Comput. Aided Geom. Des. 107 (2023) 102255

Contents lists available at ScienceDirect

Computer Aided Geometric Design

journal homepage: www.elsevier.com/locate/cagd

Detecting and parametrizing polynomial surfaces without base points ☆

Sonia Pérez-Díaz [a,*], Marian Fernández de Sevilla [a], Rafael Magdalena Benedicto [b], Li-Yong Shen [c]

[a] Universidad de Alcalá, Dpto. de Física y Matemáticas, Dpto. de Ciencias de la Computación, E-28871 Madrid, Spain
[b] Universidad de Valencia, IDAL Electronic Engineering Department, E-46100 Burjassot, Valencia, Spain
[c] School of Mathematical Sciences, University of Chinese Academy of Sciences, 100049, Beijing, China

ARTICLE INFO

Keywords:
Polynomial surfaces
Rational parametrization
Base points

ABSTRACT

Given an algebraic surface implicitly defined by an irreducible polynomial, we present a method that decides whether or not this surface can be parametrized by a polynomial parametrization without base points and, in the affirmative case, we show how to compute this parametrization.

1. Introduction

We present some absolutely novel results which allow us to decide whether or not an algebraic surface can be polynomially parameterized with the additional characteristic that the polynomial parameterization does not have base points. From these results, a constructive algorithm is derived for such a polynomial parametrization if one exists.

Although in the case of curves the problem of polynomial parametrizations is well studied and characterized (see Manocha and Canny, 1991, and Sendra et al., 2007), for the case of surfaces, as far as the authors' knowledge is concerned, only some partial results are known, such as approximate polynomial constructions (Pérez–Díaz et al., 2007, Shen and Pérez–Díaz, 2019, Wang et al., 2024) or questions related to the polynomial reparametrization and base points (see Cox et al., 2022 and Pérez-Díaz and Sendra, 2020).

In recent times, the study of these varieties has generated a great deal of interest especially for the applications of surfaces in important problems related to different areas such as engineering, industry, modeling or artificial intelligence (Hoffmann et al., 1993, Hoschek and Lasser, 1993). Also studied are the reparametrization of surfaces (see Pérez-Díaz, 2006, Pérez-Díaz, 2013), the calculation of the implicit equation that defines a surface represented by a parametrization (see Pérez-Díaz and Sendra, 2008 or Alcázar and Pérez-Díaz, 2020), calculation of the fiber of the rational application induced by the parametrization and its singularities (see Pérez-Díaz and Sendra, 2004 and Pérez–Díaz et al., 2015), etc.

The polynomial character of a variety (a surface or a curve) is interesting because of the applications it implicitly permits. For example, plotting in the polynomial case avoids many problems in computer-aided geometric design, or in the geometric modeling (Pérez-Díaz and Sendra, 2003). When one has a rational parametrization of a surface, the numerical instability when the values,

☆ Editor: Falai Chen.

* Corresponding author.
E-mail addresses: sonia.perez@uah.es (S. Pérez-Díaz), marian.fernandez@uah.es (M. Fernández de Sevilla), rafael.magdalena@uv.es (R.M. Benedicto), lyshen@ucas.ac.cn (L.-Y. Shen).

https://doi.org/10.1016/j.cagd.2023.102255
Received 16 May 2023; Received in revised form 12 November 2023; Accepted 13 November 2023

Available online 15 November 2023

substituted in the parameters of the parametrizations, get close to the poles of the rational functions is a problematic issue for the applications mentioned above since in this case, the denominators define algebraic curves whose points are all poles of the parametrization.

The problem dealt with in this paper could be of interest to the computer aided geometric design (CAGD) community. In CAGD, polynomial parametrizations are widely used to represent curves and surfaces. They provide a concise mathematical representation, are computationally efficient, and can be manipulated easily to achieve various shapes. The most commonly used polynomial parametric representations are Bézier curves and B-spline curves. For the case of surfaces, the applications of polynomial parametrizations in CAGD are very important as for instance in shape design since with the help of polynomial curves and surfaces, designers in industries such as automotive, aerospace, and product design can create and manipulate shapes efficiently. In computer graphics and animation, polynomial curves and surfaces are essential tools since they help in defining the motion paths or morphing shapes. Polynomial surfaces can be used to define the boundary of objects, and mathematical tools can quickly check for intersections or close approaches, which are critical in simulations or video games. In robotics and CNC machining, polynomial curves can represent the path that a robot arm or tool should follow, ensuring smooth and predictable motions. In computer graphics, parametric surfaces can be used to map 2D textures onto 3D objects seamlessly. In engineering simulations, polynomial parametrizations can help in creating an approximate solution domain where computations are performed (see e.g. Boehm et al., 1984, Cohen et al., 2001, Farin, 2002, Farin et al., 2002, Hoschek and Lasser, 1993, Piegl and Tiller, 1997, Rogers and Adams, 1990).

In this paper, we first present some preliminary results that essentially deal with bases points (see Section 2). The non-existence of base points is also important since, in general, these points are obstructions to obtaining properties of hypersurfaces (even in implicitness, see Pérez-Díaz and Sendra, 2008 or Pérez-Díaz and Sendra, 2013). It is thus important to study when such a parameterization can be found.

In Section 3, important results characterizing the existence of polynomial parameterizations without base points are presented. The proofs are constructive and it is shown how to construct the polynomial parametrization, if it exists. Note that sometimes the polynomial parameterization must necessarily be improper. Nevertheless, given an algebraic surface $\mathcal{V}$, one may apply Schicho's algorithm (see Schicho, 1998) and compute a rational proper parametrization $\mathcal{P}$ of $\mathcal{V}$. One could then apply Pérez-Díaz and Sendra (2020) and polynomially parameterize $\mathcal{P}$, if a proper polynomial parametrization exists. However, there are some problems with this way of proceeding. On the one hand, the method would be much less effective than the one presented in this paper in which only one system needs to be solved. On the other hand, one could only find proper polynomial parametrization, if one exists (in this paper, we are able to find also improper polynomial parametrizations). Moreover and more important, to apply the reparametrization algorithm presented in Pérez-Díaz and Sendra (2020), one needs an additional transversality condition to be satisfied by $\mathcal{P}$. This condition, which has to do with the intersection of the base points, is restrictive and in case it is not satisfied, the algorithm presented in Pérez-Díaz and Sendra (2020) cannot be applied.

Finally, once the theoretical results that support the method developed here are proved, in Section 4, we present an algorithm for polynomial parametrization which is illustrated in detail by several examples.

It would be interesting to continue this study on the polynomial character of a given algebraic surface, even if the polynomial parametrization that we could find has base points. However, as can be seen in this article, the non-existence of bases points is essential and forms the cornerstone in the initialization of the method developed here.

In addition, as the reader can check throughout the paper, the terminology *without base points* refers to a compactification of the affine space to a projective space. This is a specific choice and there are many other ways to compactify, as for instance as a product of two projective lines (e.g. tensor-product surfaces). This approach will be a very interesting idea for a future work that opens important questions such as if it could be possible that a surface does not admit a polynomial parameterization without base points in $\mathbb{P}^2$ (in the sense of this paper) but admits a polynomial parameterization without base points over a compactification other than $\mathbb{P}^2$.

2. Notation and preliminaries

In this section, we introduce some previous results related to base points and the notation that will be used throughout this paper. For more details, refer to Cox et al. (2022).

In the following, $\mathbb{K}$ is an algebraically closed field of characteristic zero, $\overline{x} = (x_1, \ldots, x_4), \overline{y} = (y_1, \ldots, y_4)$ and $\overline{t} = (t_1, t_2, t_3)$. Furthermore, $\mathbb{F}$ is the algebraic closure of $\mathbb{K}(\overline{x}, \overline{y})$, and $\mathbb{P}^k(\mathbb{K})$ denotes k–dimensional projective space.

For a rational map

$$\begin{array}{cccc} \mathcal{M}: & \mathbb{P}^{k_1}(\mathbb{K}) & \dashrightarrow & \mathbb{P}^{k_2}(\mathbb{K}) \\ & \overline{h} = (h_1 : \cdots : h_{k_1+1}) & \longmapsto & (m_1(\overline{h}) : \cdots : m_{k_2+1}(\overline{h})), \end{array}$$

where the non-zero m_i are homogeneous polynomials in $\overline{h}$ of the same degree, we denote by $\deg(\mathcal{M})$ the degree $\deg_{\overline{h}}(m_i)$, and by $\mathrm{degMap}(\mathcal{M})$ the degree of the map $\mathcal{M}$; that is, the cardinality of a generic fiber of $\mathcal{M}$ (see e.g., Harris, 1995). Recall that the rational map $\mathcal{M}$ is proper or invertible if $\mathrm{degMap}(\mathcal{M}) = 1$. Otherwise, we say that $\mathcal{M}$ is improper.

Let $g \in \mathbb{L}[t_1, t_2, t_3]$ be homogeneous and non-zero, where $\mathbb{L}$ is a field extension of $\mathbb{K}$. Then, $\mathscr{C}(g)$ denotes the projective plane curve defined by g over the algebraic closure of $\mathbb{L}$. Hence, if $\mathscr{C}(f), \mathscr{C}(g)$ are two curves in $\mathbb{P}^2(\mathbb{K})$ and $A \in \mathbb{P}^2(\mathbb{K})$, we represent by $\mathrm{mult}_A(\mathscr{C}(f), \mathscr{C}(g))$ the multiplicity of the intersection of $\mathscr{C}(f)$ and $\mathscr{C}(g)$ at A. In addition, we denote by $\mathrm{mult}(A, \mathscr{C}(f))$ the multiplicity of $\mathscr{C}(f)$ at A.

Finally, $\mathcal{V} \subset \mathbb{P}^3(\mathbb{K})$ represents a rational projective surface, and

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{P}^2(\mathbb{K}) & \dashrightarrow & \mathcal{V} \subset \mathbb{P}^3(\mathbb{K}) \\ & \bar{t} & \longmapsto & (p_1(\bar{t}) : \cdots : p_4(\bar{t})) \end{array}$$

denotes a rational parametrization of the projective rational surface $\mathcal{V}$, where p_i are homogeneous non-zero polynomials of the degree n and $\gcd(p_1, \ldots, p_4) = 1$.

Definition 1. A base point of $\mathcal{P}$ is a point $A \in \mathbb{P}^2(\mathbb{K})$ such that $p_i(A) = 0$ for every $i \in \{1, 2, 3, 4\}$. $\mathscr{B}(\mathcal{P})$ represents the set of base points of $\mathcal{P}$; i.e. $\mathscr{B}(\mathcal{P}) = \mathscr{C}(p_1) \cap \cdots \cap \mathscr{C}(p_4)$.

