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Research article

Controllability results of neutral Caputo fractional functional differential equations

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Abstract: In this paper, using the properties of the phase space on infinite delay, generalized Gronwall inequality and fixed point theorems, the existence and controllability results of neutral fractional functional differential equations with multi-term Caputo fractional derivatives were obtained under Lipschitz and non-Lipschitz conditions.

Keywords: neutral Caputo fractional functional differential equations; existence and controllability; generalized Gronwall inequality; fixed point theorems; Lipschitz and non-Lipschitz conditions
Mathematics Subject Classification: 34H05

1. Introduction

As the most important qualitative aspect of a control system, the controllability means that one can govern the state of a system to the desired final state by using a suitable control function, which has a significant role in the dynamical system. In the last 10 years, the controllability of fractional differential equations or inclusions has been studied widely, which involves Cauchy and nonlocal conditions, abstract spaces and general spaces, impulsive and non-impulsive systems and finite delay and infinite delay. The main topics are the approximate controllability [1–14], controllability [15–37], exact controllability [38, 39], total controllability [40], numerical controllability [41, 42], relative controllability [43, 44] and optimal controllability [45], et al. The fixed point method combined with the other nonlinear analysis theories are the main methods.

In [46], the authors considered the existence and attractivity dependence of solutions for a class of multi-term Caputo fractional functional differential equations with finite delay

$$\begin{cases} {}^{C}D^{\alpha}u(t) = \sum\limits_{i=1}^{m} {}^{C}D^{\alpha_i} f_i(t, u_t) + f_0(t, u_t), \ t > t_0, \alpha_i \in (0, \alpha); \\ u(t) = \varphi(t), \ t_0 - \sigma \le t \le t_0, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}, {}^{C}D^{\alpha_i}$ denote the Caputo's fractional derivative of order α, α_i with $0 \leq \alpha_i \leq \alpha$. In [47], the finite-time stability of neutral Caputo fractional functional differential system is considered

$$\begin{cases} {}^{c}D_0^{\lambda}x(t) - C{}^{c}D_0^{\mu}x(t-\tau(t)) = Ax(t) + Bx(t-\tau(t)) + Dw(t) + f(t, x(t), x(t-\tau(t)), w(t)), t \in [0, T], \\ x(t) = \phi(t), t \in [-\tau, 0], \end{cases} \tag{1.2}$$

where ${}^{C}D_0^{\lambda}, {}^{C}D_0^{\mu}$ denote the Caputo's fractional derivative of order λ, μ with $0 \leq \mu \leq \lambda < 1$. More details of existence and stability of neutral fractional functional differential equations are noted in [48–55]. For basic knowledge of fractional differential equations, see [56–58].

The controllability problem of equations similar to Eqs (1.1) and (1.2) with infinite delay is rarely studied. Motivated by the above discussion, in this paper we investigate the following classes of multi-term neutral Caputo fractional functional differential equations with infinite delay

$$\begin{cases} {}^{C}D^{\alpha}x(t) - \sum_{i=1}^{m} {}^{C}D^{\beta_i}g_i(t, x(t)) = f(t, x_t), \ t \in J = [0, T]; \\ x(t) = \phi(t), \ t \in I = (-\infty, 0], \\ {}^{C}D^{\beta_i}x(0) = \mu_i \in R^n, \end{cases} \tag{1.3}$$

and its controllability form

$$\begin{cases} {}^{C}D^{\alpha}x(t) - \sum_{i=1}^{m} {}^{C}D^{\beta_i}g_i(t, x(t)) = f(t, x_t) + Bu(t), \ t \in J = [0, T]; \\ x(t) = \phi(t), \ t \in I = (-\infty, 0], \\ {}^{C}D^{\beta_i}x(0) = \mu_i \in R^n, \end{cases} \tag{1.4}$$

where ${}^{C}D^{\alpha}, {}^{C}D^{\beta_i}$ denote Caputo fractional derivative of α, β_i order, $0 \leq \beta_i \leq \alpha < 1$; $x \in R^n$, $f \in C(J \times R^n, R^n), g_i \in C(J \times R^n, R^n)$, $\phi \in \mathcal{D}$ [59] (will be defined later). For any function x defined on $(-\infty, T]$ and any $t \in J$, $x_t = x(t+\theta), \theta \in I$ represents the history of the state from time $-\infty$ up to the present time t. The control function $u(t) \in L^{\infty}(J, R^m)$ or $u(t) \in L^2(J, R^m)$, B is a bounded linear operator.

The main contributions and difficulties of this paper are listed bellow:

(1) The Gronwall inequality is powerful tool to consider existence, stability and other qualitative and quantitative properties of solutions to differential systems. However we need to pay attention to the use of fractional order integral operators on the monotonicity of nonnegative functions [47, 60].

(2) To overcome the difficulty in investigating the priori bound of neutral fractional functional differential equations with infinite delay, the property of the phase space $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ on infinite delay in [59] and a kind of Gronwall fractional integral inequality in [61] are used together.

In this article, by using the generalized Gronwall inequality and the fixed point approach (which includes the contraction mapping principle and the Schaefer fixed point theorem), we establish some sufficient conditions for the existence results of Eq (1.3) and controllability results of Eq (1.4) under Lipschitz and non-Lipschitz conditions, respectively.

2. Preliminaries

Let $L^p(J, R^n)$ be the Banach space of all measurable functions from J into R^n, which are Lebesgue integrable with the norm $\|x\|_{L^p} = \int_0^T \|x(t)\|^p dt, 1 \leq p < +\infty$. Let $C(J, R^n)$ be the Banach space of all continuous functions from J into R^n with the norm $\|u\| := \sup\{\|u(t)\| : t \in J\}$. We define the space

$$C^{\beta}((-\infty, T], R^n) = \{u : (-\infty, T] \to R^n : u|_I \in \mathcal{D}; u(t)|_J, {}^C D^{\beta_i} u(t)|_J \in C(J, R^n)\}$$

as the space of all functions from $(-\infty, T]$ into R^n with the semi-norm $\|u\| := \|\phi\|_{\mathcal{D}} + \|u\|$.

Definition 2.1. *[56–58] For a function h given on the interval* $[a, b]$, *the Caputo fractional order derivative of order* α *of h is defined by*

$$({}^c D^{\alpha}_{a^+} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of* α.

Definition 2.2. *[56–58] The Riemann-Liouville integral of order* γ *for* f *defined on* $[a, b]$ *is denoted by*

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, t \in [a, b], \gamma > 0,$$

provided that the right side is point-wise defined on $[a, b]$, *where* $\Gamma(\cdot)$ *is the gamma function.*

Definition 2.3. *[56–58] The Riemann-Liouville fractional derivative of order* γ *for* f *defined on* $[a, b]$ *is denoted by*

$$D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\gamma-1} f(s) ds, t \in [a, b], n-1 < \gamma < n,$$

provided that the right side is point-wise defined on $[a, b]$, *where* $\Gamma(\cdot)$ *is the gamma function. In particular, if* $\gamma \in [0, 1)$, *then*

$$D^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t (t-s)^{-\gamma} f(s) ds, t \in [a, b].$$

Definition 2.4. *[56–58] The Caputo fractional order derivative of order* γ *for a function* $f : [0, +\infty) \to R$ *can be written as*

$${}^C D^{\gamma} f(t) = D^{\gamma} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)(0)} \right), t > 0, n-1 < \gamma < n.$$

Lemma 2.1. *[56–58] Let* $\alpha > 0$ *and* $m = [\alpha] + 1$ *then the general solution to the fractional differential equation* ${}^C D^{\alpha} u(t) = 0$ *is given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots c_{m-1} t^{m-1},$$

where $c_i \in R, i = 0, 1, \cdots, m-1$ *are some constants. Further assuming that* $u \in C^m([0, T], R)$, *we can get*

$$I^{\alpha C} D^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots c_{m-1} t^{m-1},$$

for $c_i \in R, i = 0, 1, \cdots, m-1$.

