Alcázar Arribas, J.G. & Quintero, E. 2020, "Affine equivalences of trigonometric curves", Acta Applicandae Mathematicae, vol. 170, pp. 691-708.

Available at https://doi.org/10.1007/s10440-020-00354-6

(Article begins on next page)

La versión final de este artículo corresponde a la publicación Alcázar J.G., Quintero E. (2020), "Affine equivalences of trigonometric curves", Acta Applicandae Mathematicae Vol. 170, pp. 691–708.

Affine equivalences of trigonometric curves.

Juan Gerardo Alcázar[a,1,2], Emily Quintero[a,3]

[a]*Departamento de Física y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain*

Abstract

We provide an efficient algorithm to detect whether two given *trigonometric curves*, i.e. two parametrized curves whose components are truncated Fourier series, in any dimension, are affinely equivalent, i.e. whether there exists an affine mapping transforming one of the curves onto the other. If the coefficients of the parametrizations are known exactly (the *exact* case), the algorithm boils down to univariate gcd computation, so it is efficient and fast. If the coefficients of the parametrizations are known with finite precision, e.g. floating point numbers (the *approximate* case), the univariate gcd computation is replaced by the computation of singular values of an appropriate matrix. Our experiments show that the method works well, even for high degrees.

1. Introduction

Two objects are affinely equivalent if there exists a nonsingular affine transformation mapping one of the objects onto the other one. Detecting whether two objects are related by an affine mapping is a classical problem in applied fields like Pattern Recognition, Image Processing and Computer Vision, and has been addressed in many papers using different strategies: see for instance [7, 10, 19, 20, 25] and the references in these papers. Essentially, the underlying problem is to be able to recognize a same image when it undergoes a smooth deformation, which is modelled as an affine mapping. Furthermore, an important particular instance of affine equivalence is the situation when the two objects are related by a similarity, in which case both objects are the same except for position and scaling.

In this paper, we consider the affine equivalence problem for parametrized curves in any dimension whose components are truncated Fourier series. In some references [13, 14], these curves receive the name of *trigonometric curves*,

Email addresses: `juange.alcazar@uah.es` (Juan Gerardo Alcázar), `emily.quintero@edu.uah.es` (Emily Quintero)

[1]Supported by the Spanish Ministerio de Economía y Competitividad and by the European Regional Development Fund (ERDF), under the project MTM2017-88796-P, and is a member of the Research Group ASYNACS (Ref. CCEE2011/R34

[2]Member of the Research Group ASYNACS (Ref. CT-CE2019/683.)

[3]Supported by a grant from the Carolina Foundation.

or generalized trigonometric curves. In other, more applied, references (see for instance [24]), these curves are called *elliptic Fourier descriptor (EFD) representations*, and are often used to describe closed planar and space curves (see for instance the references in [24]). In particular, for these curves one can compute *shape descriptors* (see [8, 9, 15], among many others), which are numbers that can be computed from the parametrization, and that can be used for curve recognition, in particular for similarity recognition.

The approach that we use here to solve the affine equivalence problem for trigonometric curves is similar to the approach recently used in papers like [2, 3, 11], coming from the fields of Symbolic Computation and Computer Aided Geometric Design, to detect symmetry [3], similarity [2], affine and even projective equivalence [11] for rational curves, i.e. curves parametrized by quotients of polynomials. Basically, in these papers it is shown that for any mapping between two rational curves (that can coincide, in the case of symmetry detection) one has another mapping in the parameter space (usually, the field of real or complex numbers) which corresponds to a Möebius transformation. Then the symmetry, similarity, affine or projective transformation itself can be written in terms of the parameters of this Möebius transformation. Solving a polynomial system whose unknowns are the parameters of the Möebius transformation leads to the mapping we are seeking, or to a proof that such a mapping does not exist, in the case when the objects are not affinely equivalent.

For trigonometric curves it is a well-known trick to compute a rational parametrization depending on one complex parameter taking values in the unit circle, and thus the techniques above are applicable. However, the rational parametrization of a trigonometric curve has special properties, that one can exploit. In particular, we can prove that the associated Möebius transformation has a predictable shape only depending in one parameter, so that the final computation boils down to computing a greatest common divisor of univariate complex polynomials. In the presence of floating point numbers (what we call the *approximate case*), however, due to numerical inaccuracies this greatest common divisor can be constant, and thus the method must be adapted. A good option is to replace greatest common divisors by *approximate common divisors* (see for instance [5, 16, 18, 17, 26]). However, and since we could not find any public implementation of approximate common divisors of univariate polynomials over the complex numbers, here we replace approximate gcds by singular value computation of certain matrices, associated to the univariate polynomials whose approximate gcd we would like to compute.

The structure of the paper is the following. In Section 2 we provide some background on trigonometric curves, and some general results that we use later. The algorithm for the exact case is presented in Section 3. The approximate case is addressed in Section 4. Finally, Section 5 contains our conclusion, together with some open problems.

2. Preliminaries on trigonometric curves.

A *trigonometric curve* $\mathcal{C} \subset \mathbb{R}^n$, following [14], is a parametric curve whose components are truncated Fourier series, i.e.

$$\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t)), \tag{1}$$

where

$$x_i(t) = \sum_{k=0}^{m_i} [a_k^{(i)} \cos(kt) + b_k^{(i)} \sin(kt)], \quad t \in [0, 2\pi],\ i = 1, \ldots, n. \tag{2}$$

We refer to a parametrization of this kind as a *trigonometric* parametrization. A trigonometric parametrization $\boldsymbol{x}(t)$ is *simple* if it is injective except for finitely many values of the parameter t; for instance, $(\sin(t), \cos(t))$ is a simple parametrization of a circle, while $(\sin(2t), \cos(2t))$ is not. Furthermore, a trigonometric parametrization $\widehat{\boldsymbol{x}}(t)$ is a *simplification* of another trigonometric parametrization $\boldsymbol{x}(t)$, if $\widehat{\boldsymbol{x}}(t)$ is simple and both $\boldsymbol{x}(t), \widehat{\boldsymbol{x}}(t)$ parametrize a same curve $\mathcal{C} \subset \mathbb{R}^n$; we also say that $\widehat{\boldsymbol{x}}(t)$ is the result of simplifying $\boldsymbol{x}(t)$. In [14] it is shown (see Theorem 2.1 in [14]) that any trigonometric curve admits either a simple, or a polynomial parametrization. Furthermore, algorithms for simplifying, implicitizing and parametrizing trigonometric curves are also provided in [14]. In our case, we will assume that the parametrizations we work with are simple. Moreover, we will assume that $\mathcal{C}$ is not contained in a hyperplane of $\mathbb{R}^n$.

Any trigonometric curve admits infinitely many simplifications. However, the following result, which is a reformulation of Theorem 2.5 in [14], shows that all simplifications of a same trigonometric curve are related by very precise transformations.

Lemma 1. *Let $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)$ be two simple trigonometric parametrizations of a same trigonometric curve $\mathcal{C} \subset \mathbb{R}^n$. Then $\boldsymbol{x}_2 = \boldsymbol{x}_1 \circ \psi$, where $\psi(t) = \alpha \pm t$.*

When dealing with trigonometric curves, a common technique is to use a rational parametrization of the curves by means of the change $z = e^{\mathbf{i}t}$, where $\mathbf{i}^2 = -1$ and z belongs to the unit circle S^1 (see for instance [13, 14, 24]). Since $e^{\mathbf{i}t} = \cos t + \mathbf{i} \sin t$, and taking into account that for $z \in S^1$ the conjugate $\overline{z}$ satisfies that $\overline{z} = \frac{1}{z}$, we deduce that

$$\cos t = \frac{z^2+1}{2z},\ \ \sin t = \frac{z^2-1}{2\mathbf{i}z},\ \ \cos(Mt) = \frac{z^{2M}+1}{2z^M},\ \ \sin(Mt) = \frac{z^{2M}-1}{2\mathbf{i}z^M}, \tag{3}$$

where $M \in \mathbb{Z}$. Substituting these relationships into Eq. (2), we get a *rational* parametrization (i.e. a parametrization whose components are quotients of polynomials)

$$\tilde{\boldsymbol{x}}(z) = (\tilde{x}_1(z), \ldots, \tilde{x}_n(z)), \tag{4}$$

where each component satisfies that

$$\tilde{x}_i(z) = \frac{P_i(z)}{z^{m_i}},\ i = 1, \ldots, n, \tag{5}$$

with $P_i(z)$ a complex polynomial of degree $2m_i$, and $z \in S^1$. We refer to $\tilde{\boldsymbol{x}}(z)$ as the rational complex parametrization *associated* with $\boldsymbol{x}(t)$. Denoting $N = \max\{m_i | i = 1, \ldots, n\}$, we say that the *degree* of $\tilde{\boldsymbol{x}}(z)$ is $2N$. Observe that not every $P_i(z)$ has degree $2N$, but there always exists $i \in \{1, \ldots, n\}$ such that the degree of $P_i(z)$ is $2N$.

