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On metric-invariant relativistic transformations under anisotropy in the context of special relativity

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In a context of special relativity under anisotropy, Burde recently presented in this journal a new group of coordinate transformations obtained using Lie group techniques from the principles of relativity and correspondence, which show conformal metric invariance. As a consequence, this author argued that conformal invariance is a necessary condition of any anisotropic transformation satisfying those principles, and rejected the adequacy of any transformation obtained from the Lorentz transformation by a coordinate change, e.g. resynchronization such as the ε-Lorentz transformation. This paper argues against these assertions, and presents, as a counterexample, a relativistic generalization of the Tangherlini transformation obtained by the same Lie group method, that satisfies the previous requirements and enjoys the metric invariance property.

Keywords: Anisotropy; conformal metric invariance; conventionality of simultaneity; Lie group of transformations; special relativity; Tangherlini transformation.

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1. Introduction

The discovery of the cosmic microwave background radiation (CMBR) has stimulated[1,2] the proposal of test theories of relativity hoping to shed light on the validity of the relativity principle by empirically contrasting the special theory of relativity (STR) with anisotropic test theories considering the CMBR frame as a privileged frame. Since the pioneering work of Mansouri and Sexl,[3] several proposals have been made (although see e.g. Refs. 4 and 5, where for different reasons this approach is not seen as acceptable), one of the most recent by Burde[6,7] (Ref. 8 is a first step and serves as a background). This author, in the context of an anisotropic world,[a] using Lie group techniques, obtains a Lie group of coordinate transformations between

[a]Unless otherwise stated, throughout this paper, by an anisotropic world we mean a world in which light propagates anisotropically with respect to any inertial frame — except an isotropic preferred

inertial frames that show conformal metric invariance, that is, they leave a scaled metric invariant rather than the metric.

Apart from the interest of the derivation method and the new transformation itself, Burde makes some assertions about his results that could be considered at least controversial, and that could be summarized in the following main claims (e.g. Ref. 6, pp. 1595–1596):

(1) In an anisotropic world, the invariance of the interval (metric) cannot be used as a first principle in the derivation of a coordinate transformation[b] between inertial frames. The invariance of the light propagation equation should be used instead.

    Since using this latter principle leads to conformal metric-invariant transformations — as is the case of the new transformation obtained by Burde — a metric-invariant transformation would not be suitable for anisotropic kinematics.

(2) Any transformation obtained from the Lorentz transformation (LT) by a coordinate change is in fact the LT using a nonstandard coordinatization. Since such a transformation would leave the metric invariant, using the above claim, such a transformation would be incompatible with a truly anisotropic propagation of light and could not serve as a basis for anisotropic kinematics. In particular, this assertion rules out any ε-Lorentz transformation (ε-LT),[9,10] to represent an anisotropic world.[c]

The above claims seem to contradict the conventionality of simultaneity (CS)[11,12] which, as is well known, is a consequence of our ignorance of the one-way speed of light. The same ignorance that had to be circumvented by Einstein stipulating by *definition*[13] the one-way speed of light underlying the synchronization of clocks.[d]

Thus, according to the above statements, any transformation obtained — such as an ε-LT — from the LT by resynchronization (i.e. a change of coordinates) in order to reflect an assumed true anisotropy cannot describe this anisotropic propagation of light, but only the isotropic world of the LT in which simultaneity is

frame (PF) — and whose anisotropy depends on its velocity with respect to this privileged frame. Invariant anisotropies, such as those discussed in Ref. 8 leading to relativity of simultaneity, are therefore excluded.

[b]In what follows, we refer only to inertial frames and linear transformations between them.

[c]As is well known, the ε-LTs are obtained from the LT by a change of coordinates that implements a resynchronization of clocks. The particular case $\varepsilon = 1/2$ corresponds to the standard or Einstein synchronization in which case the ε-LT is the LT (see below).

[d]Note that, as usual, by CS we do not mean the freedom one has to establish arbitrary simultaneity using arbitrary synchronization of clocks, since this is always possible, but the conventionality that will always pervade any claim to establish "true" simultaneity — i.e. true physical simultaneity, whether absolute or relative — insofar as, in the context of the theory of relativity, such true simultaneity is not known due to our ignorance of the one-way speed of light.

relative, hence it is unconventional. In other words, since nonstandard synchronizations under STR lead to ε-LT ($\varepsilon \neq 1/2$), ruling out ε-LT to represent an anisotropic world according to the above claims is equivalent to ruling out the possibility that such synchronizations represent the supposed true anisotropy, and this is equivalent to establishing the standard procedure as the only true one, which is not only against CS but against the anisotropic hypothesis itself.

This view that all transformations obtained from the LT by a change of coordinates are intrinsic to, and exclusive to, the isotropic world of the LT is reminiscent of that of important authors (e.g. Refs. 14–16) who reduced the importance of CS to the almost irrelevant role of a mere change of coordinates that does not change the geometry. I thought this debate[17] was on the way to resolution after contributions such as Refs. 18 and 19 clarified Malament's objection under the property of general covariance and a gauge equivalence interpretation. However, some of the motivations put forward by Burde represent a new iteration of the debate on CS under the new guise of conformal metric invariance as an essential feature of transformations under anisotropy and, therefore, a discriminating element between an isotropic and an anisotropic world.

This paper tries to clarify this conflictive panorama through two main objectives:

(1) To analyze the arguments used by Burde to support the above assertions and explain what is wrong with them.
(2) To introduce consequential modifications to the assumptions made by Burde in order to obtain a new transformation that also satisfies the principles of relativity and correspondence, but maintains the invariance of the metric. The resulting transformation thus serves as a counterexample to Burde's assertions.

To this end, in Sec. 2, we summarize the motivations, principles and results of Burde's work. In Sec. 3, we analyze these motivations and principles and their effects on the resulting Burde's transformation (BT). In Sec. 4, we apply the critique of the previous section to obtain a transformation that serves as a counterexample to Burde's claims. In Sec. 5, we carry out a final discussion about the BT and of the work done here, trying to clarify its meaning from the point of view of CS. Finally, in Sec. 6, we summarize our conclusions. Appendix A presents succinctly the BT, and Appendix B shows some transformations presented as a counterexample during the analysis of Sec. 3.

2. Burde's Transformation

In this section, we summarize the motivations, objectives and methodology that lead to the new BT. This transformation was first presented in Ref. 6. In this paper, Burde motivates and derives BT, analyzes its meaning and its relationship with other known anisotropic transformations obtained from nonstandard synchrony, and applies the new transformation to the CMBR dipole anisotropy. A previous article[8] is also relevant to understand the motivation and evolution of BT: in that article, the same Lie group methodology is presented and applied to obtain a transformation leaving the anisotropy direction invariant in all inertial frames, a case not considered here, which reduces to LT in the particular isotropic case. A more recent article applies BT to general relativity (GR) (Ref. 7, Sec. 2.3, pp. 6–7).

Of these three articles, Ref. 6 will be taken as our main reference although we will use the others when convenient — the oldest to better understand the background, and the more recent as its introduction and presentation of the transformation are more succinct.

2.1. *The physical context*

The physical context of BT is a world described by the following assumptions (Ref. 6, Sec. 3, pp. 1597–1598):

(1) The speed of light measured over closed paths is c relative to any inertial frame (*round-trip light principle*).
(2) There is a PF relative to which, the one-way speed of light is isotropic.
(3) With respect to any other inertial frame, the one-way speed of light is anisotropic and this anisotropy depends on the frame velocity with respect to the PF.

For ease of reference, we will call this world an *anisotropic world* — sometimes without making explicit that anisotropy depends on velocity. The transformation obtained by Burde using the Lie group technique ultimately depends on a free anisotropy parameter function $k(a)$ with a the group parameter, which has to be specified using the last two assumptions.

Under the round-trip light principle, the one-way speed of light relative to any inertial frame S is (Ref. 6, pp. 1597–1598)

$$c_{\boldsymbol{n}} = \frac{c}{1 + \boldsymbol{k} \cdot \boldsymbol{n}} = \frac{c}{1 + k \cos \theta_k}, \tag{1}$$

(compare with (Ref. 10, p. 127) although there, the synchronization parameter $\boldsymbol{\kappa}$ has a negative sign,[e] i.e. $\boldsymbol{k} = -\boldsymbol{\kappa}$) where $\boldsymbol{k}$ is a vector defining the anisotropy in the given frame, $k = |\boldsymbol{k}|$, and θ_k is the angle between the direction of propagation $\boldsymbol{n}$ and $\boldsymbol{k}$. Therefore, assuming that the x-axis is chosen to be along the anisotropy vector $\boldsymbol{k}$, the corresponding light propagation equation is[f]

$$ds^2 = c^2 dt^2 - 2kc dt dx - (1 - k^2) dx^2 - dy^2 - dz^2 = 0. \tag{2}$$

[e]To make it easier to track the source, we will follow Burde's notation and choose a compromise notation when his different papers diverge.
[f]Note that this equation cannot be formulated without having previously defined the frame's time. However, it is relevant that we are not informed until the end of the article (Ref. 6, Discussion section, p. 1613) that the synchronization procedure used is that based on light signals with one-way velocities according to relation (1), as if this synchronization procedure were derived from the form of the light propagation equation and not the other way around.

2.2. *Motivation and first principles*

In this context, the main motivations for the new transformations can be summarized as follows (Ref. 6, pp. 1595–1596):

(1) Coordinate transformations obtained from LT by a coordinate change are not suitable for an anisotropic world. The ε-LT is a particular case.
(2) Coordinate transformations that leave the metric invariant are not suitable to an anisotropic world. Only under isotropy the invariance of the metric is equivalent to the invariance of the light propagation equation. Since any coordinate transformation obtained from LT by a coordinate change is metric invariant, such a transformation cannot be suitable to an anisotropic world. The ε-LT are again a particular case.
(3) Under isotropy, the ε-LT do not satisfy the principle of correspondence (PC) unless $\varepsilon = 0$, therefore, LT is privileged under isotropy. Since according to Burde the ε-LT are essentially isotropic, a new transformation under anisotropy satisfying the correspondence principle is necessary.

We present these motivations in more detail below. Given the crucial role of Burde's words in this section and the length of the referenced articles, we will quote them verbatim where necessary to minimize the risk of error or ambiguity in paraphrase, and make the paper as self-sufficient as possible.

2.2.1. *Transformations obtained from LT by coordinate change are not suitable for an anisotropic world*

This motivation is best expressed directly through Burde's words (Ref. 8, pp. 1575–1576) (emphasis added)

> "*What is essential for interpretation of the ε-Lorentz transformations is that they can be obtained from the standard Lorentz transformations by a change of coordinates* (...)*. Thus, the ε-Lorentz transformations are in fact the Lorentz transformations of the standard special relativity represented using the "nonstandard" coordinatization of the four-dimensional space-time manifold.* This might be expected in view of the fact that the kinematic arguments used in the derivations of the ε-Lorentz transformations in the aforementioned works [articles by Edwards, Winnie and Ungar] are based on the assumption that, in the case of $\varepsilon = 1/2$, the relations of the special relativity theory in its standard formulation are valid. *The above implies that the ε-Lorentz transformations cannot provide a basis for kinematics of the special relativity if an anisotropy of the one-way speed of light is due to a real space anisotropy which should influence all physical processes and, in particular, propagation of light. Effects of a real space anisotropy could not be eliminated by a change of the space-time coordinatization.* The special relativity kinematics applicable to that situation should be developed based

on the first principles, without refereeing to the relations of the standard relativity theory."

Burde's view could therefore be summarized as being that any coordinate transformation like ε-LT which can be obtained from and converted back to LT by a change of coordinates, describes an isotropic kinematics and thus is not suitable for representing an anisotropic world. Consequently, a specific one has to be derived from first principles.

