Universal α-central extensions of Hom-Leibniz n-algebras

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ABSTRACT
We construct homology with trivial coefficients of Hom-Leibniz n-algebras. We introduce and characterize universal (α)-central extensions of Hom-Leibniz n-algebras. In particular, we show their interplay with the zero-th and first homology with trivial coefficients. When $n = 2$ we recover the corresponding results on universal central extensions of Hom-Leibniz algebras. The notion of non-abelian tensor product of Hom-Leibniz n-algebras is introduced and we establish its relationship with universal central extensions. A generalization of the concept and properties of unicentral Leibniz algebras to the setting of Hom-Leibniz n-algebras is developed.

KEYWORDS
Hom-Leibniz n-algebra; universal (α)-central extension; perfect Hom-Leibniz n-algebra; non-abelian tensor product; unicentral Hom-Leibniz n-algebra

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1. Introduction

Algebras endowed with an n-ary operation play important roles, among others, in Lie and Jordan theories, geometry, analysis, physics and biology. For instance, this kind of structures were considered to analyze DNA recombination [29]. Leibniz n-algebras and its corresponding skew-symmetric version, named as Lie n-algebras or Filippov algebras, arose in the setting of Nambu mechanics [26], a generalization of the Hamiltonian mechanics. The particular case $n = 3$ has found applications in string theory and M-branes [7,28] and in the M-theory generalization of the Nahm's equation proposed by Basu and Harvey [8]. It can also be used to construct solutions of the Yang-Baxter equation [27], which first appeared in statistical mechanics [9].

Deformations of algebras structures by means of endomorphisms give rise to Hom-algebra structures. They are motivated by discrete and deformed vector fields and differential calculus. Part of the reason to study Hom-algebras is its

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relation with the q-deformations of the Witt and the Virasoro algebras (see [21]).

In this way, deformations of algebras of Lie type were considered, among others, in [21,24,25,30,31]. Deformations of algebras of Leibniz type were considered, among others, in [14,18,20,22,24]. The generalizations of n-ary algebra structures, such as Hom-Leibniz n-algebras (or n-ary Hom-Nambu) and Hom-Lie n-algebras (or n-ary Hom-Nambu-Lie), have been introduced in [4] by Ataguema, Makhlouf and Silvestrov. In these Hom-type algebras, the n-ary Nambu identity is deformed using $n-1$ linear maps, called the twisting maps, given rise to the fundamental identity (n-ary Hom-Nambu identity) (see Definition 2.1). When these twisting maps are all equal to the identity map, one recovers Leibniz n-algebras (n-ary Nambu) and Lie n-algebras (Nambu-Lie algebras).

The topic of central extensions of algebraic structures is also present in many applications to Physics. For instance, the Witt algebra and its one-dimensional universal central extension, the Virasoro algebra, often appear in problems with conformal symmetry in the setting of string theory [19].

Recently in [17] was noticed an important fact concerning universal central extensions in the setting of semi-abelian categories, the so called **UCE** condition, namely if B is a perfect object of the category and, $f : B \twoheadrightarrow A$ and $g : C \twoheadrightarrow B$ are central extensions of the category, then the extension $f \circ g : C \twoheadrightarrow A$ is central. We show in this paper that the category of Hom-Leibniz n-algebras doesn't satisfy **UCE** condition (see Example 3.7). From this fact, our aim in this article is to introduce and characterize universal α-central extensions of Hom-Leibniz n-algebras. In case $n = 2$ we recover the corresponding results on universal α-central extensions of Hom-Leibniz algebras in [14,15]. Moreover, in case $\alpha = \mathsf{id}$ we recover results on universal central extensions of Leibniz n-algebras in [12]. In case $n = 2$ and $\alpha = \mathsf{id}$ we recover results from [13].

The article is organized as follows: in section 2 we introduce the necessary basic concepts on Hom-Leibniz n-algebras and construct the homology with trivial coefficients of Hom-Leibniz n-algebras. Bearing in mind [10], we endow the underlying vector space to a Hom-Leibniz n-algebra $\mathcal{L}$ with a structure of $(\mathcal{D}_{n-1}(\mathcal{L}) = \mathcal{L}^{\otimes n-1}, \alpha')$-symmetric Hom-co-representation as Hom-Leibniz algebras and define the homology with trivial coefficients of $\mathcal{L}$ as the Hom-Leibniz homology $HL_*^{\alpha}(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})$.

In section 3 we present our main results on universal central extensions. Based on the investigation initiated in [14], we generalize the concepts of (α)-central extension, universal (α)-central extension and perfection to the framework of Hom-Leibniz n-algebras. We also extend the corresponding characterizations of universal (α)-central extensions. Since Hom-Leibniz n-algebras category doesn't satisfy **UCE** condition, characterizations are divided between universal central and universal α-central (see Theorem 3.9).

In section 4 we introduce the concept of non-abelian tensor product of Hom-Leibniz n-algebras that generalizes the non-abelian tensor product of Hom-Leibniz algebras in [15] and the non-abelian tensor product of Leibniz n-algebras in [12], and we establish its relationship with the universal central extension.

The final section is devoted to develop a generalization of the concept and properties of unicentral Leibniz algebras in [13] to the setting of Hom-Leibniz

n-algebras. As a first step we show that the classical result: perfect Leibniz algebras are unicentral, doesn't hold in the framework of Hom-Leibniz n-algebras (see Example 5.1) and requires an additional condition (see Proposition 5.4). The main result in this section establishes that for two perfect Hom-Leibniz n-algebras, $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ and $(\mathcal{L}', \widetilde{\alpha}_{\mathcal{L}'})$ with both $\alpha_{\mathcal{L}}, \alpha_{\mathcal{L}'}$ injective and such that the universal central extensions $\big(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\big)$, and $\big(\mathfrak{uce}(\mathcal{L}'), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')}\big)$ satisfy condition (5) (see below), then the following statements hold:

a) If $\big(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\big) \cong \big(\mathfrak{uce}(\mathcal{L}'), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')}\big)$, then $\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))} \cong \frac{\alpha_{\mathcal{L}'}(\mathcal{L}')}{Z(\alpha_{\mathcal{L}'}(\mathcal{L}'))}$.

b) If $\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))} \cong \frac{\alpha_{\mathcal{L}'}(\mathcal{L}')}{Z(\alpha_{\mathcal{L}'}(\mathcal{L}'))}$, then $\big(\mathfrak{uce}(\alpha_{\mathcal{L}}(\mathcal{L})), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\big) \cong \big(\mathfrak{uce}(\alpha_{\mathcal{L}'}(\mathcal{L}')), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')|}\big)$.

2. Preliminaries on Hom-Leibniz n-algebras

In this section we introduce necessary material on Hom-Leibniz n-algebras, also called n-ary Hom-Nambu algebras in [2,4,32] or n-ary Hom-Nambu-Lie algebras in [3].

2.1. Basic definitions

Definition 2.1. A Hom-Leibniz n-algebra is a triple $(\mathcal{L}, [-, \dots, -], \widetilde{\alpha})$ consisting of a $\mathbb{K}$-vector space $\mathcal{L}$ equipped with an n-linear map $[-, \dots, -] : \mathcal{L}^{\times n} \longrightarrow \mathcal{L}$ and a family $\widetilde{\alpha} = (\alpha_i)\,, 1 \leq i \leq n-1$ of linear maps $\alpha_i : \mathcal{L} \longrightarrow \mathcal{L}$, satisfying the following fundamental identity:

$$
\begin{gathered}
[[x_1, x_2, \dots, x_n], \alpha_1(y_1), \alpha_2(y_2), \dots, \alpha_{n-1}(y_{n-1})] = \\
\sum_{i=1}^{n} [\alpha_1(x_1), \dots, \alpha_{i-1}(x_{i-1}), [x_i, y_1, y_{2,}, \dots, y_{n-1}], \alpha_i(x_{i+1}), \dots, \alpha_{n-1}(x_n)]
\end{gathered}
\tag{1}
$$

for all $(x_1, \dots, x_n) \in \mathcal{L}^{\times n}, y = (y_1, \dots, y_{n-1}) \in \mathcal{L}^{\times (n-1)}$.

The linear maps $\alpha_1, \dots, \alpha_{n-1}$ are called the twisting maps of the Hom-Leibniz n-algebra. When the n-ary bracket is skew-symmetric, i.e. $[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = (-1)^{\epsilon(\sigma)}\,[x_1, \dots, x_n], \sigma \in S_n$, then the structure is called Hom-Lie n-algebra (or n-ary Hom-Nambu algebra in [1,3], or n-Hom-Lie algebra [23]).

Let $x = (x_1, \dots, x_n) \in \mathcal{L}^{\times n}$, $y = (y_1, \dots, y_{n-1}) \in \mathcal{L}^{\times (n-1)}$, $\widetilde{\alpha}\,(y) = (\alpha_1\,(y_1)\,, \dots, \;\alpha_{n-1}\,(y_{n-1})) \in \mathcal{L}^{\times (n-1)}$ and define the adjoint representation as the linear map $ad_y : \mathcal{L} \longrightarrow \mathcal{L}$, such that $ad_y\,(x) = [x, y_1, \dots, y_{n-1}]$, for all $y \in \mathcal{L}$. Then identity (1) may be written as follows:

$$
ad_{\widetilde{\alpha}(y)}[x_1, \dots, x_n] = \sum_{i=1}^{n} \left[\alpha_1\,(x_1)\,, \dots, \alpha_{i-1}\,(x_{i-1})\,, ad_y\,(x_i)\,, \alpha_i\,(x_{i+1})\,, \dots, \alpha_{n-1}\,(x_n)\right]
$$

Definition 2.2. [1] A Hom-Leibniz n-algebra $(\mathcal{L}, [-, \dots, -], \widetilde{\alpha})$ is said to be multiplicative if the linear maps in the family $\widetilde{\alpha} = (\alpha_i)_{1 \leq i \leq n-1}$ are of the form

$\alpha_1 = \cdots = \alpha_{n-1} = \alpha$, and they preserve the bracket, that is, $\alpha[x_1,\ldots,x_n] = [\alpha(x_1),\ldots,\alpha(x_n)]$, for all $(x_1,\ldots,x_n) \in \mathcal{L}^{\times n}$.

Definition 2.3. [1] A homomorphism between two Hom-Leibniz n-algebras $(\mathcal{L}, [-,\ldots,-], \widetilde{\alpha})$ and $\left(\mathcal{L}', [-,\ldots,-]', \widetilde{\alpha'}\right)$ where $\widetilde{\alpha} = (\alpha_i)$ and $\widetilde{\alpha}' = (\alpha'_i)$, $1 \leq i \leq n-1$, is a linear map $f : \mathcal{L} \to \mathcal{L}'$ such that:

a) $f([x_1,\ldots,x_n]) = [f(x_1),\ldots,f(x_n)]'$;
b) $f \circ \alpha_i = \alpha'_i \circ f, i = 1,\ldots,n-1$

for all $x_1,...,x_n \in \mathcal{L}$.

We denote by $_{\mathsf{n}}\mathsf{HomLeib}$ the category of Hom-Leibniz n-algebras. In case $n = 2$, identity (1) is the Hom-Leibniz identity (2.1) in [14], so Hom-Leibniz 2-algebras are exactly Hom-Leibniz algebras and we use the notation $\mathsf{HomLeib}$ instead of $_{2}\mathsf{HomLeib}$.