Remark 1. *One can easily see that*

$$P_i(z) = \frac{1}{2} \sum_{k=0}^{m_i} [A_k z^{m_i+k} + B_k z^{m_i-k}]$$

where $A_k = a_k^{(i)} - \mathbf{i} b_k^{(i)}$, $B_k = a_k^{(i)} + \mathbf{i} b_k^{(i)}$. *In particular, since* a_{m_i}, b_{m_i} *are real and nonzero, then* $B_{m_i} \neq 0$, *so no cancellation in* $\tilde{x}_i(z) = \frac{P_i(z)}{z^{m_i}}$ *is possible.*

Remark 2. *An alternative possibility to work with trigonometric curves is to apply the classical rational change of variables*

$$(\cos(t), \sin(t)) \to \left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2} \right). \tag{6}$$

However, this produces parametrizations with more terms, and higher coefficients. For instance, consider the function

$$f(t) = \cos(t) + 3\cos(2t) - 2\cos(3t) + 4\cos(5t) - \cos(8t).$$

While the change in Eq. (6) *produces*

$$\frac{-s^{16} - 74s^{14} + 1758s^{12} - 8306s^{10} + 12780s^8 - 7838s^6 + 1842s^4 - 166s^2 + 5}{s^{16} + 8s^{14} + 28s^{12} + 56s^{10} + 70s^8 + 56s^6 + 28s^4 + 8s^2 + 1},$$

the change in Eq. (3) *yields*

$$\frac{4z^{16} + z^{14} + 2z^{12} - 2z^{10} + 4z^6 - 1}{z^{16}},$$

which is a simpler expression, with smaller coefficients and fewer terms.

Since for $t \in [0, 2\pi]$ the mapping $z = e^{\mathbf{i}t} : [0, 2\pi] \to S^1$ is invertible, and since we are assuming that Eq. (1) is a simple trigonometric parametrization, we get that Eq. (4), seen as mapping from S^1 to $\mathcal{C}$, is injective for almost all points of S^1. In other words, for $z \in S^1$ Eq. (4) defines a *birational* mapping, i.e. a rational mapping with a rational inverse. Furthermore, we get the following result as a corollary of Lemma 1.

Corollary 2. *Let* $\tilde{\boldsymbol{x}}_1(z), \tilde{\boldsymbol{x}}_2(z)$ *be two rational parametrizations of a same trigonometric curve* $\mathcal{C} \subset \mathbb{R}^n$, *associated with two simple trigometric parametrizations* $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)$ *of* $\mathcal{C}$. *Then* $\tilde{\boldsymbol{x}}_2 = \tilde{\boldsymbol{x}}_1 \circ \xi$, *where* $\xi(z) = kz$ *or* $\xi(z) = \frac{k}{z}$, *and* $k, z \in S^1$.

Proof. From Lemma 1, $\boldsymbol{x}_2 = \boldsymbol{x}_1 \circ (\alpha \pm t)$. Since $\boldsymbol{x}_j = \tilde{\boldsymbol{x}}_j \circ e^{\mathbf{i}t}$ for $j = 1, 2$, we get $\tilde{\boldsymbol{x}}_2 \circ e^{\mathbf{i}t} = \tilde{\boldsymbol{x}}_1 \circ e^{\mathbf{i}t} \circ (\alpha \pm t)$. Thus,

$$\tilde{\boldsymbol{x}}_2 \circ e^{\mathbf{i}t} = \tilde{\boldsymbol{x}}_1 \circ (e^{\mathbf{i}\alpha} \cdot e^{\pm \mathbf{i}t}).$$

Calling $k = e^{\mathbf{i}\alpha}$ and since $z = e^{\mathbf{i}t}$, the result follows. $\square$

Notice that $\varphi(z) = kz$ and $\varphi(z) = \frac{k}{z}$ are, in particular, *Möbius transformations* of S^1. Moreover, we also have the following corollary of Lemma 1, which follows from Corollary 2.

Corollary 3. *Let $\tilde{\boldsymbol{x}}_1(z), \tilde{\boldsymbol{x}}_2(z)$ be two rational parametrizations of a same trigonometric curve $\mathcal{C} \subset \mathbb{R}^n$, associated with two simple trigometric parametrizations $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)$ of $\mathcal{C}$. Then the degrees of both $\tilde{\boldsymbol{x}}_1(z), \tilde{\boldsymbol{x}}_2(z)$ are the same.*

3. Affine equivalence of trigonometric curves

Given two curves $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$, the curves $\mathcal{C}, \mathcal{D}$ are said to be *affinely equivalent* if there exists a nonsingular affine mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$,

$$f(\mathbf{x}) = \boldsymbol{A}\mathbf{x} + \boldsymbol{b}, \qquad \mathbf{x} \in \mathbb{R}^n, \tag{7}$$

with $\boldsymbol{b} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ a nonsingular square matrix, such that $f(\mathcal{C}) = \mathcal{D}$. We say that f is an *affine equivalence* between $\mathcal{C}, \mathcal{D}$, or that $\mathcal{C}, \mathcal{D}$ are *affinely equivalent*. If $\boldsymbol{A}$ is an orthogonal matrix, i.e. $\boldsymbol{A}^T\boldsymbol{A} = I$, where I denotes the $n \times n$ identity matrix, we say that f defines an *isometry* between $\mathcal{C}, \mathcal{D}$. If $\boldsymbol{A} = \lambda \boldsymbol{Q}$ where $\boldsymbol{Q}$ is orthogonal and $\lambda \neq 0$, we say that f defines a *similarity* between $\mathcal{C}, \mathcal{D}$. Furthermore, if $\mathcal{C} = \mathcal{D}$ and f defines a non-trivial isometry of $\mathcal{C}$ onto itself, we say that f is a *symmetry* of $\mathcal{C}$.

Additionally, we say that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an *involution* if $f \circ f = \mathrm{id}_{\mathbb{R}^n}$. Involutions are particularly interesting when $\mathcal{C} = \mathcal{D}$ and we consider the symmetries of $\mathcal{C}$, since notable symmetries like reflections on a plane, axial symmetries (i.e. symmetries with respect to a line, or equivalently rotations of angle π about a line) and central symmetries (i.e. symmetries with respect to a point) are involutions.

In the rest of the paper, we assume that $\mathcal{C}, \mathcal{D}$ are trigonometric curves defined by simple parametrizations

$$\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t)), \;\; \boldsymbol{y}(t) = (y_1(t), \ldots, y_n(t)) \tag{8}$$

where $x_i(t), y_i(t)$ are as in Eq. (2). We denote by $\tilde{\boldsymbol{x}}(z), \tilde{\boldsymbol{y}}(z)$, with $z \in S^1$, the rational parametrizations associated with $\boldsymbol{x}(t), \boldsymbol{y}(t)$, so the components of $\tilde{\boldsymbol{x}}(z), \tilde{\boldsymbol{y}}(z)$ are as in Eq. (5). Our goal is to detect whether $\mathcal{C}$ and $\mathcal{D}$ are affinely equivalent, i.e. to check whether they are related by a mapping like Eq. (7), and in the affirmative case to find the affine equivalences between $\mathcal{C}$ and $\mathcal{D}$. We first need the following lemma.

