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Research Article

# Asymptotic Behavior of a Parametric Algebraic Surface

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**Abstract:** Starting from the concept of infinite branches and approximation surfaces, we present a method to compute infinite branches and surfaces having the same asymptotic behavior as an input parametric surface. The results obtained in this work represent a breakthrough for the study of surfaces and their applications.

***Keywords*:** parametric algebraic surface, infinity branches, convergent branches, asymptotic behavior, approaching surfaces

**MSC:** 14J26, 14J29, 14J70, 14M20,14E08

## 1. Introduction

In the last few years, researchers have analyzed several problems related to unirational algebraic varieties because these entities are very important for practical applications (see e.g., [1-4]). In particular, in this work, we consider infinity branches for parametric surfaces which reflect the behavior of a given surface at the points with "sufficiently large coordinates". Our idea is to use infinity branches to interpolate situations in which the surface can not be approached, for instance, when one is going to the infinity. For instance, one may consider problems related with the plotting of these entities, modeling or blending of surfaces, high-dimensional interpolation, rational approximation of non-rational curves and surfaces, etc. (see [5-9]).

A branch defined at an infinity point (infinity branch) provides important techniques because it allows the study of how an algebraic curve/surface behaves at the infinity. Therefore, a branch defined at an infinity point allows for the application of symbolic computation techniques to real-world problems, of particular interest in several areas such as computer-aided geometric design (CAGD), computer-aided design (CAD), engineering, computer science or control systems (see e.g., [5-7, 10-15]).

Throughout this paper, using the concepts of branches defined at points of infinity and approaching surfaces (see [16]), we derive a method to compute infinity branches and surfaces having the same asymptotic behavior as a parametrically defined input surface. The results are a major novelty and represent an important advance for the analysis of surfaces and the study of their applications.

More precisely, the article is structured as follows: In Section 2, we provide some preliminaries, and in particular,


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we give the notions of infinity branches, convergent branches, and approaching surfaces. This section is the basis for the development of Section 3, where we consider surfaces parametrically defined and show how to compute branches, for parametric surfaces without implicitizing. Once one knows how to compute the infinity branches, it is shown how to determine surfaces that behave in a similar way at the infinity as the original input surface. Finally, we conclude this paper with some conclusions and ideas to develop in future work (see Section 4).

## 2. Infinity branches

In this section, we provide some previous notions and results that are essential in the development of Section 3. Moreover, we illustrate this with some examples. In particular, the concept of *infinity branch* was already given by the authors in previous papers for curves (see [10, 11, 17]) and for surfaces in [16].

Let $V$ be an algebraic affine irreducible surface over $\mathbb{C}$ and implicitly defined by the polynomial $f(x,y,z) \in \mathbb{R}[x,y,z]$, where $\mathbb{R}$ denotes the field of real numbers. We denote by $V^*$ its corresponding projective surface defined by $F(x,y,z,w) \in \mathbb{R}[x,y,z,w]$. Furthermore, $\mathbb{C} \ll t_2 \gg$ represents the field of formal Puiseux series in the variable $t_2$ (see e.g., [18, 19], Section 2.5 in [3, 20, 21], Chapter 4 (Section 2) in [4]), and let $p = (m_1(t_2), m_2(t_2))$, $m_1(t_2), m_2(t_2) \in \mathbb{C} \ll t_2 \gg$ be a local parametrization (see Section 2.5 in [3]) of a curve at the infinity of $V^*$ defined by $g(y, z, 0)$ is an irreducible polynomial that divides $F(1, y, z, 0)$). Observe that $F(1, m_1(t_2), m_2(t_2), 0) = 0$ and by abuse of notation, we refer to $P = (1 : m_1(t_2) : m_2(t_2) : 0)$ as an infinity point of $V^*$.

We determine the series expansion for the solutions of $g(y, z, t_1) = 0$ w.r.t. $(y, z)$ in some neighborhood of $t_1 = 0$. We get solutions defined by Puiseux series that form conjugacy classes. We consider one of these solutions defined by $\varphi(t_1,t_2) = (\varphi_1(t_1,t_2), \varphi_2(t_1,t_2)) \in \mathbb{C}(t_1,t_2) \subset \mathbb{C} \ll t_1,t_2 \gg$, where $\mathbb{C} \ll t_1,t_2 \gg$ denotes the field of Pusieux series in the variables $t_1$, $t_2$ (see [20]). Then $g(\varphi_1(t_1, t_2), \varphi_2(t_1, t_2), t_1) = 0$ in some neighborhood of $t_1 = 0$ where $\varphi(t_1, t_2)$ converges. Hence, there exists some $M \in \mathbb{R}^+$ such that

$$F(1, \varphi_1(t_1,t_2), \varphi_2(t_1,t_2), t_1) = g(\varphi_1(t_1,t_2), \varphi_2(t_1,t_2), t_1) = 0,$$

where $(t_1,t_2) \in \mathbb{C}^2$ and $|t_1| < M$, and thus,

$$F(t_1^{-1}, t_1^{-1}\varphi_1(t_1,t_2), t_1^{-1}\varphi_2(t_1,t_2) : 1) = f(t_1^{-1}, t_1^{-1}\varphi_1(t_1,t_2), t_1^{-1}\varphi_2(t_1,t_2)) = 0,$$

for $(t_1,t_2) \in \mathbb{C}^2$ and $0 < |t_1| < M$. We set $t_1^{-1} \to t_1$ and we obtain that

$$f(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) = 0, \quad (t_1,t_2) \in \mathbb{C}^2 \text{ and } |t_1| > M^{-1}, \text{ where}$$

$$r_j(t_1,t_2) = t_1\varphi_j(t_1^{-1},t_2) = m_j(t_2)t_1 + a_{1j}(t_2)t_1^{1-N_{1j}/N_1} + a_{2j}(t_2)t_1^{1-N_{2j}/N_1} + a_{3j}(t_2)t_1^{1-N_{3j}/N_1} + \cdots.$$

It is well known that one may find $N_1$ series in the class of conjugation (w.r.t. $t_1$). That is, let $\varphi_{11},\ldots,\varphi_{1N_1}$ be these series, and

$$r_{1i}(t_1,t_2) = t_1\varphi_{1i}(t_1^{-1},t_2) = m_1(t_2)t_1 + a_1(t_2)c_i^{N_{11}}t_1^{1-N_{11}/N_1} + a_2(t_2)c_i^{N_{21}}t_1^{1-N_{21}/N_1} + a_3(t_2)c_i^{N_{31}}t_1^{1-N_{31}/N_1} + \cdots \quad (1)$$

where $c_1,\ldots,c_{N_1}$ are the $N_1$ complex roots of $x^{N1} = 1$ (similarly for $r_{2i}(t_1, t_2)$). In [16], it is proved that all the properties and results hold for any of the leaves one have in the conjugacy class, so for simplicity in the explanations, in the following we consider any of its leaves in the conjugation class.

