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Affine equivalences, isometries and symmetries of ruled rational surfaces.

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Abstract

An algorithmic method is presented for computing all the affine equivalences between two rational ruled surfaces defined by rational parametrizations. The algorithm works directly in parametric rational form, i.e. without computing or making use of the implicit equation of the surface. The method translates the problem into parameter space, and relies on polynomial system solving. Geometrically, the problem is related to finding the projective equivalences between two projective curves (corresponding to the directions of the rulings of the surfaces). This problem was recently addressed in a paper by Hauer and Jüttler, and we exploit the ideas by these authors in the algorithm presented in this paper. The general idea for affine equivalences is adapted to computing the isometries between two rational ruled surfaces, and the symmetries of a given rational ruled surface. The efficiency of the method is shown through several examples.

1. Introduction

Two surfaces are *affinely equivalent* when one of these surfaces is the result of applying an invertible affine transformation to the other surface. Any invertible affine transformation is a global diffeomorphism, and therefore the transformation preserves both the topology and the differential properties of the surface. Recognizing affine equivalence is important in fields like Computer Vision or Pattern Recognition, where one often has objects stored in a database, and needs to recognize those objects up to some transformation.

While affine transformations do not, in general, preserve metric properties, *isometries* do. Isometries are well classified [10], and comprise translations, central symmetries, reflections in a plane, rotational symmetries (with axial

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symmetries as a special case), and their composites. A *symmetry* of a surface is a symmetry of 3-space that leaves the surface invariant. In particular, symmetries of 3-space are orthogonal transformations.

Additionally, knowing the symmetries of a surface is useful in order to understand the geometry of the surface and to visualize the surface correctly. Symmetries are also useful in applications like image storage and medial axis computations, or again, object detection and recognition. In the literature of applied fields like computer aided geometric design, pattern recognition or computer vision one can find many methods to detect symmetries (see for instance the Introduction to [2]). However, these methods are usually applied to objects where no specific structure is assumed, and are more oriented towards finding approximate symmetries.

The same thing can be said about methods to identify affine equivalences; see for example the paper [17] and the references provided therein. In fact, in applications the problem which has received more attention is the detection of affine equivalences between point clouds, since images and objects are often represented this way.

In contrast, in this paper we address a type of surface with a strong structure, namely rational ruled algebraic surfaces. Additionally, we make use of the structure of these surfaces in order to compute affine equivalences, isometries or symmetries. Ruled surfaces consist of straight lines, and are classical in Differential and Algebraic Geometry. A complete account of many properties of these surfaces is given, for instance, in the books [14, 16].

Some recent publications address similar problems for curves and surfaces. Projective and affine equivalences between rational curves in arbitrary dimension are considered in [12]. The same problem for rational and polynomial surfaces is considered in [13]. However, in [13] the parametrizations of the surfaces are assumed to have no projective base points, which is not the case, in general, for rational ruled surfaces. Projective equivalences between some special varieties are also studied in [7], where ruled surfaces, among others, are considered. In [7] an algorithm to compute projective equivalences, based in a strong background of Algebraic Geometry, is provided. Nevertheless, the authors of [7] are not very specific about computational questions or timings, so it is not easy to compare their methods with the one in this paper.

A naive approach to solve the problem could be to compute first the implicit equations of the surfaces, which can be done efficiently by using μ -bases [8, 9], then pick a generic affine transformation, and finally impose that the transformation maps one surface onto the other. This last step is equivalent to recovering, up to a constant, the implicit equation of the second surface. This approach leads to a polynomial system in 12 variables, the parameters of the affine transformation, where the degree of each polynomial equation is bounded by the degree of the implicit equation. However, this approach is often impractical. On one hand, the system has many variables, which makes it complicated to solve even when the polynomial equations have low total degree. On the other hand, computing the implicit equation may destroy sparsity, when present in the parametrization. The same criticism applies to computing isometries or

symmetries by this method.

In this paper we use a different approach, based on previous work of one of the authors on symmetries of rational curves [5] and polynomially parametrized surfaces [2]. The idea is to reduce the problem to computations in the parameter space, an idea also present in [12, 13]. In order to perform this reduction, we observe that whenever the parametrizations defining the surfaces are *proper*, i.e. birational, any affine equivalence of the surfaces is associated with a birational transformation of the parameter space (the plane), i.e. a Cremona transformation. Taking advantage of the structure of the surfaces, in our case of the fact that the surfaces are ruled, we prove that the corresponding Cremona transformation has a specific form.

This specific form reveals something else, namely that the matrix associated with an affine equivalence corresponds to a projective equivalence between the projective curves defined by the directions of the rulings of the surfaces. From this point, we take advantage of the detailed analysis of the computation of projective equivalences between curves carried out in [12]. In particular, in our case the main difficulty, from a computational point of view, is the solution of a polynomial system in the parameters of the Cremona transformation whose structure is studied in [12]. The core of the computational method we propose to find the affine equivalences of the surfaces is based on ideas in [12].

For isometries or symmetries, the fact that the transformation we are seeking preserves metric properties provides extra conditions that help reduce the computation time.

The structure of this paper is the following. We start with a preliminary section, Section 2, where we fix the hypotheses required on the input, together with some general notions to be used later in this paper. Several results behind the method, and the analysis of the structure of the Cremona transformations behind affine equivalences, are presented in Section 3. The computational method itself is addressed in Section 4, where we apply many results in [12]. We report on experiments in Section 5. Some brief observations on projective equivalences are provided in Section 6. A brief summary of our work is presented in Section 7.

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2. Preliminaries.

Let S_1, S_2 be real ruled surfaces, defined by means of real, rational parametrizations $\mathbf{x}_1, \mathbf{x}_2$ in standard form [15]

$$\mathbf{x}_i(t, s) = \mathbf{p}_i(t) + s \cdot \mathbf{q}_i(t). \quad (1)$$

An algorithm for writing a ruled rational surface in standard form is provided in [15], where it is shown that any ruled rational surface can be brought into an equation like Eq. (1), although the parametrization might not be real (e.g. quadrics). At each point $P = \mathbf{x}_i(t, s) \in S_i$, $i = 1, 2$, the vector $\mathbf{q}_i(t)$ defines the direction of the *ruling* through P , i.e. the line L_P through P contained in S_i .

Furthermore, we will also suppose that S is not doubly-ruled, i.e. that there are not two different families of rulings contained in S . It is well-known that the doubly-ruled surfaces are planes, hyperbolic paraboloids, and single-sheeted hyperboloids (see [11, §I.3]). For paraboloids and hyperboloids one can study affine equivalences by first computing the implicit equation, which is easy to do in the case of quadrics, and then applying matrix methods.

In the rest of this paper we will assume that for $i = 1, 2$, the $\mathbf{x}_i(t, s)$ are *proper*, i.e. that the parametrizations in Eq. (1) are injective except for, at most, a 1-dimensional subset of S_i . In particular, this assumption implies that for $i = 1, 2$, \mathbf{x}_i^{-1} exist and are rational.

Without loss of generality, we assume that the components of $\mathbf{q}_i(t)$ are three polynomials with 1 as greatest common divisor. This assumption, which is technically important, can be achieved by setting

$$\widehat{\mathbf{q}}_i(t) := \mu_i(t)\mathbf{q}_i(t), \quad \mu_i(t) = \frac{\mu_{1,i}(t)}{\mu_{2,i}(t)}, \quad (2)$$

and

$$\widehat{s} := s/\mu_i(t), \quad (3)$$

where $\mu_{1,i}(t)$ is the least common multiple of the denominators of the components of $\mathbf{q}_i(t)$, and $\mu_{2,i}(t)$ is the greatest common divisor of the numerators of the components of $\mathbf{q}_i(t)$. As

$$\mathbf{p}_i(t) + s \cdot \mathbf{q}_i(t) = \mathbf{p}_i(t) + \widehat{s} \cdot \widehat{\mathbf{q}}_i(t),$$

one has a new parametrization that satisfies the assumption. If one parametrization is proper, the other one is also proper, since the change of parameter is invertible.