Lemma 4. *Let $\boldsymbol{x}(t)$ be a simple trigonometric parametrization as in Eq.* (1), *let $\tilde{\boldsymbol{x}}(z)$ be its associated rational parametrization, and let $2N$ be the degree of $\tilde{\boldsymbol{x}}(z)$. Let $f(\mathbf{x}) = \boldsymbol{A}\mathbf{x} + \boldsymbol{b}$, $\mathbf{x} \in \mathbb{R}^n$, with $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ a nonsingular square matrix.*

(1) $\boldsymbol{x}^\star(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}$ is a simple trigonometric parametrization, with associated rational complex parametrization $\tilde{\boldsymbol{x}}^\star(z) = \boldsymbol{A}\tilde{\boldsymbol{x}}(z) + \boldsymbol{b}$.

(2) The degrees of $\tilde{\boldsymbol{x}}(z)$ and $\tilde{\boldsymbol{x}}^\star(z)$ are the same.

Proof. Let us see (1). Since the components of $\boldsymbol{x}^\star(t)$ are linear combinations of the components of $\boldsymbol{x}(t)$, it is clear that $\boldsymbol{x}^\star(t)$ is trigonometric. Furthermore, since $\boldsymbol{A}$ is regular, f is an injective mapping. Thus, $\boldsymbol{x}^\star(t)$ is simple because it is the composition of a simple trigonometric parametrization with an injective mapping. Finally, since $\tilde{\boldsymbol{x}}^\star = \boldsymbol{x}^\star \circ e^{\mathbf{i}t}$, we easily deduce that $\tilde{\boldsymbol{x}}^\star(z) = \boldsymbol{A}\tilde{\boldsymbol{x}}(z) + \boldsymbol{b}$.

Now let us see (2). It is clear that the degree of $\tilde{\boldsymbol{x}}^\star(z)$ cannot be greater than $2N$; so let us see that the degree of $\tilde{\boldsymbol{x}}^\star(z)$ cannot be less than $2N$. Following the notation in Eq. (2), for $i = 1, \ldots, n$ let $a_N^{(i)}, b_N^{(i)}$ denote the coefficients of $\cos(Nt), \sin(Nt)$ in the i-th component of $\boldsymbol{x}(t)$, $x_i(t)$. Of course $a_N^{(i)}, b_N^{(i)}$ are zero when $m_i < N$. Notice, however, that since the degree of $\tilde{\boldsymbol{x}}(z)$ is $2N$, not all the $a_N^{(i)}, b_N^{(i)}$ can vanish. The coefficients of $\cos(Nt), \sin(Nt)$ in the i-th component of $\boldsymbol{x}^\star(t)$ are

$$\begin{array}{c} \boldsymbol{A}_{i1}a_N^{(1)} + \boldsymbol{A}_{i2}a_N^{(2)} + \cdots + \boldsymbol{A}_{in}a_N^{(n)}, \\ \boldsymbol{A}_{i1}b_N^{(1)} + \boldsymbol{A}_{i2}b_N^{(2)} + \cdots + \boldsymbol{A}_{in}b_N^{(n)} \end{array}$$

Now if the degree of $\tilde{\boldsymbol{x}}^\star(z)$ is less than $2N$, then the above expressions must vanish for all $j = 1, \ldots, n$. Since not all the $a_N^{(i)}, b_N^{(i)}$ are zero, this implies that there exists $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq \mathbf{0}$, such that $\boldsymbol{A} \cdot \mathbf{v} = \mathbf{0}$. But this is impossible because $\boldsymbol{A}$ is a regular matrix. $\square$

Then we have the following result.

Theorem 5. *Let $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ be two trigonometric curves, defined by rational complex parametrizations $\tilde{\boldsymbol{x}}(z), \tilde{\boldsymbol{y}}(z)$, with $z \in S^1$, associated with simple trigonometric parametrizations $\boldsymbol{x}(t), \boldsymbol{y}(t)$. Let $f : \mathcal{C} \to \mathcal{D}$ be an affine mapping $f(\mathbf{x}) = \boldsymbol{A}\mathbf{x} + \boldsymbol{b}$, where $\mathbf{x} \in \mathbb{R}^n$, with $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ a nonsingular square matrix, such that $f(\mathcal{C}) = \mathcal{D}$. Then there exists $k \in S^1$ and $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$, such that the diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] & \mathcal{D} \\ S^1 \arrow[u, dashed, "\tilde{\boldsymbol{x}}"] \arrow[r, dashed, "\varphi"'] & S^1 \arrow[u, dashed, "\tilde{\boldsymbol{y}}"'] \end{tikzcd} \tag{9}$$

is commutative, i.e. for $z \in S^1$ we get that $f \circ \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{y}} \circ \varphi$, or equivalently

$$\boldsymbol{A}\tilde{\boldsymbol{x}}(z) + \boldsymbol{b} = \tilde{\boldsymbol{y}}(\varphi(z)). \tag{10}$$

Furthermore, the degrees of $\tilde{\boldsymbol{x}}(z)$ and $\tilde{\boldsymbol{y}}(z)$ are the same.

Proof. Since $f(\mathcal{C}) = \mathcal{D}$ and by statement (1) of Lemma 4, $\boldsymbol{x}^\star(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{b}$ is also a simple trigonometric parametrization of $\mathcal{D}$. Furthermore, also by statement (1) of Lemma 4, the rational complex parametrization $\tilde{\boldsymbol{x}}^\star(z)$ associated with $\boldsymbol{x}^\star(t)$ is $\tilde{\boldsymbol{x}}^\star(z) = \boldsymbol{A}\tilde{\boldsymbol{x}}(z) + \boldsymbol{b}$. Then the results follow from Corollary 2, Corollary 3 and the statement (2) of Lemma 4. $\square$

Theorem 5 provides the following corollary on the involutional symmetries of a trigonometric curve $\mathcal{C}$.

Corollary 6. *In the hypotheses of Theorem 5, if $\mathcal{C} = \mathcal{D}$ and f is a nontrivial involutional symmetry (i.e. different from the identity) then $\varphi(z) = -z$ or $\varphi(z) = \frac{k}{z}$ with $k \in S^1$.*

Proof. From Theorem 5, assuming $\mathcal{C} = \mathcal{D}$ we get $\varphi = \tilde{\boldsymbol{x}}^{-1} \circ f \circ \boldsymbol{x}$. Thus, if $f \circ f = \mathrm{id}_{\mathbb{R}^n}$ then $\varphi \circ \varphi = \mathrm{id}_{\mathbb{R}^n}$ as well, so φ is an involution of S^1. Now from Lemma 2, $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$. The mapping $\varphi(z) = \frac{k}{z}$ is always an involution. However, $\varphi(z) = kz$ is an involution only when $k = \pm 1$. Since $\varphi(z) = z$ implies that f is the identity, the result follows. $\square$

Theorem 5 can be exploited in order to find the affine equivalences between $\mathcal{C}$ and $\mathcal{D}$. The general idea is to write first the entries $\boldsymbol{A}_{ij}$ of the matrix $\boldsymbol{A}$ and the components of the vector $\boldsymbol{b}$ as rational functions of k by using Eq. (10), and then finding, if any, the values $k \in S^1$ such that Eq. (10) is satisfied. In particular, we get polynomial conditions

$$g_1(k), \ldots, g_r(k) \tag{11}$$

that must have a common root $k \in S^1$ for $\mathcal{C}, \mathcal{D}$ to be affinely equivalent.

The next result shows that under our hypotheses, in particular by excluding that $\mathcal{C}$ lies in a hyperplane, this is always possible.

Lemma 7. *If $\mathcal{C}$ is not contained in a hyperplane, Eq.* (10) *allows to write the $\boldsymbol{A}_{ij}$ and $\boldsymbol{b}$ in terms of k.*

Proof. We focus on proving that the entries $\boldsymbol{A}_{ij}$ of $\boldsymbol{A}$ can be written as rational functions of k. Once this is done, from Eq. (10) we get $\boldsymbol{b} = \tilde{\boldsymbol{y}}(\varphi(\mathbf{a})) - \boldsymbol{A}\tilde{\boldsymbol{x}}(\mathbf{a})$ for any $\mathbf{a} \in S^1$.