Lemma 2.2. *[60] For any nonnegative function $w \in C([a,b],[0,+\infty))$ and any $t \in [a,b]$, we have the following inequality*

$$\sup_{0\le\tau\le t}\int_0^\tau (\tau-s)^{\alpha-1}w(s)ds \le \int_0^t (t-s)^{\alpha-1}\sup_{0\le\sigma\le s} w(\sigma)ds, \alpha>0.$$

Lemma 2.3. *[61] Let u be a nonnegative continuous function defined on the interval $I=[a,b]$, and $p(t): I\to(0,\infty)$ be a nondecreasing continuous function. Suppose that $q(t): I\to[0,\infty)$ is a nondecreasing continuous function. If u satisfies the following inequality:*

$$u(t)\le p(t)+q(t)\sum_{i=1}^{n}(I_{a^+}^{\alpha_i}u)(t),\ \alpha_i>0,\ t\in I,$$

then for every $k\in N$ such that $(k+1)\min\{\alpha_1,\alpha_2,\cdots,\alpha_n\}>1$,

$$u(t)\le P_k(t)\exp\left(\int_a^t H_{k+1}(t,s)ds\right),\ t\in I,$$

where

$$P_k(t):=p(t)\left(1+\sum_{j=1}^{k}q^j(t)\sum_{\substack{i_1+\cdot+i_n=j,\\ 0\le i_1,\cdots,i_n\le j}}\begin{pmatrix} j\\ i_1,\cdots,i_n\end{pmatrix}\frac{(t-a)^{i_1\alpha_1+\cdots+i_n\alpha_n}}{\Gamma(1+i_1\alpha_1+\cdots+i_n\alpha_n)}\right),$$

$$H_{k+1}(t,s):=q^{k+1}(t)\sum_{\substack{j_1+\cdot+j_n=k+1,\\ 0\le j_1,\cdots,j_n\le k+1}}\begin{pmatrix} k+1\\ j_1,\cdots,j_n\end{pmatrix}\frac{(t-s)^{j_1\alpha_1+\cdots+j_n\alpha_n}}{\Gamma(1+j_1\alpha_1+\cdots+j_n\alpha_n)},$$

with $\begin{pmatrix} k+1\\ j_1,\cdots,j_n\end{pmatrix}$ is combination number formula.

Similar result to Lemma 2.2, one can see:

Lemma 2.4. *[47] Assume that $x(t)\in C^1([0,+\infty),[0,+\infty))$ and $x'(t)\ge 0$ and $\lambda>0$ then $\frac{\int_0^t (t-s)^{\lambda-1}x(s)ds}{\Gamma(\lambda)}$ is monotonically increasing with respect to t.*

We introduce the axiomatic definition of the phase space $(\mathcal{D},\|\cdot\|_{\mathcal{D}})$ in [59].

Let X be a linear topological space of functions from $(-\infty,0]$ to X with the semi-norm $\|\cdot\|_{\mathcal{D}}$, and called an admissible phase space $\mathcal{D}$ if the following fundamental axioms conditions hold.

(A_1) If $x:(-\infty,T]\to X$ is continuous on $[0,T]$ and $x_0\in\mathcal{D}$, then for any $t\in[0,T]$, the following conditions hold:

(a) $x_t\in\mathcal{D}$;

(b) $\|x(t)\|\le H\|x_t\|_{\mathcal{D}}$ for some positive constant H;

(c) there are functions $K(t), M(t):[0,\infty)\to[0,\infty)$ such that

$$\|x_t\|_{\mathcal{D}}\le K(t)\sup\{\|x(s)\|: s\in[0,t]\}+M(t)\|x_0\|_{\mathcal{D}},$$

where K is continuous, M is locally bounded and H,K,M are independent of x with $K_T=\sup\{K(s): s\in[0,T]\}$, $M_T=\sup\{M(s): s\in[0,T]\}$.

(A_2) For the function $x(\cdot)$ in (A_1), the function $x_t\in\mathcal{D}$ and is continuous on the interval $[0,T]$.

(A_3) The space $\mathcal{D}$ is a Banach space.

3. Main results

By Lemma 2.1 and some basic theory of fractional calculus, we can get the following two lemmas. Lemmas 3.1 and 3.2 are proved in a similar way, and only the detailed proof of Lemma 3.2 is given.

Lemma 3.1. *The function $x \in C^{\beta}((-\infty, T], R^n)$ is a solution of Eq* (1.3) *if, and only if, $x(t)$ satisfies the following integral equation*

$$x(t) = \begin{cases} \phi(0) - \sum\limits_{i=1}^{m} \dfrac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \sum\limits_{i=1}^{m} \dfrac{\int_0^t (t-s)^{\alpha-\beta_i-1} g_i(s,x(s))ds}{\Gamma(\alpha-\beta_i)} + \dfrac{\int_0^t (t-s)^{\alpha-1} f(s,x_s)]ds}{\Gamma(\alpha)}, t \in J; \\ \phi(t),\ t \in I. \end{cases}$$

Lemma 3.2. *The function $x \in C^{\beta}((-\infty, T], R^n)$ is a solution of Eq* (1.4) *if, and only if, $x(t)$ satisfies the following integral equation*

$$x(t) = \begin{cases} \phi(0) - \sum\limits_{i=1}^{m} \dfrac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \sum\limits_{i=1}^{m} \dfrac{\int_0^t (t-s)^{\alpha-\beta_i-1} g_i(s,x(s))ds}{\Gamma(\alpha-\beta_i)} \\ + \dfrac{\int_0^t (t-s)^{\alpha-1}[f(s,x_s)+Bu(s)]ds}{\Gamma(\alpha)}, t \in J; \\ \phi(t),\ t \in I. \end{cases}$$

Proof. Necessity: For $t \in J$, on both sides of (1.4) by using the fractional integral operator I_0^{α}, then we get

$$I_0^{\alpha}\Big[{}^{C}D_0^{\alpha}x(t) - \sum_{i=1}^{m} {}^{C}D_0^{\beta_i} g_i(t,x(t))\Big] = I_0^{\alpha}[f(t,x_t) + Bu(t)].$$

By Lemma 2.1, we have

$$x(t) - \phi(0) - \Big[\sum_{i=1}^{m} I_0^{\alpha-\beta_i} g_i(t,x(t)) - \sum_{i=1}^{m} \frac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\Big] = I_0^{\alpha}[f(t,x_t)+Bu(t)],\ t \in J,$$

and so

$$x(t) = \phi(0) - \sum_{i=1}^{m} \frac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \sum_{i=1}^{m} I_0^{\alpha-\beta_i} g_i(t,x(t)) + I_0^{\alpha}[f(s,x_t)+Bu(t)],\ t \in J.$$

Sufficiency. If $x(t)$ is a solution of Eq (1.4) on both sides of

$$x(t) = \phi(0) - \sum_{i=1}^{m} \frac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \sum_{i=1}^{m} I_0^{\alpha-\beta_i} g_i(t,x(t)) + I_0^{\alpha}[f(s,x_t)+Bu(t)],\ t \in J,$$

by using the Caputo fractional derivative operator ${}^{C}D_0^{\alpha}$, Definitions 2.1 and 2.4 and the fact that $I_0^{\alpha-\beta_i} g_i(t,x(t))|_{t=0} = 0$, we get

$${}^{C}D_0^{\alpha}x(t) = -\sum_{i=1}^{m} \frac{g_i(0,\phi(0))^{C}D_0^{\alpha}t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \sum_{i=1}^{m} {}^{C}D_0^{\alpha} I_0^{\alpha-\beta_i} g_i(t,x(t)) + {}^{C}D_0^{\alpha} I_0^{\alpha}[f(t,x_t)+Bu(t)],\ t \in J,$$

where

$$\begin{aligned}\sum_{i=1}^{m}\frac{g_i(0,\phi(0))^C D_0^{\alpha}t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} &= \sum_{i=1}^{m}\frac{g_i(0,\phi(0))D_0^{\alpha}t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} - \sum_{i=1}^{m}\frac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}|_{t=0}\frac{t^{-\beta_i}}{\Gamma(1-\beta_i)}\\ &= \sum_{i=1}^{m}\frac{g_i(0,\phi(0))D_0^{\alpha}t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\\ &= \sum_{i=1}^{m}\frac{g_i(0,\phi(0))t^{-\beta_i}}{\Gamma(1-\beta_i)},\ t\in J;\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m}{}^C D_0^{\alpha}I_0^{\alpha-\beta_i}g_i(t,x(t)) &= \sum_{i=1}^{m}D_0^{\alpha}I_0^{\alpha-\beta_i}g_i(t,x(t)) - \sum_{i=1}^{m}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}I_0^{\alpha-\beta_i}g_i(t,x(t))|_{t=0}\\ &= \sum_{i=1}^{m}D_0^{\alpha}I_0^{\alpha-\beta_i}g_i(t,x(t))\\ &= \sum_{i=1}^{m}I_0^{\beta_i}g_i(t,x(t)),\ t\in J;\end{aligned}$$

$$\begin{aligned}{}^C D_0^{\alpha}I_0^{\alpha}[f(t,x_t)+Bu(t)] &= D_0^{\alpha}I_0^{\alpha}[f(t,x_t)+Bu(t)] - I_0^{\alpha}[f(t,x_t)+Bu(t)]|_{t=0}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\\ &= D_0^{\alpha}I_0^{\alpha}[f(t,x_t)+Bu(t)] = f(t,x_t)+Bu(t),\ t\in J.\end{aligned}$$