Now, we give the notion of the infinity branch which is uniquely determined, up to conjugation. The general point is that any point at the infinity (i.e., with large coordinates) belongs to some infinity branch (see Lemma 1 in [16]).

**Definition 1.** The set

$$B = \{(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M\}$$

is an *infinity branch* of the surface $V$.

As one may note, the previous construction is considered for an infinity point $(1 : b(t_2) : c(t_2) : 0)$, where $(b(t_2), c(t_2))$ defines a local parametrization of a curve at the infinity. For the case of an infinity point of the form $(a(t_2) : 1 : c(t_2) : 0)$, we reason as above by taking the surface defined by $F(x, 1, z, w)$. Finally, for the infinity points of the form $(a(t_2) : b(t_2) : 1 : 0)$, we consider the surface defined by $F(x, y, 1, w)$.

In the following, we work with infinity branches defined from the point at infinity $P = (1 : m_1 : m_2 : 0), m_1, m_2 \in \mathbb{C} \ll t_2 \gg$. Similarly, one reason for the other cases.

In Example 1, we show how to determine the branches for an algebraic implicit surface.

**Example 1.** We consider the polynomial

$$f(x,y,z) = x^2 + z^2x^2 + zy^3 \in \mathbb{R}[x,y,z]$$

defining the surface $V$. Hence, its projective surface $V^*$ is given by

$$F(x,y,z,w) = x^2w^2 + z^2x^2 + zy^3.$$

It holds that $p_1 = (t_2, -t_2^3)$ and $P_2 = (1 : t_2 : 0 : 0)$ are the points at the infinity. We compute the infinity branches to $P_1$ and $P_2$ and for this purpose, we work with the curve implicitly defined by the polynomial $g(y, z, w) = F(1, y, z, w)$ (note that $g(p_i, 0) = 0$, where $p_1 = (t_2, -t_2^3)$ and $p_2 = (t_2, 0)$).

We determine the series expansion that define the solutions of $g(y, z, t_1) = 0$ w.r.t. $(y, z)$ around $t_1 = 0$. In this particular case, the curve defined by $g(y, z, t_1)$ is not rational over $\mathbb{C}(t_1)$ and hence, we compute a parametrization which is local.

First, we get $\varphi_1(t_1, t_2) = (\varphi_{11}(t_1, t_2), \varphi_{12}(t_1, t_2))$, where

$$\varphi_{11}(t_1,t_2) = t_2 \in \mathbb{C}(t_1,t_2) \subset \mathbb{C} \ll t_1,t_2 \gg \text{ and}$$

$$\varphi_{12}(t_1,t_2) = -t_2^3 + t_1^2/t_2^3 + t_1^4/t_2^9 + \cdots \in \mathbb{C}(t_1,t_2) \subset \mathbb{C} \ll t_1,t_2 \gg.$$

It holds that $\varphi_1(0, t_2) = p_1$, and $g(\varphi_1(t_1, t_2), t_1) = 0$. Hence, the first infinity branch is given as

$$B_1 = \{(t_1, r_{11}(t_1,t_2), r_{12}(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M\}, \text{ where}$$

$$r_{11}(t_1,t_2) = t_1\varphi_{11}(t_1^{-1},t_2) = t_1t_2, \quad r_{12}(t_1,t_2) = t_1\varphi_{12}(t_1^{-1},t_2) = -t_1t_2^3 + t_1^{-1}t_2^{-3} + t_1^{-3}t_2^{-9} + \cdots.$$

In Figure 1, the surfaces $V$ and $V_1$ are plotted. Remind that $V_1$ is established from the infinity branch $B_1$. In Section 2, we will see that $V_1$ approaches $V$.

Secondly, we get $\varphi_2(t_1, t_2) = (\varphi_{21}(t_1, t_2), \varphi_{22}(t_1, t_2))$, where

$$\varphi_{21}(t_1,t_2) = t_2 \in \mathbb{C}(t_1,t_2) \subset \mathbb{C} \ll t_1,t_2 \gg \text{ and}$$

$$\varphi_{22}(t_1,t_2) = -t_1^2/t_2^3 - t_1^4/t_2^9 + \cdots \in \mathbb{C}(t_1,t_2) \subset \mathbb{C} \ll t_1,t_2 \gg.$$

Observe that $\varphi_2(0, t_2) = p_2$, and $g(\varphi_2(t_1, t_2), t_1) = 0$. Thus, the second infinity branch is $B_2 = \{(t_1, r_{21}(t_1,t_2), r_{22}(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M\}$, where

$$r_{21}(t_1,t_2) = t_1\varphi_{21}(t_1^{-1},t_2) = t_1t_2, \quad r_{22}(t_1,t_2) = t_1\varphi_{22}(t_1^{-1},t_2) = -t_1^{-1}t_2^{-3} - t_1^{-3}t_2^{-9} + \cdots.$$

Figure 2 represents $V$ and a surface, $V_2$, constructed from the infinity branch $B_2$ (we will see that $V_2$ approaches $V$). Figure 3 represents $V$ and $V_1$ and $V_2$ together.