2.2.2. *Metric-invariant transformations are only valid in an isotropic world*

The nonapplicability of ε-LT to an anisotropic system is reinforced by the equivalence, under isotropy (STR), between the properties of metric invariance and light propagation equation invariance, of linear transformations. Under anisotropy such an equivalence does not hold, thus Burde concludes that, since the ε-LT — like any other transformations obtained from LT by a coordinate change — leave the metric invariant, they are not suitable for an anisotropic world.

Burde bases these considerations on arguments such as those of Pauli's (Ref. 20, pp. 9–10), which we reproduce here due to their interest (Ref. 8, pp. 1589–1590). If, in the context of STR, S, S' and S'' are inertial frames in standard configuration such that S' moves with velocity β relative to S, and S'' moves with velocity $-\beta$ relative to S' (i.e. S'' is at rest with S), then

(1) For a linear transformation[g] (in particular LT) to leave the light equation invariant, i.e. $(ds')^2 = 0 = ds^2$, it must satisfy $(ds')^2 = \lambda\,(\beta)\,ds^2$, with $\lambda\,(\beta)$ associated to a possible change of lengths in the directions transverse to motion (Ref. 20, p. 2).
(2) Applying LT to motion in the opposite direction: $(ds'')^2 = ds^2 = \lambda(-\beta)(ds')^2 = \lambda(-\beta)\lambda(\beta)ds^2$, therefore, $\lambda(-\beta)\lambda(\beta) = 1$.
(3) For reasons of symmetry, $\lambda(\beta)$ must be independent of the direction of the velocity, i.e. $\{\lambda(\beta) = \lambda(-\beta)$ and $\lambda(\beta) \geq 0\} \Rightarrow \lambda(\beta) = 1$.

In this context, Burde concludes (Ref. 6, pp. 1595–1596) (emphasis added)

"*Thus, in general, not the invariance of the interval but invariance of the equation of light propagation should be a starting point for derivation of the transformations.* (...) Therefore the use of the interval invariance is usually preceded by a proof of its validity (...) based on invariance of the equation of light propagation. However, those proofs are not valid if an anisotropy is present and the same arguments lead to the conclusion that, *in the presence of anisotropy, the interval is not invariant but modified by a conformal factor. The "ε-Lorentz transformations", like the standard*

[g]Throughout this discussion, we refer to linear transformations.

> *Lorentz transformations, leave the interval invariant and therefore they are applicable only to the case of no anisotropy.*"

Also (Ref. 8, pp. 1591–1592) (emphasis added)

> "[The] *effects of a real space anisotropy cannot be made vanishing by a coordinate change and so the conformal factor should appear in all physically important relations independently of the synchrony chosen.* In particular, the conformal factor remains in the relations describing the length contraction and time dilation effects after that those relations have been converted to the standard variables, with Einstein synchrony assumed."

These considerations lead Burde to conclude that, in an anisotropic world, the invariance of the light propagation equation — rather than the invariance of the metric — must be used as the first principle to derive a relativistic linear transformation.

2.2.3. *The principle of correspondence*

Burde's formulation of the PC affects only to the x-transformation (Ref. 8, p. 1580)

> "*Correspondence principle.* In the limit of small velocities $v \ll 1$ (small values of the group parameter $a \ll 1$), the formula for transformation of the coordinate x turns into that of the Galilean transformation:
>
> $$x = X - vT \tag{2.4}$$
>
> It should be noted that the relations $t = T$, $y = Y$ and $z = Z$, which are commonly included into the system of equations called the Galilean transformations, are not required to be valid in the limit of small velocities. (...) So only the relation 2.4, which does contain the first order term, provides a reliable basis for specifying the group generators based on the correspondence principle."

This formulation gives Burde another reason to reject the ε-LT (Ref. 8, footnote 3, p. 1578) (see also Ref. 6, p. 1596)

> "It is worth noting that the "ε-Lorentz transformations" do not satisfy the correspondence principle unless the standard (Einstein) synchrony is used. In the limit of small velocities, the coordinate transformation contains additional terms including the synchronization parameter and light speed which are alien to the framework of the Galilean kinematics. So the correspondence principle applied to the ε-Lorentz transformations singles out the Einstein synchrony as a "natural" one in a sense."

In summary, the considerations of the previous two sections restrict the validity of the ε-LT to the isotropic world, but in this kind of world the LT is privileged since, according to Burde, the PC is satisfied only if Reichenbach's $\varepsilon = 1/2$. Therefore, in an anisotropic world, another transformation must be selected in order to satisfy the PC.

2.2.4. *First principles*

In accordance with all of the above, Burde aims to construct a relativistic anisotropic kinematics by developing from first principles a coordinate transformation between inertial frames. Given two arbitrary inertial frames S and S' with respective coordinate systems $\{X, Y, Z, T\}$ and $\{x, y, z, t\}$, the coordinate transformation sought has the following general form:

$$\begin{aligned} &t = q(X, T, K; a), \quad x = f(X, T, K; a), \quad y = g(Y, Z, K; a), \\ &z = h(Y, Z, K; a), \quad k = p(K; a). \end{aligned} \tag{3}$$

Note that in this general form, the anisotropy parameter k is one more state variable subject to the transformation (Ref. 6, p. 1560). This permits the new transformation to satisfy the principle of relativity in an anisotropic scenario (Ref. 6, p. 1598; Ref. 7, pp. 2–3) (emphasis added)

> "*In such a framework, the principle of relativity is preserved since the privileged frame, in which the anisotropy parameter is zero, enters the analysis on equal footing with other frames* — transformations from/to that frame are not distinguished from other members of the group of transformations‡. As a matter of fact, *the symmetry underlying the transformation group is the same spacetime symmetry as that expressed by the Lorentz invariance but extended by allowing the one-way speed of light to be anisotropic.*"

In the footnote on that page, Burde makes it clear that

> "‡ In the standard SR, validity of the relativity principle is achieved simply by assuming that the one-way speed of light is isotropic and equal to c in all inertial frames which allows the study of relative motion without any concern with the study of absolute motion. *Nevertheless, this restricting assumption is required neither by the experiment nor by the relativity principle or the group property of the transformations.*"

The above general form also assumes that frames are arranged in standard configuration (e.g. Ref. 21, p. 45): S' moves inertially with velocity v relative to S along their common x-axis, with the y and z-axes of the two frames remaining parallel. Consequently, based on symmetry arguments, the transformations of the variables x and t do not involve the variables y and z and vice versa.[h] In this context, since anisotropy is attributed to motion with respect to the PF, the direction of

[h] Later, during the resolution of the problem (Ref. 6, p. 1601), it is also assumed that there are no rotations in the (y, z) plane, i.e. $y = g(Y, K; a)$ and $z = h(Z, K; a)$.

motion is assumed to coincide with the direction of anisotropy (Ref. 7, p. 7; Ref. 6, p. 1599).

Under these assumptions, the equations of light propagation in S and S' are, respectively (Ref. 6, pp. 1598 and 1600)

$$dS^2 = c^2dT^2 - 2KcdTdX - (1-K^2)dX^2 - dY^2 - dZ^2 = 0, \tag{4}$$

$$ds^2 = c^2dt^2 - 2kcdtdx - (1-k^2)dx^2 - dy^2 - dz^2 = 0, \tag{5}$$

where K and k reflect the true anisotropy in both frames.

In this framework, the first principles imposed to the sought transformation are

(1) The invariance of the light propagation equation as it would be required by the principle of relativity, i.e. the transformation should transform[i] Eq. (4) into Eq. (5).
(2) Group structure (Ref. 6, p. 1599): the transformation must form a one-parameter group with parameter $a = a(v)$ reflecting the anisotropy dependence on motion.
(3) The PC, i.e. the condition that (Ref. 6, pp. 1560 and 1600) *in the limit of small velocities* $v \ll c$ *(...), the formula for transformation of the coordinate* x *turns into that of the Galilean transformation* (GT)

$$x = X - vT. \tag{6}$$

2.3. *Derivation and further specification*

In order to better understand the effect of the assumptions on the resulting transformation, it is important to know the derivation strategy and their order of application. We do not go into the details of the derivation here, which are perfectly described in the references (Ref. 6, pp. 1598 and 1599ff; Ref. 7, pp. 7–9). The derivation is structured in five steps that can be summarized in the following two main stages:

(1) Stage 1 (steps 1–4 of the derivation). From the principle of invariance of the light propagation equation, by means of the Lie group technique,[22,23] a general transformation is obtained that relates any pair of inertial systems with arbitrary anisotropy parameters. The PC is applied at this stage to specify the Lie infinitesimal generator of the spatial transformation from Eq. (6). The resulting transformation depends on an *indefinite* function (Ref. 6, p. 1603) $k = k(a)$ representing the anisotropy of the inertial frames.
(2) Stage 2 (step 5 of the derivation). A further specification (Ref. 6, Sec. 5) of the above general transformation is carried out by using the remaining anisotropy assumptions (Sec. 2.1): (a) the existence of an isotropic PF ($k = 0$), and (b) the

[i]Note that the anisotropy parameter is potentially different for each frame, hence this *invariance* should be interpreted as *covariance* of the light equation.

anisotropy of each inertial frame is originated by its motion, and depends on its velocity β, relative to the PF. These assumptions lead Burde to postulate the anisotropy parameter as a function $k(a) = F(\beta)$, as explained below, in order to obtain a more specific transformation.

2.3.1. *Postulated anisotropy model*

Since the general transformation obtained in the first stage depends on a free function $k(a)$, Burde reduces this degree of freedom with the assumption that anisotropy is due to motion with respect to the PF, thus postulating a function (Ref. 6, p. 1603) $k = F(\beta)$ with $\beta = v/c$ where v is the velocity of the inertial frame with respect to the PF. Later, he postulates the form of the function $k = F(\beta)$ (*ibid.* p. 1605)

> "Although the function $F(\beta)$ is not known, a further specification can be made based on the argument that an expansion of the function $F(\beta)$ in a series with respect to β should not contain a quadratic term since it is expected that a direction of the anisotropy vector changes to the opposite if a direction of a motion with respect to a preferred frame is reversed: $F(\beta) = -F(-\beta)$. Thus, with accuracy up to the third order in β, the dependence of the anisotropy parameter on the velocity with respect to a preferred frame can be approximated by:"

$$k = F(\beta) \approx q\beta, \quad \beta = f(k) \approx \frac{k}{q}.$$

Since the function $\beta = \beta(a)$ had been previously obtained in the derivation, the model of anisotropy just postulated permits the specification of the free function as $k(a) = F(\beta(a)) = q\beta(a)$. Accordingly, the final BT (Appendix A) depends on the *universal constant*[j] q such that the case $q = 0$ corresponds to STR. In this context, Burde hopes that the measurable effects of his theory can provide estimates for q, thus determining deviations from the STR in the manner of a test theory (*ibid.* p. 1593).

3. Analysis of the BT Transformation

In this section, we review BT as well as his motivations and first principles, with a focus on the following aspects that decisively determine its final form and physical meaning:

(1) The relationship between the natural synchronization assumed in the BT transformation and its anisotropy model.
(2) The specific formulation of the correspondence principle and its relation to the relative simultaneity shown by the BT.

[j]Note that q is denoted by b in p. 11 of Ref. 7 because in this paper q is the time transformation function.

(3) The relationship between anisotropy, invariance of the light propagation equation and the conformal invariance property.

3.1. *Anisotropy, natural synchronization and velocity*

Reading the quote of Sec. 2.2.1, among others, one gets the impression that Burde does not attach much importance to synchronization of clocks, since it can always be modified by a coordinate change. However, without this synchronization, we can neither define time nor give meaning to the light propagation equation (2) because the mere formulation of that equation relies on a synchronization method. It is relevant that, as we have indicated in Sec. 2.1, we had not been informed until the Discussion section (Ref. 6, p. 1613) about the method chosen. It is true however that, since k represents the true anisotropy of the inertial frame, we can deduce that all inertial frames use their respective value of this parameter and use it to synchronize their clocks. This synchrony can be implemented for example with light signals using the equation[k]:

$$t_B = t_A + \varepsilon(t'_A - t_A), \tag{7}$$

with Reichenbach's

$$\varepsilon_k = \frac{1}{2}(1 + k), \tag{8}$$

where k is the true anisotropy parameter of the inertial frame (Ref. 11, Sec. II.19, p. 127; Ref. 10, Sec. 1.5.1, p. 106; Ref. 14, Sec. IV.7, p. 167).