Example 2.4.

a) When the maps $(\alpha_i)_{1\leq i\leq n-1}$ in Definition 2.1 are all of them the identity maps, then one recovers the definition of Leibniz n-algebra [16]. Hence Hom-Leibniz n-algebras include Leibniz n-algebras as a full subcategory, thereby motivating the name "Hom-Leibniz n-algebras" as a deformation of Leibniz n-algebras twisted by homomorphisms. Moreover it is a multiplicative Hom-Leibniz n-algebra.

b) Hom-Lie n-algebras are Hom-Leibniz n-algebras whose bracket satisfies the condition $[x_1,\ldots,x_i,x_{i+1},\ldots,x_n] = 0$ as soon as $x_i = x_{i+1}$ for $1 \leq i \leq n-1$. So the category $_{\mathsf{n}}\mathsf{HomLie}$ of Hom-Lie n-algebras can be considered as a full subcategory of $_{\mathsf{n}}\mathsf{HomLeib}$. For any multiplicative Hom-Leibniz n-algebra $(\mathcal{L}, [-,\ldots,-], \widetilde{\alpha})$ there is associated the Hom-Lie n-algebra $(\mathcal{L}_{\mathrm{Lie}}, [-,\ldots,-], \widetilde{\widetilde{\alpha}})$, where $\mathcal{L}_{\mathrm{Lie}} = \mathcal{L}/\mathcal{L}^{\mathrm{ann}}$, the bracket is the canonical bracket induced on the quotient and $\widetilde{\widetilde{\alpha}}$ is the homomorphism naturally induced by $\widetilde{\alpha}$. Here $\mathcal{L}^{\mathrm{ann}} = \langle\{[x_1,\ldots,x_i,x_{i+1},\ldots,x_n]$, as soon as $x_i = x_{i+1}, 1 \leq i \leq n-1, x_j \in \mathcal{L}, j = 1,\ldots,n\}\rangle$.

c) Any Hom-vector space V together with the trivial n-ary bracket $[-,-,\ldots,-]$ (i.e. $[x_1,x_2,\ldots,x_n] = 0$ for all $x_i \in V, 1 \leq i \leq n$) and any collection of linear maps $\widetilde{\alpha}_V = (\alpha_i : V \to V)_{1\leq i\leq n-1}$, is a Hom-Leibniz n-algebra, called abelian Hom-Leibniz n-algebra.

d) Hom-Lie triple systems [5,32] are Hom-Leibniz 3-algebras $\mathcal{L}$ satisfying the following properties:
- $[x,y,z] = -[y,x,z]$,
- $[x,y,z] + [y,z,x] + [z,x,y] = 0$,

for all $x,y,z \in \mathcal{L}$.

e) 1-dimensional Hom-Leibniz n-algebras over a field $\mathbb{K}$, whose characteristic in not a factor of $n-1$, are abelian Hom-Leibniz n-algebras or Hom-Leibniz n-algebras with any bracket and the collection $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$ contains at least one trivial map $\alpha_i, 1 \leq i \leq n-1$.

In the sequel we refer to multiplicative Hom-Leibniz n-algebras as Hom-Leibniz n-algebras and we shall use the shortened notation $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ when there is not confusion with the bracket operation.

Definition 2.5. Let $(\mathcal{L}, [-, \ldots, -], \widetilde{\alpha}_{\mathcal{L}})$ be a Hom-Leibniz n-algebra. A Hom-Leibniz n-subalgebra $(\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}})$ is a linear subspace $\mathcal{H}$ of $\mathcal{L}$, which is closed for the bracket and invariant by $\widetilde{\alpha}_{\mathcal{L}}$, that is,

a) $[x_1, \ldots, x_n] \in \mathcal{H}$, for all $x_1, \ldots, x_n \in \mathcal{H}$,
b) $\alpha_{\mathcal{H}}(x) \in \mathcal{H}$, for all $x \in \mathcal{H}$ $(\alpha_{\mathcal{H}} = \alpha_{\mathcal{L}|})$.

A Hom-Leibniz n-subalgebra $(\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}})$ of $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is said to be an n-sided Hom-ideal if $[x_1, x_2, \ldots, x_n] \in \mathcal{H}$ as soon as $x_i \in \mathcal{H}$ and $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathcal{L}$, for all $i = 1, 2, \ldots, n$.

If $(\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}})$ is an n-sided Hom-ideal of $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$, then the quotient $\left(\mathcal{L}/\mathcal{H}, \widetilde{\alpha}_{\mathcal{L}/\mathcal{H}}\right)$ naturally inherits a structure of Hom-Leibniz n-algebra, which is said to be the quotient Hom-Leibniz n-algebra.

Definition 2.6. Let $(\mathcal{M}_i, \widetilde{\alpha}_{\mathcal{L}|}), 1 \leq i \leq n$, be subalgebras of a Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$. We call commutator subspace corresponding to the subalgebras $\mathcal{M}_i, 1 \leq i \leq n$, to the vector subspace of $\mathcal{L}$

$$[\mathcal{M}_1, \ldots, \mathcal{M}_n] = \langle \big\{ [x_1, \ldots, x_n], x_i \in \mathcal{M}_{\sigma(i)}, 1 \leq i \leq n, \sigma \in S_n \big\} \rangle$$

Definition 2.7. Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a Hom-Leibniz n-algebra. The subspace

$$\begin{aligned} Z(\mathcal{L}) \quad = \quad & \{x \in \mathcal{L} \mid [x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n] = 0, \\ & \forall x_j \in \mathcal{L}, j \in \{1, \ldots, \widehat{i}, \ldots, n\}, i \in \{1, \ldots, n\}\} \end{aligned}$$

is said to be the center of $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$.

When the endomorphism $\alpha : \mathcal{L} \to \mathcal{L}$ is surjective, then $Z(\mathcal{L})$ is an n-sided Hom-ideal of $\mathcal{L}$.

Proposition 2.8. *[33, Theorem 4.8 (2)] Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a Hom-Leibniz $(n+1)$-algebra. Then $(\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}, [-, -], \alpha')$ is a Hom-Leibniz algebra with respect to the bracket*

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^{n} \alpha\,(a_1) \otimes \cdots \otimes [a_i, b_1, \ldots, b_n] \otimes \cdots \otimes \alpha\,(a_n)$$

and endomorphism $\alpha' = \mathcal{D}_n(\mathcal{L}) \to \mathcal{D}_n(\mathcal{L})$ given by

$$\alpha'(a_1 \otimes ... \otimes a_n) = \alpha(a_1) \otimes ... \otimes \alpha(a_n).$$

2.2. Homology with trivial coefficients of Hom-Leibniz n-algebras

Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a Hom-Leibniz n-algebra, then $\mathcal{L}$ (as a $\mathbb{K}$-vector space) is endowed with a symmetric Hom-co-representation structure [14, Definition 3.1] over $(\mathcal{D}_{n-1}(\mathcal{L}) = \mathcal{L}^{\otimes(n-1)}, \alpha')$ as Hom-Leibniz algebras with respect to the following actions

$$\begin{aligned} [-, -] \quad &: \quad \mathcal{L} \times \mathcal{D}_{n-1}(\mathcal{L}) \longrightarrow \mathcal{L} \\ [-, -] \quad &: \quad \mathcal{D}_{n-1}(\mathcal{L}) \times \mathcal{L} \longrightarrow \mathcal{L} \end{aligned}$$

given by

$$\begin{array}{rcl} [l, l_1 \otimes \cdots \otimes l_{n-1}] & := & [l, l_1, \ldots, l_{n-1}] \\ [l_1 \otimes \cdots \otimes l_{n-1}, l] & := & -[l, l_1, \ldots, l_{n-1}] \end{array} \tag{2}$$

and endomorphism $\alpha : \mathcal{L} \to \mathcal{L}$ such that $\widetilde{\alpha}_{\mathcal{L}} = (\alpha_i), \alpha_i = \alpha, 1 \leq i \leq n-1$.

Now we construct a chain complex for Hom-Leibniz n-algebras in order to compute its homology with trivial coefficients. Firstly we recall this complex for Hom-Leibniz homology [14]. Let $(L, [-,-], \alpha_L)$ be a Hom-Leibniz algebra and (M, α_M) be a Hom-co-representation over $(L, [-,-], \alpha_L)$. The Hom-Leibniz complex $(CL_*^\alpha(L, M), d_*)$ is given by setting $CL_n^\alpha(L, M) := M \otimes L^{\otimes n}, n \geq 0$, and by differentials the $\mathbb{K}$-linear maps $d_n : CL_n^\alpha(L, M) \to CL_{n-1}^\alpha(L, M)$ defined by

$$d_n(m \otimes x_1 \otimes \cdots \otimes x_n) = [m, x_1] \otimes \alpha_L(x_2) \otimes \cdots \otimes \alpha_L(x_n) +$$

$$\sum_{i=2}^{n} (-1)^i [x_i, m] \otimes \alpha_L(x_1) \otimes \cdots \otimes \widehat{\alpha_L(x_i)} \otimes \cdots \otimes \alpha_L(x_n) +$$

$$\sum_{1 \leq i < j \leq n} (-1)^{j+1} \alpha_M(m) \otimes \alpha_L(x_1) \otimes \cdots \otimes \alpha_L(x_{i-1}) \otimes [x_i, x_j] \otimes \cdots \otimes \widehat{\alpha_L(x_j)} \otimes \cdots \otimes \alpha_L(x_n).$$

The homology of the chain complex $(CL_*^\alpha(L, M), d_*)$ is called homology of the Hom-Leibniz algebra $(L, [-,-], \alpha_L)$ with coefficients in the Hom-co-representation (M, α_M) [14] and is denoted by $HL_*^\alpha(L, M) := H_*(CL_*^\alpha(L, M), d_*)$.

In order to construct the chain complex $({}_nCL_*^\alpha(\mathcal{L}), \delta_\star)$ which allows the computation of the homology with trivial coefficients of a Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$, we only need to have in mind that (2) endows $(\mathcal{L}, \alpha)$ with a Hom-co-representation structure over $(\mathcal{D}_{n-1}(\mathcal{L}), \alpha')$, so it makes sense the construction of its Hom-Leibniz complex, hence we define

$${}_nCL_*^\alpha(\mathcal{L}) := CL_*^\alpha(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})$$

thus, by definition, the homology with trivial coefficients for the Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is

$${}_nHL_*^\alpha(\mathcal{L}, \mathbb{K}) := HL_*^\alpha(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})$$

and we will use the short notation ${}_nHL_*^\alpha(\mathcal{L})$ instead of ${}_nHL_*^\alpha(\mathcal{L}, \mathbb{K})$.