Two surfaces S_1, S_2 are *affinely equivalent* if there exists an invertible affine mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$,

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}^3, \quad (4)$$

with $\mathbf{b} \in \mathbb{R}^3$ and $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ a nonsingular square matrix of order 3, such that $f(S_1) = S_2$. In this case, the mapping f is an *affine equivalence* between S_1, S_2 .

If \mathbf{A} is an orthogonal matrix, i.e. $\mathbf{A}^T \mathbf{A} = I$, where I denotes the 3×3 identity matrix, then f defines an *isometry* between S_1, S_2 , in which case S_1, S_2 are *isometric*. If $\mathbf{A} = \lambda \mathbf{Q}$ where \mathbf{Q} is orthogonal and $\lambda \neq 0$, f defines a *similarity* between S_1, S_2 , and S_1, S_2 are *similar*. Furthermore, if $S_1 = S_2$ and f defines a non-trivial isometry of S onto itself, then f is a *symmetry* of S , and S is *symmetric*.

Finally, $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an *involution* if $f \circ f = \text{id}_{\mathbb{R}^3}$. Not every involution is affine, but when we consider $S_1 = S_2 = S$ certain notable symmetries of S are affine involutions. In particular, reflections in a plane, axial symmetries (i.e. symmetries with respect to a line, or equivalently rotations around a line of angle π), and central symmetries (i.e. symmetries with respect to a point) are affine involutions. Rotational symmetries, i.e. rotations leaving S invariant, with a rotation angle different from π , however, are not involutions.

3. Affine equivalences of ruled surfaces.

Let S_1, S_2 be real rational ruled surfaces parametrized by $\mathbf{x}_1, \mathbf{x}_2$ as in Eq. (1), satisfying the conditions of Section 2, i.e. for $i = 1, 2$ we assume that: $\mathbf{x}_i(t, s)$ is proper; $\mathbf{q}_i(t)$ is polynomial with relatively prime components; S_i is not doubly-ruled. Our goal in this section is to present a method to detect whether S_1, S_2 are affinely equivalent, and in the affirmative case to compute the affine equivalences between S_1, S_2 . In order to develop our method we will assume that S_1, S_2 are not cylindrical. Nevertheless, we will address cylindrical surfaces in Subsection 4.2. Notice that our hypotheses on $\mathbf{q}_i(t)$ imply that S_i is cylindrical if and only if $\mathbf{q}_i(t)$ is constant.

The following result is crucial for us.

Theorem 1. *Let S_1, S_2 be two rational real ruled surfaces properly parametrized by $\mathbf{x}_1, \mathbf{x}_2$ as in Eq. (1). A mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, with $\mathbf{A} \in \mathbb{R}^{3 \times 3}$, $\mathbf{b} \in \mathbb{R}^3$ and \mathbf{A} nonsingular, satisfies $f(S_1) = S_2$, so that S_1, S_2 are affinely equivalent, if and only if there exists a birational transformation $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that the diagram*

$$\begin{array}{ccc}
 S_1 & \xrightarrow{f} & S_2 \\
 \uparrow & & \uparrow \\
 \mathbf{x}_1 & & \mathbf{x}_2 \\
 \downarrow & & \downarrow \\
 \mathbb{R}^2 & \xrightarrow{\varphi} & \mathbb{R}^2
 \end{array} \tag{5}$$

is commutative. In particular, for a generic point $(t, s) \in \mathbb{R}^2$

$$f \circ \mathbf{x}_1 = \mathbf{x}_2 \circ \varphi. \tag{6}$$

Proof. “ \Rightarrow ” Since \mathbf{x}_2 is proper by hypothesis, \mathbf{x}_2^{-1} exists and is rational. Therefore, $\varphi = \mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1$ is birational, because φ is the composite of birational transformations. “ \Leftarrow ” Since $f \circ \mathbf{x}_1 = \mathbf{x}_2 \circ \varphi$, whenever $\mathbf{x}_1(t, s)$ and $(\mathbf{x}_2 \circ \varphi)(t, s)$ are well-defined ($f \circ \mathbf{x}_1(t, s) \in S_2$, so $f(S_1) \subset S_2$). Since f is nonsingular, $f(S_1)$

defines a rational surface, i.e. $f(S_1)$ does not degenerate into a curve. Additionally $f(S_1), S_2$ are both rational, and therefore irreducible; since $f(S_1) \subset S_2$ and $f(S_1), S_2$ are irreducible, $f(S_1) = S_2$, i.e. S_1, S_2 are affinely equivalent. \square

Additionally, from Eq. (6) one can easily see that each affine mapping f is associated with a different φ .

From Theorem 1 we observe that φ is a birational transformation of the plane. Such a transformation is called a *Cremona transformation*. However, unlike the birational transformations of the line, which are the well-known *Möbius transformations*, i.e. the transformations of the type

$$\psi : \mathbb{R} \dashrightarrow \mathbb{R}, \quad \psi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \quad (7)$$

Cremona transformations do not have a generic closed form. Therefore, in order to describe φ , we need to make use of the properties of the surfaces we are investigating, in this case, the fact that they are ruled. The following result provides a first clue in this direction.

Proposition 2. *Let S_1, S_2 be rational ruled surfaces properly parametrized as in Eq. (1), which are not doubly ruled. Let $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ be a nonsingular affine mapping satisfying $f(S_1) = S_2$, and let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the birational transformation making the diagram in Eq. (5) commutative. Then*

$$\varphi(t, s) = (\psi(t), a(t) \cdot s + c(t)), \quad (8)$$

where $\psi(t)$ is a Möbius transformation and $a(t), c(t)$ are rational functions.

Proof. Since f is an affine mapping, f maps rulings of S_1 onto rulings of S_2 . Let $\varphi(t, s) = (\varphi_1(t, s), \varphi_2(t, s))$. A generic ruling of S_i , with $i = 1, 2$ is defined by $\mathbf{x}_i(t_{a_i}, s)$, where t_{a_i} is a constant. Since S_2 is not doubly ruled, the ruling parametrized by $\mathbf{x}_1(t_{a_1}, s)$ is mapped by f onto the ruling parametrized by $\mathbf{x}_2(t_{a_2}, s)$. Using Eq. (6), we get

$$f(\mathbf{x}_1(t_{a_1}, s)) = \mathbf{x}_2(\varphi(t_{a_1}, s)) = \mathbf{x}_2(\varphi_1(t_{a_1}, s), \varphi_2(t_{a_1}, s)),$$

so $\varphi_1(t_{a_1}, s) = t_{a_2}$, i.e. $\varphi_1(t_{a_1}, s)$ does not depend on s . Since this independence happens for a generic t_{a_1} , we deduce that $\varphi_1(t, s) = \varphi_1(t)$. Since φ is birational, φ_1 is birational as well; in particular, we deduce that φ_1 is a birational transformation of the line, so φ_1 must be a Möbius transformation, which we represent by $\psi(t)$. The rest of the theorem follows from Eq. (6), taking into account that $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$. \square