However, if we further assume the existence of an isotropic PF S_0, if inertial frames follow this procedure using their respective k, then they all end up adopting a common and *true* simultaneity: true, because they use their respective true one-way speed of light, and common, because they all arrive at the same simultaneity relation since all inertial frames anisotropies — PF isotropy included — form a coherent system determined by the universal propagation of light and their respective relative velocities, and are thus interdependent.

Specifically, according to Redhead,[24] (Ref. 25, p. 90ff), if an inertial frame S moves relative to a standard frame S_0 with velocity β as seen by S_0, the adoption by S of the simultaneity of S_0 is equivalent to synchronizing its clocks with[l]

$$\varepsilon = \frac{1}{2}(1 + \beta). \tag{9}$$

Combining both results (Eqs. (8) and (9)), the resulting anisotropy factor of S is $k = \beta$, i.e. its standard velocity relative to the PF S_0. Therefore, given the assumed

[k] A light signal is emitted from A at time t_A, is reflected at B at time t_B, and returns to A at time t'_A.

[l] Actually, Redhead presents the situation under the point of view of S assuming that it also has standard clocks, in which case, $m = -\beta$ is the slope of the simultaneity hyperplane of S_0, according to S. However, here we have that S_0 has standard clocks and S wants to define its own time using the simultaneity of S_0. In any case, if S had adopted the standard synchrony and decided later to adopt the — under our hypotheses, *true* — simultaneity of S_0, the final result would be the same.

anisotropic world with isotropic PF, we arrive univocally at a specific anisotropy model, the one defined by: $k = F(\beta) = \beta$. This result seems to be ignored by Burde in postulating the function $F(\beta) = q\beta$.

Furthermore, as it is well known, this anisotropic inertial kinematics is described by the Tangherlini transformations (TT)[26,27,m]

$$\begin{cases} t = \dfrac{T}{\gamma}, \\ x = \gamma(X - \beta T). \end{cases} \tag{10}$$

Nevertheless, Burde seems to conceive of TT as the mere result of a synchronization of convenience with no physical basis, performed solely for the purpose of establishing a common simultaneity in all frames (Ref. 6, p. 1612) (emphasis added)

> "*Thus, using the synchronization method,* [signals travelling with infinite or arbitrarily large velocity, or by external synchronization, as assumed by Tangherlini[26,27]] *that is different from synchronization by light signals,* yields the transformations [TT] which exhibit absolute simultaneity. They also exhibit non-invariant one-way speed of light so that, *in that approach, the anisotropy of the velocity of light in a moving inertial frame is a feature that emerges due to synchronization procedure designed to keep simultaneity unchanged between all inertial frames of reference.*"

The quote already shown in Sec. 2.2.1 confirms this view which, however, is not correct. The main difference between Burde's view and our previous account is that we have effectively used the anisotropy hypothesis — which acts here as a general hypothesis — in its entirety. In this physical scenario (Sec. 2.1) TT can be obtained from synchronizing clocks either with the true anisotropic light signals[n] using $\varepsilon_k = \frac{1}{2}(1 + k)$ with $k = \beta$, or by any other *natural* method — i.e. one that respects the true anisotropy — such as the *external method* (Ref. 3, p. 502) consisting in synchronizing clocks with PF's clocks when they pass each other.

Therefore, TT can be derived directly from the anisotropy assumption without necessarily resulting from any resynchronization of standard clocks. This means that TT is logically independent from LT. Of course, this does not preclude that if clocks had been previously synchronized with the standard method, such a natural synchrony could also be obtained by a coordinate change implementing a resynchronization of clocks with $k = \beta$. However, since our scenario is that of an anisotropic propagation of light, that resynchronization would not make the resulting TT unsuitable for an anisotropic world — nor, for that matter, would it make LT unsuitable for an isotropic one, just because LT can be obtained from TT

[m] First proposed by Tangherlini in the context of general covariance (Ref. 26, p. 46ff), who called it Absolute LT, this transformation was later discovered by other authors.[28] In fact, it is a particular case of Winnie's ε-LT[9] with $\varepsilon = 1/2$ and $\varepsilon' = \varepsilon_k$.

[n] Tangherlini eventually justified his original idea of synchronization by infinite speed signals as a mathematical exercise (Ref. 27, p. 32).

in this scenario — but on the contrary, depending on the hypotheses, it may be the only suitable solution for it (more on this in the Discussion section).

BT versus TT. What place is left for BT in this context? In principle, by the above argument, only the value $q = 1$ would have physical meaning. However, the fact is that TT is not the same transformation as BT with $q = 1$ and $K = 0$, since the former possesses the property of metric invariance instead of conformal metric invariance (compare (10) with (A.12) for $K = 0$ in Appendix A).

On the other hand, values $q \neq 1$ are not compatible with the true anisotropy parameter $k = \beta$. If $q = 0$, BT reduces to LT, i.e. to STR, and in general, synchronization of clocks at A and B using light signals with anisotropy $k = F(\beta) = q\beta$ would follow the equation:

$$t_B = t_A + \varepsilon(t'_A - t_A) = t_A + \frac{1}{2}(1 + q\beta)(t'_A - t_A), \tag{11}$$

which for $q \neq 1$ would assign an earlier ($q < 1$) or later ($q > 1$) time than the *true* time $t_B = t_A + \frac{1}{2}(1 + \beta)(t'_A - t_A)$ corresponding to the true one-way speed of light.

Therefore, we have to conclude that values $q \neq 1$ specified in the function $k = F(\beta) = q\beta$ would correspond to transformations mathematically satisfying the first principles stated in the previous section, with none of them having physical meaning compatible with an isotropic PF because their implicit synchronization would contradict the existence of such a PF imposed in stage 2 of the derivation (p. 2) — i.e. the anisotropic parameter k used in the synchronization would be different from the velocity β relative to that PF.

Another relevant question is: where does this freedom of the k-function arise during the BT derivation? Alternative values $k = \beta$ and $k = 0$ represent freedom by themselves, but the general freedom of function k arises in stage 1 of the derivation when the PF hypothesis had not yet been imposed. At that time k revealed itself as fully free. However, after adding an isotropic PF to the general anisotropy hypothesis, the propagation of light becomes fully specified and k can no longer be free. After the assumption of an isotropic PF, the freedom that k enjoys after the first stage of the derivation is completely eliminated by the univocal relation between the anisotropy of an inertial frame and its velocity relative to the PF. This is precisely the simple but deep import of Redhead's result: the true anisotropy of any inertial frame is faithfully represented by adopting the standard time of another inertial frame: the isotropic PF.

Finally, if the only value with compatible physical meaning is $q = 1$, why not simply choose TT as the sought transformation? The fact is that TT does not relate arbitrary pairs of inertial frames since PF is always one of them. Therefore, TT does not enjoy the sought group structure in order to satisfy the anisotropic principle of relativity.

In any case, one question remains in the air: is conformal metric invariance a necessary condition for a relativistic transformation under anisotropy? We will try to answer this question below, but first we will look at another puzzling property of the BT.

3.2. *Relative simultaneity and the PC*

The above considerations are related to another surprising feature of BT: the relative simultaneity it implicitly shows. After stage 1 of the derivation — before the anisotropy model $k = F(\beta)$ was postulated — the time transformation of BT is (Eq. (A.3)) (Ref. 6, Eq. (27), p. 1602)

$$ct = \frac{R}{\sqrt{(1-K\beta)^2-\beta^2}}[(1-K\beta-k\beta)cT-((1-K^2)\beta+K-k)X], \tag{12}$$

whose x coefficient $(1-K^2)\beta+K-k$ is not easily made zero since β is any velocity, and k, K are arbitrary anisotropy values of an undefined function k. If $K = 0$ (the PF), this x coefficient is zero only if $\beta = k$, which excludes any anisotropy model $F(\beta) = q\beta$ with $q \neq 1$ as incompatible with the PF assumption and the anisotropy dependence on velocity confirming our analysis of the previous section.

However, the relevant point here is that BT shows relative simultaneity just after being derived from first principles (light equation invariance and PC) as if these principles caused by themselves the relativity of simultaneity. It is as if, under these prima facie reasonable first principles, the anisotropies of the inertial frames were in tension with one another producing relative simultaneity in the same way that conflicting isotropies originate relative simultaneity in STR. However, while in STR, isotropies are clearly in conflict, in an anisotropic world with an isotropic PF and anisotropy depending on the velocity, anisotropies are interdependent.

It could be answered here that this discrepancy is not important as it is always possible to resynchronize clocks by a coordinate change to adopt a common simultaneity. However, this resynchronization would not change the fact that the true anisotropies produced relative simultaneity. Thus, one would expect all inertial frames, each using its true anisotropy k, to define the same simultaneity relation, unless these first principles determine a strange pattern of conflicting anisotropies. In this sense, only one hypothesis remains that could be responsible for this: the specific formulation adopted for the PC expressed by equation $x = X - vT$. This formulation is applied to obtain the infinitesimal generator of space $\xi(X, T, K)$ (Eq. (A.1)) (Ref. 6, pp. 1596 and 1600)

$$\xi = \left.\frac{\partial x}{\partial a}\right|_{a=0} = \left.\frac{\partial(X-v(a)T)}{\partial a}\right|_{a=0} = -bT, \tag{13}$$

with $b = v'(0)$, and using $b = 1$ without loss of generality since this constant can be eliminated by redefining the group parameter, the generator becomes

$$\xi(X, T, K) = -T. \tag{14}$$

Therefore, this PC formulation only affects the transformation of space and lets the infinitesimal generator of time $\tau(X, T, K)$ depend on space X (Eq. (A.2)).

This produces an infinitesimal exchange between time and space leading to a finite transformation with relative simultaneity. This formulation is used in Ref. 8 (see first quote of Sec. 2.2.3) for deriving LT by the same Lie group technique. For that purpose, it was reasonable to use the LT approximation to GT for small velocities. However, this same LT approximation is used by Burde for the BT despite the differences between LT and BT respective underlying worlds.

Ghosal *et al.* (Ref. 29, p. 256ff) highlighted the differences between LT and TT regarding their approximation to GT for small velocities. Given the great repercussion of this matter in what follows, we reproduce his words verbatim (emphasis added):

> "If β^2 is neglected in the Lorentz factor, the LT reduces to the Approximate Lorentz Transformation (ALT)
>
> $$\bar{x} = x - vt, \quad \bar{t} = t - \left(\frac{vx}{c^2}\right)$$
>
> (...) Thus, [this equation] for all v in general, does not represent a Galilean World (GW). Of course one may choose $\beta^2 \ll 1$ (...) and it becomes clear that [this equation] represents a GW approximately. But then there is a subtle point that must be carefully noted. *The resulting GW is not a GW in totality but it is limited by the very approximation.* To exemplify this point, consider *the Tangherlini Transformation* (*TT*), *which represents an Einstein World* (*EW*) *with absolute* (*Galilean*) *synchrony.* (...) *Note here that if* $\beta^2 \ll 1$, *the resulting transformations represent a GT in totality.* Obviously, this fact is absent in [the above equation]. *Thus we have demonstrated that the LT does not lead under the small velocity approximation to Galilean* (*absolute*) *synchrony.*"

In addition, Baierlein (Ref. 30, p. 193) — a reference also cited by Burde — after recalling that the composition of two LTs is an LT, says referring to infinitesimal transformations

> "Consequently, any Lorentz transformation with finite speed can be constructed by iterating a Lorentz transformation with a small (and ultimately infinitesimal) ratio v/c. If the Lorentz transformation for infinitesimal v/c were to reduce to the Galilean transformation, then the iterative process could never generate a finite Lorentz transformation that is radically different from the Galilean transformation. But the finite transformations are indeed radically different, and so — however subtly — the infinitesimal Lorentz transformation must differ significantly from the Galilean transformation."