In particular, we have

$${}_nHL_0^\alpha(\mathcal{L}) = HL_0^\alpha(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L}) = \mathsf{Coker}(d_1 : \mathcal{L}^{\otimes n} \to \mathcal{L}) = \mathcal{L}_{\mathrm{ab}}$$

If $\mathcal{L}$ is an abelian Hom-Leibniz n-algebra, then $\mathcal{L}$ is endowed with a trivial

Hom-co-representation structure from $\mathcal{D}_{n-1}(\mathcal{L})$, then

$$ {}_nHL_1^\alpha(\mathcal{L}) = HL_1^\alpha(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L}) = \frac{\mathcal{L} \otimes \mathcal{L}^{\otimes(n-1)}}{\alpha(\mathcal{L}) \otimes [\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(n-1)}]} $$

When $\mathcal{L}$ is a Hom-Leibniz 2-algebra, that is, a Hom-Leibniz algebra, then we have that

$$ {}_2CL_*^\alpha(\mathcal{L}) = CL_*^\alpha(\mathcal{L}, \mathcal{L}) \cong CL_{*+1}^\alpha(\mathcal{L}) $$

(see the proof of Proposition 3.4 in [14]). Thus ${}_2HL_k^\alpha(\mathcal{L}) \cong HL_{k+1}^\alpha(\mathcal{L})$, for all $k \geq 1$. In particular, ${}_2HL_0^\alpha(\mathcal{L}) \cong HL_1^\alpha(\mathcal{L}) \cong \mathcal{L}_{\text{ab}}$. When $\alpha = \mathsf{id}$, then the corresponding results for Leibniz n-algebras in [10,11] are recovered.

3. Universal central extensions

Definition 3.1. A short exact sequence of Hom-Leibniz n-algebras $(K) : 0 \to (\mathcal{M}, \widetilde{\alpha}_\mathcal{M}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_\mathcal{K}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_\mathcal{L}) \to 0$ is said to be central if $[\mathcal{M}, \mathcal{K}, \stackrel{n-1}{\ldots}, \mathcal{K}] = 0$. Equivalently, $\mathcal{M} \subseteq Z(\mathcal{K})$.

We say that (K) is α-central if $[\alpha_\mathcal{M}(\mathcal{M}), \stackrel{n-1}{\ldots}, \alpha_\mathcal{M}(\mathcal{M}), \mathcal{K}] = 0$.

Remark 1. Let us observe that the notion of central extension in case $\widetilde{\alpha}_\mathcal{K} = (\mathsf{id}_\mathcal{K})$ coincides with the notion of central extension of Leibniz n-algebras given in [10]. Nevertheless, the notion of α-central extension in case $\widetilde{\alpha}_\mathcal{K} = (\mathsf{id}_\mathcal{K})$ gives rise to a new notion of central extension of Leibniz n-algebras. In particular, this kind of central extensions are abelian extensions of Leibniz n-algebras [16].

In case $n = 2$, we recover the notions of central and α-central extension of a Hom-Leibniz algebra introduced in [14].

Obviously every central extension is an α-central extension, but the converse doesn't hold as the following counterexample shows:

Let $(\mathcal{L}, \widetilde{\alpha}_\mathcal{L})$ be the Hom-Leibniz 3-algebra where $\mathcal{L}$ is the two-dimensional vector space with basis $\{a_1, a_2\}$, the bracket operation is given by $[a_i, a_i, a_i] = a_i, i = 1, 2$ and zero elsewhere, and endomorphism $\alpha_\mathcal{L} = 0$.

On the other hand, let $(\mathcal{K}, \widetilde{\alpha}_\mathcal{K})$ be the Hom-Leibniz 3-algebra where $\mathcal{K}$ is the three-dimensional vector space with basis $\{b_1, b_2, b_3\}$, the bracket operation is given by $[b_i, b_i, b_i] = b_i, i = 1, 2, 3$ and zero elsewhere, and endomorphism $\alpha_\mathcal{K} = 0$.

The surjective homomorphism $\pi : (\mathcal{K}, \widetilde{\alpha}_\mathcal{K}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_\mathcal{L})$ given by $\pi(b_1) = 0$, $\pi(b_2) = a_1, \pi(b_3) = a_2$, is an α-central extension, since $\mathsf{Ker}(\pi) = \langle \{b_1\} \rangle$ and $[\alpha_\mathcal{K}(\mathsf{Ker}(\pi)), \alpha_\mathcal{K}(\mathsf{Ker}(\pi)), \mathcal{K}] = 0$, but is not a central extension since $Z(\mathcal{K}) = 0$.

Definition 3.2. A central extension $(K) : 0 \to (\mathcal{M}, \widetilde{\alpha}_\mathcal{M}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_\mathcal{K}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_\mathcal{L}) \to 0$ is said to be universal if for every central extension $(K') : 0 \to (\mathcal{M}', \widetilde{\alpha}_{\mathcal{M}'}) \xrightarrow{i'} (\mathcal{K}', \widetilde{\alpha}_{\mathcal{K}'}) \xrightarrow{\pi'} (\mathcal{L}, \widetilde{\alpha}_\mathcal{L}) \to 0$ there exists a unique homomorphism of Hom-Leibniz n-algebras $h : (\mathcal{K}, \widetilde{\alpha}_\mathcal{K}) \to (\mathcal{K}', \widetilde{\alpha}_{\mathcal{K}'})$ such that $\pi' \circ h = \pi$.

The central extension $(K) : 0 \to (\mathcal{M}, \widetilde{\alpha}_\mathcal{M}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_\mathcal{K}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_\mathcal{L}) \to 0$ is said to be universal α-central extension if for every α-central extension $(K') : 0 \to$

$(\mathcal{M}',\widetilde{\alpha}_{\mathcal{M}'}) \stackrel{i'}{\to} (\mathcal{K}',\widetilde{\alpha}_{\mathcal{K}'}) \stackrel{\pi'}{\to} (\mathcal{L},\widetilde{\alpha}_{\mathcal{L}}) \to 0$ there exists a unique homomorphism of Hom-Leibniz n-algebras $h : (\mathcal{K},\widetilde{\alpha}_{\mathcal{K}}) \to (\mathcal{K}',\widetilde{\alpha}_{\mathcal{K}'})$ such that $\pi' \circ h = \pi$.

Remark 2. Obviously, every universal α-central extension is a universal central extension. Note that in the case $\widetilde{\alpha}_{\mathcal{K}} = (\mathsf{id}_{\mathcal{K}})$ both notions coincide. In case $n = 2$ we recover the corresponding notions of universal (α-)central extension of Hom-Leibniz algebras given respectively in [14, Definition 4.3].

Definition 3.3. A Hom-Leibniz n-algebra $(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})$ is said to be perfect if $\mathcal{L} = [\mathcal{L},\dots,\mathcal{L}]$.

Lemma 3.4. *Let $\pi : (\mathcal{K},\widetilde{\alpha}_{\mathcal{K}}) \twoheadrightarrow (\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})$ be a surjective homomorphism of Hom-Leibniz n-algebras. If $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra, then $(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})$ is a perfect Hom-Leibniz n-algebra as well.*

Lemma 3.5. *If $0 \to (\mathcal{M},\widetilde{\alpha}_{\mathcal{M}}) \stackrel{i}{\to} (\mathcal{K},\widetilde{\alpha}_{\mathcal{K}}) \stackrel{\pi}{\to} (\mathcal{L},\widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a universal central extension, then $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ and $(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})$ are perfect Hom-Leibniz n-algebras.*

Proof. Assume that $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ is not a perfect Hom-Leibniz n-algebra, then $[\mathcal{K},\dots,\mathcal{K}] \subsetneq \mathcal{K}$, thus $\left(\mathcal{K}/[\mathcal{K},\dots,\mathcal{K}],\widetilde{\alpha}_{|}\right)$, where $(\widetilde{\alpha}_{|})$ is the induced natural homomorphism, is an abelian Hom-Leibniz n-algebra (see Example 2.4 *c)*).

Consider the central extension $0 \to (\mathcal{K}_{\text{ab}},\widetilde{\alpha}_{|}) \to \left(\mathcal{K}_{\text{ab}} \times \mathcal{L},\widetilde{\alpha}_{|} \times \widetilde{\alpha}_{\mathcal{L}}\right) \stackrel{pr}{\to} (\mathcal{L},\widetilde{\alpha}_{\mathcal{L}}) \to 0$, then the homomorphisms of Hom-Leibniz n-algebras $\varphi,\psi : (\mathcal{K},\widetilde{\alpha}_{\mathcal{K}}) \to (\mathcal{K}_{\text{ab}} \times \mathcal{L},\ \widetilde{\alpha}_{|} \times \widetilde{\alpha}_{\mathcal{L}})$ given by $\varphi(k) = \left(\overline{k},\pi(k)\right)$ and $\psi(k) = (0,\pi(k))$ ($\overline{k}$ denotes the coset $k + [\mathcal{K},\dots,\mathcal{K}]$) are two distinct homomorphisms of Hom-Leibniz n-algebras such that $pr \circ \varphi = \pi = pr \circ \psi$, which contradicts the universality of the given extension.

Lemma 3.4 completes the proof. □

Lemma 3.6. *Let $0 \to (\mathcal{M},\widetilde{\alpha}_{\mathcal{M}}) \stackrel{i}{\to} (\mathcal{K},\widetilde{\alpha}_{\mathcal{K}}) \stackrel{\pi}{\to} (\mathcal{L},\widetilde{\alpha}_{\mathcal{L}}) \to 0$ be an α-central extension and $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra. If there exists a homomorphism of Hom-Leibniz n-algebras $f : (\mathcal{K},\widetilde{\alpha}_{\mathcal{K}}) \to (\mathcal{A},\widetilde{\alpha}_{\mathcal{A}})$ such that $\tau \circ f = \pi$, where $0 \to (\mathcal{N},\widetilde{\alpha}_{\mathcal{N}}) \stackrel{j}{\to} (\mathcal{A},\widetilde{\alpha}_{\mathcal{A}}) \stackrel{\tau}{\to} (\mathcal{L},\widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a central extension, then f is unique.*

Proof. Assume that there are two homomorphisms $f_1, f_2 : (\mathcal{K},\widetilde{\alpha}_{\mathcal{K}}) \to (\mathcal{A},\widetilde{\alpha}_{\mathcal{A}})$ such that $\tau \circ f_1 = \pi = \tau \circ f_2$, then $f_1 - f_2 \in \mathsf{Ker}(\tau) = \mathcal{N}$, i.e. $f_1(k) = f_2(k) + n_k$, $n_k \in \mathcal{N}$.

Since $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra, it is enough to show that f_1 and f_2 coincide on $[\mathcal{K},\dots,\mathcal{K}]$. Indeed

$$f_1[k_1,\dots,k_n] = [f_2(k_1) + n_{k_1},\dots,f_2(k_n) + n_{k_n}] = [f_2(k_1),\dots,f_2(k_n)] + A = f_2[k_1,\dots,k_n],$$

since a typical summand in A is of the form $\left[n_{k_1},\dots,n_{k_j},f_2(k_{j+1}),\dots,f_2(k_n)\right]$ which vanishes because $\mathcal{N} \subseteq Z(\mathcal{A})$. □

The category ${}_{\mathsf{n}}\mathsf{HomLeib}$ is a semi-abelian category that doesn't satisfy the so called in [17] **UCE** condition, namely if B is a perfect object of the category and, $f : B \twoheadrightarrow A$ and $g : C \twoheadrightarrow B$ are central extensions of the category, then the extension $f \circ g : C \twoheadrightarrow A$ is central, as the following example shows:

Example 3.7. Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be the two-dimensional Hom-Leibniz 3-algebra with basis $\{b_1, b_2\}$, bracket given by $[b_2, b_1, b_1] = b_2, [b_2, b_2, b_2] = b_1$ and zero elsewhere, and endomorphism $\widetilde{\alpha}_{\mathcal{L}} = (0)$.