Then, we get

$$\begin{aligned}\sum_{i=1}^{m}{}^C D_0^{\alpha}I_0^{\alpha-\beta_i}g_i(t,x(t)) - \sum_{i=1}^{m}{}^C D^{\alpha}\frac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} &= \sum_{i=1}^{m}I_0^{\beta_i}g_i(t,x(t)) - \sum_{i=1}^{m}\frac{g_i(0,\phi(0))t^{-\beta_i}}{\Gamma(1-\beta_i)}\\ &= \sum_{i=1}^{m}{}^C D_0^{\alpha}g_i(t,x(t)),\ t\in J.\end{aligned}$$

On both sides of the above equality, using the Caputo fractional operator ${}^C D^{\alpha}$, we get

$$ {}^C D_0^{\alpha}x(t) - \sum_{i=1}^{m}{}^C D_0^{\beta_i}g_i(t,x(t)) = f(t,x_t)+Bu(t),\ t\in J. $$

The proof is completed.

Definition 3.1. *Equation* (1.4) *is said to be controllable on the interval J if for every $\phi\in\mathcal{D}, x_T = x(T)\in R^n$, there is a control function $u(t)\in L^{\infty}(J,R^m)$ or $u(t)\in L^2(J,R^m)$ such that the mild solution $x(t)$ of Eq* (1.4) *satisfies $x(T)=x_T$ and $x_0=\phi\in\mathcal{D}$.*

We make the following assumptions throughout the paper.

(H_1) The functions $g_i\in C(J\times R^n,R^n), i=1,\cdots,m, f\in C(J\times\mathcal{D},R^n)$ and some nonnegative continuous functions $c_{i1}(t), i=1,\cdots,m, c_3(t), c_4(t)$ exist such that

$$\begin{aligned}\|g_i(t,u)-g_i(t,v)\| &\le c_{i1}(t)\|u-v\|,\\ \|f(t,u)-f(t,v)\| &\le c_3(t)\|u-v\|_{\mathcal{D}},\end{aligned}$$

with $g_i(0,0)=f(0,0)=0$. Furthermore, we have

$$\begin{aligned}\|g_i(t,u)\| &\le \|g_i(t,u)-g_i(t,0)\|+\|g_i(t,0)\| \le c_{i1}(t)\|u\|+c_{i2}(t), t\in J, u\in R^n,\\ \|f(t,u)\| &\le \|f(t,u)-f(t,0)\|+\|f(t,0)\| \le c_3(t)\|u\|_{\mathcal{D}}+c_4(t), t\in J, u\in\mathcal{D},\end{aligned}$$

and we denote

$$c_{i2}(t) = \sup\{\|g_i(t,0)\|, t \in J\}, c_4(t) = \sup\{\|f(t,0)\|, t \in J\},$$
$$\hat{c}_{ij} = \sup\{c_{ij}(t), j = 1,2, t \in J\}, \hat{c}_l = \sup\{c_l(t), l = 3,4, t \in J\}.$$

(H_1') The functions $g_i \in C(J \times R^n, R^n), f \in C(J \times \mathcal{D}, R^n)$, there exists some functions $c_{ij}(t) \in L^{\frac{1}{q_{ij}}}(J, R^+), q_{ij} \in [0, \alpha - \beta_i), j = 1,2; c_j(t) \in L^{\frac{1}{q_j}}(J, R^+), j = 3,4, q_3, q_4 \in [0, \alpha)$, such that

$$\|g_i(t,u) - g_i(t,v)\| \leq c_{i1}(t)\|u - v\|, u, v \in R^n,$$
$$\|f(t,u) - f(t,v)\| \leq c_3(t)\|u - v\|_{\mathcal{D}}, u, v \in \mathcal{D}.$$

(H_1'') The functions $g_i \in C(J \times R^n, R^n), f \in C(J \times \mathcal{D}, R^n)$, there exists nonnegative continuous functions $c_{i1}(t), c_{i2}(t), i = 1, \cdots, m, c_3(t), c_4(t)$ such that

$$\|g_i(t,u)\| \leq c_{i1}(t)\|u\| + c_{i2}(t), t \in J, u \in R^n,$$
$$\|f(t,u)\| \leq c_3(t)\|u\|_{\mathcal{D}} + c_4(t), t \in J, u \in \mathcal{D}.$$

(H_2) $(1 + \frac{M_1 M_2 T^\alpha}{\Gamma(\alpha+1)})\Big\{\sum_{i=1}^m \frac{\hat{c}_{i1} T^{\alpha-\beta_i}}{\Gamma(\alpha - \beta_i + 1)} + \frac{\hat{c}_3 K_T T^\alpha}{\Gamma(\alpha+1)}\Big\} < 1.$

(H_2') $(1 + \frac{M_1 M_2 T^\alpha}{\Gamma(\alpha+1)})\Big[\sum_{i=1}^m \frac{T^{\beta_i - q_{i1}}}{\Gamma(\alpha - \beta_i)}\Big(\frac{1 - q_{i1}}{\beta_i - q_{i1}}\Big)^{1-q_{i1}}\|c_{i1}\|_{L_J^{\frac{1}{q_{i1}}}} + \frac{T^{\alpha - q_3}}{\Gamma(\alpha+1)}\Big(\frac{1 - q_3}{\alpha - q_3}\Big)^{1-q_3}\|c_3\|_{L_J^{\frac{1}{q_3}}}\Big] < 1,\ t \in J.$

(H_3) $B : L^\infty(J, R^m) \to R^n$ is a bounded linear operator. The operator $\mathbf{W}_u : L^\infty(J, R^m) \to R^n$ is defined as

$$\mathbf{W}_u = \frac{1}{\Gamma(\alpha)}\int_0^T (T - s)^{\alpha-1} B u_u(s) ds,$$

with inverse operator $\mathbf{W}_u^{-1}$ (in $L^\infty(J, R^m)/ker\mathbf{W}_u$), and there exists two positive constants $M_i, i = 1,2$ such that $\|B\| \leq M_1, \|\mathbf{W}_u^{-1}\| \leq M_2$.

(H_3') $B : L^\infty(J, R^m) \to R^n$ is a linear operator. The operator $\mathbf{W}_u : L^2(J, R^m) \to R^n$ is defined as

$$\mathbf{W}_u = \frac{1}{\Gamma(\alpha)}\int_0^T (T - s)^{\alpha-1} B u_u(s) ds$$

with inverse operator $\mathbf{W}_u^{-1}$ (in $L^2(J, R^m)/ker\mathbf{W}_u$), and there are two positive constants $M_i, i = 1,2$ such that $\|B\| \leq M_1, \|\mathbf{W}_u^{-1}\| \leq M_2$.

Using the Banach contraction mapping principle, we can get the existence result of Eq (1.3) and the controllability of Eq (1.4). The method of proving the following theorems is standard. For the integrity of the paper, the details of the proof of the relevant conclusion are given below.

Theorem 3.1. *If the assumptions* (H_1)–(H_3) *hold, then Eq* (1.4) *is controllable on J.*

Proof. Define tha space $S(T) = \{x \in C^\beta((-\infty, T], R^n) : x_0 = \phi\}$. For any $x \in S(T)$, define the control function $u_x(\cdot)$ as

$$u_x(t) = \mathbf{W}_u^{-1}\Bigg\{x_T - \Bigg[\phi(0) - \sum_{i=1}^m \frac{g_i(0, \phi(0))}{\Gamma(\alpha - \beta_i + 1)} T^{\alpha-\beta_i} + \sum_{i=1}^m \frac{\int_0^T (T - s)^{\alpha-\beta_i-1} g_i(s, x(s)) ds}{\Gamma(\alpha - \beta_i)} + \frac{\int_0^T (T - s)^{\alpha-1} f(s, x_s)] ds}{\Gamma(\alpha)}\Bigg]\Bigg\}, \tag{3.1}$$

where x_T denotes the final state of Eq (1.4) at T.

