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*Research article*

## Efficient control chart-based monitoring of scale parameter for a process with heavy-tailed non-normal distribution

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**Abstract:** Statistical process control is a procedure of quality control that is widely used in industrial processes to enable monitoring by using statistical techniques. All production processes are faced with natural and unnatural variations. To maintain the stability of the production process and reduce variation, different tools are used. Control charts are significant tools to monitor a production process. In this article, we design an extended exponentially weighted moving average (EEWMA) chart under the assumption of inverse Maxwell (IM) distribution, an IM EEWMA (IMEEWMA) control chart. We have estimated the performance of the proposed chart in terms of various run-length (RL) properties, including the average RL, standard deviation of the RL and median RL. We have also carried out a comparative analysis of the proposed chart with the existing Shewhart-type chart for IM distribution (VIM chart) and IM exponential weighted moving average (IMEWMA) chart. We observed that the proposed IMEEWMA chart performed better than the VIM chart and IMEWMA chart in terms of the ability to detect small and moderate shifts. To demonstrate its practical application, we have applied the IMEEWMA chart, along with existing control charts, to monitor the lifetime of car brake pad data. This real-world example illustrates the superiority of the IMEEWMA chart over its counterparts in industrial scenarios.

**Keywords:** statistical process control; exponentially weighted moving average; extended exponentially weighted moving average; Inverse Maxwell distribution; maximum likelihood estimator; average run length

## 1. Introduction

Statistical process control (SPC) is a widely utilized quality control procedure to monitor industrial processes by using statistical techniques. In any production process, both natural and unnatural variations occur. To maintain process stability and reduce variation, various tools are employed. Control charts play a significant role in monitoring production processes. These charts can be categorized as memory-less control charts and memory-type control charts. The fundamental memory-type control chart is the exponentially weighted moving average (EWMA) control chart. Additionally, there are other control charts such as the extended-EWMA (EEWMA), cumulative sum (CUSUM) and hybrid-EWMA (HEWMA), which have been developed for the quick detection of small to moderate shifts. Several variations of the EWMA control chart have been introduced to detect small shifts, some of which can monitor changes in both mean and variance simultaneously.

Literature suggests that EWMA control charts primarily rely on the assumption of normality for process distribution. For instance, Chen et al. [1] monitored variations in the mean and variance of a normal process. Yeh et al. [2] developed an improved EWMA control chart for monitoring the variability in multivariate normal processes. Other studies have proposed various applications of normal distribution-based EWMA control charts. Zhang and Chen [3] introduced the double EWMA (DEWMA) control chart as an enhancement of the traditional EWMA control chart. Khoo and Wong [4] extended the DEWMA methodology by introducing the double moving average approach. Haq [5] proposed a new control chart named the HEWMA control chart, which combines concepts from the traditional EWMA control chart. Shabbir and Awan [6] presented a Shewhart-type control chart to monitor moderate-sized shifts effectively in the process mean. Naveed et al. [7] introduced EEWMA-based and EWMA-based monitoring charts for tracking the process location parameters by using the moving average statistic, considering both known and unknown additional information.

One favorable property of the EWMA chart, along with the mentioned charts, is its robustness to non-normality. However, in reality, there are cases in which the assumption of normality may not hold, making non-normality an important issue in ongoing process monitoring. Therefore, the development of control charts based on non-normality conditions is essential. Borrer et al. [8] and Maravelakis et al. [9] demonstrated that EWMA control charts can robustly monitor the mean and variance of a process when the normality assumption is violated. They also provided useful transformation schemes for handling non-normal data. Several other studies have highlighted the critical role of a normality assumption in Phase I, while non-normality becomes a concern in Phase II monitoring. Specifically, when the sample size is not sufficiently large, non-normality becomes problematic. To address this, monitoring schemes based on control charts have been developed for non-normal symmetric and skewed distributions. In recent years, scholars have focused on using EWMA and CUSUM control charts to monitor non-normal processes. For example, Borrer et al. [10] developed a Poisson EWMA control chart, and Gan [11,12] applied an EWMA control chart to monitor binomial observations and Poisson observations. Yu et al. [13] used EWMA control chart schemes to monitor the water quality in the negative binomial distribution. Gan [14] provided an EWMA control chart with reflecting boundaries. Maravelakis et al. [9] investigated the use of EWMA chart as a tool to monitor dispersion in non-normal distributions. Abbasi et al. [15] studied EWMA control charts for both normal and non-normal processes, exploring several non-normal distributions such as the logistic and Student's *t* distributions. Raza et al. [16] developed EWMA charts under the multivariate student's *t* distribution

and multivariate gamma distribution. Zhang and Chen [3] designed EWMA charts for monitoring the mean of censored Weibull lifetimes. Pascual [17] developed an EWMA control chart for monitoring location parameters, and he assumed Weibull distribution parameters. Arif et al. [18] introduced the EWMA-np control chart for Weibull distribution.

In real-life scenarios, non-normal datasets are commonly encountered across various applied fields such as the engineering, agricultural and industrial sectors. While statistics offers a range of established distribution models to characterize these non-normal datasets, practical situations often arise whereby these conventional distributions prove inadequate. In such cases, alternative statistical distributions, including the inverse counterparts of typical distributions, become necessary to effectively capture the data's behavior. One notable candidate for modeling non-normal distributions is the Maxwell distribution. Particularly in the context of lifetime modeling, this distribution exhibits significant distributional properties and finds applications not only in the realm of statistics but also in fields of chemistry, physics and statistical mechanics. It is important to recognize that utilizing the inverse of a distribution is often less parsimonious compared to using the original distribution itself. Consequently, Singh and Srivastava [19] introduced the IM distribution, derived from the Maxwell distribution, to expand its applicability. They conducted a comprehensive exploration of various statistical properties, including moments, survival functions, and parameter estimation techniques. However, it is worth noting that none of the aforementioned research endeavors delved into practical applications of this distribution. This absence of applied usage, coupled with the unavailability of some crucial statistical properties, served as the primary motivation for our current study. Furthermore, in the existing literature, there is a notable absence of memory-type control charting methods apart from that associated with IM EWMA (IMEWMA). Therefore, an attempt was made to address this gap by introducing another memory-type control chart, known as the IM extended EWMA (IMEEWMA).

While the aforementioned distributions are commonly considered for data modeling, there is a need to explore other non-normal distributions in theoretical and real-life fields. Karlis and Santourian [20] demonstrated an improvement in the parsimony of real distributions by considering the inverse of a distribution, using the example of the inverse Gaussian distribution. In the literature, several studies have derived and applied different forms of inverse distributions, such as inverse exponential, inverse gamma, inverse Rayleigh, inverse Lindley, inverse Weibull and IM distributions [21–34]. The IM distribution is another positively skewed non-normal distribution. Essentially, the IM distribution is derived from the Maxwell distribution and has heavier tails than the Maxwell distribution, which means that it has a higher probability of observing extreme values than the Maxwell distribution. This statement was presented by Karlis and Santourian [20] who compared the coefficient of skewness for both the Maxwell and IM distributions. They also suggest that the IM distribution enhances the applicability of the Maxwell family of distributions. Although the theoretical properties of the IM distribution have been investigated, no real-life applications of this distribution have been attempted yet. Arafat et al. [25] conducted a study on the survival times of breast cancer patients and found that these times followed an IM distribution. As part of their research, they developed an EWMA control chart to enable the early detection of breast cancer cells for timely patient treatment. This study is motivated to consider the IM distribution because its application in control charts has many potential advantages over other heavy-tailed distributions. These advantages include the following. i) The IM distribution can be a suitable choice when your data follow a distribution that is characterized by a heavy tail, which means that it has a higher probability of extreme values. This distribution allows you to model data with longer tails, endowing it with the flexibility to capture extreme events or outliers in a process under study. ii) If data closely follows the characteristics of an IM distribution, using it in control charts can result in a better fit to the data than other heavy-tailed distributions. A better fit can

lead to more accurate statistical process control. iii) Control limits in control charts are typically based on the assumed distribution of the data. If the data follow an IM distribution, control limits derived from this distribution may be more appropriate for identifying significant deviations from the mean or median of the process. iv) In situations wherein a process yields extreme values or outliers that are essential to detect, a heavy-tailed distribution like the IM distribution can provide increased sensitivity to these events. It can help to identify process shifts or variations that might be missed when using control charts based on distributions with thinner tails. An IM distribution is more sensitive to quick shift or outlier than the Maxwell distribution and other heavy-tailed distributions. The distribution is widely used in statistical mechanics, lifetime modeling and chemistry applications.

Hossain et al. [21] proposed the Shewhart-type chart for the Maxwell distribution (VM chart) specifically for the purpose of analyzing and monitoring data that follows the Maxwell distribution. The VM control chart is based on a sample mean and range, and its statistical properties were evaluated by performing a simulation study. The authors demonstrated that the proposed control chart performs well in terms of monitoring process parameters. Further Hossain et al. [22] provided an improved and efficient version of the VM chart by introducing a CUSUM control chart for monitoring the failure rate in the boring process. Hossain and Omar [23] conducted a study on Shewhart-type control charts for observing processes that adhere to a Rayleigh distribution. They developed a chart designed to effectively monitor the scale parameter of the Rayleigh-distributed process. Morales and Panza [24] presented an adaptive version of the EWMA control chart with rigorous limits that allow one to obtain the mean from an asymmetric normal distribution. Lin et al. [25] utilized a Bayesian approach to construct a Phase II EWMA control chart for monitoring the variance of distribution-free service processes. There is a wealth of literature available that explores the applications of EWMA control charts based on different non-normal distributions. Al-Omari et al. [26] investigated the theoretical properties of the IM distribution. The IM distribution provides a suitable option for modeling various types of life or survival time data and genetic data, with potential applications in various fields, including statistical mechanics and chemistry.

The primary objective of this study is to address the need for effective monitoring of processes when the underlying distribution deviates from normality and follows a heavy-tailed IM distribution. The focus is specifically on detecting changes in the scale parameter of an ongoing process. The motivation for this research stems from the recognition that non-normality is prevalent in real-life processes, and assuming normality can lead to inadequate process monitoring. While existing literature has explored control charts based on various non-normal distributions, there is a lack of research that specifically targets the IM distribution by using the EEWMA control chart, which yields heavier tails than the Maxwell distribution and has potential applications in life or survival time data and genetic data analysis.