Accordingly, it seems that Burde's PC formulation that leaves the infinitesimal time generator space-dependent, is biased towards the relative time of LT despite that the hypothesized world is not the isotropic world of STR. The problem, however, is how to translate the above considerations into the general form of the infinitesimal transformations of the Lie technique.

3.2.1. *Exploring the effect of space in the Lie infinitesimal time transformation*

Studying in depth the possibilities of the Lie group technique to obtain anisotropic transformations is beyond the scope of this work. However, we have done a brief exploration of the influence of the stipulated general form of the infinitesimal transformations on the type of transformations finally obtained. As a result, we have observed that the list of variables initially allowed as arguments of such infinitesimal transformation determine — as could be expected — the relativistic effects of the finite transformation obtained — i.e. relative/absolute simultaneity, time contraction/dilation, length contraction/dilation, etc. Therefore, these lists of allowed variables constitute an important degree of freedom in the modeling process.[o]

In this context, we concluded that, for infinitesimal transformations to be compatible with the Galilean world in the terms expressed by Baierlein and Ghosal, we had to prevent any infinitesimal change in time resulting from its exchange with space. This could be achieved by excluding the variable x from the infinitesimal transformation of time so we considered the following alternatives to Burde's PC formulation:

(1) Exclude the variable x as an argument of the infinitesimal transformation of time — i.e. of its infinitesimal generator — in order to avoid the exchange between space and time at each infinitesimal change of the group parameter.
(2) Exclude the variable x from both time and space infinitesimal transformations.[p]

The transformations resulting from the application of Burde's methodology to these alternative specifications, show the following properties (Appendix B shows the transformations):

(1) Both transformations show absolute simultaneity directly from first principles — i.e. without further resynchronization

[o] In Burde's methodology the initial list of arguments of the general form of the infinitesimal transformation is not a list of *candidate* variables from which the relevant variables are eventually selected. The light propagation equation invariance principle implies equating to zero all the monomials that can be constructed with all the variables in the specified list, without exception. If there are too many variables, the method may even lead to infeasible results. That is why the initial specification of variables in the infinitesimal transformations is so important: it determines from the outset the type of transformation we will obtain. The methodology does not act as a variable selection method.
[p] Note that this does not entail that the corresponding finite transformation will be independent from the excluded variable.

(a) Excluding x from the time infinitesimal transformation results in a GT for time (Eq. (B.8)). The transformation from the PF shows length dilation in all directions (Eq. (B.12)).
(b) Excluding x from the infinitesimal transformations of time and space results in a GT for space (Eq. (B.15)). The transformation from the PF shows time dilation and length contraction along transverse directions (Eq. (B.18)).

(2) For both transformations, the transformation from the PF shows $k = \beta$ as anisotropy transformation[q] where β is the velocity of the moving inertial frame relative to the isotropic PF (in which $K = 0$). This anisotropy model is obtained automatically from the first principles by means of the derivation process, without the need for additional anisotropy postulates. This result coincides with the considerations made in the previous sections.
(3) Both transformations enjoy conformal metric invariance with their respective conformal factors inverse to each other (Eqs. (B.10) and (B.17)).

The above results show that an anisotropic world with absolute simultaneity is possible under the Lie group technique. In addition, these results also highlight the modeling freedom of this methodology, which, if not properly controlled, can produce results easily misinterpreted as objective physical impositions.

Consequently, the relative simultaneity shown by the BT is a consequence of the initial stipulations about the desired transformation — list of arguments of its infinitesimal transformations — and the LT bias of the PC formulation used, but not of an intrinsic property of the anisotropic world.

Nonetheless, the obtained absolute simultaneity transformations described above keep showing conformal metric invariance rather than plain metric invariance. Of course, these transformations can be further transformed, by scaling, into metric-invariant transformations using the inverse square root of the conformal factor as scale factor. However, our next important question is whether the property of conformal metric invariance is a necessary property to faithfully represent the anisotropic nature of the assumed world.

3.3. *Anisotropy and conformal invariance*

The conformal metric invariance of BT seems to confirm Burde's arguments in favor of the need to use the invariance of the light propagation equation, rather than metric invariance, as a first principle in an anisotropic scenario. These arguments can be summarized in the following claims (recall the quotes in Sec. 2.2.2):

(1) In an anisotropic world, the symmetry arguments that lead to the invariance of the metric are no longer valid since (Ref. 8, p. 1589) "*as a physical phenomenon it* [anisotropy] *influences all the processes*". Consequently, "*not the invariance*

[q]Remember that the anisotropy parameter k is also a state variable that participates in the transformations.

of the interval but invariance of the equation of light propagation should be a starting point for derivation of the transformations".

(2) The metric invariance property of ε-LT or any other linear transformation obtained by a coordinate change from LT, is a proof of its adherence to STR isotropic world and thus of its incompatibility with an anisotropic one.

(3) Conversely, the fact that the conformal factor of a transformation cannot be made to disappear by using standard coordinates, is a proof of the specificity of conformal transformations for an anisotropic world.

However, the coefficient structure of a metric also determines the symmetry or asymmetry of the measurements of physical quantities, and this prompt us to ask why the metric is not sufficient to reflect anisotropic situations. Looking for example at Eqs. (4) and (5), we easily discern an anisotropic pattern in the quadratic structure of the metric whose effect is precisely to deform the spherical propagation of light by flattening it along its x-axis as a consequence of its slowing down in that direction. Consequently, it is not clear what additional physical significance a scaled metric can provide in this context in which scale and scale invariance play no theoretical role.

Furthermore, these claims seem clearly in tension with our discussion so far: as we have noted in Sec. 3.1, TT is a particular case of ε-LT with $\varepsilon = 1/2$ in the isotropic PF frame S_0 and $\varepsilon' = \frac{1}{2}(k+1)$, with $k = \beta$ in any inertial frame S' moving with velocity β relative to S_0. However, TT is metric invariant, thus we have obtained a metric-invariant transformation under anisotropy — that, of course, also leaves the light propagation equation invariant (isotropic in S_0, and anisotropic with anisotropy $k = \beta$ in S'). It is true that TT does not satisfy the group structure, but its metric invariance property seems to contradict that anisotropy, as a physical phenomenon, affects the metric. On the contrary, it suggests that conformal invariance may not be necessary in this anisotropic scenario.

In order to evaluate this fact, we have applied the Lie group technique as Burde did but replacing the principle of light equation invariance by the principle of metric invariance $ds^2 = dS^2$ with ds^2 and dS^2 defined like in Eqs. (4) and (5). Under these conditions, postulating infinitesimal transformations with the same form as those of BT (Eq. (A.1)), the Lie technique did not provide a feasible solution.

However, the restriction made in the previous section in the infinitesimal transformation of time as a result of the analysis of the relation between anisotropy and simultaneity and its reflection in the PC formulation, produced a metric-invariant transformation that shows that anisotropy and metric invariance are perfectly compatible. In the next section, we give the details of the obtained transformation.

4. A Metric-Invariant Anisotropic Transformation: The Relativistic TT

In this section, we present the transformation obtained by the Lie technique under the same anisotropic world hypothesized by Burde, but: (a) establishing metric invariance instead of light equation invariance as first principle as motivated in the previous section, and (b) excluding the variable x from the infinitesimal time generator in accordance with the rationale in Sec. 3.2.

4.1. *An anisotropic metric-invariant transformation*

Given two inertial frames S and S' under the same initial conditions established in Sec. 2.2.4, p. 8 — standard configuration, S' moving with velocity $\beta = v/c$ relative to S along the x-axis, etc. — the infinitesimal transformations are

$$\begin{cases} ct \approx cT + \tau(T, K)a, \\ x \approx X + \xi(X, T, K)a, \\ y \approx Y + \eta(Y, Z, K)a, \\ z \approx Z + \zeta(Y, Z, K)a, \\ k \approx K + a\chi(K), \end{cases} \tag{15}$$

where the infinitesimal time generator is

$$\tau(T, K) = c\,\frac{\partial t}{\partial a}\bigg|_{a=0} = c\,\frac{\partial T}{\partial a}\bigg|_{a=0}, \tag{16}$$

and, analogously, the other generators ξ, η, ζ and χ are the partials of x, y, z and k, respectively, with respect to the group parameter a, valued at zero.

Following Burde's methodology, we substitute the differential of these transformations in the anisotropic metric that we want to keep invariant

$$ds^2 = c^2dt^2 - 2kcdtdx - (1 - k^2)dx^2 - dy^2 - dz^2, \tag{17}$$

and linearize the function $f(a) = ds^2(a)$ around zero, i.e. we impose $f'(0) = 0$ with boundary condition $f(0) = ds^2(0) = dS^2$ or, equivalently, use $c^2dT^2 = dS^2 + (1 - K^2)dX^2 + dY^2 + dZ^2$ in $f'(0)$. Equating to zero the corresponding monomials, we obtain the infinitesimal generators leading to the following Lie boundary problem:

$$\begin{cases} k' = \chi(k), & \text{with } k(0) = K, \\ ct' = -\dfrac{k\chi}{1 - k^2}ct, & \text{with } ct(0) = cT, \\ x' = \dfrac{k\chi}{1 - k^2}x - \dfrac{\chi}{1 - k^2}ct, & \text{with } x(0) = X, \\ y' = 0, & \text{with } y(0) = Y, \\ z' = 0, & \text{with } z(0) = Z, \end{cases} \tag{18}$$

where $\chi(K)$ is a free function, the prime symbol means derivative with respect to the Lie group parameter a, and it was required (Ref. 6, p. 1601) that the (x, z) and (X, Z) planes coincide at all times excluding rotations in the plane (y, z), i.e. the

y and z generators only depend on their respective variable and k. The solution to this problem is the sought finite transformation

$$\begin{cases} k(a) = K + a\chi, \\ ct(a) = \sqrt{\dfrac{K^2-1}{k^2-1}}cT, \\ x(a) = \sqrt{\dfrac{K^2-1}{k^2-1}}X + \dfrac{1}{\sqrt{(K^2-1)(k^2-1)}}a\chi cT, \\ y(a) = Y; \;\; z(a) = Z. \end{cases} \tag{19}$$

Substituting $a\chi = k(a) - K$ from the k transformation into the x transformation, we obtain

$$x(a) = \frac{1}{\sqrt{(K^2-1)(k^2-1)}}((K^2-1)X + (k-K)cT). \tag{20}$$

Following Burde, in this transformation, we use the condition $x = 0$ for $X = vT = \beta cT$ and we obtain

$$0 = x = \frac{1}{\sqrt{(K^2-1)(k^2-1)}}((K^2-1)\beta + (k-K))cT \tag{21}$$

$$\Rightarrow k(a) - K = (1-K^2)\beta(a) \quad \text{for } k(a)^2 \neq 1, \quad \forall\, a \tag{22}$$

$$\Rightarrow \begin{cases} k(a) = (1-K^2)\beta(a) + K \quad \text{or} \quad \beta(a) = \dfrac{k(a)-K}{1-K^2}, \\ \chi(K) = \chi(k(0)) = \left.\dfrac{dk(a)}{da}\right|_{a=0} = (1-K^2)\beta'(0), \end{cases} \tag{23}$$

for $k(a)^2 \neq 1, \forall\, a$. Using these results in the first equation of (19) we get

$$a = \frac{k(a)-K}{\chi(K)} = \frac{(1-K^2)\beta(a) + K - K}{(1-K^2)\beta'(0)} = \frac{\beta(a)}{\beta'(0)} \propto \beta(a). \tag{24}$$

The final transformation becomes

$$\begin{cases} k = K + a\chi = K + \beta(1-K^2) = K + \Gamma^{-2}\beta, \\ ct = \sqrt{\dfrac{1-k^2}{1-K^2}}cT = \dfrac{\Gamma}{\gamma}cT, \\ x = \sqrt{\dfrac{1-K^2}{1-k^2}}(X - \beta cT) = \dfrac{\gamma}{\Gamma}(X - \beta cT), \\ y = Y; \;\; z = Z, \end{cases} \tag{25}$$

where we have defined $\gamma = (1-k^2)^{-1/2}$ and $\Gamma = (1-K^2)^{-1/2}$.