In order to deal with the base points of the parametrization, we introduce the following auxiliary polynomials:

$$W_1(\bar{x}, \bar{t}) := \sum_{i=1}^{4} x_i\, p_i(t_1, t_2, t_3), \qquad W_2(\bar{y}, \bar{t}) := \sum_{i=1}^{4} y_i\, p_i(t_1, t_2, t_3),$$

where x_i, y_i are new variables. We will work with the projective plane curves $\mathscr{C}(W_i)$ in $\mathbb{P}^3(\mathbb{F})$. Under these conditions, using the multiplicity of intersection of these two curves, we define the multiplicity of a base point as follows.

Definition 2. The *multiplicity of a base point* $A \in \mathscr{B}(\mathcal{P})$ is $\mathrm{mult}_A(\mathscr{C}(W_1), \mathscr{C}(W_2))$, that is, is the multiplicity of intersection at A of $\mathscr{C}(W_1)$ and $\mathscr{C}(W_2)$; we denote this multiplicity by

$$\mathrm{mult}(A, \mathscr{B}(\mathcal{P})) := \mathrm{mult}_A(\mathscr{C}(W_1), \mathscr{C}(W_2))$$

In addition, we define the *multiplicity of the base points locus of* $\mathcal{P}$, denoted $\mathrm{mult}(\mathscr{B}(\mathcal{P}))$, as

$$\mathrm{mult}(\mathscr{B}(\mathcal{P})) := \sum_{A \in \mathscr{B}(\mathcal{P})} \mathrm{mult}(A, \mathscr{B}(\mathcal{P})) = \sum_{A \in \mathscr{B}(\mathcal{P})} \mathrm{mult}_A(\mathscr{C}(W_1), \mathscr{C}(W_2)).$$

Note that, since $gcd(p_1, \ldots, p_4) = 1$, the set $\mathscr{B}(\mathcal{P})$ is either empty of finite.

We have the following degree formula relating degrees and base point locus multiplicity. For this purpose, in the following, we denote by $\deg(\mathcal{V})$ the algebraic degree of $\mathcal{V}$, and $\mathrm{Content}_{\upsilon}(p)$ represents the content of a polynomial p with respect to a set of variables υ (that is, the greatest common divisor of the coefficients of p with respect to the variables υ).

Theorem 1.

$$\mathrm{mult}(\mathscr{B}(\mathcal{P})) = \deg(\mathcal{P})^2 - \deg(\mathcal{V}) \cdot \mathrm{degMap}(\mathcal{P}).$$

Furthermore, one may compute $\mathrm{mult}(\mathscr{B}(\mathcal{P}))$ *as*

$$\mathrm{mult}(\mathscr{B}(\mathcal{P})) = \deg_{\bar{t}}(\mathrm{Content}_{\{\bar{x}, \bar{y}\}}(\mathrm{Res}_{t_3}(W_1, W_2))).$$

The above formulae shows that the multiplicity of the base point locus of a projective rational surface parametrization can be expressed as the degree of the content of a univariate resultant (see Cox et al., 2022). The equality $\mathrm{mult}(\mathscr{B}(\mathcal{P})) = \deg(\mathcal{P})^2 - \deg(\mathcal{V}) \cdot \mathrm{degMap}(\mathcal{P})$ relates the degree of the surface, the degree of the parametrization, the base point multiplicity and the degree of the rational map induced by the parametrization.

The following corollary is a direct consequence of Theorem 1.

Corollary 1.

1. *If* $\mathcal{P}$ *is birational, then* $\deg(\mathcal{P})^2 - \mathrm{mult}(\mathscr{B}(\mathcal{P})) = \deg(\mathcal{V})$.
2. *A rational surface whose degree is not the square of a natural number cannot be birationally parametrized without base points in* $\mathbb{P}^2(\mathbb{K})$.

3. Polynomial parametrizations without base points

In this section, we deal with the problem of computing, if it exists, a polynomial parametrization without base points of a given implicitly defined algebraic surface.

For this purpose, let $\mathcal{V}$ be a projective surface defined by the homogeneous irreducible polynomial

$$F(\bar{x}) = f_d(x_1, x_2, x_3) - (x_4 f_{d-1}(x_1, x_2, x_3) + \cdots + x_4^d f_0) \in \mathbb{K}[\bar{x}],$$

where $d := \deg(\mathcal{V})$ and $f_j(x_1, x_2, x_3)$ are homogeneous polynomials of degree j, for $j = 0, \ldots, d$. Recall that the corresponding affine surface is obtained by dehomogenizing w.r.t. the variable x_4. We denote the corresponding affine surface by $\mathcal{V}_a$ defined by the irreducible polynomial $f(x_1, x_2, x_3) = F(x_1, x_2, x_3, 1) \in \mathbb{K}[x_1, x_2, x_3]$.

Furthermore, if $\mathcal{P}(\bar{t}) = (p_1(\bar{t}) : \cdots : p_4(\bar{t}))$ is a parametrization of $\mathcal{V}$, the affine rational parametrization given by

$$\mathcal{P}_a(t_1,t_2) = (p_1(t_1,t_2)/p_4(t_1,t_2), p_2(t_1,t_2)/p_4(t_1,t_2), p_3(t_1,t_2)/p_4(t_1,t_2))$$

parametrizes $\mathcal{V}_a$.

Recall that a surface $\mathcal{V}$ is *polynomial* if $\mathcal{V}$ can be parametrized by a polynomial parametrization $\mathcal{P}$ of the form

$$\mathcal{P}(\bar{t}) = (p_1(\bar{t}) : p_2(\bar{t}) : p_3(\bar{t}) : p_4(\bar{t})) \in \mathbb{P}^3(\mathbb{K}), \quad p_4(\bar{t}) = t_3^n,$$

where $n := \deg(\mathcal{P})$. Observe that this is equivalent to saying that $\mathcal{V}_a$ is polynomial since in this case, $\mathcal{V}_a$ can be parametrized by

$$\mathcal{P}_a(t_1,t_2) = (p_1(t_1,t_2,1), p_2(t_1,t_2,1), p_3(t_1,t_2,1)) \in \mathbb{K}[t_1,t_2]^3.$$

We **assume** without loss of generality that $\deg(p_i(t_1,t_2,1)) = n$, $i = 1,2,3$. Otherwise, we apply a linear change of coordinates on $\mathcal{V}_a$. Clearly this new surface is polynomial if and only if $\mathcal{V}_a$ is polynomial. Furthermore, if one computes a polynomial parametrization of this new surface, undoing the change of coordinates, we get a polynomial parametrization for $\mathcal{V}_a$. We also assume that $n \geq 2$ since for the case $n = 1$, $\mathcal{V}$ is a plane (see Corollary 1) and for this case, we may easily compute a proper polynomial parametrization.

As we state in the introduction, one may apply Schicho's algorithm (Schicho, 1998) to determine a proper rational parametrization $\mathcal{P}$ of $\mathcal{V}$. Afterwards, one could apply Pérez-Díaz and Sendra (2020), and polynomially parameterize $\mathcal{P}$, if a suitable polynomial parametrization exists. However, using this process some problems arise. First, the method would be much less efficient than the one presented in this paper, in which we only have to solve a linear system. Second, only proper polynomial parameterizations could be found, if they exist, but in this work, we are able to find improper polynomial parameterizations as well. Moreover, in order to apply the reparametrization algorithm presented in Pérez-Díaz and Sendra (2020), $\mathcal{P}$ is required to satisfy an additional transversality *the transversality* condition. This transversality condition, which has to do with the intersection of the base points, is restrictive and in case it is not fulfilled, the algorithm in Pérez-Díaz and Sendra (2020) can not be applied.

We start with the following theorem, where properties related to the desired polynomial parametrization without base points are proved.

Theorem 2. *Let $\mathcal{P}$ be a polynomial parametrization of degree n of a given surface $\mathcal{V}$ of degree d. Let*

$$\mathcal{Q}_{\mathcal{P}}(t_1,t_2) = (q_1(t_1,t_2) : q_2(t_1,t_2) : q_3(t_1,t_2)), \qquad q_i(t_1,t_2) = p_i(t_1,t_2,0),\ i = 1,2,3.$$

Then

1. *$\mathcal{P}$ does not have base points if and only if $n^2 = d \cdot \mathrm{degMap}(\mathcal{P})$. Furthermore, $\mathcal{P}$ is proper if and only if $n^2 = d$.*
2. *$\mathcal{P}$ does not have base points if and only if $\deg(\mathcal{Q}_{\mathcal{P}}) = n$.*
3. *Let $\deg(\mathcal{Q}_{\mathcal{P}}) = k \geq 1$. Then $\mathcal{Q}_{\mathcal{P}}$ is a homogeneous parametrization of a projective plane curve, $\mathcal{C}_{\mathcal{V}}$, defined by an irreducible component of f_d.*
4. *Let $\mathcal{P}$ be without base points. $\mathcal{Q}_{\mathcal{P}}$ is proper if and only if $f_d(x_1,x_2,x_3) = g(x_1,x_2,x_3)^n$, where $g(x_1,x_2,x_3)$ is the irreducible polynomial of degree n defining the curve $\mathcal{C}_{\mathcal{V}}$.*
5. *Let $\mathcal{P}$ be without base points. $\mathcal{Q}_{\mathcal{P}}$ is improper if and only if $f_d(x_1,x_2,x_3) = h(x_1,x_2,x_3)^n$, where $h(x_1,x_2,x_3)$ is the irreducible polynomial of degree $\ell < n$ defining the curve $\mathcal{C}_{\mathcal{V}}$.*

Proof.

1. The first statement follows from the degree formula

$$\mathrm{mult}(\mathscr{B}(\mathcal{P})) = \deg(\mathcal{P})^2 - \deg(\mathcal{V}) \cdot \mathrm{degMap}(\mathcal{P})$$

introduced in Theorem 1. If $\mathcal{P}$ does not have base points then $\mathrm{mult}(\mathscr{B}(\mathcal{P})) = \emptyset$ and we get

$$\deg(\mathcal{P})^2 = n^2 = \deg(\mathcal{V}) \cdot \mathrm{degMap}(\mathcal{P}) = d \cdot \mathrm{degMap}(\mathcal{P}).$$

In addition, $\mathcal{P}$ is proper if and only if $\mathrm{degMap}(\mathcal{P}) = 1$ which is equivalent to $n^2 = d$.

2. The condition $\mathrm{mult}(\mathscr{B}(\mathcal{P})) = \emptyset$ is equivalent to

$$\gcd(p_1(t_1,t_2,0), p_2(t_1,t_2,0), p_3(t_1,t_2,0)) = 1.$$

Note that

$$\deg(\gcd(p_1(t_1,t_2,0), p_2(t_1,t_2,0), p_3(t_1,t_2,0))) > 1$$

if and only if there exists a point $A \in \mathbb{P}^2(\mathbb{K})$ such that $p_i(A) = 0$ for every $i \in \{1,2,3,4\}$.

3. Since

$$0 = F(\mathcal{P}(\bar{t})) = F(\mathcal{P}(t_1,t_2,0)) = f_d(\mathcal{Q}_{\mathcal{P}})$$

we deduce that $\mathcal{Q}_{\mathcal{P}}$ parametrizes the projective plane curve, $\mathcal{C}_{\mathcal{V}}$, defined by an irreducible component of f_d.