The crux of this paper lies in the development of an EEWMA control chart tailored to the IM distribution, aiming to provide valuable insights for accurate process monitoring and contribute to improved quality control practices. By addressing the limitations of assuming normality and focusing on the IM distribution, this study was proposed to enhance understanding in the field of statistical process monitoring and facilitate more robust and reliable quality control techniques.

In Section 2, we discuss the existing control charts related to the IM and IMEWMA control charts. Section 3 presents the development of the proposed control chart, while Section 4 describes the different performance measures associated with the proposed control chart. In Section 5, a comparison between the IMEEWMA control charts and the existing V and IMEWMA charts is presented through the result of simulation. Furthermore, Section 6 includes a real-life application to illustrate the application of the proposed control chart. Finally, in Section 7, the manuscript provides a general discussion of the findings.

## 2. Materials and methods

In order to investigate the properties and performance of the proposed chart, we present the methodology and related details for the chart's development in different sub-sections. Subsection 2.1 offers a concise introduction to a heavy-tailed IM distribution, including its properties. Additionally, subsection 2.2 provides a reproduction of existing control charts that are based on the IM distribution.

### 2.1. IM distribution and its properties

Let us consider that  $Y$  is a continuous random variable that follows the Maxwell distribution with the scale parameter  $\sigma$ . The probability distribution function is given as follows:

$$f(y, \sigma) = \sqrt{\frac{2}{\pi}} \sigma^{-3} y^2 e^{-\frac{y^2}{2\sigma^2}}; \quad y > 0, \sigma > 0 \quad (1)$$

where  $X = Y^{-1}$  then, then  $Z$  can be defined as an IM random variable. The probability density function (PDF) of the IM distribution with scale parameter  $\sigma$  is defined as follows:

$$f(x, \sigma) = \sqrt{\frac{2}{\pi}} \sigma^{-3} x^{-4} e^{-\frac{1}{2x^2\sigma^2}}; \quad x > 0, \sigma > 0. \quad (2)$$

The IM distribution has one scale parameter  $\sigma$ . The maximum likelihood estimator (MLE)  $\hat{\sigma}$  of  $\sigma$  is given below

$$\hat{\sigma} = \sqrt{3n^{-1} \sum_{i=1}^n \frac{1}{x_i^2}}. \quad (3)$$

### 2.2. Some existing control charts for IM distribution

If it is assumed that an ongoing production process follows an IM distribution, and interest is in monitoring the process scale parameter  $\sigma$ , then there are two control charts available for monitoring the process for a possible shift in the value of its scale parameter. This section presents methods for developing the available control charts for an IM distributed process, i.e., A memory-less Shewhart-type (VIM) control chart and a memory-based EWMA control chart. A VIM control chart is given by Omar et al. [31], and a memory-based EWMA control chart is given by Arafat et al. [32] in order to monitor the scale parameter of a positively skewed process.

#### 2.2.1. VIM control chart for IM distribution

Omar et al. [31] proposed an IM distribution-based control chart for monitoring the scale parameter  $\sigma$ , and their suggested control chart is named the  $V_{IM}$  control chart. Their plotting statistic for the IM distribution-based control chart is given by

$$V_{IM} = (3n)^{-1} \sum_{i=1}^n \frac{1}{r_i^2}. \quad (4)$$

The expected mean and variance of  $V_{IM}$  are given, respectively, by  $E[V_{IM}] = \sigma^2$  and

$$\text{Var}(V_{IM}) = \frac{2\sigma^4}{3n}$$

The control limits for the proposed chart can be expressed as follows,

$$LCL = R_1 \sigma_0^2, \quad CL = \sigma_0^2, \quad UCL = R_2 \sigma_0^2,$$

$$\text{where, } R_1 = \left[ 1 - L \sqrt{\frac{2}{3n}} \right], \quad \text{and } R_2 = \left[ 1 + L \sqrt{\frac{2}{3n}} \right].$$

$R_1$  and  $R_2$  factors may be used to simplify the calculations involving the  $L$  control chart coefficient. Utilizing these factors will significantly contribute to reduce the complexity involved in the construction of the VIM control chart for monitoring a scale parameter.

During the monitoring of the IM scale parameter, two situations regarding the availability of required process parameters may need to be addressed in different practical situations:

- i) when  $\sigma^2$  is known,
- ii) when  $\sigma^2$  is unknown.

For an unknown  $\sigma^2$ , the control limits are given below,

$$LCL = R_1 \bar{V}_{IM}, \quad CL = \bar{V}_{IM}, \quad UCL = R_2 \bar{V}_{IM}.$$

If any data point exceeds the upper control limit (UCL) or falls below the lower control limit (LCL), it indicates that the process is out of control. On the other hand, if the data points remain within these control limits, we can conclude that the process is in control.

### 2.2.2. Arafat et al. [32] EWMA control chart for IM distribution

The EWMA control chart was first introduced by Roberts [35] but this chart is based on the normality assumption and considering Arafat et al. [32] established a new EWMA control chart in the case of non-normal distribution. In their case, the non-normal distribution is the IM distribution and the proposed control chart is the IMEWMA chart. The chart is reproduced as given below.

The plotting statistic for the IMEWMA control chart is defined as follows

$$Z_i = \lambda V_{IM_i} + (1 - \lambda)Z_{i-1}. \quad (5)$$

In the above statistic,  $V_{IM_i}$  is the current observation for  $(i = 1, 2, 3, \dots)$ .  $Z_{i-1}$  represents the previous value of the statistic. The smoothing constant  $\lambda$  lies between 0 and 1 (i.e.,  $0 < \lambda \leq 1$ ).

The initial value  $Z_0$  is taken to be  $\sigma_0^2$ . If we do not have any information about the target scale parameter, then it can be estimated from phase I sample. An alternative form of the EWMA statistic can be written as follows

$$Z_i = \lambda(1 - \lambda)^j V_{IM_{i-j}} + (1 - \lambda)^j Z_0. \quad (6)$$

The mean and variance of IMEWMA statistics, respectively, are given by;

$$E(Z_i) = \sigma^2, \quad \text{and } \text{Var}(Z_i) = \frac{2\sigma^4}{3n} \left\{ \frac{\lambda}{2-\lambda} (1 - (1 - \lambda)^{2i}) \right\},$$

where,  $\sigma^2$  is the square of the scale parameter. So, the control structure of an IMEWMA chart is given as follows

$$UCL = P_1 \sigma^2, \quad CL = \sigma^2, \quad UCL = P_2 \sigma^2, \quad (7)$$

where  $P_1 = \left[ 1 - L \sqrt{\frac{2}{3n} \times \frac{\lambda}{2-\lambda} (1 - (1-\lambda)^{2i})} \right]$ , and  $P_2 = \left[ 1 + L \sqrt{\frac{2}{3n} \times \frac{\lambda}{2-\lambda} (1 - (1-\lambda)^{2i})} \right]$ .

$P_1$  and  $P_2$  factors may be used to simplify the calculations involving the  $L$  control chart coefficient. Utilizing these factors will significantly contribute to reduce the complexity involved in the construction of the IMEWMA control chart for monitoring a scale parameter.

As previously mentioned two different scenarios may need to be present to monitor the scale parameter in the IM distribution, and these scenarios are stated as follows

- i) when  $\sigma^2$  is known,
- ii) when  $\sigma^2$  is unknown.

For an unknown  $\sigma^2$ , involved control limits are given below

$$LCL = P_1 \bar{V}_{IM}, \quad CL = \bar{V}_{IM}, \quad UCL = P_2 \bar{V}_{IM}.$$

If any data point is beyond the UCL or falls below the LCL, it signifies that the process is out of control. Conversely, if the data points consistently remain within these control limits, we can deduce that the process is under control.

### 3. Development of the proposed EEWMA control chart for IM distribution

Naveed et al. [36] presented an extended form of the Robert [33] EWMA control chart known as an EEWMA control chart. Like an EWMA control chart, the EEWMA control chart is also based on the assumption that the distribution of an ongoing process is a normal distribution. This study was designed to develop the Naveed et al. [36] EEWMA control chart for a non-normal distribution by following the methodology proposed by Arafat et al. [32], and Omer et al. [31]. For this study, the non-normal distribution is a positively skewed IM distribution, and the proposed control chart is referred to as the IMEEWMA control chart. The structure of the proposed IMEEWMA chart is designed and illustrated in the following text.

Considering a random variable  $Q$ , where,  $Q = \frac{1}{2x^2\sigma^2}$ , the PDF of  $Q$  is given by

$$f_Q(q) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} q^{\frac{3}{2}-1} e^{-q}. \quad (8)$$

The above expression is a pdf of the gamma distribution which shows that the shape and scale parameters are  $3/2$  and  $1$ , respectively. As per the additive property of the gamma distribution, we can write the IM variates as  $\sum_{i=1}^n Q_i \sim \text{gamma}\left(\frac{3n}{2}, 1\right)$ . Here, we introduce another variable  $V_{IM} = (3n)^{-1} \sum_{i=1}^n \frac{1}{x_i^2}$  which is the square of the MLE of scale parameter  $\sigma$ . Finally, we consider the pivotal quantity  $S$  which follows a gamma distribution with scale and shape parameters as  $3/2$  and  $1$ . Here,  $S$  is a gamma-distributed random variable, so its mean is  $E[S] = E\left[\frac{3nV_{IM}}{2\sigma^2}\right] = \frac{3n}{2}$ . Thus we can write the mean and variance of  $V_{IM}$  as  $E[V_{IM}] = \sigma^2$  and  $Var(V_{IM}) = \frac{2\sigma^4}{3n}$  respectively.

The plotting statistic for the proposed EEWMA control chart which is an extension of the EWMA chart for the IM distribution that gives positive weight to the current observation and negative weight to the proceeding observations, is defined as

$$u_i = \varphi_1 V_{IM_i} - \varphi_2 V_{IM_{i-1}} + (1 - \varphi_1 + \varphi_2) u_{i-1}. \quad (9)$$

The smoothing constants of  $\varphi_1$  and  $\varphi_2$  lie between  $0 < \varphi_1 \leq 1$  and  $0 < \varphi_2 \leq \varphi_1$  respectively.