According to Lemma 3.2, it follows that z is a solution of Eq (1.4) if, and only if,

$$z(t)=\begin{cases}\phi(0)-\sum\limits_{i=1}^{m}\dfrac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}+\sum\limits_{i=1}^{m}\dfrac{\int_0^t(t-s)^{\alpha-\beta_i-1}g_i(s,z(s))ds}{\Gamma(\alpha-\beta_i)}\\ +\dfrac{\int_0^t(t-s)^{\alpha-1}[f(s,z_s)+Bu(s)]ds}{\Gamma(\alpha)},\ t\in J,\\ \phi(t),\ t\in I.\end{cases}\tag{3.2}$$

For any $\phi:(-\infty,0]\to R^n\in\mathcal{D}$, let $\tilde{\phi}$ be the extension of ϕ. $\tilde{\phi}$ is denoted by

$$\tilde{\phi}(t)=\begin{cases}\phi(0)-\sum\limits_{i=1}^{m}\dfrac{g_i(0,\phi(0))}{\Gamma(\alpha-\beta_i+1)}t^{\alpha-\beta_i},\ t\in J,\\ \phi(t),\ t\in I.\end{cases}\tag{3.3}$$

For any $x\in C(J,R^n)$, let $\tilde{x}$ be the extension of x. $\tilde{x}$ is denote by

$$\tilde{x}(t)=\begin{cases}x(t)-\phi(0)+\sum\limits_{i=1}^{m}\dfrac{g_i(0,\phi(0))}{\Gamma(\alpha-\beta_i+1)}t^{\alpha-\beta_i},\ t\in J,\\ 0,\ t\in I.\end{cases}\tag{3.4}$$

It can be seen that if $z(t)=\tilde{\phi}(t)+\tilde{x}(t), z_t=\tilde{\phi}_t+\tilde{x}_t, t\in J$ satisfies the following integral equation

$$z(t)=\phi(0)-\sum_{i=1}^{m}\frac{g_i(0,\phi(0))t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}+\sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}g_i(s,z(s))ds}{\Gamma(\alpha-\beta_i)}+\frac{\int_0^t(t-s)^{\alpha-1}[f(s,z_s)+Bu(s)]ds}{\Gamma(\alpha)},\ t\in J,$$

then x satisfies the following integral equation

$$\begin{aligned}x(t)&=\sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}g_i(s,\tilde{x}(s)+\tilde{\phi}(s))ds}{\Gamma(\alpha-\beta_i)}+\frac{\int_0^t(t-s)^{\alpha-1}[f(s,\tilde{x}_s+\tilde{\phi}_s)+Bu(s)]ds}{\Gamma(\alpha)}\\&=\sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}g_i(s,x(s))ds}{\Gamma(\alpha-\beta_i)}+\frac{\int_0^t(t-s)^{\alpha-1}[f(s,\tilde{x}_s+\tilde{\phi}_s)+Bu(s)]ds}{\Gamma(\alpha)},\ t\in J.\end{aligned}$$

Consider the following Banach space $\Omega=\{x\in S(T);\|x\|\le r: x_0=0\}$ with the norm $\|x\|=\sup_{t\in J}\|x(t)\|$, where r is a sufficiently large positive number. The operator $N:\Omega\to\Omega$ is defined as

$$(Nx)(t)=\begin{cases}\sum\limits_{i=1}^{m}\dfrac{\int_0^t(t-s)^{\alpha-\beta_i-1}g_i(s,x(s))ds}{\Gamma(\alpha-\beta_i)}+\dfrac{\int_0^t(t-s)^{\alpha-1}[f(s,\tilde{x}_s+\tilde{\phi}_s)+Bu(s)]ds}{\Gamma(\alpha)},\ t\in J,\\ 0,\ t\in I.\end{cases}\tag{3.5}$$

So, the fixed point of N is equal to the controllable of Eq (1.4) on J.

(1) $N\Omega\subseteq\Omega$. For any $x\in\Omega$, by the assumption (H_1), we get $\|Nx\|=0\le r, t\in I$.

By the property $(A_1(c))$ on the phase space $(\mathcal{D},\|\cdot\|_{\mathcal{D}})$, we have

$$\begin{aligned}&\|\tilde{x}_t+\tilde{\phi}_t\|_{\mathcal{D}}\\ \le\ & K(t)\sup\{\|\tilde{x}(s)\|:s\in[0,t]\}+M(t)\|\tilde{x}(0)\|_{\mathcal{D}}+K(t)\sup\{\|\tilde{\phi}(s)\|:s\in[0,t]\}+M(t)\|\tilde{\phi}(0)\|_{\mathcal{D}}\\ \le\ & K_T\sup\{\|\tilde{x}(s)\|:s\in[0,t]\}+K_T\sup\{\|\phi(0)-\sum_{i=1}^{m}\frac{g_i(0,\phi(0))s^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\|:s\in[0,t]\}+M_T\|\phi\|_{\mathcal{D}}\\ \le\ & K_T\sup\{\|x(s)\|:s\in[0,t]\}+K_T\sum_{i=1}^{m}\frac{\|g_i(0,\phi(0))\|}{\Gamma(\alpha-\beta_i+1)}T^{\alpha-\beta_i}+(K_T+M_T)\|\phi\|_{\mathcal{D}}\\ \le\ & K_Tr+K_T\sum_{i=1}^{m}\frac{\|g_i(0,\phi(0))\|}{\Gamma(\alpha-\beta_i+1)}T^{\alpha-\beta_i}+(K_T+M_T)\|\phi\|_{\mathcal{D}}=\mathcal{A}, t\in J.\end{aligned}\tag{3.6}$$

By the assumption (H_1) and the control function u_x, for $t \in J$, we get

$$
\begin{aligned}
\|u_x\| &\le M_2\Big[\|x_T\| + \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{m} \frac{\|g_i(0,\phi(0))\|T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} \\
&\quad + \sum_{i=1}^{m} \frac{\int_0^T (T-s)^{\alpha-\beta_i-1}(c_{i1}(s)\|x(s)\| + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \frac{\int_0^T (T-s)^{\alpha-1}(c_3(s)\|\tilde{x}_s + \tilde{\phi}_s\|_{\mathcal{D}} + c_4(s))ds}{\Gamma(\alpha)}\Big] \\
&\le M_2\Big[\|x_T\| + \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{m} \frac{(c_{i1}(0)\|\phi(0)\|_{\mathcal{D}} + c_{i2}(0))T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} \\
&\quad + \sum_{i=1}^{m} \frac{\int_0^T (T-s)^{\alpha-\beta_i-1}(c_{i1}(s)r + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} + \frac{\int_0^T (T-s)^{\alpha-1}(c_3(s)\mathcal{A} + c_4(s))ds}{\Gamma(\alpha)}\Big] \\
&\le M_2\Big[\|x_T\| + \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{m} \frac{((c_{i1}(0)\|\phi(0)\|_{\mathcal{D}} + c_{i2}(0)) + \hat{c}_{i1}r + \hat{c}_{i2})T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \frac{(\hat{c}_3\mathcal{A} + \hat{c}_4)T^{\alpha}}{\Gamma(\alpha+1)}\Big] = \mathcal{B}.
\end{aligned} \tag{3.7}
$$

So,

$$
\begin{aligned}
\|(Nx)(t)\| &\le \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}(c_{i1}(s)\|x(s)\| + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \frac{\int_0^t (t-s)^{\alpha-1}[c_3(s)(\|\tilde{x}_s + \tilde{\phi}_s\|_{\mathcal{D}}) + c_4(s) + \|B\|\|u_x(s)\|]ds}{\Gamma(\alpha)} \\
&\le \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}c_{i1}(s)r ds}{\Gamma(\alpha-\beta_i)} + \sum_{i=1}^{m} \frac{\hat{c}_{i2}t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} \\
&\quad + \frac{\int_0^t (t-s)^{\alpha-1}c_3(s)\mathcal{A} ds}{\Gamma(\alpha)} + \frac{(\hat{c}_4 + M_1\mathcal{B})t^{\alpha}}{\Gamma(\alpha+1)} \\
&\le \sum_{i=1}^{m} \frac{(\hat{c}_{i1}r + \hat{c}_{i2})T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \frac{(\hat{c}_3\mathcal{A} + \hat{c}_4 + M_1\mathcal{B})T^{\alpha}}{\Gamma(\alpha+1)} = C, t \in J,
\end{aligned} \tag{3.8}
$$

then $\|Nx\| \le C \le r,\ t \in J$, i.e., $N\Omega \subseteq \Omega$.