4. Since $\deg(\mathcal{Q}_\mathcal{P}) = n$ (statement 2), $\mathcal{Q}_\mathcal{P}$ is proper if and only if $\mathcal{Q}_\mathcal{P}$ parametrizes an irreducible plane curve of degree n (see Sendra et al., 2007). Thus, since $\deg(\mathcal{Q}_\mathcal{P}) = n = \deg(p_1) = \deg(p_2) = \deg(p_3)$ and $d = n^2/\mathrm{degMap}(\mathcal{P})$,

$$f_d(x_1,x_2,x_3) = g(x_1,x_2,x_3)^{n/\mathrm{degMap}(\mathcal{P})},$$

where $g(x_1,x_2,x_3)$ is the irreducible polynomial of degree n defining the curve $\mathcal{C}_\mathcal{V}$ (see Pérez-Díaz and Sendra, 2004 and Chapter 4 in Sendra et al., 2007).
5. First, observe that $\deg(\mathcal{Q}_\mathcal{P}) = n$ (statement 2). Therefore $\mathcal{Q}_\mathcal{P}$ is improper if and only if $\mathcal{Q}_\mathcal{P} = \overline{\mathcal{Q}}_\mathcal{P}(R)$, where $\deg(R) = r > 1$ and $\deg(\overline{\mathcal{Q}}_\mathcal{P}) = \ell < n$. Hence $\deg(\mathcal{Q}_\mathcal{P}) = r \cdot \ell = n$, and hence $\overline{\mathcal{Q}}_\mathcal{P}$ parametrizes a plane curve of degree ℓ (see Sendra et al., 2007). Therefore,

$$f_d(x_1,x_2,x_3) = h(x_1,x_2,x_3)^{r\cdot n/\mathrm{degMap}(\mathcal{P})},$$

where $h(x_1,x_2,x_3)$ is the irreducible polynomial of degree $\ell < n$ defining the curve $\mathcal{C}_\mathcal{V}$ (we recall that from Sendra et al., 2007, if we compute the implicit equation from $\mathcal{Q}_\mathcal{P}$ we get h^r). □

From Theorem 2, in particular statements 4 and 5, we easily get the following corollary.

Corollary 2. *Let $\mathcal{V}$ be a rational surface of degree d. If $\mathcal{V}$ can be parametrized by a polynomial parametrization $\mathcal{P}$ of degree n without base points then*

$$f_d(x_1,x_2,x_3) = h(x_1,x_2,x_3)^\beta$$

where $h(x_1,x_2,x_3)$ is an irreducible polynomial of degree $\alpha \le n$, and $d = \alpha \cdot \beta = n^2/\mathrm{degMap}(\mathcal{P})$.

Remark 1. Recall that in the case of algebraic plane curves, the problem of polynomial parametrizations is thoroughly studied and characterized (see Manocha and Canny, 1991, and Chapter 6 of Sendra et al., 2007). In particular and in order to compare with Corollary 2, a plane curve of degree d can be parametrized by a polynomial parametrization if and only if $f_d(x_1,x_2) = (bx_1 - ax_2)^d, a, b \in \mathbb{K}$ and there exists a unique infinity branch at the infinity point $(a : b : 0)$.

For the case of surfaces, as far as the authors' knowledge is concerned, only some partial results are known, such as approximate polynomial constructions (Pérez–Díaz et al., 2007, Shen and Pérez–Díaz, 2019, Wang et al., 2024) or questions concerning polynomial reparametrization and base points (see Cox et al., 2022 and Pérez-Díaz and Sendra, 2020). In particular, Pérez-Díaz and Sendra (2020) presents an algorithm for reparametrizing birational surface parametrizations into birational polynomial surface parametrizations without base points, if they exist. However, no results relating the highest order form of the implicit polynomial are provided.

In the following, and in order to simplify the reasoning, we assume that we are given a surface $\mathcal{V}$ of degree d such that

$$f_d(x_1,x_2,x_3) = g(x_1,x_2,x_3)^{n/\mu},$$

where $g(x_1,x_2,x_3)$ is an irreducible polynomial of degree n, and $\mu \in \mathbb{N}$, $\mu \neq 0$. Observe that if $\mathcal{V}$ can be parametrized by a parametrization $\mathcal{P}$ of degree n without base points then $\mu = \mathrm{degMap}(\mathcal{P})$. However, note that some surfaces could be parameterized polynomially by an improper parametrization (see Example 4).

Under these conditions, we may compute $\mathcal{Q}_\mathcal{P}$ a proper parametrization of the curve $\mathcal{C}_\mathcal{V}$ represented by the homogeneous polynomial $g(x_1,x_2,x_3)$. Observe that if $\mathcal{P}$ is a polynomial parametrization without base points of degree n of a given surface $\mathcal{V}$ of degree d, then $\mathcal{Q}_\mathcal{P}$ exists. The converse of this statement is not true that is, one may have $\mathcal{Q}_\mathcal{P}$ and still $\mathcal{V}$ can not be parametrized polynomially by a parametrization without base points (see Example 3).

For the case of a given surface $\mathcal{V}$ of degree d such that

$$f_d(x_1,x_2,x_3) = h(x_1,x_2,x_3)^{r\cdot n/\mu},$$

where $h(x_1,x_2,x_3)$ is an irreducible polynomial of degree ℓ, and $\mu \in \mathbb{N}$, $\mu \neq 0$, we will see that the theoretical reasoning is similar to Remark 3. In this case, a rational improper parametrization of degree n of the curve defined by h has to be computed (see Remark 3 and Example 2). However, we will see that some technical difficulties arise.

Finally, by abuse of notation we represent the components p_i, $i = 1,2,3$, of the parametrization we are looking for, $\mathcal{P} = (p_1 : p_2 : p_3 : t_3^n)$, by

$$(p_1(\bar{t}), p_2(\bar{t}), p_3(\bar{t})) = \mathcal{P}_n(t_1,t_2) + t_3\mathcal{P}_{n-1}(t_1,t_2) + t_3^2/2!\mathcal{P}_{n-2}(t_1,t_2) + \cdots + t_3^n/n!\mathcal{P}_0(t_1,t_2),$$

where $\mathcal{P}_i(t_1,t_2) \in \mathbb{K}[t_1,t_2]^3$ are homogeneous and $\deg(\mathcal{P}_i) = i$, for $i = 1,\ldots,n$. Note that $\mathcal{P}_n(t_1,t_2) = \mathcal{Q}_\mathcal{P}(t_1,t_2)$.

Furthermore, in the following, $\mathcal{T}_{\rho,t_3=0,s}(\bar{t})$ will denote the Taylor polynomial of $\rho(\bar{t})$ at $t_3 = 0$ of order $s \in \mathbb{N}$.

Theorem 3 presents a necessary and sufficient condition for the existence of a homogeneous polynomial parametrization of $\mathcal{V}$ without base points. The proof of this theorem is constructive and then a method solving the problem of polynomial parametrization is derived.

Theorem 3. *$\mathcal{P}$ is a homogeneous polynomial parametrization of $\mathcal{V}$ with $\mu = \text{degMap}(\mathcal{P})$ and without base points if and only if*

$$\mathcal{T}_{U,t_3=0,n^2}(\bar{t}) = t_3^{\mu}\, \mathcal{T}_{V^{\mu/n},t_3=0,n^2}(\bar{t}) \tag{3.1}$$

where

$$U(\bar{t}) := g(\mathcal{P}), \qquad V(\bar{t}) := f_{d-1}(\mathcal{P}) + t_3^n f_{d-2}(\mathcal{P}) + \cdots + t_3^{n(n^2-1)} f_0.$$

In particular, equality (3.1) is equivalent to the system

$$\begin{cases} U(t_1,t_2,0) = 0 \\ \frac{\partial^{\ell} U}{\partial t_3}(t_1,t_2,0) = 0, \;\; \ell = 1,\ldots,\mu-1 \\ \frac{\partial^{\mu} U}{\partial^{\mu} t_3}(t_1,t_2,0) = \mu! \cdot V(t_1,t_2,0)^{\frac{\mu}{n}} \\ \frac{\partial^{\mu+1} U}{\partial^{\mu+1} t_3}(t_1,t_2,0) = \mu! \cdot \frac{\mu}{n} V(t_1,t_2,0)^{\frac{\mu-n}{n}} \cdot \frac{\partial V}{\partial t_3}(t_1,t_2,0) \\ \vdots \\ \frac{\partial^{\mu+k} U}{\partial^{\mu+k} t_3}(t_1,t_2,0) = \frac{\mu!\cdot k\cdot \mu}{n} \cdots \frac{\mu-(k-2)n}{n} V(t_1,t_2,0)^{\frac{\mu-(k-1)n}{n}} \cdot (\frac{\partial V}{\partial t_3}(t_1,t_2,0))^{k-1} + \cdots + \\ \frac{\mu!\cdot\mu\cdot k}{n} V(t_1,t_2,0)^{\frac{\mu-n}{n}} \cdot \frac{\partial^{k-1} V}{\partial^{k-1} t_3}(t_1,t_2,0) \end{cases} \tag{3.2}$$

for $k = 1,\ldots,n^2-\mu$, $n \geq 2$.

Proof. Since $F(\mathcal{P}) = 0$, where

$$F(\bar{x}) = f_d(x_1,x_2,x_3) - (x_4 f_{d-1}(x_1,x_2,x_3) + \cdots + x_4^d f_0),$$
$$f_d(\mathcal{P}) = t_3^n f_{d-1}(\mathcal{P}) + \cdots + t_3^{nd} f_0 = t_3^n (f_{d-1}(\mathcal{P}) + \cdots + t_3^{n(d-1)} f_0).$$

From Theorem 2, we get that the previous equality is equivalent to

$$U(\bar{t}) = t_3^{\mu}\, V(\bar{t})^{\mu/n}$$

which implies that

$$\mathcal{T}_{U,t_3=0,n^2}(\bar{t}) = t_3^{\mu}\, \mathcal{T}_{V^{1/n},t_3=0,n^2}(\bar{t}).$$

On the other hand, observe that $U(\bar{t}) = t_3^{\mu} V(\bar{t})^{\mu/n}$ is equivalent to $U(\bar{t})^{n/\mu} = t_3^n V(\bar{t})$ and $\deg(U) = \deg(t_3^{\mu} V^{\mu/n}) = n^2$. Hence, if $\mathcal{T}_{U,t_3=0,n^2}(\bar{t}) = t_3^{\mu}\, \mathcal{T}_{V^{\mu/n},t_3=0,n^2}(\bar{t})$ by the properties of Taylor polynomial we deduce that $U(\bar{t})^{n/\mu} = t_3^n V(\bar{t})$ and then, $F(\mathcal{P}) = 0$, and $\mathcal{P}$ is polynomial. Note that $\deg(\mathcal{P}) = \deg(\mathcal{Q}_{\mathcal{P}}) = n$ and $n^2 = d \cdot \mu$ hence, $\mathcal{P}$ does not have base points.