The transformation obtained is, by construction, metric invariant, enjoys group structure, and satisfies the correspondence principle (Sec. 3.2) showing absolute simultaneity. Note also how the function $k(a)$ ends up fully specified in the derivation, with no need for further stipulations.

For further reference, in what follows, we will refer to this transformation as the AMI transformation — for anisotropic metric-invariant transformation — or simply, AMIT.

4.2. *AMI inverse transformation*

The AMI inverse transformation can be obtained from the above noting that the velocity β' of S relative to S' is

$$\beta' = \frac{dx}{cdt} = \frac{\frac{\gamma}{\Gamma}(dX - \beta cdT)}{\frac{\Gamma}{\gamma}cdT} = -\frac{\gamma^2}{\Gamma^2}\beta. \tag{26}$$

Therefore, the inverse transformation is

$$\begin{cases} K = k + \gamma^{-2}\beta' = k + \gamma^{-2}\left(-\frac{\gamma^2}{\Gamma^2}\beta\right) = k - \Gamma^{-2}\beta, \\ cT = \sqrt{\frac{1-K^2}{1-k^2}}ct = \frac{\gamma}{\Gamma}ct, \\ X = \sqrt{\frac{1-k^2}{1-K^2}}(x - \beta' ct) = \frac{\Gamma}{\gamma}\left(x + \frac{\gamma^2}{\Gamma^2}\beta ct\right), \\ Y = y;\ \ Z = z. \end{cases} \tag{27}$$

4.3. *AMIT velocity-addition formula*

If S_1, S_2 and S_3 are three inertial frames with S_j moving with velocity β_{ij} relative to S_i as seen by the latter, $(x_i^{(j)}, t_i^{(j)})$ is the path of frame j as seen by frame i, and $\gamma_i = (1-k_i^2)^{-1/2}$, then using the AMIT (Eq. (25)) between S_1 and S_2 we get the following formula for addition of velocities:

$$\beta_{23} = \frac{dx_2^{(3)}}{cdt_2^{(3)}} = \frac{\frac{\gamma_2}{\gamma_1}(dx_1^{(3)} - \beta_{12}cdt_1^{(3)})}{\frac{\gamma_1}{\gamma_2}cdt_1^{(3)}} = \left(\frac{\gamma_2}{\gamma_1}\right)^2\beta_{13} - \left(\frac{\gamma_2}{\gamma_1}\right)^2\beta_{12} \tag{28}$$

$$\Rightarrow \beta_{13} = \beta_{12} + \left(\frac{\gamma_1}{\gamma_2}\right)^2\beta_{23}. \tag{29}$$

4.4. *AMIT from/to a PF: The TT*

If, in addition, we assume an isotropic ($K = 0$) PF S_0, and S is an inertial frame moving with velocity β relative to S_0, and we consider the AMIT from S_0 to S, the S anisotropy parameter is

$$k(a) = (1-K^2)\beta(a) + K = \beta(a), \tag{30}$$

and since $\Gamma = 1$, such a transformation (Eq. (25)) becomes

$$\begin{cases} k = \beta & \text{(anisotropy due to velocity)}, \\ ct = \dfrac{cT}{\gamma} & \text{(absolute simultaneity and time dilation)}, \\ x = \gamma(X - \beta cT) & \text{(length contraction)}, \\ y = Y; \;\; z = Z, \end{cases} \tag{31}$$

which we can recognize as the TT.

The inverse of this transformation is obtained from Eq. (25) using the reciprocal velocity (Eq. (26)) $\beta' = -\gamma^2\beta$

$$K = k - \gamma^{-2}\beta' = \beta - \gamma^{-2}(-\gamma^2\beta) = 0, \tag{32}$$

thus

$$\begin{cases} K = 0, \\ cT = \sqrt{\dfrac{1-K^2}{1-k^2}}\, ct = \gamma ct, \\ X = \sqrt{\dfrac{1-k^2}{1-K^2}}(x - \beta' ct) = \gamma^{-1}(x + \gamma^2\beta ct), \\ Y = y; \;\; Z = z. \end{cases} \tag{33}$$

4.5. *AMIT with PF: A relativistic TT*

If we consider any two frames S and S' moving with velocities β and β' relative to the PF S_0, we can think of the respective inverse AMITs from S and S', to S_0 (i.e. two TTs), which are obtained from (27) using the (reciprocal) velocities $-\gamma^2\beta$ and $-\gamma'^2\beta'$ of S_0 relative to S and S', respectively

$$\begin{cases} cT = \gamma ct, \\ X = \gamma^{-1}(x + \gamma^2\beta ct), \\ Y = y; \quad Z = z, \end{cases} ; \quad \begin{cases} cT = \gamma' ct', \\ X = (\gamma')^{-1}(x' + (\gamma')^2\beta' ct'), \\ Y = y'; \;\; Z = z', \end{cases} \tag{34}$$

where as usual $\gamma = (1-\beta^2)^{-1/2}$ and $\gamma' = (1-(\beta')^2)^{-1/2}$. Therefore

$$\left.\begin{cases} \gamma' ct' = \gamma ct, \\ (\gamma')^{-1}(x' + (\gamma')^2\beta' ct') = \gamma^{-1}(x + \gamma^2\beta ct), \\ y' = y; \;\; z' = z, \end{cases}\right| \Rightarrow \begin{cases} ct' = \dfrac{\gamma}{\gamma'} ct, \\ x' = \dfrac{\gamma'}{\gamma}(x - \gamma^2(\beta' - \beta)ct), \\ y' = y; \;\; z' = z. \end{cases} \tag{35}$$

However, this transformation is the AMIT between S and S': the velocity β'_{S} of S' relative to S can be obtained from the fact that β' is the velocity of S' relative to S_0 as follows:

$$\beta'_{\mathsf{S}} = \frac{dx}{cdt} = \frac{\gamma(dX - \beta c dT)}{\dfrac{cdT}{\gamma}} = \gamma^2(\beta' - \beta). \tag{36}$$

Substituting this velocity into Eq. (25) with $\gamma \to \gamma'$ and $\Gamma \to \gamma$ we get the AMIT with the following transformation for k:

$$k' = k + \gamma^{-2}\beta'_{\mathsf{S}} = \beta + \gamma^{-2}\gamma^2(\beta' - \beta) = \beta'. \tag{37}$$

Of course, this result is a particular case of the inverse and transitive properties of the AMIT group structure. Therefore, under the assumption of a PF, the AMIT is a kind of *relativistic TT* (RTT), i.e. a more general TT not restricted to relationships between inertial frames and the PF, but *directly* relating arbitrary inertial frames endowed with Tangherlini coordinate systems.

In addition, the RTT group of transformations is a particular case of ε-LT with $\varepsilon = \frac{1}{2}(k+1)$ and $k = \beta$, and $\varepsilon' = \frac{1}{2}(k'+1)$ with $k' = \beta'$, thus serving as a counterexample to Burde's claim that (metric-invariant) transformations than can be obtained from the LT by a coordinate change are not valid for an anisotropic world.

By construction, the AMIT is a Lie group of metric-invariant transformations which shows that conformal invariance is not necessary for a transformation to leave the anisotropic light propagation equation invariant. Furthermore, the AMIT also qualifies in an important way Burde's claim (Ref. 8, footnote 1, p. 1575) that "*Winnie's ε-LTs do not possess the group properties if $\varepsilon_1 \neq \varepsilon_2$ but, in the case of $\varepsilon_1 = \varepsilon_2$ (...) do form a group*".

Finally, the RTT transformation is relevant not only for a true anisotropic physical world but also for a conventional situation in which a set of inertial systems decide to establish a common simultaneity based on privileging one of them by convention.

Consulting the literature,[r] we have found the RTT transformation for the first time in Eq. (34) of Ref. 28 where Selleri calls it *inertial transformation* and indicates that it "*can be shown to form a group*". However, in pp. 659–660 of Ref. 32, Selleri says

> "Notice however that it is not possible to multiply any two transformations of the set, but only two such that the second velocity of the first one coincides with the first velocity of the second one. For this reason the inertial transformations do not form a group."

[r] Ungar[31] obtained a transformation that looks like AMIT but is not: it has group structure and leaves the metric and an anisotropic propagation of light invariant, yet all inertial frames have the same anisotropy (without PF) — as in the STR all inertial frames are isotropic — in the vein of Burde's 2016 paper[8] which is not relevant here.

Therefore, the above derivation based on the Lie group technique settles this question of the RTT group structure. According to p. 43ff of Ref. 33, the RTT have been later rediscovered by Homem (Ref. 34, p. 6ff) where its inverse transformation is derived.

All these references to the RTT are restricted to motions along the x-axis so in the next section we present its vector form.

4.6. *Vector form of the RTT*

As we have noted above, the RTT can be seen of *transversal* transformation that directly relates two inertial frames by pivoting in the PF. We will use this fact to easily obtain an RTT vector form. We start defining the RTT as follows (Ref. 26, p. 53; Ref. 35, p. 22):

Definition 4.1. If S_0 is a privileged frame, and S, S' are two arbitrary inertial frames with spacetime vectors $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, and absolute velocities $\boldsymbol{\beta}$, $\boldsymbol{\beta}'$, respectively, the RTT $\mathscr{T}_{\boldsymbol{\beta},\boldsymbol{\beta}'}$ is the transformation between two arbitrary inertial frames defined as follows:

$$\boldsymbol{\xi}' = \mathscr{T}_{\boldsymbol{\beta},\boldsymbol{\beta}'}(\boldsymbol{\xi}) = (T_{\boldsymbol{\beta}'} \circ T_{\boldsymbol{\beta}}^{-1})(\boldsymbol{\xi}), \tag{38}$$

where $T_{\boldsymbol{\beta}}$ and $T_{\boldsymbol{\beta}'}$ are the TTs between S_0 and S, and S_0 and S', respectively.

This definition corresponds to the following composition of TTs:

$$\begin{gathered} \mathsf{S} \xrightarrow{T_{\boldsymbol{\beta}}^{-1}} \mathsf{S}_0 \xrightarrow{T_{\boldsymbol{\beta}'}} \mathsf{S}' \\ \boldsymbol{\xi} \longrightarrow \boldsymbol{\xi}_0 \longrightarrow \boldsymbol{\xi}'. \end{gathered} \tag{39}$$

Using this definition, we can obtain the vector form of the RTT from both the TT and its inverse vector forms as obtained in p. 484 of Ref. 36

$$\boldsymbol{\xi}' = (ct'\boldsymbol{r}') = T_{\boldsymbol{\beta}'}(\boldsymbol{\xi}_0) = \begin{pmatrix} (\gamma')^{-1}cT \\ \boldsymbol{R} + \left(\dfrac{\gamma'-1}{(\boldsymbol{\beta}')^2}(\boldsymbol{R}\cdot\boldsymbol{\beta}') - \gamma' cT\right)\boldsymbol{\beta}' \end{pmatrix}, \tag{40}$$

with

$$\boldsymbol{\xi}_0 = (cT\boldsymbol{R}) = T_{\boldsymbol{\beta}}^{-1}(\boldsymbol{\xi}) = \begin{pmatrix} \gamma ct \\ \boldsymbol{r} + \left(\dfrac{\gamma^{-1}-1}{\boldsymbol{\beta}^2}(\boldsymbol{r}\cdot\boldsymbol{\beta}) + \gamma ct\right)\boldsymbol{\beta} \end{pmatrix}, \tag{41}$$

where $\boldsymbol{R}$, $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the spatial vectors of S_0, S and S', respectively.