(2) N is a contractive mapping. Choose any $x, y \in \Omega$ with $x_0 = y_0, t \in I$, then

$$
\begin{aligned}
\|(Nx)(t) - (Ny)(t)\| &\le \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}c_{i1}(s)\|x(s) - y(s)\|ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \frac{\int_0^t (t-s)^{\alpha-1}c_3(s)\|\tilde{x}_s - \tilde{y}_s\|_{\mathcal{D}}ds}{\Gamma(\alpha)} + \frac{\int_0^t (t-s)^{\alpha-1}\|Bu_x(s) - Bu_y(s)\|ds}{\Gamma(\alpha)} \\
&\le \Big(1 + \tfrac{M_1M_2T^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big\{\sum_{i=1}^{m} \frac{\hat{c}_{i1}T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \frac{\hat{c}_3K_TT^{\alpha}}{\Gamma(\alpha+1)}\Big\}\|x - y\|,\ t \in J.
\end{aligned} \tag{3.9}
$$

Thus,

$$
\|Nx - Ny\| < \|x - y\|,\ t \in J,
$$

i.e., N is a contraction operator and N has a unique fixed point. By the correspondence

$$
z(t) = \tilde{\phi}(t) + \tilde{x}(t), \quad z_t = \tilde{\phi}_t + \tilde{x}_t, t \in J,
$$

and Eqs (3.1) and (3.2), it follows that Eq (1.4) is controllable on J.

Theorem 3.2. *Suppose that the assumptions $(H_1'), (H_2'), (H_3')$ hold with $\alpha > \frac{1}{2}$, then Eq* (1.4) *is controllable on J.*

Proof. Define the space $S(T)$ as Theorem 3.2. For any $x \in S(T)$, choose the control function $u_x(\cdot)$ as in (3.3). Consider $\Omega = \{x \in S(T); \|x\| \leq r\}$, r is a sufficiently large positive number. Define the operator $N : \Omega \to \Omega$ as in (3.5). We show that N is a contraction mapping and has a fixed point.

(1) $N\Omega \subseteq \Omega$. For any $x \in \Omega$, by the assumption (H_1'), we have $\|Nx\| = 0 \leq r, t \in I$.

By the property $(A_1(c))$ on the phase space $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$, similar to (3.6), we have $\|x_t + \tilde{\phi}_t\|_{\mathcal{D}} \leq \mathcal{A}'$, $t \in J$.

By the assumption (H_1'), similar to the assumption (H_1), we have

$$\begin{aligned}
&\|g_i(t,u)\| \leq c_{i1}(t)\|u\| + c_{i2}(t), t \in J, u \in R^n,\\
&\|f(t,u)\| \leq c_3(t)\|u\|_{\mathcal{D}} + c_4(t), t \in J, u \in \mathcal{D},
\end{aligned}$$

where $c_{ij}(t), c_3(t), c_4(t)$ are defined as in the assumption (H_1). So, by the control function u_x and Hölder inequality, we get

$$\begin{aligned}
\|u_x\| \leq\ & M_2\Big[\|x_T\| + \|\phi(0)\|_{\mathcal{D}} + \sum_{i=1}^{m} \frac{\|g_i(0,\phi(0))\|T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} \\
&+ \sum_{i=1}^{m} \frac{\int_0^T (T-s)^{\alpha-\beta_i-1}(c_{i1}(s)r + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} + \frac{\int_0^T (T-s)^{\alpha-1}(c_3(s)\mathcal{A} + c_4(s))ds}{\Gamma(\alpha)}\Big] \\
\leq\ & M_2\Big[\|x_T\| + \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{m} \frac{(c_{i1}(0)\|\phi(0)\|_{\mathcal{D}} + c_{i2}(0))T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\Big] \\
&+ M_2 \sum_{i=1}^{m} \frac{\int_0^T (T-s)^{\alpha-\beta_i-1}(c_{i1}(s)r + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} \\
&+ M_2 \frac{\int_0^T (T-s)^{\alpha-1}(c_3(s)\mathcal{A} + c_4(s))ds}{\Gamma(\alpha)} \\
\leq\ & M_2\Big[\|x_T\| + \|\phi(0)\|_{\mathcal{D}} + \sum_{i=1}^{m} \frac{(c_{i1}(0)\|\phi(0)\|_{\mathcal{D}} + c_{i2}(0))T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\Big] \\
&+ M_2 \sum_{i=1}^{m} \frac{r\Big(\int_0^T (T-s)^{\frac{\alpha-\beta_i-1}{1-q_{i1}}}ds\Big)^{1-q_{i1}}\|c_{i1}\|_{L_J^{\frac{1}{q_{i1}}}} + \Big(\int_0^T (T-s)^{\frac{\alpha-\beta_i-1}{1-q_{i2}}}ds\Big)^{1-q_{i2}}\|c_{i2}\|_{L_J^{\frac{1}{q_{i2}}}}}{\Gamma(\alpha-\beta_i)} \\
&+ M_2 \frac{\mathcal{A}\Big(\int_0^T (T-s)^{\frac{\alpha-1}{1-q_3}}ds\Big)^{1-q_3}\|c_3\|_{L_J^{\frac{1}{q_3}}} + \Big(\int_0^T (T-s)^{\frac{\alpha-1}{1-q_4}}ds\Big)^{1-q_4}\|c_4\|_{L_J^{\frac{1}{q_4}}}}{\Gamma(\alpha)} \\
\leq\ & M_2\Big[\|x_T\| + \|\phi(0)\|_{\mathcal{D}} + \sum_{i=1}^{m} \frac{(c_{i1}(0)\|\phi(0)\| + c_{i2}(0))T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\Big] \\
&+ M_2 \sum_{i=1}^{m} \frac{r\Big(\frac{1-q_{i1}}{\beta_i-q_{i1}}\Big)^{1-q_{i1}} T^{\beta_i-q_{i1}}\|c_{i1}\|_{L_J^{\frac{1}{q_{i1}}}} + \Big(\frac{1-q_{i2}}{\beta_i-q_{i2}}\Big)^{1-q_{i2}} T^{\beta_i-q_{i2}}\|c_{i2}\|_{L_J^{\frac{1}{q_{i2}}}}}{\Gamma(\alpha-\beta_i)} \\
&+ M_2 \frac{\mathcal{A}T^{\alpha-q_3}\Big(\frac{1-q_3}{\alpha-q_3}\Big)^{1-q_3}\|c_3\|_{L_J^{\frac{1}{q_3}}} + T^{\alpha-q_4}\Big(\frac{1-q_4}{\alpha-q_4}\Big)^{1-q_4}\|c_4\|_{L_J^{\frac{1}{q_4}}}}{\Gamma(\alpha)} = \mathcal{B}', \ t \in J,
\end{aligned} \tag{3.10}$$

and $\|u_x\|_{L^2(J)} \leq \mathcal{B}'\sqrt{T}$.

Therefore, we have

$$
\begin{aligned}
\|(Nx)(t)\| &\leq \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}(c_{i1}(s)r + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}(c_3(s)\mathcal{A} + c_4(s))}{\Gamma(\alpha)}ds + \frac{M_1}{\Gamma(\alpha+1)}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\|u_x\|_{L^2(J)} \\
&\leq \sum_{i=1}^{m} \frac{M_2 r T^{\beta-q_1}}{\Gamma(\beta)}\Big(\frac{1-q_{i1}}{\beta-q_{i1}}\Big)^{1-q_{i1}}\|c_{i1}\|_{L_J^{\frac{1}{q_{i1}}}} + \sum_{i=1}^{m} \frac{M_2 T^{\beta-q_{i2}}}{\Gamma(\beta)}\Big(\frac{1-q_{i2}}{\beta-q_{i2}}\Big)^{1-q_{i2}}\|c_{i2}\|_{L_J^{\frac{1}{q_{i2}}}} \\
&\quad + \frac{M_2\mathcal{A}' T^{\alpha-q_3}}{\Gamma(\alpha)}\Big(\frac{1-q_3}{\alpha-q_3}\Big)^{1-q_3}\|c_3\|_{L_J^{\frac{1}{q_3}}} + \frac{M_2 T^{\alpha-q_4}}{\Gamma(\alpha)}\Big(\frac{1-q_4}{\alpha-q_4}\Big)^{1-q_4}\|c_4\|_{L_J^{\frac{1}{q_4}}} \\
&\quad + \frac{M_1}{\Gamma(\alpha+1)}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\mathcal{B}'\sqrt{T} = C', t \in J.
\end{aligned}
$$

Thus, $\|Nx\| \leq C' \leq r$, i.e., $N\Omega \subseteq \Omega$.