Finally, we observe that equality (3.1) is equivalent to the system (3.2), which is obtained by equating the coefficients w.r.t. the variable t_3. □

From the proof of the previous theorem, we deduce the following corollaries where the main result is that Theorem 3 provides a unique solution (if it exists) depending on at most two undetermined parameters.

In the following, we denote by $\Gamma_{\mathcal{N}}$ the set of undetermined parameters of a certain parametrization $\mathcal{N}$, and $\text{Card}(\Gamma_{\mathcal{N}})$ represents the cardinal of this set.

Corollary 3. *Let $\mathcal{N}(\bar{t}, \Gamma_{\mathcal{N}})$ be a parametrization satisfying*

$$\mathcal{T}_{U,t_3=0,k}(\bar{t}) = t_3^{\mu}\, \mathcal{T}_{V^{\mu/n},t_3=0,k}(\bar{t})$$

for some $k \in \mathbb{N}$ with $1 \leq k \leq n^2$ and $Card(\Gamma_{\mathcal{N}}) \leq 2$. If $\mathcal{V}$ can be parametrized by a polynomial parametrization, $\mathcal{P}$ with $\deg(\mathcal{P}) = n$, $\text{degMap}(\mathcal{P}) = \mu$ and $\mathcal{P}$ does not have base points, then $\mathcal{N}$ is a polynomial parametrization of $\mathcal{V}$ without base points.

Proof. $\mathcal{N}(\bar{t}, \Gamma_{\mathcal{N}})$ denotes a solution of the system obtained from the equality $\mathcal{T}_{U,t_3=0,k}(\bar{t}) = t_3^{\mu}\, \mathcal{T}_{V^{\mu/n},t_3=0,k}(\bar{t})$ for some $k \in \mathbb{N}$ with $1 \leq k \leq n^2$, with $\text{Card}(\Gamma_{\mathcal{N}}) \leq 2$. Observe that by construction $\deg(\mathcal{N}) = n$ and $\mathcal{N}$ does not have base points. Furthermore $\text{degMap}(\mathcal{N}) = \mu$.

Since $\mathcal{V}$ can be parametrized by $\mathcal{P}$, which is a polynomial parametrization of degree n, with $\text{degMap}(\mathcal{P}) = \mu$ and without base points, then $\mathcal{P}(R) = \mathcal{N}$, where $R = (r_1, r_2, t_3)$, $r_i \in \mathbb{K}[\bar{t}]$ and $\deg(r_1) = \deg(r_2) = 1$ (see Pérez-Díaz, 2006 or Pérez-Díaz, 2013). Thus, $r_1 = t_1 + at_2$ and $r_2 = t_2 + bt_1$, which implies that $\text{Card}(\Gamma_{\mathcal{N}}) \leq 2$ and $\mathcal{N}$ is a polynomial parametrization of $\mathcal{V}$. □

Corollary 4. *Let $\mathcal{N}(\bar{t}, \Gamma_{\mathcal{N}})$ be a parametrization satisfying the equalities in the system (3.2). Then, $Card(\Gamma_{\mathcal{N}}) \leq 2$ and $\mathcal{N}$ is the unique solution of the system (3.2).*

Proof. Let $\mathcal{P}_i(\bar{t}, \Gamma_{\mathcal{P}_i})$ two solutions of the system (3.2) where $\text{Card}(\Gamma_{\mathcal{P}_i}) \leq 2$ (see Corollary 3). By construction, $\mathcal{P}_i$ do not have base points, $\text{degMap}(\mathcal{P}_i) = \mu$ and $\deg(\mathcal{P}_i) = n$ for $i = 1, 2$. Hence $\mathcal{P}_1(R) = \mathcal{P}_2$, where $R = (r_1, r_2, t_3)$, $r_i \in \mathbb{K}[\bar{t}]$ and $\deg(r_1) = \deg(r_2) = 1$ (see Pérez-Díaz, 2006 or Pérez-Díaz, 2013). Thus, $r_1 = t_1 + at_2$ and $r_2 = t_2 + bt_1$, which implies that $\Gamma_{\mathcal{P}_1} = \Gamma_{\mathcal{P}_2} \subseteq \{a, b\}$. Therefore, the system provides a unique solution that depends at most on two unknown parameters $\{a, b\}$. □

The system (3.2) can be expressed by the equations we present in Remark 2, which involve only the derivatives of $U(\bar{t})$ and $V(\bar{t})$ with respect to the variable t_3.

In order to present these equations, we introduce the following notation: let $H(\overline{x})$ be a homogeneous polynomial and a projective parametrization $\mathcal{M} = (m_1, m_2, m_3, m_4)$ depending on some variables, in particular in the variable s. The polynomial H_{x_k} represents the partial derivative w.r.t. x_k of H. We will consider $H(\mathcal{M})$, and

$$\nabla(H) \cdot \frac{\partial \mathcal{M}}{\partial s} = H_{x_1}(\mathcal{M})\frac{\partial m_1}{\partial s} + H_{x_2}(\mathcal{M})\frac{\partial m_2}{\partial s} + H_{x_3}(\mathcal{M})\frac{\partial m_3}{\partial s} + H_{x_4}(\mathcal{M})\frac{\partial m_4}{\partial s}.$$

Furthermore, if we differentiate again, we obtain

$$H_{x_1x_1}(\mathcal{M})(\frac{\partial m_1}{\partial s})^2 + 2H_{x_1x_2}(\mathcal{M})\frac{\partial m_1}{\partial s}\frac{\partial m_2}{\partial s} + 2H_{x_1x_3}(\mathcal{M})\frac{\partial m_1}{\partial s}\frac{\partial m_3}{\partial s} + 2H_{x_1x_4}(\mathcal{M})\frac{\partial m_1}{\partial s}\frac{\partial m_4}{\partial s} + H_{x_2x_2}(\mathcal{M})(\frac{\partial m_2}{\partial s})^2$$
$$+ 2H_{x_2x_3}(\mathcal{M})\frac{\partial m_2}{\partial s}\frac{\partial m_3}{\partial s} + 2H_{x_2x_4}(\mathcal{M})\frac{\partial m_2}{\partial s}\frac{\partial m_4}{\partial s} + H_{x_3x_3}(\mathcal{M})(\frac{\partial m_3}{\partial s})^2 + 2H_{x_3x_4}(\mathcal{M})\frac{\partial m_3}{\partial s}\frac{\partial m_4}{\partial s} + H_{x_4x_4}(\mathcal{M})(\frac{\partial m_4}{\partial s})^2,$$

and we represent this polynomial by

$$\nabla^2(H) \cdot \left(\frac{\partial \mathcal{M}}{\partial s}\right)^2 + \nabla(H) \cdot \frac{\partial^2 \mathcal{M}}{\partial^2 s}.$$

If we differentiate again, we would obtain

$$\nabla^3(H) \cdot \left(\frac{\partial \mathcal{M}}{\partial s}\right)^3 + 4\nabla^2(H) \cdot \left(\frac{\partial \mathcal{M}}{\partial s}\right)\left(\frac{\partial^2 \mathcal{M}}{\partial^2 s}\right) + \nabla(H) \cdot \frac{\partial^3 \mathcal{M}}{\partial^3 s}.$$

In a similar way these polynomials would be represented for higher order derivatives.

Remark 2.

$$\begin{cases} U(t_1, t_2, 0) = g(\mathcal{Q}_{\mathcal{P}}) \\ \frac{\partial U}{\partial t_3}(t_1, t_2, 0) = \nabla g(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-1} \\ \frac{\partial^2 U}{\partial^2 t_3}(t_1, t_2, 0) = \nabla^2 g(\mathcal{Q}_{\mathcal{P}}) \cdot (\mathcal{P}_{n-1})^2 + \nabla g(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-2} \\ \vdots \\ \frac{\partial^k U}{\partial^k t_3}(t_1, t_2, 0) = \nabla^k g(\mathcal{Q}_{\mathcal{P}}) \cdot (\mathcal{P}_{n-1})^k + \cdots + \nabla g(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-k} \end{cases} \tag{3.3}$$

and

$$\begin{cases} V(t_1, t_2, 0) = f_{d-1}(\mathcal{Q}_{\mathcal{P}}) \\ \frac{\partial V}{\partial t_3}(t_1, t_2, 0) = \nabla f_{d-1}(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-1} \\ \frac{\partial^2 V}{\partial^2 t_3}(t_1, t_2, 0) = \nabla^2 f_{d-1}(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-1} + \nabla f_{d-1}(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-2} \\ \vdots \\ \frac{\partial^k V}{\partial^k t_3}(t_1, t_2, 0) = \nabla^k f_{d-1}(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-1}^k + \cdots + \nabla f_{d-1}(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-k} \end{cases} \tag{3.4}$$

for $k=1,\dots,n^2$ $(n\geq 2)$.

From the previous equalities and Theorem 3, we get

$$\begin{cases} g(\mathcal{Q}_{\mathcal{P}})=0 \\ \nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-1}=0 \\ \nabla^{\ell} g(\mathcal{Q}_{\mathcal{P}})\cdot(\mathcal{P}_{n-1})^{\ell}+\cdots+\nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-\ell}=0, \qquad \ell=1,\dots,\mu-1 \\ \nabla^{\mu} g(\mathcal{Q}_{\mathcal{P}})\cdot(\mathcal{P}_{n-1})^{\mu}+\cdots+\nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-\mu}=\mu!\cdot f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu}{n}} \\ \nabla^{\mu+1} g(\mathcal{Q}_{\mathcal{P}})\cdot(\mathcal{P}_{n-1})^{\mu+1}+\cdots+\nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-\mu-1}=\mu!\cdot\frac{\mu}{n}f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu-n}{n}}\nabla f_{d-1}(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-1} \\ \vdots \\ \nabla^{\mu+k} g(\mathcal{Q}_{\mathcal{P}})\cdot(\mathcal{P}_{n-1})^{\mu+k}+\cdots+\nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-\mu-k}= \\ \mu!\cdot\frac{\mu\cdot k}{n}\cdots\frac{\mu-(k-2)n}{n}f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu-(k-1)n}{n}}(\nabla f_{d-1}(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-1})^{k-1}+\cdots+ \\ \mu!\cdot\frac{\mu\cdot k}{n}f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu-n}{n}}\frac{\partial^{k-1}f_{d-1}(\mathcal{P})}{\partial^{k-1}t_3}(0), \qquad k=1,\dots,n^2-\mu. \end{cases} \tag{3.5}$$

Note that since $\deg(g)=n$,

$$\nabla^k g(x_1,x_2,x_3)=0,\ k>n.$$

In addition, since $\deg(f_{d-1})=d-1$, we get

$$\nabla^k f_{d-1}(x_1,x_2,x_3)=0,\ k>d-1.$$

In fact,

$$\nabla^k f_j(x_1,x_2,x_3)=0,\ k>j$$

because $\deg(f_j)=j$, for $j=0,\dots,d-1$.

