im_1^2}{(1+\rho+\tau+5\rho\tau)^2(n+1)^2} + \frac{2|\delta|\Im_1}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}. \tag{3.30}$$

Similarly, if we use (3.24) in (3.28), we obtain

$$\begin{aligned}\frac{2(1+2\rho+2\tau+10\rho\tau)}{\delta}\sigma(n,3)a_3 = {}& \frac{1}{2}\Im_1(b_2+c_2) - \frac{1}{4}\Im_1(b_1^2+c_1^2)\\ &+ \frac{1}{4}\Im_2(b_1^2+c_1^2) + \frac{1}{2}\Im_1(b_2-c_2).\end{aligned} \tag{3.31}$$

Further computation on (3.31) yields that

$$a_3 = \frac{\delta[2\Im_1(b_2+b_2) - \Im_1(b_1^2+c_1^2) + \Im_2(b_1^2+c_1^2)]}{8(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}. \tag{3.32}$$

Finally, by taking the absolute value of (3.32) and employing Lemma 3.1 once more for the coefficients b_1, b_2 and c_1, we can firmly establish that

$$|a_3| \le \frac{2|\delta|[\Im_1 + |\Im_1 - \Im_2|]}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}. \tag{3.33}$$

Now, from (3.30) and (3.33), we can find the bound for $|a_3|$. □

4. Corollaries and consequences

In this section, we consider some Ma-Minda type functions, $\varphi(z)$, which provide several corollaries for the scenario that $0 \le \Im_2 \le \Im_1$.

If we put

$$\varphi(z) = \frac{1+\mathrm{A}z}{1+\mathrm{B}z}$$

in Theorem 3.1, then Corollary 4.1 can be obtained.

Corollary 4.1. *If* $f(z) \in \Upsilon_\Sigma(\delta,\rho,\tau,n;\mathrm{A},\mathrm{B})$ *is given by* (1.1), *then*

$$\begin{aligned}|a_2| \le \min\Bigg[&\frac{|\delta|(\mathrm{A}-\mathrm{B})}{(1+\rho+\tau+5\rho\tau)(n+1)}, \sqrt{\frac{2|\delta|(2+\mathrm{B})(\mathrm{A}-\mathrm{B})}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}},\\ &\frac{\sqrt{2}|\delta|(\mathrm{A}-\mathrm{B})}{\sqrt{|2(1+\mathrm{B})(1+\rho+\tau+5\rho\tau)^2(n+1)^2 + \delta(\mathrm{A}-\mathrm{B})(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)|}}\Bigg]\end{aligned}$$

and

$$|a_3| \le \min\left[\frac{|\delta|^2(\mathrm{A}-\mathrm{B})^2}{(1+\rho+\tau+5\rho\tau)^2(n+1)^2}+\frac{2|\delta|(\mathrm{A}-\mathrm{B})}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}, \frac{2|\delta|(2+\mathrm{B})(\mathrm{A}-\mathrm{B})}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}\right].$$

If we set

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\zeta}$$

in Theorem 3.1, then Corollary 4.2 can be obtained.

Corollary 4.2. *If* $f(z) \in \mathcal{Q}_{\Sigma_1}(\delta,\rho,\tau,n;\zeta)$ *is given by* (1.1)*, then*

$$|a_2| \le \min\left[\frac{2|\delta|\zeta}{(1+\rho+\tau+5\rho\tau)(n+1)}, \sqrt{\frac{4|\delta|\zeta(2-\zeta)}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}}, \frac{2|\delta|\zeta}{\sqrt{|(1-\zeta)(1+\rho+\tau+5\rho\tau)^2(n+1)^2+\delta\zeta(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)|}}\right]$$

and

$$|a_3| \le \min\left[\frac{4|\delta|^2\zeta^2}{(1+\rho+\tau+5\rho\tau)^2(n+1)^2}+\frac{4|\delta|\zeta}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}, \frac{4|\delta|\zeta(2-\zeta)}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}\right].$$

Remark 4.1. *The bounds for* $|a_2|$ *and* $|a_3|$ *obtained in Corollary 4.2 are improvements of the results that are given in [14, Corollary 5].*

If we put

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z}$$

in Theorem 3.1, then Corollary 4.3 can be obtained.