In order to obtain a closed vector form in terms of the relative velocity of S' with respect to S, recall that a TT is a transformation from S_0 to an inertial frame S which results from the composition of a Lorentz boost $L_{\boldsymbol{\beta}}$ and a synchrony transformation $C_{\boldsymbol{\beta}}$ (e.g. Ref. 14, p. 169ff; Ref. 10) which adopts the simultaneity of the PF S_0

$$\boldsymbol{\xi} = T_{\boldsymbol{\beta}}(\boldsymbol{\xi}_0) = C_{\boldsymbol{\beta}} \circ L_{\boldsymbol{\beta}}(\boldsymbol{\xi}_0) = C_{\boldsymbol{\beta}}(\bar{\boldsymbol{\xi}}), \tag{42}$$

where $\bar{\boldsymbol{\xi}} = L_{\boldsymbol{\beta}}(\xi_0)$ is the state vector of a standard inertial frame with clocks synchronized by Einstein's method, and a synchrony transformation $C_{\boldsymbol{k}}$ transforms coordinates as follows:

$$\begin{cases} ct = c\bar{t} + \boldsymbol{k} \cdot \bar{\boldsymbol{r}}, \\ \boldsymbol{r} = \bar{\boldsymbol{r}}, \end{cases} \tag{43}$$

where $c\bar{t}$ and $\bar{\boldsymbol{r}}$ are standard variables. Accordingly, the RTT is a particular case of the *Generalized LT* (Ref. 10, Eqs. (39) and (40), p. 129) between two inertial frames S and S' in which clocks have been resynchronized with synchrony vectors $\boldsymbol{\kappa} = -\boldsymbol{k} = -\boldsymbol{\beta}$ and $\boldsymbol{\kappa}' = -\boldsymbol{k}' = -\boldsymbol{\beta}'$, respectively (note that Anderson *et al.*'s $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ have a minus sign in the resynchronization coordinate change of Eq. (43))

$$\begin{aligned} cdt' = {} & \tilde{\gamma}(1 - \boldsymbol{k} \cdot \boldsymbol{\beta}'_{\mathsf{S}} + \boldsymbol{k}' \cdot \boldsymbol{\beta}'_{\mathsf{S}})cdt + (\tilde{\gamma}\boldsymbol{\beta}'_{\mathsf{S}} - \boldsymbol{k}') \cdot d\boldsymbol{r} \\ & + (\tilde{\gamma}(1 - \boldsymbol{k} \cdot \boldsymbol{\beta}'_{\mathsf{S}}) - 1)\frac{\boldsymbol{k} \cdot \boldsymbol{\beta}'_{\mathsf{S}}}{(\boldsymbol{\beta}'_{\mathsf{S}})^2}\boldsymbol{\beta}'_{\mathsf{S}} \cdot d\boldsymbol{r} - \tilde{\gamma}\boldsymbol{k} \cdot \boldsymbol{\beta}'_{\mathsf{S}}(\boldsymbol{k} \cdot d\boldsymbol{r}), \end{aligned} \tag{44a}$$

$$d\boldsymbol{r}' = -\tilde{\gamma}\boldsymbol{\beta}'_{\mathsf{S}}cdt + d\boldsymbol{r} + (\tilde{\gamma}(1 - \boldsymbol{k} \cdot \boldsymbol{\beta}'_{\mathsf{S}}) - 1)\frac{\boldsymbol{\beta}'_{\mathsf{S}} \cdot d\boldsymbol{r}}{(\boldsymbol{\beta}'_{\mathsf{S}})^2} + \tilde{\gamma}\boldsymbol{\beta}'_{\mathsf{S}}(\boldsymbol{k} \cdot d\boldsymbol{r}), \tag{44b}$$

where $\tilde{\gamma} = \gamma(1 + \boldsymbol{k} \cdot \boldsymbol{\beta}'_{\mathsf{S}})$ and $\boldsymbol{\beta}'_{\mathsf{S}}$ is the velocity of S' relative to S as seen by the latter with its resynchronized clocks. This velocity is given by Eq. (26) in p. 128 of Ref. 10 in terms of $\tilde{\boldsymbol{v}} \equiv c\boldsymbol{\beta}'_{\mathsf{S}}$

$$\tilde{\boldsymbol{v}} = \frac{\boldsymbol{v}}{1 - \kappa \cdot \dfrac{\boldsymbol{v}}{c}}, \tag{45}$$

where $\boldsymbol{v}$ is the relative velocity of S' with respect to S before resynchronization (i.e. with standard variables in S) which can be obtained by relativistic addition of velocities $\boldsymbol{v} = c\boldsymbol{\beta} \oplus c\boldsymbol{\beta}' \equiv \boldsymbol{v}/c = \boldsymbol{\beta} \oplus \boldsymbol{\beta}'$ with $\oplus$ the relativistic addition. Such a relative velocity can be obtained more easily here noting that it is the reciprocal velocity (as seen by S under the TT from S_0) of the *absolute* velocity $\boldsymbol{\beta}'$ of S'. Using Eq. (40) without *primes* for the TT from S_0 to S, this velocity is

$$\begin{aligned} \boldsymbol{\beta}'_{\mathsf{S}} &= \frac{d\boldsymbol{r}}{cdt} \\ &= \frac{d}{\gamma^{-1}cdT}\left(\boldsymbol{R} + \left(\frac{\gamma - 1}{\boldsymbol{\beta}^2}(\boldsymbol{R} \cdot \boldsymbol{\beta}) - \gamma cT\right)\boldsymbol{\beta}\right) && (46) \\ &= \gamma^2\left(\gamma^{-1}\boldsymbol{\beta}' - \frac{\gamma^{-1} - 1}{\boldsymbol{\beta}^2}(\boldsymbol{\beta}' \cdot \boldsymbol{\beta})\boldsymbol{\beta} - \boldsymbol{\beta}\right). && (47) \end{aligned}$$

This is Eq. (36) generalized to vector form. In particular, if $\mathsf{S}' = \mathsf{S}_0$ (i.e. $\boldsymbol{\beta}' = 0$) we get $\boldsymbol{\beta}'_{\mathsf{S}} = -\gamma^2\boldsymbol{\beta}$ which is the reciprocal velocity under TT, the vector version of Eq. (26) for motion along the x-axis as the anisotropy direction.

Therefore, like the RTT of Sec. 4.5 is a particular case of the ε-LT with anisotropy and motion along the x-axis, the vectorial RTT is a particular case of the vectorial version of the ε-LT, in which ε and ε' are now functions of direction (Ref. 37, p. 790)

$$\varepsilon(\boldsymbol{n}) = \frac{1}{2}(1 + \boldsymbol{k} \cdot \boldsymbol{n}),$$

(and analogously for ε') where $\boldsymbol{n}$ is the direction-determining unitary vector of Eq. (1), and $\boldsymbol{k} = \boldsymbol{\beta}$ and $\boldsymbol{k}' = \boldsymbol{\beta}'$ are the respective anisotropy vectors of the frames.

5. Discussion

The RTT presented in the previous section confirms that Burde's assertions mentioned in the introduction are false. On the contrary

(1) Anisotropy is not incompatible with transformations between inertial frames possessing the property of metric invariance.
(2) Under anisotropy, the RTT satisfies the first principles of group structure, relativity and correspondence, despite being a particular case of ε-LT and thus convertible from and to LT by a change of coordinates.

The fundamental idea behind Burde's initiative is that any transformation between inertial frames, ε-LT in particular, that can be obtained from, and converted back to, LT by a change of coordinates is no more than an LT in disguise and is not suitable for an anisotropic world. In fact (Ref. 8, p. 1591), the ε-LT, "*being converted to the standard variables coincide with the LTs* (...) *and thus show no traces of the anisotropy and, in particular, leave the anisotropic interval* [17] *invariant*".

From a mathematical point of view, the conformal character of BT comes from combining the invariance of the light equation as a first principle with the relative time of Burde's PC formulation — which is due to the inclusion of space as an argument of the infinitesimal transformation for time.[s] If, in these circumstances, the invariance of the equation of light is replaced by metric invariance as a first principle, the problem becomes infeasible. However, if in addition space is eliminated as an argument of the infinitesimal transformation of time (i.e. we formulate PC with absolute rather than relative simultaneity) then the problem becomes feasible and we obtain the AMI transformation as a solution.

From a physics perspective, it is very important to keep in mind two fundamental assumptions of Burde

[s]Since the technique obtains the infinitesimal generators from equating all monomials to zero in the linearization (Sec. 4.1 and p. 1601 of Ref. 6), all postulated arguments of the infinitesimal generators (in particular, space in the time generator) contribute to the equations for obtaining them — the technique does not perform variable selection but all postulated variables actively participate. Thus, the finite time transformation ends up showing relative simultaneity.

(1) **Anisotropy.** The one-way speed of light is anisotropic with respect to any inertial frame except for an isotropic PF.
(2) **Synchronization.** Each frame synchronizes its clocks using this true one-way speed of light.

These two assumptions can be paraphrased in the following terms[t]: if k is the anisotropy parameter of an inertial frame, this frame synchronizes its clocks using what we can call the *natural* or *true* method, i.e. the method using the true one-way speed of light. This method is equivalent to using the following Reichenbach's ε:

$$\varepsilon_k = \frac{1}{2}(k+1) \Leftrightarrow k = 2\varepsilon_k - 1, \tag{48}$$

where $k = \beta$ (see Sec. 3.1) or the following one-way speeds of light:

$$c_+ = \frac{c}{2\varepsilon_k} = \frac{c}{1+k}; \quad c_- = \frac{c}{2(1-\varepsilon_k)} = \frac{c}{1-k}. \tag{49}$$

In line with the *true* character of c_+, c_- and k, we can call ε_k the *natural* or *true* Reichenbach's ε. However, this *true* synchronization leads to TT. In fact, if in the derivation of LT by Einstein in 1905 from $\varepsilon = \varepsilon' = 1/2$ (forgive the anachronism) we replaced the value $\varepsilon' = 1/2$ by $\varepsilon' = \varepsilon_{k'}$, then we would obtain TT. If further we replaced $\varepsilon = 1/2$ by $\varepsilon = \varepsilon_k$ we would obtain RTT. The fact that this same result can also be obtained by nonstandard resynchronization of standard clocks is irrelevant to this. Therefore, one could have directly proposed TT and its relativistic extension, RTT, as a counterexample to Burde's claims without the need to use the Lie group technique. This was actually the initial intention of this work before discovering that RTT could be also obtained by that technique.

Burde could reply that an ε-LT cannot be appropriate for the anisotropic world because it could be converted back to LT by a mere resynchronization of standard clocks without leaving any trace of anisotropy. But this answer requires specifying under which hypothetical world it was made: if the world were isotropic, the RTT would hide this isotropy and the LT would be the *natural* transformation; but if the world were really anisotropic, it is the LT that would hide this anisotropy and the RTT would be the *natural* transformation corresponding to Eq. (49). This shows that, regardless of whether RTT and LT can be converted into each other by a change of coordinates, RTT and LT are logically independent, and ε-LT is as legitimate in the anisotropic scenario as LT is in the isotropic one.

5.1. *Polysemy of a coordinate system: The importance of assumptions*

Although the assumption about the world determines the character (*true* or arbitrary) of a synchronization; under coordinatization freedom, a given synchronization

[t]For the discussion, we assume the simplest case of motion and anisotropy of the inertial frames along their common x-axis.

can be both *true* (in the above sense) in one hypothetical world and arbitrary in another.

In these circumstances, if the assumption about the world is not made explicit enough or goes unnoticed, a kind of polysemy can occur in synchronization (i.e. in Reichenbach's ε) because it can be used with two different meanings: as the *true* synchronization (Eqs. (48) and (49)) or as an arbitrary one.[u] Both meanings are often conflated in the CS literature, suggesting the need to use two parameters: k — and its corresponding ε_k — on the one hand, and any freely chosen ε on the other — the former as an element of the geometry, the latter belonging to the realm of coordinate systems. With this notation, setting $\varepsilon = \varepsilon_k$ means that the *true* synchronization has been selected.