For the proposed statistic,  $V_{IMi}$  is the current observation for  $(i = 1, 2, 3, \dots)$ . The quantity  $V_{IMi-1}$  represents the previous value of the variable and  $U_{i-1}$  represents the previous value of the statistic. The initial value  $u_0$  is taken to be  $\sigma_0^2$ . If we do not have any information about the target scale parameter, then it can be estimated from Phase I sample. The mean and variance of IMEEWMA statistics are respectively as follows.

$$E(u_i) = \sigma^2,$$

and

$$\text{Var}(u_i) = \frac{2\sigma^4}{3n} \{\varphi_1^2 + \varphi_2^2\} \left[ \left\{ \frac{1-a^{2i}}{2(\varphi_1-\varphi_2) - (\varphi_1-\varphi_2)^2} \right\} - \left\{ \frac{1-a^{2i-2}}{2(\varphi_1-\varphi_2) - (\varphi_1-\varphi_2)^2} \right\} \right], \quad (10)$$

where,  $\sigma^2$  is the square of the scale parameter. So, the control structure of an IMEEWMA chart is given follows:

$$LCL = W_1\sigma^2, \quad CL = \sigma^2, \quad UCL = W_2\sigma^2, \quad (11)$$

where,

$$W_1 = \left[ 1 - K \sqrt{\frac{2}{3n} \{\varphi_1^2 + \varphi_2^2\} \left[ \left\{ \frac{1-a^{2i}}{2(\varphi_1-\varphi_2) - (\varphi_1-\varphi_2)^2} \right\} - \left\{ \frac{1-a^{2i-2}}{2(\varphi_1-\varphi_2) - (\varphi_1-\varphi_2)^2} \right\} \right]} \right]$$

and

$$W_2 = \left[ 1 + K \sqrt{\frac{2}{3n} \{\varphi_1^2 + \varphi_2^2\} \left[ \left\{ \frac{1-a^{2i}}{2(\varphi_1-\varphi_2) - (\varphi_1-\varphi_2)^2} \right\} - \left\{ \frac{1-a^{2i-2}}{2(\varphi_1-\varphi_2) - (\varphi_1-\varphi_2)^2} \right\} \right]} \right].$$

In the process of monitoring the scale parameter of the IM distribution, we may need to encounter two situations:

- i) when  $\sigma^2$  is known,
- ii) when  $\sigma^2$  is unknown.

In the case of a known  $\sigma^2$ , the control limits can be written as follows

$$LCL = W_1\sigma_0^2, \quad CL = \sigma_0^2, \quad UCL = W_2\sigma_0^2.$$

But in the second case when  $\sigma^2$  is unknown, we estimate  $V_{IM}$  and use it as follows

$$LCL = W_1\bar{V}_{IM}, \quad CL = \bar{V}_{IM}, \quad UCL = W_2\bar{V}_{IM}. \quad (12)$$

$K$  control chart coefficients (see, Tables 1–6) are used to calculate  $W_1$  and  $W_2$ . The factor  $K$  which is mentioned in Eq (11) was obtained by conducting Monte Carlo simulation with  $\sigma_0 = 1$  in such a manner that the expected in-control average run length ( $ARL_0$ ) can be obtained.  $K$  determines the width of the control limits and its value can be selected according to  $ARL_0$ .

**Table 1.** Values of  $K$  coefficient for  $ARL_0 = 370$ .

Sample size	$\varphi_1 = 0.1,$ $\varphi_2 = 0.03$	$\varphi_1 = 0.3,$ $\varphi_2 = 0.05$	$\varphi_1 = 0.5,$ $\varphi_2 = 0.07$	$\varphi_1 = 0.1,$ $\varphi_2 = 0.0005$	$\varphi_1 = 0.25,$ $\varphi_2 = 0.1$
2	4.990	4.011	4.125	2.799	5.093
3	5.530	4.070	4.069	2.769	5.532
5	6.420	4.224	4.065	2.757	6.281
6	6.801	4.306	4.087	2.753	6.599
9	7.799	4.540	4.173	2.749	7.459



**Table 2.** Factors for IMEEWMA control chart with  $\varphi_1=0.1$ ,  $\varphi_2=0.03$  and  $ARL_0=370$ .

Time (i)	Sample size ( $n$ )									
	2		3		5		6		9	
	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$
1	0.69922	1.30078	0.72784	1.27216	0.75525	1.24475	0.76332	1.23668	0.77839	1.22161
2	0.72027	1.27973	0.74689	1.25311	0.77239	1.22761	0.77989	1.22011	0.79390	1.20610
3	0.73985	1.26015	0.76460	1.23540	0.78832	1.21168	0.79529	1.20471	0.80833	1.19167
4	0.75806	1.24194	0.78108	1.21892	0.80314	1.19686	0.80962	1.19038	0.82175	1.17825
5	0.77500	1.22500	0.79641	1.20359	0.81692	1.18308	0.82295	1.17705	0.83423	1.16577
6	0.79075	1.20925	0.81066	1.18934	0.82973	1.17027	0.83534	1.16466	0.84583	1.15417
7	0.80540	1.19460	0.82391	1.17609	0.84165	1.15835	0.84687	1.15313	0.85662	1.14338
8	0.81902	1.18098	0.83624	1.16376	0.85274	1.14726	0.85759	1.14241	0.86666	1.13334
9	0.83169	1.16831	0.84770	1.15230	0.86304	1.13696	0.86756	1.13244	0.87599	1.12401
10	0.84347	1.15653	0.85836	1.14164	0.87263	1.12737	0.87683	1.12317	0.88467	1.11533
11	0.85443	1.14557	0.86828	1.13172	0.88155	1.11845	0.88545	1.11455	0.89275	1.10725
12	0.86462	1.13538	0.87750	1.12250	0.88984	1.11016	0.89347	1.10653	0.90025	1.09975
13	0.87409	1.12591	0.88607	1.11393	0.89755	1.10245	0.90093	1.09907	0.90724	1.09276
14	0.88291	1.11709	0.89405	1.10595	0.90472	1.09528	0.90786	1.09214	0.91373	1.08627
15	0.89110	1.10890	0.90146	1.09854	0.91139	1.08861	0.91431	1.08569	0.91977	1.08023
16	0.89873	1.10127	0.90836	1.09164	0.91759	1.08241	0.92031	1.07969	0.92538	1.07462
17	0.90582	1.09418	0.91478	1.08522	0.92336	1.07664	0.92589	1.07411	0.93061	1.06939
18	0.91241	1.08759	0.92074	1.07926	0.92873	1.07127	0.93108	1.06892	0.93546	1.06454
19	0.91854	1.08146	0.92629	1.07371	0.93372	1.06628	0.93590	1.06410	0.93998	1.06002
20	0.92424	1.07576	0.93145	1.06855	0.93836	1.06164	0.94039	1.05961	0.94418	1.05582

**Table 3.** Factors for IMEEWMA control chart with  $\phi_1=0.3$ ,  $\phi_2=0.05$  and  $ARL_0 = 370$ .

Time (i)	Sample size ( $n$ )									
	2		3		5		6		9	
	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$
1	0.29569	1.70431	0.41648	1.58352	0.53090	1.46910	0.56346	1.43654	0.62420	1.37580
2	0.47177	1.52823	0.56236	1.43764	0.64818	1.35182	0.67260	1.32740	0.71815	1.28185
3	0.60383	1.39617	0.67177	1.32823	0.73613	1.26387	0.75445	1.24555	0.78861	1.21139
4	0.70287	1.29713	0.75383	1.24617	0.80210	1.19790	0.81583	1.18417	0.84146	1.15854
5	0.77715	1.22285	0.81537	1.18463	0.85157	1.14843	0.86188	1.13812	0.88109	1.11891
6	0.83286	1.16714	0.86153	1.13847	0.88868	1.11132	0.89641	1.10359	0.91082	1.08918
7	0.87465	1.12535	0.89615	1.10385	0.91651	1.08349	0.92231	1.07769	0.93312	1.06688
8	0.90599	1.09401	0.92211	1.07789	0.93738	1.06262	0.94173	1.05827	0.94984	1.05016
9	0.92949	1.07051	0.94158	1.05842	0.95304	1.04696	0.95630	1.04370	0.96238	1.03762
10	0.94712	1.05288	0.95619	1.04381	0.96478	1.03522	0.96722	1.03278	0.97178	1.02822
11	0.96034	1.03966	0.96714	1.03286	0.97358	1.02642	0.97542	1.02458	0.97884	1.02116
12	0.97025	1.02975	0.97535	1.02465	0.98019	1.01981	0.98156	1.01844	0.98413	1.01587
13	0.97769	1.02231	0.98152	1.01848	0.98514	1.01486	0.98617	1.01383	0.98810	1.01190
14	0.98327	1.01673	0.98614	1.01386	0.98886	1.01114	0.98963	1.01037	0.99107	1.00893
15	0.98745	1.01255	0.98960	1.01040	0.99164	1.00836	0.99222	1.00778	0.99330	1.00670
16	0.99059	1.00941	0.99220	1.00780	0.99373	1.00627	0.99417	1.00583	0.99498	1.00502
17	0.99294	1.00706	0.99415	1.00585	0.99530	1.00470	0.99562	1.00438	0.99623	1.00377
18	0.99471	1.00529	0.99561	1.00439	0.99647	1.00353	0.99672	1.00328	0.99718	1.00282
19	0.99603	1.00397	0.99671	1.00329	0.99736	1.00264	0.99754	1.00246	0.99788	1.00212
20	0.99702	1.00298	0.99753	1.00247	0.99802	1.00198	0.99815	1.00185	0.99841	1.00159