Let us now investigate the structure of the function $a(t)$ in Eq. (8). Recall that $\mathbf{x}_i(t, s) = \mathbf{p}_i(t) + s \cdot \mathbf{q}_i(t)$, where $\mathbf{q}_i(t) = (q_{i,1}(t), q_{i,2}(t), q_{i,3}(t))$, each $q_{i,j}(t)$ is polynomial and $\gcd(q_{i,1}, q_{i,2}, q_{i,3}) = 1$. Also, let

$$n_i = \max\{\deg(q_{i,1}(t)), \deg(q_{i,2}(t)), \deg(q_{i,3}(t))\}, \quad (9)$$

and let us write

$$a(t) = \frac{A(t)}{B(t)}, \quad \psi(t) = \frac{\alpha t + \beta}{\gamma t + \delta},$$

where $A, B \in \mathbb{R}[t]$, $\gcd(A, B) = 1$, and $\alpha\delta - \beta\gamma \neq 0$. Combining Eq. (8) and Eq. (6) with $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, and comparing the coefficients of s , we get

$$\mathbf{A} \cdot \mathbf{q}_1(t) = a(t) \cdot \mathbf{q}_2(\psi(t)). \quad (10)$$

Since $\mathbf{q}_i(t)$, $i = 1, 2$, is polynomial, the left hand-side of Eq. (10) is polynomial as well, so the right hand-side of Eq. (10) must also be polynomial. This observation yields the following results; here, we denote the entries of the matrix \mathbf{A} by \mathbf{A}_{ij} .

Lemma 3. $(\gamma t + \delta)^{n_2}$ divides $A(t)$.

Proof. From Eq. (10), for $i = 1, 2, 3$ we get

$$\mathbf{A}_{i1} \cdot q_{1,1}(t) + \mathbf{A}_{i2} \cdot q_{1,2}(t) + \mathbf{A}_{i3} \cdot q_{1,3}(t) = a(t) \cdot q_{2,i}(\psi(t)), \quad (11)$$

where $q_{2,i}(t) = a_{\ell_i} t^{\ell_i} + a_{\ell_i-1} t^{\ell_i-1} + \dots + a_0$, with $\ell_i \leq n_2$ for $i \in \{1, 2, 3\}$. Furthermore, $\ell_i = n_2$ for at least one $i \in \{1, 2, 3\}$. Additionally,

$$q_{2,i}(\psi(t)) = \frac{a_{\ell_i} (\alpha t + \beta)^{\ell_i} + a_{\ell_i-1} (\alpha t + \beta)^{\ell_i-1} (\gamma t + \delta) + \dots + a_0 (\gamma t + \delta)^{\ell_i}}{(\gamma t + \delta)^{\ell_i}}. \quad (12)$$

Since $\gamma t + \delta$ does not divide $\alpha t + \beta$, the numerator and denominator of $q_{2,i}(\psi(t))$ are relatively prime. Since the left hand-side of Eq. (11) is a polynomial, $a(t) \cdot q_{2,i}(\psi(t))$ must be a polynomial as well, so $(\gamma t + \delta)^{\ell_i}$ divides $A(t)$. Since $\ell_i = n_2$ for some $i \in \{1, 2, 3\}$, the statement follows. \square

Lemma 4. $B(t)$ is a constant.

Proof. Recall that $\gcd(q_{2,1}, q_{2,2}, q_{2,3}) = 1$. Let $N_i(t)$ be the numerator of $q_{2,i}(\psi(t))$. Since the left hand-side of Eq. (11) is a polynomial, $B(t) | N_i(t)$ for $i = 1, 2, 3$. Thus, $B(t) | G(t)$, where $G = \gcd(N_1, N_2, N_3)$. Now suppose that $G(t)$ is not constant. Then N_1, N_2, N_3 have a common root t_0 . Moreover, since the numerators and denominators of the $q_{2,i}(\psi(t))$ are relative prime, $\gamma t_0 + \delta \neq 0$. Therefore, $\psi(t_0)$ is well defined and $\psi(t_0)$ is a common root of the $q_{2,i}(t)$, because $q_{2,i}(\psi(t_0)) = \frac{N_i(t_0)}{(\gamma t_0 + \delta)^{\ell_i}}$. But this contradicts the fact that $\gcd(q_{2,1}, q_{2,2}, q_{2,3}) = 1$. Thus, $G(t)$ is constant and since $B(t) | G(t)$, $B(t)$ must be a constant. \square

Finally, we get the following proposition on the form of the function $a(t)$.

Proposition 5. *The function $a(t)$ satisfies $a(t) = k \cdot (\gamma t + \delta)^{n_2}$, where k is a nonzero constant.*

Proof. From the two previous lemmas we have $a(t) = k(t) \cdot (\gamma t + \delta)^{n_2}$ for some polynomial $k(t)$. Additionally, from Eq. (10)

$$\mathbf{A} \cdot \mathbf{q}_1(t) = k(t) \cdot (\gamma t + \delta)^{n_2} \cdot \mathbf{q}_2(\psi(t)). \quad (13)$$

Taking Eq. (12) into account, we observe that $(\gamma t + \delta)^{n_2} \cdot \mathbf{q}_2(\psi(t))$ is polynomial. If $k(t)$ is not a constant, then the components of $\mathbf{A} \cdot \mathbf{q}_1(t)$ are not relatively prime, i.e. $\mathbf{A} \cdot \mathbf{q}_1(t) = r(t)\mathbf{q}_1^*(t)$, with $r(t)$ nonconstant, and $\mathbf{q}_1^*(t)$ a polynomial parametrization with relatively prime components. However, since \mathbf{A} is nonsingular, in that case we have $\mathbf{q}_1(t) = r(t)\mathbf{A}^{-1}\mathbf{q}_1^*(t)$, which implies that the components of $\mathbf{q}_1(t)$ are not relatively prime either. Since by hypothesis the components of $\mathbf{q}_1(t)$ are relatively prime, $k(t)$ must be a constant k . Finally, since \mathbf{A} is nonsingular, from Eq. (13) we get that $k \neq 0$. \square

Taking Proposition 5 and Eq. (13) into account, we get the following corollary.

Corollary 6. *If S_1, S_2 are affinely equivalent, then $n_1 = n_2$.*

We summarize the previous results in the following theorem. In the rest of this paper, we denote, according to Corollary 6, $n_1 = n_2 = n$.

Theorem 7. *Let S_1, S_2 be two rational ruled surfaces, which are not doubly ruled, properly parametrized as in Eq. (1). Let $\mathbf{q}_i(t) = (q_{i,1}(t), q_{i,2}(t), q_{i,3}(t))$, with $q_{i,j}(t) \in \mathbb{R}[t]$ for $i = 1, 2$ and $j = 1, 2, 3$, and $n_1 = n_2 = n$. Let $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, with \mathbf{A} nonsingular, such that $f(S_1) = S_2$, and let $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the birational transformation making the diagram in Eq. (5) commutative. Then*

$$\varphi(t, s) = (\psi(t), k \cdot (\gamma t + \delta)^n \cdot s + c(t)), \quad (14)$$

where $\psi(t)$ is a Möbius transformation, k is a constant, and $c(t)$ is a rational function. Moreover,

$$\mathbf{A} \cdot \mathbf{q}_1(t) = k \cdot (\gamma t + \delta)^n \cdot \mathbf{q}_2(\psi(t)). \quad (15)$$

Eq. (15) can be interpreted in geometric terms. In order to do this, it is preferable to write Eq. (15) projectively. Let $\tilde{\mathbf{q}}_i(t, \omega) = [q_{i,1}(t, \omega) : q_{i,2}(t, \omega) : q_{i,3}(t, \omega)] \in \mathbb{P}^2(\mathbb{R})$, where $i = 1, 2$ and ω is a homogenization variable. Then Eq. (15) can be written as

$$\mathbf{A} \cdot \tilde{\mathbf{q}}_1(t, \omega) = k \cdot \tilde{\mathbf{q}}_2(\alpha t + \beta \omega, \gamma t + \delta \omega). \quad (16)$$