Corollary 4.3. *If* $f(z) \in \Theta_{\Sigma_1}(\delta,\rho,\tau,n;\eta)$ *is given by* (1.1)*, then*

$$|a_2| \le \min\left[\frac{2|\delta|(1-\eta)}{(1+\rho+\tau+5\rho\tau)(n+1)}, \sqrt{\frac{4|\delta|(1-\eta)}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}}\right]$$

and

$$|a_3| \le \frac{4|\delta|(1-\eta)}{(1+2\rho+2\tau+10\rho\tau)(n+1)(n+2)}.$$

By setting $n = 0$ and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\zeta}$$

in Theorem 3.1, Corollary 4.4 can be obtained.

Corollary 4.4. *If $f(z) \in \mathcal{WS}_{\Sigma}(\delta,\rho,\tau;\zeta)$ is given by* (1.1)*, then*

$$|a_2| \leq \min\left[\frac{2|\delta|\zeta}{1+\rho+\tau+5\rho\tau}, \sqrt{\frac{2|\delta|\zeta(2-\zeta)}{1+2\rho+2\tau+10\rho\tau}}, \frac{2|\delta|\zeta}{\sqrt{|(1-\zeta)(1+\rho+\tau+5\rho\tau)^2+2\delta\zeta(1+2\rho+2\tau+10\rho\tau)|}}\right]$$

and

$$|a_3| \leq \min\left[\frac{4|\delta|^2\zeta^2}{(1+\rho+\tau+5\rho\tau)^2}+\frac{2|\delta|\zeta}{1+2\rho+2\tau+10\rho\tau}, \frac{2|\delta|\zeta(2-\zeta)}{1+2\rho+2\tau+10\rho\tau}\right].$$

Remark 4.2. *The upper bounds for $|a_2|$ and $|a_3|$ that are obtained in Corollary* 4.4 *are improvements of the results that are given in* [19, Corollary 2.1].

If we set $n = 0$ and

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z}$$

in Theorem 3.1, Corollary 4.5 can be obtained.

Corollary 4.5. *If $f(z) \in \mathcal{WS}^{*}_{\Sigma}(\delta,\rho,\tau;\eta)$ is given by* (1.1)*, then*

$$|a_2| \leq \min\left[\frac{2|\delta|(1-\eta)}{1+\rho+\tau+5\rho\tau}, \sqrt{\frac{2|\delta|(1-\eta)}{1+2\rho+2\tau+10\rho\tau}}\right]$$

and

$$|a_3| \leq \frac{2|\delta|(1-\eta)}{1+2\rho+2\tau+10\rho\tau}.$$

Remark 4.3. *The bound for $|a_2|$ obtained in Corollary* 4.5 *is smaller than the bound obtained in* [19, Corollary 3.1] *because*

$$\frac{2|\delta|(1-\eta)}{1+\rho+\tau+5\rho\tau} \leq \sqrt{\frac{2|\delta|(1-\eta)}{1+2\rho+2\tau+10\rho\tau}}; \quad \eta \geq \frac{(1+\rho+\tau+5\rho\tau)^2}{2|\delta|(1+2\rho+2\tau+10\rho\tau)}.$$

Also, the upper bound on $|a_3|$ given in Corollary 4.5 *is smaller than the upper bound given in* [19, Corollary 3.1] *because*

$$\frac{2|\delta|(1-\eta)}{1+2\rho+2\tau+10\rho\tau} \leq \frac{2|\delta|(1-\eta)}{1+2\rho+2\tau+10\rho\tau}+\frac{4|\delta|^2(1-\eta)^2}{1+\rho+\tau+5\rho\tau}.$$

By setting $n = 0$, $\tau = 0$, $\delta = 1$ and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\zeta}$$

in Theorem 3.1, Corollary 4.6 can be obtained.

Corollary 4.6. *If $f(z) \in \mathcal{B}_\Sigma(\zeta,\rho)$ is given by* (1.1)*, then*

$$|a_2| \leq \min\left[\frac{2\zeta}{1+\rho}, \sqrt{\frac{2\zeta(2-\zeta)}{1+2\rho}}, \frac{2\zeta}{\sqrt{(1-\zeta)(1+\rho)^2+2\zeta(1+2\rho)}}\right]$$

and

$$|a_3| \leq \min\left[\frac{4\zeta^2}{(1+\rho)^2}+\frac{2\zeta}{1+2\rho}, \frac{2\zeta(2-\zeta)}{1+2\rho}\right].$$

Remark 4.4. *The upper bounds for $|a_2|$ and $|a_3|$ obtained in Corollary* 4.6 *are improvements of the results that are given in* [8, Theorem 2.2].