Let $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ be the three-dimensional Hom-Leibniz 3-algebra with basis $\{a_1, a_2, a_3\}$, bracket given by $[a_2, a_2, a_2] = a_1, [a_3, a_2, a_2] = a_3, [a_3, a_3, a_3] = a_2$ and zero elsewhere, and endomorphism $\widetilde{\alpha}_{\mathcal{K}} = (0)$.

Obviously $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz 3-algebra and $Z(\mathcal{K}) = \langle\{a_1\}\rangle$. The linear map $\pi : \left(\mathcal{K}, \widetilde{0}\right) \to \left(\mathcal{L}, \widetilde{0}\right)$ given by $\pi(a_1) = 0, \pi(a_2) = b_1, \pi(a_3) = b_2$, is a central extension since π is a surjective homomorphism of Hom-Leibniz 3-algebras and $\mathsf{Ker}(\pi) = \langle\{a_1\}\rangle \subseteq Z(\mathcal{K})$.

Now consider the four-dimensional Hom-Leibniz 3-algebra $(\mathcal{F}, \widetilde{\alpha}_{\mathcal{F}})$ with basis $\{e_1, e_2, e_3, e_4\}$, bracket given by $[e_3, e_2, e_2] = e_1$, $[e_3, e_3, e_3] = e_2, [e_4, e_3, e_3] = e_4$, $[e_4, e_4, e_4] = e_3$ and zero elsewhere, and endomorphism $\widetilde{\alpha}_{\mathcal{F}} = (0)$.

The linear map $\rho(e_1) = 0, \rho(e_2) = a_1, \rho(e_3) = a_2, \rho(e_4) = a_3$ is a central extension since ρ is a surjective homomorphism of Hom-Leibniz 3-algebras and $\mathsf{Ker}(\rho) = \langle\{e_1\}\rangle = Z(\mathcal{F})$.

The composition $\pi \circ \rho : \left(\mathcal{F}, \widetilde{0}\right) \to \left(\mathcal{L}, \widetilde{0}\right)$ is given by $\pi \circ \rho(e_1) = \pi(0) = 0, \pi \circ \rho(e_2) = \pi(a_1) = 0, \pi \circ \rho(e_3) = \pi(a_2) = b_1, \pi \circ \rho(e_4) = \pi(a_3) = b_2$. Consequently, $\pi \circ \rho$ is a surjective homomorphism, but is not a central extension, since $\mathsf{Ker}(\pi \circ \rho) = \langle\{e_1, e_2\}\rangle \nsubseteq Z(\mathcal{F})$. However, $\pi \circ \rho : \left(\mathcal{F}, \widetilde{0}\right) \to \left(\mathcal{L}, \widetilde{0}\right)$ is an α-central extension.

Lemma 3.8. *Let* $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \stackrel{i}{\to} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \stackrel{\pi}{\to} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ *and* $0 \to (\mathcal{N}, \widetilde{\alpha}_{\mathcal{N}}) \stackrel{j}{\to} (\mathcal{F}, \widetilde{\alpha}_{\mathcal{F}}) \stackrel{\rho}{\to} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \to 0$ *be central extensions with* $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ *a perfect Hom-Leibniz* n*-algebra. Then the composition extension* $0 \to (\mathcal{P}, \widetilde{\alpha}_{\mathcal{P}}) = \mathsf{Ker}(\pi \circ \rho) \to (\mathcal{F}, \widetilde{\alpha}_{\mathcal{F}}) \stackrel{\pi\circ\rho}{\to} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ *is an* α*-central extension.*

Moreover, if $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \stackrel{i}{\to} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \stackrel{\pi}{\to} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ *is a universal* α*-central extension, then* $0 \to (\mathcal{N}, \widetilde{\alpha}_{\mathcal{N}}) \stackrel{j}{\to} (\mathcal{F}, \widetilde{\alpha}_{\mathcal{F}}) \stackrel{\rho}{\to} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \to 0$ *is split.*

Proof. Since $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra, then every element $f \in \mathcal{F}$ can be written as $\sum\limits_k \lambda_k [f_{k1}, ..., f_{kn}] + n$, $n \in \mathcal{N}$, $f_{k1}, ..., f_{kn} \in \mathcal{F}$. So, for any element in $[\alpha_{\mathcal{P}}(\mathcal{P}), \stackrel{n-1}{\dots}, \alpha_{\mathcal{P}}(\mathcal{P}), \mathcal{F}]$ we have

$$\begin{aligned}[\alpha_{\mathcal{P}}(p_1), \dots, f_i, \dots, \alpha_{\mathcal{P}}(p_{n-1})] \quad = \quad & \sum_k \lambda_k ([\alpha_{\mathcal{P}}(p_1), \dots, [f_{i_{k1}}, \dots, f_{i_{kn}}], \dots, \\ & \alpha_{\mathcal{P}}(p_{n-1})] + [\alpha_{\mathcal{P}}(p_1), \dots, n, \dots, \alpha_{\mathcal{P}}(p_{n-1})])\end{aligned}$$

which vanishes by application of the fundamental identity and bearing in mind that $[\mathcal{P}, \mathcal{F}, \dots, \mathcal{F}] \in \mathsf{Ker}(\rho) = \mathcal{N}$ and $\mathcal{N} \subseteq Z(\mathcal{F})$.

For the second statement, if $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \stackrel{i}{\to} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \stackrel{\pi}{\to} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a universal α-central extension, then by the first statement, $0 \to (\mathcal{P}, \widetilde{\alpha}_{\mathcal{P}}) = \mathsf{Ker}(\pi \circ \rho) \to (\mathcal{F}, \widetilde{\alpha}_{\mathcal{F}}) \stackrel{\pi\circ\rho}{\to} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ is an α-central extension, then there exists a unique homomorphism of Hom-Leibniz algebras $\sigma : (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \to (\mathcal{F}, \widetilde{\alpha}_{\mathcal{F}})$ such that $\pi \circ \rho \circ \sigma = \pi$. On the other hand, $\pi \circ \rho \circ \sigma = \pi = \pi \circ \mathsf{id}$ and $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is perfect, then Lemma 3.6 implies that $\rho \circ \sigma = \mathsf{id}$. $\square$

Theorem 3.9.

a) If a central extension $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a universal α-central extension, then $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra and every central extension of $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is split.

b) Let $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ be a central extension.

If $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra and every central extension of $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is split, then $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a universal central extension.

c) A Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ admits a universal central extension if and only if $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is perfect. Furthermore, the kernel of the universal central extension is canonically isomorphic to ${}_nHL_1^{\alpha}(\mathcal{L})$.

d) If $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a universal α-central extension, then ${}_nHL_0^{\alpha}(\mathcal{K}) = {}_nHL_1^{\alpha}(\mathcal{K}) = 0$.

e) If ${}_nHL_0^{\alpha}(\mathcal{K}) = {}_nHL_1^{\alpha}(\mathcal{K}) = 0$, then any central extension $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a universal central extension.

Proof. a) If $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \xrightarrow{i} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ is a universal α-central extension, then is a universal central extension by Remark 2, so $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra by Lemma 3.5 and every central extension of $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is split by Lemma 3.8.

b) Let us consider any central extension $0 \to (\mathcal{N}, \widetilde{\alpha}_{\mathcal{N}}) \xrightarrow{j} (\mathcal{A}, \widetilde{\alpha}_{\mathcal{A}}) \xrightarrow{\tau} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$. Construct the pull-back extension $0 \to (\mathcal{N}, \widetilde{\alpha}_{\mathcal{N}}) \xrightarrow{\chi} (\mathcal{Q}, \widetilde{\alpha}_{\mathcal{Q}}) \xrightarrow{\overline{\tau}} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \to 0$, where $\mathcal{Q} = \mathcal{A} \times_{\mathcal{L}} \mathcal{K} = \{(a,k) \in \mathcal{A} \times \mathcal{K} \mid \tau(a) = \pi(k)\}$ and $\alpha_{\mathcal{Q}}(a,k) = (\alpha_{\mathcal{A}}(a), \alpha_{\mathcal{K}}(k))$, which is central, consequently is split, that is, there exists a homomorphism $\sigma : (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \to (\mathcal{Q}, \widetilde{\alpha}_{\mathcal{Q}})$ such that $\overline{\tau} \circ \sigma = \mathsf{id}$.

Then $\overline{\pi} \circ \sigma$, where $\overline{\pi} : (\mathcal{Q}, \widetilde{\alpha}_{\mathcal{Q}}) \to (\mathcal{A}, \widetilde{\alpha}_{\mathcal{A}})$ is induced by the pull-back construction, satisfies $\tau \circ \overline{\pi} \circ \sigma = \pi$. Lemma 3.6 concludes the proof.

c) For a Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$, let be the chain homology complex ${}_nC_*^{\alpha}(\mathcal{L}, \mathbb{K})$, where $\mathbb{K}$ is endowed with a trivial Hom-co-representation structure.

$${}_nC_*^{\alpha}(\mathcal{L}, \mathbb{K}) : \cdots \to \mathcal{L}^{\otimes k(n-1)+1} \xrightarrow{\delta_k} \mathcal{L}^{\otimes (k-1)(n-1)+1} \xrightarrow{\delta_{k-1}} \cdots \to \mathcal{L}^{\otimes 2n-1} \xrightarrow{\delta_2} \mathcal{L}^{\otimes n} \xrightarrow{\delta_1} \mathcal{L}$$

The low differentials are given by

$$\begin{aligned} \delta_1(x_1 \otimes \cdots \otimes x_n) &= [x_1, \dots, x_n] \\ \delta_2(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_{n-1}) &= [x_1, \dots, x_n] \otimes \alpha_{\mathcal{L}}(y_1) \otimes \cdots \otimes \alpha_{\mathcal{L}}(y_{n-1}) - \\ & \sum_{i=1}^{n} \alpha_{\mathcal{L}}(x_1) \otimes \cdots \otimes [x_i, y_1, \dots, y_{n-1}] \otimes \dots \\ & \otimes \alpha_{\mathcal{L}}(x_n) \end{aligned}$$

As a $\mathbb{K}$-vector space, let $I_{\mathcal{L}}$ be the subspace of $\mathcal{L}^{\otimes 2n-1}$ spanned by the elements of the form

$$\begin{aligned}&[x_1,\dots,x_n]\otimes\alpha_{\mathcal{L}}(y_1)\otimes\cdots\otimes\alpha_{\mathcal{L}}(y_{n-1})-\\&\sum_{i=1}^{n}\alpha_{\mathcal{L}}(x_1)\otimes\cdots\otimes[x_i,y_1,\dots,y_{n-1}]\otimes\cdots\otimes\alpha_{\mathcal{L}}(x_n)\end{aligned}$$

that is $I_{\mathcal{L}}=\mathsf{Im}\left(\delta_2:\mathcal{L}^{\otimes 2n-1}\to\mathcal{L}^{\otimes n}\right)$. Let $\mathfrak{uce}(\mathcal{L})$ be the quotient vector space $\frac{\mathcal{L}^{\otimes n}}{I_{\mathcal{L}}}$. Every coset $(x_1\otimes\cdots\otimes x_n)+I_{\mathcal{L}}$ is denoted by $\{x_1,\dots,x_n\}$.

