SERIES SOLUTION OF TYPHOID FEVER MODEL USING DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT

This paper presents an analysis of PSI_uI_eTR type model, which are used to study the transmission dynamics of typhoid fever diseases in a population. Basic idea of typhoid fever disease transmission using compartmental modeling is discussed. Differential Transformation Method (DTM) is discussed in detail, which is used to compute the series solution of the non-linear system of differential equation governing the model equations. The validity of the (DTM) in solving the proposed model is established by classical fourth-order Runge-Kutta method which is implemented in Maple 18. Graphical results confirm that (DTM) is in good agreement with RK-4 and this produced correctly same behaviour of the model, thus validating the efficiency and accuracy of (DTM) in finding the series solution of an epidemic model.

Keywords: *Typhoid Fever , Differential Transform Method, Runge-Kutta Method.*

1. Introduction

Typhoid fever, a communicable disease which infects human only and occurs due to systemic infection mainly by salmonella typhi organism that causes symptoms. It is an acute generalized infectious disease of the intestinal lymphoid tissue and the gall bladder. Incubation period of typhoid fever, usually 10-14 days but it may be as short as 3 days or as long as 21 days. The disease is transmitted from person-to-person as a result of improper hygiene and unsafe food and water handling practices. Recent report, however, suggests that individuals may be indirectly infected with typhoid through contact with fecal and urine contamination in their immediate environment. (Shanahan, 1998).

The disease is endemic in many developing countries where water supply sanitation and waste treatment is inadequate. The disease remains a substantial public health problem. Globally, the disease burden was estimated to be over 16 million cases of illness each year, resulting in over 600,000 mortality rates Mushayabasa (2011).

Several mathematical models has been developed on the transmission dynamics of typhoid fever, these includes, (Adetunde, 2008; Cvjetanovic et al, 2014; Kalajdzievska, 2011; Lauria et al 2009; Moatlhodl & Gosaamang, 2017; Chamuchi et al, 2014; Joshua, 2011; Muhammad, et al 2015; Mushayabasa, 2011; Nthiiri, 2016; Virginia et al, 2014; Watson & Edmunds, 2015; Peter, et al 2017). In this paper, we extend previous efforts by introducing a model that includes educated infected and uneducated infected class.

This work focused on the application of differential transform method to the proposed model and to verify the validity of the method in solving the model equations using computer in-built Maple 18 classical fourth-order Runge-Kutta method as a base. In recent years, the differential transform method (DTM) is mostly used for solving non-linear ordinary and partial differential equations. It is a semi-analytic technique that formalizes the Taylor series in a totally different approach. The concept of (DTM) was first introduced by Zhou, (1986) in a study to solve nonlinear problems of electrical circuits. The DTM obtains an analytical solution in form of polynomial. DTM has been successfully applied to solve many nonlinear problems arising in engineering, mathematics, physics and mechanics. Abazari et al, (2010). The main advantage of DTM is that it can be applied directly to solve linear and nonlinear Ordinary Differential Equations without requiring linearization, discretization or perturbation. (Hassan, 2008; Peter & Ibrahim, 2017; Akinboro, et al 2014).

We employ the (DTM) to the system of differential equations which describe the proposed model and approximate the solutions in a sequence of time intervals. In other to verify the accuracy and validity of the (DTM), we compared the obtained results with fourth-order Runge-Kutta Method.