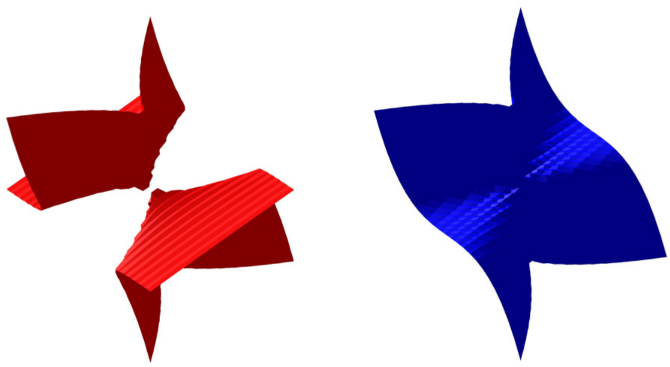

**Figure 1.** Surfaces $V$ (left) and $V_1$ (right)

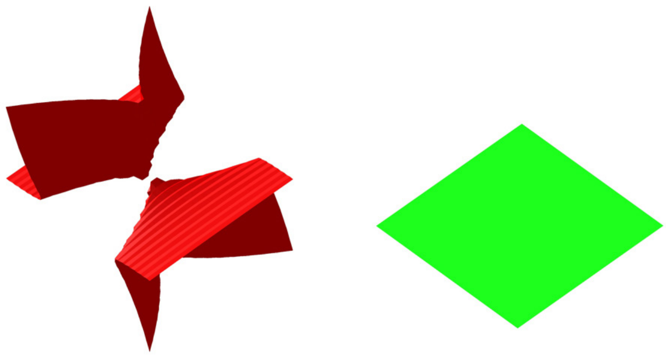

**Figure 2.** Surfaces $V$ (left) and $V_2$ (right)

**Remark 1.** One should note that the computation of the series $\varphi_i(t_1, t_2) \in \mathbb{C} \ll t_1, t_2 \gg, i = 1, 2,$ is not easy. In some particular cases, this question can be solved as in the previous example. Also, in the parametric case, we will see how to solve this problem (see Section 3). More general cases will be studied in some future works.

Now, we introduce the notions of *convergent branches* and approaching surfaces. The intuitive idea is that, two infinity branches converge if they get closer when one is tending to the infinity. This concept will permit us to examine whether two surfaces approach each other at infinity and to check their asymptotic behavior.

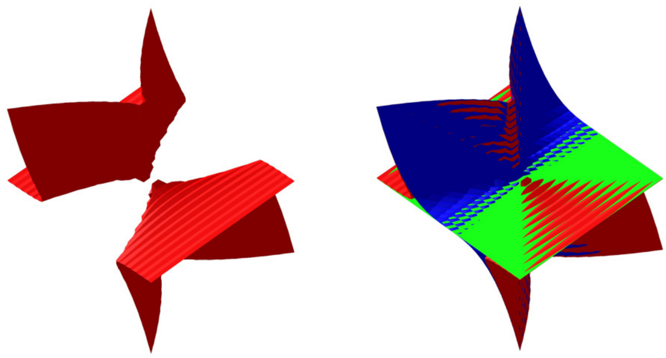

**Figure 3.** $V$ (left), and $V$, $V_1$, and $V_2$ (right)

**Definition 2.** Let $B = \{(t_1, r_1(t_1, t_2), r_2(t_1, t_2)) \in \mathbb{C}^3 : (t_1, t_2) \in \mathbb{C}^2,$ and $\overline{B} = \{(t_1, \overline{r_1}(t_1, t_2), \overline{r_2}(t_1, t_2)) \in \mathbb{C}^3 : (t_1, t_2) \in \mathbb{C}^2,$ $|t_1| > \overline{M}\}$ be two infinity branches. It is said that they are convergent if $\lim_{t_1 \to \infty}(\overline{r_i}(t_1, t_2) - r_i(t_1, t_2)) = 0,$ for $i = 1, 2$.

**Proposition 1.** Two branches $B = \{(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M\}$ and $\overline{B} = \{(t_1, \overline{r_1}(t_1,t_2), \overline{r_2}(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > \overline{M}\}$ are convergent if the monomials on the variable $t_1$ that have a non negative exponent in the series $r_i(t_1, t_2)$ and $r_i(t_1, t_2)$ are the same (for $i = 1, 2$).

From Proposition 1, we deduce that $m_i(t_2) = \overline{m}_i(t_2)$, $i = 1, 2$, and then, $B$ and $\overline{B}$ are associated to the same infinity point. In addition, we note that the number of monomials, w.r.t. $t_1$ having a positive exponent in both series is finite. Furthermore, it holds that the value $n_1 \in \mathbb{N}$ which is obtained when one simplifies the non-negative exponents w.r.t. $t_1$ is the same in both convergent infinity branches. We say that $n_1$ is the degree w.r.t. $t_1$ of the infinity branch.

Two convergent branches at the infinity could be contained in the same surface or in different surfaces. In this last case, we say that the surfaces are approaching surfaces. To introduce this notion, we define the distance from $p$ to a given surface $V$ as $d(p, V) = \min\{d(p, q) : q \in V\}$.

**Definition 3.** Let $V$ be a surface with an infinity branch $B = \{(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M\}$. A surface $\overline{V}$ approaches $V$ at $B$ if $\lim_{t_1 \to \infty} d((t_1, r_1(t_1,t_2), r_2(t_1,t_2)), \overline{V}) = 0$.

**Theorem 1.** Let $V$ be a surface and let $B$ be an infinity branch of $V$. It holds that a new surface $\overline{V}$ approaches $V$ at $B$ if $\overline{V}$ has an infinity branch, $\overline{B}$ satisfying that $B$ and $\overline{B}$ are convergent. Furthermore, $V$ and $\overline{V}$ have a common infinity point.

From Theorem 1, we get that $\overline{V}$ approaches $V$ at $B$ ($B$ is an infinity branch of $V$) if $V$ approaches $\overline{V}$ at $\overline{B}$ where is some infinity branch of $\overline{V}$. In this case, we say that $V$ and $\overline{V}$ are approaching surfaces. Furthermore, if $V$ approaches at all of its infinity branches and reciprocally, we say that both surfaces have the same *asymptotic behavior*.