This means (see Section 3 of [12]) that the projective curves defined by $\tilde{\mathbf{q}}_1(t, \omega)$ and $\tilde{\mathbf{q}}_2(t, \omega)$ are *projectively equivalent*, and even more, that \mathbf{A} defines a projectivity mapping the projective curve defined by $\tilde{\mathbf{q}}_1(t, \omega)$ onto the projective curve defined by $\tilde{\mathbf{q}}_2(t, \omega)$ (or $k \cdot \tilde{\mathbf{q}}_2(t, \omega)$, since projectively $\tilde{\mathbf{q}}_2(t, \omega)$ and $k \cdot \tilde{\mathbf{q}}_2(t, \omega)$ can be identified). This observation makes perfect sense from a geometric point of view: affine equivalences map rulings of S_1 onto rulings of S_2 , as observed in the proof of Proposition 2, and $\tilde{\mathbf{q}}_1(t, \omega)$, $\tilde{\mathbf{q}}_2(t, \omega)$ define the *directions* of these rulings. The matrix \mathbf{A} defines the map sending the direction of each ruling of S_1 onto the direction of a ruling of S_2 .

Projective equivalences between curves in any dimension, and in particular systems of equations like Eq. (16) (and therefore Eq. (15)) are studied in great

detail in [12]. We will benefit from the study carried out in [12] in the next section, where we address the computation of the affine equivalences between S_1, S_2 .

Finally, from Eq. (1), Eq. (6), and Eq. (14) we get the relationship

$$\mathbf{A}\mathbf{p}_1(t) + \mathbf{b} = \mathbf{p}_2(\psi(t)) + c(t)\mathbf{q}_2(\psi(t)). \quad (17)$$

We will see how to exploit Eq. (15) and Eq. (17) in the next section.

4. Computation of the affine equivalences.

The computation of the affine equivalences between S_1, S_2 is based on the following result, which in turn follows from the results of the previous section.

Proposition 8. *The affine equivalences $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ between S_1, S_2 correspond to the $\mathbf{A} \in \mathbb{R}^{3 \times 3}$, $\mathbf{b} \in \mathbb{R}^3$ satisfying Eq. (15) and Eq. (17), where $\det(\mathbf{A}) \neq 0$, $k \neq 0$, $\psi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}$ and $\alpha\delta - \beta\gamma \neq 0$.*

Notice that since the components of $\mathbf{q}_2(t)$ are polynomials of degree at most n , Eq. (15) involves only polynomials, and provides equations which are linear in the entries \mathbf{A}_{ij} of the matrix \mathbf{A} . Furthermore, the coefficients of the \mathbf{A}_{ij} in these linear equations are constants, while the constant terms of these linear equations depend on $\alpha, \beta, \gamma, \delta$ and k . However, Eq. (17) involves rational functions, i.e. polynomial denominators. Additionally, since $\alpha\delta - \beta\gamma \neq 0$ we can always assume either that $\alpha\delta - \beta\gamma = 1$, or separate the analysis in two different cases, namely the case $\gamma = 1$, and the case $\gamma = 0$, $\delta = 1$. This last possibility allows us to perform the computation with fewer variables (although twice).

The computation proceeds in three different steps, (A), (B), (C). Let us describe these steps in detail.

(A) *Writing \mathbf{A} in terms of $\alpha, \beta, \gamma, \delta$, and k .* At this step we exploit Eq. (15), which has been studied in great detail in Section 3 of [12]. Writing Eq. (15) in components, we get

$$\begin{cases} \mathbf{A}_{11} \cdot q_{1,1}(t) + \mathbf{A}_{12} \cdot q_{1,2}(t) + \mathbf{A}_{13} \cdot q_{1,3}(t) & = k(\gamma t + \delta)^n q_{2,1}(\psi(t)), \\ \mathbf{A}_{21} \cdot q_{1,1}(t) + \mathbf{A}_{22} \cdot q_{1,2}(t) + \mathbf{A}_{23} \cdot q_{1,3}(t) & = k(\gamma t + \delta)^n q_{2,2}(\psi(t)), \\ \mathbf{A}_{31} \cdot q_{1,1}(t) + \mathbf{A}_{32} \cdot q_{1,2}(t) + \mathbf{A}_{33} \cdot q_{1,3}(t) & = k(\gamma t + \delta)^n q_{2,3}(\psi(t)). \end{cases} \quad (18)$$

Since the $q_{2,j}(t)$ have degree at most n , the expressions on the right hand-side of Eq. (18) are, in fact, polynomials. Equating the coefficients of t^ℓ , for $\ell = 0, 1, \dots, n$, on both sides of Eq. (18), we get a system \mathcal{L} , linear in the \mathbf{A}_{ij} , where the coefficients of the \mathbf{A}_{ij} are constant numbers, and where the constant terms are polynomials in $\alpha, \beta, \gamma, \delta$ and k . Let us write $\mathbf{q}_1(t)$ as

$$\mathbf{q}_1(t) = \mathbf{v}_0 + \mathbf{v}_1 t + \dots + \mathbf{v}_n t^n, \quad (19)$$

where $\mathbf{v}_\ell \in \mathbb{R}^3$, for $\ell = 0, \dots, n$, is a numeric row vector whose components are the coefficients in t^ℓ of $q_{1,1}(t), q_{1,2}(t)$ and $q_{1,3}(t)$. Then the system \mathcal{L} has the form:

rameters in terms of the others, we can reduce the number of parameters in the subsequent computations.

Notice that when $n = 2$, we get $3n - 6 = 3 \cdot 2 - 6 = 0$, so no extra conditions in $\alpha, \beta, \gamma, \delta$ are generated. However, we can still write the \mathbf{A}_{ij} in terms of $\alpha, \beta, \gamma, \delta$ and k .

- (2) **Case $r = 2$:** in this case, since $r = 2$ applying the Gauss-Jordan method to the system \mathcal{L} we get $3(n + 1) - 3 \cdot 2 = 3n - 3$ additional conditions on $\alpha, \beta, \gamma, \delta, k$ that must hold for \mathcal{L} to be consistent, with the same properties as in the previous case. As before, we denote the collection of all these polynomial conditions by $P_{\mathcal{A}}$. However, since $\text{rank}(\mathcal{A}) = 6$ is less than the number of \mathbf{A}_{ij} , we cannot, in this case, write all the \mathbf{A}_{ij} only in terms of $\alpha, \beta, \gamma, \delta, k$, i.e. three of the \mathbf{A}_{ij} must be considered as parameters as well. This observation makes sense from a geometric point of view: if $r = 2$ then $\mathbf{q}_1(t), \mathbf{q}_2(t)$ parametrize projective lines, and there are infinitely many projective transformations mapping a projective line onto another projective line.

Observe that when the components of $\mathbf{q}_1(t)$ are linear we are always either in the case $r = 1$, or in the case $r = 2$. In this last case, we do not get any extra conditions on $\alpha, \beta, \gamma, \delta$, because since $n = 1$ the number $3n - 3$ of extra conditions vanishes.

Summarizing, at this step we write either all the \mathbf{A}_{ij} , when $r = 3$, or only six of the \mathbf{A}_{ij} , when $r = 2$, in terms of $\alpha, \beta, \gamma, \delta, k$. Furthermore, except in the case $r = 3, n = 2$ and the case $n = 1$, we get polynomial conditions on $\alpha, \beta, \gamma, \delta$, which may help either to detect that the surfaces are not affine equivalent (when these conditions are not compatible), or to reduce the number of parameters.