2.   Methodology

This section describes the formulation of the model. The human populations $N(t)$ is divided into six sub-populations namely; protected class, $P(t)$, susceptible class, $S(t)$, uneducated infectious class, $Iu(t)$, educated infectious class $Ie(t)$, treated class $T(t)$ and recovered individuals class, $R(t)$. Individuals are recruited into the susceptible class by either immigration or birth at the rate Λ . Susceptible individuals acquire typhoid infection at per capita rate λ . We assume that proportion α progress to educated infectious class, while the compliment $1-\alpha$ progress to uneducated infectious compartment class. Susceptible individuals received vaccination to protect themselves against the disease at the rate τ . Since vaccine wanes with time, then after its expiry, the protected individuals return back to susceptible class at the rate ω . We assume that individuals in each compartment undergo a natural death at the rate μ . Let β_1 , β_2 , and β_3 be transmission rates for uneducated infectious, educated infectious and treated individuals respectively then the susceptible population S(t), is exposed to force of infection denoted by $\lambda = \beta_1 I_u + \beta_2 I_e + \beta_3 T$. Detailed description of parameters and variables are shown in Table 1 while the compartmental flow diagram of the model is shown in Figure 1.
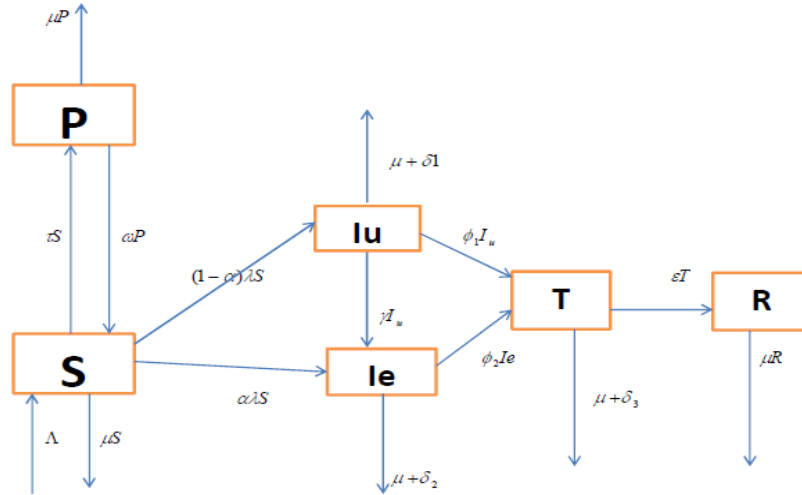
Figure 1. Pictorial Representation of Model

2.1   Model Assumptions

1. Recovered individuals may become susceptible again at the rate ω , this is due to the fact that typhoid does not confer permanent immunity on recovery.
2. Susceptible individuals receive vaccination to protect themselves against infection at the rate τ .
3. Susceptible individual can be infected through a direct contact with educated infected   or uneducated infected.
4. All parameters are non-negative.
5. No treatment failure. All treated individuals recoverd.

$$
\left.
\begin{aligned}
\frac{dP}{dt} &= \tau S - (\mu + \omega)P \\
\frac{dS}{dt} &= \Lambda - \lambda S + \omega P - (\tau + \mu)S \\
\frac{dIu}{dt} &= (1 - \alpha)\lambda S - (\mu + \delta_1 + \phi_1 + \gamma)I_u \\
\frac{dIe}{dt} &= \alpha\lambda S - (\mu + \delta_2 + \phi_2)I_e + \gamma Iu \\
\frac{dT}{dt} &= \phi_1 I_u + \phi_2 I_e - (\mu + \delta_3 + \varepsilon)T \\
\frac{dR}{dt} &= \varepsilon T - \mu R
\end{aligned}
\right\} \tag{1}
$$

Where the force of infection λ is given as:

$$\lambda = \beta_1 I_u + \beta_2 I_e + \beta_3 T \qquad (2)$$

Substituting the value of force of infection

$$\left.\begin{aligned}
\frac{dP}{dt} &= \tau S - (\mu + \omega)P - \mu S \\
\frac{dS}{dt} &= \Lambda - (\beta_1 I_u + \beta_2 I_e + \beta_3 T)S + \omega P - (\tau + \mu)S \\
\frac{dIu}{dt} &= (1-\alpha)(\beta_1 I_u + \beta_2 I_e + \beta_3 T)S - (\mu + \delta_1 + \phi_1 + \gamma)I_u \\
\frac{dIe}{dt} &= \alpha S(\beta_1 I_u + \beta_2 I_e + \beta_3 T) - (\mu + \delta_2 + \phi_2)I_e + \gamma I_u \\
\frac{dT}{dt} &= \phi_1 I_u + \phi_2 I_e - (\mu + \delta_3 + \varepsilon)T \\
\frac{dR}{dt} &= \varepsilon T - \mu R
\end{aligned}\right\} \qquad (3)$$

Table 1. Description of Variables and Parameters for the Model

Variables	Description
$P(t)$	protected individuals at time t
$S(t)$	susceptible individuals at time t
Iu	uneducated infectious individuals at time t
Ie	educated infectious individuals at time t t
T	treated individuals at time t t
$R(t)$	recovered individuals at time t
Parameters	Interpretation
Λ	recruitment rate of susceptible individuals
μ	natural death rate
δ_1	disease induced death rate for I_u class
δ_2	disease induced death rate for I_e
δ_3	disease induced death rate for T
ϕ_1	treatment rate for I_u
ϕ_2	treatment rate for I_e
ω	wanning rate of vaccine
ε	rate of recovery from treatment
γ	rate of educating or counseling uneducated infectives
τ	rate of vaccinating individual in the susceptible class
β_1	transmission rate between S and I_u class
β_2	transmission rate between S and I_e class
β_3	transmission rate between S and T class

2.2 Existence and Uniqueness of Solution

The implementation of any mathematical model largely based on whether the given system of equations has a solution, and if the solution is unique, we shall use the Lipchitz condition to verify the existence and uniqueness of solution for the system of the model equation 3.