## 3. Parametric surfaces

Some previous work concerning the problem of studying the asymptotic behavior for surfaces deal with algebraic surfaces implicitly defined (see [16]). In this section, we study surfaces parametrically defined and we show how to compute branches for these surfaces without implicitizing.

When the infinity branches are determined, one is interested in computing surfaces that have the same asymptotic behavior as the input surface at each infinity branch.

For this, one simply has to remove in $r_1$ and $r_2$ the terms having negative exponent w.r.t. $t_1$.

### 3.1 *Computation of branches for parametric surfaces*

Let $V$ be a surface defined by the parametrization

$$P(s_1, s_2) = (p_1(s_1,s_2), p_2(s_1,s_2), p_3(s_1,s_2)) \in \mathbb{R}(s_1,s_2)^3,$$

where $p_i(s_1,s_2) = p_{i1}(s_1,s_2)/p_{i2}(s_1,s_2), i = 1, 2, 3,$ with $\gcd(p_{i1}, p_{i2}) = 1$. If $V^*$ represents the projective surface of $V$, a parametrization of $V^*$ is

$$P^*(s_1,s_2) = \left(1 : \frac{p_{21}(s_1,s_2)}{p_{22}(s_1,s_2)} \cdot \frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)} : \frac{p_{31}(s_1,s_2)}{p_{32}(s_1,s_2)} \cdot \frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)} : \frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)}\right).$$

We may assume w.l.o.g that $x = 0$ is not a curve at infinity of $V^*$ (otherwise, one may apply a linear change of coordinates).

To determine surfaces having the same asymptotic behavior as $V$, first we compute the branches of $V$. That is,

$$B = \{(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M\},$$

where $r_i(t_1,t_2) = t_1 \varphi_i(t_1^{-1}, t_2)$ for $i = 1, 2$. For this, from Definition 1, we get

$$f(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) = F(t_1, t_1\varphi_1(t_1^{-1}, t_2), t_1\varphi_2(t_1^{-1}, t_2), 1) =$$
$$F(1, \varphi_1(t_1^{-1}, t_2), \varphi_2(t_1^{-1}, t_2), t_1^{-1}) = F(1, \varphi_1(t_1, t_2), \varphi_2(t_1, t_2), t_1) = 0$$

around $t_1 = 0$, and $F$ is the implicit polynomial of $V^*$. Observe that in this section, we are given the parametrization $P^*$ of $V^*$ and then, $F(P^*(s_1, s_2)) =$

$$F\left(1, \frac{p_{21}(s_1,s_2)}{p_{22}(s_1,s_2)} \cdot \frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)}, \frac{p_{31}(s_1,s_2)}{p_{32}(s_1,s_2)} \cdot \frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)}, \frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)}\right) = 0.$$

Hence, to compute the branches of $V$, and in particular $\varphi_i$, $i = 1, 2$, one rewrite $P^*(s_1, s_2)$ in the form of $(1 : \varphi_1(t_1, t_2) : \varphi_2(t_1, t_2) : t_1)$ around $t_1 = 0$. This is a search for a value of the parameters $(t_1, t_2)$, say $\ell(t_1, t_2) \in \mathbb{C} \ll t_1, t_2 \gg^2$, such that $P^*(\ell(t_1, t_2)) = (1 : \varphi_1(t_1, t_2) : \varphi_2(t_1, t_2) : t_1)$ around $t_1 = 0$.

There exist solutions $\ell_1(t_1,t_2), \ell_2(t_1,t_2), \ldots, \ell_k(t_1,t_2) \in \mathbb{C} \ll t_1, t_2 \gg^2$ such that, $p_{12}(\ell_i(t_1, t_2)) - t_1 p_{11}(\ell_i(s, t)) = 0$, $i = 1, ..., k$, in a neighborhood of $t_1 = 0$. Note that

$$\ell_i(t_1,t_2) = (\ell_{1i}(t_1,t_2), \ell_{2i}(t_1,t_2)) = (\ell_{1,i,0}(t_2) + \ell_{1,i,1}(t_2)t_1^{n_{1,i,1}} + \ell_{1,i,2}(t_2)t_1^{n_{1,i,2}} + \cdots,$$
$$\ell_{2,i,0}(t_2) + \ell_{2,i,1}(t_2)t_1^{n_{2,i,1}} + \ell_{2,i,2}(t_2)t_1^{n_{2,i,2}} + \cdots), 0 < n_{r,i,1} < n_{r,i,2} < \cdots, r = 1, 2$$

where $h(\ell_{1,i,0}(t_2), \ell_{2,i,0}(t_2)) = 0$ and $\ell_{r,i,j}(t_2) \in \mathbb{C} \ll t_2 \gg^2$ for $i = 1, ..., k$ and $j = 0, 1, ..., r = 1, 2$.

Hence, for $i = 1, ..., k$, there exists $M_i \in \mathbb{R}^+$ such that the points $(1 : \varphi_1(t_1, t_2) : \varphi_2(t_1, t_2) : t_1)$ or similarly, the points $(t_1^{-1} : t_1^{-1}\varphi_1(t_1,t_2) : t_1^{-1}\varphi_2(t_1,t_2) : 1)$, where

$$\varphi_{1i}(t_1,t_2) = \frac{p_{21}(\ell_i(t_1,t_2))}{p_{22}(\ell_i(t_1,t_2))}, \quad \varphi_{2i}(t_1,t_2) = \frac{p_{31}(\ell_i(t_1,t_2))}{p_{32}(\ell_i(t_1,t_2))}, \tag{2}$$

are in $V^*$ for $|t_1| < M_i$ ($P^*(\ell_i(t_1,t_2)) \in V^*$ since $P^*$ is a parametrization of $V^*$). Observe that $\varphi_{ji}(t_1, t_2)$, $j = 1, 2$, is Puiseux series, since $p_{k1}(\ell_i(t_1,t_2))$ and $p_{k2}(\ell_i(t_1,t_2)), k = 2, 3$, can be expressed as Puiseux series and $\mathbb{C} \ll t_1, t_2 \gg$ is a field.