Finally observe that from the equation

$$\nabla^{\mu} g(\mathcal{Q}_{\mathcal{P}})\cdot(\mathcal{P}_{n-1})^{\mu}+\cdots+\nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_{n-\mu}=\mu!\cdot f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu}{n}}$$

we deduce that if V can n be parametrized by a polynomial parametrization without base points then $f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu}{n}}$ should be a homogeneous polynomial in the variables t_1,t_2.

In the following theorem, we prove that the computation of the polynomial parametrization, if it exists, is independent of the rational parametrization, $\mathcal{P}_n(t_1,t_2)$, of the curve $\mathcal{C}_{\mathcal{V}}$.

Theorem 4. *Let $\mathcal{M}(t_1,t_2)$ be a homogeneous proper parametrization of $\mathcal{C}_{\mathcal{V}}$. If $\mathcal{V}$ can be parametrized by a polynomial parametrization, $\mathcal{P}$, without base points, then there exists a reparametrization of $\mathcal{P}$, $\mathcal{N}=\mathcal{P}(r_1,r_2,t_3)$ where $\deg(r_i)=1, r_i\in\mathbb{K}[t_1,t_2]$, and $\mathcal{N}$ is a polynomial parametrization of $\mathcal{V}$ without base points. Furthermore, $\mathcal{Q}_{\mathcal{P}}=\mathcal{Q}_{\mathcal{N}}=\mathcal{M}$.*

Proof. First observe that since $\mathcal{M}$ is a proper parametrization of $\mathcal{C}_{\mathcal{V}}$, we have $\deg(\mathcal{M})=n$ (see Chapter 4 in Sendra et al., 2007).

On the other hand, since $\mathcal{V}$ can be parametrized by a polynomial parametrization $\mathcal{P}$ of degree n without base points, using statement 2 of Theorem 2, we obtain $\mathcal{M}=\mathcal{Q}_{\mathcal{P}}(r_1,r_2)$, where $r_i(t_1,t_2)\in\mathbb{K}[t_1,t_2]$ are homogeneous polynomials of degree 1 (note that $\deg(\mathcal{Q}_{\mathcal{P}})=n$ and see Pérez-Díaz, 2006). Therefore, $\mathcal{N}=\mathcal{P}(r_1,r_2,t_3)$ is a polynomial reparametrization of $\mathcal{V}$ and $\deg(\mathcal{N})=\deg(\mathcal{P})=n$ and $\mathrm{degMap}(\mathcal{N})=\mathrm{degMap}(\mathcal{P})$. Hence, $n^2=d\cdot\mathrm{degMap}(\mathcal{N})$, which allows us to conclude that $\mathcal{N}$ does not have base points. Finally, we also get $\mathcal{Q}_{\mathcal{P}}=\mathcal{Q}_{\mathcal{N}}=\mathcal{M}$. □

Once we have computed a rational parametrization of $g(x_1,x_2,x_3)$, namely $\mathcal{P}_n(t_1,t_2)$, we solve the system (3.2) (see also Remak 2) which satisfies the following important property that allows one to easily solve the equations involved. The key is that $f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\mu/n}$ should be a homogeneous polynomial in the variables t_1,t_2 since otherwise $\mathcal{V}$ can not be parametrized by a polynomial parametrization without base points (see Remark 2).

Lemma 1. *The system (3.2) is a triangular system in the undetermined coefficients of $\mathcal{P}_i$, for $i=1,\dots,n$. Furthermore, in order to compute $\mathcal{P}_i$, in general, we do not need to solve the whole system.*

Algorithm Polynomial parametrization.

Given an implicit algebraic projective surface $\mathcal{V}$ defined by the polynomial, $F(\overline{x}) = f_d(x_1, x_2, x_3) - (x_4 f_{d-1}(x_1, x_2, x_3) + \cdots + x_4^d f_0)$, $\deg(F) = d$, this algorithm decides whether $\mathcal{V}$ can be parametrized by a polynomial parametrization without base points.

1. Check whether $f_d(x_1, x_2, x_3) = g(x_1, x_2, x_3)^{n/\mu}$, where $g(x_1, x_2, x_3)$ is an irreducible polynomial of degree n defining a rational plane curve $\mathcal{C}_{\mathcal{V}}$, and $\mu \in \mathbb{N}$, $\mu \neq 0$. In the affirmative case, go to Step 2 (note that if $\mathcal{V}$ can be parametrized by a polynomial parametrization, $\mathcal{P}$, without base points then $\mu = \text{degMap}(\mathcal{P})$ and $\deg(\mathcal{P}) = n$). Otherwise, check Remark 3 or RETURN $\mathcal{V}$ *can not parametrized by a polynomial parametrization without base points* (see Theorem 2).
2. Compute a rational proper parametrization of $\mathcal{C}_{\mathcal{V}}$, namely $\mathcal{P}_n(t_1, t_2) = \mathcal{Q}_{\mathcal{P}}(t_1, t_2)$.
3. Represent the components of the parametrization we are looking for, $\mathcal{P} = (p_1 : p_2 : p_3 : t_3^n)$, by

$$(p_1(\overline{t}), p_2(\overline{t}), p_3(\overline{t})) = \mathcal{P}_n(t_1, t_2) + t_3 \mathcal{P}_{n-1}(t_1, t_2) + t_3^2/2! \mathcal{P}_{n-2}(t_1, t_2) + \cdots + t_3^n/n! \mathcal{P}_0(t_1, t_2),$$

   where

$$\mathcal{P}_{n-k}(t_1, t_2) = (a_{1,n-k} t_1^{n-k} + \cdots + a_{n+1-k,n-k} t_2^{n-k}, b_{1,n-k} t_1^{n-k} + \cdots + b_{n+1-k,n-k} t_2^{n-k}, c_{1,n-k} t_1^{n-k} + \cdots + c_{n+1-k,n-k} t_2^{n-k}) \in \mathbb{K}[t_1, t_2]^3,\ k = 1, \ldots, n$$

   are homogeneous and $\deg(\mathcal{P}_i) = i$, for $i = 1, \ldots, n$.
4. For k from 1 to s, where $s \geq n$ is such that $\text{Card}(\Gamma_{\mathcal{P}}) > 2$, solve the system

$$\begin{cases} \nabla^{\ell} g(\mathcal{Q}_{\mathcal{P}}) \cdot (\mathcal{P}_{n-1})^{\ell} + \cdots + \nabla g(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-\ell} = 0, \qquad \ell = 1, \ldots, \mu - 1 \\[2ex] \nabla^{\mu+k} g(\mathcal{Q}_{\mathcal{P}}) \cdot (\mathcal{P}_{n-1})^{\mu+k} + \cdots + \nabla g(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-\mu-k} = \\ \mu! \cdot \frac{\mu \cdot k}{n} \cdots \frac{\mu-(k-2)n}{n} f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu-(k-1)n}{n}} (\nabla f_{d-1}(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_{n-1})^{k-1} + \cdots + \\ \mu! \cdot \frac{\mu \cdot k}{n} f_{d-1}(\mathcal{Q}_{\mathcal{P}})^{\frac{\mu-n}{n}} \frac{\partial^{k-1} f_{d-1}(\mathcal{P})}{\partial^{k-1} t_3}(0), \qquad k = 1, \ldots, n^2 - \mu. \end{cases} \tag{4.1}$$

5. Check whether

$$\mathcal{T}_{U, t_3=0, n^2}(\overline{t}) = t_3^{\mu}\, \mathcal{T}_{V^{\mu/n}, t_3=0, n^2}(\overline{t}), \quad U = g(\mathcal{P}),\ V = f_{d-1}(\mathcal{P}) + \cdots + t_3^{n(n^2-1)} f_0$$

   or $F(\mathcal{P}) = 0$. In the affirmative case, RETURN $\mathcal{V}$ *can be parametrized by a polynomial parametrization defined by* $\mathcal{P}$ *with* $\text{degMap}(\mathcal{P}) = \mu$ *and without base points*. Otherwise, check Remark 3 or RETURN $\mathcal{V}$ *can not parametrized by a polynomial parametrization without base points* (see Theorem 2).

Proof. In order to simplify the proof, we may assume with lost of generality that $\mu = 1$. Under these conditions, in the first equality, which is homogeneous of degree $(n-1)n + (n-1) = n^2 - 1$, we determine $\mathcal{P}_{n-1} =$

$$(a_{1,n-1} t_1^{n-1} + \cdots + a_{n,n-1} t_2^{n-1}, b_{1,n-1} t_1^{n-1} + \cdots + b_{n,n-1} t_2^{n-1}, c_{1,n-1} t_1^{n-1} + \cdots + c_{n,n-1} t_2^{n-1})$$

that has $3n$ undetermined parameters. Maybe some additional equations should be considered to compute completely $\mathcal{P}_{n-1}$ depending on the independent equations obtained from this first equality. These equations are clearly linear in $a_{i,n-1}, b_{i,n-1}, c_{i,n-1}$, $i = 1, \ldots, n$.

In the second equality, which is homogeneous of degree $(n-2)n + 2(n-1) = n^2 - 2$, we determine

$$\mathcal{P}_{n-2} = (a_{1,n-2} t_1^{n-2} + \cdots + a_{n-1,n-2} t_2^{n-2}, b_{1,n-2} t_1^{n-2} + \cdots + b_{n-1,n-2} t_2^{n-2}, c_{1,n-2} t_1^{n-2} + \cdots + c_{n-1,n-2} t_2^{n-2})$$

which has $3(n-1)$ undetermined parameters. Maybe some additional equations should be considered to compute completely $\mathcal{P}_{n-2}$ depending on the independent equations obtained from the second equality. These equations are clearly linear in $a_{i,n-2}, b_{i,n-2}, c_{i,n-2}$, $i = 1, \ldots, n-1$.

In general, in the k-th equality (for $k = 1, \ldots, n$), which is homogeneous of degree $(n-k)n + k(n-1) = n^2 - k$, we determine

$$\mathcal{P}_{n-k} = (a_{1,n-k} t_1^{n-k} + \cdots + a_{n-k+1,n-k} t_2^{n-k}, b_{1,n-k} t_1^{n-k} + \cdots + b_{n-k+1,n-k} t_2^{n-k}, c_{1,n-k} t_1^{n-k} + \cdots + c_{n-k+1,n-k} t_2^{n-k})$$

which has $3(n-k+1)$ undetermined parameters. Maybe some additional equations should be considered to compute completely $\mathcal{P}_{n-k}$ depending on the independent equations obtained from this equality. These equations are clearly linear in $a_{i,n-k}, b_{i,n-k}, c_{i,n-k}$, $i = 1, \ldots, n-k+1$.

For $k = n+1, \ldots, n^2$, we obtain equalities of degree $(n-k)n + k(n-1) = n^2 - k$, and the undetermined parameters previously introduced that have not yet been calculated that is, we do not have new undetermined parameters.