Let the system of equations of the model be as follows:

$$A_1 = \tau S - (\mu + \omega)P$$
$$A_2 = \Lambda - (\beta_1 I_u + \beta_2 I_e + \beta_3 T)S + \omega P - (\tau + \mu)S$$
$$A_3 = (1 - \alpha)(\beta_1 I_u + \beta_2 I_e + \beta_3 T)S - (\mu + \delta_1 + \phi_1 + \gamma)I_u$$
$$A_4 = \alpha S(\beta_1 I_u + \beta_2 I_e + \beta_3 T) - (\mu + \delta_2 + \phi_2)I_e + \gamma I_u$$
$$A_5 = \phi_1 I_u + \phi_2 I_e - (\mu + \delta_3 + \varepsilon)$$
$$A_6 = \varepsilon T - \mu R$$

Theorem 1. (Derrick and Groosman,1976)

Let B denote the region
$$|t - t_0| \leq b, \|x - x_0\| \leq 1, x = (x_1, x_2 \ldots \ldots, x_n), x_o = (x_{10}, x_{20}, \ldots \ldots x_{no})$$
And suppose that $a(t, x)$ satisfies the Lipchitz condition

$$\|a(t, x_1) - a(t, x_2)\| \leq k\|x_1 - x_2\|$$

whenever the pairs (t, x_1) and (t, x_2) belong to B where k is a positive constant. Then, there is a constant $\delta \geq 0$ such that there exists a unique continuous vector solution of $x(t)$ of the system in the interval $t - t_o \leq \delta$. It is important to note that the condition is satisfied by the requirement that $\dfrac{\partial a_i}{\partial x_j}, i, j = 1,2,3\Lambda$, be continuous and bounded in B Considering the model equation 3, we are interested in the region $0 \leq \alpha \leq R$. We look for the bounded solution in the region and whose partial derivatives satisfy $a \leq \alpha \leq 0$. where α and δ are positive constants.

Theorem 2

Let B denote the region $0 \leq \alpha \leq R$, then equation 3 have a unique solution. We show that
$$\frac{\partial a_i}{\partial x_j}, i, j = 1,2,3,4,5,6$$

Are continuous and bounded in B For A_1

$$\left|\frac{\partial a_1}{\partial P}\right| = | -(\mu + \omega) | < \infty, \left|\frac{\partial a_1}{\partial S}\right| = | \tau | < \infty \left|\frac{\partial a_1}{\partial I_u}\right| = 0 < \infty, \left|\frac{\partial a_1}{\partial I_e}\right| = 0 < \infty,$$

$$\left|\frac{\partial a_1}{\partial T}\right| = 0 < \infty, \left|\frac{\partial a_1}{\partial R}\right| = 0 < \infty$$

For A_2

$$\left|\frac{\partial a_2}{\partial P}\right| = |\ \omega\ | < \infty \quad \left|\frac{\partial a_2}{\partial S}\right| = |\ -(\beta_1 I_u + \beta_2 I_e + \beta_3 I_T) - (\tau + \mu)\ | < \infty$$

$$\left|\frac{\partial a_2}{\partial I_u}\right| = |\ \beta_1 S\ | < \infty, \quad \left|\frac{\partial a_2}{\partial I_e}\right| = |\ \beta_2 S\ | < \infty, \quad \left|\frac{\partial a_2}{\partial T}\right| = |\ \beta_3 S\ | < \infty \quad \left|\frac{\partial a_2}{\partial R}\right| = 0 < \infty,$$