(B) *Writing \mathbf{b} in terms of $\alpha, \beta, \gamma, \delta$, and k , and computing $c(t)$.* Writing Eq. (17) in components, we get

$$\begin{cases} \mathbf{A}_{11} \cdot p_{1,1}(t) + \mathbf{A}_{12} \cdot p_{1,2}(t) + \mathbf{A}_{13} \cdot p_{1,3}(t) + b_1 & = p_{2,1}(\psi(t)) + c(t)q_{2,1}(\psi(t)), \\ \mathbf{A}_{21} \cdot p_{1,1}(t) + \mathbf{A}_{22} \cdot p_{1,2}(t) + \mathbf{A}_{23} \cdot p_{1,3}(t) + b_2 & = p_{2,2}(\psi(t)) + c(t)q_{2,2}(\psi(t)), \\ \mathbf{A}_{31} \cdot p_{1,1}(t) + \mathbf{A}_{32} \cdot p_{1,2}(t) + \mathbf{A}_{33} \cdot p_{1,3}(t) + b_3 & = p_{2,3}(\psi(t)) + c(t)q_{2,3}(\psi(t)), \end{cases} \quad (21)$$

where we assume that the \mathbf{A}_{ij} , or some of the \mathbf{A}_{ij} , have already been written in terms of $\alpha, \beta, \gamma, \delta, k$. Now we proceed as follows:

- (i) Eliminating $c(t)$ between the first and second equations of Eq. (21) provides an equation E_1 linear in b_1, b_2 , with coefficients that are rational functions of t .
- (ii) Proceeding in the same way with the second and third equations, we get an equation E_2 , linear in b_2, b_3 .
- (iii) Evaluating E_1 and E_2 at several random t -values we get a linear system in b_1, b_2, b_3 , whose solution provides \mathbf{b} .

(iv) Finally, we compute $c(t)$ from any equation of Eq. (21).

We will refer later to this procedure as “the steps (i)-(iv)”.

(C) *Deriving a polynomial system \mathcal{S} , and computing the affine equivalences.* Substituting the expressions for \mathbf{A} , \mathbf{b} and $c(t)$ computed in steps (A) and (B) into Eq. (6), we get a polynomial system \mathcal{S} . If $r = 3$, the unknowns of \mathcal{S} are, at most, $k, \alpha, \beta, \gamma, \delta$, and we can have fewer unknowns if the polynomial conditions $P_{\mathcal{A}}$ in step (A) allow us to write some of these variables in terms of the others. If $r = 2$, we can have at most three more unknowns besides $k, \alpha, \beta, \gamma, \delta$, namely three of the \mathbf{A}_{ij} . Again, the polynomial system $P_{\mathcal{A}}$ may help reduce the total number of parameters, and therefore of unknowns in \mathcal{S} . Thus, the number of unknowns in \mathcal{S} is ≤ 5 , if $r = 3$, and ≤ 8 , if $r = 2$.

The solutions of this polynomial system provide the affine equivalences between S_1, S_2 . We summarize the whole procedure to find the affine equivalences between S_1, S_2 in Algorithm **Affine-Eq-Ruled**.

Algorithm 1 Affine-Eq-Ruled

Require: Two ruled surfaces S_1, S_2 , properly parametrized by $\mathbf{x}_i(t, s) = \mathbf{p}_i(t) + s\mathbf{q}_i(t)$, $i = 1, 2$, where each $\mathbf{q}_i(t)$ is polynomial with relatively prime components of degree $\leq n$.

Ensure: The affine equivalences $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$ between S_1, S_2 .

- 1: Compute the system \mathcal{L} in Eq. (20).
 - 2: Apply the Gauss-Jordan method on the system \mathcal{L} .
 - 3: **if** $r = 3$ and $n \geq 3$, or $r = 2$ and $n \geq 2$ **then**
 - 4: solve the polynomial system $P_{\mathcal{A}}$ in $\alpha, \beta, \gamma, \delta$.
 - 5: **if** $P_{\mathcal{A}}$ is not consistent **then**
 - 6: **return** S_1 and S_2 are not affinely equivalent, and **stop**
 - 7: **end if**
 - 8: **end if**
 - 9: Solve the system \mathcal{L}
 - 10: Write the solutions of \mathcal{L} with as few variables as possible, using, if any, the solutions of $P_{\mathcal{A}}$
 - 11: Follow steps (i)-(iv) to write \mathbf{b} in terms of the variables in step 10, and to compute $c(t)$
 - 12: Substitute $\mathbf{A}, \mathbf{b}, c(t)$ and the φ in Eq. (14) into Eq. (6)
 - 13: Derive from the preceding substitution a polynomial system \mathcal{S} in the parameters appearing in step 9
 - 14: **if** no solution is found **then**
 - 15: **return** S_1 and S_2 are not affinely equivalent.
 - 16: **else**
 - 17: **for** each solution found **do**
 - 18: compute the corresponding mapping $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$
 - 19: **end for**
 - 20: **end if**
-

4.1. The special case of conical surfaces.

We say that S is a *conical* surface if all the rulings of S intersect at a point $\mathbf{p}_0 \in S$, called the *vertex*. The vertex can be computed by using the results in [3]. By applying a translation if necessary, we can always assume that \mathbf{p}_0 is the origin. Therefore, if S is rational and properly parametrized we can assume that S is given by means of a parametrization $\mathbf{x}(t, s) = s\mathbf{q}(t)$, where $\mathbf{q}(t)$ is polynomial.

Now given two rational conical surfaces S_1, S_2 parametrized by $\mathbf{x}_i(t, s) = s\mathbf{q}_i(t)$, with $\mathbf{q}_i(t)$ polynomial for $i = 1, 2$, any affine equivalence between S_1, S_2 has the form $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, so $\mathbf{b} = \mathbf{0}$. Since $p_1(t), p_2(t)$ are identically zero, the function $c(t)$ is identically zero as well, and therefore Eq. (17) is reduced to $0 = 0$. Thus, the computation of the affine equivalences between S_1, S_2 reduces to solving Eq. (15). Notice as well that the system derived from Eq. (15) is homogeneous in k and the entries of the matrix \mathbf{A} , which implies that \mathbf{A} is defined only up to a multiplicative constant. This observation makes perfect sense, since any conical surface is invariant by homotheties where the homothety center is the vertex.

4.2. The special case of cylindrical surfaces.

Under the assumption that $\mathbf{q}_1(t), \mathbf{q}_2(t)$ are polynomials with relatively prime components, S_1, S_2 are cylindrical iff the $\mathbf{q}_i(t)$ are constant vectors. These vectors define the directions of all the rulings of S_1, S_2 . Then in order to check whether S_1, S_2 are affinely equivalent, it suffices to check whether the planar curves $\mathcal{C}_1, \mathcal{C}_2$, obtained by intersecting S_1, S_2 with planes Π_1, Π_2 respectively normal to $\mathbf{q}_1(t), \mathbf{q}_2(t)$, are affinely equivalent. This can be done, for instance, by using the algorithm in [12]. Notice that the affine equivalences of S_1, S_2 are, in this case, the affine equivalences of the plane sections followed by any translations along the direction of the rulings of S_2 , and any dilatation in the same direction.