(2) N is a contractive operator. Choose any $x, y \in \Omega$ with $x_0 = y_0, t \in I$, as (3.9), we get

$$
\begin{aligned}
\|(Nx)(t) - (Ny)(t)\| &\leq \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}c_{i1}(s)\|x(s)-y(s)\|ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \frac{\int_0^t (t-s)^{\alpha-1}c_3(s)\|x_s - y_s\|_{\mathcal{D}}ds}{\Gamma(\alpha)} + \frac{\int_0^t (t-s)^{\alpha-1}\|Bu_x(s) - Bu_y(s)\|ds}{\Gamma(\alpha)} \\
&\leq \Big(1 + \frac{M_1 M_2 T^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big[\sum_{i=1}^{m} \frac{T^{\beta_i-q_1}}{\Gamma(\alpha-\beta_i)}\Big(\frac{1-q_{i1}}{\beta_i-q_{i1}}\Big)^{1-q_{i1}}\|c_{i1}\|_{L_J^{\frac{1}{q_{i1}}}} \\
&\quad + \frac{T^{\alpha-q_2}}{\Gamma(\alpha+1)}\Big(\frac{1-q_2}{\alpha-q_2}\Big)^{1-q_2}\|c_3\|_{L_J^{\frac{1}{q_2}}}\Big]\|x-y\|,
\end{aligned}
$$

and $\|Nx - Ny\| < \|x - y\|$, so N is a contraction mapping and has a fixed point. Thus, (1.4) is controllable on the interval J.

Theorem 3.3. *Suppose that the assumptions (H_1''), (H_3) hold, then Eq* (1.4) *is controllable on J.*

Proof. For any $x \in S(T)$, define the control function $u_x(\cdot)$ and N as in Theorem 3.1. We will show that N is continuous and completely continuous.

(1) N is continuous. Let $\{x_n\}$ be a sequence in $S(T)$ such that $\lim_{n\to\infty} x_n = x$ with $x_{nt} = \phi(t), t \in I$, then $\|Nx_n - Nx\| = 0, t \in I$. When $t \in J$, by the property $(A_1(c))$ on the phase space $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$, we have

$$
\begin{aligned}
\|x_{ns} - x_s\|_{\mathcal{D}} = \|\tilde{x}_{ns} - \tilde{x}_s\|_{\mathcal{D}} &\leq K(t)\sup\{\|x_n(s) - x(s)\| : s \in [0,t]\} + M(t)\|\phi_0 - \phi_0\|_{\mathcal{D}} \\
&\leq K_T \sup\{\|x_n(s) - x(s)\| : s \in [0,t]\}, t \in J,
\end{aligned}
$$

and by the assumption (H_1''), we get

$$
\begin{aligned}
&\|(Nx_n)(t) - (Nx)(t)\| \\
&\leq \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}\|g_i(s, x_n(s)) - g_i(s, x(s))\|ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \frac{\int_0^t (t-s)^{\alpha-1}[\|f(s, \tilde{x}_{ns} + \tilde{\phi}_s) - f(s, \tilde{x}_s + \tilde{\phi}_s)\| + \|Bu_{x_n}(s) - Bu_x(s)\|]ds}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}\|g_i(s,x_n(s)) - g_i(s,x(s))\|ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \frac{\int_0^t (t-s)^{\alpha-1}[\|f(s,x_{ns}) - f(s,x_s)\| + \|Bu_{x_n}(s) - Bu_x(s)\|]ds}{\Gamma(\alpha)} \\
&\leq \sum_{i=1}^{m} \frac{\varepsilon \int_0^t (t-s)^{\alpha-\beta_i-1}\|ds}{\Gamma(\alpha-\beta_i)} + \frac{\int_0^t (t-s)^{\alpha-1}(\varepsilon + \|B\|\|u_{x_n}(s) - u_x(s)\|)ds}{\Gamma(\alpha)} \qquad (3.11) \\
&\leq \sum_{i=1}^{m} \frac{\varepsilon t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \frac{\varepsilon t^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_2 \int_0^t (t-s)^{\alpha-1}\|u_{x_n}(s) - u_x(s)\|ds}{\Gamma(\alpha)} \\
&\leq \sum_{i=1}^{m} \frac{\varepsilon t^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \frac{\varepsilon t^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_1 M_2 T^{\alpha}\varepsilon}{\Gamma(\alpha+1)} \to 0,
\end{aligned}
$$

as $x_n \to x$, i.e.,

$$\lim_{n\to\infty} \sup\{\|(Nx_n)(t) - (Nx)(t)\|\} = 0,\ t \in J.$$

In all $\lim_{n\to\infty} \|N(x_n) - Nx\| = 0$, i.e., N is continuous.

(2) N maps a bounded set $B_r = \{x \in S(T) : \|x\| \leq r\}, r > 0$ into a bounded set, i.e., $NB_r = \{y : \|y\| \leq L\}, L > 0$, where r, L are sufficiently large positive numbers. We get easily that $\|Nx\| = \|\phi\|_{\mathcal{D}} = 0 \leq r,\ t \in I$. From the assumptions $(H_1''), (H_3)$, similar to the proof of Theorem 3.2, we get Nx is bounded.

(3) N maps $B_r \subset S(T)$ into equi-continuous, i.e., as $\hat{t}_1 \to \hat{t}_2$, $\hat{t}_1, \hat{t}_2 \in I$,