The system whose undetermined parameters are $a_{i,j}, b_{i,j}, c_{i,j}$ only provides a solution in each step (see Corollary 4) and we have to consider new steps till $\text{Card}(\Gamma_{\mathcal{P}}) \leq 2$. □

4. Algorithm and examples

In this section, we summarize the method presented in Section 3 in the following algorithm, which decides whether a given implicit algebraic projective surface can be parametrized by a polynomial parametrization without base points.

Furthermore, we illustrate this algorithm with several examples where different situations are considered.

Remark 3. For the case of a given surface $\mathcal{V}$ of degree d such that

$$f_d(x_1,x_2,x_3)=h(x_1,x_2,x_3)^{r\cdot n/\mu},$$

where $h(x_1,x_2,x_3)$ is an irreducible polynomial of degree ℓ, and $\mu\in\mathbb{N}$, $\mu\neq 0$, we may apply the previous algorithm which is based in Theorem 3.

In *Step 1*, we have to factor the highest order form and to decide if the irreducible factor represents a rational curve. For this purpose, one may use previous results and algorithms some of them are implemented in well known mathematical software (in this paper we use Maple 2022). More precisely, for factorizing multivariate polynomials, the *factor* command in Maple 2022 offers two algorithms: Wang's algorithm (see Wang, 1978) and the algorithm by Monagan and Tuncer (Monagan and Tuncer, 2016 and Monagan and Tuncer, 2018). The problem of deciding the rationality of a plane curve can be approached using for instance the results of Chapters 3 and 4 in Sendra et al. (2007).

In *Step 2*, we compute a non-proper rational parametrization of $C_{\mathcal{V}}$ of degree n. For this purpose, we determine a proper parametrization $\mathcal{M}$ of degree ℓ and we consider $\mathcal{M}(t_1^r,t_2^r)=\mathcal{Q}_{\mathcal{P}}(t_1,t_2)$. Let $\mathcal{P}_n(t_1,t_2)=\mathcal{Q}_{\mathcal{P}}(t_1,t_2)$. One may prove as in Theorem 4 that if a polynomial parametrization exists, it is independent of the rational parametrization of the curve $C_{\mathcal{V}}$.

Step 4 is similar but the systems obtained are not linear. In this case, and reasoning as in Theorem 3, we consider

$$U(\bar{t}):=f_d(\mathcal{P}),\qquad V(\bar{t}):=f_{d-1}(\mathcal{P})+\cdots+t_3^{d-1}f_0$$

and we impose $\mathcal{T}_{U,t_3=0,d\cdot n}(\bar{t})=t_3\,\mathcal{T}_{V,t_3=0,d\cdot n}(\bar{t})$ (note that some of the coefficients of these Taylor polynomials are identically zero). Since the system obtained is not linear, in this case the computations are more effective if we do not use radicals.

In the following examples, we illustrate the method presented in this paper for computing a polynomial parametrization without base points of a given surface $\mathcal{V}$, if one exists. In the first example (Example 1) we obtain a proper polynomial parametrization but in the last example (Example 4), the polynomial parametrization is improper. Additionally, we also show an example where the algorithm returns that the input surface can not be parametrized by a polynomial parametrization without base points (Example 3), and in Example 2 we illustrate Remark 3.

Example 1. Let $\mathcal{V}$ be an algebraic surface of degree $d=9$ defined by the irreducible homogeneous polynomial

$$F(\overline{x})=f_9(x_1,x_2,x_3)-(x_4f_8(x_1,x_2,x_3)+\cdots+x_4^9f_0),$$

where

$$\begin{aligned}
f_9&=(x_1^3-3x_1^2x_2+3x_1x_2^2+x_1x_3^2-x_2^3)^3,\\
f_8&=-(7x_1^8-56x_1^7x_2+6x_1^7x_3+196x_1^6x_2^2-18x_1^6x_2x_3+8x_1^6x_3^2-392x_1^5x_2^3+18x_1^5x_2^2x_3-40x_1^5x_2x_3^2-2x_1^5x_3^3\\
&\quad+490x_1^4x_2^4-6x_1^4x_2^3x_3+80x_1^4x_2^2x_3^2+x_1^4x_3^4-392x_1^3x_2^5-80x_1^3x_2^3x_3^2-2x_1^3x_2x_3^4+196x_1^2x_2^6+40x_1^2x_2^4x_3^2+x_1^2x_2^2x_3^4\\
&\quad-56x_1x_2^7-8x_1x_2^5x_3^2+7x_2^8),\\
f_7&=-(22x_1^7-147x_1^6x_2+11x_1^6x_3+441x_1^5x_2^2-22x_1^5x_2x_3+2x_1^5x_3^2-735x_1^4x_2^3+11x_1^4x_2^2x_3-8x_1^4x_2x_3^2+735x_1^3x_2^4+12x_1^3x_2^2x_3^2\\
&\quad-441x_1^2x_2^5-8x_1^2x_2^3x_3^2+147x_1x_2^6+2x_1x_2^4x_3^2-21x_2^7-x_3^7),\\
f_6&=-(35x_1^6-210x_1^5x_2+4x_1^5x_3+525x_1^4x_2^2-4x_1^4x_2x_3-12x_1^4x_3^2-700x_1^3x_2^3+36x_1^3x_2x_3^2+525x_1^2x_2^4-36x_1^2x_2^2x_3^2+2x_1^2x_3^4\\
&\quad-210x_1x_2^5+12x_1x_2^3x_3^2+35x_2^6),\\
f_5&=-(35x_1^5-175x_1^4x_2-x_1^4x_3+350x_1^3x_2^2-13x_1^3x_3^2-350x_1^2x_2^3+26x_1^2x_2x_3^2+175x_1x_2^4-13x_1x_2^2x_3^2-35x_2^5),\\
f_4&=-(-x_2+x_1)(21x_1^3-63x_1^2x_2+63x_1x_2^2-4x_1x_3^2-21x_2^3),\\
f_3&=-7(-x_2+x_1)^3,\quad f_2=-(-x_2+x_1)^2,\quad f_1=0,\quad f_0=0.
\end{aligned}$$

In *Step 1*, we observe that

$$f_9(x_1,x_2,x_3)=g(x_1,x_2,x_3)^3=(x_1^3-3x_1^2x_2+3x_1x_2^2+x_1x_3^2-x_2^3)^3$$

and the curve $C_{\mathcal{V}}$ defined by the irreducible polynomial $g(x_1,x_2,x_3)$ with $\deg(g)=n=3$ is rational. This implies, that if a polynomial parametrization $\mathcal{P}=(p_1:p_2:p_3:t_3^n)$ without base points exists, then $\mathcal{P}$ would be proper and $\deg(\mathcal{P})=n=3$.

In *Step 2*, we compute a rational proper parametrization of $C_{\mathcal{V}}$. We get,

$$\mathcal{P}_3(t_1,t_2)=\mathcal{Q}_{\mathcal{P}}(t_1,t_2)=\left(t_1^3,t_1^3+t_1t_2^2,t_2^3\right).$$

In *Step 3*, we represent the components of the parametrization we are looking for, $\mathcal{P}$, as

$$(p_1(\bar{t}),p_2(\bar{t}),p_3(\bar{t}))=\mathcal{P}_3(t_1,t_2)+t_3\mathcal{P}_2(t_1,t_2)+t_3^2/2!\mathcal{P}_1(t_1,t_2)+t_3^3/3!\mathcal{P}_0(t_1,t_2),$$

where

$$\mathcal{P}_{3-k}(t_1,t_2) = (a_{1,3-k}t_1^{3-k} + \cdots + a_{4-k,3-k}t_2^{3-k}, b_{1,3-k}t_1^{3-k} + \cdots + b_{4-k,3-k}t_2^{3-k}, c_{1,3-k}t_1^{3-k} + \cdots + c_{4-k,3-k}t_2^{3-k}) \in \mathbb{K}[t_1,t_2]^3,$$
$$k = 1,2,3$$

are homogeneous and $\deg(\mathcal{P}_i) = i$ for $i = 1, \ldots, 3$.

In the *Step 4*, we compute $\mathcal{P}_2(t_1,t_2)$, $\deg(\mathcal{P}_2) = 2$. For this purpose, we consider the equation

$$\nabla g(Q_{\mathcal{P}}) \cdot \mathcal{P}_2 = f_8(Q_{\mathcal{P}})^{\frac{1}{3}}.$$

Observe that the system generated from the coefficients of the previous equality w.r.t. $\{t_1,t_2\}$ is linear in the undetermined coefficients of $\mathcal{P}_2$. We find that

$$\mathcal{P}_2(t_1,t_2) = (3b_{3,2}t_1^2, b_{1,2}t_1^2 + b_{3,2}t_2^2 + b_{2,2}t_1t_2, t_1^2 + 3/2b_{2,2}t_2^2 - 9/2t_1t_2b_{3,2} + 3/2t_1t_2b_{1,2}).$$

Now, we compute $\mathcal{P}_1(t_1,t_2)$, $\deg(\mathcal{P}_1) = 1$. We consider the equation

$$\nabla^2 g(Q_{\mathcal{P}}) \cdot (\mathcal{P}_2)^2 + \nabla g(Q_{\mathcal{P}}) \cdot \mathcal{P}_1 = \nabla f_8(Q_{\mathcal{P}}) \cdot \mathcal{P}_2.$$

Observe that the system generated from the coefficients of the previous equality w.r.t. $\{t_1,t_2\}$ is linear in the undetermined coefficients of $\mathcal{P}_1$. We find that

$$\mathcal{P}_1(t_1,t_2) = (6b_{3,2}^2t_1 + 2t_2, 2b_{3,2}b_{2,2}t_2 + b_{1,1}t_1 + 2t_2, 4b_{3,2}t_1 - 9t_2b_{3,2}^2 + 3/4t_2b_{2,2}^2 + 3/2t_2b_{1,1})$$
$$\mathcal{P}_2(t_1,t_2) = (3b_{3,2}t_1^2, 3b_{3,2}t_1^2 + b_{3,2}t_2^2 + b_{2,2}t_1t_2, t_1^2 + 3b_{2,2}t_2^2/2).$$

We compute $\mathcal{P}_0(t_1,t_2)$, $\deg(\mathcal{P}_0) = 0$. We consider the equation

$$\nabla^3 g(Q_{\mathcal{P}}) \cdot (\mathcal{P}_2)^3 + 3\nabla^2 g(Q_{\mathcal{P}}) \cdot \mathcal{P}_2\mathcal{P}_1 + \nabla g(Q_{\mathcal{P}}) \cdot \mathcal{P}_0 = \nabla^2 f_8(Q_{\mathcal{P}}) \cdot \mathcal{P}_2 + \nabla f_8(Q_{\mathcal{P}}) \cdot \mathcal{P}_1.$$