For A_3

$$\left|\frac{\partial a_3}{\partial P}\right| = 0 < \infty, \quad \left|\frac{\partial a_3}{\partial S}\right| = |\ (1-\alpha)(\beta_1 I_u + \beta_2 I_e + \beta_3 T)\ | < \infty \quad \left|\frac{\partial a_3}{\partial I_e}\right| = |\ (1-\alpha)\beta_2 S\ | < \infty$$

$$\left|\frac{\partial a_3}{\partial I_u}\right| = |\ (1-\alpha)\beta_1 S - (\mu + \delta_1 + \phi_1 + \gamma)\ | < \infty \quad \left|\frac{\partial a_3}{\partial T}\right| = |\ (1-\alpha)\beta_3 S\ | < \infty$$

$$\left|\frac{\partial a_3}{\partial R}\right| = 0 < \infty$$

These partial derivative exist, continuous and are bounded, similarly for A_4 through to A_6. Hence, by theorem 2, the model has a unique solution

3.   Result and Analysis

The processes involved in DTM is given as follows: Given an arbitrary function of x, suppose $e(x)$ is a non-linear funtion of x, then $e(x)$ can be expanded in a Taylor series about a point $x = 0$ as

$$e(x) = \sum_{k=0}^{\infty} x^k \frac{1}{k!}\left[\frac{d^k}{dx^k} e(x)\right]_{x=0}$$

Thus, the differential Transform of $y(x)$ is given as:

$$E(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k} e(x)\right]_{x=0}$$

and the inverse differential Transform is given as

$$e(x) = \sum_{k=0}^{\infty} E(k) x^k$$

Table 2 illustrate some operational properties of DTM. In the table, $m(x)$ and $n(x)$ are arbitary functions with transformed into $M(k)$ and $N(k)$ respectively.

Table 2. Basic operation properties of the DTM

S/No	Original Function	Transformed Function
1	$e(x) = m(x) \pm n(x)$	$E(k) = M(k) \pm N(k)$
2	$e(x) = \alpha m(x)$	$E(k) = \alpha M(k)$, α is a constant
3	$e(x) = \dfrac{dm(x)}{dx}$	$E(k) = (k+1)M(k+1)$
4	$e(x) = \dfrac{d^2 c(x)}{dx^2}$	$E(k) = (k+1)(k+2)C(k+2)$
5	$e(x) = \dfrac{d^a m(x)}{dx^a}$	$E(k) = (k+1)(k+2)\Lambda\ (k+n)C(k+n)$
6	$e(x) = 1$	$E(k) = \delta(k)$
7	$e(x) = x$	$E(k) = \delta(k-1)$, δ is the Kronecker delta
8	$e(x) = e^{(\lambda x)}$	$E(k) = \dfrac{\lambda^k}{k!}$
9	$e(x) = c(x)d(x)$	$E(k) = \sum_{n=0}^{k} D(a)C(k-a)$
10	$e(x) = (1+x)^a$	$E(k) = \dfrac{a(a-1)(a-2)\Lambda\ (a-k+1)}{k!}$

3.1   Solution of the Model

In this section, we apply the steps involved in differential transform method as follows: Using the operational properties (1), (2), (6) and (7) in Table 2 and applying them to the systyem of differential equations in (1) we obtain the following system of transformed equations below,

where
$$K_2 = \mu + \tau,\ K_1 = \omega + \mu,\ K_3 = \mu_3 + \gamma,\ K_4 = \mu_3 + \delta_1 + \phi_2,\ K_5 = \mu_3 + \delta_1 + \varepsilon$$

$$P(k+1) = \frac{1}{k+1}[\tau S(k) - K_1 P(k)]$$

$$S(k+1) = \frac{1}{k+1}[\Lambda b(k,0) + \omega P(k) - \sum_{l=0}^{k} S(l)(\beta_1 I_u(k-l) + \beta_2 I_e(k-l) + \beta_3 T(k-l)) - K_2 S(k)]$$

$$I_u(k+1) = \frac{1}{k+1}[(1-\alpha)\sum_{l=0}^{k} S(l)(\beta_1 I_u(k-l) + \beta_2 I_e(k-l) + \beta_3 T(k-l)) - K_3 I_u(k)]$$

$$I_e(k+1) = \frac{1}{k+1}[\alpha \sum_{l=0}^{k} S(l)(\beta_1 I_u(k-l) + \beta_2 I_e(k-l) + \beta_3 T(k-l)) + \gamma I_u(k) - K_4 I_e(k)]$$

$$T(k+1) = \frac{1}{k+1}[\phi_1 I_u(k) + \phi_2 I_e(k) - K_5 T(k)]$$

$$R(k+1) = \frac{1}{k+1}[\varepsilon T(k) - \mu R(k)]$$

Subject to the following initial conditions $P(0) = 100$, $S(0) = 200$, $I_u(0) = 140$, $I_e(0) = 120$, $T(0) = 80$, $R(0) = 60$. Using the initial conditions and the parameter values in the table we have the following series solutions.