4.3. Computing isometries and symmetries.

Let us address now the case when the affine mapping $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is an isometry, that is, \mathbf{A} is an orthogonal matrix. For finding the isometries between S_1, S_2 we can certainly apply Algorithm **Affine-Eq-Ruled**, with the extra condition that \mathbf{A} is orthogonal. However, in this case we have additional conditions, which may be an advantage for simplifying the computation. Indeed, since orthogonal mappings preserve norms, taking norms in Eq. (10), with k a constant, we get the condition

$$\|\mathbf{q}_1(t)\|^2 - k^2 \cdot (\gamma t + \delta)^{2n} \cdot \|\mathbf{q}_2(\psi(t))\|^2 = 0. \quad (22)$$

Equating to zero all the coefficients in t at the left hand-side of Eq. (22), we get a polynomial system \mathcal{P} of $2n + 1$ equations, each one consisting of a homogeneous polynomial of degree $2n$ in the variables $\alpha, \beta, \gamma, \delta$ multiplied by k^2 , plus a constant. These equations have a higher degree than the equations of the polynomial system $P_{\mathcal{A}}$, all of degree n . However, collecting the equations

in $P_{\mathcal{A}}$ and \mathcal{P} provides a bigger polynomial system in $\alpha, \beta, \gamma, \delta, k$, which may help to reduce the total number of parameters in the polynomial system \mathcal{S} , and/or the number of tentative values for $\alpha, \beta, \gamma, \delta, k$. In particular, in the cases $r = 3, n = 2$ and $n = 1$ applying Algorithm **Affine-Eq-Ruled** does not provide extra conditions on $\alpha, \beta, \gamma, \delta, k$; however, Eq. (22) does.

If $S_1 = S_2 = S$, the isometries leaving S invariant are the symmetries of S . We can find the symmetries of S by proceeding as before with $S_1 = S_2$. However, recall from Section 2 that certain notable symmetries, like central symmetries, axial symmetries and reflections in a plane, are affine involutions, i.e. affine mappings f satisfying $f \circ f = \text{id}_{\mathbb{R}^3}$. If we are interested only in affine involutions (isometric, or non-isometric) we can improve the computation as follows. First, from Eq. (6) one can see that $f \circ f = \text{id}_{\mathbb{R}^3}$ iff the corresponding φ satisfies $\varphi \circ \varphi = \text{id}_{\mathbb{R}^2}$. By Theorem 7, one has

$$\varphi(t, s) = (\varphi_1(t, s), \varphi_2(t, s)) = (\psi(t), s \cdot k(\gamma t + \delta)^n + c(t)),$$

and the condition $(\varphi \circ \varphi)(t, s) = (t, s)$ introduces two constraints:

- (i) $(\varphi_1 \circ \varphi_1)(t, s) = t$, i.e. $(\psi \circ \psi)(t) = t$. In turn, this constraint implies that

$$\alpha^2 - \delta^2 = 0, \quad \beta(\alpha + \delta) = 0, \quad \gamma(\alpha + \delta) = 0.$$

Therefore, either $\alpha = -\delta$, or $\alpha + \delta \neq 0$ and $\alpha = \delta, \beta = \gamma = 0$.

- (ii) $\varphi_2(\varphi_1(t), \varphi_2(t, s)) = s$, which implies

$$[s \cdot k(\gamma t + \delta)^n + c(t)] \cdot k \cdot \left[\gamma \cdot \frac{\alpha t + \beta}{\gamma t + \delta} + \delta \right]^n + c(\psi(t)) = s.$$

Comparing coefficients of s , we deduce that

$$k^2 \cdot [\gamma(\alpha + \delta)t + (\gamma\beta + \delta^2)]^n = 1,$$

which in turn yields

$$\gamma(\alpha + \delta) = 0, \quad k^2(\gamma\beta + \delta^2)^n = 1.$$

Thus, either $\alpha = -\delta$ and $k^2(\gamma\beta + \delta^2)^n = 1$, or $\alpha = \delta, \gamma = 0$ and $k^2\delta^{2n} = 1$.

Putting (i) and (ii) together, we get the following result, which allows decreasing the total number of parameters, and therefore of unknowns in the polynomial system \mathcal{S} . Notice that this result is applicable to any affine involution (in particular, isometric involutions).

Theorem 9. *Let S be a rational ruled surface, which is not doubly ruled, properly parametrized as in Eq. (1). Let $\mathbf{q}(t) = (q_1(t), q_2(t), q_3(t))$, with $q_i(t) \in \mathbb{R}[t]$ for $i = 1, 2, 3$, and*

$$n = \max\{\deg(q_1(t)), \deg(q_2(t)), \deg(q_3(t))\}.$$

Finally, let $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, with $\mathbf{A} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{R}^3$, be an affine involution leaving S invariant. With the notation of Theorem 7, one has:

(I) $\alpha = -\delta$, $k^2(\gamma\beta + \delta^2)^n = 1$, or

(II) $\varphi(t, s) = (t, -s + c(t))$, with $c(t)$ a rational function.

Observe that in case (II) f fixes each line of the ruling, and acts on these lines as an affine involution.

Remark 1. Since any similarity can be written as $f(\mathbf{x}) = \lambda\mathbf{Q} + \mathbf{b}$, where $\lambda \neq 0$ is the scaling constant, taking norms in Eq. (10), with k a constant, we get the condition

$$\lambda^2 \|\mathbf{q}_1(t)\|^2 - k^2 \cdot (\gamma t + \delta)^{2n_2} \cdot \|\mathbf{q}_2(\psi(t))\|^2 = 0. \quad (23)$$

The analysis in this case is very similar to that of isometries, although the polynomial system has one more variable, namely λ .

4.4. Two examples

We illustrate the previous ideas in the following examples, one corresponding to the case $r = 3$, and the other one to the case $r = 2$.

Example 1. Let S_1 and S_2 be the rational ruled surfaces parametrized by $\mathbf{x}_1(t, s) = \mathbf{p}_1(t) + s \cdot \mathbf{q}_1(t)$ and $\mathbf{x}_2(t, s) = \mathbf{p}_2(t) + s \cdot \mathbf{q}_2(t)$, where

$$\begin{aligned} \mathbf{p}_1(t) &= (t^4 + t^2 + t, t^6 + t^3, t^5 + t^3 + t^2 + 3t), \\ \mathbf{q}_1(t) &= (t^3 + t, t^5, t^4 + t^2 + 3), \\ \mathbf{p}_2(t) &= (5t^4 + 5t^2 + 5t - 1, 3t^5 + 3t^3 + 3t^2 + 9t + 5, -t^6 + t^4 - t^3 + t^2 + t), \\ \mathbf{q}_2(t) &= (5t^3 + 5t, 3t^4 + 3t^2 + 9, -t^5 + t^3 + t). \end{aligned}$$

In this case, $n = 5$. Furthermore, when we write $\mathbf{q}_1(t)$ as in Eq. (19), we observe that we fall in the case $r = 3$. The surfaces S_1, S_2 are shown in Fig. 1.

We consider first the symmetries of S_1 . If we apply Algorithm 1 directly, with $S_1 = S_2$, the solvability of the linear system derived from Eq. (15) in the entries of the matrix \mathbf{A} yields, after factoring out k , the following three conditions on the coefficients $\alpha, \beta, \gamma, \delta$:

$$(a) \quad 2\alpha^3\delta\gamma - 3\alpha^2\beta\delta^2 + 3\alpha^2\beta\gamma^2 - 6\alpha\beta^2\delta\gamma - 4\alpha\delta^3\gamma + 4\alpha\delta\gamma^3 - \beta^3\gamma^2 - 6\beta\delta^2\gamma^2 + \beta\gamma^4 = 0.$$

$$(b) \quad 5\alpha^4\beta - 10\alpha^2\beta^3 = 0.$$

$$(c) \quad \alpha^4\delta + 4\alpha^3\beta\gamma - 6\alpha^2\beta^2\delta - \alpha^2\delta^3 + 3\alpha^2\delta\gamma^2 - 4\alpha\beta^3\gamma - 6\alpha\beta\delta^2\gamma + 2\alpha\beta\gamma^3 - 3\beta^2\delta\gamma^2 - 30\delta^3\gamma^2 + 15\delta\gamma^4 = 0.$$