Observe that the system generated from the coefficients of the previous equality w.r.t. $\{t_1,t_2\}$ is linear in the undetermined parameters of $\mathcal{P}_0$. We find that

$$\mathcal{P}_0(t_1,t_2) = (6b_{3,2}^3 + 3b_{2,2}, -12b_{3,2}^3 + 3b_{3,2}b_{1,1} + 3b_{2,2}, -27/2b_{3,2}^2b_{2,2} - 3/8b_{2,2}^3 + 6b_{3,2}^2 + 9/4b_{2,2}b_{1,1}).$$
$$\mathcal{P}_1(t_1,t_2) = (6b_{3,2}^2t_1 + 2t_2, 2b_{3,2}b_{2,2}t_2 + b_{1,1}t_1 + 2t_2, 4b_{3,2}t_1 - 9t_2b_{3,2}^2 + 3/4t_2b_{2,2}^2 + 3/2t_2b_{1,1})$$
$$\mathcal{P}_2(t_1,t_2) = (3b_{3,2}t_1^2, 3b_{3,2}t_1^2 + b_{3,2}t_2^2 + b_{2,2}t_1t_2, t_1^2 + (3b_{2,2}t_2^2)/2).$$

Note that the number of undetermined coefficients in $\mathcal{P}$ w.r.t. the variables $\{t_1,t_2\}$ is 3. So, we consider more equations till the number of undetermined parameters decreases (see Corollary 3). Hence, we consider the equation

$$\nabla^4 g(Q_{\mathcal{P}}) \cdot (\mathcal{P}_2)^4 + 6\nabla^3 g(Q_{\mathcal{P}}) \cdot (\mathcal{P}_2)^2\mathcal{P}_1 + 3\nabla^2 g(Q_{\mathcal{P}}) \cdot (\mathcal{P}_1)^2 + 4\nabla^2 g(Q_{\mathcal{P}}) \cdot \mathcal{P}_2\mathcal{P}_0$$
$$= \nabla^3 f_8(Q_{\mathcal{P}}) \cdot (\mathcal{P}_2)^2 + 2\nabla^2 f_8(Q_{\mathcal{P}}) \cdot \mathcal{P}_1 + \nabla f_8(Q_{\mathcal{P}}) \cdot \mathcal{P}_0,$$

from where we get a system that is linear in b_4, and we obtain

$$\mathcal{P} = (b_{3,2}^3t_3^3 + 1/2t_3^3b_{2,2} + 3b_{3,2}t_1^2t_3 + 3b_{3,2}^2t_1t_3^2 + t_2t_3^2 + t_1^3, b_{3,2}^3t_3^3 + 1/2t_3^3b_{2,2} + 1/4b_{3,2}b_{2,2}^2t_3^3 + 3b_{3,2}t_1^2t_3 + b_{3,2}t_2^2t_3 + b_{2,2}t_1t_2t_3$$
$$+ 3b_{3,2}^2t_1t_3^2 + 1/4t_3^2t_1b_{2,2}^2 + t_2t_3^2 + b_{3,2}b_{2,2}t_2t_3^2 + t_1^3 + t_1t_2^2, 1/8t_3^3b_{2,2}^3 + t_3^3b_{3,2}^2 + t_3t_1^2 + 3/2t_3b_{2,2}t_2^2 + 2b_{3,2}t_1t_3^2 + 3/4t_3^2t_2b_{2,2}^2 + t_2^3).$$

Note that $\mathrm{Card}(\Gamma_{\mathcal{P}}) = 2$. We go to *Step 5* and we check that $F(\mathcal{P}) = 0$. Therefore, the algorithm RETURNS that $\mathcal{V}$ *can be parametrized by a proper polynomial parametrization defined by* $\mathcal{P}$ *without base points*.

For the case of a given surface $\mathcal{V}$ of degree d such that

$$f_d(x_1,x_2,x_3) = h(x_1,x_2,x_3)^{r\cdot n/\mu},$$

where $h(x_1,x_2,x_3)$ is an irreducible polynomial of degree ℓ, and $\mu \in \mathbb{N}$, $\mu \neq 0$, we reason as we state in Remark 3. For this purpose, we consider

$$U(\bar{t}) := f_d(\mathcal{P}), \qquad V(\bar{t}) := f_{d-1}(\mathcal{P}) + \cdots + t_3^{d-1}f_0$$

and we impose $\mathcal{T}_{U,t_3=0,d\,n}(\bar{t}) = t_3\,\mathcal{T}_{V,t_3=0,d\,n}(\bar{t})$.

Example 2. Let $\mathcal{V}$ be an algebraic surface of degree $d = 4$ defined by the irreducible homogeneous polynomial

$$F(\bar{x}) = f_4(x_1,x_2,x_3) - (x_4f_3(x_1,x_2,x_3) + \cdots + x_4^4f_0),$$

where

$$f_4 = (x_1 - x_3)^4,$$
$$f_3 = -4(x_1 - x_3)^2(3x_1 - 8x_2 - 3x_3),$$
$$f_2 = -(-98x_1^2 - 192x_1x_2 + 196x_1x_3 + 256x_2^2 + 192x_2x_3 - 98x_3^2),$$
$$f_1 = -(-5604x_1 - 796x_3 + 2144x_2), \quad f_0 = -4489.$$

In *Step 1*, we observe that

$$f_4(x_1, x_2, x_3) = h(x_1, x_2, x_3)^4 = (x_1 - x_3)^4$$

and the curve $\mathcal{C}_{\mathcal{V}}$ defined by the polynomial $h(x_1, x_2, x_3) = (x_1 - x_3)$ with $\deg(h) = 1$ is rational. This implies, that if a polynomial parametrization $\mathcal{P}$ without base points exists, then $\mathcal{P}$ would be proper with $\deg(\mathcal{P}) = n = 2$ and we have to compute a non-proper parametrization $\mathcal{Q}_{\mathcal{P}}$ of the line $\mathcal{C}_{\mathcal{V}}$ defined by $h(x_1, x_2, x_3)$.

In *Step 2*, we compute a rational improper parametrization of degree $n = 2$ of the curve $\mathcal{C}_{\mathcal{V}}$. We get

$$\mathcal{P}_2(t_1, t_2) = \mathcal{Q}_{\mathcal{P}}(t_1, t_2) = \left(t_1^2, t_2^2, t_1^2\right).$$

In *Step 3*, we represent the components of the parametrization $\mathcal{P}$ we are looking for

$$(p_1(\bar{t}), p_2(\bar{t}), p_3(\bar{t})) = \mathcal{P}_2(t_1, t_2) + t_3\mathcal{P}_1(t_1, t_2) + t_3^2/2!\mathcal{P}_0(t_1, t_2),$$

where

$$\mathcal{P}_{2-k}(t_1, t_2) = (a_{1,2-k}t_1^{2-k} + \cdots + a_{3-k,2-k}t_2^{2-k}, b_{1,2-k}t_1^{2-k} + \cdots + b_{3-k,2-k}t_2^{2-k}, c_{1,2-k}t_1^{2-k} + \cdots + c_{3-k,2-k}t_2^{2-k}) \in \mathbb{K}[t_1, t_2]^3, \quad k = 1, 2$$

are homogeneous and $\deg(\mathcal{P}_i) = i$ for $i = 1, 2$.

In the *Step 4*, we compute $\mathcal{P}_1(t_1, t_2)$, $\deg(\mathcal{P}_1) = 1$. For this purpose, we consider

$$U(\bar{t}) := f_4(\mathcal{P}), \qquad V(\bar{t}) := f_3(\mathcal{P}) + \cdots + t_3^3 f_0$$

and we impose $\mathcal{T}_{U,t_3=0,8}(\bar{t}) - t_3\,\mathcal{T}_{V,t_3=0,8}(\bar{t}) = 0$. From the first non-zero coefficient of the previous equality we get a system (constructed from the coefficients w.r.t. $\{t_1, t_2\}$) in the undetermined coefficients of $\mathcal{P}_1$. We obtain,

$$\mathcal{P}_1(t_1, t_2) = (a_{1,1}t_1 + a_{2,1}t_2, b_{1,1}t_1 + b_{2,1}t_2, a_{1,1}t_1 + a_{2,1}t_2 - 4t_2).$$

Now, we compute $\mathcal{P}_0(t_1, t_2)$, $\deg(\mathcal{P}_0) = 1$ considering the second non-zero coefficient of the equality $\mathcal{T}_{U,t_3=0,8}(\bar{t}) - t_3\,\mathcal{T}_{V,t_3=0,8}(\bar{t}) = 0$. We find that

$$\mathcal{P}_0(t_1, t_2) = (c_{1,0}, b_{1,0}, c_{1,0}),$$

and

$$\mathcal{P} = (1/2c_{1,0}t_3^2 + a_{1,1}t_1t_3 + a_{2,1}t_2t_3 + t_1^2, 1/2b_{1,0}t_3^2 + b_{1,1}t_1t_3 + b_{2,1}t_2t_3 + t_2^2, 1/2c_{1,0}t_3^2 + a_{1,1}t_1t_3 + a_{2,1}t_2t_3 - 4t_2t_3 + t_1^2)$$

Note that the number of undetermined coefficients in $\mathcal{P}$ w.r.t. the variables $\{t_1, t_2\}$ is 6 (i.e. $\mathrm{Card}(\Gamma_{\mathcal{P}}) = 6$). So, we consider more equations till the number of undetermined parameters decreases (see Corollary 3). Hence, in the next step, we get a solution in the undetermined coefficients $b_{1,1}, b_{2,1}$, and

$$\mathcal{P} = (1/2c_{1,0}t_3^2 + a_{1,1}t_1t_3 + a_{2,1}t_2t_3 + t_1^2, 1/2b_{1,0}t_3^2 + 5t_1t_3 - 3/2t_3t_2 + t_2^2, 1/2c_{1,0}t_3^2 + a_{1,1}t_1t_3 + a_{2,1}t_2t_3 + 4t_2t_3 + t_1^2).$$

We observe that the number of undetermined coefficients in $\mathcal{P}$ w.r.t. the variables $\{t_1, t_2\}$ is 4 (i.e. $\mathrm{Card}(\Gamma_{\mathcal{P}}) = 4$). So, we consider more equations and we get a system that is linear in $a_{1,1}, a_{2,1}, b_{1,0}$. We find that

$$\mathcal{P} = (1/2c_{1,0}t_3^2 + a_{1,1}t_1t_3 - t_2t_3 + t_1^2, -67/16t_3^2 + 5/2t_3^2a_{1,1} + 5t_1t_3 - 3/2t_3t_2 + t_2^2, 1/2c_{1,0}t_3^2 + a_{1,1}t_1t_3 + 3t_2t_3 + t_1^2).$$

We consider two more equations defined from the Taylor polynomial, and we get the value for $b_{1,0}$ and $c_{1,0}$. We find that

$$\mathcal{P} = (1/4a_{1,1}^2t_3^2 + a_{1,1}t_1t_3 - t_2t_3 + t_1^2, -67/16t_3^2 + 5/2t_3^2a_{1,1} + 5t_1t_3 - 3/2t_3t_2 + t_2^2, 1/4a_{1,1}^2t_3^2 + a_{1,1}t_1t_3 + 3t_2t_3 + t_1^2).$$

Note that $\mathrm{Card}(\Gamma_{\mathcal{P}}) = 1$. We go to *Step 5* and we get $F(\mathcal{P}) = 0$. Hence, the algorithm RETURNS that $\mathcal{V}$ *can be parametrized by a proper polynomial parametrization* $\mathcal{P}$ *without base points*.