A possibility to write $\boldsymbol{A}$ in terms of k is to choose $n+1$ distinct complex numbers $z_0, z_1, \ldots, z_n \in S^1$, and then consider the matrix equations $\boldsymbol{A}\tilde{\boldsymbol{x}}(z_i) + \boldsymbol{b} = \tilde{\boldsymbol{y}}(\varphi(z_i))$, $i = 0, 1, \ldots, n$. By subtracting the first equation from the last n equations, we get n matrix equations of the form

$$\boldsymbol{A}(\tilde{\boldsymbol{x}}(z_i) - \tilde{\boldsymbol{x}}(z_0)) = \tilde{\boldsymbol{y}}(\varphi(z_i)) - \tilde{\boldsymbol{y}}(\varphi(z_0)). \tag{12}$$

Let W be the $n \times n$ matrix whose columns are the vectors $\mathbf{v}_i = \tilde{\boldsymbol{x}}(z_i) - \tilde{\boldsymbol{x}}(z_0)$, for $i = 1, \ldots, n$, and let Z be the matrix whose columns are the vectors $\mathbf{w}_i = \tilde{\boldsymbol{y}}(\varphi(z_i)) - \tilde{\boldsymbol{y}}(\varphi(z_0))$. From Eq. (12), we get the matrix equation $\boldsymbol{A} \cdot W = Z$. If the $\mathbf{v}_i$ are linearly independent, then W^{-1} exists, and $\boldsymbol{A} = Z \cdot W^{-1}$; thus, all the $\boldsymbol{A}_{ij}$ can be written as rational functions of k.

So the only possibility for not succeeding in writing the $\boldsymbol{A}_{ij}$ in terms of k, is that we fail to find n vectors $\mathbf{v}_i$ which are linearly independent. In this case, for any choosing of distinct complex numbers $z_0, z_1, \ldots, z_{n-1} \in S^1$, the vector $\tilde{\boldsymbol{x}}(z) - \tilde{\boldsymbol{x}}(z_0)$ is linearly dependent with the $\mathbf{v}_i = \tilde{\boldsymbol{x}}(z_i) - \tilde{\boldsymbol{x}}(z_0)$, for $i = 1, \ldots, n-1$. In turn, this implies that there exist functions $\lambda_1(z), \ldots, \lambda_n(z)$ such that

$$\lambda_1(z)\mathbf{v}_1 + \ldots + \lambda_{n-1}(z)\mathbf{v}_{n-1} + \lambda_n(z)(\tilde{\boldsymbol{x}}(z) - \tilde{\boldsymbol{x}}(z_0)) = \mathbf{0}$$

for $z \in S^1$. But then $\tilde{\boldsymbol{x}}(z)$ belongs to the hyperplane through $\tilde{\boldsymbol{x}}(z_0)$, spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$, i.e. $\mathcal{C}$ is contained in a hyperplane. $\square$

The proof of Lemma 7 suggests a strategy to write $\boldsymbol{A}$, and then $\boldsymbol{b}$, in terms of k by substituting random values $z \in S^1$ in Eq. (10). However, in order to write $\boldsymbol{A}, \boldsymbol{b}$ in terms of k we can proceed directly from Eq. (10). In order to make the process more clear, let us write

$$\tilde{\boldsymbol{x}}(z) = \left(\frac{\widehat{P}_1(z)}{z^N}, \ldots, \frac{\widehat{P}_n(z)}{z^N} \right). \tag{13}$$

The $\widehat{P}_k(z)$ are polynomials of degree at most $2N$, although there is some k for which the degree of $\widehat{P}_k(z)$ is exactly $2N$. Because of this, some of the $\widehat{P}_k(z)$, but not all of them, can have z as a factor, with some multiplicity. Also, let us write

$$\tilde{\boldsymbol{y}}(z) = \left(\frac{\widehat{Q}_1(z)}{z^N}, \ldots, \frac{\widehat{Q}_n(z)}{z^N} \right). \tag{14}$$

Again, the $\widehat{Q}_k(z)$ are polynomials of degree at most $2N$ and not all of them can have z as a factor with some multiplicity. Furthermore, by Theorem 5 we have $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$. Thus, we get

$$\tilde{\boldsymbol{y}}(\varphi(z)) = \left(\frac{Q_1(k,z)}{z^N}, \ldots, \frac{Q_n(k,z)}{z^N} \right), \tag{15}$$

where the $Q_j(k,z)$ are polynomials in z, of degree $2N$, with coefficients polynomially depending on k, regardless of whether $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$. Also, for $i = 1, \ldots, n$ let us write

$$\begin{array}{rcl} \widehat{P}_i(z) & = & \alpha_0^{(i)} + \alpha_1^{(i)} z + \ldots + \alpha_N^{(i)} z^N + \cdots + \alpha_{2N}^{(i)} z^{2N}, \\ Q_i(k,z) & = & \beta_0^{(i)}(k) + \beta_1^{(i)}(k) z + \ldots + \beta_N^{(i)}(k) z^N + \cdots + \beta_{2N}^{(i)}(k) z^{2N}, \end{array} \tag{16}$$

where the coefficients of $Q_i(k,z)$, seen as a polynomial in z, are polynomials in k of degree at most $2N$.

Now Eq. (10) can be written as

$$\boldsymbol{A} \cdot \left(\frac{\widehat{P}_1(z)}{z^N}, \ldots, \frac{\widehat{P}_n(z)}{z^N} \right)^T + \boldsymbol{b} = \left(\frac{Q_1(k,z)}{z^N}, \ldots, \frac{Q_n(k,z)}{z^N} \right)^T. \tag{17}$$

Multiplying by z^N, we get

$$\boldsymbol{A} \cdot \left(\widehat{P}_1(z), \dots, \widehat{P}_n(z)\right)^T + z^N \boldsymbol{b} = (Q_1(k,z), \dots, Q_n(k,z))^T . \tag{18}$$

From Eq. (18), equaling the coefficients of the terms in z^ℓ, $\ell \neq N$, at both sides of the equation, we get linear equations

$$\boldsymbol{A}_{i1}\alpha_\ell^{(1)} + \dots + \boldsymbol{A}_{in}\alpha_\ell^{(n)} = \beta_\ell^{(i)}(k) \tag{19}$$

where $i = 1, \dots, n$, $\ell = 0, 1, \dots, N-1, N+1, \dots, 2N$. Thus, we get $2Nn$ linear equations of this type. Additionally, also from Eq. (18), equaling the coefficients of the terms in z^N at both sides of the equation, we get linear equations

$$\boldsymbol{A}_{i1}\alpha_N^{(1)} + \dots + \boldsymbol{A}_{in}\alpha_N^{(n)} + b_i = \beta_N^{(i)}(k) \tag{20}$$

where $i = 1, \dots, n$. Thus, we get n linear equations of this type. Putting together the equations Eq. (19) and Eq. (20), we get a linear system $\mathcal{S}$, whose unknowns are the n^2 entries $\boldsymbol{A}_{ij}$ of the matrix $\boldsymbol{A}$, and the n coordinates of the vector $\boldsymbol{b}$, that must be consistent for some values $k \in S^1$ in the event that the curves $\mathcal{C}, \mathcal{D}$ are affinely equivalent. We refer to $\mathcal{S}$ as the linear system *associated* with Eq. (10). Moreover, the coefficient matrix $\mathcal{A}$ of the system $\mathcal{S}$ has the following block structure:

$$\mathcal{A} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{1} \end{bmatrix} \tag{21}$$

The block $\mathbf{B}_1$ is block diagonal, and consists of n copies of the $2N \times n$ submatrix

$$\begin{bmatrix} \alpha_0^{(1)} & \cdots & \alpha_0^{(n)} \\ \vdots & \ddots & \vdots \\ \alpha_{2N}^{(1)} & \cdots & \alpha_{2N}^{(n)} \end{bmatrix} \tag{22}$$

where the row corresponding to the subindex N is missing. The block $\mathbf{B}_2$ is also block diagonal, and consists of n copies of the row matrix

$$\begin{bmatrix} \alpha_N^{(1)} & \cdots & \alpha_N^{(n)} \end{bmatrix} . \tag{23}$$

The block $\mathbf{0}$ is corresponds to a $2Nn \times n$ null matrix, and the block $\mathbf{1}$ is the identity matrix of dimension n.