Since $\alpha\delta - \beta\gamma \neq 0$, we add the equation $(\alpha\delta - \beta\gamma)u - 1 = 0$. Additionally, from Eq. (22) we get 11 equations of degree 10 in $\alpha, \beta, \gamma, \delta$ and degree 2 in k . These 11 conditions, (a), (b), (c) and $(\alpha\delta - \beta\gamma)u - 1 = 0$ imply

$$\{\alpha = \pm 1, \beta = 0, \gamma = 0, \delta = 1, k = \pm 1\}. \quad (24)$$

Then we compute (numeric) values for \mathbf{A} , \mathbf{b} and explicit forms for $c(t)$. Finally, using Eq. (6) as a test, we get one nontrivial symmetry for S_1 , corresponding to $\{\alpha = -1, \beta = 0, \delta = 1, \gamma = 0, k = 1\}$, with $\varphi(t, s) = (-t, s + 2t)$. The symmetry is defined by $f_0(\mathbf{x}) = \mathbf{A}_0\mathbf{x} + \mathbf{b}_0$, where

$$\mathbf{A}_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{b}_0 = (0 \ 0 \ 0)^T, \quad (25)$$

so S_1 is symmetric with respect to the z -axis. The total computation time here is 3.744 seconds. An alternative possibility to using the above 11 conditions, is the following: (1) write the entries of \mathbf{A} in terms of $\alpha, \beta, \gamma, \delta$; (2) compute the six conditions (of degree 10) derived from the fact that \mathbf{A} is orthogonal; (3) solve the polynomial system consisting of these six conditions, and (a), (b), (c). Solving the polynomial system yields $\{\alpha = \pm 1, \beta = 0, \gamma = 0, \delta = 1, k = k\}$, and the corresponding expressions for $\mathbf{b}, c(t)$ depend on k as well. Finally, using Eq. (6) the symmetry is computed. However, the total computation time of this strategy is slightly slower, namely 4.602 seconds.

In order to check whether S_1, S_2 are affinely equivalent, we apply **Algorithm 1**. The solvability conditions of the linear systems derived from Eq. (13) in the entries of the matrix \mathbf{A} are:

- $10\alpha^3\delta\gamma - 15\alpha^2\beta\delta^2 + 15\alpha^2\beta\gamma^2 - 30\alpha\beta^2\delta\gamma - 20\alpha\delta^3\gamma + 20\alpha\delta\gamma^3 - 5\beta^3\gamma^2 - 30\beta\delta^2\gamma^2 + 5\beta\gamma^4 = 0$.
- $3\alpha^4\delta + 12\alpha^3\beta\gamma - 18\alpha^2\beta^2\delta - 3\alpha^2\delta^3 + 9\alpha^2\delta\gamma^2 - 12\alpha\beta^3\gamma - 18\alpha\beta\delta^2\gamma + 6\alpha\beta\gamma^3 - 9\beta^2\delta\gamma^2 - 90\delta^3\gamma^2 + 45\delta\gamma^4 = 0$.
- $-5\alpha^4\beta + 2\alpha^3\delta\gamma + 10\alpha^2\beta^3 - 3\alpha^2\beta\delta^2 + 3\alpha^2\beta\gamma^2 - 6\alpha\beta^2\delta\gamma - 4\alpha\delta^3\gamma + 4\alpha\delta\gamma^3 - \beta^3\gamma^2 - 6\beta\delta^2\gamma^2 + \beta\gamma^4 = 0$.

Since $\alpha\delta - \beta\gamma \neq 0$, we add the equation $(\alpha\delta - \beta\gamma)u - 1 = 0$, obtaining expressions for $\alpha, \beta, \gamma, \delta$ depending only on k ; the same thing happens with \mathbf{b} and $c(t)$. Finally, we get two φ s corresponding to affine equivalences, namely

$$\varphi_1(t, s) = (t, s), \quad \varphi_2(t, s) = (-t, s + 2t).$$

The mapping $\varphi_1(t, s)$ corresponds to the affine mapping $f_1(\mathbf{x}) = \mathbf{A}_1\mathbf{x} + \mathbf{b}_1$, where

$$\mathbf{A}_1 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 0 \end{pmatrix}, \quad \mathbf{b}_1 = (-1 \ 5 \ 0)^T. \quad (26)$$

The mapping $\varphi_2(t, s)$ corresponds to the affine mapping $f_2(\mathbf{x}) = \mathbf{A}_2\mathbf{x} + \mathbf{b}_2$, where

$$\mathbf{A}_2 = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}, \quad \mathbf{b}_2 = (-1 \ 5 \ 0)^T. \quad (27)$$

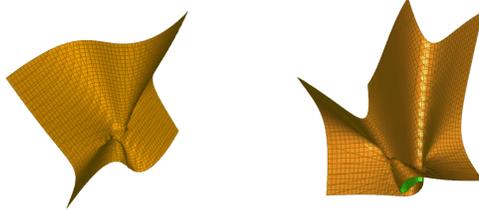


Figure 1: S_1 (left) and S_2 (right).

Therefore, we conclude that S_1, S_2 are related by two affine mappings f_1, f_2 . Notice that this result is consistent with the fact that S_1 has a non-trivial symmetry; in fact, one can check that $f_2 = f_1 \circ f_0$. The computation time is 5.179 seconds.

Example 2. Let S_1 and S_2 be the rational ruled surfaces parametrized by $\mathbf{x}_1(t, s) = \mathbf{p}_1(t) + s \cdot \mathbf{q}_1(t)$ and $\mathbf{x}_2(t, s) = \mathbf{p}_2(t) + s \cdot \mathbf{q}_2(t)$, where

$$\begin{aligned} \mathbf{p}_1(t) &= \left(t + \frac{3}{4}, 4t^2 + 3, t \right), \\ \mathbf{q}_1(t) &= (t^3 + 2t^2 + 1, -t^3 + t^2 + t, -t^3 + t^2 + t), \\ \mathbf{p}_2(t) &= \left(\frac{\sqrt{3}+1}{2}t + \frac{3\sqrt{3}}{8} - \frac{1}{2}, 4t^2 + 5, \frac{\sqrt{3}-1}{2}t - \frac{\sqrt{3}}{2} - \frac{3}{8} \right), \end{aligned}$$

and

$$\begin{aligned} \mathbf{q}_2(t) &= \left(\frac{\sqrt{3}-1}{2}t^3 + \left(\sqrt{3} + \frac{1}{2} \right)t^2 + \frac{t}{2} + \frac{\sqrt{3}}{2}, -t^3 + t^2 + t, \right. \\ &\quad \left. - \left(\frac{\sqrt{3}+1}{2} \right)t^3 + \left(\frac{\sqrt{3}}{2} - 1 \right)t^2 + \frac{\sqrt{3}}{2}t - \frac{1}{2} \right), \end{aligned}$$

Here, $n = 3$. Furthermore, when we write $\mathbf{q}_1(t)$ as in Eq. (19), we observe that we are in the case $r = 2$.

In this case, we analyze the isometries mapping S_1 onto S_2 . There is only one isometry, associated with $\varphi(t, s) = (t, s)$, defined by $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -\frac{1}{2} & 2 & -\frac{\sqrt{3}}{2} \end{pmatrix}^T, \quad (28)$$

corresponding to a rotation of $\frac{\pi}{6}$ around the y -axis. Applying Algorithm 1 with the additional equations corresponding to Eq. (22), we need to test only two tentative solutions, and the computation time is 3.588 seconds. If, instead of Eq. (22), we use the orthogonality conditions on the columns of the matrix \mathbf{A} , we need to test four tentative solutions, and the computation time is 4.696 seconds.