By construction, the following identity holds

$$\begin{aligned}&\{[x_1,\dots,x_n],\alpha_{\mathcal{L}}(y_1),\dots,\alpha_{\mathcal{L}}(y_{n-1})\}=\\&\sum_{i=1}^{n}\{\alpha_{\mathcal{L}}(x_1),\dots,[x_i,y_1,\dots,y_{n-1}],\dots,\alpha_{\mathcal{L}}(x_n)\}\end{aligned}\tag{3}$$

Since δ_1 vanishes on $I_{\mathcal{L}}$, it induces a linear map $u_{\mathcal{L}}:\mathfrak{uce}(\mathcal{L})\to\mathcal{L}$, given by $u_{\mathcal{L}}(\{x_1,\dots,x_n\})=[x_1,\dots,x_n]$, and $(\widetilde{\alpha}_{\mathcal{L}})$ induces $(\widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})})$, where $\alpha_{\mathfrak{uce}(\mathcal{L})}(\{x_1,\dots,x_n\})=\{\alpha_{\mathcal{L}}(x_1),\dots,\alpha_{\mathcal{L}}(x_n)\}$.

The bracket operation

$$[\{x_{1,1},...,x_{n,1}\},\dots,\{x_{1,n},\dots,x_{n,n}\}]=\{[x_{1,1},\dots,x_{n,1}],\dots,[x_{1,n},\dots,x_{n,n}]\}$$

endows $\left(\mathfrak{uce}(\mathcal{L}),\widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)$ with a structure of Hom-Leibniz n-algebra and $u_{\mathcal{L}}:\left(\mathfrak{uce}(\mathcal{L}),\widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)\to(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})$ becomes a surjective homomorphism of Hom-Leibniz n-algebras when $(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})$ is perfect because $\mathsf{Im}(u_{\mathcal{L}})=[\mathcal{L},\dots,\mathcal{L}]$.

From the construction immediately follows that $\mathsf{Ker}\,(u_{\mathcal{L}})={}_nHL_1^{\alpha}(\mathcal{L})$, so we have the central extension

$$0\to\left({}_nHL_1^{\alpha}(\mathcal{L}),\widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\right)\to\left(\mathfrak{uce}(\mathcal{L}),\widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)\stackrel{u_{\mathcal{L}}}{\to}(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})\to 0$$

which is universal, because for any central extension $0\to(\mathcal{M},\widetilde{\alpha}_{\mathcal{M}})\to(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})\stackrel{\pi}{\to}(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})\to 0$ there exists the homomorphism of Hom-Leibniz n-algebras $\beta:\left(\mathfrak{uce}(\mathcal{L}),\ \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)\to(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ given by $\beta(\{x_1,\dots,x_n\})=[k_1,...,k_n],\pi(k_i)=x_i$, such that $\pi\circ\beta=u_{\mathcal{L}}$.

A direct checking shows that $\left(\mathfrak{uce}(\mathcal{L}),\widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)$ is perfect, then Lemma 3.6 guarantees the uniqueness of β.

d) If $0\to(\mathcal{M},\widetilde{\alpha}_{\mathcal{M}})\to(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})\stackrel{\pi}{\to}(\mathcal{L},\widetilde{\alpha}_{\mathcal{L}})\to 0$ is a universal α-central extension, then $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ is perfect by Remark 2 and Lemma 3.5, so ${}_nHL_0^{\alpha}(\mathcal{K})=0$. By Lemma 3.8 and statement *c)*, the universal central extension corresponding to $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ is split, so ${}_nHL_1^{\alpha}(\mathcal{K})=0$.

e) ${}_nHL_0^{\alpha}(\mathcal{K})=0$ implies that $(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$ is a perfect Hom-Leibniz n-algebra.

${}_nHL_1^{\alpha}(\mathcal{K})=0$ implies that $(\mathfrak{uce}(\mathcal{K}),\widetilde{\alpha}_{\mathfrak{uce}(\mathcal{K})})\stackrel{\sim}{\to}(\mathcal{K},\widetilde{\alpha}_{\mathcal{K}})$. Statement *b)* ends the proof. □

Remark 3. When $n=2$, the above results recover the corresponding ones for Hom-Leibniz algebras in [14].

4. Non-abelian tensor product

Let $(\mathcal{M}_i, \widetilde{\alpha}_{\mathcal{M}_i}), 1 \leq i \leq n$, be n-sided Hom-ideals of a Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$. We denote by $\mathcal{M}_1 * \cdots * \mathcal{M}_n$ the vector space spanned by all the symbols $m_{\sigma(1)} * \cdots * m_{\sigma(n)}$, where $m_i \in \mathcal{M}_i, i \in \{1, 2, \dots, n\}, \sigma \in S_n$.

We claim that $(\mathcal{M}_1 * \cdots * \mathcal{M}_n, \widetilde{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n})$ is a Hom-vector space, where $(\widetilde{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n})$ is induced by $\alpha_{\mathcal{M}_i}, 1 \leq i \leq n$, i.e.

$$\alpha_{\mathcal{M}_1 * \cdots * \mathcal{M}_n} \left(m_{\sigma(1)} * \cdots * m_{\sigma(n)}\right) = \alpha_{\mathcal{M}_{\sigma(1)}} \left(m_{\sigma(1)}\right) * \cdots * \alpha_{\mathcal{M}_{\sigma(n)}} \left(m_{\sigma(n)}\right).$$

We denote by $\mathcal{DL}_n(\mathcal{M}_1, \dots, \mathcal{M}_n)$ the vector subspace spanned by the elements of the form:

a) $\lambda \left(m_{\sigma(1)} * \cdots * m_{\sigma(n)}\right) = \left(\lambda m_{\sigma(1)}\right) * m_{\sigma(2)} * \cdots * m_{\sigma(n)} = \cdots = m_{\sigma(1)} * \cdots * \left(\lambda m_{\sigma(n)}\right)$.

b) $m_{\sigma(1)} * \cdots * \left(m'_{\sigma(i)} + m''_{\sigma(i)}\right) * \cdots * m_{\sigma(n)} = m_{\sigma(1)} * \cdots * m'_{\sigma(i)} * \cdots * m_{\sigma(n)} + m_{\sigma(1)} * \cdots * m''_{\sigma(i)} * \cdots * m_{\sigma(n)}$, for any $i \in \{1, 2, \dots, n\}$.

c) $\left[m_{\tau(1)}, \dots, m_{\tau(n)}\right] * \alpha_{\mathcal{M}_{\tau(n+1)}} \left(m_{\tau(n+1)}\right) * \cdots * \alpha_{\mathcal{M}_{\tau(2n-1)}} \left(m_{\tau(2n-1)}\right) - \sum_{i=1}^{n} \alpha_{\mathcal{M}_{\tau(1)}}(m_{\tau(1)}) * \cdots * \left[m_{\tau(i)}, m_{\tau(n+1)}, \dots, m_{\tau(2n-1)}\right] * \cdots * \alpha_{\mathcal{M}_{\tau(n)}}(m_{\tau(n)})$.

d) $[m_{\sigma(1)}, \dots, m_{\sigma(n)}] * \alpha_{\mathcal{M}_{n+1}}(m_{n+1}) * \cdots * \alpha_{\mathcal{M}_{2n-1}}(m_{2n-1}) - (-1)^{\epsilon(\sigma)}[m_1, \dots, m_n] * \alpha_{\mathcal{M}_{n+1}}(m_{n+1}) * \cdots * \alpha_{\mathcal{M}_{2n-1}}(m_{2n-1})$.

for all $\lambda \in \mathbb{K}$, $m_i \in \mathcal{M}_i, 1 \leq i \leq n$, $\sigma \in S_n, \tau \in S_{2n-1}$.

Moreover, it can be readily checked that $\alpha_{\mathcal{M}_1 * \cdots * \mathcal{M}_n} (\mathcal{DL}_n(\mathcal{M}_1, \dots, \mathcal{M}_n)) \subseteq \mathcal{DL}_n(\mathcal{M}_1, \dots, \mathcal{M}_n)$, hence we can construct the quotient Hom-vector space

$$(\mathcal{M}_1 * \cdots * \mathcal{M}_n / \mathcal{DL}_n(\mathcal{M}_1, \dots, \mathcal{M}_n), \overline{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n})$$

which is endowed with a structure of Hom-Leibniz n-algebra with respect to the bracket

$$\begin{array}{l} [m_{11} * \cdots * m_{n1}, m_{12} * \cdots * m_{n2}, \dots, m_{1n} * \cdots * m_{nn}] := \\ [m_{11}, \dots, m_{n1}] * [m_{12}, \dots, m_{n2}] * \cdots * [m_{1n}, \dots, m_{nn}] \end{array} \tag{4}$$

where we abbreviate a coset $\overline{m_{1i} * \cdots * m_{ni}}$ by $m_{1i} * \cdots * m_{ni}$ and the endomorphism $\overline{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n}$ by $\alpha_{\mathcal{M}_1 * \cdots * \mathcal{M}_n}$.

Definition 4.1. The above Hom-Leibniz n-algebra structure on

$$(\mathcal{M}_1 * \cdots * \mathcal{M}_n / \mathcal{DL}_n(\mathcal{M}_1, \dots, \mathcal{M}_n), \overline{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n})$$

is called the non-abelian tensor product of the n-sided Hom-ideals $(\mathcal{M}_i, \widetilde{\alpha}_{\mathcal{M}_i}), 1 \leq i \leq n$, and it will be denoted by $(\mathcal{M}_1 * \cdots * \mathcal{M}_n, \widetilde{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n})$.

Remark 4. If $\widetilde{\alpha}_{\mathcal{L}} = (\mathsf{id}_{\mathcal{L}})$, then $(\mathcal{M}_1 * \cdots * \mathcal{M}_n, \widetilde{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n})$ coincides with the non-abelian tensor product of Leibniz n-algebras introduced in [12]. In case $n = 2$, we recover a particular case of the non-abelian tensor product of Hom-Leibniz algebras given in [15].

For any n-sided Hom-ideals $(\mathcal{M}_i, \widetilde{\alpha}_{\mathcal{M}_i}), 1 \leq i \leq n$, of a Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$, there exists a homomorphism of Hom-Leibniz n-algebras

$$\psi : (\mathcal{M}_1 * \cdots * \mathcal{M}_n, \widetilde{\alpha}_{\mathcal{M}_1 * \cdots * \mathcal{M}_n}) \to \left(\bigcap_{i=1}^{n} \mathcal{M}_i, \widetilde{\alpha}_{\cap} \right)$$

given by

$$\psi(m_{\sigma(1)} * \cdots * m_{\sigma(n)}) = [m_{\sigma(1)}, \ldots, m_{\sigma(n)}]$$

for any $m_{\sigma(i)} \in \mathcal{M}_{\sigma(i)}, i = 1, \ldots, n, \sigma \in S_n$.

In particular, when $\mathcal{M}_i = \mathcal{L}, 1 \leq i \leq n$, from relation (4) immediately follows that $\psi : (\mathcal{L} * \cdots * \mathcal{L}, \widetilde{\alpha}_{\mathcal{L} * \cdots * \mathcal{L}}) \twoheadrightarrow \left([\mathcal{L}, \ldots, \mathcal{L}], \widetilde{\alpha}_{\mathcal{L}|}\right)$ is a central extension.