when $k = 5$ the solution to the system (3) in closed form is obtained as

$$P(t) = \sum_{n=0}^{k} P(k)t^k = 100 - 44.200t - 2175.731800t^2 - 56687.74153t^3$$
$$- 2.278939978*10^7 t^4 + 1.467445243*10^9 t^5 + \Lambda$$

$$S(t) = \sum_{n=0}^{k} S(k)t^k = 200 - 43798.400t - 1.714600444*10^6 t^2 + 9.112120560*10^8 t^3$$
$$+ 7.351857008*10^{10} t^4 - 2.180288358*10^{13} t^5 + \Lambda$$

$$Iu(t) = \sum_{n=0}^{k} Iu(k)t^k = 140 + 43573.7200t + 1.690468474*10^6 t^2 - 9.049550227*10^8 t^3$$
$$- 7.2867056*10^{10} t^4 + 2.165376880*10^{13} t^5 + \Lambda$$

$$Ie(t) = \sum_{n=0}^{k} I(k)t^k = 120 + 338.7600t + 25367.74734t^2 - 6.226011593*10^6 t^3$$
$$- 6.646692210*10^8 t^4 + 1.482798385*10^5 t^5 + \Lambda$$

$$T(t) = \sum_{n=0}^{k} T(k)t^k = 80 - 7.760t - 924.6823600t^2 + 24886.17805t^3$$
$$- 9.520339790*10^6 t^4 - 6.216417920*10^8 t^5 + \Lambda$$

$$R(t) = \sum_{n=0}^{k} R(k)t^k = 60 + 23.480t + 1.t^2 + 123.4433511t^3$$
$$2484.235566t^4 - 7.616977354*10^5 t^5 + \Lambda$$

4.   Numerical Simulation and Graphical Illustration of the Model.

We demonstrated the numerical simulation which illustrate the analytical results for the proposed model . This is achieved by using some set of parameter values given in the table 3 below and whose source are mainly from literature and as well as assumptions . The DTM is demostrated against mapple buit-in fourth order Runge-Kutta Procedure for the solution of the model. Figures 2 to 7 shows the combined plots of the solutions of $P(t)$ $S(t)$, $I_u(t)$ $I_e(t)$ and $R(t)$ by DTM aand RK4