**Table 4.** Factors for IMEEWMA control chart with  $\varphi_1=0.5$ ,  $\varphi_2=0.07$  and  $ARL_0=370$ .

Time (i)	Sample size ( $n$ )									
	2		3		5		6		9	
	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$
1	00000	2.20240	00000	2.20240	0.25060	1.74940	0.31219	1.68781	0.42659	1.57341
2	0.31463	1.68537	0.31463	1.68537	0.57284	1.42716	0.60795	1.39205	0.67316	1.32684
3	0.60934	1.39066	0.60934	1.39066	0.75652	1.24348	0.77653	1.22347	0.81370	1.18630
4	0.77732	1.22268	0.77732	1.22268	0.86122	1.13878	0.87262	1.12738	0.89381	1.10619
5	0.87307	1.12693	0.87307	1.12693	0.92089	1.07911	0.92739	1.07261	0.93947	1.06053
6	0.92765	1.07235	0.92765	1.07235	0.95491	1.04509	0.95862	1.04138	0.96550	1.03450
7	0.95876	1.04124	0.95876	1.04124	0.97430	1.02570	0.97641	1.02359	0.98033	1.01967
8	0.97649	1.02351	0.97649	1.02351	0.98535	1.01465	0.98655	1.01345	0.98879	1.01121
9	0.98660	1.01340	0.98660	1.01340	0.99165	1.00835	0.99234	1.00766	0.99361	1.00639
10	0.99236	1.00764	0.99236	1.00764	0.99524	1.00476	0.99563	1.00437	0.99636	1.00364
11	0.99565	1.00435	0.99565	1.00435	0.99729	1.00271	0.99751	1.00249	0.99792	1.00208
12	0.99752	1.00248	0.99752	1.00248	0.99845	1.00155	0.99858	1.00142	0.99882	1.00118
13	0.99859	1.00141	0.99859	1.00141	0.99912	1.00088	0.99919	1.00081	0.99933	1.00067
14	0.99919	1.00081	0.99919	1.00081	0.99950	1.00050	0.99954	1.00046	0.99962	1.00038
15	0.99954	1.00046	0.99954	1.00046	0.99971	1.00029	0.99974	1.00026	0.99978	1.00022
16	0.99974	1.00026	0.99974	1.00026	0.99984	1.00016	0.99985	1.00015	0.99988	1.00012
17	0.99985	1.00015	0.99985	1.00015	0.99991	1.00009	0.99991	1.00009	0.99993	1.00007
18	0.99991	1.00009	0.99991	1.00009	0.99995	1.00005	0.99995	1.00005	0.99996	1.00004
19	0.99995	1.00005	0.99995	1.00005	0.99997	1.00003	0.99997	1.00003	0.99998	1.00002
20	0.99997	1.00003	0.99997	1.00003	0.99998	1.00002	0.99998	1.00002	0.99999	1.00001

**Table 5.** Factors for IMEEWMA control chart with  $\phi_1=0.1$ ,  $\phi_2=0.0005$  and  $ARL_0=370$ .

Time (i)	Sample size ( $n$ )									
	2		3		5		6		9	
	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$
1	0.83840	1.16160	0.86947	1.13053	0.89933	1.10067	0.90820	1.09180	0.92518	1.07482
2	0.85448	1.14552	0.88245	1.11755	0.90934	1.09066	0.91733	1.08267	0.93263	1.06737
3	0.86896	1.13104	0.89415	1.10585	0.91836	1.08164	0.92556	1.07444	0.93933	1.06067
4	0.88200	1.11800	0.90468	1.09532	0.92649	1.07351	0.93297	1.06703	0.94537	1.05463
5	0.89374	1.10626	0.91417	1.08583	0.93380	1.06620	0.93964	1.06036	0.95080	1.04920
6	0.90431	1.09569	0.92271	1.07729	0.94039	1.05961	0.94564	1.05436	0.95570	1.04430
7	0.91383	1.08617	0.93040	1.06960	0.94632	1.05368	0.95105	1.04895	0.96011	1.03989
8	0.92241	1.07759	0.93732	1.06268	0.95166	1.04834	0.95592	1.04408	0.96407	1.03593
9	0.93013	1.06987	0.94356	1.05644	0.95647	1.04353	0.96031	1.03969	0.96765	1.03235
10	0.93708	1.06292	0.94918	1.05082	0.96080	1.03920	0.96426	1.03574	0.97087	1.02913
11	0.94334	1.05666	0.95423	1.04577	0.96470	1.03530	0.96781	1.03219	0.97377	1.02623
12	0.94898	1.05102	0.95879	1.04121	0.96821	1.03179	0.97102	1.02898	0.97638	1.02362
13	0.95405	1.04595	0.96289	1.03711	0.97138	1.02862	0.97390	1.02610	0.97873	1.02127
14	0.95863	1.04137	0.96658	1.03342	0.97422	1.02578	0.97650	1.02350	0.98084	1.01916
15	0.96274	1.03726	0.96991	1.03009	0.97679	1.02321	0.97883	1.02117	0.98275	1.01725
16	0.96645	1.03355	0.97290	1.02710	0.97910	1.02090	0.98094	1.01906	0.98447	1.01553
17	0.96979	1.03021	0.97560	1.02440	0.98118	1.01882	0.98284	1.01716	0.98601	1.01399
18	0.97279	1.02721	0.97802	1.02198	0.98305	1.01695	0.98454	1.01546	0.98740	1.01260
19	0.97550	1.02450	0.98021	1.01979	0.98474	1.01526	0.98608	1.01392	0.98866	1.01134
20	0.97794	1.02206	0.98218	1.01782	0.98626	1.01374	0.98747	1.01253	0.98979	1.01021

**Table 6.** Factors for IMEEWMA control chart with  $\phi_1=0.1$ ,  $\phi_2=0.25$  and  $ARL_0=370$ .

Time (i)	Sample size ( $n$ )									
	2		3		5		6		9	
	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$	$W_1$	$W_2$
1	0.20826	1.79174	0.29783	1.70217	0.38246	1.61754	0.40772	1.59228	0.45338	1.54662
2	0.32702	1.67298	0.40315	1.59685	0.47509	1.52491	0.49656	1.50344	0.53538	1.46462
3	0.42797	1.57203	0.49268	1.50732	0.55383	1.44617	0.57208	1.42792	0.60507	1.39493
4	0.51377	1.48623	0.56878	1.43122	0.62075	1.37925	0.63627	1.36373	0.66431	1.33569
5	0.58671	1.41329	0.63346	1.36654	0.67764	1.32236	0.69083	1.30917	0.71466	1.28534
6	0.64870	1.35130	0.68844	1.31156	0.72599	1.27401	0.73720	1.26280	0.75746	1.24254
7	0.70140	1.29860	0.73518	1.26482	0.76709	1.23291	0.77662	1.22338	0.79384	1.20616
8	0.74619	1.25381	0.77490	1.22510	0.80203	1.19797	0.81013	1.18987	0.82477	1.17523
9	0.78426	1.21574	0.80866	1.19134	0.83173	1.16827	0.83861	1.16139	0.85105	1.14895
10	0.81662	1.18338	0.83736	1.16264	0.85697	1.14303	0.86282	1.13718	0.87339	1.12661
11	0.84413	1.15587	0.86176	1.13824	0.87842	1.12158	0.88340	1.11660	0.89239	1.10761
12	0.86751	1.13249	0.88250	1.11750	0.89666	1.10334	0.90089	1.09911	0.90853	1.09147
13	0.88738	1.11262	0.90012	1.09988	0.91216	1.08784	0.91575	1.08425	0.92225	1.07775
14	0.90427	1.09573	0.91510	1.08490	0.92534	1.07466	0.92839	1.07161	0.93391	1.06609
15	0.91863	1.08137	0.92784	1.07216	0.93654	1.06346	0.93913	1.06087	0.94382	1.05618
16	0.93084	1.06916	0.93866	1.06134	0.94606	1.05394	0.94826	1.05174	0.95225	1.04775
17	0.94121	1.05879	0.94786	1.05214	0.95415	1.04585	0.95602	1.04398	0.95941	1.04059
18	0.95003	1.04997	0.95568	1.04432	0.96102	1.03898	0.96262	1.03738	0.96550	1.03450
19	0.95753	1.04247	0.96233	1.03767	0.96687	1.03313	0.96823	1.03177	0.97068	1.02932
20	0.96390	1.03610	0.96798	1.03202	0.97184	1.02816	0.97299	1.02701	0.97507	1.02493

$W_1$  and  $W_2$  are used directly to simplify the calculations involving the  $K$  control chart coefficient. Utilizing these factors will significantly contribute to reduce the complexity involved in the construction of the IMEEWMA control chart for monitoring a scale parameter.

Regarding monitoring the process scale parameter for the case of the IM distribution, the hypotheses are given as follows:

$H_0 = \sigma^2 = \sigma_0^2$ ; or,  $\delta = 1$ ; (no shift in the process)

$H_1 = \sigma^2 = \sigma_1^2 = \delta\sigma_0^2$ ; or,  $\delta \neq 1$ ; (shift occurs in the process)

here  $\delta$  indicates the process shift.

#### 4. Performance evaluation of the IMEEWMA control chart

We can evaluate the performance of control charts by using various measures such as the average run length (ARL), the standard deviation of run length (SDRL) and the median run length (MRL). These measurements are famous standards for comparing the performances of different control charts. The RL is the number of sample points on the control chart before signaling the first out of control (OOC) point. The ARL is the expected number, and SDRL is the standard deviation of the RL. The  $ARL_0$  for an in-control process (IC) should be larger than the  $ARL_1$  for an OOC signal when a shift in the process parameter is observed. The ARL decreases when the shift increases so an alarm for an OOC situation can be raised quickly [37]. There are several ways to conduct a performance evaluation for control charts such as the integral equation method, Markov chain method and Monte Carlo method [38,39]. Here we use the ARL along with the SDRL to evaluate the performance of the proposed IMEEWMA control chart and other relevant control charts. The Monte Carlo simulation was performed by using R language to compute the ARLs and SDRLs. In order to compute control chart coefficients for the proposed control chart and other control charts,  $ARL_0$  was fixed at 370 (see Tables 7–11).