5. Experimentation and performance of the method.

We have implemented the method described in Section 4 in the computer algebra system Maple 18, and we have tried several examples in an Intel(R) Core(TM) 2, Quad CPU Q6600, with 2.40 GHz and 4 Gb RAM; this is also the machine used in the examples of Subsection 4.4. We have analyzed both affine equivalences and isometries. For isometries, we used the conditions derived from Eq. (22), because this tends to speed up the computation.

The results for affine equivalences of some representative examples (“Ex.”) are summarized in Table 1. When the surfaces are affinely equivalent, the second surface is the result of applying to the first surface an affine equivalence with matrix

$$\begin{pmatrix} -1/2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

For each example, we have included: (1) the degree of $\mathbf{p}_i(t)$ (“deg($\mathbf{p}_i(t)$)”) for the surfaces; (2) the degree of $\mathbf{q}_i(t)$ (“deg($\mathbf{q}_i(t)$)”); (3) the computation time (in CPU seconds) of the method for all the affine mappings, and the computation time using the implicit equation of the surfaces (in red); (4) the number of affine equivalences between the two surfaces. The parametrizations of the surfaces in the examples of Table 1 can be found in [1]. In Figure 2 (left) we provide the surface in Ex. 7, Table 1. It is easy to observe that this surface is symmetric with respect to a plane, which explains why we get two affine equivalences between the surface, and its image under an affine mapping. The pictures corresponding to the rest of the surfaces in Table 1 can be found in [1].

The examples in Table 1 with more than one affine equivalence correspond to surfaces with symmetries. Furthermore, in some cases we identify infinitely many equivalences, implying that the surfaces are invariant under infinitely many affine mappings. In the column of timings, we highlight in red the worst time between our method, and the naive method mentioned in the Introduction using the implicit equation. This last timing does not include the time for computing the implicit equation, i.e. we assume that the implicit equation is already known. Only in one of the examples shown, where the implicit equation is very simple ($F(x, y, z) = x^3 - 27yz^2$) is the method using the implicit equation faster.

The results for symmetries and isometries for several representative examples are summarized in Table 2: for each example, we include data also included in the affine equivalences table, plus the computation time (in CPU seconds) of

our method for computing all the symmetries of the surface (“all sym.”), for computing only the involutive symmetries of the surface (“involutions”), and for computing the isometries (“isometries”) between each surface and its image under an orthogonal transformation with associated matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 4/5 & 0 & -3/5 \\ 3/5 & 0 & 4/5 \end{pmatrix}. \quad (29)$$

The parametrizations of the surfaces in the examples of Table 2 can be found in [1]. We also include the computation time using the implicit equation of the surface (“implicit”), assuming that this equation is available. Maple was able to provide an answer with this last method in less than 90 seconds in only two of the examples. In one of the examples, as observed before, the implicit equation turns out to be very simple, which explains why the method using the implicit equation is faster. Finally, we also include the type of symmetries found. In some cases, the symmetries detected are composites of rotations and reflections, denoted as “rotation+reflection”. The surface in Ex. 1 of Table 2 is shown in Fig. 2 (right). One can observe that this surface has several symmetry axes and symmetry planes. To be more precise, this surface has three axial symmetries and two symmetry planes, and is invariant under two different symmetries, composites of a rotation and a reflection. The pictures of the surfaces in the remaining examples of Table 2 are shown in [1].



Figure 2: Surfaces in Ex. 7, Table 1 (left) and Ex. 1, Table 2 (right)

Ex.	$\deg(\mathbf{p}_i(t))$	$\deg(\mathbf{q}_i(t))$	computation time (secs.)/ implicit	Affine equivalences
1	5	3	2.309 / > 90	2
2	1	2	16.676 / > 90	∞
3	0	3	2.870 / 0.687	∞
4	3	3	3.167 / > 90	2
5	4	4	0.452 / > 90	2
6	3	3	6.583 / > 90	1
7	4	4	0.608 / > 90	2
8	7	6	11.563 / > 90	2
9	4	3	0.469 / 7.800	0

Table 1

Ex.	$\deg(\mathbf{p}_i(t))$	$\deg(\mathbf{q}_i(t))$	computation time (secs.) all sym / involutions / isometries / implicit	symmetries and isometries
1	9	6	9.640 / 7.410 / 9.267 / > 90	2 mirror 3 axial 2 rotational + reflect 8 isometries
2	7	4	1.981 / 1.812 / 1.996 / > 90	1 reflect 2 isometries
3	7	6	1.888 / 1.778 / 2.184 / > 90	1 reflect. 2 isometries
4	5	3	1.684 / 1.607 / 2.043 / > 90	1 mirror 2 isometries
5	0	6	4.587 / 3.291 / 10.280 / > 90	5 reflect. 5 axial central 2 rotational 2 rotational + reflect 16 isometries
6	1	2	3.448 / 0.390 / 3.978 / 63.648	1 axial 2 isometries
7	0	3	1.451 / 1.185 / 2.901 / 0.296	central 1 reflection 1 axial 4 isometries
8	7	7	1.935 / 1.809 / 2.372 / > 90	central 2 isometries
9	6	5	1.716 / 1.653 / 2.200 / > 90	1 axial 2 isometries
10	17	6	9.828 / 6.973 / 10.124 / > 90	4 reflect. 1 axial sym. 2 rotational 8 isometries

Table 2

6. Observations on the computation of projective equivalences.

Projective equivalences between S_1, S_2 correspond to rational mappings $f(\mathbf{x})$ from \mathbb{R}^3 to \mathbb{R}^3 satisfying $f(S_1) = S_2$, where the components of f have the form

$$\frac{a_{i1}x + a_{i2}y + a_{i3}z + b_i}{a_{41}x + a_{42}y + a_{43}z + b_4}, \quad (30)$$

for $i = 1, 2, 3$. Whenever f is invertible, Theorem 1 is also valid for this case, so each projective equivalence between S_1, S_2 has an associated mapping $\varphi(t, s) = (\varphi_1(t, s), \varphi_2(t, s))$ in parameter space. Additionally, projective mappings are collineations, i.e. they map lines to lines. Thus, we can argue as in the first part of Proposition 2 to conclude that $\varphi_1(t, s) = \psi(t)$, where $\psi(t)$ is a Möbius transformation. However, in general, the form of $\varphi_2(t, s)$ is not the same as in Proposition 2. Indeed, using Eq. (30) one has that

$$\varphi_2(t, s) = \frac{\xi_1(t) + s\xi_2(t)}{\xi_3(t) + s\xi_4(t)},$$

where the $\xi_j(t)$ are polynomials. As a consequence, the remaining results of Section 3, and in particular the form of φ predicted by Theorem 7, cannot be easily generalized. Therefore, an approach analogous to the one in this paper for projective equivalences requires further work.

7. Conclusion.

We have presented a unifying method to compute affine equivalences, isometries and symmetries of ruled rational surfaces, working directly on the rational parametric form. In order to do this computation, we reduce the problem to parameter space, taking advantage of the fact that, under our hypotheses, these transformations have an associated birational transformation of the real plane with a specific structure. From here, we observe that the matrix defining any affine equivalence (resp. isometry or symmetry) between the surfaces defines a projective equivalence between two projective curves corresponding to the directions of the rulings of the surfaces. Thus, we take advantage of the ideas in [12], where projective equivalences of curves in any dimension are considered, to solve our problem. In the case of isometries or symmetries, we have extra conditions coming from the fact that orthogonal transformations preserve norms. The algorithm is efficient, as shown in several nontrivial examples. For projective equivalences, we provide only a small hint on the form of the birational transformation of the plane behind such equivalences: giving a complete description, and generalizing the method to also cover these equivalences, requires more effort.

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