$$(Nx)(\hat{t}_2) - (Nx)(\hat{t}_1) = \phi(\hat{t}_2) - \phi(\hat{t}_1) = 0.$$

As $\hat{t}_1, \hat{t}_2 \in J, \hat{t}_1 \to \hat{t}_2$, we have

$$
\begin{aligned}
&\|(Nx)(\hat{t}_2) - (Nx)(\hat{t}_1)\| \\
&\leq \sum_{i=1}^{m} \frac{\int_0^{\hat{t}_1} [(\hat{t}_2-s)^{\alpha-\beta_i-1} - (\hat{t}_1-s)^{\alpha-\beta_i-1}]\|g_i(s,x(s))\|ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \sum_{i=1}^{m} \frac{\int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2-s)^{\alpha-\beta_i-1}\|g_i(s,x(s))\|ds}{\Gamma(\alpha-\beta_i)} + \frac{\int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2-s)^{\alpha-1}\|f(s,\tilde{x}_s+\tilde{\phi}_s)\|ds}{\Gamma(\alpha)} \\
&\quad + \frac{\int_{\hat{t}_1}^{\hat{t}_2} [(\hat{t}_2-s)^{\alpha-1} - (\hat{t}_1-s)^{\alpha-1}]\|f(s,\tilde{x}_s+\tilde{\phi}_s)\|ds}{\Gamma(\alpha)} + \frac{\int_0^{\hat{t}_1} (\hat{t}_2-s)^{\alpha-1}\|Bu_x(s)\|ds}{\Gamma(\alpha)} \\
&\quad + \frac{\int_{\hat{t}_1}^{\hat{t}_2} [(\hat{t}_2-s)^{\alpha-1} - (\hat{t}_1-s)^{\alpha-1}]\|Bu_x(s)\|ds}{\Gamma(\alpha)} \\
&\leq \sum_{i=1}^{m} \frac{\int_0^{\hat{t}_1} [(\hat{t}_2-s)^{\alpha-\beta_i-1} - (\hat{t}_1-s)^{\alpha-\beta_i-1}](c_{i1}(s)\|x(s)\| + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \sum_{i=1}^{m} \frac{\int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2-s)^{\alpha-\beta_i-1}(c_{i1}(s)\|x(s)\| + c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)} \\
&\quad + \frac{\int_0^{\hat{t}_1} [(\hat{t}_2-s)^{\alpha-1} - (\hat{t}_1-s)^{\alpha-1}](c_3(s)\|\tilde{x}_s+\tilde{\phi}_s\|_{\mathcal{D}} + c_4(s))ds}{\Gamma(\alpha)} \\
&\quad + \frac{\int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2-s)^{\alpha-1}(c_3(s)\|\tilde{x}_s+\tilde{\phi}_s\|_{\mathcal{D}} + c_4(s))ds}{\Gamma(\alpha)} + \frac{M_1(\hat{t}_2^{\alpha} - \hat{t}_1^{\alpha})}{\Gamma(\alpha+1)}\|u_x\|_{L^\infty(J,R^m)}
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{i=1}^{m}\frac{(\hat{c}_{i1}r+\hat{c}_{i2})\int_0^{\hat{t}_1}[(\hat{t}_2-s)^{\alpha-\beta_i-1}-(\hat{t}_1-s)^{\alpha-\beta_i-1}]ds}{\Gamma(\alpha-\beta_i)}\\
&\quad+\sum_{i=1}^{m}\frac{(\hat{c}_{i1}r+\hat{c}_{i2})\int_{\hat{t}_1}^{\hat{t}_2}(\hat{t}_2-s)^{\alpha-\beta_i-1}ds}{\Gamma(\alpha-\beta_i)}\\
&\quad+\frac{(\hat{c}_3\mathcal{A}+\hat{c}_4)\int_0^{\hat{t}_1}[(\hat{t}_2-s)^{\alpha+\beta-1}-(\hat{t}_1-s)^{\alpha+\beta-1}]ds}{\Gamma(\alpha+\beta)}\\
&\quad+\frac{(\hat{c}_3\mathcal{A}+\hat{c}_4)\int_{\hat{t}_1}^{\hat{t}_2}(\hat{t}_2-s)^{\alpha+\beta-1}ds}{\Gamma(\alpha+\beta)}+\frac{M_1(\hat{t}_2^{\alpha+\beta}-\hat{t}_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)}\|u_x\|_{L^\infty(J,R^m)}\to 0,
\end{aligned} \tag{3.12}$$

as $\hat{t}_2\to\hat{t}_1$, so

$$\lim_{\hat{t}_2\to\hat{t}_1}\sup\{\|(Nx)(\hat{t}_2)-(Nx)(\hat{t}_1)\|\}=0,\ t\in J$$

in all

$$\lim_{\hat{t}_2\to\hat{t}_1}\|(Nx)(\hat{t}_2)-(Nx)(\hat{t}_1)\|=0,\ \hat{t}_1,\hat{t}_2\in I\cup J.$$

By steps one through three and the Ascoli-Arzela theorem, it follows that $N:S(T)\to S(T)$ is continuous and completely continuous.

(4) Let $K=\{x\in S(T): x=\lambda Nx, 0<\lambda<1\}$. We show that the set K is bounded. For any $x\in K$, then $x=\lambda Nx, 0<\lambda<1$, so for $t\in J$, we get

$$x(t)=\lambda\Big[\sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}g_i(s,x(s))ds}{\Gamma(\alpha-\beta_i)}+\frac{\int_0^t(t-s)^{\alpha-1}[f(s,\tilde{x}_s+\tilde{\phi}_s)+Bu(s)]ds}{\Gamma(\alpha)}\Big]. \tag{3.13}$$

By the assumption (H_1''), similar to the proof of the first step of Theorem 3.2, for $t\in J$ we have

$$\begin{aligned}
\|x(t)\| &\leq \sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}\|g_i(s,x(s))\|ds}{\Gamma(\alpha-\beta_i)}+\frac{\int_0^t(t-s)^{\alpha-1}[\|f(s,\tilde{x}_s+\tilde{\phi}_s)\|+\|Bu_x(s)\|]ds}{\Gamma(\alpha)}\\
&\leq \sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}(c_{i1}(s)\|x(s)\|+c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)}\\
&\quad+\frac{\int_0^t(t-s)^{\alpha-1}(c_3(s)\|\tilde{x}_s+\tilde{\phi}_s\|_{\mathcal{D}}+c_4(s))ds}{\Gamma(\alpha)}+\frac{M_1T^\alpha}{\Gamma(\alpha+1)}\|u_x\|_{L^\infty(J,R^n)}\\
&\leq \frac{M_1T^\alpha}{\Gamma(\alpha+1)}\|u_x\|_{L^\infty(J,R^n)}+\sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}(c_{i1}(s)\|x(s)\|+c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)}\\
&\quad+\frac{\int_0^t(t-s)^{\alpha-1}[c_3(s)(K(s)\sup\{\|\tilde{x}(\theta)+\tilde{\phi}(\theta)\|:\theta\in[0,s]\}+M(s)\|\phi\|_{\mathcal{D}})+c_4(s)]ds}{\Gamma(\alpha)}\\
&\leq \frac{M_1T^\alpha}{\Gamma(\alpha+1)}\|u_x\|_{L^\infty(J,R^n)}+\sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}(c_{i1}(s)\|x(s)\|+c_{i2}(s))ds}{\Gamma(\alpha-\beta_i)}\\
&\quad+\frac{\int_0^t(t-s)^{\alpha-1}[c_3(s)(K(s)\sup\{\|\tilde{x}(\theta)\|+\|\tilde{\phi}(\theta)\|:\theta\in[0,s]\}+M(s)\|\phi\|_{\mathcal{D}})+c_4(s)]ds}{\Gamma(\alpha)}\\
&\leq \frac{M_1T^\alpha}{\Gamma(\alpha+1)}\mathcal{B}+\sum_{i=1}^{m}\frac{\int_0^t(t-s)^{\alpha-\beta_i-1}[c_{i1}(s)\|x(s)\|+c_{i2}(s)]ds}{\Gamma(\alpha-\beta_i)}\\
&\quad+\frac{\int_0^t(t-s)^{\alpha-1}[c_3(s)(K(s)\sup\{\|\tilde{x}(\theta)\|+\|\tilde{\phi}(\theta)\|:\theta\in[0,s]\}+M(s)\|\phi\|_{\mathcal{D}})+c_4(s)]ds}{\Gamma(\alpha)}.
\end{aligned} \tag{3.14}$$

Let

$$\nu(t) = \sup\{\|x(s)\| : s \in [0,t]\}, t \in J, \tag{3.15}$$

then by Lemma 2.2, we get

$$\begin{aligned}
\nu(t) \leq\ & \frac{M_1 T^\alpha}{\Gamma(\alpha+1)}\mathcal{B} + \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}[c_{i1}(s)\nu(s) + c_{i2}(s)]ds}{\Gamma(\alpha-\beta_i)} \\
& + \frac{\int_0^t (t-s)^{\alpha-1}[c_3(s)(K(s)\nu(s) + K(s)\|\phi(0) - \sum_{i=1}^{m} \frac{g_i(0,\phi(0))s^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\| + K(s)\|\tilde{\phi}(s)\| + M(s)\|\phi\|_{\mathcal{D}})]ds}{\Gamma(\alpha)} \\
& + \frac{\int_0^t (t-s)^{\alpha-1}c_4(s)ds}{\Gamma(\alpha)} \\
\leq\ & \frac{M_1 T^\alpha}{\Gamma(\alpha+1)}\mathcal{B} + \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}[\hat{c}_{i1}\nu(s) + c_{i2}(s)]ds}{\Gamma(\alpha-\beta_i)} \\
& + \frac{\int_0^t (t-s)^{\alpha-1}[\hat{c}_3(K_T\nu(s) + K_T \sum_{i=1}^{m} \frac{\|g_i(0,\phi(0))\|}{\Gamma(\alpha-\beta_i+1)}T^{\alpha-\beta_i} + (K_T + M_T)\|\phi\|_{\mathcal{D}} + c_4(s)]ds}{\Gamma(\alpha)} \\
=\ & \frac{M_1 T^\alpha}{\Gamma(\alpha+1)}\mathcal{B} + \sum_{i=1}^{m} \frac{\int_0^t (t-s)^{\alpha-\beta_i-1}c_{i2}ds}{\Gamma(\alpha-\beta_i)} \\
& + \frac{\int_0^t (t-s)^{\alpha-1}(K_T \sum_{i=1}^{m} \frac{\|g_i(0,\phi(0))\|}{\Gamma(\alpha-\beta_i+1)}T^{\alpha-\beta_i} + (K_T + M_T)\|\phi\|_{\mathcal{D}} + \hat{c}_4)ds}{\Gamma(\alpha)} \\
& + \sum_{i=1}^{m} \frac{\hat{c}_{i1}\int_0^t (t-s)^{\alpha-\beta_i-1}\nu(s)ds}{\Gamma(\alpha-\beta_i)} + \frac{\hat{c}_3 K_T \int_0^t (t-s)^{\alpha-1}\nu(s)ds}{\Gamma(\alpha)} \\
\leq\ & \frac{M_1 T^\alpha \mathcal{B}}{\Gamma(\alpha+1)} + \sum_{i=1}^{m} \frac{\hat{c}_{i2}T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)} + \frac{(K_T \sum_{i=1}^{m} \frac{\|g_i(0,\phi(0))\|}{\Gamma(\alpha-\beta_i+1)}T^{\alpha-\beta_i} + (K_T + M_T)\|\phi\|_{\mathcal{D}} + \hat{c}_4)T^\alpha}{\Gamma(\alpha+1)} \\
& + \sum_{i=1}^{m} \frac{\hat{c}_{i1}K_T \int_0^t (t-s)^{\alpha-\beta_i-1}\nu(s)ds}{\Gamma(\alpha-\beta_i)} + \frac{\hat{c}_3 K_T \int_0^t (t-s)^{\alpha-1}\nu(s)ds}{\Gamma(\alpha)}, t \in J.
\end{aligned} \tag{3.16}$$