**Table 7.** ARL and SDRL results for IMEEWMA chart with  $\phi_1 = 0.1$ ,  $\phi_2 = 0.03$  and  $ARL_0 = 370$ .

Sample size and $K$ coefficient										
$\delta$	$n = 2$ & $K=4.990$		$n = 3$ & $K=5.530$		$n = 5$ & $K=6.420$		$n = 6$ & $K=6.801$		$n = 9$ & $K=7.799$	
	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
1	372.94	349.87	368.13	342.32	368.70	332.83	368.62	332.51	368.86	332.37
1.25	34.21	22.29	28.16	15.86	22.72	10.49	21.20	9.08	18.47	6.67
1.5	14.57	8.69	12.29	6.48	10.16	4.48	9.51	3.93	8.42	3.00
1.75	8.79	5.40	7.40	4.03	6.13	2.84	5.76	2.50	5.10	1.96
2	6.14	3.81	5.14	2.90	4.24	2.05	4.00	1.85	3.55	1.44
2.25	4.68	2.96	3.93	2.24	3.24	1.61	3.03	1.44	2.69	1.13
2.5	3.76	2.40	3.15	1.81	2.60	1.31	2.45	1.17	2.16	0.93
2.75	3.16	2.01	2.65	1.53	2.19	1.10	2.06	0.98	1.82	0.79
3	2.74	1.73	2.32	1.30	1.91	0.94	1.80	0.85	1.59	0.67
5	1.53	0.80	1.33	1	1.16	0.39	1.11	0.33	1.04	0.20

**Table 8.** ARL and SDRL results for IMEEWMA chart with  $\varphi_1 = 0.3$ ,  $\varphi_2 = 0.05$  and  $ARL_0 = 370$ .

Sample size and $K$ coefficient										
$\delta$	$n = 2$ & $K=4.011$		$n = 3$ & $K=4.070$		$n = 5$ & $K=4.224$		$n = 6$ & $K=4.306$		$n = 9$ & $K=4.540$	
	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>
1	369.97	369.34	367.74	364.25	368.58	364.85	368.41	363.79	371.05	365.71
1.25	38.41	35.63	28.15	25.20	18.71	15.31	16.22	12.75	11.74	8.29
1.5	13.10	11.02	9.48	7.35	6.40	4.36	5.65	3.66	4.34	2.49
1.75	7.17	5.59	5.29	3.76	3.71	2.32	3.31	1.97	2.62	1.39
2	4.89	3.63	3.69	2.47	2.64	1.56	2.37	1.32	1.91	0.95
2.25	3.73	2.66	2.83	1.83	2.08	1.17	1.87	0.99	1.54	0.71
2.5	3.05	2.09	2.34	1.45	1.74	0.91	1.59	0.78	1.33	0.55
2.75	2.60	1.72	2.02	1.19	1.53	0.75	1.41	0.64	1.21	0.44
3	2.28	1.47	1.79	1.01	1.40	0.64	1.29	0.53	1.13	0.35
5	1.39	0.68	1.19	0	1.05	0.22	1.03	0.16	1.00	0.06

**Table 9.** ARL and SDRL results for IMEEWMA chart with  $\varphi_1 = 0.5$ ,  $\varphi_2 = 0.07$ , and  $ARL_0 = 370$ .

Sample size & $K$ Coefficient										
$\delta$	$n=2$ & $k=4.125$		$n=3$ & $k=4.069$		$n=5$ & $k=4.065$		$n=6$ & $k=4.087$		$n=9$ & $k=4.173$	
	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>	<i>ARL</i>	<i>SDRL</i>
1	367.43	364.80	367.58	365.35	368.35	365.24	367.33	364.78	367.64	365.22
1.25	49.28	47.91	36.39	34.77	23.45	21.60	20.04	18.06	13.86	11.80
1.5	16.28	14.91	11.34	9.92	7.03	5.49	6.00	4.54	4.33	2.90
1.75	8.37	7.15	5.87	4.66	3.83	2.64	3.31	2.15	2.48	1.43
2	5.44	4.38	3.88	2.83	2.62	1.65	2.33	1.38	1.79	0.92
2.25	4.03	3.08	2.94	2.00	2.04	1.18	1.84	1.00	1.46	0.68
2.5	3.22	2.33	2.39	1.53	1.71	0.92	1.56	0.77	1.27	0.51
2.75	2.72	1.90	2.05	1.25	1.51	0.75	1.38	0.62	1.17	0.40
3	2.36	1.58	1.81	1.04	1.37	0.62	1.27	0.51	1.10	0.31
5	1.40	0.70	1.19	0	1.04	0.21	1.02	0.15	1.00	0.05

**Table 10.** ARL and SDRL results for IMEEWMA chart taking  $\varphi_1 = 0.1$ ,  $\varphi_2 = 0.0005$ , and  $ARL_0 = 370$ .

Sample size & $K$ Coefficient										
$\delta$	$n = 2$ & $K=2.799$		$n = 3$ & $K=2.769$		$n = 5$ & $K=2.757$		$n = 6$ & $K=2.753$		$n = 9$ & $K=2.749$	
	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
1	370.23	378.65	368.68	376.47	368.72	374.11	368.30	376.02	369.83	375.15
1.25	26.88	25.38	19.79	17.80	13.39	11.22	11.63	9.50	8.50	6.53
1.5	9.67	8.34	6.99	5.71	4.76	3.61	4.17	3.05	3.10	2.10
1.75	5.45	4.46	4.04	3.13	2.78	1.94	2.48	1.64	1.91	1.12
2	3.83	3.03	2.84	2.07	2.03	1.28	1.81	1.07	1.46	0.73
2.25	2.96	2.25	2.24	1.52	1.65	0.93	1.50	0.78	1.24	0.50
2.5	2.44	1.76	1.89	1.19	1.43	0.71	1.32	0.60	1.13	0.37
2.75	2.11	1.44	1.67	0.98	1.29	0.57	1.21	0.47	1.07	0.27
3	1.90	1.24	1.50	0.81	1.13	0.37	1.14	0.39	1.04	0.21
5	1.26	0.55	1.11	0.34	1.01	0.09	1.01	0.09	1.00	0.02

**Table 11.** ARL and SDRL results for IMEEWMA chart with  $\varphi_1 = 0.25$ ,  $\varphi_2 = 0.1$  and  $ARL_0 = 370$ .

Sample size & $K$ Coefficient										
$\delta$	$n=2$ & $K=5.093$		$n=3$ & $K=5.532$		$n=5$ & $K=6.281$		$n=6$ & $K=6.599$		$n=9$ & $K=7.459$	
	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
1	371.15	362.81	369.69	358.55	367.60	350.12	367.63	352.58	368.90	353.59
1.25	35.59	28.22	27.28	19.45	20.15	12.06	18.15	10.08	14.85	6.92
1.5	13.92	9.36	10.98	6.43	8.53	4.14	7.86	3.56	6.70	2.58
1.75	8.28	5.23	6.65	3.72	5.27	2.50	4.89	2.19	4.22	1.65
2	5.81	3.63	4.74	2.64	3.78	1.82	3.50	1.60	3.04	1.23
2.25	4.49	2.80	3.64	2.05	2.91	1.43	2.72	1.26	2.38	0.98
2.5	3.63	2.26	2.98	1.66	2.40	1.17	2.24	1.04	1.95	0.81
2.75	3.11	1.90	2.53	1.41	2.04	0.98	1.92	0.89	1.68	0.69
3	2.72	1.65	2.22	1.21	1.80	0.85	1.69	0.77	1.48	0.60
5	1.55	0.80	1.32	1	1.14	0.36	1.09	0.30	1.03	0.18

## 5. Simulation results and discussion

In a simulation study, we generate a sample of random data in such a way that it replicates a real problem. The steps are taken as follows

- Generate data from IM distribution.
- Fix the sample size  $n$  for each random sample.
- Generate a random sample of size  $n$  from  $T \sim \text{gamma}(3/2, 2\sigma_0^2)$ .
- Take the square root of  $T$  to get a sample with a Maxwell distribution.
- Obtain a sample from the IM random variable  $R$  of size  $n$  by setting  $R=1/x$ .
- Compute IEEWMA computing statistic, i.e.,  $u_i$
- Repeat the first five steps until the expected number of subgroups is obtained.
- Develop the control limits as proposed in the previous section.



- Plot all values of the  $u_i$  statistic with respect to the control limits.

We consider  $\sigma_0=1$  and generated 120 sample observations by adopting the K-S test, we then found that the null hypothesis cannot be rejected for the generated data.

Here, for the graphical representation of the IMEEWMA control chart in the case of the simulated data, we consider the design parameters as  $n = 6$ ,  $\lambda = 0.25$ ,  $\phi_1 = 0.1$ ,  $\phi_2 = 0.0005$  and  $ARL_0 = 370$ . To check the ability of the chart to detect the process variations when it is OOC, we have considered a shift in the process scale parameter. We increased the scale parameter by 25% (i.e.,  $\delta = 1.25$ ) and 30% (i.e.,  $\delta = 1.3$ ) after the 10th sample and constructed the proposed IMEEWMA with the existing VIM charts (see Figures 1 and 2). It is obvious that for a 25% increment, there was no OOC signal for the VIM chart and the IMEWMA chart yielded the OOC signal after 13 data points whereas the IMEEWMA chart yielded the OOC signal after 4 data points. Similarly, for a 30% increment ( $\delta = 1.3$ ) we noticed that all points were within the control limits for the VIM chart and the IMEWMA chart yielded the OOC signal after 11 points whereas the control limits for the IMEEWMA chart were exceeded after 4 data points (see Figure 2). Hence the proposed IMEEWMA chart shows better detection ability than the V and IMEWMA charts for both small and moderate shifts.

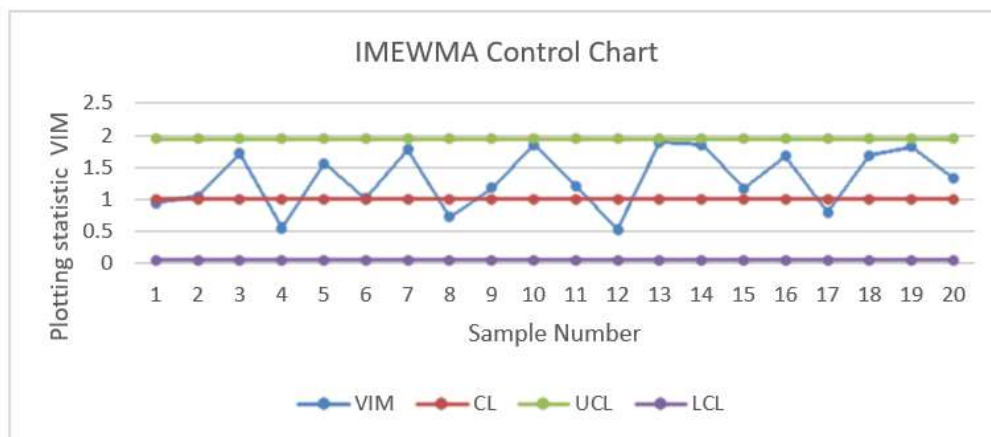


Figure 1(a). VIM control chart with  $\delta = 1.25$ .