We are now in a position to better understand Burde's rejection of ε-LT as a candidate to represent anisotropic kinematics because it has its origin in this polysemy of synchronization. For example, Burde interprets Winnie's ε-LT as the result of a mere resynchronization of standard clocks (second meaning), while his own approach (Ref. 7, pp. 3–4) "*stands apart from the ample literature on CS and clock synchronization*, [since] *anisotropy is governed by a physical law which is not influenced by changing the synchronization procedure*" (this is the first meaning). However, Winnie's explicit purpose carries the first meaning (Ref. 9, p. 82) (italics in the original)

> "The purpose of this paper, then, is to systematically explore the kinematics of the Special Theory when the assumption of the equality of the speed of light in all directions is *not* made to begin with."

In this context, Burde's view that the conversion from ε-LT to LT by resynchronization leaves no trace of anisotropy (thus invalidating the former to represent anisotropic kinematics), assigns to the resulting LT the ability to reflect the true one-way speed of light, i.e. isotropic, (first meaning), and thus at the same time considers the ε-LT to be the result of arbitrary synchronization (second meaning). Therefore, this assertion tacitly assumes an isotropic world and considers any nonstandard synchronization as arbitrary.[v]

However, there is no justification as to why isotropy has been hypothesized instead of anisotropy. It is true, of course, that in an isotropic world only standard synchrony reflects the existing relative simultaneity and that no nonstandard resynchronization will change that actual isotropy. But this is not the fault of the ε-LT resulting from the resynchronization — as if it were not capable of representing anisotropy — but is due to our underlying isotropy hypothesis. However, here, our main hypothesis is anisotropy, so we can no longer draw conclusions from an isotropic world to an anisotropic one.

[u] Note, however, that synchronization is not completely arbitrary (see Sec. 4 of Ref. 18).
[v] Note that the same argument could be directed against LT under an anisotropic world: the passage from LT to ε-LT leaves no trace of isotropy, so LT is not suitable for an isotropic world.

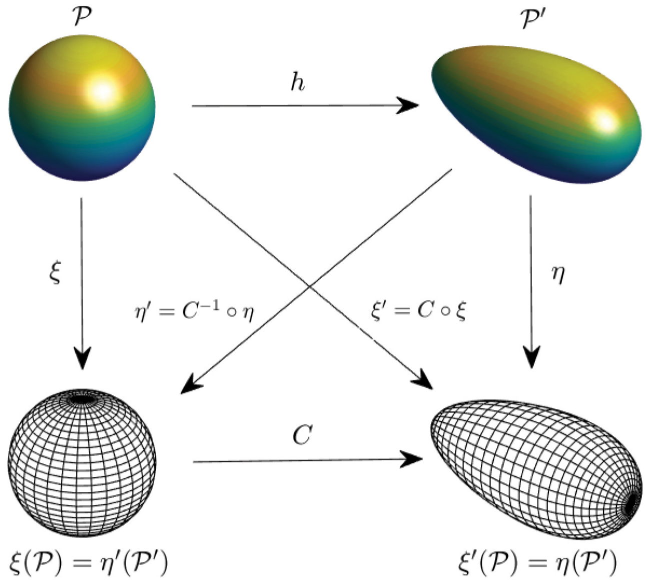

Fig. 1. *Polysemy of coordinate systems.* The lower right coordinate system (e.g. a Tangherlini coordinate system) can be a faithful representation $\boldsymbol{\eta}$ of the upper right perspective P' (e.g. that of a moving observer in an anisotropic world), or an anisotropic representation $\boldsymbol{\xi}'$ of the upper left perspective P (e.g. that of an observer in an isotropic world). Reciprocally, the lower left coordinate system (e.g. a standard one) can be either a faithful representation $\boldsymbol{\xi}$ of the upper left perspective P or a distorted representation $\boldsymbol{\eta}'$ of the upper right perspective P'. Consequently, the result of a change of coordinates C (e.g. a resynchronization) can be interpreted as a new coordinate system of the same observer (*passively*) or as a coordinate system of a different observer (*actively*).

The situation has been metaphorically depicted in Fig. 1. This figure represents the polysemy of any representation (here, a coordinate system): each coordinate system can be interpreted as a faithful representation of the perspective of one observer, but also as a distorted representation of the perspective of a different observer.

The left-hand side of the figure metaphorically represents an isotropic perspective (top) and a *natural* coordinate system defined on it (e.g. standard Einstein coordinates) (bottom). The right-hand side of the figure metaphorically represents an anisotropic perspective and a *natural* coordinate system representing it (e.g. Tangherlini coordinates).

In this context, the lower right representation can be seen as a faithful representation of the upper right perspective (e.g. a moving observer in an anisotropic world) but, at the same time, as a distorted anisotropic representation of the upper left perspective (e.g. an observer in an isotropic world, or the PF in an anisotropic world).

Following Burde's argument, the lower right coordinate system would not be suitable to represent the upper right perspective because it can be converted back to the lower left coordinate system without leaving any trace of anisotropy.

Reciprocally, however, the lower left coordinate system would also be unsuitable for representing the upper left isotropic perspective.

The above considerations show that the only way to dissolve this polysemy of coordinate systems is to make clear our prior assumption about the world. In our anisotropic case, there is a *natural* synchronization by light signals that reflects the *true* simultaneity: that specified by Eqs. (48) and (49) that give rise to the RTT.

5.2. *The CS debate footprint*

Therefore, all these positions that try to privilege LT and STR under CS do so because in essence they are implicitly assuming isotropy and, under such an assumption, any change of coordinates cannot change that hypothesis. This explains, for example, Friedman's words on nonstandard ($\varepsilon \neq 1/2$) simultaneity (Ref. 14, p. 312)

> "[T]his additional structure has no explanatory power and no useful purpose is served by introducing it into Minkowski space-time."

From another perspective, the STR symmetry group is the Lorentz group (e.g. Ref. 38, p. 832), and a resynchronization of clocks of the kind $c\bar{t} = ct + \kappa x$, $\bar{x} = x$, does not belong to such a group. Therefore, such a resynchronization implies abandoning the theory, so we cannot say that a resynchronization of clocks like those leading to the ε-LT, is a mere coordinate change that leads to LT *represented using the "nonstandard" co-ordination.*

Perhaps the consideration of a more general covariant formulation of STR (Ref. 38, pp. 833–834) or viewing this theory as a particular case of GR,[w] led Burde to the same conclusion as many prominent figures of the CS debate,[17] who, after having acknowledged that (Ref. 14, pp. 175–176)

> "Minkowski space-time can be described equally well from the point of view of any coordinate system (...) Therefore, the equivalence of ε-systems and inertial systems in this sense reveals no deep facts about Minkowski space-time and special relativity; rather, it is simply a trivial consequence of general covariance."

[w]In the context of GR, a generalization of equations (48) and (49) are provided by p. 441 of Ref. 18. Since STR spacetime can be seen as a special case of flat spacetime, for which the Riemann–Christoffel curvature tensor vanishes, general covariance makes it possible to consider all possible metrics that preserve the conformal structure of the theory, i.e. $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$ (Greek letters range from 0 to 3 with x^0 the time coordinate and Latin letters range from 1 to 3 indicating the spatial coordinates). In this setting, Havas shows that any metric that satisfies Hilbert's conditions (*ibid.* Eq. (6), p. 438), maintains such a conformal structure. For such a metric, it is satisfied that

$$c(n^i) = \frac{c}{\left(\frac{1}{c}\right) g_{i0}n^i + 1} \quad \text{and} \quad \varepsilon(n^i) = \frac{1}{2}\left(\frac{1}{c}g_{i0}n^i + 1\right),$$

where if $\vec{n}$ goes in the x-direction, we can see Eqs. (49) and (48) with $k = \frac{1}{c}g_{10}n^i = v/c = \beta$.

Subsequently privileged LT over any other transformation obtained from it by a nonstandard resynchronization mainly (Ref. 14, p. 310) on the basis of Malament's arguments.[16] These arguments considered the conformal structure as implicitly isotropic (standard light cone) and then concluded, unsurprisingly, that only standard orthogonal hyperplanes of simultaneity were implicitly definable from the corresponding causal structure[x] (e.g. Ref. 18; Ref. 10, p. 124ff; Ref. 39; Ref. 19, p. 94).

Once accepted that (Ref. 14, p. 176) "*inertial systems* [i.e. standard systems] *and ε-systems are equally good coordinate representations*" of the STR (isotropic) spacetime, it is a simple additional step to consider ε-systems inappropriate to represent an anisotropic world, since, otherwise, different synchronizations (in the first meaning of the previous section) would correspond to different worlds, which jeopardize Minkowski's world under the threat of CS.

Under CS (i.e. under our current ignorance about the one-way speed of light) both coordinatizations and their respective LT and RTT transformations, i.e. both kinematic theories, are equally good to represent our current empirical evidence (the *hard evidence* and the *Selleri's ensemble* of theories mentioned in Ref. 40). However, under anisotropy assumptions, only one theory is correct (recall Fig. 1): the one based on a synchronization with Eqs. (48) and (49) representing true simultaneity which give rise to the RTT.

6. Conclusion

The line of argument followed by Burde throughout his work from the anisotropy hypothesis can be summarized as follows:

(1) Any transformation obtained from LT by a coordinate change is not suitable for an anisotropic world since it can be converted back to LT without leaving any trace of anisotropy.

The ε-LT is obtained from LT by resynchronization, therefore, it is not suitable for an anisotropic world.

(2) Under anisotropy, the invariance of the metric must be replaced by the invariance of the light propagation equation as a starting condition to obtain a transformation between inertial frames. This leads to transformations between inertial frames possessing the property of conformal invariance instead of metric invariance.

The ε-LT enjoys the property of metric invariance so it is not suitable for an anisotropic world.

[x] In this sense, it is paradoxical that Malament's proof would be correct only if the reference observer path O were that of a privileged isotropic frame. In that case, the standard simultaneity Sim_O would be the only one implicitly definable from O and the causal connectability relation associated to its isotropic light cone. However, Sim_O would be privileged also for other inertial observers.

(3) Under anisotropy, the application of the Lie group technique from the first principles of light propagation equation invariance and of correspondence produces a conformal metric-invariant transformation, which corroborates the previous claims.

The analysis of the previous sections and, in particular, the RTT obtained using the Lie group technique, which serves as a counterexample, confirms that all the above statements are false. This line of argument may have its main motivation in the ambiguity of the concept of covariance (reduced or general (Ref. 38, pp. 833–834)) with which one can approach the STR: while the former privileges the classical formulation of the Minkowski metric, the latter includes the covariance of the form of the metric — which encompasses the isotropic and anisotropic cases.