Theorem 4.2. *If $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is a perfect Hom-Leibniz n-algebra, then ψ : $(\mathcal{L} * \cdots * \mathcal{L},\ \widetilde{\alpha}_{\mathcal{L} * \cdots * \mathcal{L}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is a universal central extension.*

Proof. Let $0 \to \left(Ker\,(\chi), \widetilde{\alpha}_{\mathcal{C}|}\right) \xrightarrow{i} (\mathcal{C}, \widetilde{\alpha}_{\mathcal{C}}) \xrightarrow{\chi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ be a central extension of $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$. Since $\mathsf{Ker}\,(\chi) \subseteq Z\,(\mathcal{C})$ we get a well-defined homomorphism of Hom-Leibniz n-algebras $f : \mathcal{L} * \cdots * \mathcal{L} \to \mathcal{C}$ given on generators by $f(l_{\sigma(1)} * \cdots * l_{\sigma(n)}) = [c_{l_{\sigma(1)}}, \ldots, c_{l_{\sigma(n)}}]$, where $c_{l_{\sigma(i)}}$ is an element in $\chi^{-1}\left(l_{\sigma(i)}\right), i = 1, \ldots, n, \sigma \in S_n$.

On the other hand, relation (4) implies that $(\mathcal{L} * \cdots * \mathcal{L}, \widetilde{\alpha}_{\mathcal{L} * \cdots * \mathcal{L}})$ is perfect, then the homomorphism f is unique by Remark 1 and Lemma 3.6. $\square$

Remark 5. If $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is a perfect Hom-Leibniz n-algebra, then $\mathsf{Ker}(\psi) \cong {}_nHL_1^{\alpha}(\mathcal{L})$ by Theorem 3.9 *c)*.

Since universal central extensions of perfect Hom-Leibniz n-algebras are unique up to isomorphisms, then $(\mathcal{L} * \cdots * \mathcal{L}, \widetilde{\alpha}_{\mathcal{L} * \cdots * \mathcal{L}}) \cong \left(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)$ by means of the isomorphism $\varphi : (\mathcal{L} * \cdots * \mathcal{L}, \widetilde{\alpha}_{\mathcal{L} * \cdots * \mathcal{L}}) \to \left(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right), \varphi(l_{\sigma(1)} * \cdots * l_{\sigma(n)}) = \{l_{\sigma(1)}, \ldots, l_{\sigma(n)}\}, \sigma \in S_n$.

In case $n = 2$, the universal central extension in Theorem 4.2 provides the universal central extension of a Hom-Leibniz algebra given in [15].

Proposition 4.3. *If $(\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}})$ is an n-sided Hom-ideal of a perfect Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$, then there is an exact sequence of vector spaces*

$$\mathsf{Ker}\Big(\bigoplus_{i=1}^{n} \mathcal{L} * \cdots * \overbrace{\mathcal{M}}^{i} * \cdots * \mathcal{L} \xrightarrow{\psi_|} \mathcal{M}\Big) \to {}_nHL_1^{\alpha}(\mathcal{L}) \to {}_nHL_1^{\alpha}(\mathcal{L}/\mathcal{M}) \to \frac{\mathcal{M}}{\bigoplus_{i=1}^{n} [\mathcal{L}, \ldots, \overbrace{\mathcal{M}}^{i}, \ldots, \mathcal{L}]} \to 0$$

Proof. Consider the following commutative diagram of Hom-Leibniz n-

algebras where π denotes the canonical projection on the quotient

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
(\bigoplus\limits_{i=1}^{n} \mathcal{L} * \cdots * \overbrace{\mathcal{M}}^{i} * \cdots * \mathcal{L}, \widetilde{\alpha}_{\mathcal{L}*\cdots*\mathcal{L}_{|}}) \arrow[r, "\psi_{|}"] \arrow[d] & (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \arrow[d] \\
(\mathcal{L} * \cdots * \mathcal{L}, \widetilde{\alpha}_{\mathcal{L}*\cdots*\mathcal{L}}) \arrow[r, "\psi"] \arrow[d, "\pi*\cdots*\pi"] & (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \arrow[d, "\pi"] \\
(\frac{\mathcal{L}}{\mathcal{M}} * \cdots * \frac{\mathcal{L}}{\mathcal{M}}, \widetilde{\alpha}_{\frac{\mathcal{L}}{\mathcal{M}}*\cdots*\frac{\mathcal{L}}{\mathcal{M}}}) \arrow[r, "\overline{\psi}"] \arrow[d] & (\frac{\mathcal{L}}{\mathcal{M}}, \overline{\widetilde{\alpha}}_{\mathcal{L}}) \arrow[d] \\
0 & 0
\end{tikzcd}
$$

where $\psi(l_{\sigma(1)} * \cdots * l_{\sigma(n)}) = [l_{\sigma(1)}, \ldots, l_{\sigma(n)}], \sigma \in S_n$. Then, forgetting the Hom-Leibniz n-algebra structures, by using the Snake Lemma for the same diagram of vector spaces, we obtain the following exact sequence,

$$\mathsf{Ker}(\psi_{|}) \to \mathsf{Ker}(\psi) \to \mathsf{Ker}(\overline{\psi}) \to \mathsf{Coker}(\psi_{|}) \to \mathsf{Coker}(\psi) \to \mathsf{Coker}(\overline{\psi}) \to 0$$

where $\mathsf{Ker}\left(\psi_{|}\right) \cong \mathsf{Ker}(\bigoplus\limits_{i=1}^{n} \mathcal{L} * \cdots * \overbrace{\mathcal{M}}^{i} * \cdots * \mathcal{L} \to \mathcal{M})$; $\mathsf{Ker}\left(\psi\right) \cong {}_nHL_1^{\alpha}(\mathcal{L})$ and $\mathsf{Ker}\left(\overline{\psi}\right) \cong {}_nHL_1^{\alpha}(\mathcal{L}/\mathcal{M})$ by Remark 5; $\mathsf{Coker}\left(\psi_{|}\right) \cong \dfrac{\mathcal{M}}{\bigoplus\limits_{i=1}^{n}[\mathcal{L}, \ldots, \overbrace{\mathcal{M}}^{i}, \ldots, \mathcal{L}]}$ and $\mathsf{Coker}\left(\psi\right) = \mathsf{Coker}\left(\overline{\psi}\right) = 0$. □

5. Unicentrality of Hom-Leibniz n-algebras

Our goal in this section is the generalization of the concept and properties of unicentral Leibniz algebras to the setting of Hom-Leibniz n-algebras. Namely (see [13]), a Leibniz algebra $\mathfrak{q}$ is said to be unicentral if $\pi(Z(\mathfrak{g})) = Z(\mathfrak{q})$ for every central extension $\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$. In particular, perfect Leibniz algebras are unicentral (see [13, Proposition 4]).

As a first step, we show in the following example, that perfect Hom-Leibniz n-algebras are not generally unicentral.

Example 5.1. Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be the three-dimensional Hom-Leibniz 3-algebra with basis $\{e_1, e_2, e_3\}$, bracket operation given by $[e_1, e_1, e_1] = e_1; [e_1, e_1, e_2] = e_2$; $[e_1, e_2, e_1] = e_3$ and zero elsewhere, and $\widetilde{\alpha}_{\mathcal{L}} = (0)$. Obviously, $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is a perfect Hom-Leibniz 3-algebra.

Consider the four-dimensional Hom-Leibniz 3-algebra $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ with basis $\{a_1, a_2, a_3, a_4\}$, bracket operation given by $[a_3, a_3, a_3] = a_3; [a_3, a_3, a_1] = a_1; [a_3, a_1, a_3] = a_2; [a_3, a_3, a_2] = a_4$ and zero elsewhere, and $\widetilde{\alpha}_{\mathcal{K}} = (0)$.

The surjective homomorphism $f : (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ given by $f(a_1) = e_2; f(a_2) = e_3; f(a_3) = e_1; f(a_4) = 0$, is a central extension since $\mathsf{Ker}(f) = \langle \{a_4\} \rangle$ and $Z(\mathcal{K}) = \langle \{a_4\} \rangle$. Moreover, $f(Z(\mathcal{K})) = 0$, but $Z(\mathcal{L}) = \langle \{e_3\} \rangle$, hence $f(Z(\mathcal{K})) \subsetneqq Z(\mathcal{L})$.

By this fact, in what follows we show some results concerning the generalization of properties of unicentral Leibniz algebras.

Definition 5.2. A perfect Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is said to be centrally closed if its universal central extension is

$$0 \to 0 \to (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \stackrel{\sim}{\to} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$$

i.e. ${}_nHL_1^{\alpha}(\mathcal{L}) = 0$ and $(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}) \cong (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$

Lemma 5.3. *Let $f : (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a central extension of a perfect Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$. Then the following statements hold:*

a) $\mathcal{K} = [\mathcal{K}, \dots, \mathcal{K}] + \mathsf{Ker}(f)$.

b) If $\alpha_{\mathcal{L}}(l) \in Z(\alpha_{\mathcal{L}}(\mathcal{L}))$, then $[l_1, \dots, l_{i-1}, l, l_{i+1}, \dots, l_n] \in \mathsf{Ker}(\alpha_{\mathcal{L}})$, for all $l_j \in \mathcal{L}, i \in \{1, 2, \dots, n\}, j \in \{1, \dots, \widehat{i}, \dots, n\}$.

Proof. a) For any $k \in \mathcal{K}$, $f(k) \in \mathcal{L} = [\mathcal{L}, \dots, \mathcal{L}]$, then $f(k) = [f(k_1), \dots, f(k_n)]$, hence $k - [k_1, \dots, k_n] \in \mathsf{Ker}(f)$.

b) If $\alpha_{\mathcal{L}}(l) \in Z(\alpha_{\mathcal{L}}(\mathcal{L}))$, then $[\alpha_{\mathcal{L}}(l_1), \dots, \alpha_{\mathcal{L}}(l), \dots, \alpha_{\mathcal{L}}(l_n)] = 0$. □

Proposition 5.4. *Let $f : (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a central extension of a perfect Hom-Leibniz n-algebra $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ with $\alpha_{\mathcal{L}}$ injective, such that $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ satisfies the following condition*

$$[\alpha(k), \alpha(k), \alpha(k_3), \dots, \alpha(k_n)] = 0, \textit{for all } k, k_3, \dots, k_n \in \mathcal{K} \tag{5}$$

Then

$$f(Z(\alpha_{\mathcal{K}}(\mathcal{K}))) = Z(\alpha_{\mathcal{L}}(\mathcal{L}))$$

Proof. Let $\alpha_{\mathcal{K}}(k) \in Z(\alpha_{\mathcal{K}}(\mathcal{K}))$, then $f(\alpha_{\mathcal{K}}(k)) \in Z(\alpha_{\mathcal{L}}(\mathcal{L}))$ since

$$\begin{aligned}
[\alpha_{\mathcal{L}}(l_1), \dots, f(\alpha_{\mathcal{K}}(k)), \dots, \alpha_{\mathcal{L}}(l_n)] &= [\alpha_{\mathcal{L}}(f(k_1)), \dots, f(\alpha_{\mathcal{K}}(k)), \dots, \alpha_{\mathcal{L}}(f(k_n))] \\
&= f[\alpha_{\mathcal{K}}(k_1), \dots, \alpha_{\mathcal{K}}(k), \dots, \alpha_{\mathcal{K}}(k_n)] = 0
\end{aligned}$$