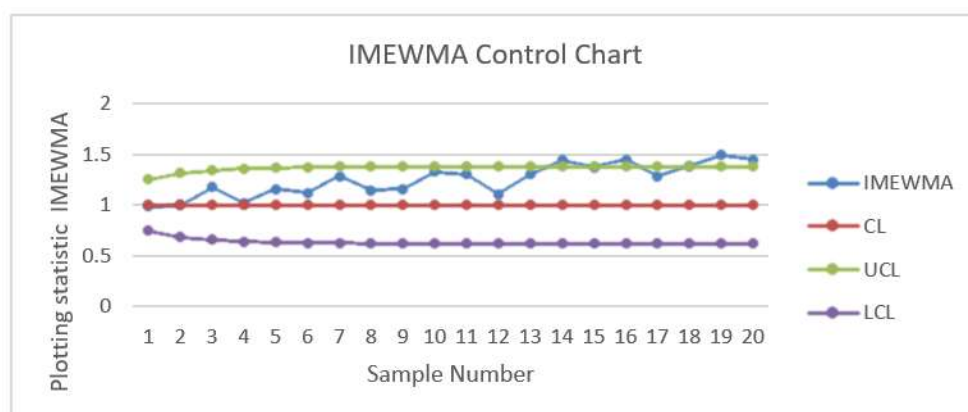
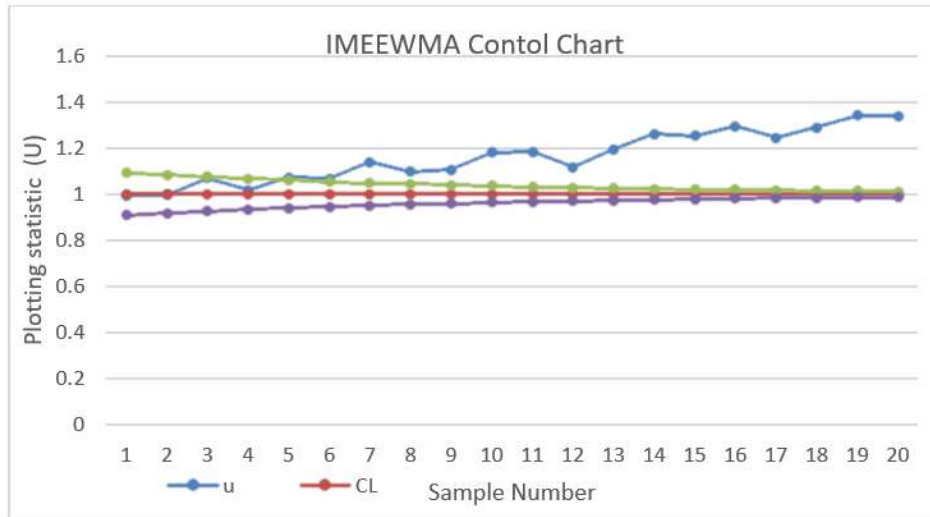
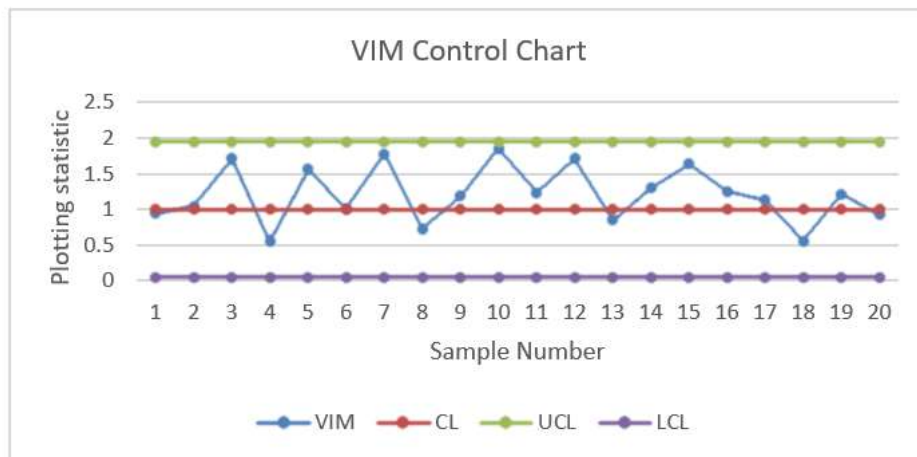


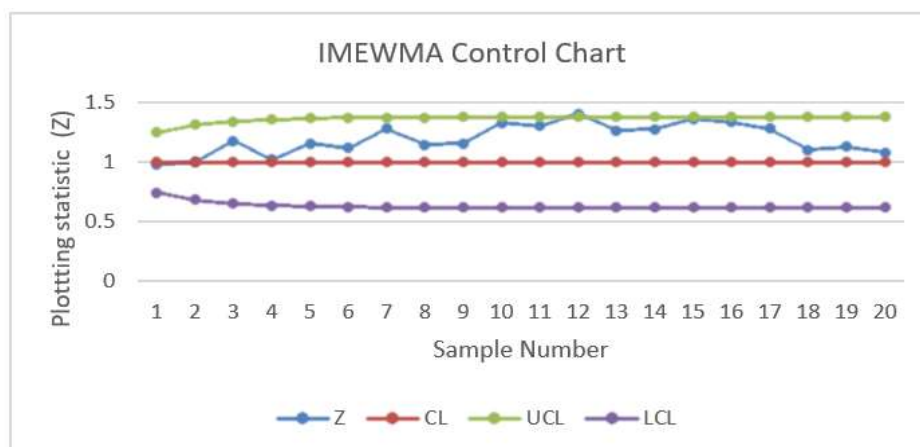
Figure 1(b). IMEWMA control chart with  $\delta = 1.25$ .



**Figure 1(c).** IMEEWMA control chart with  $\delta = 1.25$ ,  $\varphi_1 = 0.1$ , &  $\varphi_2 = 0.0005$ .



**Figure 2(a).** VIM control chart with  $\delta = 1.3$ .



**Figure 2(b).** IMEEWMA control chart with  $\delta = 1.3$ ,  $\varphi_1 = 0.1$ , &  $\varphi_2 = 0.0005$ .

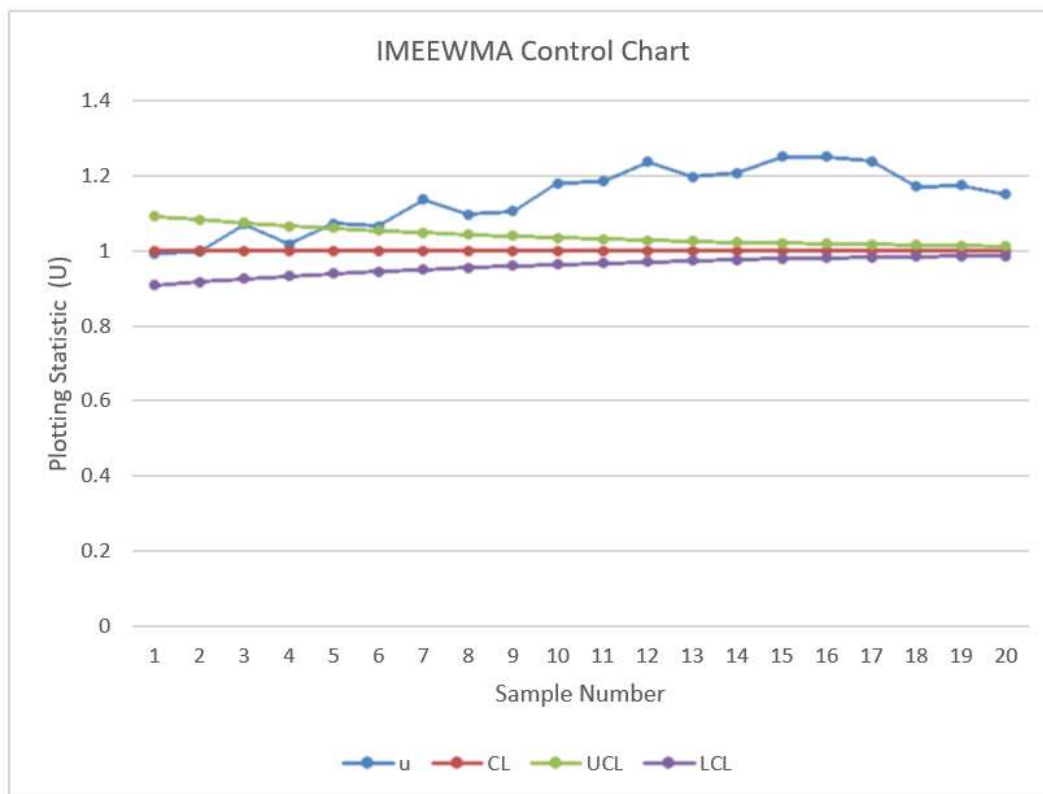


Figure 2(c). IMEEWMA control chart with  $\delta = 1.3$ ,  $\varphi_1 = 0.1$ , &  $\varphi_2 = 0.0005$ .

## 6. Real-life application-based results and discussion

To demonstrate the performance of our proposed IMEEWMA chart, a real data set was collected. The brake pad is a crucial component of a vehicle's braking system, and its lifetime refers to the distance covered by the vehicle while the brake pads are installed. In this particular dataset, we have information on the brake pad lifetimes of 98 vehicles. Therefore, we were able to utilize this dataset to construct our proposed chart, which, after truncation, included 84 samples, with each sample having a subgroup size of seven.

Based on the design of the IMEEWMA control chart, the estimated value of the MLE was  $\bar{V}_{IM} = 9.08 \times 10^{-10}$  for the car's data sets. To create time-varying control limits for the three control charts mentioned in the text, we computed the  $R$ ,  $P$  and  $W$  coefficients for  $n = 7$  the results are provided in Tables 12–15. Figures 3–5 present a graphical representation of the IMEEWMA chart using these values.

We are aware that there are multiple criteria for detecting a lack of control. For instance, in Figure 5, if a point falls outside of the control limits or point 8 exceeds the control limit, it indicates a lack of control.

Observing Figures 6–8 respectively, we note that the V, EWMA and IMEWMA charts for brake pad lifetime data use  $\delta = 1.25$  after the seventh sample. Despite a 25% increase in the shift, the VIM chart and IMEWMA chart still indicate that the process is in control. However, the IMEEWMA chart reveals that the process goes out of control since, among the seven points, four points fall outside of the control limits. Therefore, based on our car brake pad lifetime data, we can conclude that our proposed IMEEWMA control chart can detect small shifts more effectively than both the VIM chart and the IMEWMA chart.

**Table 12.** W coefficients for IMEEWMA, VIM and IMEWMA control charts based on cars' brake pad lifetimes with  $\delta = 1.25$  and  $n=7$ .