In particular, notice that the number of linear equations we get is $2Nn+n = (2N+1)n$, and the number of unknowns is $n^2 + n$, so $\mathcal{A} \in \mathcal{M}_{(2N+1)n \times (n^2+n)}$.

Lemma 8. *If $\mathcal{C}$ is not contained in a hyperplane, then $2N \geq n$.*

Proof. The vector $\tilde{\boldsymbol{x}}(z)$ is parallel to the vector

$$\tilde{\boldsymbol{x}}^\star(z) = z^N \tilde{\boldsymbol{x}}(z) = (\widehat{P}_1(z), \dots, \widehat{P}_n(z)).$$

In turn, we can write $\tilde{\boldsymbol{x}}^\star(z)$ as

$$\tilde{\boldsymbol{x}}^\star(z) = \mathbf{a}_0 + \mathbf{a}_1 z + \cdots + \mathbf{a}_{2N} z^{2N},$$

where $\mathbf{a}_j \in \mathbb{R}^{2N}$ for $j = 0, \ldots, 2N$. If $2N < n$, then $\tilde{\boldsymbol{x}}^\star(z)$, and therefore, for every $z \in S^1$, $\tilde{\boldsymbol{x}}(z)$ belongs to a subspace of $\mathbb{R}^n$ of dimension less or equal than $n-1$. Thus, $\mathcal{C}$ is contained in a hyperplane. □

Proposition 9. *Assume that $\mathcal{C} \subset \mathbb{R}^n$ is not contained in a hyperplane. The linear system $\mathcal{S}$ associated with Eq.* (5) *provides $0 \leq r \leq (2N-n)n$ nonzero polynomial conditions in k as in Eq.* (11)*, of degree bounded by $2N$.*

Proof. From Lemma 7, the system $\mathcal{S}$ is solvable. Since $\mathcal{S}$ has n^2+n unknowns (the $\boldsymbol{A}_{ij}$ and the components of $\boldsymbol{b}$), we need n^2+n equations of $\mathcal{S}$ to write the $\boldsymbol{A}_{ij}$ and the components of $\boldsymbol{b}$ in terms of k. Substituting these expressions for the entries of $\boldsymbol{A}$ and $\boldsymbol{b}$ in the remaining equations of $\mathcal{S}$, we get at most $(2N+1)n-(n^2+n) = (2N-n)n$ nonzero polynomial conditions on k, $g_1(k), \ldots, g_r(k)$, where $2N-n \geq 0$ because of Lemma 8. Since the constant terms of $\mathcal{S}$, i.e. the $\beta_\ell^{(i)}(k)$ (see Eq. (19) and Eq. (20)), are polynomials in k of degree $\leq 2N$, by Cramer's rule the $\boldsymbol{A}_{ij}$ and the components of $\boldsymbol{b}$ are polynomials of degree $\leq 2N$. Thus, when substituted in the remaining equations of $\mathcal{S}$, we get polynomials in k of degree $\leq 2N$. □

Remark 3. *If the number r of polynomial conditions in k in Proposition 9 is $r=0$, then the curves are related by infinitely many affine transformations. Indeed, in this case after using the equations of $\mathcal{S}$ to write $\boldsymbol{A}$ and $\boldsymbol{b}$ in terms of k, we get that Eq.* (10) *is satisfied for all values of k. Hence, every affinity $f(\mathbf{x}) = \boldsymbol{A}\mathbf{x}+\boldsymbol{b}$, with $\boldsymbol{A} = \boldsymbol{A}(k)$ and $\boldsymbol{b} = \boldsymbol{b}(k)$, maps $\mathcal{C}$ onto $\mathcal{D}$.*

Corollary 10. *Any two trigonometric curves $\mathcal{C}, \mathcal{D}$ in $\mathbb{R}^n$ defined by rational complex parametrizations of degree N with $2N = n$, are related by infinitely many affine equivalences.*

Proof. Since $2N = n$, by Proposition 9 the number r of polynomial conditions is $r = 0$. Then the result follows from Remark 3. □

The preceding ideas are summarized in Algorithm `Affine-Trigonometric`, and illustrated in Example 1.

Algorithm 1 Affine-Trigonometric

Require: Two trigonometric curves $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$, defined by simple parametrizations $\boldsymbol{x}(t), \boldsymbol{y}(t)$, of the same degree $2N$.

Ensure: The affine equivalences $f(\mathbf{x}) = \boldsymbol{A}\mathbf{x} + \boldsymbol{b}$ between $\mathcal{C}, \mathcal{D}$.

1: Compute the rational complex parametrizations $\tilde{\boldsymbol{x}}(z), \tilde{\boldsymbol{y}}(z)$ associated with the curves.
2: Set $\varphi(z) = kz$
3: Write $\boldsymbol{A}, \boldsymbol{b}$ in terms of k using Eq. (10), i.e. solving the corresponding n^2+n equations of the system $\mathcal{S}$ stemming from Eq. (10).
4: Substitute $\boldsymbol{A}, \boldsymbol{b}$ in terms of k into the remaining equations of $\mathcal{S}$, to compute the polynomial conditions $g_1(k) = 0, \ldots, g_r(k) = 0$
5: Compute the complex roots $k \in S^1$ of the greatest common divisor of $g_1(k), \ldots, g_r(k)$.
6: **return**, if any, the affine equivalences corresponding to the k found in the step before
7: Set $\varphi(z) = \frac{k}{z}$, and repeat steps (3-6)
8: **if** no value $k \in S^1$ is found **then**
9:     **return** $\mathcal{C}$ `and` $\mathcal{D}$ `are not affinely equivalent.`
10: **end if**

The complexity of Algorithm `Affine-Trigonometric` is provided in the following proposition. Here we use the standard Big O notation $\mathcal{O}$ for the time complexity analysis, and the Soft O notation $\tilde{\mathcal{O}}$ to ignore logarithmic factors.

Proposition 11. *Let $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ be two trigonometric curves of degree N, not contained in a hyperplane. The complexity of Algorithm* `Affine-Trigonometric` *is $\tilde{\mathcal{O}}(N^3)$.*

Proof. Writing $\boldsymbol{A}, \boldsymbol{b}$ in terms of k implies solving the linear system $\mathcal{S}$ stemming from Eq. (10). This can be done by applying Gaussian Elimination to the system $\mathcal{S}$. The coefficient matrix of this system is $\mathcal{A} \in \mathcal{M}_{(2N+1)n \times (n^2+n)}$. Since $2N \geq n$ because of Lemma 8, the rank of $\mathcal{A}$ is bounded by $(2N+1)n$ and thus the complexity of Gaussian Elimination on $\mathcal{S}$ is $\mathcal{O}(N^3 n^3)$ (see for instance [4]). Computing the polynomials $g_1(k), \ldots, g_r(k)$ does not increase the complexity. The degrees of the $g_i(k)$ are bounded by $2N$, and thus computing the gcd of the $g_i(k)$ can be done in $\tilde{\mathcal{O}}(N)$ time (see Corollary 11.6 in [23]). The roots of the gcd can be computed in $\tilde{\mathcal{O}}(N^3)$ time (see [6]), so we get an overall complexity of $\tilde{\mathcal{O}}(N^3)$. $\square$

Example 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be the plane trigonometric curves parametrized by*

$\boldsymbol{x}(t), \boldsymbol{y}(t)$ *respectively, with* $t \in [0, 2\pi]$*, where*

$$
\begin{aligned}
x_1(t) &= -\frac{1}{3}\sin(3t) + \frac{2}{3}\cos(t),\\
x_2(t) &= -\sin(5t) - 2\sin(t) - \frac{1}{3}\cos(t),\\
y_1(t) &= -\frac{1}{6}\sin(5t) + \frac{1}{4}\sin(3t) - \frac{1}{3}\sin(t) - \frac{5}{9}\cos(t) + 4,\\
y_2(t) &= -\frac{\sqrt{3}}{2}\sin(5t) + \frac{2}{15}\sin(3t) - \sqrt{3}\sin(t) - \frac{8+5\sqrt{3}}{30}\cos(t) - 2.
\end{aligned}
$$