Acknowledgments

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Appendix A. Burde's Transformation

A.1. *The transformation*

BT has the general form given by Eq. (3) with a the parameter of the transformation. The corresponding infinitesimal transformations are

$$\begin{cases} ct \approx cT + \tau(X, T, K)a, \\ x \approx X + \xi(X, T, K)a, \\ y \approx Y + \eta(Y, Z, K)a, \\ z \approx Z + \zeta(Y, Z, K)a, \\ k \approx K + a\chi(K), \end{cases} \tag{A.1}$$

where the infinitesimal generator for time is

$$\tau(X, T, K) = c\,\frac{\partial t}{\partial a}\bigg|_{a=0} = c\,\frac{\partial T}{\partial a}\bigg|_{a=0}, \tag{A.2}$$

and similarly for the other generators ξ, η, ζ and χ. The direct and inverse BT transformations are as follows:

- Direct transformation:

$$\begin{cases} ct = \dfrac{R}{\sqrt{(1 - K\beta)^2 - \beta^2}}[(1 - K\beta - k\beta)cT - ((1 - K^2)\beta + K - k)X], \\ x = \dfrac{R}{\sqrt{(1 - K\beta)^2 - \beta^2}}(X - \beta cT), \\ y = RY; \;\; z = RZ. \end{cases} \tag{A.3}$$

- Inverse transformations:

$$\begin{cases} cT = \dfrac{R^{-1}}{\sqrt{(1-K\beta)^2-\beta^2}}[ct + ((1-K^2)\beta + K - k)x], \\ X = \dfrac{R^{-1}}{\sqrt{(1-K\beta)^2-\beta^2}}[(1-K\beta-k\beta)x + \beta ct], \\ Y = R^{-1}y; \ \ Z = R^{-1}z, \end{cases} \tag{A.4}$$

with $k(a)$ undefined and

$$R = e^{-\int_0^a k(\alpha)d\alpha}, \tag{A.5}$$

$$a = \frac{1}{2}\ln\frac{1+\beta-K\beta}{1-\beta-K\beta}. \tag{A.6}$$

Assuming an isotropic PF and that the anisotropy of the inertial frames depends on their velocity with respect to the PF through the anisotropy model $k = F(\beta) = q\beta$ with q a universal constant to be determined empirically, the above transformations are further specified with

$$k = \frac{q[K + \beta(q-K^2)]}{q + \beta K(1-q)}, \tag{A.7}$$

$$\beta = \frac{\sinh a}{K\sinh a + q\cosh a} \quad \text{with } \beta = \frac{v}{c}, \tag{A.8}$$

$$R = e^{-\int_0^a k(\alpha)d\alpha} = \left(\frac{q^2(1+\beta(1-K))(1-\beta(1+K))}{(q+\beta K(1-q))^2}\right)^{q/2}. \tag{A.9}$$

The conformal relation between metrics is

$$ds^2 = R^2 dS^2. \tag{A.10}$$

A.2. *Transformations to/from the PF*

Transformations to/from the PF ($K = 0$) in which the other frame moves with velocity β relative to the PF

$$R = \left(\frac{q^2(1-\beta^2)}{q^2}\right)^{q/2} = (1-\beta^2)^{q/2} = \gamma^{-q} = \begin{cases} \gamma, & \text{if } q = -1, \\ 1, & \text{if } q = 0, \\ \gamma^{-1/2}, & \text{if } q = \dfrac{1}{2}, \\ \gamma^{-1}, & \text{if } q = 1. \end{cases} \tag{A.11}$$

- If $q = 0$, the LT.

- If $q = 1$

$$\begin{cases} ct = \dfrac{cT}{\gamma^2}, \\ x = X - \beta cT, \\ y = \dfrac{Y}{\gamma}; \;\; z = \dfrac{Z}{\gamma}, \end{cases} \tag{A.12}$$

  i.e. a kind of TT but with greater time dilation, no length contraction in the direction of motion, and length dilation in transversal directions.
- If $q = -1$

$$\begin{cases} ct = \gamma^2(1+\beta^2)cT - 2\beta X, \\ x = \gamma^2(X - \beta cT), \\ y = \gamma Y; \;\; z = \gamma Z, \end{cases} \tag{A.13}$$

  i.e. absolute simultaneity is lost, with length dilation in all directions but greater in the direction of motion.

Appendix B. Light Equation Invariant Anisotropic Lie Transformations with Absolute Simultaneity

We have tried to obtain absolute simultaneity transformations compatible with an anisotropic world with isotropic PF, by replacing the relativistic PC proposed by Burde, with the restriction of the initial list of arguments of the transformation. Specifically, we explored two alternatives, both keeping the same arguments for the y, z and k transformations.

(1) Excluding the x variable from the infinitesimal generator of time (Appendix A, Eq. (A.1)) $\tau(X, T, K) \to \tau(T, K)$ in order to avoid the space and time swapping characteristic of relative simultaneity. The corresponding infinitesimal transformation for time becomes

$$ct \approx cT + a\tau(T, K). \tag{B.1}$$

(2) Excluding the x variable also from the space transformation $\xi(X, T, K) \to \xi(T, K)$. The corresponding infinitesimal transformation for space becomes

$$x \approx X + a\xi(T, K). \tag{B.2}$$

The arguments of the y, z and k transformations remain the same as do their corresponding infinitesimal transformations in (A.1).

B.1. *Excluding the x variable from the time generator*

Substituting (B.1) and the other infinitesimal transformations (A.1) in the light propagation equation (2) linearized in a, using the boundary condition

$c^2dT^2 = (1-K^2)dX^2 + 2KdXdT + dY^2 + dZ^2$ and equating monomials to zero, we determine the infinitesimal generators leading to the following Lie boundary problem:

$$\begin{cases} k' = \chi(K), & \text{with } k(0) = K, \\ ct' = 0, & \text{with } ct(0) = cT, \\ x' = \dfrac{2k\chi}{1-k^2}x - \dfrac{\chi}{1-k^2}ct, & \text{with } x(0) = X, \\ y' = \dfrac{k\chi}{1-k^2}Y, & \text{with } y(0) = Y, \\ z' = \dfrac{k\chi}{1-k^2}Z, & \text{with } z(0) = Z, \end{cases} \tag{B.3}$$

where $\chi(K)$ is a free function and prime means derivative with respect to the Lie group parameter a. The solution to this problem is the sought finite transformation

$$\begin{cases} k = K + a\chi(K), \\ ct = cT, \\ x = \dfrac{1}{1-k^2}((1-K^2)X - a\chi cT), \\ y = \sqrt{\dfrac{1-K^2}{1-k^2}}Y; \;\; z = \sqrt{\dfrac{1-K^2}{1-k^2}}Z. \end{cases} \tag{B.4}$$

Substituting $\chi(K) = \frac{k(a)-K}{a}$ in the x transformation we get

$$\begin{cases} k = K + a\chi(K), \\ ct = cT, \\ x = \dfrac{1-K^2}{1-k^2}X - \dfrac{k-K}{1-k^2}cT, \\ y = \sqrt{\dfrac{1-K^2}{1-k^2}}Y; \;\; z = \sqrt{\dfrac{1-K^2}{1-k^2}}Z. \end{cases} \tag{B.5}$$

We follow Burde's path: using condition $x = 0$ for $X = vT = vT = \beta cT$ in the space transformation we obtain

$$\begin{cases} k(a) = (1-K^2)\beta(a) + K \Leftrightarrow \beta(a) = \dfrac{k(a)-K}{1-K^2}, \\ \chi(K) = \chi(k(0)) = \left.\dfrac{dk(a)}{da}\right|_{a=0} = (1-K^2)\beta'(0), \end{cases} \tag{B.6}$$

for $k(a)^2 \neq 1, \forall\, a$. Thus

$$a = \frac{k(a) - K}{\chi(K)} = \frac{(1-K^2)\beta(a) + K - K}{(1-K^2)\beta'(0)} = \frac{\beta(a)}{\beta'(0)} \propto \beta(a), \tag{B.7}$$

and the transformations become

$$\begin{cases} k = K + \beta(1-K^2), \\ ct = cT, \\ x = \dfrac{1-K^2}{1-k^2}(X - \beta cT), \\ y = \sqrt{\dfrac{1-K^2}{1-k^2}}Y; \quad z = \sqrt{\dfrac{1-K^2}{1-k^2}}Z. \end{cases} \tag{B.8}$$

The inverse transformation uses the velocity β' of S relative to S'. If x is the position of S relative to S', then

$$\beta' = \frac{dx}{cdt} = \frac{\dfrac{1-K^2}{1-k^2}(dX - \beta c dT)}{cdT} = -\frac{1-K^2}{1-k^2}\beta. \tag{B.9}$$

B.1.1. *Conformal invariance*

This transformation satisfies the metric conformal invariance property

$$ds^2 = \Omega^2 dS^2 \quad \text{with } \Omega = \sqrt{\frac{1-K^2}{1-k^2}}, \tag{B.10}$$

where dS^2 and ds^2 are given by (4) and (5), respectively.

B.1.2. *The transformation under a PF ($K = 0$) assumption*

Finally, if we assume an isotropic PF S_0, i.e. $K = 0$, and S is an inertial frame moving relative to S_0 with velocity β, then the anisotropy parameter becomes

$$k(a) = (1-K^2)\beta(a) + K = \beta(a), \tag{B.11}$$

and the transformation from S_0 to S becomes

$$\begin{cases} k = \beta & \text{(anisotropy due to velocity)}, \\ ct = cT & \text{(absolute time)}, \\ x = \gamma^2(X - \beta cT) & \text{(length contraction)}, \\ y = \gamma Y; \quad z = \gamma Z & \text{(length contraction)}, \end{cases} \tag{B.12}$$

with $\gamma = (1-k^2)^{-1/2} = (1-\beta^2)^{-1/2}$ the usual relativistic factor.

B.2. *Excluding the x variable from the time and space generators*

If we postulate time and space generators without x in their arguments, following the above procedure, we obtain the following Lie boundary problem:

$$\begin{cases} k' = \chi(K), & \text{with } k(0) = K, \\ ct' = -\dfrac{2k}{1-k^2}\chi ct, & \text{with } ct(0) = cT, \\ x' = -\dfrac{1}{1-k^2}\chi ct, & \text{with } x(0) = X, \\ y' = -\dfrac{k}{1-k^2}\chi y, & \text{with } y(0) = Y, \\ z' = -\dfrac{k\chi}{1-k^2}Z, & \text{with } z(0) = Z, \end{cases} \tag{B.13}$$

whose solution, after substituting $\chi(K)$ in the space transformation, is the finite transformation

$$\begin{cases} k = K + a\chi(K), \\ ct = \dfrac{1-k^2}{1-K^2}cT, \\ x = X - \dfrac{k-K}{1-K^2}cT, \\ y = \sqrt{\dfrac{1-k^2}{1-K^2}}Y; \;\; z = \sqrt{\dfrac{1-k^2}{1-K^2}}Z. \end{cases} \tag{B.14}$$

The condition $x = 0$ for $X = vT = c\beta T$ produces the same results as in the previous case (Eqs. (B.6) and (B.7)), and the transformation becomes

$$\begin{cases} k = K + \beta(1-K^2), \\ ct = \dfrac{1-k^2}{1-K^2}cT, \\ x = X - \beta cT, \\ y = \sqrt{\dfrac{1-k^2}{1-K^2}}Y; \;\; z = \sqrt{\dfrac{1-k^2}{1-K^2}}Z. \end{cases} \tag{B.15}$$

The inverse transformation uses the velocity β' of S relative to S'. If x tracks the position of S relative to S', then

$$\beta' = \frac{dx}{cdt} = \frac{dX - \beta cdT}{\dfrac{1-k^2}{1-K^2}cdT} = -\frac{1-K^2}{1-k^2}\beta. \tag{B.16}$$

B.2.1. *Conformal invariance*

This transformation satisfies the metric conformal invariance property with an inverse conformal factor Ω^2 to the previous case

$$ds^2 = \Omega^2 dS^2 \quad \text{with } \Omega = \sqrt{\frac{1-k^2}{1-K^2}}, \tag{B.17}$$

where dS^2 and ds^2 are given by (4) and (5), respectively.

B.2.2. *The transformation with a PF ($K = 0$)*

Finally, if we assume an isotropic PF S_0, i.e. $K = 0$, and S is an inertial frame moving relative to S_0 with velocity β, then the anisotropy coefficient becomes again that of Eq. (B.11), and the transformation from the PF becomes

$$\begin{cases} k = \beta & \text{(anisotropy due to velocity)}, \\ ct = \dfrac{cT}{\gamma^2} & \text{(absolute simultaneity and time dilation)}, \\ x = X - \beta cT & \text{(Galilean space transformation)}, \\ y = \gamma Y; \;\; z = \gamma Z & \text{(length contraction)}, \end{cases} \tag{B.18}$$

with $\gamma = (1-k^2)^{-1/2} = (1-\beta^2)^{-1/2}$ the usual relativistic factor.

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