$I$	IMEEWMA		VIM		IMEWMA	
	$W_1$	$W_2$	$R_1$	$R_2$	$P_1$	$P_2$
1	0.914978	1.085022	0	2.1367	0.767696	1.232304
2	0.923438	1.076562	0	2.1367	0.70962	1.29038
3	0.931055	1.068945	0	2.1367	0.681574	1.318426
4	0.937915	1.062085	0	2.1367	0.666834	1.333166
6	0.944093	1.055907	0	2.1367	0.658822	1.341178
7	0.959176	1.040824	0	2.1367	0.650554	1.349446
8	0.963238	1.036762	0	2.1367	0.649781	1.350219
9	0.966896	1.033104	0	2.1367	0.649347	1.350653
10	0.970189	1.029811	0	2.1367	0.649103	1.350897
11	0.973156	1.026844	0	2.1367	0.648966	1.351034
12	0.975827	1.024173	0	2.1367	0.648889	1.351111
13	0.978232	1.021768	0	2.1367	0.648846	1.351154

**Table 13.** ARL results for IMEEWMA, IMEWMA and VIM charts with  $ARL_0 = 370$  and  $n= 6$ .

$\delta \rightarrow$ Shift		1	1.05	1.1	1.15	1.2	1.25	1.35	1.5	1.75	2
V		370.59	288.19	200.54	130.61	88.72	61.01	31.86	14.49	5.93	3.39
IMEWMA	$\lambda = 0.75$	368.97	194.87	110.04	67.69	43.70	29.79	15.99	8.01	3.84	2.49
	$\lambda = 0.50$	368.97	172.76	90.06	51.65	32.85	21.85	11.90	6.33	3.36	2.34
	$\lambda = 0.25$	371.20	143.43	66.41	36.41	22.68	15.69	9.18	5.50	3.35	2.46
IMEEWMA	$\phi_1 = 0.1, \phi_2 = 0.03$	367.19	114.11	57.90	37.11	27.15	21.24	14.50	9.53	5.76	4.00
	$\phi_1 = 0.3, \phi_2 = 0.05$	367.80	139.26	65.10	36.68	19.18	16.05	9.53	5.64	3.32	2.37
	$\phi_1 = 0.5, \phi_2 = 0.07$	370.38	163.59	82.81	47.10	29.68	19.86	11.07	6.02	3.31	2.32
	$\phi_1 = 0.1, \phi_2 = 0.0005$	369.08	118.40	48.91	26.07	16.60	11.65	6.93	4.13	2.48	1.82

**Table 14.** SDRL results for IMEEWMA, IMEWMA and VIM charts with  $ARL_0 = 370$  and  $n = 6$ .

$\delta$		1	1.05	1.1	1.15	1.2	1.25	1.35	1.5	1.75	2
<b>V</b>		369.30	282.50	203.72	131.35	88.68	61.04	31.68	13.93	5.41	2.84
<b>IMEWMA</b>	$\lambda = 0.75$	366.39	194.13	110.24	66.63	42.75	29.11	15.11	7.21	3.04	1.72
	$\lambda = 0.50$	367.53	170.38	88.23	50.16	31.36	20.17	10.46	4.99	2.25	1.39
	$\lambda = 0.25$	369.58	140.45	63.31	33.06	19.65	12.79	6.64	3.44	1.80	1.20
<b>IMEEWA</b>	$\phi_1 = 0.1, \phi_2 = 0.03$	362.72	85.76	35.22	19.07	12.37	9.13	5.95	3.93	2.52	1.84
	$\phi_1 = 0.3, \phi_2 = 0.05$	364.90	134.78	60.10	32.33	19.18	12.65	6.81	3.61	1.98	1.33
	$\phi_1 = 0.5, \phi_2 = 0.07$	367.10	161.18	80.84	44.70	27.73	17.90	9.34	4.53	2.17	1.37
	$\phi_1 = 0.1, \phi_2 = 0.0005$	372.70	118.29	46.70	23.32	14.11	9.50	5.38	3.02	1.64	1.08

**Table 15.** MDRL for IMEEWMA, IMEWMA and VIM charts with  $ARL_0 = 370$  and  $n = 6$ .

$\delta$		1	1.05	1.1	1.15	1.2	1.25	1.35	1.5	1.75	2
<b>V</b>		256	204	139	91	63	42	22	10	4	3
<b>IMEWA</b>	$\lambda = 0.75$	257	135	76	48	30	21	11	6	3	2
	$\lambda = 0.50$	257	121	63	36	23	16	9	5	3	2
	$\lambda = 0.25$	258	100	47	26	17	12	7	5	3	2
<b>IMEEWMA</b>	$\phi_1 = 0.1, \phi_2 = 0.03$	265	89	49	33	25	20	14	9	6	4
	$\phi_1 = 0.3, \phi_2 = 0.05$	256	98	47	27	17	13	8	5	3	2
	$\phi_1 = 0.5, \phi_2 = 0.07$	258	114	58	33	21	15	8	5	3	2
	$\phi_1 = 0.1, \phi_2 = 0.0005$	256	82	35	20	13	9	6	3	2	1

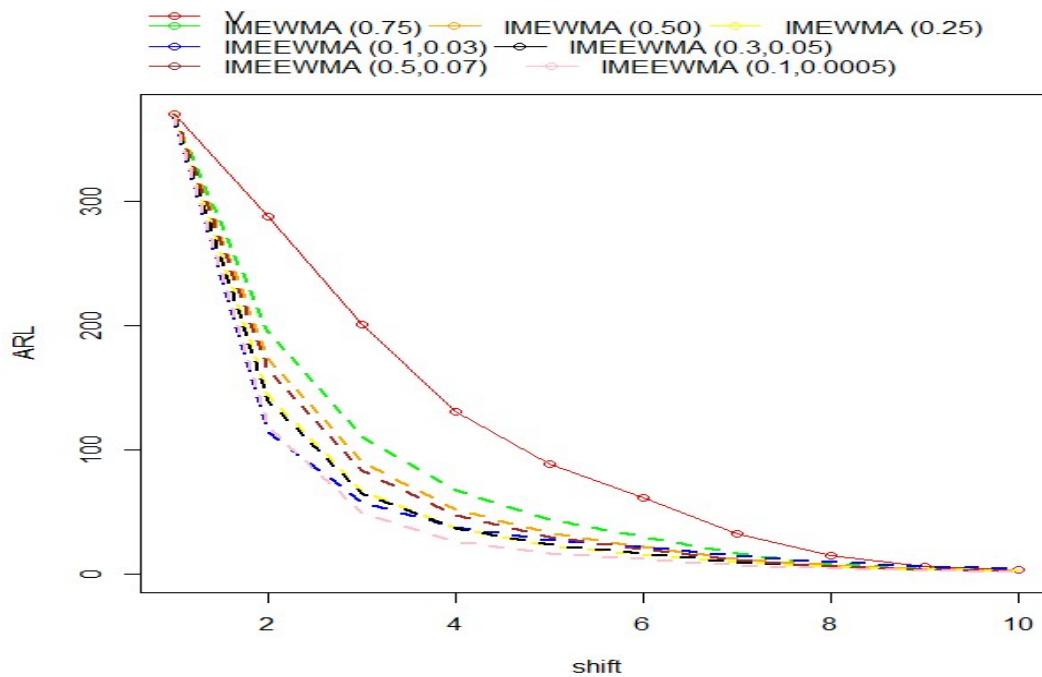


Figure 3. ARL curves for IMEEWMA, IMEWMA and VIM chart for  $n=6$ .