By Lemma 2.3, it follows that there is a positive constant $\hat{L}$ such that $\|x(t)\| \leq \nu(t) \leq \hat{L}, t \in J$ and $\|x\| \leq \hat{L}$, i.e., K is a bounded set. By using the Schaefer fixed point theorem, the operator N has a fixed point, which is corresponding to the controllable of Eq (1.4) on the interval J.

Theorem 3.4. *If the assumptions* $(H_1), (H_2)$ *hold, then Eq* (1.3) *has a unique solution on* J.

Proof. The proof process of Theorem 3.4 is similar to the proof of (3.9) in Theorem 3.1, so we omit it.

Theorem 3.5. *If the assumptions* $(H_1'), (H_2')$ *hold, then Eq* (1.3) *has at least one solution on* J.

Proof. The proof process of Theorem 3.5 is similar to the proof of Theorem 3.2, so we omit it.

Theorem 3.6. *If the assumptions* (H_1'') *holds, then Eq* (1.3) *has at least one solution on* J.

Proof. The proof process of Theorem 3.6 is similar to the proof of Theorem 3.4, so we omit it.

Remark. These theorems are also valid for Eqs (1.3) and (1.4) with finite delays.

4. An example

Let $\mathcal{D} = \{y \in C((-\infty, 0], R) : \lim_{s\to-\infty} \exp(\lambda s)y(s) \text{ exists}\}$ with the norm $\|y\|_{\mathcal{D}} = \sup_{s\le 0}\{\exp(\lambda s)|y(s)|\}$, where $\lambda > 0$ is a constant, then $\mathcal{D}$ satisfies the assumptions in [59] with $K(t) = M(t) = H = 1$.

Example 1. Consider the controllability of the following neutral Caputo fractional functional differential equations as a special case of (1.4) with $n = 1$

$$\begin{cases} {}^{C}D^{0.8}x(t) - {}^{C}D^{0.6}\dfrac{x(t)}{100+t} - {}^{C}D^{0.5}\dfrac{x(t)}{100+\exp(t)} = \dfrac{\exp(0.5t)x_t}{(\exp(t)+\exp(-t))(200+\|x_t\|_{\mathcal{D}})} + \dfrac{u(t)}{100}, t \in [0,1], \\ x(t) = \phi(t) \in \mathcal{D},\ t \in (-\infty, 0], \end{cases} \tag{4.1}$$

where $\lambda = 0.5$ and

$$\begin{aligned} g_1(t,x) &= \frac{x(t)}{100+t},\ g_2(t,x) = \frac{x(t)}{100+\exp(t)}, t \in J, x \in R, \\ f(t,x) &= \frac{\exp(0.5t)x}{(\exp(t)+\exp(-t))(200+\|x\|_{\mathcal{D}})}, t \in J, x \in \mathcal{D}; \\ Bu(t) &= \frac{u(t)}{100}, t \in R. \end{aligned}$$

Then, we get

$$\begin{aligned} \|g_1(t,x) - g_1(t,y)\| &= \frac{1}{100+t}\|x-y\| \le \frac{1}{100}\|x-y\|, \\ \|g_2(t,x) - g_2(t,y)\| &= \frac{1}{100+\exp(t)}\|x-y\| \le \frac{1}{101}\|x-y\|, \\ \|f(t,x) - f(t,y)\| &= \frac{\exp(0.5t)}{(\exp(t)+\exp(-t))}\left\|\frac{x}{200+\|x\|_{\mathcal{D}}} - \frac{y}{200+\|y\|_{\mathcal{D}}}\right\| \\ &\le \frac{\exp(0.5t)}{(\exp(t)+\exp(-t))}\frac{\|x-y\|_{\mathcal{D}}}{200} \le \frac{1}{200}\|x-y\|_{\mathcal{D}}, \end{aligned}$$

and the conditions $(H_1), (H_2), (H_3)$ in Theorem 3.1 hold, so Eq (4.1) is controllable on J.

Example 2. Consider the controllability of the following neutral Caputo fractional functional differential equations as a special case of (1.4) with $n = 1$

$$\begin{cases} {}^{C}D^{0.8}x(t) - {}^{C}D^{0.6}\dfrac{x(t)}{2\sqrt[5]{10^5+\exp(t)}} - {}^{C}D^{0.5}\dfrac{x(t)}{\sqrt[4]{100+t^2}} = \dfrac{\exp(-0.5t)x_t}{(t+100)(t+2000)} + \dfrac{u(t)}{100}, t \in [0,1], \\ x(t) = \phi(t) \in \mathcal{D},\ t \in (-\infty, 0], \end{cases} \tag{4.2}$$

where $\lambda = 0.5$ and

$$\begin{aligned} g_1(t,x) &= \frac{x(t)}{2\sqrt[5]{10^5+\exp(t)}}, t \in J, x \in R, \\ g_2(t,x) &= \frac{x(t)}{\sqrt[4]{10^4+t^2}}, t \in J, x \in R; \\ f(t,x) &= \frac{\exp(-0.5t)x_t}{\sqrt{(t+100)(t+200)}}, t \in J, x \in \mathcal{D}; \\ Bu(t) &= \frac{u(t)}{100}, t \in R. \end{aligned}$$

Then, we get

$$\|g_1(t,x)-g_1(t,y)\| = \frac{1}{2\sqrt[5]{10^5+\exp(t)}}\|x(t)-y(t)\| \le \frac{1}{20}\|x-y\|,$$
$$\|g_2(t,x)-g_2(t,y)\| = \frac{1}{\sqrt[4]{10^4+t^2}}\|x(t)-y(t)\| \le \frac{1}{10}\|x-y\|,$$
$$\|f(t,x_t)-f(t,y_t)\| = \frac{\exp(-0.5t)\|x_t-y_t\|}{\sqrt{(t+100)(t+200)}} \le \frac{1}{100\sqrt{2}}\|x-y\|_{\mathcal{D}},$$

and the conditions (H_1'), (H_2'), (H_3') in Theorem 3.2 hold, so Eq (4.2) is controllable on J.

5. Conclusions

We considered a class of deterministic Caputo fractional functional differential equations with infinite delay and multiple Caputo fractional derivatives. The controllability of Eq (1.4) and existence of solution to Eq (1.3) were obtained by using the properties of the phase space $\mathcal{D}$ on infinite delay, Gronwall inequality and the monotone properties of fractional order operators, and some were fixed point theorems under Lipschitz and non-Lipschitz conditions. We gave two examples to explain the main results.

In view of the wide application prospect of stochastic fractional differential systems [62–65], we will extended the results of this paper to relevant stochastic fractional derivative systems. Currently, fractional calculus is defined in various forms, such as the following Riemann-Liouville, Hilfer, Caputo-Hadamard and more. Can the results of this manuscript and some methods such as averaging principles for fractional differential equations be extended to the above cases? In the future, we will strengthen the research in the above directions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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