Table 3. Parameters values for the model

Parameter	Initial Value	Source
Λ	200	Assumed
μ	0.142	Mushayabasa, (2011)
ε	0.4	Kariuki, C. (2011)
γ	0.6	Assumed
ω	0.5	Assumed
τ	0.1	Estimated
α	0.0072	Assumed
β_1	1.5	Kalajdzievska, D. (2011),
β_2	0.05	Assumed
β_3	0.05	Assumed
δ	0.075	Lawi, (2011)
ϕ_1	0.04	Estimated
ϕ_2	0.3	Estimated
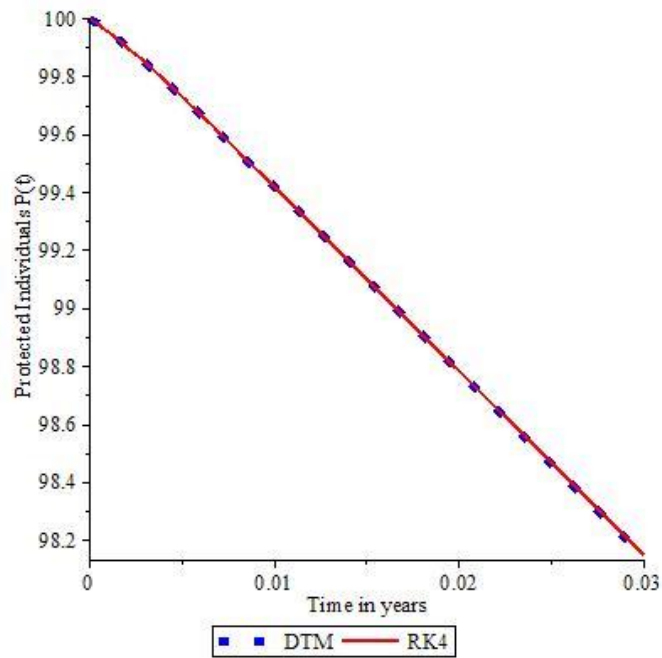
Figure 2. Solution of Protected Population by DTM and RK4
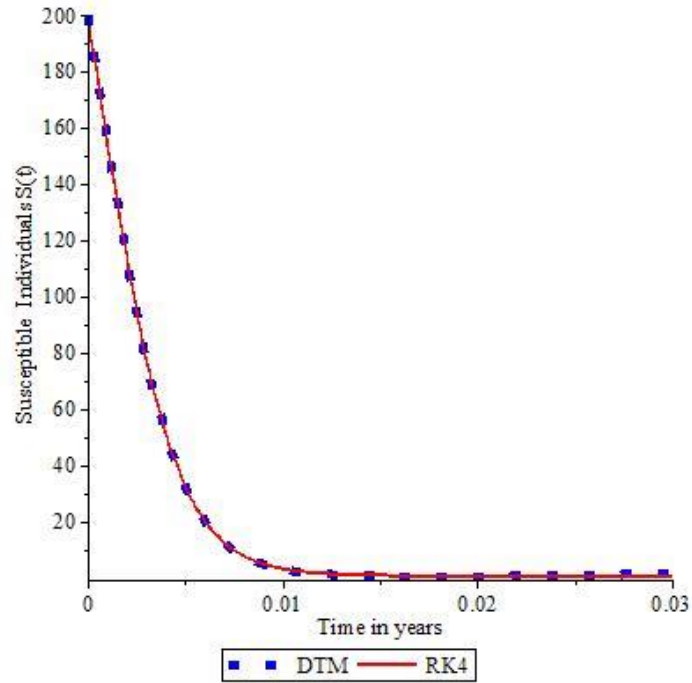
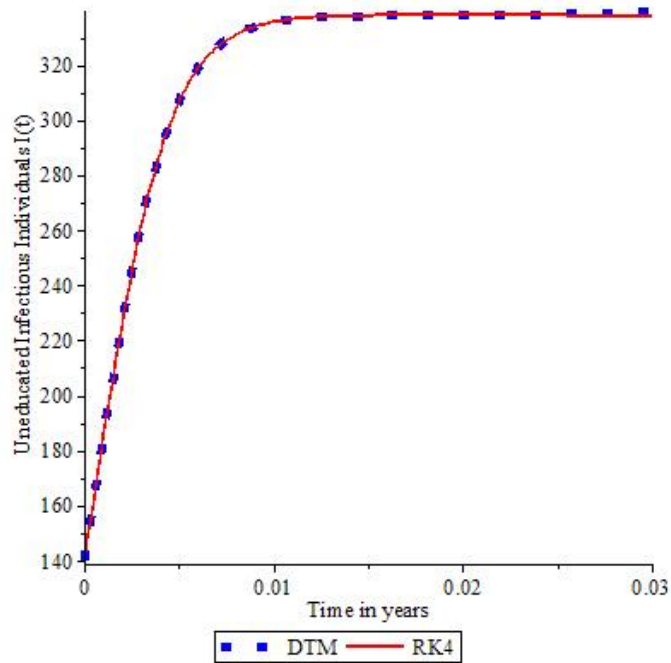Figure 3. Solution of Susceptible Population by DTM and RK4

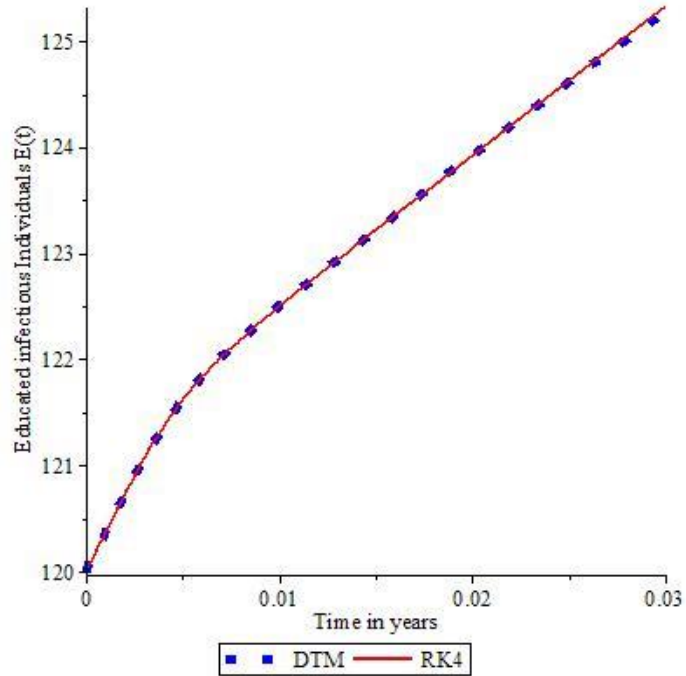Figure 4. Solution of Uneducated Infected Population by DTM and RK4