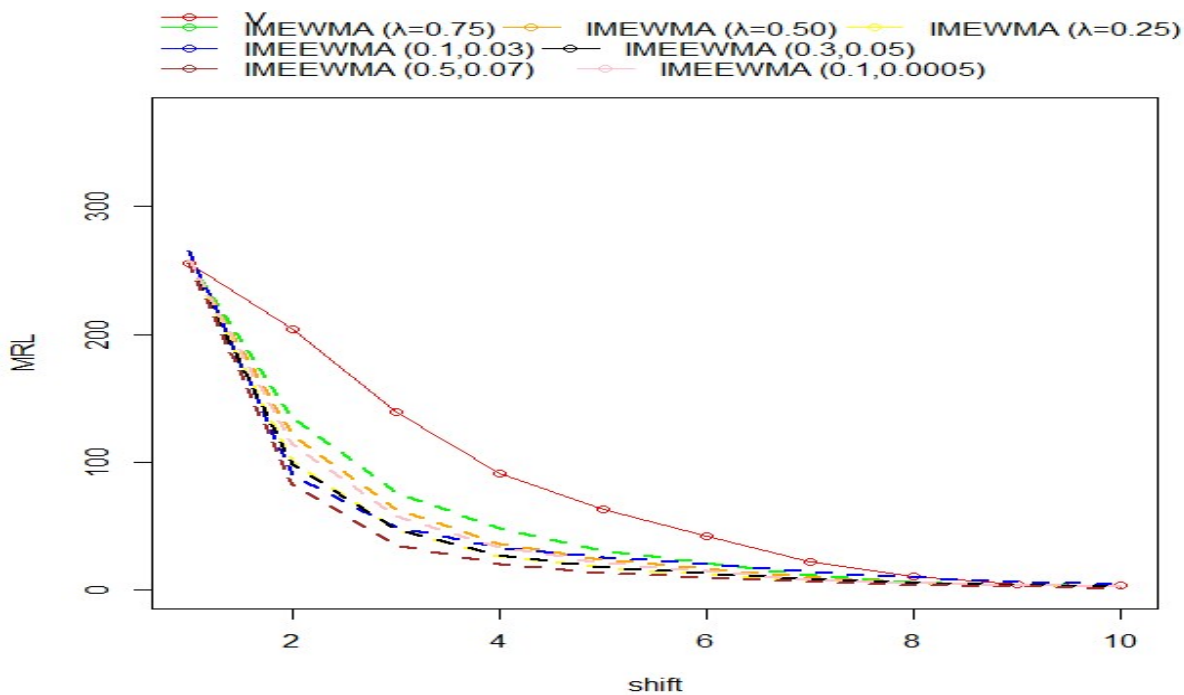
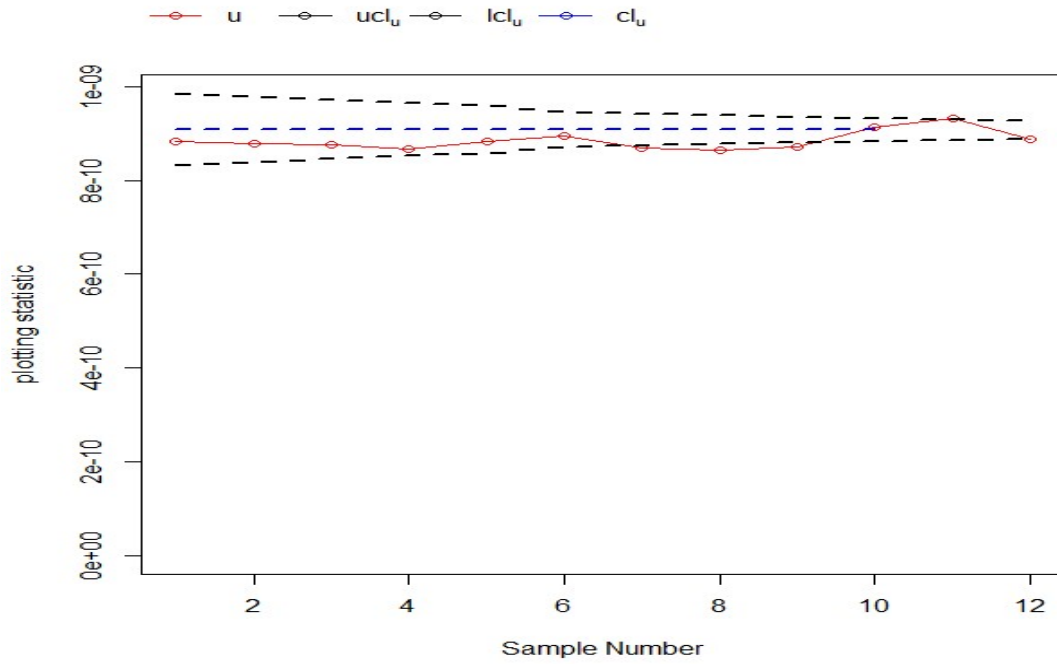
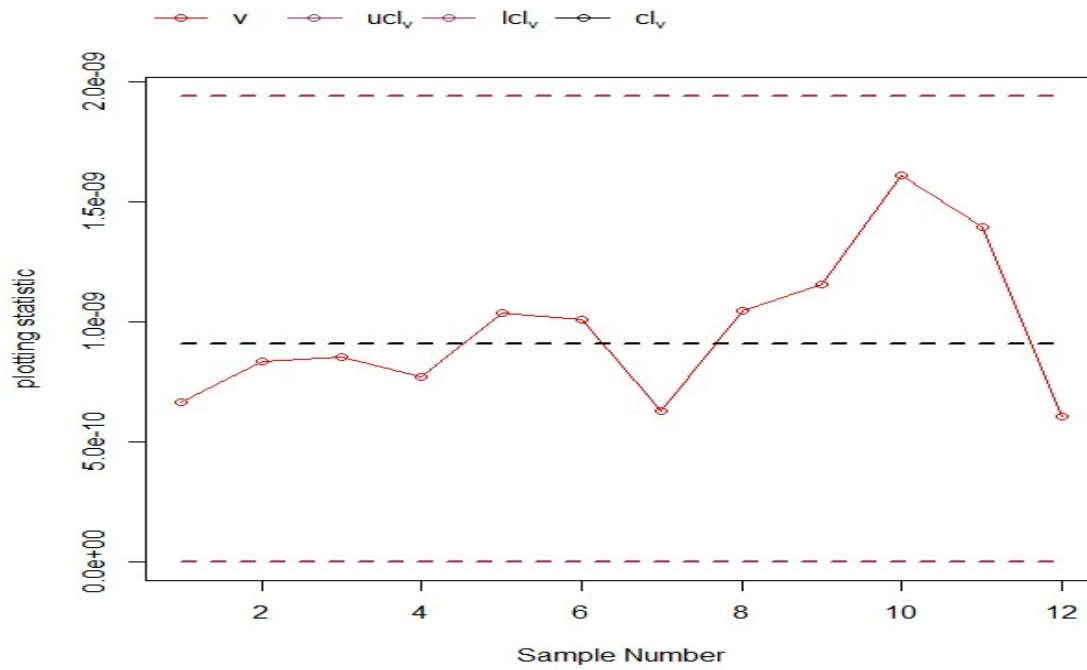


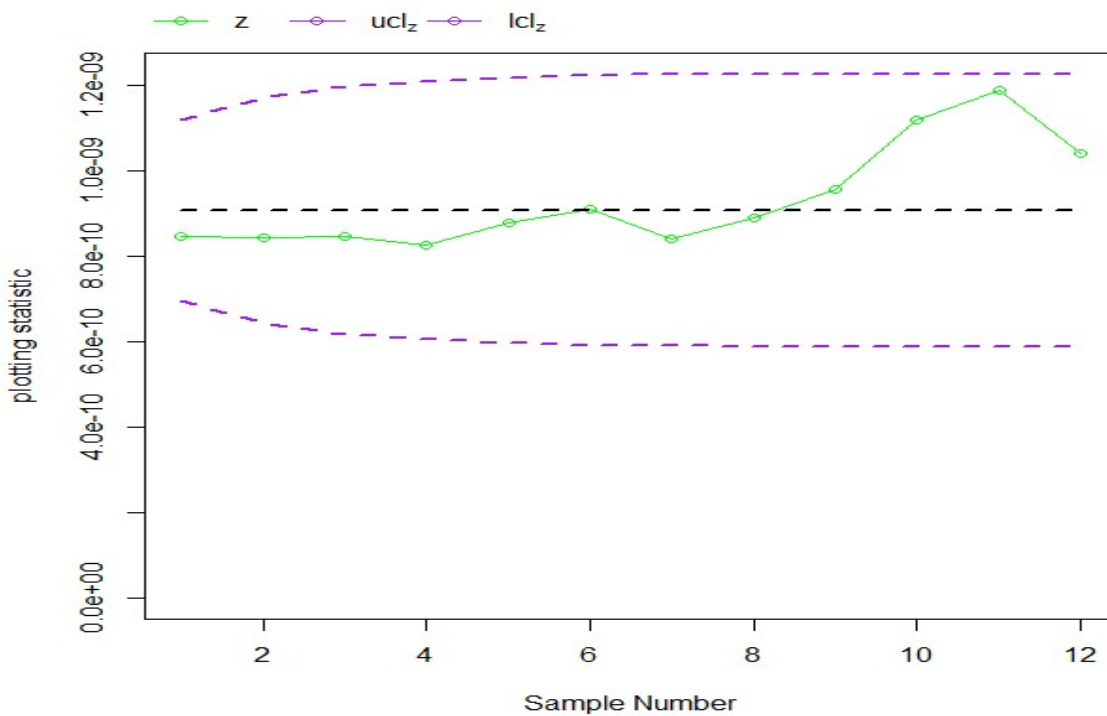
Figure 4. MRL curves for IMEEWMA, IMEWMA and VIM chart for  $n=6$ .



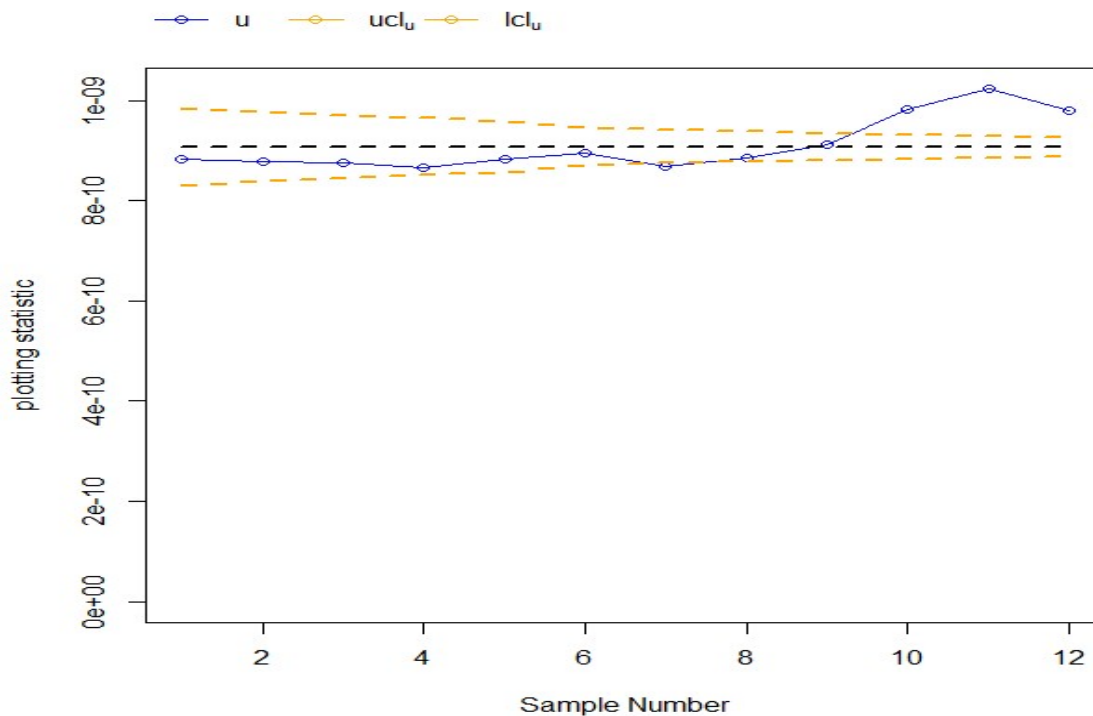
**Figure 5.** IMEEWMA chart results for cars' brake pad lifetimes at  $\delta = 1$ .



**Figure 6.** IM chart results for cars' brake pad lifetimes at  $\delta = 1.25$ .



**Figure 7.** IMEWMA charts for cars' brake pad lifetimes at  $\delta = 1.25$ .



**Figure 8.** IMEEWMA charts for cars' brake pad lifetimes at  $\delta = 1.25$ .



## 7. Conclusions

In this study, we designed a control chart, named the IMEEWMA chart, to monitor a process following the IM distribution. The proposed chart is based on the MLE of the scale parameter of the IM distribution. The performance of the IMEEWMA chart was investigated by using RL characteristics like the ARL, SDRL and MRL. Based on these measures, it has been observed that the best performance of the IMEEWMA chart is obtained when the smoothing parameter value is small, and it keeps improving with the increase in sample size. The simulation study suggests that our proposed chart is more effective than the existing VIM chart and IMEWMA chart for small shifts, whereas the VIM chart and IMEWMA chart demonstrated better performance for larger shifts and small to moderate shifts respectively. We have also considered real-life scenarios and applied our designed IMEEWMA chart to monitor the process of car brake pad lifetimes.

### Use of AI tools declaration

The authors declare that they have used Chat GPT as an artificial intelligence tool only to improve the readability of this article.

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### Conflict of interest

The authors declare no conflict of interest.

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