The associated rational commplex parametrizations are

$$
\begin{aligned}
\tilde{\boldsymbol{x}}(z) &= \left(\frac{\mathbf{i}z^6 + 2z^4 + 2z^2 - \mathbf{i}}{6z^3}, \frac{3\mathbf{i}z^{10} - (1-6\mathbf{i})z^6 - (1+6\mathbf{i})z^4 - 3\mathbf{i}}{6z^5}\right),\\
\tilde{\boldsymbol{y}}(z) &= (\tilde{y}_1(z), \tilde{y}_2(z)),
\end{aligned}
$$

$$\tilde{y}_1(z) = \frac{6\mathbf{i}z^{10} - 9\mathbf{i}z^8 - (20-12\mathbf{i})z^6 + 288z^5 - (20+12\mathbf{i})z^4 + 9\mathbf{i}z^2 - 6\mathbf{i}}{72z^5},$$

$$\tilde{y}_2(z) = \frac{15\mathbf{i}\sqrt{3}z^{10} - 4\mathbf{i}z^8 - (5\sqrt{3}+8-30\sqrt{3}\mathbf{i})z^6 - 120z^5 - (5\sqrt{3}+8+30\sqrt{3}\mathbf{i})z^4 + 4\mathbf{i}z^2 - 15\sqrt{3}\mathbf{i}}{60z^5}.$$

We consider first the case $\varphi(z) = kz$*. Step 4 of Algorithm 1 provides 12 polynomials; we show 4 of them below:*

$$-(18+168\mathbf{i})k^{10} + (27+54\mathbf{i})k^8 - (156-12\mathbf{i})k^6 + (156+12\mathbf{i})k^4 - (27-54\mathbf{i})k^2 + 18 + 36\mathbf{i},$$

$$(-54+6\mathbf{i})k^{10} + (81+315\mathbf{i})k^8 - (128+168\mathbf{i})k^6 + (128-168\mathbf{i})k^4 - (81-9\mathbf{i})k^2 + 54 + 6\mathbf{i},$$

$$(-36+174\mathbf{i})k^{10} + (54-45\mathbf{i})k^8 + (28-180\mathbf{i})k^6 - (28+180\mathbf{i})k^4 - (54-261\mathbf{i})k^2 + 36 - 30\mathbf{i},$$

$$(36+30\mathbf{i})k^{10} - (54+261\mathbf{i})k^8 - (28-180\mathbf{i})k^6 + (28+180\mathbf{i})k^4 + (54+45\mathbf{i})k^2 - 36 - 174\mathbf{i}.$$

The gcd of all of the 12 polynomials is $k^2 - 1$*. Thus, we get* $k = \pm 1$*, i.e.*

$$\varphi_1(z) = z, \ \varphi_2(z) = -z.$$

The mapping $\varphi_1(z)$ *corresponds to the affine mapping* $f_1(\mathbf{x}) = \boldsymbol{A}_1\mathbf{x} + \boldsymbol{b}_1$*, where*

$$\boldsymbol{A}_1 = \begin{pmatrix} -\frac{3}{4} & \frac{1}{6} \\ -\frac{2}{5} & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{24}$$

The mapping $\varphi_2(z)$ *corresponds to the affine mapping* $f_2(\mathbf{x}) = \boldsymbol{A}_2\mathbf{x} + \boldsymbol{b}_2$*, where*

$$\boldsymbol{A}_2 = \begin{pmatrix} \frac{3}{4} & -\frac{1}{6} \\ \frac{2}{5} & -\frac{\sqrt{3}}{2} \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{25}$$

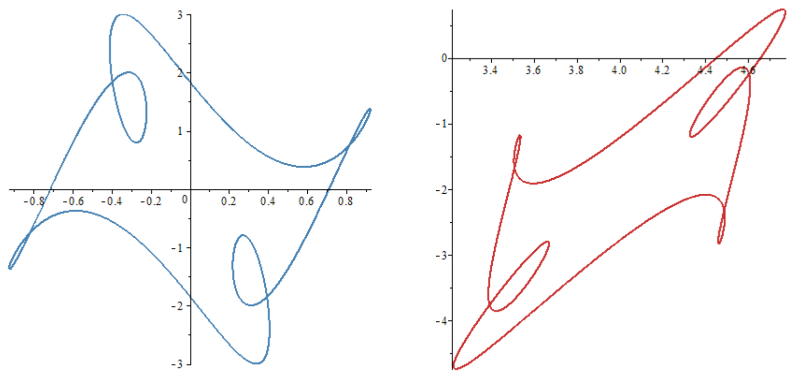

Figure 1: $\mathcal{C}$ (left) and $\mathcal{D}$ (right).

When $\varphi(z) = \dfrac{k}{z}$ *we obtain no solution. Therefore, we conclude that* $\mathcal{C}, \mathcal{D}$ *are related by two affine mappings* f_1, f_2*. The total computation time was 2.714 seconds, in an Intel(R) Core(TM) i5-7500, CPU 3.40 GHz and 32 Gb RAM. Notice that both* $\mathcal{C}, \mathcal{D}$ *have a nontrivial symmetry* τ *with respect to a point, which is the reason why we get two affine equivalences.*

4. Approximate affine equivalences.

In this section we consider the case when the curves $\mathcal{C}$ and $\mathcal{D}$ are defined by means of simple trigonometric parametrizations $\boldsymbol{x}(t), \boldsymbol{y}(t)$ as in Eq. (1) and Eq. (2), but where the coefficients of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are given with finite precision, i.e. as floating point numbers. In this case, and even if the curves $\mathcal{C}$ and $\mathcal{D}$ are very close to being related by an affinity, applying the same procedure as in the exact case yields polynomial conditions $g_1(k), \dots, g_r(k)$ with a constant gcd, so, even though these polynomials have some roots which are very close to each other, no common root of $g_1(k), \dots, g_r(k)$ is computed.

Thus, in this case we focus not on the affine equivalences between $\mathcal{C}$ and $\mathcal{D}$, but on *approximate* affine equivalences. In order to do it, we proceed as in the exact case to compute $g_1(k), \dots, g_r(k)$, and then we find the *approximate* common roots of $g_1(k), \dots, g_r(k)$.

A first possibility to do this is to compute an *approximate gcd*, also called ϵ-*gcd* of $g_1(k), \dots, g_r(k)$. Although there is a wide variety of methods to compute an approximate gcd (see, among others, [5, 16, 18, 17, 26]), we could not find a public implementation for our case, where the polynomials $g_i(k)$ have complex coefficients. Because of this, we have used another method, based on the SVD decomposition.

Our method proceeds as follows. Given the polynomials $g_1(k), \dots, g_r(k)$, we can write the univariate polynomial system consisting of the $g_i(k)$ as a matrix system

$$\mathbf{M} \cdot (1 \;\; k \;\; k^2 \;\; \cdots \;\; k^{2N})^T = \mathbf{0}, \tag{26}$$

where $\mathbf{M} \in \mathcal{M}_{r\times(2N+1)}$ is a rectangular numeric matrix whose entries are the coefficients of the $g_i(k)$.

Now, in the exact case the dimension of the kernel of the matrix $\mathbf{M}$ equals the number of zero singular values of $\mathbf{M}$. Additionally, in the exact case, a common root $k_0 \in \mathbb{C}$ of the $g_i(k)$ corresponds to a vector $\mathbf{v} = (1; k_0; k_0^2; \cdots; k_0^{2N}) \in \mathrm{Ker}(\mathbf{M})$. Thus, if $\dim(\mathrm{Ker}(\mathbf{M})) = 1$, i.e. if $\mathbf{M}$ has exactly one zero singular value, and the $g_i(k)$ have a common root k_0, then we can compute k_0 by determining the element of $\mathrm{Ker}(\mathbf{M})$ whose first coordinate is 1. Furthermore, let $\mathbf{M} = U \cdot S \cdot V^\star$ be the SVD decomposition of $\mathbf{M}$, where * denotes conjugate transposition and S is a $r \times (2N+1)$ matrix with the singular values in the main diagonal (the remaining elements of S are zero). If the singular value 0 is the last entry of the q-th column of S, then the q-th column of V provides a generator for $\mathrm{Ker}(\mathbf{M})$.