Finally, we set $t_1 \to t_1^{-1}$ and we have that the points $(t_1, r_{1i}(t_1, t_2), r_{2i}(t_1, t_2))$, where $r_{ji}(t_1,t_2) = t_1\varphi_{ji}(t_1^{-1}, t_2)$, are in $V$ for $|t_1| > M_i^{-1}$. Hence, the infinity branches of $V$ are the sets $B_i = \{(t_1, r_{1i}(t_1,t_2), r_{2i}(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M_i^{-1}\}$, for $i = 1, ..., k$.

**Remark 2.** The series $\ell_i(t_1, t_2)$ satisfy that

$$p_{12}(\ell_i(t_1,t_2)) / p_{11}(\ell_i(t_1,t_2)) = t_1,$$

for $i = 1, ..., k$. Therefore, from (2), we get

$$\varphi_{1i}(t_1,t_2) = \frac{p_{22}(\ell_i(t_1,t_2))}{p_{21}(\ell_i(t_1,t_2))} \cdot \frac{p_{12}(\ell_i(t_1,t_2))}{p_{11}(\ell_i(t_1,t_2))} = \frac{p_{22}(\ell_i(t_1,t_2))}{p_{21}(\ell_i(s,t))} t_1 = p_2(\ell_i(t_1,t_2))t_1,$$

and

$$r_{i1}(t_1,t_2) = t_1\varphi_{1i}(t_1^{-1}, t_2) = p_2(\ell_i(t_1^{-1}, t_2)).$$

Similarly, $\varphi_{2i}(t_1, t_2) = p_3(\ell_i(t_1, t_2))t_1$ and

$$r_{i2}(t_1,t_2) = t_1\varphi_{2i}(t_1^{-1}, t_2) = p_3(\ell_i(t_1^{-1}, t_2)).$$

**Example 2.** Let the surface $V$ be parametrically defined by

$$P(s_1,s_2) = (s_1^2 / s_2^2, (s_1^2 - s_2^2 - 5s_1 + 1) / (s_2^2 - s_1), (s_1^2 + s_2^2 - s_2 - 5) / (s_2^2 + 3)).$$

Let us determine the branches of $V$ from $P$. For that, we first compute the solutions of

$$p_{12}(s_1, s_2) - t_1 p_{11}(s_1, s_2) = 0,$$

around $t_1 = 0$. We get

$$\ell(t_1, t_2) = (t_2 t_1^{-1/2}, t_2) \in \mathbb{C} \ll t_1, t_2 \gg^2 .$$

For $\ell(t_1, t_2) \in \mathbb{C} \ll t_1, t_2 \gg^2$, we compute the corresponding infinity branch of $V$,

$$B = \{(t_1, r_1(t_1, t_2), r_2(t_1, t_2)) \in \mathbb{C}^3 : (t_1, t_2) \in \mathbb{C}^2, | t_1 |> M_i^{-1}\},$$

where $r_j(t_1, t_2) = p_{(j+1)}(\ell(t_1^{-1}, t_2)), j = 1, 2,$ is given as Puiseux series. We get

$$\begin{aligned} r_1(t_1, t_2) &= -t_2 t_1^{1/2} + 5 - t_2^2 + 6t_2 t_1^{-1/2} - t_2^{-1} t_1^{-1/2} - t_2^3 t_1^{-1/2} + 6t_2^2 t_1^{-1} - t_1^{-} - t_2^4 t_1^{-1} + 6t_2^3 t_1^{-3/2} - t_2 t_1^{-3/2} - t_2^5 t_1^{-3/2} + \cdots, \\ r_2(t_1, t_2) &= t_2^2 t_1 / (t_2^2 + 3) + t_2^2 / (t_2^2 + 3) - t_2 / (t_2^2 + 3) - 5 / (t_2^2 + 3). \end{aligned}$$

**Remark 3.** If a surface $V$ has degree $d$ and it cannot be approached by a new surface of degree less than $d$, we say that $V$ is a perfect surface. In addition, we say that $\overline{V}$ is a g-asymptote if it is a perfect surface that approaches $V$ at branch at the infinity. In a future work, we will analyze if the approaching surfaces computed in this paper are perfect surfaces.

**Example 3.** Let the surface $V$ be parametrically defined by

$$P(s_1, s_2) = (s_1^3, s_1^2(s_2^2 + 1), s_1^2 s_2).$$

As in Example 2, we obtain one infinity branch

$$B = \{(t_1, r_1(t_1, t_2), r_2(t_1, t_2)) \in \mathbb{C}^3 : (t_1, t_2) \in \mathbb{C}^2, | t_1 |> M\}, \quad \text{where}$$

$$r_1(t_1, t_2) = t_1^{2/3}(t_2^2 + 1), \qquad r_2(t_1, t_2) = t_1^{2/3} t_2.$$

Note that $P(t_1, t_2) = (t_1^3, r_1(t_1^3, t_2), r_2(t_1^3, t_2))$. Thus, we could deduce that $V$ cannot be approached by any surface of degree less than the degree of $V$. Thus, we would say that $V$ is a perfect surface (see Remark 3).

## 3.2 *Asymptotic behavior of parametric surfaces*

Once we know how to determine the infinity branches, we could compute surfaces that have the same asymptotic behavior as the given surface at each of them by simply removing the terms with negative exponent in the variable $t_1$ from $r_1$ and $r_2$, and to compute a new surface having the same asymptotic behavior that the input surface for each of them.

More precisely, in Example 2, we observe that once we have the infinity branch, we may determine a surface having the same asymptotic behavior as the input surface at each of them by simply removing the terms with negative exponent in the variable $t_1$ from $r_1$ and $r_2$. In that case, we get

$$\tilde{r}_1(t_1, t_2) = -t_2 t_1^{1/2} + 5 - t_2^2,$$

$$\tilde{r}_2(t_1, t_2) = t_2^2 t_1 / (t_2^2 + 3) + t_2^2 / (t_2^2 + 3) - t_2 / (t_2^2 + 3) - 5 / (t_2^2 + 3).$$

Observe that, for this particular example, $(t_1^2, \tilde{r}_1(t_1^2, t_2), \tilde{r}_2(t_1^2, t_2)) \in \mathbb{R}(t_1, t_2)$ is, in fact, a rational parametrization that defines a surface $\overline{V}$ that has the same asymptotic behavior as $V$ at $B$.