In the following example we conclude that the given input surface $\mathcal{V}$ does not have a polynomial parametrization without base points.

Example 3. Let $\mathcal{V}$ be an algebraic surface of degree $d=4$ defined by the irreducible homogeneous polynomial

$$F(\overline{x})=f_4(x_1,x_2,x_3)-(x_4f_3(x_1,x_2,x_3)+\cdots+x_4^4f_0),$$

where

$$f_4=(x_1^2-2x_1x_2-x_1x_3+x_2^2)^2,$$
$$f_3=-(4x_1^3-13x_1^2x_2-3x_1^2x_3+11x_1x_2^2+x_1x_2x_3-3x_2^3-x_3^3),$$
$$f_2=-(5x_1^2-9x_1x_2+x_1x_3+3x_2^2+3x_3^2),\quad f_1=-(-3x_3-x_2),\quad f_0=-1.$$

In *Step 1*, we observe that

$$f_4(x_1,x_2,x_3)=g(x_1,x_2,x_3)^2=(x_1^2-2x_1x_2-x_1x_3+x_2^2)^2$$

and the curve $\mathcal{C}_{\mathcal{V}}$ defined by the polynomial $g(x_1,x_2,x_3)$ with $\deg(g)=n=2$ is rational. This implies that if a polynomial parametrization $\mathcal{P}$ without base points exists, then $\mathcal{P}$ would be proper and $\deg(\mathcal{P})=n=2$.

In *Step 2*, we compute a rational proper parametrization

$$\mathcal{P}_2(t_1,t_2)=\mathcal{Q}_{\mathcal{P}}(t_1,t_2)=\left(t_1^2,t_1^2+t_1t_2,t_2^2\right).$$

In *Step 3*, we represent the components of the parametrization $\mathcal{P}$ we are looking for

$$(p_1(\bar{t}),p_2(\bar{t}),p_3(\bar{t}))=\mathcal{P}_2(t_1,t_2)+t_3\mathcal{P}_1(t_1,t_2)+t_3^2/2!\mathcal{P}_0(t_1,t_2),$$

where

$$\mathcal{P}_{2-k}(t_1,t_2)=(a_{1,2-k}t_1^{2-k}+\cdots+a_{3-k,2-k}t_2^{2-k},b_{1,2-k}t_1^{2-k}+\cdots+b_{3-k,2-k}t_2^{2-k},c_{1,2-k}t_1^{2-k}+\cdots+c_{2-k,2-k}t_2^{2-k})\in\mathbb{K}[t_1,t_2]^3,\quad k=1,2$$

are homogeneous and $\deg(\mathcal{P}_i)=i$ for $i=1,2$.

In the *Step 4*, we compute $\mathcal{P}_1(t_1,t_2)$, $\deg(\mathcal{P}_1)=2$. For this purpose, we consider the equation

$$\nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_1=f_4(\mathcal{Q}_{\mathcal{P}})^{\frac{1}{2}}.$$

Observe that the system generated from the coefficients of the previous equality w.r.t $\{t_1,t_2\}$ is linear in the undetermined coefficients of $\mathcal{P}_1$. We find that

$$\mathcal{P}_1(t_1,t_2)=((2+2b_{2,1})t_1-t_2,b_{1,1}t_1+b_{2,1}t_2,-t_1+(-4-4b_{2,1}+2b_{1,1})t_2).$$

Now, we compute $\mathcal{P}_0(t_1,t_2)$, $\deg(\mathcal{P}_0)=1$. We consider the equation

$$\nabla^2g(\mathcal{Q}_{\mathcal{P}})\cdot(\mathcal{P}_1)^2+\nabla g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_0=\nabla f_3(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_1.$$

Observe that the system generated from the coefficients of the previous equality w.r.t. $\{t_1,t_2\}$ is linear in the undetermined coefficients of $\mathcal{P}_0$. We find that

$$\mathcal{P}_0(t_1,t_2)=(2b_{2,1}^2-2b_{1,1}+8b_{2,1}+6,2b_{1,1}b_{2,1}-2b_{2,1}^2+2,2b_{1,1}^2-8b_{1,1}b_{2,1}+8b_{2,1}^2-8b_{1,1}+14b_{2,1}+8)$$

and

$$\mathcal{P}=(b_{2,1}^2t_3^2-b_{1,1}t_3^2+2b_{2,1}t_1t_3+4b_{2,1}t_3^2+t_1^2+2t_1t_3-t_2t_3+3t_3^2,b_{1,1}b_{2,1}t_3^2-b_{2,1}^2t_3^2+b_{1,1}t_1t_3+b_{2,1}t_2t_3+t_1^2+t_1t_2+t_3^2,b_{1,1}^2t_3^2$$
$$-4b_{1,1}b_{2,1}t_3^2+4b_{2,1}^2t_3^2+2b_{1,1}t_2t_3-4b_{1,1}t_3^2-4b_{2,1}t_2t_3+7b_{2,1}t_3^2-t_1t_3+t_2^2-4t_2t_3+4t_3^2).$$

One may check that there is no solution to the next equation given by

$$\nabla^3g(\mathcal{Q}_{\mathcal{P}})\cdot(\mathcal{P}_1)^3+3\nabla^2g(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_1\mathcal{P}_0=\nabla^2f_3(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_1+\nabla f_3(\mathcal{Q}_{\mathcal{P}})\cdot\mathcal{P}_0.$$

Thus the algorithm RETURNS that $\mathcal{V}$ *can not be parametrized by a polynomial parametrization without base points*.

In the following example we conclude that the given input surface $\mathcal{V}$ has a polynomial parametrization without base points which is improper.

Example 4. Let $\mathcal{V}$ be an algebraic surface of degree $d=2$ defined by the irreducible homogeneous polynomial

$$F(\overline{x})=f_2(x_1,x_2,x_3)-(x_4f_1(x_1,x_2,x_3)+x_4^2f_0),$$

where

$$f_2 = x_1^2 - x_2x_3, \quad f_1 = 2x_1, \; f_0 = -1.$$

In *Step 1*, we observe that

$$f_2(x_1, x_2, x_3) = g(x_1, x_2, x_3) = (x_1^2 - x_2x_3)$$

and the curve $\mathcal{C}_{\mathcal{V}}$ defined by the polynomial $g(x_1, x_2, x_3)$ with $\deg(g) = n = 2$ is rational. We observe that $n/\mu = 1$ which implies that if a polynomial parametrization $\mathcal{P}$ without base points exists, then $\mu = \text{degMap}(\mathcal{P}) = 2$ and $\deg(\mathcal{P}) = n = 2$.

In *Step 2*, we compute a rational proper parametrization

$$\mathcal{P}_2(t_1, t_2) = \mathcal{Q}_{\mathcal{P}}(t_1, t_2) = \left(t_1t_2, t_2^2, t_1^2\right).$$

In *Step 3*, we represent the components of the parametrization $\mathcal{P}$ we are looking for

$$(p_1(\bar{t}), p_2(\bar{t}), p_3(\bar{t})) = \mathcal{P}_2(t_1, t_2) + t_3\mathcal{P}_1(t_1, t_2) + t_3^2/2!\mathcal{P}_0(t_1, t_2),$$

where

$$\mathcal{P}_{2-k}(t_1, t_2) = (a_{1,2-k}t_1^{2-k} + \cdots + a_{3-k,2-k}t_2^{2-k}, b_{1,2-k}t_1^{2-k} + \cdots + b_{3-k,2-k}t_2^{2-k}, c_{1,2-k}t_1^{2-k} + \cdots + c_{3-k,2-k}t_2^{2-k}) \in \mathbb{K}[t_1, t_2]^3, \; k = 1, 2$$

are homogeneous and $\deg(\mathcal{P}_i) = i$ for $i = 1, 2$.

In the *Step 4*, we compute $\mathcal{P}_1(t_1, t_2)$, $\deg(\mathcal{P}_1) = 1$. For this purpose, we consider the equation

$$\nabla g(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_1 = f_1(\mathcal{Q}_{\mathcal{P}}).$$

Observe that this system is linear in the undetermined coefficients of $\mathcal{P}_1$. We find that

$$\mathcal{P}_1(t_1, t_2) = (a_{1,1}t_1 + a_{2,1}t_2, 2a_{1,1}t_2, 2a_{2,1}t_1).$$

Now, we compute $\mathcal{P}_0(t_1, t_2)$, $\deg(\mathcal{P}_0) = 1$. We consider the equation

$$\nabla^2 g(\mathcal{Q}_{\mathcal{P}}) \cdot (\mathcal{P}_1)^2 + \nabla g(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_0 = \nabla f_1(\mathcal{Q}_{\mathcal{P}}) \cdot \mathcal{P}_1.$$

Observe that this system is linear in the undetermined coefficients of $\mathcal{P}_0$. We find that

$$\mathcal{P}_0(t_1, t_2) = (2a_{1,1}a_{2,1} + 2, 2a_{1,1}^2, 2a_{2,1}^2),$$

and

$$\mathcal{P}(t_1, t_2) = (a_{1,1}a_{2,1}t_3^2 + a_{1,1}t_1t_3 + a_{2,1}t_2t_3 + t_1t_2 + t_3^2, (a_{1,1}t_3 + t_2)^2, (a_{2,1}t_3 + t_1)^2).$$

Note that $\text{Card}(\Gamma_{\mathcal{P}}) = 2$. We go to *Step 5* and we check that $F(\mathcal{P}) = 0$. Therefore, the algorithm RETURNS that $\mathcal{V}$ *can be parametrized by a polynomial parametrization without base points defined by* $\mathcal{P}$. In this case, $\mathcal{P}$ is not proper and in fact $\text{degMap}(\mathcal{P}) = 2$.

CRediT authorship contribution statement

The authors have contributed equally to this work.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Sonia Perez-Diaz reports was provided by University of Alcala.

Data availability

No data was used for the research described in the article.

Acknowledgements

The author S. Pérez-Díaz is partially supported by Ministerio de Ciencia, Innovación y Universidades - Agencia Estatal de Investigación/PID2020-113192GB-I00 (Mathematical Visualization: Foundations, Algorithms and Applications). The author R. Magdalena Benedicto is partially supported by the State Plan for Scientific and Technical Research and Innovation of the Spanish MCI (PID2021-127946OB-I00) and by the European Union, Digital Europe Program 21-22 Call Cloud Data and TEF (CitCom.ai 101100728). The author Li-Yong Shen is partially supported by the National Key Research and Development Program of China under Grant 2020YFA0713703, Beijing Natural Science Foundation under Grant Z190004, NSFC (Nos. 11688101, 61872332) and Fundamental Research Funds for the Central Universities.

The author S. Pérez-Díaz belongs to the Research Group ASYNACS (Ref.CCEE2011/R34).

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