If the $g_i(k)$ have several common roots, which corresponds to the case when there are several affine equivalences relating the curves $\mathcal{C}$ and $\mathcal{D}$, the problem is more complicated: in this case, $\dim(\mathrm{Ker}(\mathbf{M})) = p > 1$ and the common roots k_j of the $g_i(k)$ must be determined by first computing a basis for $\mathrm{Ker}(\mathbf{M})$, and then computing the vectors $(1; k_i; k_i^2; \cdots; k_i^{2N}) \in \mathrm{Ker}(\mathbf{M})$. This amounts to imposing that there exist $\lambda_1, \ldots, \lambda_p \in \mathbb{C}$ such that

$$\lambda_1 \vec{v}_1 + \cdots + \lambda_p \vec{v}_p = (1 \;\; k_j \;\; \cdots \;\; k_j^{2N})^T, \tag{27}$$

where $\vec{v}_1, \ldots, \vec{v}_p$ form a basis of $\mathrm{Ker}(\mathbf{M})$. In turn, this yields to polynomial conditions on the k_j. In the exact case, this is perfectly feasible; in the approximate case, though, in practice this case is more complicated. Thus, we will focus on the first case, where $\dim(\mathrm{Ker}(\mathbf{M})) = 1$, corresponding to the situation where there is at most one affine equivalence relating the curves.

In order to perform the translation of the above ideas to the approximate case, a first observation is that singular values behave well in presence of perturbations (see for instance [22]). Thus, if we introduce a small perturbation in a matrix, we will get small perturbations in the singular values of the matrix too. Hence, in practice we build the matrix $\mathbf{M}$, and compute the smallest singular value. Assuming there is just one singular value really close to zero, and therefore that $\dim(\mathrm{Ker}(\mathbf{M}))$ is "almost" 1, proceed as before computing the column of the matrix V corresponding to the smallest singular value, and finding k_0 from there. The idea is illustrated in the following example.

Example 2. *Let us consider the curves $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^3$, parametrized by*

$$\boldsymbol{x}(t) = (9\cos(t) - 5\sin(t), \sin(3t) + 15\cos(2t) - \cos(t) - 8\sin(t), -2\sin(4t) + \cos(3t))$$

and $\boldsymbol{y}(t) = (y_1(t), y_2(t), y_3(t))$, with $t \in [0, 2\pi]$, where

$$\begin{aligned}
y_1(t) &= 6\sin(4t) - 3\cos(3t) + 5\sin(3t) + 75\cos(2t) - 59\cos(t) - 10\sin(t) + 1,\\
y_2(t) &= -8\sin(4t) + 4\cos(3t) - 9\cos(t) + 5\sin(t),\\
y_3(t) &= -2\sin(3t) - 30\cos(2t) + 11\cos(t) + 11\sin(t) + 1.
\end{aligned}$$

The curves $\mathcal{C}, \mathcal{D}$ are related by one affine equivalence $f(\mathbf{x}) = \boldsymbol{A}\mathbf{x} + \boldsymbol{b}$ corresponding to $\varphi_1(z) = z$, where

$$\boldsymbol{A} = \begin{pmatrix} -6 & 5 & -3 \\ -1 & 0 & 4 \\ 1 & -2 & 0 \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Now, we apply a random perturbation of order 10^{-2} to all the coefficients of the parameterizations $\boldsymbol{x}(t), \boldsymbol{y}(t)$. Consider now $\boldsymbol{x}^*(t)$ given by

$$x_1^*(t) = 9.0018\cos(t) - 4.998\sin(t),$$

$$x_2^*(t) = 0.0001\cos(3t) + 1.002\sin(3t) + 15.0014\cos(2t) + 0.00009\sin(2t) \\ -0.9995\cos(t) - 7.9988\sin(t),$$

$$x_3^*(t) = 0.0009\cos(4t) - 1.999\sin(4t) + 0.999\cos(3t) + 0.0009\sin(3t) + 0.0009\cos(2t) \\ -0.0002\sin(2t) + 0.0005\cos(t) - 0.001\sin(t).$$

and $\boldsymbol{y}^*(t)$ given by

$$y_1^*(t) = 0.0009\cos(4t) + 6.0021\sin(4t) - 2.9982\cos(3t) + 5.0022\sin(3t) \\ +75.0014\cos(2t) - 0.00005\sin(2t) - 58.9981\cos(t) - 9.9979\sin(t) + 1.0015,$$

$$y_2^*(t) = 0.0017\cos(4t) - 7.9984\sin(4t) + 4.0008\cos(3t) + 0.0014\sin(3t) \\ +0.0003\cos(2t) + 0.0016\sin(2t) - 9.00006\cos(t) + 5.0005\sin(t),$$

$$y_3^*(t) = -0.00003\cos(3t) - 1.9999\sin(3t) - 29.9982\cos(2t) + 0.0015\sin(2t) \\ +11.0006\cos(t) + 11.0021\sin(t) + 0.9999.$$

For $\varphi(z) = kz$, applying Algorithm 1 we get the 8 polynomial equations in k of degree 8. Thus, we get a system as in Eq. (26) *where $\mathbf{M} \in \mathcal{M}_{8\times 9}$. The smallest singular value is, approximately, $8.6235 \cdot 10^{-4}$, and the remaining singular values are not close to zero.*

Hence, writing $\mathbf{M} = U \cdot S \cdot V^*$, the eighth column of V is a good candidate to be the multiple of a solution. From here, we get $k = 0.999798 + 0.000032\mathbf{i}$, which gives rise to $\tilde{f}(\mathbf{x}) = \tilde{\boldsymbol{A}}\mathbf{x} + \tilde{\boldsymbol{b}}$, where

$$\tilde{\boldsymbol{A}} \approx \begin{pmatrix} -5.9895 - 0.0099\mathbf{i} & 4.9987 + 0.0194\mathbf{i} & -3.0013 - 0.0137\mathbf{i} \\ -1.0004 - 0.0052\mathbf{i} & 0.00501 + 0.0016\mathbf{i} & 3.9994 + 0.0169\mathbf{i} \\ 0.9967 + 0.0022\mathbf{i} & -2.00002 - 0.0067\mathbf{i} & 0 \end{pmatrix},$$

$$\tilde{\boldsymbol{b}} \approx \begin{pmatrix} 1.00148896437165846329264695668830 \\ 0 \\ 0.99994267059320696144908424685127 \end{pmatrix}.$$

The relative errors are $\dfrac{\|\boldsymbol{b} - \tilde{\boldsymbol{b}}\|}{\|\boldsymbol{b}\|} \approx 0.00083$ and $\dfrac{\|\boldsymbol{A} - \tilde{\boldsymbol{A}}\|_2}{\|\boldsymbol{A}\|_2} \approx 0.00063$, where $\|\cdot\|$ is the Euclidean norm of vectors and $\|\cdot\|_2$ is the spectral norm (the largest singular value) of matrices.

With $\varphi(z) = \dfrac{k}{z}$, the smallest singular value is 1.2876. In this case, we conclude that there is no approximate solution.

We have implemented the method described above with the help of the computer algebra systems MATLAB R2019a and Maple 18. In Fig. 2 we show the CPU time, in seconds, run in the machine of Example 1, for some representative examples of growing degrees. One can observe that for these data the cubic polynomial $p(x) = -0.0023x^3 + 0.2094x^2 - 0.6193x + 6.6161$ fits very well, with a coefficient of determination (R^2) equal to 98.04 %.