**Theorem 2**. Let $P$ be a given parametrization of a surface $V$ such that $h(s_1, s_2)$ divides $p_{12}$ and $\gcd(h, p_{i2}) = 1$, $i = 2, 3$.

Then, $V$ and $\overline{V}$ have the same asymptotic behavior, where $\overline{V}$ is defined by the local parametrization

$$Q(t) = (t_1, p_2(\ell_{10}(t_2), \ell_{20}(t_2)), p_3(\ell_{10}(t_2), \ell_{20}(t_2))) \in (\mathbb{C} \ll t_2 \gg)[t_1],$$

and $h(\ell_{10}(t_2), \ell_{20}(t_2)) = 0$. Under these conditions, $V$ is a cylinder over the $x$ axis which implicit equation can be computed using eliminating techniques and more precisely, by eliminating the variables $s_1$, $s_2$ from the system

$$p_{21}(s_1, s_2) - yp_{22}(s_1, s_2) = p_{31}(s_1, s_2) - zp_{32}(s_1, s_2) = h(s_1, s_2) = 0.$$

*Proof.* First, we observe that since

$$p_{12}(\ell(t_1, t_2)) - t_1 p_{11}(\ell(t_1, t_2)) = 0$$

around $t_1 = 0$, we have that

$$\ell(t_1, t_2) = [(\ell_{10}(t_2) + \ell_{11}(t_2)t_1^{n_{11}} + \ell_{12}(t_2)t_1^{n_{12}} + \cdots, \ell_{20}(t_2) + \ell_{21}(t_2)t_1^{n_{21}} + \ell_{22}(t_2)t_1^{n_{22}} + \cdots),$$

for $0 < n_{r1} < n_{r2} < \ldots$, $r = 1, 2$, and where $h(\ell_{10}(t_2), \ell_{20}(t_2)) = 0$ and $\ell_{rj}(t_2) \in \mathbb{C} \ll t_2 \gg, j = 0, 1, \ldots, r = 1, 2.$ Taking into account that $\gcd(h, p_{i2}) = 1$, $i = 2, 3$, we have that

$$p_i(s_1, s_2) = p_i(\ell_{10}(t_2), \ell_{20}(t_2)) + \frac{\partial p_i}{\partial s_1}(\ell_{10}(t_2), \ell_{20}(t_2))(s_1 - (\ell_{10}(t_2)) + \frac{\partial p_i}{\partial s_2}(\ell_{10}(t_2), \ell_{20}(t_2))(s_2 - (\ell_{20}(t_2)) + \cdots$$

for $i = 2, 3$. Thus, $p_i(\ell(t_1, t_2)) =$

$$p_i(\ell_{10}(t_2), \ell_{20}(t_2)) + \frac{\partial p_i}{\partial s_1}(\ell_{10}(t_2), \ell_{20}(t_2))(\ell_{11}(t_2)t_1^{n_1} + \ell_{12}(t_2)t_1^{n_2} + \cdots) + \cdots, i = 2, 3,$$

and $r_i(t_1, t_2) = p_i(\ell(t_1^{-1}, t_2)) =$

$$p_i(\ell_{10}(t_2), \ell_{20}(t_2)) + t_1^{-n_{11}}(\frac{\partial p_i}{\partial s_1}(\ell_{10}(t_2), \ell_{20}(t_2))(\ell_{11}(t_2) + \ell_{12}(t_2)t_1^{n_{11}-n_{12}} + \cdots) + \cdots,$$

for $i = 2, 3$. Therefore, since $-n_{11} < 0$, $n_{11} - n_{12} < 0\ldots$, we get that

$$Q(t) = (t_1, p_2(\ell_{10}(t_2), \ell_{20}(t_2)), p_3(\ell_{10}(t_2), \ell_{20}(t_2))) \in (\mathbb{C} \ll t_2 \gg)[t_1],$$

where $h(\ell_{10}(t_2), \ell_{20}(t_2)) = 0$.

**Remark 4.** One reason similarly for the case that $h(s_1, s_2)$ divides $p_{22}$ and $\gcd(h, p_{i2}) = 1$, $i = 1, 3$. In this case, we obtain that $\overline{V}$ is a cylinder over the $y$ axis.

If $h(s_1, s_2)$ divides $p_{32}$ and $\gcd(h, p_{i2}) = 1$, $i = 1, 2$, we obtain that $\overline{V}$ is a cylinder over the $z$ axis.

We observe that the situation presented in Theorem 2 is very usual when we are considering applied problems or rational surface parametrization obtained from numerical problems from CAGD or CAD.

**Example 4.** Let the surface $V$ be parametrically defined by

$$\begin{aligned} P(s_1, s_2) &= (p_{11}(s_1, s_2)/p_{12}(s_1, s_2), p_{21}(s_1, s_2)/p_{22}(s_1, s_2), p_{31}(s_1, s_2)/p_{32}(s_1, s_2)) \\ &= (s_1^2/(s_2^2 + s_1^2 + s_1 + s_2 - 3), (s_1^2 - s_2^2 - 5s_1 + 1)/(s_2^2 - 1), (s_1^2 - s_2 - 1)/(s_2 + 2)). \end{aligned}$$

Observe that we are in the conditions of Theorem 2, and thus we compute $\overline{V}$ a cylinder over the $x$ axis that has the same asymptotic behavior as $V$. More precisely, by eliminating the variables $s_1$, $s_2$ from the system

$$p_{21}(s_1, s_2) - yp_{22}(s_1, s_2) = p_{31}(s_1, s_2) - zp_{32}(s_1, s_2) = h(s_1, s_2) = 0,$$

where $h(s_1, s_2) = s_2^2 + s_1^2 + s_1 + s_2 - 3,$ we get the implicit equation

$$\overline{f}_x(y,z) = -3093 - 5022z - 176z^4 - 9y^4 + 372y^3 + 666y^3z + 34y^4z - 200z^4y - 32z^4y^2 + 59y^4z^2 - 457y^3z^3 - 11y^3z^2 + 9y^4z^4 - 60y^4z^3 + 30y^3z^4 - 1972z^3y - 1382z^3y^2 - 3341yz - 1318yz^2 - 762y^2z - 506y^2z^2 - 1496z^2 - 648y^2 - 1112z^3 - 2241y$$

that defines a cylinder, $V_1$, over the $x$ axis that has the same asymptotic behavior as $V$. In Figure 4, we plot $V$ and $V_1$ together (left), $V$ (center), and $V_1$ (right).