Conversely, for any $\alpha_{\mathcal{L}}(l) \in Z(\alpha_{\mathcal{L}}(\mathcal{L}))$, there exists any $k \in \mathcal{K}$ such that $f(k) = l$, hence $\alpha_{\mathcal{L}}(l) = \alpha_{\mathcal{L}}(f(k)) = f(\alpha_{\mathcal{K}}(k))$. We must show that $\alpha_{\mathcal{K}}(k) \in Z(\alpha_{\mathcal{K}}(\mathcal{K}))$. Indeed,

$$\begin{aligned}
[\alpha_{\mathcal{K}}(k), \alpha_{\mathcal{K}}(k_2), \dots, \alpha_{\mathcal{K}}(k_n)] &= -[\alpha_{\mathcal{K}}(k_2), \alpha_{\mathcal{K}}(k), \alpha_{\mathcal{K}}(k_3), \dots, \alpha_{\mathcal{K}}(k_n)] \\
&= -[\alpha_{\mathcal{K}}[k_{21}, \dots, k_{2n}] + \mathsf{Ker}(f), \alpha_{\mathcal{K}}(k), \alpha_{\mathcal{K}}(k_3), \dots, \\
&\quad \alpha_{\mathcal{K}}(k_n)]
\end{aligned}$$

by condition (5) and Lemma 5.3 *a)*. Applying the fundamental identity (1) and having in mind that $\mathsf{Ker}(f) \subseteq Z(\mathcal{K})$, the above equality reduces to

$$\begin{array}{l} -[[\alpha_{\mathcal{K}}(k_{21}), k, k_3, \dots, k_n], \alpha_{\mathcal{K}}^2(k_{22}), \dots, \alpha_{\mathcal{K}}^2(k_{2n})] \\ -[\alpha_{\mathcal{K}}^2(k_{21}), [\alpha_{\mathcal{K}}(k_{22}), k, k_3, \dots, k_n], \alpha_{\mathcal{K}}^2(k_{23}), \dots, \alpha_{\mathcal{K}}^2(k_{2n})] - \dots \\ -[\alpha_{\mathcal{K}}^2(k_{21}), \dots, \alpha_{\mathcal{K}}^2(k_{2(n-1)}), [\alpha_{\mathcal{K}}(k_{2n}), k, k_3, \dots, k_n]] \end{array}$$

which vanishes since the brackets of the form $[\alpha_{\mathcal{K}}(k_{2i}), k, k_3, \dots, k_n]$ are in $\mathsf{Ker}(f) \subseteq Z(\mathcal{K})$ because $f[\alpha_{\mathcal{K}}(k_{2i}), k, k_3, \dots, k_n] = [f(\alpha_{\mathcal{K}}(k_{2i})), l, f(k_3), \dots, f(k_n)] \in \mathsf{Ker}(\alpha_{\mathcal{L}})$ by Lemma 5.3 *b)*, and $\alpha_{\mathcal{L}}$ is injective.

The vanishing of the other possible brackets is completely analogous to the last arguments, so we omit it. □

Remark 6. Hom-Lie n-algebras are examples of Hom-Leibniz n-algebras satisfying condition (5). Also Hom-Lie triple systems satisfy condition (5) in case $n = 3$ (see Example 2.4 *d)*).

Example 5.5. In the following we present a concrete example of central extension satisfying the conditions established in Proposition 5.4.

Consider the four-dimensional $\mathbb{C}$-vector space $\mathcal{L}$ with basis $\{e_1, e_2, e_3, e_4\}$ endowed with the ternary bracket operation given by $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = e_2; [e_1, e_2, e_4] = e_3; [e_1, e_2, e_3] = e_4$, together with the corresponding skew-symmetric ones and zero elsewhere. By Lemma 2.2 in [6], $(\mathcal{L}, [-,-,-])$ is a Lie 3-algebra.

Now consider the homomorphism of Lie 3-algebras $\alpha : \mathcal{L} \to \mathcal{L}$ given by $\alpha(e_1) = e_1; \alpha(e_2) = -e_2; \alpha(e_3) = e_3; \alpha(e_4) = -e_4$. Then Theorem 3.4 in [4] endows $\mathcal{L}$ with a structure of Hom-Leibniz 3-algebra with bracket operation given by $\{e_2, e_3, e_4\} = e_1; \{e_1, e_3, e_4\} = -e_2; \{e_1, e_2, e_4\} = e_3; \{e_1, e_2, e_3\} = -e_4$, together with the corresponding skew-symmetric ones and zero elsewhere, and $\widetilde{\alpha}_{\mathcal{L}} = (\alpha, \alpha)$. This Hom-Leibniz 3-algebra is perfect and $\alpha_{\mathcal{L}}$ is injective.

Consider the four-dimensional $\mathbb{C}$-vector space $\mathcal{K}$ with basis $\{a_1, a_2, a_3, a_4\}$ endowed with the ternary bracket operation given by $[a_2, a_3, a_4] = a_1; [a_1, a_3, a_4] = a_2; [a_1, a_2, a_4] = a_3; [a_1, a_2, a_3] = a_4$, together with the corresponding skew-symmetric ones and zero elsewhere. By Lemma 2.2 in [6], $(\mathcal{K}, [-,-,-])$ is a Lie 3-algebra.

Now consider the homomorphism of Lie 3-algebras $\beta : \mathcal{K} \to \mathcal{K}$ given by $\beta(a_1) = -a_1; \beta(a_2) = a_2; \beta(a_3) = -a_3; \beta(a_4) = a_4$. Then Theorem 3.4 in [4] endows $\mathcal{K}$ with a structure of Hom-Leibniz 3-algebra with bracket operation given by $\{a_2, a_3, a_4\} = -a_1; \{a_1, a_3, a_4\} = a_2; \{a_1, a_2, a_4\} = -a_3; \{a_1, a_2, a_3\} = a_4$, together with the corresponding skew-symmetric ones and zero elsewhere, and $\widetilde{\alpha}_{\mathcal{K}} = (\beta, \beta)$. This Hom-Leibniz 3-algebra obviously satisfies condition (5).

The surjective homomorphism $f : (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ defined by $f(a_1) = e_2; f(a_2) = e_1; f(a_3) = e_4; f(a_4) = -e_3$, is a central extension since $\mathsf{Ker}(f)$ and $Z(\mathcal{K})$ are both trivial.

Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a perfect Hom-Leibniz n-algebra with $\alpha_{\mathcal{L}}$ injective. Assume that $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ satisfies condition (5). Then $\big(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\big)$ satisfies condition (5) provided that $\{\alpha_{\mathcal{L}}(l), \alpha_{\mathcal{L}}(l), \alpha_{\mathcal{L}}(l_3), \dots, \alpha_{\mathcal{L}}(l_n)\} \in {}_nHL_1^{\alpha}(\mathcal{L})$, for all $l, l_3, \dots, l_n \in \mathcal{L}$, is the zero coset. This fact occurs, for instance, when $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is centrally closed.

From now on, we assume that $\left(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)$ satisfies condition (5) when $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ does. Then Proposition 5.4 gives the following equality:

$$u_{\mathcal{L}}(Z(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L})))) = Z(\alpha_{\mathcal{L}}(\mathcal{L})) \tag{6}$$

Theorem 5.6. *Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a perfect Hom-Leibniz n-algebra with $\alpha_{\mathcal{L}}$ injective such that $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ and $\left(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right)$ satisfy condition (5). Then there is an isomorphism*

$$\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))} \cong \frac{\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L}))}{Z(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L})))}$$

Proof. The universal central extension of $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ induces the central extension

$$\begin{aligned} 0 \to \ & \left(\alpha_{\mathfrak{uce}(\mathcal{L})}\left({}_nHL_1^{\alpha}(\mathcal{L})\right), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\right) \to \\ & \to \left(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L})), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\right) \to \left(\alpha_{\mathcal{L}}(\mathcal{L}), \widetilde{\alpha}_{\mathcal{L}|}\right) \to 0 \end{aligned}$$

when $\alpha_{\mathcal{L}}$ is injective. Moreover condition (5) is preserved by the terms in this central extension.

Bearing in mind (6), the kernels of the horizontal arrows in the commutative diagram

$$\begin{tikzcd} Z(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L}))) \arrow[r, two heads, "u_{\mathcal{L}|}"] \arrow[d, tail] & Z(\alpha_{\mathcal{L}}(\mathcal{L})) \arrow[d, tail] \\ \alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L})) \arrow[r, two heads, "u_{\mathcal{L}|}"] & \alpha_{\mathcal{L}}(\mathcal{L}) \end{tikzcd}$$

coincide, then the cokernels of the vertical homomorphisms are isomorphic. □

Proposition 5.7. *Let $0 \to (\mathcal{M}, \widetilde{\alpha}_{\mathcal{M}}) \to (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ and $0 \to (\mathcal{N}, \widetilde{\alpha}_{\mathcal{N}}) \to (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \xrightarrow{\tau} (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \to 0$ be central extensions of Hom-Leibniz n-algebras. Then the following statements hold:*

a) If $\pi \circ \tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is a universal α-central extension, then $\tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a universal central extension.

b) If $\tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a universal central extension, then $\pi \circ \tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is an α-central extension which is universal over central extensions, that is, for any central extension $0 \to (\mathcal{A}, \widetilde{\alpha}_{\mathcal{A}}) \to (\mathcal{P}, \widetilde{\alpha}_{\mathcal{P}}) \xrightarrow{\omega} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$ there exists a unique homomorphism $\Phi : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \to (\mathcal{P}, \widetilde{\alpha}_{\mathcal{P}})$ such that $\omega \circ \Phi = \pi \circ \tau$.

Proof. a) If $\pi \circ \tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is a universal α-central extension, then ${}_nHL_0^{\alpha}(\mathcal{H}) = {}_nHL_1^{\alpha}(\mathcal{H}) = 0$ by Theorem 3.9 *d)*. Hence $\tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a universal central extension by Theorem 3.9 *e)*.

b) If $\tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a universal central extension, then $(\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}})$ and $(\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ are perfect Hom-Leibniz n-algebras by Lemma 3.5, hence $\pi \circ \tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ is an α-central extension by Lemma 3.8. Moreover $\pi \circ \tau$ is universal over central extensions. Indeed, for any central extension $0 \to (\mathcal{A}, \widetilde{\alpha}_{\mathcal{A}}) \to$

$(\mathcal{P}, \widetilde{\alpha}_{\mathcal{P}}) \stackrel{\omega}{\to} (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \to 0$, construct the pull-back extension

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{A}, \widetilde{\alpha}_{\mathcal{A}}) \arrow[r] \arrow[d, equal] & (\mathcal{P} \times_{\mathcal{L}} \mathcal{K}, \widetilde{\alpha}_{\mathcal{P}} \times \widetilde{\alpha}_{\mathcal{K}}) \arrow[r, "\overline{\omega}"] \arrow[d, "\overline{\pi}"] & (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}}) \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & (\mathcal{A}, \widetilde{\alpha}_{\mathcal{A}}) \arrow[r] & (\mathcal{P}, \widetilde{\alpha}_{\mathcal{P}}) \arrow[r, "\omega"] & (\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}) \arrow[r] & 0
\end{tikzcd}$$

which is central. Since $\tau : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \twoheadrightarrow (\mathcal{K}, \widetilde{\alpha}_{\mathcal{K}})$ is a universal central extension, then there exists a unique homomorphism $\varphi : (\mathcal{H}, \widetilde{\alpha}_{\mathcal{H}}) \to (\mathcal{P} \times_{\mathcal{L}} \mathcal{K}, \widetilde{\alpha}_{\mathcal{P}} \times \widetilde{\alpha}_{\mathcal{K}})$ such that $\overline{\omega} \circ \varphi = \tau$. Then $\Phi = \overline{\pi} \circ \varphi$ satisfies the required universal property thanks to Lemma 3.6. □

Corollary 5.8. *Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}), (\mathcal{L}', \widetilde{\alpha}_{\mathcal{L}'})$ be perfect Hom-Leibniz n-algebras with both $\alpha_{\mathcal{L}}, \alpha_{\mathcal{L}'}$ injective and such that $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}}), (\mathcal{L}', \widetilde{\alpha}_{\mathcal{L}'}), \big(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\big)$, and $\big(\mathfrak{uce}(\mathcal{L}'), \ \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')}\big)$ satisfy condition (5). Then*

a) If $\big(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\big) \cong \big(\mathfrak{uce}(\mathcal{L}'), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')}\big)$, then $\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))} \cong \frac{\alpha_{\mathcal{L}'}(\mathcal{L}')}{Z(\alpha_{\mathcal{L}'}(\mathcal{L}'))}$.

b) If $\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))} \cong \frac{\alpha_{\mathcal{L}'}(\mathcal{L}')}{Z(\alpha_{\mathcal{L}'}(\mathcal{L}'))}$, then $\big(\mathfrak{uce}(\alpha_{\mathcal{L}}(\mathcal{L})), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\big) \cong \big(\mathfrak{uce}(\alpha_{\mathcal{L}'}(\mathcal{L}')), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')|}\big)$.