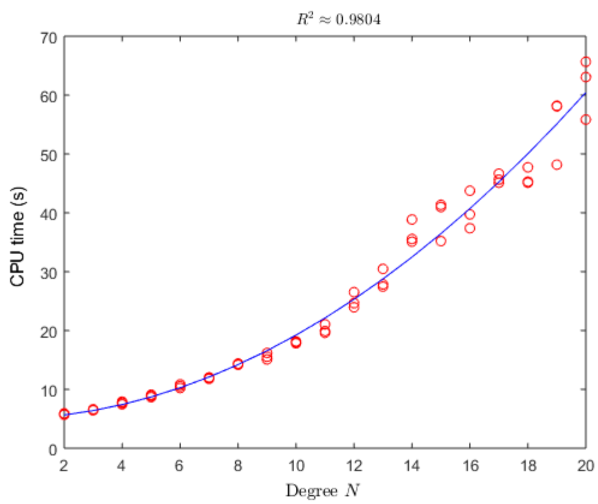

Figure 2: CPU time versus degree

The features of some of the examples used in Fig. 2 are provided in Table 1. In all these examples we considered two curves $\mathcal{C}, \mathcal{D}$ where $\mathcal{D}$ was the result of applying to $\mathcal{C}$ the affine transformation $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1/5 & -3 \\ -1 & 0 & -4 \\ 3 & 5 & \sqrt{3} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix},$$

and introducing afterwards a perturbation of order 10^{-2}. The transformation computed by our method is denoted by $\tilde{f}(\boldsymbol{x}) = \tilde{\boldsymbol{A}}\boldsymbol{x} + \tilde{\boldsymbol{b}}$. In Table 1 we can see the degree N of each example, and the values of $\dfrac{\|\boldsymbol{A} - \tilde{\boldsymbol{A}}\|_2}{\|\boldsymbol{A}\|_2}$ and $\dfrac{\|\boldsymbol{b} - \tilde{\boldsymbol{b}}\|}{\|\boldsymbol{b}\|}$ for each example, where $\| \bullet \|_2$ represents the 2-norm, which measure the relative error in each case.

Degree N	$\dfrac{\|\boldsymbol{A}-\tilde{\boldsymbol{A}}\|_2}{\|\boldsymbol{A}\|_2}$	$\dfrac{\|\boldsymbol{b}-\tilde{\boldsymbol{b}}\|}{\|\boldsymbol{b}\|}$	CPU time (secs.)
3	0.00065	0.00636	6.0837
5	0.00081	0.00921	8.6786
8	0.00209	0.00975	13.6119
9	0.00042	0.00624	15.0914
10	0.00019	0.00294	17.8637
12	0.00060	0.00789	25.4754
14	0.00018	0.00299	35.0920
15	0.00022	0.00287	40.9761
17	0.00063	0.00918	45.1100
20	0.00009	0.00173	65.1659

Table 1

5. Conclusion.

In this paper we have presented algorithms, both exact and approximate, to solve the affine equivalence problem for curves parametrized by truncated Fourier series. In the exact case, the algorithm boils down to linear system solving and univariate gcd computation, and is efficient and fast, with cubic complexity in the degree of the curves. In the approximate case, univariate gcd computation is replaced by the singular value decomposition of an appropriate matrix; again, in this case the practical results are good.

One can wonder what the generalization could be for computing not affine equivalence, but projective equivalence. We conjecture that two trigonometric curves are projectively equivalent iff they are affinely equivalent, although a proof for this fact seems elusive. An argument in favour of the conjecture is the fact that while the affine mapping of a trigonometric curve is another trigonometric curve, this is no longer true for a projective mapping. In the case of projective transformations, one can also prove that a Möbius transformation of the unit circle is also involved. However, we could not find a proof that this transformation must be of the type kz or k/z; in fact, one can prove that if this is the case, the projective transformation is an affinity. Thus, we leave this question here as an open problem.

References

[1] Alcázar J.G., Hermoso C. (2016), *Involutions of polynomially parametrized surfaces*, Journal of Computational and Applied Mathematics Vol. 294, pp. 23–38.

[2] Alcázar J.G., Hermoso C., Muntingh G. (2014), *Detecting similarity of Rational Plane Curves*, Journal of Computational and Applied Mathematics vol. 269, pp. 1-13.

[3] Alcázar J.G., Hermoso C., Muntingh G. (2015), *Symmetry detection of rational space curves from their curvature and torsion*, Computer Aided Geometric Design Vol. 33, pp. 51–65.

[4] Basu S., Pollack R., Roy M.F. (2003), *Algorithms in real algebraic geometry*, Springer.

[5] Batselier K., Dreesen P., De Moor B. (2013), *A geometrical approach to finding multivariate approximate LCMs and GCDs*, Linear Algebra and its Applications, Vol. 438, Issue 9, pp. 3618-3628.

[6] Becker R., Sagraloff M., Sharma V., Yap C. (2018), *A near-optimal subdivision algorithm for complex root isolation based on the Pellet test and Newton iteration*, Journal of Symbolic Computation Vol. 86, pp. 51–96.

[7] Boutin M. (2000), *Numerically Invariant Signature Curves*, International Journal of Computer Vision Vol. 40, pp. 235–248.

[8] Crimmins T. (1982), *A complete set of Fourier descriptors for two-dimensional shapes*, IEE Transactions on Systems, Man and Cybernetics Vol. 12, pp. 848–855.

[9] Dalitz C., Brandt C., Goebbels S., Kolanus D. (2013), *Fourier descriptors for broken shapes*, Journal on Advances in Signal Processing Vol. 161.

[10] Feng S., Kogan I., Krim H. (2010), *Classification of Curves in 2D and 3D via Affine Integral Signatures*, Acta Applicandae Mathematicae Vol. 109, pp. 903–937.

[11] Hauer M., Jüttler B. (2018), *Projective and affine symmetries and equivalences of rational curves in arbitrary dimension*, Journal of Symbolic Computation Vol. 87, pp. 68–86.

[12] Hauer M., Jüttler B., Schicho J. (2018), *Projective and affine symmetries and equivalences of rational and polynomial surfaces*, Journal of Computational and Applied Mathematics, in Press.

[13] Hong H. (1995), *Implicitization of curves parametrized by generalized trigonometric polynomials*, Proceedings of Applied Algebra, Algebraic Algorithms and Error Correcting Codes (AAECC-11), pp. 285–296.

[14] Hong H., Schicho J. (1998), *Algorithms for Trigonometric Curves (Simplification, Implicitization, Parameterization)*, Journal of Symbolic Computation Vol. 26, Issue 3, pp. 279-300.

[15] Ghorbel F., Derrode S., Mezhoud R., Bannour T., Dhahbi S. (2006), *Image reconstruction from a complete set of similarity invariants extracted from complex moments*, Pattern Recognition Letters Vol. 27, pp. 1361–1369.

[16] Karmarkar N., Lakshman Y. N. (1998), *On Approximate GCDs of UnivariatePolynomials*, Journal of Symbolic Computation Vol. 26 (6), pp. 653–666.

[17] Noda M.T., Sasaki T. (1991), *Approximate GCD and its application to ill-conditioned equations*, Journal of Computational and Applied Mathematics Vol. 38, pp. 335–351.

[18] Pan, V. (2001), *Numerical computation of a polynomial gcd and extensions*, Information and computation Vol. 167, pp. 71-85.

[19] Rahtu E., Salo M., Heikkilä J. (2005), *Affine invariant pattern recognition using Multiscale Autoconvolution*, IEEE Transactions on Pattern Analysis and Machine Intelligence Vol. 27, No. 6, pp. 908–918.

[20] Rothganger F., Lazebnik S., Schmid C., Ponce J. (2006), *3D Object Modeling and Recognition Using Local Affine-Invariant Image Descriptors and Multi-View Spatial Constraints*, International Journal of Computer Vision Vol. 66, pp. 231–259.

[21] Sendra J.R., Winkler F., Perez-Diaz S. (2008), *Rational Algebraic Curves*, Springer-Verlag.

[22] Stewart, G. W. (2006), *Perturbation of the SVD in the presence of small singular values*, Linear Algebra and its Applications Vol. 419. pp. 53–77.

[23] Von Zur Gathen J., Gerhard J. (1999), *Modern Computer Algebra*, Cambridge University Press, Cambridge.

[24] Yalcin H., Unel M., Wolowich W. (2003), *Implicitization of Parametric Curves by Matrix Annihilation*, International Journal of Computer Vision 54, pp. 105-115.

[25] Yang Z., Cohen F. (1999), *Image Registration and Object Recognition Using Affine Invariants and Convex Hulls*, IEEE Transactions on Image Processing Vol. 8, No. 7, pp. 934–946.

[26] Zeng Z., (2004), *The approximate gcd of inexact polynomials. Part I: a univariate algorithm*, Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation (ISSAC), ACM, (2004), pp. 320–327.