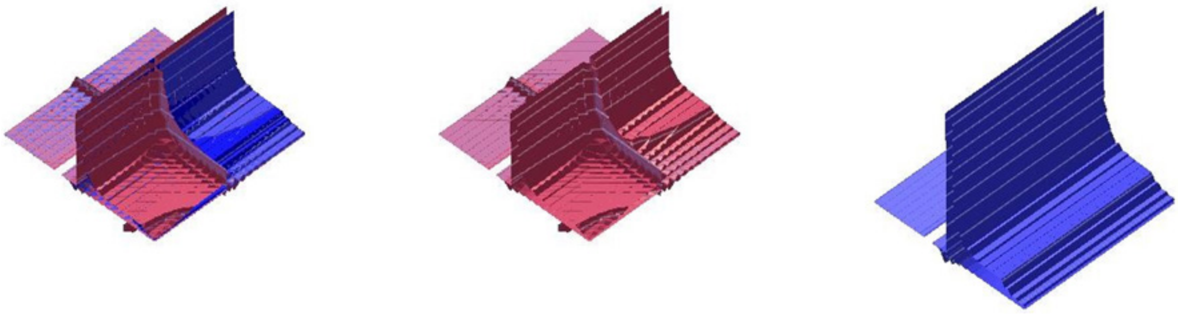

**Figure 4.** $V$ and $V_1$ together (left), $V$ (center), and $V_1$ (right)

Reasoning similarly (see Remark 4), we obtain two new cylinders over the $y$ axis and $z$ axis, respectively, that have the same asymptotic behavior as the input surface. More precisely, we have that by eliminating the variables $s_1$, $s_2$ from the system $p_{11}(s_1, s_2) - xp_{12}(s_1, s_2) = p_{31}(s_1, s_2) - zp_{32}(s_1, s_2) = h(s_1, s_2) = 0$, where $h(s_1, s_2) = s_2^2 - 1,$ we get two implicit equations

$$\overline{f}_y^2(x,z) = 9x^2z^2 + 3x^2z - x^2 - 4x + 4 - 18xz + 12z - 18xz^2 + 9z^2$$

and

$$\overline{f}_y^2(x,z) = 9x^2z^2 + 3x^2z - x^2 - 4x + 4 - 18xz + 12z - 18xz^2 + 9z^2$$

that defines two cylinders, $V_1$ and $V_2$, over the $y$ axis that has the same asymptotic behavior as $V$ (note that the polynomial $h$ factorizes as $h = (s_2 - 1)(s_2 + 1)$). In Figure 5, we plot $V$ and $V_i$, $i = 1, 2$ (left), $V_1$ (center), and $V_2$ (right).

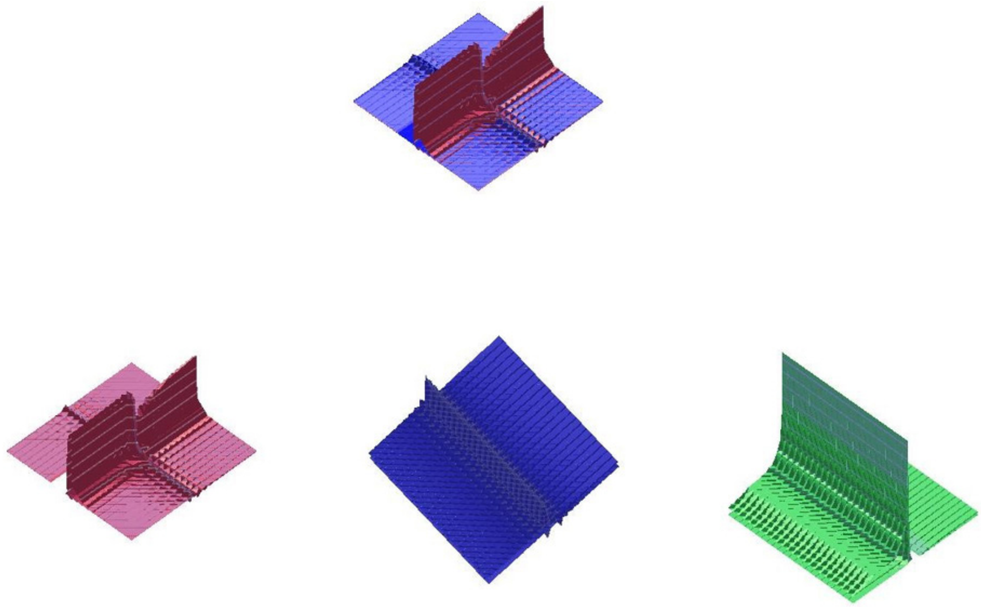

**Figure 5.** First row: $V$ and $V_i$, $i = 1, 2$ together. Second row: surfaces $V$ (right), $V_1$ (center) and $V_2$ (right)

Similarly, by eliminating the variables $s_1$, $s_2$ from the system $p_{21}(s_1, s_2) - yp_{22}(s_1, s_2) = p_{11}(s_1, s_2) - xp_{12}(s_1, s_2) = h(s_1, s_2) = 0$, where $h(s_1, s_2) = s_2 + 2$, we get the implicit equation

$$\overline{f}_z(x, y) = 44x^2 - 9x^2y^2 + 6x^2y + 18xy^2 - 28x + 15xy - 9 - 18y - 9y^2$$

that defines a cylinder over the $z$ axis that has the same asymptotic behavior as $V$. In Figure 6, we plot the surface $V$ and $V_1$ together (left), surface $V$ (center), and surface $V_1$ (right).