Proof. *a)* If $\big(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\big) \cong \big(\mathfrak{uce}(\mathcal{L}'), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')}\big)$, then $\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))} \cong \frac{\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L}))}{Z(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L})))} \cong \frac{\alpha_{\mathfrak{uce}(\mathcal{L}')}(\mathfrak{uce}(\mathcal{L}'))}{Z(\alpha_{\mathfrak{uce}(\mathcal{L}')}(\mathfrak{uce}(\mathcal{L}')))} \cong \frac{\alpha_{\mathcal{L}'}(\mathcal{L}')}{Z(\alpha_{\mathcal{L}'}(\mathcal{L}'))}$ by Theorem 5.6.

b) If $\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))} \cong \frac{\alpha_{\mathcal{L}'}(\mathcal{L}')}{Z(\alpha_{\mathcal{L}'}(\mathcal{L}'))}$, then $\left(\mathfrak{uce}\left(\frac{\alpha_{\mathcal{L}}(\mathcal{L})}{Z(\alpha_{\mathcal{L}}(\mathcal{L}))}\right), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\right) \cong \left(\mathfrak{uce}\left(\frac{\alpha_{\mathcal{L}'}(\mathcal{L}')}{Z(\alpha_{\mathcal{L}'}(\mathcal{L}'))}\right), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L}')}\right)$.

Now, applying Proposition 5.7 *b)* to the central extensions $u_{\mathcal{L}|} : \big(\mathfrak{uce}(\alpha_{\mathcal{L}}(\mathcal{L})), \widetilde{\alpha}_{\mathfrak{uce}\mathcal{L})|}\big) \to (\alpha_{\mathcal{L}}(\mathcal{L}), \widetilde{\alpha}_{\mathcal{L}|})$ and $p : (\alpha_{\mathcal{L}}(\mathcal{L}), \widetilde{\alpha}_{\mathcal{L}|}) \to (\alpha_{\mathcal{L}}(\mathcal{L})/Z(\alpha_{\mathcal{L}}(\mathcal{L})), \widetilde{\widetilde{\alpha}}_{\mathcal{L}})$, we conclude that $\big(\mathfrak{uce}(\alpha_{\mathcal{L}}(\mathcal{L})), \widetilde{\alpha}_{\mathfrak{uce}\mathcal{L})|}\big) \cong \Big(\mathfrak{uce}(\alpha_{\mathcal{L}}(\mathcal{L})/Z(\alpha_{\mathcal{L}}(\mathcal{L}))), \widetilde{\widetilde{\alpha}}_{\mathfrak{uce}(\mathcal{L})|}\Big)$. □

Corollary 5.9. *Let $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ be a centerless perfect Hom-Leibniz n-algebra with $\alpha_{\mathcal{L}}$ injective such that $(\mathcal{L}, \widetilde{\alpha}_{\mathcal{L}})$ and $\big(\mathfrak{uce}(\mathcal{L}), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})}\big)$ satisfy condition (5). Then $Z(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L}))) \cong {}_nHL_1^{\alpha}(\alpha_{\mathcal{L}}(\mathcal{L}))$ and the universal central extension of $(\alpha_{\mathcal{L}}(\mathcal{L}), \ \widetilde{\alpha}_{\mathcal{L}|})$ is*

$$0 \to \big(Z(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L}))), \widetilde{\alpha}_{\mathfrak{uce}\mathcal{L})|}\big) \to \big(\mathfrak{uce}(\alpha_{\mathcal{L}}(\mathcal{L})), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\big) \to (\alpha_{\mathcal{L}}(\mathcal{L}), \widetilde{\alpha}_{\mathcal{L}|}) \to 0$$

Proof. $Z(\mathcal{L}) = 0$ and $\alpha_{\mathcal{L}}$ injective implies that $Z(\alpha_{\mathcal{L}}(\mathcal{L})) = 0$. Then Theorem 5.6 implies that $0 \to \big(Z(\alpha_{\mathfrak{uce}(\mathcal{L})}(\mathfrak{uce}(\mathcal{L}))), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\big) \to \big(\alpha_{\mathcal{L}}(\mathfrak{uce}(\mathcal{L})), \widetilde{\alpha}_{\mathfrak{uce}(\mathcal{L})|}\big) \to (\alpha_{\mathcal{L}}(\mathcal{L}), \widetilde{\alpha}_{\mathcal{L}|}) \to 0$ is isomorphic to the universal central extension of $(\alpha_{\mathcal{L}}(\mathcal{L}), \widetilde{\alpha}_{\mathcal{L}|})$. □

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References

[1] Ammar F, Mabrouk S, Makhlouf A. Representations and cohomology of n-ary multiplicative Hom-Nambu-Lie algebras. J. of Geometry and Physics 2011; 61: 1898–1913.

[2] Ammar F, Mabrouk S, Makhlouf A. Cohomology of Hom-Leibniz and n-ary Hom-Nambu-Lie superalgebras. 2014; arXiv 1406.3776.

[3] Arnlind J, Makhlouf A, Silvestrov S. Construction of n-Lie algebras and n-ary Hom-Nambu-Lie. J. Math. Phys. 2011; 52 (12): 123502 (13 pp.).

[4] Ataguema H, Makhlouf A, Silvestrov S. Generalization of n-ary Nambu algebras and beyond. J. Math. Phys. 2009; 50 (8): 083501 (15 pp.).

[5] Attan S, Issa AN. Hom-Bol algebras. Quasigroups Related Systems 2013; 21 (2): 131–146.

[6] Bai R, Zhang L, Wu Y, Li Z. On 3-Lie algebras with abelian ideals and subalgebras. Linear Algebra Appl. 2013; 438 (5): 2072–2082.

[7] Bagger J, Lambert N. Gauge Symmetry and Supersymmetry of Multiple M2-Branes. Phys. Rev. D 2008; 77 (6): 065008 (6 pp.).

[8] Basu A, Harvey JA. The M2-M5 brane system and a generalized Nahm's equation. Nucl. Phys. B 2005; 713 (1-3): 136–150.

[9] Baxter RJ. Exactly solved models in statistical mechanics. London: Academic Press, Inc.; 1982.

[10] Casas JM. Homology with trivial coefficientes of Leibniz n-algebras. Comm. Algebra 2003; 31 (3): 1377–1386.

[11] Casas JM. Homology with coefficientes of Leibniz n-algebras. C. R. Acad. Sci. Paris, Ser. I 2009; 347: 595–598.

[12] Casas JM. A non-abelian tensor product and universal central extensions of Leibniz n-algebra. Bull. Belg. Math. Soc. Simon Stevin 2004; 11 (2): 259–270.

[13] Casas JM, Corral N. On universal central extensions of Leibniz algebras. Comm. Algebra 2009; 37 (6): 2104–2120.

[14] Casas JM, Insua MA, Pacheco Rego N. On universal central extensions of Hom-Leibniz algebras. J. Algebra Appl. 2014; 13 (8): 1450053 (22 pp.).

[15] Casas JM, Khmaladze E, Pacheco Rego N. A non-abelian Hom-Leibniz tensor product and applications. Linear Multilinear Algebra 2017; (to appear): DOI 10.1080/03081087.2017.1338651.

[16] Casas JM, Loday JL, Pirashvili T. Leibniz n-algebras. Forum Math. 2002; 14: 189–207.

[17] Casas JM., Van der Linden T. Universal central extensions in semi-abelian categories. Appl. Categor. Struct. 2014; 22 (1): 253–268.

[18] Cheng YS, Su YC. (Co)homology and universal central extension of Hom-Leibniz algebras. Acta Math. Sin. (Engl. Ser.) 2011; 27 (5): 813–830.

[19] Gannon T. Moonshine beyond the Monster. The bridge connecting algebra, modular forms and physics. Cambridge: Cambridge Monographs on Mathematical Physics, Cambridge University Press; 2006.

[20] Gaparayi D, Issa AN. Hom-Lie-Yamaguti structures on Hom-Leibniz algebras. Extracta Math. 2013; 28 (1): 1–12.

[21] Hartwig JT, Larson D, Silvestrov SD. Deformations of Lie algebras using σ-derivations. J. Algebra 2006; 295 (2): 314–361.

[22] Issa AN. Some characterizations of Hom-Leibniz algebras. Int. Electron. J. Algebra 2013; 14: 1–9.

[23] Kitouni A, Makhlouf A, Silvestrov S. On (n+1)-Hom-Lie algebras induced by n-Hom-Lie algebras. Georgian Math. J. 2016; 23 (1): 75–95.

[24] Makhlouf A, Silvestrov S. Hom-algebra structures. J. Gen. Lie Theory Appl. 2008; 2 (2): 51–64.

[25] Makhlouf A, Silvestrov S. Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras. Forum Math. 2010; 22 (4): 715–739.

[26] Nambu Y. Generalized Hamiltonian dynamics. Phys. Rev. D 1973; (3) 7: 2405–2412.

[27] Okubo S. Introduction to octonion and other non-associative algebras in physics. Cambridge: Montroll Memorial Lecture Series in Mathematical Physics 2, Cambridge University Press; 1995.

[28] Palmer S, Smann Ch. M-brane models from non-abelian gerbes. J. High Energy Phys. 2012; 7: 010 (16 pp.).

[29] Sverchkov SR. The structure and representation of n-ary algebras of DNA recombination. Cent. Eur. J. Math. 2011; 9 (6): 1193–1216.

[30] Yau D. Hom-algebras and homology. J. Lie Theory 2009; 19 (2): 409–421.

[31] Yau D. Enveloping algebras of Hom-Lie algebras. J. Gen. Lie Theory Appl. 2008; 2 (2): 95–108.

[32] Yau D. On n-ary Hom-Nambu and Hom-Nambu-Lie algebras. J. Geom. Phys. 2012; 6 (2): 506–522.

[33] Zhao J, Chen L. n-ary Hom-Nambu algebras. 2015; ArXiv: 1505.08168v1.