By setting $n = 0$, $\tau = 0$, $\delta = 1$ and

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z}$$

in Theorem 3.1, Corollary 4.7 can be obtained.

Corollary 4.7. *If $f(z) \in \mathcal{B}_\Sigma(\eta,\rho)$ is given by* (1.1)*, then*

$$|a_2| \leq \min\left[\frac{2(1-\eta)}{1+\rho}, \sqrt{\frac{2(1-\eta)}{1+2\rho}}\right]$$

and

$$|a_3| \leq \frac{2(1-\eta)}{1+2\rho}.$$

Remark 4.5. *The upper bound for $|a_2|$ obtained in Corollary* 4.7 *is more accurate than the upper bound obtained in* [8, Theorem 3.2] *since*

$$\frac{2(1-\eta)}{1+\rho} \leq \sqrt{\frac{2(1-\eta)}{1+2\rho}}; \quad \eta \geq 1-\frac{(1+\rho)^2}{1+2\rho}.$$

Also, the upper bound of $|a_3|$ obtained in Corollary 4.7 *is smaller than the upper bound obtained in* [8, Theorem 3.2]*, since*

$$\frac{2(1-\eta)}{1+2\rho} \leq \frac{2(1-\eta)}{1+2\rho}+\frac{4(1-\eta)^2}{(1+\rho)^2}.$$

By setting $n = 0$, $\tau = 0$, $\rho = 1$, $\delta = 1$ and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^\zeta$$

in Theorem 3.1, Corollary 4.8 can be obtained.

Corollary 4.8. *Let $f(z) \in \mathcal{H}_\Sigma(\zeta)$ be given by* (1.1)*; then,*

$$|a_2| \le \min\left[\zeta,\ \sqrt{\frac{2}{3}\zeta(2-\zeta)},\ \frac{2\zeta}{\sqrt{4+2\zeta}}\right]$$

and

$$|a_3| \le \min\left[\zeta\left(\frac{2}{3}+\zeta\right),\ \frac{2}{3}\zeta\,(2-\zeta)\right].$$

Remark 4.6. *The upper bounds for $|a_2|$ and $|a_3|$ obtained in Corollary* 4.8 *are improvements of the upper bounds obtained in* [18, Theorem 1].

If we set $n = 0$, $\tau = 0$, $\rho = 1$, $\delta = 1$ and

$$\varphi(z) = \frac{1+(1-2\eta)z}{1-z}$$

in the Theorem 3.1, Corollary 4.9 can be obtained.

Corollary 4.9. *If $f(z) \in \mathcal{H}_\Sigma(\eta)$ is given by* (1.1)*, then*

$$|a_2| \le \min\left[1-\eta,\ \sqrt{\frac{2}{3}(1-\eta)}\right]$$

and

$$|a_3| \le \frac{2}{3}(1-\eta).$$

Remark 4.7. *The upper bound for $|a_2|$ obtained in Corollary* 4.9 *is more accurate than the upper bound obtained in* [18, Theorem 2] *since*

$$1-\eta \le \sqrt{\frac{2}{3}(1-\eta)}; \quad \eta \ge \frac{1}{3}.$$

Also, the upper bound on $|a_3|$ given in Corollary 4.9 *is smaller than the upper bound given in* [18, Theorem 2] *because*

$$\frac{2}{3}(1-\eta) \le \frac{(1-\eta)(5-3\eta)}{3}.$$

5. Conclusions

In the present paper, we have successfully established estimates for the Taylor-Maclaurin coefficients of functions within a novel general subclass, denoted as $\Upsilon_\Sigma(\delta,\rho,\tau,n;\varphi)$, which pertains to bi-univalent functions within the open unit disk. Notably, our results exhibit superior accuracy compared to prior research efforts, and they serve to both generalize and enhance the outcomes achieved by previous researchers. It is worth noting that, under specific conditions, the derived bounds are even more stringent than those put forth in earlier studies. Moreover, the article sheds light on the significant implications that emerge when specific parameterizations are applied within this subclass.

In conclusion, the field of normalized analytic and bi-univalent functions, revived by Srivastava et al. [18], continues to expand in various directions, including q-analysis and q-theory. We support q-results and their potential extensions to the defined class. Our investigation also encourages further research, aligning with recent developments for our function class that were focused on finding the upper bounds of Hankel and Toeplitz determinants, as well as works related to the Fekete-Szegö functional.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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