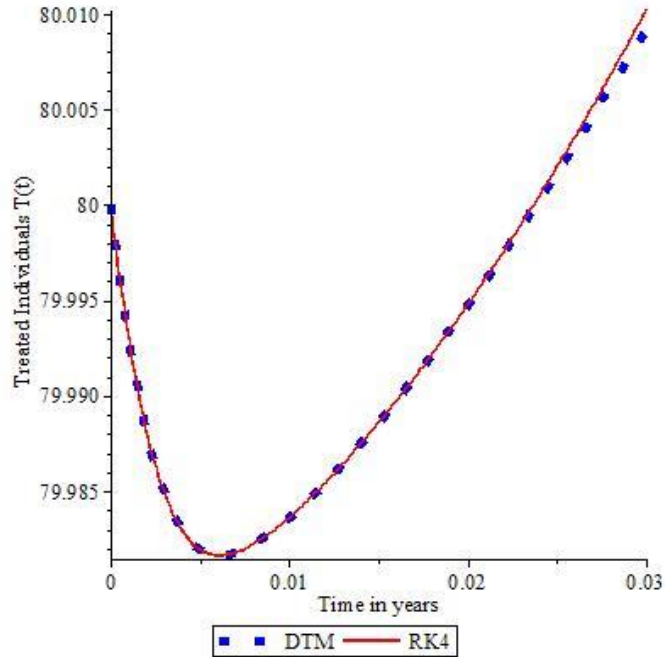Figure 5. Solution of Educated Infected Population by DTM and RK4

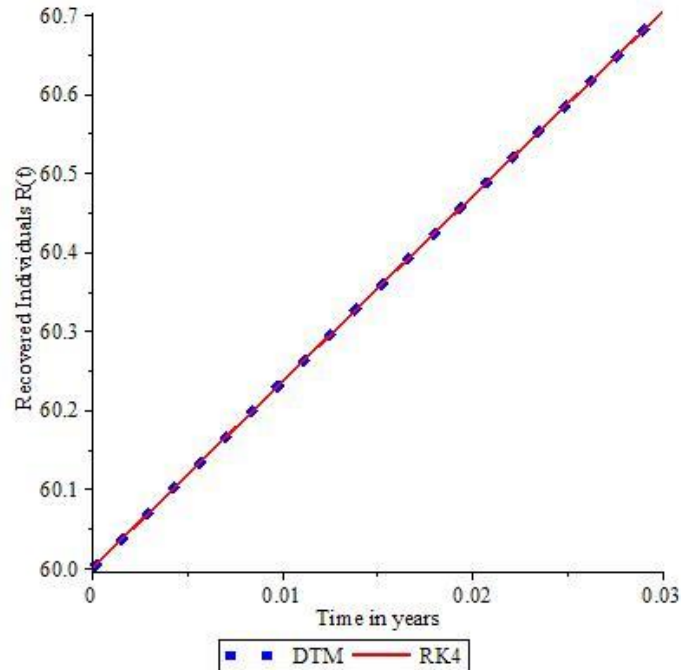Figure 6. Solution of Treated Population by DTM and RK4

Figure 7. Solution of Recovered Population by DTM and RK4

4.1   Discussion of Results

The solutions obtained by using Differential Transform Method with given initial conditions compared favourably with the solution obtained by using classical fouth-other Runge-Kuta method. The solutions of the two methods follows the same pattern and behaviour. This shows that Differential Transform Method is suitable and efficient to conduct the analysis of epidemic models.

5.   Conclusion

In this paper, (DTM) is employed to attempt the series solution of the model. Numerical simulations were carried out to compare the results obtained by (DTM) with the result of classical fourth-order Runge-Kutta method. The results of the simulations were displayed graphically.The results shows that (DTM) is in good agreement with RK-4 and produced accurately the same behaviour thus, validating the reliability of (DTM) in finding the approximate solution of epidemic model.

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