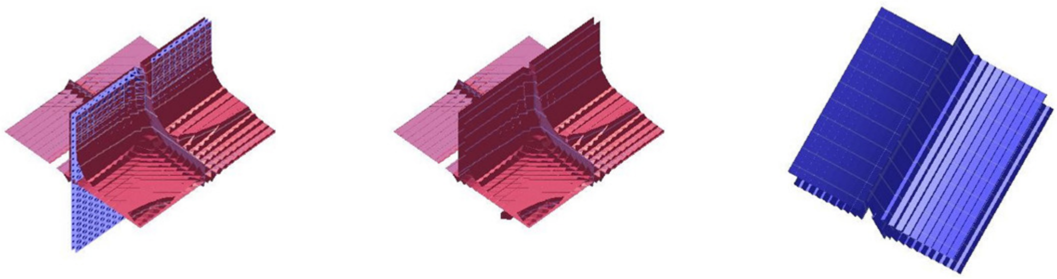

**Figure 6.** Surface $V$ and $V_1$ together (left), surface $V$ (center) and surface $V_1$ (right)

## 4. Conclusion

In this work, using some previous concepts introduced in [16] as branches at infinity, we obtain a method to compute infinity branches and surfaces having the same asymptotic behavior as an input surface that is parametrically defined. The results are a great novelty and represent an important advance for the analysis of surfaces and the study of their applications. In fact, since infinity branches reflect the status of a given surface at the infinity, in future work, our idea is to use these entities to deal with some important problems, such as plotting surfaces, problems of modeling or blending, high-dimensional interpolation, rational approximation of non-rational curves and surfaces, etc. (see [5-9]).

As in the case of curves, some important questions that should be answered remain open (see [11, 17]). More precisely, we would be interested in formally introducing the notion of perfect surface as well as some properties and effective algorithms for computing generalized asymptotes from implicit and parametrically defined surfaces.

Furthermore, as we stated previously, one should deeply study the computation of $\varphi_i$, $i = 1, 2$. In addition, we note that in the approach we present in this paper, once we have computed the branches, we determine surfaces that have the same asymptotic behavior as the original surface by simply removing the terms with negative exponent from $r_1$ and $r_2$ (w.r.t. $t_1$). This question should be carefully analyzed since if we remove these terms in the variable $t_1$, we could be removing terms in the variable $t_2$ that could be necessary for the approximation between surfaces.

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## Conflict of interest

All authors declare that they have no conflicts of interest.

## References

[1] Hoffmann CM, Sendra JR, Winkler F. Special issue on parametric algebraic-curves and applications - foreword. *Journal of Symbolic Computation*. 1997; 23(2-3): 133-135.

[2] Hoschek J, Lasser D. *Fundamentals of computer aided geometric design.* Wellesley, MA: A K Peters; 1993.

[3] Sendra JR, Winkler F, Pérez-Díaz S. *Rational algebraic curves: A computer algebra approach*. Berlin, Heidelberg: Springer; 2007.

[4] Walker RJ. *Algebraic curves*. New York: Springer; 1978.

[5] Pérez-Díaz S, Sendra J, Sendra JR. Parametrization of approximate algebraic curves by lines. *Theoretical Computer Science*. 2004; 315(2-3): 627-650.

[6] Pérez-Díaz S, Sendra J, Sendra JR. Parametrization of approximate algebraic surfaces by lines. *Computer Aided Geometric Design*. 2005; 22(92): 147-181.

[7] Pérez-Díaz S, Sendra JR. Rueda SL, Sendra J. Approximate parametrization of plane algebraic curves by linear systems of curves. *Computer Aided Geometric Design*. 2010; 27(2): 212-231.

[8] Pérez-Díaz S, Sendra JR. Computing all parametric solutions for blending parametric surfaces. *Journal of Symbolic Computation*. 2003; 36(6): 925-964.

[9] Shen L.Y., Wang X-Y., Pérez-Díaz S., Yuan C-M. Globally Certified C1 approximation of planar algebraic curves. *Journal of Computational and Applied Mathematics*. 2024; 436: 115399.

[10] Blasco A, Pérez-Díaz S. Asymptotic behavior of an implicit algebraic plane curve. *Computer Aided Geometric Design.* 2014; 31(7-8): 345-357.

[11] Blasco A, Pérez-Díaz S. Asymptotes and perfect curves. *Computer Aided Geometric Design*. 2014; 31(2): 81-96.

[12] Gao B, Chen Y. Finding the topology of implicitly defined two algebraic plane curves. *Journal of Systems Science and Complexity*. 2012; 25(2): 362-374.

[13] González-Vega L, Necula I. Efficient topology determination of implicitly defined algebraic plane curves. *Computer Aided Geometric Design*. 2002; 19(9): 719-743.

[14] Hong H. An effective method for analyzing the topology of plane real algebraic curves. *Mathematics and Computers in Simulation*. 1996; 42(4-6): 571-582.

[15] Zeng G. Computing the asymptotes for a real plane algebraic curve. *Journal of Algebra*. 2007; 316(2): 680-705. Available from: https://doi.org/10.1016/j.jalgebra.2007.03.030.

[16] Campo-Moltalvo E, de Sevilla MF, Pérez-Díaz S. Asymptotic behavior of a surface implicitly defined. *Mathematics*. 2022; 10(9): 1445.

[17] Blasco A. Pérez-Díaz S. Asymptotes of space curves. *Journal of Computational and Applied Mathematics*. 2015; 278: 231-247.

[18] Adrovic D, Verschelde J. Computing Puiseux series for algebraic surfaces. In: *ISSAC '12: Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation*. New York, USA: Association for Computing Machinery; 2012. p.20-27.

[19] Duval D. Rational Puiseux expansions. *Compositio Mathematica*. 1989; 70: 119-154.

[20] Soto MJ, Vicente JL. The Newton Procedure for several variables. *Linear Algebra and its Applications*. 2011; 435: 255-269. Available from: 10.1016/j.laa.2011.01.033.

[21] Stadelmeyer P. *On the computational complexity of resolving curve singularities and related problems*. PhD thesis. Johannes Kepler University Linz; 2000.